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Uniform Sample Generation in ℓ_p Balls for Probabilistic Robustness Analysis[1]

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Abstract

A number of recent papers focused on probabilistic robustness analysis and design of control systems subject to bounded uncertainty. In this work, we continue this line of research and show how to generate samples uniformly distributed in ℓ_p balls in real and complex vector spaces.

1 Preliminaries

Recently, the robust control community studied several problems related to the so-called *probabilistic approach* for robustness analysis of uncertain control systems. For parametric uncertainty it can be easily shown that the number N of randomly generated samples required to meet prespecified accuracy and confidence is independent of the number of real parameters entering into the control system. This fact is an immediate consequence of the Law of Large Numbers and it is often used in Monte Carlo simulation; see [1, 2]. More generally, if the control system is subject to structured and unstructured uncertainty, real or complex, the same conclusion holds. Therefore, in the standard M-Δ configuration [3], we can say that N is independent of the number and size of the blocks δ_i and Δ_k. Furthermore, bounds on the sample size can be easily computed [4, 5] for probability estimation and worst-case performance problems. A subsequent line of research, aiming towards probabilistic design, utilizes concepts from Learning Theory [6]. A crucial advantage of this setting is the fact that the *problem structure* is used. That is, on the contrary of the Monte Carlo approach where the sample size depends only on the accurary and confidence required, in this framework the sample size is a function of other quantities such as, for example, the Mc Millan degree of the plant and compensator. More precisely, the problem structure is taken into account through a parameter called the VC-dimension; see [7] for further details.

In both Monte Carlo and Learning Theory approaches, however, a crucial problem is to find an algorithm for sample generation in various sets. That is, given a certain multivariate probability density function f with compact support $\mathcal{S}$, a key issue is the generation of N samples within $\mathcal{S}$ distributed according to f. In this paper we provide specialized algorithms for the case when $\mathcal{S}$ is an ℓ_p ball. This case is of interest in robustness analysis and has not been explicitely treated in the existing Monte Carlo literature, see e.g. [12]. As a starting point, the distribution f is assumed to be uniform over $\mathcal{S}$. This assumption is also supported by the fact that the uniform distribution coincides with the worst-case distribution in a certain class [8, 9]. Additional properties of the uniform distribution have been shown in [10]. Extensions of the methods proposed to different distributions are discussed in the conclusions.

We now describe more precisely the problem studied in this paper: Given a real or complex ball of radius $r > 0$

$$B(r) = \{x \in \mathbb{F}^n : \|x\|_p \leq r\}$$

where $\|\cdot\|_p$ is the standard ℓ_p norm, the objective is to generate samples uniformly in $B(r)$.

In Section 2, we introduce some preliminary concepts about probability densities and random sample generation as well as measures of ℓ_p balls. In Section 3, we define ℓ_p radially symmetric random vectors and provide basic results on ℓ_p radial and uniform distributions. In Section 4, we study the case $\mathbb{F} \equiv \mathbb{R}$ and present an algorithm for uniform generation of samples of the random vector $\mathbf{x}$ in ℓ_p balls. The algorithm consists of two main steps. In the first step, we generate samples of the components of $\mathbf{x}$ according to the so-called generalized gamma density [11]. Then, we compute $\mathbf{y} = r\mathbf{w}^{1/n}\mathbf{x}/\|\mathbf{x}\|_p$, where $\mathbf{w}$ is a real random variable uniformly distributed in $[0, 1]$. In Corollary 3.2, we prove that the samples $\mathbf{y}$ generated according to the above algorithm are indeed uniformly distributed in $B(r)$. In Section 5, we study the case $\mathbb{F} \equiv \mathbb{C}$ and we propose a similar algorithm for uniform generation in the complex ball $B(r)$. Finally, in Section 6 we show three examples of sample generation in real and complex ℓ_p balls and, in Section 7 we discuss open problems and extensions to the matrix case.

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2 Definitions and notation

2.1 Probability densities and random sample generation

In this subsection, we define two probability density functions that are used throughout the paper. We also recall a standard method for generating samples according to a given univariate distribution. In the sequel, the notation $\mathbf{x} \sim f$ means that $\mathbf{x}$ is a random element with probability density f.

Gamma density: A random variable $\mathbf{x} \in \mathbb{R}$ is gamma distributed with parameters (a, b), $\mathbf{x} \sim G(a, b)$, when it has density function

$$f_{\mathbf{x}}(x) = \frac{1}{\Gamma(a) b^a} x^{a-1} e^{-x/b}, \quad x \geq 0.$$

Efficient algorithms for the generation of gamma distributed random variables are described for instance in [12] and are available in most statistical packages, such as the Matlab Statistical Toolbox. A related density function is the so-called generalized gamma density [11].

Generalized Gamma density: A random variable $\mathbf{x} \in \mathbb{R}$ is generalized gamma distributed with parameters (a, c), $\mathbf{x} \sim \bar{G}(a, c)$, when it has density function

$$f_{\mathbf{x}}(x) = \frac{c}{\Gamma(a)} x^{ca-1} e^{-x^c}, \quad x \geq 0.$$

If a random generator for $G(a, b)$ is available, a random variable $\mathbf{x} \sim \bar{G}(a, c)$ can be simply obtained as $\mathbf{x} = \mathbf{z}^{1/c}$, where $\mathbf{z} \sim G(a, 1)$. This fact can be easily proved by means of a change of variable, see e.g. [13] for details. A standard method for generating a univariate random variable with a given density function f, or with distribution function F, is the well-known *inversion method* [13, 14] which is stated below.

Inversion method. Let $\mathbf{w} \in \mathbb{R}$ be a random variable with uniform distribution in $[0, 1]$. Let $G : [0, 1] \to \mathbb{R}$ be a function such that its inverse G^{-1} is a continuous distribution function over $\mathbb{R}$, and consider the random variable $\mathbf{z} = G(\mathbf{w})$. Then, $\mathbf{z}$ has distribution function

$$F_{\mathbf{z}}(z) = G^{-1}(z). \tag{1}$$

2.2 Measures of ℓ_p balls

Let $\mathbb{F}$ be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, and let $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$, $p \in [1, \infty]$, be the standard ℓ_p norm. Then,

$$B(r) \doteq \{x \in \mathbb{F}^n : \|x\|_p \leq r\},$$

$$S(r) \doteq \{x \in \mathbb{F}^n : \|x\|_p = r\}$$

are the d-dimensional ℓ_p ball and its boundary, respectively. Clearly, the dimension d of $\mathbb{F}^n$ is n when $\mathbb{F} = \mathbb{R}$ and $2n$ when $\mathbb{F} = \mathbb{C}$. We denote as $\mu[B(r)]$ the d-dimensional measure (volume) of $B(r)$, defined as

$$\mu[B(r)] = \int_{B(r)} dV. \tag{2}$$

Similarly, $\mu[S(r)]$ is the (d-1)-dimensional measure (surface) of $S(r)$, defined as

$$\mu[S(r)] = \int_{S(r)} dV.$$

Clearly, $\mu[B(r)] = \mu[B(1)] r^d$ and $\mu[S(r)] = \mu[S(1)] r^{d-1}$. For radially defined sets as $B(r)$ the volume can be computed using a radially defined infinitesimal element of volume $dV = \nu \mu[S(r)] dr$, where ν is a constant independent of r but function of d and the ℓ_p norm used. Therefore, (2) becomes

$$\mu[B(r)] = \int_0^r \nu \mu[S(1)] \rho^{d-1} d\rho \tag{3}$$

from which it follows that

$$\mu[B(r)] = \nu \mu[S(1)] \frac{r^d}{d}, \quad \mu[B(1)] = \nu \frac{\mu[S(1)]}{d}. \tag{4}$$

3 Radial density functions

In this section we introduce radial symmetric density functions and state some fundamental results for this class of densities.

Definition 3.1 *A random vector $\mathbf{x} \in \mathbb{F}^n$ is ℓ_p radially symmetric if its density function can be written as*

$$f_{\mathbf{x}}(x) = g(r), \quad r = \|x\|_p \tag{5}$$

where $g(r)$ is called the defining function *of $\mathbf{x}$.*

In other words, for radially symmetric random vectors the density function is uniquely determined by its radial shape, which is described by $g(r)$. For given r, the level set of the density function is an equal probability set represented by $S(r)$.

Examples of radial densities for the real case. The classical multivariate normal density [14] with identity covariance matrix and mean value equal to zero is a radial density in ℓ_2 norm. In fact, setting $r = \|x\|_2$, we can write

$$f_{\mathbf{x}}(x) = \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{1}{2} x^T x} = \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{1}{2} r^2} = g(r).$$

The so-called multivariate Laplace density [14] with zero mean value is a radial density in ℓ_1 norm. Indeed, setting $r = \|x\|_1$, we have

$$f_{\mathbf{x}}(x) = \frac{1}{2^n} e^{-\sum_{i=1}^{n} |x_i|} = \frac{1}{2^n} e^{-r} = g(r).$$

For a radially symmetric random vector $\mathbf{x}$ it is of interest to study how the random variable $\mathbf{r} = \|\mathbf{x}\|_p$ is distributed. This is analyzed in the next lemma.

Lemma 3.1 *If $\mathbf{x}$ is ℓ_p radially symmetric, then the random variable $\mathbf{r} = \|\mathbf{x}\|_p$ has density function $f_{\mathbf{r}}(r)$ given by*

$$f_{\mathbf{r}}(r) = \mu[B(1)]r^{d-1}g(r)d \tag{6}$$

where $f_{\mathbf{r}}(r)$ is called the norm density function.

Proof. Notice that

$$F_{\mathbf{r}}(r) = \text{Prob}\{\|x\|_p \leq r\} = \int_{B(r)} f_{\mathbf{x}}(x)dx = \int_{B(r)} g(\|x\|_p)dx.$$

Letting $r = \|x\|_p$ and performing a radial integration as in (3),

$$F_{\mathbf{r}}(r) = \int_0^r \nu\mu[S(1)]\rho^{d-1}g(\rho)d\rho.$$

Differentiating with respect to r and using (4), we obtain (6).

The next lemma relates ℓ_p radially symmetric random vectors and uniform distributions on the level sets $S(r)$.

Lemma 3.2 *If $\mathbf{x} \in \mathbb{F}^n$ is ℓ_p radially symmetric with density function $f_{\mathbf{x}}(x) = g(r)$, $r = \|x\|_p$, then*

a) The conditional density $f_{\mathbf{x}|\mathbf{r}}$ of $\mathbf{x}$ given $\|\mathbf{x}\|_p = \mathbf{r}$ is given by

$$f_{\mathbf{x}|\mathbf{r}}(x|r) = \frac{1}{\mu[S(r)]}. \tag{7}$$

That is, $\mathbf{x}$ is uniformly distributed on $S(r)$.

b) The random vector $\mathbf{y} = \frac{\mathbf{x}}{\|\mathbf{x}\|_p}$ is uniformly distributed on $S(1)$.

Proof. a) If the random vector $\mathbf{x}$ is ℓ_p radially symmetric, then the conditional density is

$$f_{\mathbf{x}|\mathbf{r}}(x|r) = g(r)$$

and, consequently, $f_{\mathbf{x}|\mathbf{r}}$ is constant for given r. Since $f_{\mathbf{x}|\mathbf{r}}$ is a pdf

$$\int_{S(r)} f_{\mathbf{x}|\mathbf{r}}dV = f_{\mathbf{x}|\mathbf{r}} \int_{S(r)} dV = f_{\mathbf{x}|\mathbf{r}}\mu[S(r)] = 1$$

and this proves (7).

b) Clearly, $y = x/\|x\|_p \in S(1)$ for all x and we need to show that $\mathbf{y}$ is uniformly distributed on $S(1)$. Let $f_{\mathbf{y},\mathbf{r}}$ be the joint pdf of $\mathbf{y}$ and $\mathbf{r} = \|\mathbf{x}\|_p$, and $f_{\mathbf{y}|\mathbf{r}}$ the conditional pdf of $\mathbf{y}$ given $\mathbf{r}$, then

$$f_{\mathbf{y},\mathbf{r}}(y,r) = f_{\mathbf{y}|\mathbf{r}}(y|r)f_{\mathbf{r}}(r). \tag{8}$$

Writing $f_{\mathbf{y}|\mathbf{r}}$ as

$$f_{\mathbf{y}|\mathbf{r}}(y|r) = f_{\mathbf{y}|\mathbf{r}}(\frac{x}{\|x\|_p} \,|\, \|x\|_p = r) = f_{\mathbf{x}|\mathbf{r}}(\frac{x}{r} \,|\, \|x\|_p = r),$$

with the change of variable $z = x/r$ and using a)

$$f_{\mathbf{y}|\mathbf{r}}(y|r) = f_{\mathbf{z}|1}(z|\, \|\mathbf{z}\|_p = 1) = \frac{1}{\mu[S(1)]}.$$

Substituting in (8) and using (4) and (6), we obtain

$$f_{\mathbf{y},\mathbf{r}}(y,r) = \frac{1}{\mu[S(1)]}\mu[B(1)]r^{d-1}g(r)d = \nu r^{d-1}g(r).$$

Integrating with respect to r we obtain the marginal density

$$f_{\mathbf{y}}(y) = \int_0^\infty \nu\rho^{d-1}g(\rho)d\rho.$$

Consider the norm density function $f_{\mathbf{r}}(r)$ in (6) and observe that its integral over $[0,\infty]$ is equal to one. Therefore

$$\int_0^\infty \rho^{d-1}g(\rho)d\rho = \frac{1}{\mu[B(1)]d}.$$

From this relation and (4) we finally obtain

$$f_{\mathbf{y}}(y) = \frac{\nu}{\mu[B(1)]d} = \frac{1}{\mu[S(1)]}.$$

We are now ready to state the key result of this section which shows how uniform distributions can be obtained from radially symmetric distributions and vice versa.

Theorem 3.1 *The following two conditions are equivalent:*

a) $\mathbf{x} \in \mathbb{F}^n$ is ℓ_p radially symmetric with norm density function $f_{\mathbf{r}}(r) = r^{d-1}d$, $r \in [0,1]$.

b) $\mathbf{x} \in \mathbb{F}^n$ is uniformly distributed in $B(1)$.

Proof. a)$\rightarrow$b). Since $\mathbf{x}$ is ℓ_p radially symmetric, its norm density is given by (6), then

$$g(r) = \frac{1}{\mu[B(1)]} = f_{\mathbf{x}}(x).$$

Therefore, $\mathbf{x}$ is uniformly distributed in $B(1)$.

b)$\rightarrow$a). Since $\mathbf{x}$ is uniform in $B(1)$ then

$$f_{\mathbf{x}}(x) = \begin{cases} \frac{1}{\mu[B(1)]}, & \text{if } \|x\|_p \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $f_{\mathbf{x}}(x)$ depends only on $\|x\|_p$; therefore $\mathbf{x}$ is ℓ_p radially symmetric with defining function $g(r) = f_{\mathbf{x}}(x)$, $r = \|x\|_p$. Substituting in (6) the claim is proved.

The next corollary shows how to obtain an ℓ_p radial distribution with given defining function, or with given norm density, starting from any arbitrary ℓ_p radial distribution.

Corollary 3.1 *Let $\mathbf{x} \in \mathbb{F}^n$ be ℓ_p radially symmetric and let $\mathbf{z} \in \mathbb{R}^+$ be an independent random variable with density $f_{\mathbf{Z}}(z)$, then*

$$\mathbf{y} = \mathbf{z}\frac{\mathbf{x}}{\|\mathbf{x}\|_p}$$

is ℓ_p radially symmetric and has norm density function $f_{\mathbf{r}}(r) = f_{\mathbf{Z}}(r)$, $r = \|y\|_p$. The defining function $g(r)$ is consequently given by

$$g(r) = \frac{1}{\mu[B(1)]r^{d-1}d} f_{\mathbf{r}}(r). \tag{9}$$

Proof. Clearly, $\mathbf{y}$ is ℓ_p radially symmetric and $\|\mathbf{y}\|_p = \mathbf{z}$. Therefore, the norm density function $f_{\mathbf{r}}(r)$ of $\mathbf{y}$ coincides with the density function $f_{\mathbf{Z}}(z)$ of $\mathbf{z}$. Relation (9) follows immediately from (6).

The previous corollary may be used to generate ℓ_p radially symmetric random vectors with given defining functions. In the next corollary, we specialize this result to uniform distributions.

Corollary 3.2 *Let $\mathbf{x} \in \mathbb{F}^n$ be ℓ_p radially symmetric and let $\mathbf{w} \in \mathbb{R}$ be an independent random variable uniformly distributed in $[0, 1]$. Then,*

$$\mathbf{y} = r\mathbf{z}\frac{\mathbf{x}}{\|\mathbf{x}\|_p}, \quad \mathbf{z} = \mathbf{w}^{1/d}$$

is uniformly distributed in $B(r)$.

Proof. By the inversion method, it follows that the distribution of $\mathbf{z}$ is $F_{\mathbf{Z}}(z) = z^d$, therefore $f_{\mathbf{Z}}(z) = dz^{d-1}$. For $r = 1$, the statement is immediately proved by means of Theorem 3.1. With a rescaling, it is immediate to show that $\mathbf{y}$ is uniformly distributed in $B(r)$.

This result can be interpreted as follows: First, an ℓ_p radially symmetric random vector $\mathbf{x}$ is normalized to obtain a uniform distribution on the surface $S(r)$ of the set $B(r)$, then each sample is forced into $B(r)$ by the *volumetric factor* $\mathbf{z}$. Therefore, the problem of uniform generation is reduced to that of generation of ℓ_p radially symmetric random vectors. This is discussed in the next section.

4 Generation of uniform random real vectors

In this section, we propose an algorithm to generate real vectors $\mathbf{x} \in \mathbb{R}^n$ uniformly distributed in ℓ_p balls. This algorithm is based on the results of the previous sections and, in particular, on Corollary 3.2. Let $\mathbf{x} = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots \mathbf{x}_n]^T$, where $\mathbf{x}_i \in \mathbb{R}$ are independent identically distributed (i.i.d) random variables with density function

$$f_{\mathbf{x}_i}(x_i) = k_R \mathrm{e}^{-|x_i|^p}, \quad k_R = \frac{p}{2\Gamma(\frac{1}{p})}. \tag{10}$$

Notice that the density function (10) is a bilateral generalized gamma density with parameters $(\frac{1}{p}, p)$. In particular, for $p = 1$ equation (10) is the Laplace density and for $p = 2$ it is the well-known normal density with zero mean value and variance equal to 1/2.

Since the $\mathbf{x}_i$'s are independent, the joint density $f_{\mathbf{X}}(x)$ can be written as

$$f_{\mathbf{X}}(x) = \prod_{i=1}^{n} k_R \mathrm{e}^{-|x_i|^p} = k_R^n \mathrm{e}^{-\|x\|_p^p} = g(r), \ r = \|x\|_p.$$

From the above expression, we notice that $\mathbf{x}$ is ℓ_p radially symmetric and therefore Corollary 3.2 can be immediately applied. In addition, since $f_{\mathbf{X}}(x)$ is a density function, its integral is equal to one. If we evaluate this integral using a radially infinitesimal element, we obtain

$$\int_0^\infty k_R^n \nu\mu[S(1)]\rho^{n-1}\mathrm{e}^{-\rho^p} d\rho = k_R^n \nu\mu[S(1)]\frac{\Gamma(\frac{n}{p})}{p} = 1.$$

Next, using (4), a closed form relation for the volume of the ℓ_p ball can be obtained

$$\mu[B(r)] = \mu[B(1)]r^n, \quad \mu[B(1)] = 2^n\frac{\Gamma^n(\frac{1}{p}+1)}{\Gamma(\frac{n}{p}+1)}.$$

This formula may be used for computing the acceptance rate of a rejection method; [12, 13]. To explain this method, one first observes that the set $B(r)$ can be overbounded with the n-dimensional hypercube $\{x \in \mathbb{R}^n : \|x\|_\infty \leq r\}$. A uniform distribution in the hypercube is easily obtained generating independently each component of the vector $\mathbf{x}$ uniformly in $[0, r]$. Finally, one rejects the samples which are outside the ℓ_p ball. Clearly, the acceptance rate R_a is equal to the ratio of the volume of the two balls

$$R_a = \frac{\Gamma^n(\frac{1}{p}+1)}{\Gamma(\frac{n}{p}+1)}.$$

This formula immediately shows the inefficacy of the rejection method for large n, see Table 1. We are now

	$p=1$	$p=1.5$	$p=2$
$n=2$	0.5	0.68	0.79
$n=3$	0.17	0.37	0.52
$n=4$	0.042	0.17	0.31
$n=5$	8.33e-3	6.47e-2	0.16
$n=10$	2.76e-7	1.39e-4	2.49e-3
$n=20$	4.11e-19	8.68e-12	2.46e-8
$n=30$	3.77e-33	1.91e-20	2.04e-14

Table 1: Acceptance rate of the rejection algorithm for the real case

ready to present the algorithm for uniform sample generation in real ℓ_p balls.

4.1 Algorithm for real uniform generation

Given n, p, and r, the algorithm returns a real random vector $\mathbf{y}$ which is uniformly distributed in $B(r)$.

1. Generate n independent random real scalars $\xi_i \sim \bar{G}(\frac{1}{p}, p)$.
2. Construct the vector $\mathbf{x} \in \mathbb{R}^n$ of components $\mathbf{x}_i = s_i \xi_i$, where s_i are independent random signs.
3. Generate $\mathbf{z} = \mathbf{w}^{1/n}$, where $\mathbf{w}$ is a random variable uniformly distributed in the interval $[0, 1]$.
4. Return $\mathbf{y} = r\mathbf{z}\frac{\mathbf{x}}{\|\mathbf{x}\|_p}$.

5 Generation of uniform random complex vectors

In this section, we propose an algorithm based on Corollary 3.2 to generate complex vectors $\mathbf{x} \in \mathbb{C}^n$ uniformly distributed in ℓ_p balls. Let $\mathbf{x} = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots \mathbf{x}_n]^T$, where $\mathbf{x}_i \in \mathbb{C}$. Each component $\mathbf{x}_i$ can be considered as a two-dimensional real vector $\bar{\mathbf{x}}_i \in \mathbb{R}^2$ formed by its real and imaginary parts; the absolute value of $\mathbf{x}_i$ coincides with the ℓ_2 norm of $\bar{\mathbf{x}}_i$. Let each $\bar{\mathbf{x}}_i$ be an i.i.d. ℓ_2 radially symmetric vector with defining function

$$f_{\bar{\mathbf{x}}_i}(\bar{x}_i) = k_C \mathrm{e}^{-r_i^p} = g_i(r_i), \quad r_i = \|\bar{\mathbf{x}}_i\|_2,$$

where $k_C = \frac{p}{2\pi\Gamma(\frac{2}{p})}$. Therefore, using (6), the corresponding norm density function is

$$f_{\mathbf{r}_i}(r_i) = 2\pi k_C r_i \mathrm{e}^{-r_i^p}, \quad r_i \geq 0. \tag{11}$$

The above density is indeed a generalized gamma density with parameters $(\frac{2}{p}, p)$. The density function $f_{\mathbf{X}}(x)$ is the joint density function of the $\mathbf{x}_i$'s. Since the $\mathbf{x}_i$'s are independent

$$f_{\mathbf{X}}(x) = \prod_{i=1}^{n} k_C \mathrm{e}^{-|x_i|^p} = k_C^n \mathrm{e}^{-\|x\|_p^p} = g(r),\ r = \|x\|_p.$$

As in the real case, from the above expression it follows that the random vector $\mathbf{x}$ is ℓ_p radially symmetric and the results of Corollary 3.2 can be applied. Following the same considerations of Section 4, we obtain a closed form relation for the volume of the complex ℓ_p ball

$$\mu[B(r)] = \mu[B(1)]r^{2n}, \ \ \mu[B(1)] = \pi^n \frac{\Gamma^n(\frac{2}{p}+1)}{\Gamma(\frac{2n}{p}+1)}.$$

In this case, a rejection method can be based on the complex hypercube $\{x \in \mathbb{C}^n : \|x\|_\infty \leq r\}$. A random vector $\mathbf{x}$ uniformly distributed in this set is trivially obtained generating independently each component x_i uniformly in the complex disk of radius r. The acceptance rate R_a of this method is given by

$$R_a = \frac{\Gamma^n(\frac{2}{p}+1)}{\Gamma(\frac{2n}{p}+1)}.$$

In Table 2 several values of R_a are reported, showing the inefficiency of the rejection method for large n.

	$p=1$	$p=1.5$	$p=2$
$n=2$	0.17	0.35	0.5
$n=3$	0.011	0.070	0.17
$n=4$	3.97e-4	9.39e-3	4.17e-2
$n=5$	8.82e-6	9.23e-4	8.33e-3
$n=10$	4.21e-16	3.85e-10	2.76e-7
$n=20$	1.29e-42	9.07e-27	4.11e-19
$n=30$	1.29e-73	2.30e-46	3.77e-33

Table 2: Acceptance rate of the rejection algorithm for the complex case.

5.1 Algorithm for complex uniform generation

Given n, p, and r, the algorithm returns a complex random vector $\mathbf{y}$ with uniform distribution in $B(r)$.

1. Generate n independent complex numbers $\xi_i = \mathrm{e}^{j\theta}$, where θ is uniform in $[0, 2\pi]$ (the ξ_i's are uniformly distributed on the complex unit circle.)
2. Construct the vector $\mathbf{x} \in \mathbb{C}^n$ of components $\mathbf{x}_i = \eta_i \xi_i$, where the η_i's are independent random variables $\eta_i \sim \bar{G}(\frac{2}{p}, p)$.
3. Generate $\mathbf{z} = \mathbf{w}^{1/(2n)}$, where $\mathbf{w}$ is uniformly distributed in $[0, 1]$.
4. Return $\mathbf{y} = r\mathbf{z}\frac{\mathbf{x}}{\|\mathbf{x}\|_p}$.

6 Examples

In this section, we show three examples of uniform generation in various sets. For illustrative purposes, we consider the case $n = 2$. In the first example, we take $p = 1.5$ and generate $N = 5{,}000$ samples of real two-dimensional vectors uniformly distributed in $B(1)$, using the algorithm presented in Subsection 4.1. Figure 1 shows the sample generation.

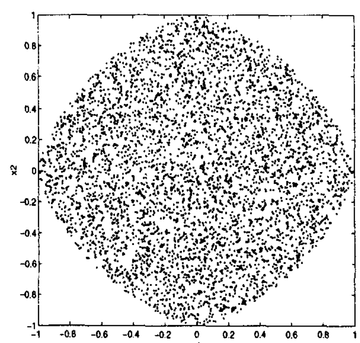

Fig. 1: Two dimensional real vectors uniformly distributed in $B(1)$, for $p = 1.5$.

We remark that the algorithm of Subsection 4.1 can be also used when $p \in (0, 1)$. However, in this case $\|x\|_p^p = \sum_{i=1}^n |x_i|^p$ is not a norm and the set

$$B(r) = \{x \in \mathbb{R}^n : \|x\|_p \leq r\}$$

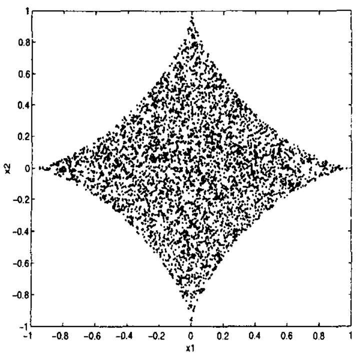

Fig. 2: Two dimensional real vectors uniformly distributed in $B(1)$, for $p = 0.7$.

is not convex. In Figure 2, we show the generation of $N = 5,000$ samples of real two-dimensional vectors uniformly distributed in $B(1)$ for $p = 0.7$.

Finally, we consider the generation of $N = 5,000$ samples of vectors $x \in \mathbb{C}^2$ uniformly distributed in $B(1)$ for $p = 1$. Notice that in this case the samples belong to a linear space of dimension $d = 4$, therefore in Figure 3 we show two-dimensional projections of the samples.

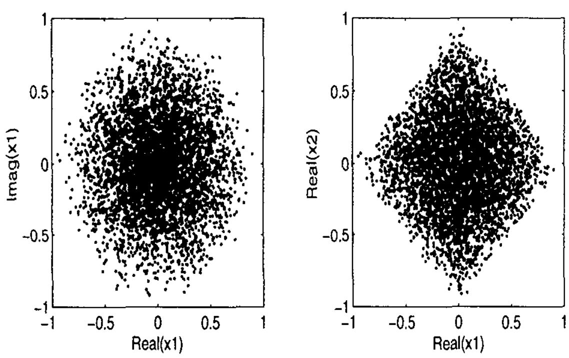

Fig. 3: Two-dimensional projections of complex vectors $x = [x_1, x_2]^T$, $x_1, x_2 \in \mathbb{C}$, uniformly distributed in $B(1)$, for $p = 1$.

7 Conclusions

In this paper, we studied uniform sample generation in the ball $B(r) = \{x \in \mathbb{F}^n : \|x\|_p \leq r\}$ where $\mathbb{F}$ is either the real or the complex field.

We have shown that this task can be performed by means of simple algorithms which use the generalized gamma density. In Corollary 3.2, we proved that the samples generated in this way are indeed uniformly distributed. Also, the basics for generation of ℓ_p radially symmetric random vectors with desired defining function, not necessarly uniform, have been provided. The algorithms can be easily implemented using Corollary 3.1.

A number of open problems will be subject of future research. In particular, the uniform generation of matrix samples in the spectral (singular value) norm ball

$$\{A \in \mathbb{F}^{n,m} : \|A\|_2 \leq r\}$$

is presently an open problem which is of paramount interest when studying, for example, the probabilistic version of the real stability radius problem [15] or when performing probabilistic design of MIMO systems.

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