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Sigma-Algebras on Spaces of Probability Measures

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ABSTRACT. This study is of interest as it relates to the choice of sigma-algebras to be used in non-parametric decision theoretic problems. Three sigma-algebras on the set of all probability measures on a metric space are considered, and the relationships among these are explored. General conditions, which ensure measurability of "well-behaved" parametric families in two of these sigma-algebras, are derived. Necessary conditions are given for a parametric family, with the Borel structure it inherits from the parameter space, to be Borel isomorphic to its image in two of these sigma-algebras. It is also shown that the set of all probability measures on a complete, separable metric space, which are absolutely continuous with respect to a given positive measure on the space, is measurable in these two sigma-algebras, and that the mapping from probability densities in $\mathscr{L}_1$ to the set of absolutely continuous probability measures is a Borel isomorphism.

Key words: sigma-algebras, non-parametric decision theory, parametric families, Borel isomorphism, Bayesian approach

1. Introduction

The use of second-order probability measures in the Bayesian approach requires that probability measures be defined over collections of probability measures on the underlying sample space. Such a collection is usually a parametric family, indexed by parameter values in some parameter space. The parameter space is typically a subset of $\mathscr{R}^n$, and the second-order probability measure is defined in terms of a probability measure on the Borel sets of $\mathscr{R}^n$. In non-parametric Bayesian approaches (Antoniak, 1974; Doksum, 1974; Ferguson, 1973) one seeks to define second-order probability measures which have a broader or "non-parametric" support. In fact, it seems desirable to consider second-order probability measures defined over the entire collection of probability measures on the underlying sample space. This requires specifying a sigma-alegebra for the collection of first-order probability measures. There are several natural choices for this sigma-algebra. This paper explores the relationships among three of these, and addresses questions of measurability for parametric families and absolutely continuous probability measures.

Let $(X, \mathscr{F})$ be a measurable space, and denote by $\mathscr{P}$ the set of all probability measures on $(X, \mathscr{F})$. We shall specify three sigma-algebras on $\mathscr{P}$. For each event $D \in \mathscr{F}$, define ϕ_D: $\mathscr{P} \to [0, 1]$ to be the evaluation map $\phi_D(P) = P(D)$.

1. Let Σ denote the sigma-algebra on $\mathscr{P}$ generated by the maps ϕ_D, for $D \in \mathscr{F}$. Thus Σ is the sigma-algebra generated by sets of the form $\{P \in \mathscr{P} | a \leq P(D) \leq b\}$, for $D \in \mathscr{F}$, and $a, b \in Q$.

2. Let ϱ denote the weak topology on $\mathscr{P}$ generated by the maps ϕ_D, $D \in \mathscr{F}$, namely, the smallest topology on $\mathscr{P}$ for which the evaluation maps are continuous. Let $\mathscr{B}_\varrho$ denote the Borel sets of ϱ.

3. The weak-star topology on $\mathscr{P}$ is the weakest topology such that the maps T_f: $\mathscr{P} \to \mathscr{R}$, defined by $T_f(P) = \int_X f \, dP$, are continuous for all bounded, continuous, real-valued functions f on X. We denote the weak-star topology on $\mathscr{P}$ by τ, and its Borel sets by $\mathscr{B}_\tau$.

According to most Bayesian interpretations, those subsets of $\mathscr{P}$ which one defines to be measurable sets should include those sets of probability measures on $(X, \mathscr{F})$ in which one is capable of harbouring degreees of belief on the basis of experimental evidence. It seems that, at the very least, subsets of $\mathscr{P}$ of the form $\{P \in \mathscr{P} | a \leq P(D) \leq b\}$, where $D \in \mathscr{F}$, $a, b \in Q$, should have this property. Since Σ is the sigma-algebra generated by these sets, it represents what could be considered a "minimal" sigma-algebra for $\mathscr{P}$. Note that Σ is the restriction to $\mathscr{P}$ of the sigma-algebra generated by the Borel cylinders in $[0,1]^{\mathscr{F}}$. (This is the sigma-algebra on which Ferguson's "finitely additive random probabilities" are defined (see Ferguson, 1973).) The weak topology ϱ is of theoretical interest, and so we consider the sigma-algebra which it generates. Since the topology τ is of great importance, particularly in robustness considerations, the sigma-algebra generated by τ is of interest.

In this paper our interest will centre on the case where X is a metric space and $\mathscr{F}$ its collection of Borel sets. Section 2 explores the relationships among Σ, $\mathscr{B}_\varrho$, and $\mathscr{B}_\tau$ in this setting. We show that $\Sigma \subseteq \mathscr{B}_\tau \subseteq \mathscr{B}_\varrho$, that $\Sigma = \mathscr{B}_\tau$ when X is separable, and that, if X is complete and separable, $\mathscr{B}_\tau = \mathscr{B}_\varrho$ if and only if X is countable. Section 3 focuses on how parametric families in $\mathscr{P}$ "fit into" Σ and $\mathscr{B}_\tau$. We give general conditions which ensure that a parametric family is $\mathscr{B}_\tau$ or Σ measurable, and conditions which guarantee that a parametric family is Borel isomorphic to its image in Σ and $\mathscr{B}_\tau$. In Section 4 we show that if X is complete and separable, the set of probability measures absolutely continuous with respect to a given positive measure μ on X is $\mathscr{B}_\tau$ or Σ measurable, and that the subset of $\mathscr{L}_1(\mu)$ corresponding to probability density functions is Borel isomorphic to the absolutely continuous probability measures in $\mathscr{P}$ with the sigma-algebra structure $\mathscr{B}_\tau$ or Σ.

2. Sigma-algebras on $\mathscr{P}$

For the remainder of the paper, X will be a metric space, and $\mathscr{F}$ its collection of Borel sets.

Theorem 2.1
$\mathscr{B}_\tau \subseteq \mathscr{B}_\varrho$.

Proof. The topology τ is generated by taking, as basic neighbourhoods of P, sets of the form

$$\{Q \in \mathscr{P} | Q(F_i) < P(F_i) + \varepsilon,\ i=1, \ldots, k\}$$

where the F_i are closed (Billingsley, 1968, theorem 3, p. 236). But for any $F \in \mathscr{F}$,

$$\{Q \in \mathscr{P} | Q(F) < P(F) + \varepsilon\} = \phi_F^{-1}[(-\infty,\ P(F)+\varepsilon)] \in \varrho,$$

and so any finite intersection of such sets is also in ϱ. □

Theorem 2.2
$\Sigma \subseteq \mathscr{B}_\tau$.

Proof. Recall that if $F \subseteq X$ is any closed set, ϕ_F is $\mathscr{B}_\tau$-measurable: $\{Q \in \mathscr{P} | Q(F) < a\} \in \mathscr{B}_\tau$, all a. Let $\mathscr{D} = \{F \in \mathscr{F} | F \text{ is closed}\}$ and $\mathscr{C} = \{D \in \mathscr{F} | \phi_D \text{ is } \mathscr{B}_\tau \text{ measurable}\}$. Then $X \in \mathscr{D} \subseteq \mathscr{C}$, where $\mathscr{D}$ is closed under finite intersections, and where $\mathscr{C}$ is a Dynkin system (Bauer, 1981). Hence, by the theory of Dynkin systems, $\mathscr{F} = \sigma(\mathscr{D}) \subseteq \mathscr{C}$. □

It seems desirable to identify those X for which $\Sigma = \mathscr{B}_\tau$. We next show that this holds whenever X is separable. Recall that the space of measures over $(X, \mathscr{F})$, namely $\mathscr{P}$, is metrizable with respect to τ as a separable metric space if and only if X is a separable metric space (Parthasarathy, 1967, p. 43).

Theorem 2.3
If X is separable, then $\mathscr{B}_\tau=\Sigma$.

Proof. From theorem 2.2, $\Sigma\subseteq\mathscr{B}_\tau$. We must show that $\tau\subseteq\Sigma$. Since X is a separable metric space, $\mathscr{P}$ is a separable metric space with respect to τ, and so it is Lindelöf. Thus any set in τ is a countable union of basic open sets of the form $\{Q\in\mathscr{P}|Q(F_i)<P(F_i)+\varepsilon, i=1, \dots, k\}$, with F_i closed. But any such set is in Σ, and since Σ is closed under countable unions, $\tau\subseteq\Sigma$. □

Summarizing our results so far, we have that $\Sigma\subseteq\mathscr{B}_\tau\subseteq\mathscr{B}_\varrho$, and that $\Sigma=\mathscr{B}_\tau$ when X is separable. The next result shows that, when X is separable and complete, $\mathscr{B}_\varrho$ is usually bigger than Σ and $\mathscr{B}_\tau$.

Theorem 2.4
Let X be a complete separable metric space. Then $\Sigma=\mathscr{B}_\varrho$ if and only if X is countable.

Proof. Suppose first that $\Sigma=\mathscr{B}_\varrho$. We shall show, by contradiction, that X must be countable. Since $\mathscr{P}$ is metrizable with respect to τ as a separable metric space, $\mathscr{B}_\tau$ is at most of cardinality c (Kuratowski, 1966, p. 347). Hence, by theorem 2.2, the cardinality of Σ is less than or equal to c. Let $\Delta=\{P_x\in\mathscr{P}|x\in X\}$, where P_x denotes the point mass at $\{x\}$, and let $\Lambda_x=\{P\in\mathscr{P}|P(\{x\})>\frac{1}{2}\}$. Consider the relative topology on Δ induced by ϱ. Note that $\Lambda_x\in\varrho$ for each x, and that $\Delta\cap\Lambda_x=\{P_x\}$ for each $x\in X$. It follows that the relative topology on Δ is discrete. Thus $|\mathscr{B}_\varrho|\geq|\varrho|\geq 2^\Delta=2^{|X|}$. Suppose to the contrary that X is not countable. But then, since X is complete, it follows that $|X|\geq c$ (Kuratowski, 1966, corollary 1, p. 445), so that $|\mathscr{B}_\varrho|\geq 2^{|X|}\geq 2^c>c$. But this contradicts the assumption that $\Sigma=\mathscr{B}_\varrho$ and the fact that the cardinality of Σ is at most c.

Suppose now that X is countable. Let $\mathscr{C}$ denote the collection of all finite intersections of sets of the form $\phi_{\{x\}}^{-1}[(a, b)]$, where $x\in X$, a, $b\in Q$. We shall show that $\mathscr{C}$ is a base for ϱ.

The subbasic open sets which generate ϱ are of the form $\phi_D^{-1}(O)$, where $O\subseteq\mathscr{R}$ is open, and $D\in\mathscr{F}$. A basic open set for ϱ is a finite intersection of such sets. Clearly $\mathscr{C}\subseteq\varrho$. Consider any basic ϱ open set, call it G, and let P be an element of G. The set G is a finite intersection of sets of the form $\phi_D^{-1}(O)$. We shall show that, for any $P\in\mathscr{P}$ and any set of the form $\phi_D^{-1}(O)$ containing P, there is a set in $\mathscr{C}$ containing P and contained in $\phi_D^{-1}(O)$. From this, it follows that $\mathscr{C}$ is a base for ϱ.

Accordingly, let $P\in\phi_D^{-1}(O)$, where $O\subseteq\mathscr{R}$ is open and $D\in\mathscr{F}$. Since O is open, there exist a, $b\in Q$ such that $P\in\phi_D^{-1}[(a, b)]$ and $(a, b)\subseteq O$. Let $\varepsilon=\frac{1}{2}\min\{P(D)-a, b-P(D)\}$. Note that, since $P(D)\in(a, b)$, we have $\varepsilon>0$. Since X is countable, there exists an integer n, and n elements $x_1, \dots, x_n\in D$, such that $P(D)-P(\{x_1, \dots, x_n\})<\varepsilon$. Since $P(D)<b-\varepsilon$, $P(D^c)>1-b+\varepsilon$, and it follows that there exists an integer m, and m elements $y_1, \dots, y_m\in D^c$, such that $P(\{y_1, \dots, y_m\})>1-b+\varepsilon$. Choose $r_1, \dots, r_n, s_1, \dots, s_m\in Q$ such that $P(\{x_i\})-\varepsilon/n<r_i<P(\{x_i\})$, $i-1, \dots, n$, and $P(\{y_j\})-\varepsilon/m<s_j<P(\{y_j\})$, $j=1, \dots, m$. Choose $t\in Q$ such that $t>1$. Define

$$C=\left(\bigcap_{i=1}^{n}\phi_{\{x_i\}}^{-1}[(r_i, t)]\right)\cap\left(\bigcap_{j=1}^{m}\phi_{\{y_j\}}^{-1}[(s_j, t)]\right).$$

Then $P\in C\in\mathscr{C}$. If $Q\in C$, then

$$Q(D)\geq\sum_{i=1}^{n}Q(\{x_i\})>\sum_{i=1}^{n}r_i>P(\{x_i, \dots, x_n\})-\varepsilon>P(D)-2\varepsilon\geq a,$$

and

$$Q(D^c) \geq \sum_{j=1}^{m} Q(\{y_j\}) > \sum_{j=1}^{m} s_j > P(\{y_1, \ldots, y_m\}) - \varepsilon > 1 - b,$$

so that $Q(D) < b$. Thus $C \subseteq \phi_D^{-1}[(a, b)] \subseteq \phi_D^{-1}(O)$.

Thus $\mathscr{C}$ is a base for ϱ. Note that $\mathscr{C} \subseteq \Sigma$. Also, $\mathscr{C}$ is countable. Thus any set in ϱ is a countable union of sets in Σ. Since $\Sigma \subseteq \mathscr{B}_\varrho$ (theorems 2.1 and 2.2), we obtain $\Sigma = \mathscr{B}_\varrho$. □

Note that the assumption of completeness in the hypothesis of theorem 2.4 is unnecessary if one assumes the continuum hypothesis.

3. Necessary conditions for Σ and $\mathscr{B}_\tau$ measurability

Consider for a moment X equal to $\mathscr{R}^n$, with $\mathscr{F}$ equal to the Borel subsets of $\mathscr{R}^n$. Then $\mathscr{P}$ is the collection of all probability measures on this measurable space. Bayesians often entertain second-order probability measures which are supported on families of probability measures on $\mathscr{R}^n$, namely subsets of $\mathscr{P}$. Usually, these second-order probability measures are defined on a parameter space, which is in bijective correspondence with the set of probability measures of interest. Thus, the parameter space and its Borel subsets comprise the measurable space on which the second-order probability measure is actually defined; the bijective correspondence defines a sigma-algebra on the "parametric family" of probability measures, and defines the second-order probability measure on the parametric family. From the viewpoint of unifying the Bayesian theory, it would be very appealing if such measurable spaces were Borel isomorphic to Σ or $\mathscr{B}_\tau$ measurable subsets of $\mathscr{P}$.

This is the problem addressed in this section. The following theorems give conditions, which apply in very general settings, ensuring that "well-behaved" parametric families are Σ or $\mathscr{B}_\tau$ measurable. Theorem 3.2 describes conditions under which parametric families are Borel isomorphic to Σ or $\mathscr{B}_\tau$ measurable subsets of $\mathscr{P}$.

In the following theorems, Θ represents a parameter space, μ is a measure on $(X, \mathscr{F})$, and ψ is a map which assigns a probability density function (with respect to μ) to each $\theta \in \Theta$.

Theorem 3.1

Let Θ be a σ-compact, first countable topological space, let μ be a positive measure on $(X, \mathscr{F})$, where X is an arbitrary metric space, and let ψ be a mapping of Θ into the real-valued measurable functions on $(X, \mathscr{F})$ which satisfies:

(1) *for every $\theta \in \Theta$, $\psi(\theta) \geq 0$ and $\int_X \psi(\theta)\, d\mu = 1$;*
(2) *for fixed $x \in X$, the map $\theta \rightarrow [\psi(\theta)](x)$ is continuous;*
(3) *for every $\theta_0 \in \Theta$, there is a neighbourhood V of θ_0 such that $\sup_{\theta \in V} \psi(\theta) \in \mathscr{L}^1(\mu)$.*

Define $P_\theta \in \mathscr{P}$ by $P_\theta(D) = \int_D \psi(\theta)\, d\mu$ for $D \in \mathscr{F}$. Then $\{P_\theta \in \mathscr{P} \mid \theta \in \Theta\} \in \mathscr{B}_\tau$.

Proof. We will show that the function $\Phi\colon \Theta \rightarrow \mathscr{P}$ defined by $\Phi(\theta) = P_\theta$ is τ-continuous. Since the continuous image of a σ-compact set is σ-compact, and since $\mathscr{P}$ is a Hausdorff space (Varadarajan, 1965, theorem 1, p. 181), and compact sets in Hausdorff spaces are closed, it will follow that the image of Θ under Φ is an F_σ, so that $\Phi(\Theta) = \{P_\theta \in \mathscr{P} \mid \theta \in \Theta\}$ is in $\mathscr{B}_\tau$.

Let $\theta_n \rightarrow \theta_0$. We must show that $P_{\theta_n} \rightarrow P_{\theta_0}$ in the weak-star topology, namely, that $\int_X f\, dP_{\theta_n} \rightarrow \int_X f\, dP_{\theta_0}$ for every bounded, continuous, real-valued function f on X. Now $\int_X f\, dP_{\theta_n} = \int_X f\psi(\theta_n)\, d\mu$. By condition (2), $[f\psi(\theta_n)](x) \rightarrow [f\psi(\theta_0)](x)$, for all x. Also, by condition (3) and the fact that f is bounded, there is a neighbourhood V of θ_0 where, for n sufficiently large,

$$|[f\cdot\psi(\theta_n)](x)|\leq|[f\cdot\sup_{\theta\in V}\psi(\theta)](x)|$$

and

$$f\cdot\sup_{\theta\in V}\psi(\theta)\in\mathscr{L}^1(\mu).$$

It follows by the Dominated Convergence Theorem that $\int_X f\psi(\theta_n)\,d\mu\to\int_X f\psi(\theta_0)\,d\mu=\int_X f\,dP_{\theta_0}$. □

Theorem 3.2

Let X be a complete separable metric space, let Θ be a Borel subset of a complete separable metric space, let μ be a positive measure on $(X,\mathscr{F})$, and let ψ be a mapping of Θ into the real-valued measurable functions on $(X,\mathscr{F})$ which satisfies conditions (1), (2), *and* (3) *in theorem* 3.1, *and in addition:*

(4) *$\theta\to P_\theta$ is one-to-one, where $P_\theta\in\mathscr{P}$ is defined by*

$$P_\theta(D)=\int_D\psi(\theta)\,d\mu,\qquad \text{for } D\in\mathscr{F}.$$

Define Φ: $\Theta\to\mathscr{P}$ by $\Phi(\theta)=P_\theta$. Then $\Phi(\Theta)\in\Sigma$ and Φ is a Borel isomorphism (see Arveson, 1976, p. 69, for a definition) of Θ onto $\Phi(\Theta)$, endowed with the Borel structure inherited from Σ.

Proof. We shall prove this theorem by applying a generalization of Souslin's theorem (Arveson, 1976, p. 70) which states the following: If X is a standard Borel space, Q a Polish space, and f a one-to-one Borel map of X into Q, then $f(X)$ is a Borel set in Q and f is an isomorphism of X onto $f(X)$. Now a *Polish space* is a topological space which is homomorphic to a complete separable metric space, and a *standard Borel space* is a Borel space which is isomorphic to a Borel subset of a Polish space.

Since X is separable, $\mathscr{B}_\tau=\Sigma$. The map Φ is τ-continuous by the argument presented in the proof of theorem 3.1 and so Φ is a Borel map with respect to $\mathscr{B}_\tau$, and hence Σ. Since Θ is a standard Borel space, $\mathscr{P}$ a Polish space with respect to the metric induced by τ (Parthasarathy, 1967, theorem 6.2, p. 43, and theorem 6.5, p. 46), and Φ a one-to-one Borel map of Θ into $\mathscr{P}$, it follows that $\Phi(\Theta)$ is Σ measurable and that Φ is a Borel isomorphism of Θ onto $\Phi(\Theta)$ with the Borel structure it inherits from $\mathscr{B}_\tau$, and hence Σ. □

We close with an example. Let $(X,\mathscr{F})=\{\mathscr{R},\mathscr{B}(\mathscr{R})\}$, and let $\mathscr{N}\subseteq\mathscr{P}$ denote the set of normal probability measures on $\mathscr{R}$, namely, $\mathscr{N}=\{P_{\mu,\sigma}\in\mathscr{P}|P_{\mu,\sigma}\sim N(\mu,\sigma^2)\}$. Take $\Theta=\mathscr{R}\times\mathscr{R}^+$, and define Φ: $\Theta\to\mathscr{P}$ by $\Phi(\mu,\sigma)=P_{\mu,\sigma}$. Then by theorem 3.2, $\mathscr{N}\in\Sigma$ and Φ is a Borel isomorphism of Θ on to $\mathscr{N}$ (with respect to Σ). Thus, a second-order probability measure supported on the measurable space on $\mathscr{N}$ determined by Θ and its Borel sets is equivalent to a second-order probability measure defined on $(\mathscr{P},\Sigma)$ and supported on $\mathscr{N}$.

One can check that theorem 3.2 applies for most of the common parametric families. Thus any second-order probability measure for such a family which is defined in the usual way (using a parameter space), is equivalent to a probability measure defined on $(\mathscr{P},\Sigma)$ and supported on the set consisting of the probability measures in the parametric family.

4. The set of absolutely continuous probability measures

In this section we study the set of all probability measures which are absolutely continuous with respect to a positive measure μ on a complete separable metric space $(X, \mathscr{F})$. We show that this set is measurable with respect to $\mathscr{B}_\tau$, or equivalently Σ. As usual, we denote by $\mathscr{L}_1(\mu)$ the class of all real valued functions on X which are integrable with respect to μ, identifying as equivalent functions which are equal μ almost everywhere, and consider it as a metric space with metric induced in the usual way by the norm $\|\cdot\|$.

We show that the subset of "probability density functions" in $\mathscr{L}_1(\mu)$, with the Borel structure this subset inherits from the sigma-algebra on $\mathscr{L}_1(\mu)$ generated by the usual metric, is Borel isomorphic to the subset of $\mathscr{P}$ consisting of the probability measures absolutely continuous with respect to μ, with the Borel structure on this subset induced by $\mathscr{B}_\tau$, or equivalently Σ.

Theorem 4.1

Let X be a complete separable metric space, let μ be a positive measure on $(X, \mathscr{F})$, let $\mathscr{N} = \{P \in \mathscr{P} | P \ll \mu\}$, and let

$$\Delta = \left\{ f \in \mathscr{L}_1(\mu) \,\middle|\, \int_X f\, d\mu = 1, \qquad f \geqslant 0 \text{ } a.e. \right\}.$$

Define Φ: $\Delta \to \mathscr{P}$ by

$$[\Phi(f)](D) = \int_D f\, d\mu,$$

any $D \in \mathscr{F}$. Then $\mathscr{N} \in \Sigma$ and Φ is a Borel isomorphism from Δ onto $\mathscr{N}$ endowed with the Borel structure inherited from Σ.

Proof. Since X is a complete separable metric space, then $\mathscr{P}$ with the metric induced by τ is also, and so $\mathscr{P}$ with this metric is a Polish space. Also, by the Radon–Nikodym theorem, Φ: $\Delta \to \mathscr{P}$ is one-to-one with $\Phi(\Delta) = \mathscr{N}$. We must show that Δ is a standard Borel space, and that Φ is a Borel map. The conclusion of the theorem will then follow from an application of the generalization of Souslin's theorem mentioned in the proof of theorem 3.2.

We begin by arguing that $\mathscr{L}_1(\mu)$ is Polish. Since Δ, being the intersection of the closed sets $\{f \in \mathscr{L}_1(\mu) | \, \|f\| = 1\}$ and $\{f \in \mathscr{L}_1(\mu) | f \geqslant 0 \, \mu \text{ a.e.}\}$, is closed and hence a Borel set, this will prove that Δ is a standard Borel space.

Since X is separable, a countable base for the topology on X exists, and it generates the collection of Borel sets $\mathscr{F}$. Thus $\mathscr{F}$ is countably generated. The set of all finite linear combinations with rational coefficients of the characteristic functions of the sets in the countable base provides a countable dense subset for $\mathscr{L}_1(\mu)$, and so $\mathscr{L}_1(\mu)$ is separable. Since $\mathscr{L}_1(\mu)$ is complete, it follows that $\mathscr{L}_1(\mu)$ is a Polish space.

It remains to show that Φ is a Borel map. But for any continuous real-valued function h on X, which is absolutely bounded by $M < \infty$, and $f, g \in \Delta$, we have

$$\left| \int_X h\, d\Phi(f) - \int_X h\, d\Phi(g) \right| = \left| \int_X hf\, d\mu - \int_X hg\, d\mu \right| \leqslant \int_X |h| \, |f-g| \, d\mu \leqslant \mathrm{M} \cdot \|f-g\|.$$

Hence Φ is τ-continuous and therefore in particular a Borel map. □

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