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C-Uniform Distribution on Compact Metric Spaces

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1. INTRODUCTION

Let (X, d) denote a compact, arcwise connected, metric space and μ a positive, regular, normalized Borel measure on X. A continuous function $\omega: \mathbb{R}_0^+ \to X$ is said to be C-uniformly distributed with respect to μ (for short C, μ-u.d.) if

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\omega(t))\, dt = \int_X f(x)\, d\mu(x) \tag{1.1}$$

holds for all continuous, real-valued functions f on X.

In the case of the k-dimensional torus $(\mathbb{R}/\mathbb{Z})^k$ this concept was introduced by H. Weyl [9]. (See also [5, 6].) Later E. Hlawka [3] introduced a quantitative measure for C-uniform distribution on $(\mathbb{R}/\mathbb{Z})^k = [0, 1)^k$ with respect to the k-dimensional Lebesgue measure λ_k: the function $\omega: \mathbb{R}_0^+ \to X$ is C, λ_k-u.d. if and only if the discrepancy

$$D_T^*(\omega) := \sup_{x \in [0,1]^k} \left| \frac{1}{T} \int_0^T \chi_{[0,x)}(\omega(t))\, dt - \lambda_k([0, x)) \right| \tag{1.2}$$

tends to 0 if T tends to infinity, $\chi_{[0,x)}$ denotes the characteristic function of the k-dimensional interval $[0, x) = [0, x_1) \times \cdots \times [0, x_k)$ for $x = (x_1, ..., x_k)$. Adapting K. F. Roth's method [7], R. J. Taschner [8] established the following lower bound for $k \geqslant 2$ (c_k denotes a constant depending only on k):

$$D_T^*(\omega) \geqslant c_k \left(\frac{1}{s(T)} \right)^{k/(k-1)} \qquad (\text{for } T \geqslant T_0) \tag{1.3}$$

provided that $\omega(t)$ is continuous and of bounded variation. $s(T)$ denotes the length of the curve $\omega(t)$ $(0 \leqslant t \leqslant T)$.

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In the two-dimensional case this bound is best possible in the following sense. For every given arclength $s_0 \geqslant 1$ there is a continuous function $\omega: [0, T] \to \mathbb{R}^2$ with $s(T) = s_0$ and $D_T^*(\omega) \leqslant cs(T)^{-2}$. In the one-dimensional case such a bound cannot exist. For every real number $s_0 \geqslant 1$ there exists a function $\omega: [0, T] \to \mathbb{R}$ with $s(T) = s_0$ and $D_T^*(\omega) = 0$. This is a consequence of the fact that the linear function $\omega(t) = t$ satisfies $D_n^*(\omega) = 0$ for every positive integer n.

In Section 2 we give a simpler proof of (1.3) and a generalisation to C-uniform distribution on compact, arcwise connected, metric spaces. In Section 3 we show that the C, μ-u.d. functions constitute a meager set with respect to uniform or compact convergence in the space of all continuous functions $\omega: \mathbb{R}_0^+ \to X$.

2. Lower Bounds for the Discrepancy

In this section we consider a metric space (X, d) as in the Introduction; for the Borel measure μ we suppose the following additional property:

> There is a real number $k > 1$ and an absolute constant $\alpha > 0$ such that for every open ball $B(x, r)$ with center $x \in X$ and radius $r > 0$,
>
> $$\mu(B(x, r)) \leqslant \alpha \cdot r^k. \tag{2.1}$$

A system $\mathscr{D}$ of measurable sets $E \subseteq X$ is called a *discrepancy system* if the following condition is satisfied:

> For every open ball $B(x, r)$ with $x \in X$ and $r > 0$ there exist a set $E \in \mathscr{D}$ and a ball $B(x, R)$ such that
>
> $$B(x, r) \subseteq E \subseteq B(x, R)$$
>
> and
>
> $$\frac{R}{r} \leqslant \beta$$
>
> with an absolute constant β. (2.2)

Then for every continuous function $\omega: \mathbb{R}_0^+ \to X$, the discrepancy (with respect to $\mathscr{D}$) can be defined by

$$D_T(\mathscr{D}, \omega) := \sup_{E \in \mathscr{D}} \left| \frac{1}{T} \int_0^1 \chi_E(\omega(t))\, dt - \mu(E) \right|. \tag{2.3}$$

Remark 1. If the linear space generated by the characteristic functions χ_E $(E \in \mathscr{D})$ is dense (with respect to uniform convergence) in the space of all continuous functions f on X, $\lim_{T \to \infty} D_T(\mathscr{D}, \omega) = 0$ implies that $\omega(t)$ is C, μ-u.d. The converse statement is true if there is a real number $m > 0$ with:

> For every $\varepsilon > 0$ and every set $E \in \mathscr{D}$ there exists an open set $V \subseteq X$ and a compact set $K \subseteq X$ with $K \subseteq E \subseteq V$, $\mu(V \setminus K) \leqslant \varepsilon^m$, and
>
> $$d(K, X \setminus E) \geqslant 2\varepsilon \qquad \text{if} \quad K \neq \varnothing$$
>
> and
>
> $$d(E, X \setminus V) \geqslant 2\varepsilon \qquad \text{if} \quad V \neq X,$$
>
> where $d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$. (2.4)

We note that (2.4) is satisfied for all the following examples (cf. Remarks 2, 3, 4, and 5).

In the following we consider a continuous function $\omega : \mathbb{R}_0^+ \to X$ with finite arclength

$$s(T) := \sup_{\mathscr{Z}} \sum_{i=1}^{N-1} d(\omega(t_i), \omega(t_{i+1})),$$

where the supremum is taken over all partitions $\mathscr{Z} : 0 = t_0 < t_1 < \cdots < t_N = T$.

THEOREM 1. *Let $\mathscr{D}$ be a discrepancy system on the compact, arcwise connected metric space (X, d) and μ a positive normalized Borel measure with property* (2.1). *For every continuous function $\omega : \mathbb{R}_0^+ \to X$ with finite arclength $s(T)$ $(T > 0)$ and $\lim_{T \to \infty} s(T) = \infty$ there exist a constant $c_k(\mathscr{D})$ such that*

$$D_T(\mathscr{D}, \omega) \geqslant c_k(\mathscr{D}) \left(\frac{1}{s(T)}\right)^{k/(k-1)} \qquad (\text{for } T \geqslant T_0). \tag{2.5}$$

Proof. As $s(T)$ is continuous in T, a simple argument shows that for every $r > 0$ there exists a subinterval $[a, b] \subseteq [0, T]$ with

$$\omega([a, b]) \subseteq B(\omega(a), r)$$

and

$$b - a \geqslant \frac{T}{[s(T)/r] + 1},$$

where $[x]$ denotes the greatest integer $\leqslant x$. (We only have to subdivide the arc $\omega([0, T])$ into $[s(T)/r]+1$ pieces of length $<r$.) By (2.2) there are balls $B(\omega(a), r)$, $B(\omega(a), R)$ and a set $E \in \mathscr{D}$ such that $B(\omega(a), r) \subseteq E \subseteq B(\omega(a), R)$ with $R/r \leqslant \beta$. Applying (2.1) we obtain

$$\begin{aligned}\left|\frac{1}{T}\int_0^T \chi_E(\omega(t))\,dt - \mu(E)\right| &\geqslant \frac{1}{T}\int_0^T \chi_E(\omega(t))\,dt - \mu(E)\\ &\geqslant \frac{1}{T}\int_0^T \chi_{B(\omega(a),\,r)}(\omega(t))\,dt - \alpha R^k\\ &\geqslant \frac{1}{T}\int_a^b dt - \alpha R^k \geqslant \frac{1}{T}\frac{T}{s(T)/r+1} - \alpha\beta^k r^k.\end{aligned}$$

Choosing $r = (s(T)\,k\alpha\beta^k)^{-1/(k-1)}$ yields

$$D_T(\mathscr{D}, \omega) \geqslant \left|\frac{1}{T}\int_0^T \chi_E(\omega(t))\,dt - \mu(E)\right| \geqslant c_k(\mathscr{D})\left(\frac{1}{s(T)}\right)^{k/(k-1)}$$

for $T \geqslant T_0$, and the proof of Theorem 1 is complete. ∎

Remark 2. In the case of the torus $(\mathbb{R}/\mathbb{Z})^k$ $(k \geqslant 2)$ with Lebesgue measure $\mu = \lambda_k$, the family $\mathscr{D}$ of all k-dimensional intervals is a discrepancy system. Hence we obtain by our theorem

$$D_T(\omega) := D_T(\mathscr{D}, \omega) \geqslant c'_k\left(\frac{1}{s(T)}\right)^{k/(k-1)} \qquad (\text{for } T \geqslant T_0). \tag{2.6}$$

Notice that $4^k \cdot D_T^*(\omega) \geqslant D_T(\omega) \geqslant D_T^*(\omega)$, thus (2.6) yields

$$D_T^*(\omega) \geqslant c_k\left(\frac{1}{s(T)}\right)^{k/(k-1)} \qquad (\text{for } T \geqslant T_0). \tag{2.7}$$

Taschner [8] derived this bound by estimating the L^2-discrepancy, which is smaller than $D_T^*(\omega)$. Furthermore, we note the lower bound (2.6) is also true for other discrepancy systems, e.g., for the system of all cubes or for the system of all balls.

Remark 3. In the case of the k-dimensional sphere S^k $(k \geqslant 2)$ with normalized surface mesure $\mu = \sigma_k$, the system $\mathscr{D}$ of all spherical caps is a discrepancy system. Hence we obtain by our theorem

$$D_T(\mathscr{D}, \omega) \geqslant c'_k\left(\frac{1}{s(T)}\right)^{k/(k-1)} \qquad (\text{for } T \geqslant T_0). \tag{2.8}$$

Remark 4. Let (X, d) be a threefold continuously differentiable surface with positive Gauss–Kronecker curvature of a convex body in $\mathbb{R}^{k+1}$ $(k \geqslant 2)$, where d denotes the inner (geodesic) metric. By arguments similar to those used in the proof of Theorem 4 in [1] it can be shown that (2.1) and (2.2) hold for the normalized surface measure $\mu = \sigma_k$ and the system $\mathscr{D}$ of all caps. Hence (2.8) holds even in this more general case.

Remark 5. Let $X \subseteq \mathbb{R}^k$ $(k \geqslant 2)$ be a compact, arcwise connected set in the Euclidean space $\mathbb{R}^k$ $(k \geqslant 2)$ (e.g., a convex body) with non-empty interior and μ the normalized k-dimensional Lebesgue measure on X. Then the system $\mathscr{D}$ of all balls with center in X is a discrepancy system. Thus we can apply our Theorem 1 in this case, too.

3. Topological Properties

In this section we investigate the topological properties of the family $\mathscr{S}$ of all C, μ-u.d. functions as a subset of the space $\mathscr{C}$ of all continuous functions $\omega: \mathbb{R}_0^+ \to X$ with respect to uniform convergence and compact convegence. (Compact convergence means uniform convergence on every compact interval $K \subseteq \mathbb{R}$.)

First we remark that for every compact, arcwise connected, metric space (X, d) and every positive normalized Borel measure on X there exists a C, μ-u.d. function $\omega(t)$: by [6, p. 183, Theorem 2.2] there exists a sequence $(\omega_n)_{n=1}^{\infty}$ which is uniformly distributed in X with respect to μ. Now define

$$\omega(t) = \begin{cases} \omega_1 & \text{for } 0 \leqslant t \leqslant \frac{3}{2} \\ \omega_n & \text{for } n - 2^{-n} \leqslant t \leqslant n + 1 - 2^{-n} \quad (n \geqslant 2) \\ \kappa_n(t) & \text{for } n - 2^{-n+1} \leqslant t \leqslant n - 2^{-n} \quad (n \geqslant 2), \end{cases} \tag{3.1}$$

where $\kappa_n(t)$ is a continuous curve with $\kappa_n(n - 2^{-n+1}) = \omega_{n-1}$ and $\kappa_n(n - 2^{-n}) = \omega_n$. Hence $\omega: \mathbb{R}_0^+ \to X$ is a continuous function and we have, for an arbitrary continuous function f on X,

$$\left| \frac{1}{[T]} \sum_{n=1}^{[T]} f(\omega_n) - \frac{1}{T} \int_0^T f(\omega(t))\, dt \right| \leqslant \frac{3}{T} \max_{x \in X} |f(x)|.$$

Since $(\omega_n)_{n=1}^{\infty}$ is uniformly distributed with respect to μ, $\omega(t)$ is C, μ-u.d.

If X contains more than one element then there exists a non-constant continuous real-valued function f on X; thus by (1.1) we can find a constant function $\omega(t) = x_0 \in X$ which is not C, μ-u.d. Therefore we always assume in the following that X contains more than one element, i.e., $\varnothing \neq \mathscr{S} \neq \mathscr{C}$.

THEOREM 2. *$\mathscr{S}$ is a closed and nowhere dense subset in the space $\mathscr{C}$ of all continuous functions $\omega: \mathbb{R}_0^+ \to X$ with respect to uniform convergence.*

Proof. Put $\bar{d}(\omega', \omega'') = \sup_{t \in \mathbb{R}_0^+} d(\omega'(t), \omega''(t))$ and let $\omega_n \in \mathscr{S}$ be a sequence of C, μ-u.d. functions such that $\lim_{n\to\infty} \bar{d}(\omega_n, \omega) = 0$ for a function $\omega \in \mathscr{C}$. We have to show that $\omega \in \mathscr{S}$. Let f denote a continuous real-valued function on X; then f is uniformly continuous on X. Let $\varepsilon > 0$ be an arbitrary positive number; then there exists a $\delta(\varepsilon) > 0$ with $|f(x) - f(y)| < \varepsilon$ for $d(x, y) < \delta$. Hence there is a positive integer $N = N(\delta(\varepsilon))$ such that $\bar{d}(\omega_n, \omega) < \delta$ for $n \geqslant N$. Thus we have, for $T > 0$ and $n \geqslant N$,

$$\left| \frac{1}{T} \int_0^T f(\omega(t))\, dt - \frac{1}{T} \int_0^T f(\omega_n(t))\, dt \right| < \varepsilon.$$

Choose $n \geqslant N$. Since $\omega_n(t)$ is C, μ-u.d., we obtain for sufficiently large T

$$\begin{aligned} &\left| \frac{1}{T} \int_0^T f(\omega(t))\, dt - \int_X f\, d\mu \right| \\ &\quad \leqslant \left| \frac{1}{T} \int_0^T f(\omega(t))\, dt - \frac{1}{T} \int_0^T f(\omega_n(t))\, dt \right| \\ &\qquad + \left| \frac{1}{T} \int_0^T f(\omega_n(t))\, dt - \int_X f\, d\mu \right| < 2\varepsilon. \end{aligned}$$

Thus $\mathscr{S}$ is closed.

In the following we will prove that $\mathscr{S}$ has empty interior. We proceed indirectly and assume an open ball $\{\omega: \bar{d}(\omega_0, \omega) < r\} \subseteq \mathscr{S}$ ($r <$ diameter of X). Choose $z \in X$ such that $\mu(B(z, r/4)) > 0$, which is possible since the compact space X can be covered by finitely many open balls $B(x, r/4)$ with radius $r/4$. Take an Urysohn function f with $f(x) = 1$ for $d(x, z) \leqslant r/4$ and $f(x) = 0$ for $d(x, z) \geqslant r/3$; then

$$\int_X f\, d\mu \geqslant \mu\left(B\left(z, \frac{r}{4}\right)\right) > 0. \tag{3.2}$$

First we assume that for every $t \geqslant t_0 > 0$, $\omega_0(t) \in B(z, r/3)$ and $\omega_0(t_0)$ is a point of the boundary of $B(z, r/3)$. Now define $\omega_1(t)$ by $\omega_1(t) = \omega_0(t)$ for $0 \leqslant t \leqslant t_0$ and $\omega_1(t) = \omega_0(t_0)$ for $t \geqslant t_0$. If $\omega_0(t) \in B(z, r/3)$ for every $t \in \mathbb{R}_0^+$, then define $\omega_1(t) = y_0$ with an arbitrary point y_0 contained in the boundary of $B(z, r/3)$. Thus we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\omega_1(t))\, dt = 0, \tag{3.3}$$

which contradicts (3.2), since ω_1 is C, μ-u.d. because of $\bar{d}(\omega_0, \omega_1) < r$.

It remains to consider the case where $\omega_0(t) \in B(z, r/3)$ for $t \in (t_i', t_i'')$ ($i \in \mathbb{N}$, $\lim_{i \to \infty} t_i = \infty$) and $\omega_0(t) \notin B(z, r/3)$ if t is contained in the complement of $\bigcup_{i=1}^{\infty} (t_i', t_i'')$. Now we define

$$\omega_1(t) = \begin{cases} \omega_0(t) & \text{for } t \notin \bigcup_{i=1}^{\infty} (t_i', t_i'') \\ \omega_0(t) & \text{for } t \in (t_i', t_i'') \text{ and } t_i'' - t_i' \leqslant 2^{-i} \\ \omega_0(t_i') & \text{for } t \in (t_i', t_i'' - 2^{-i}] \text{ and } t_i'' - t_i' > 2^{-i} \\ \omega_0(2^i(t_i'' - t_i')\, t + (1 + 2^i(t_i' - t_i''))\, t_i'') & \\ & \text{for } t \in (t_i'' - 2^{-i}, t_i'') \text{ and } t_i'' - t_i' > 2^{-i}. \end{cases}$$

As $\bar{d}(\omega_1, \omega_0) < r$ and (3.3) holds even in this case we have a contradiction to (3.2). Thus there is no open ball contained in $\mathscr{S}$ and the proof of Theorem 2 is established. ∎

Remark 6. The complement of $\mathscr{S}$ is dense in $\mathscr{C}$. Since $\mathscr{S}$ is closed, $\mathscr{S}$ is not dense in $\mathscr{C}$.

THEOREM 3. *$\mathscr{S}$ is an everywhere dense and meager subset in the space of all continuous functions $\omega: \mathbb{R}_0^+ \to X$ with respect to compact convergence.*

Proof. For arbitrary $\omega_0 \in \mathscr{C}$ and $\omega_s \in \mathscr{S}$ and an arbitrary compact interval $[a, b]$ we put

$$\omega_1(t) = \begin{cases} \omega_0(t) & \text{for } 0 \leqslant t \leqslant b \\ \kappa(t) & \text{for } b \leqslant t \leqslant b+1 \\ \omega_s(t) & \text{for } t \geqslant b+1, \end{cases}$$

where $\kappa(t)$ denotes a continuous curve with $\kappa(b) = \omega_0(b)$ and $\kappa(b+1) = \omega_s(b+1)$. Let $f: X \to \mathbb{R}$ denote an arbitrary continuous function. Then we obtain

$$\begin{aligned} &\left| \frac{1}{T} \int_0^T f(\omega_1(t))\, dt - \int_X f\, d\mu \right| \\ &\quad = \left| \frac{1}{T} \int_0^T f(\omega_s(t))\, dt - \int_X f\, d\mu + \frac{1}{T} \int_0^{b+1} (f(\omega_1(t)) - f(\omega_s(t)))\, dt \right| \\ &\quad \leqslant \left| \frac{1}{T} \int_0^T f(\omega_s(t))\, dt - \int_X f\, d\mu \right| + 2\, \frac{b+1}{T} \max_{x \in X} |f(x)|. \end{aligned}$$

As ω_s is C, μ-u.d., it follows immediately that $\mathscr{S}$ is everywhere dense in $\mathscr{C}$.

Now consider a non-constant continuous function $f: X \to \mathbb{R}$. Then there exists a point $y \in X$ with

$$\left| \int_X f \, d\mu - f(y) \right| = c > 0. \tag{3.4}$$

Since the mapping

$$\omega \mapsto \left| \frac{1}{T} \int_0^T f(\omega(t)) \, dt - \int_X f \, d\mu \right|$$

is continuous with respect to compact convergence, the sets

$$S_{m,N} = \bigcap_{T \geqslant N} \left\{ \omega \in \mathscr{C} : \left| \frac{1}{T} \int_0^T f(\omega(t)) \, dt - \int_X f \, d\mu \right| \leqslant \frac{1}{m} \right\}$$

with, m, $N \in \mathbb{N}$ are closed in $\mathscr{C}$. Furthermore, $\mathscr{S} \subseteq \bigcup_{N=1}^{\infty} S_{m,N}$ for every $m \in \mathbb{N}$. In order to prove that $\mathscr{S}$ is meager we proceed indirectly and assume, for some $\omega_0 \in \mathscr{C}$,

$$\{\omega : \sup_{t \in [a,b]} d(\omega(t), \omega_0(t)) < r\} \subseteq S_{m,N} \tag{3.5}$$

with $m > 2/c$. Now we define

$$\omega_1(t) = \begin{cases} \omega_0(t) & \text{for} \quad 0 \leqslant t \leqslant b \\ \kappa(t) & \text{for} \quad b \leqslant t \leqslant b+1 \\ y & \text{for} \quad t \geqslant b+1, \end{cases}$$

where $\kappa(t)$ denotes a continuous curve with $\kappa(b) = \omega_0(b)$ and $\kappa(b+1) = y$. Then we have for sufficiently large T

$$\begin{aligned} &\left| \frac{1}{T} \int_0^T f(\omega_1(t)) \, dt - \int_X f \, d\mu \right| \\ &\quad = \left| \frac{1}{T} \int_0^T f(y) \, dt - \int_X f \, d\mu + \frac{1}{T} \int_0^{b+1} (f(\omega_1(t)) - f(y)) \, dt \right| \\ &\quad \geqslant \left| f(y) - \int_X f \, d\mu \right| - \frac{1}{T} \int_0^{b+1} |f(\omega_1(t)) - f(y)| \, dt \\ &\quad \geqslant c - 2 \frac{b+1}{T} \max_{x \in X} |f(x)| \geqslant \frac{c}{2} > \frac{1}{m} \end{aligned}$$

in contradiction to (3.5). Hence the proof of Theorem 3 is complete. ∎

Remark 7. As $\mathscr{C}$ is a complete metric space with respect to compact convergence, Baire's theorem can be applied. Thus the complement of $\mathscr{S}$ is everywhere dense in $\mathscr{C}$.

Remark 8. The preceding theorems show that, from a topological point of view, there are only few C, μ-u.d. functions. But from a measure theoretical point of view there are many C, μ-u.d. functions. In the case of the torus $(\mathbb{R}/\mathbb{Z})^k$ it is proved (cf. [4, 2]) that almost all functions (in the sense of Wiener measure) are C, μ-u.d. In a forthcoming paper the authors will prove this result in the case of compact, connected, homogeneous manifolds.

Note added in proof. In the case of the k-dimensional sphere the lower bound for the discrepancy in Theorem 1 can be improved by an application of W. Schmidt's integral equation method to

$$D_T \geqslant \left(\frac{1}{s(T)}\right)^{1/2 + 1/(k-1) + \varepsilon}.$$

A detailed proof of this estimate is given in a forthcoming paper by the first author.

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