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Ordering fuzzy sets over the real line: An approach based on decision making under uncertainty

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Abstract: The ordering of fuzzy sets over the real line is approached from the point of view of ordering intervals rather than ordering numbers. First, the maximax and maximin criteria which are commonly used for ordering intervals are expressed in terms of characteristic functions. These criteria and the Hurwicz criterion for decision making under complete ignorance are then reformulated in a manner that allows for their generalization to the fuzzy case. Transitivity is established for these ordering rules. A criterion based on the principle of diminishing marginal utility is also presented.

Keywords: Fuzzy sets; decision criteria; preference criteria; ordering; comparison; transitivity; decision making; uncertainty; utility.

1. Introduction

The problem of ordering fuzzy sets over the real line has received considerable attention in the fuzzy set theory literature. It has been addressed in connection with introducing fuzziness into some special types of classical decision problems (see [1, 11, 19]), and it has also been dealt with independently of any specific application. All the ordering methods which have been proposed possess advantages as well as disadvantages. Most of them represent attempts to order fuzzy sets over the real line in such a manner that the optimally selected set best satisfies an intuitive notion of being the greatest.

The present study has been motivated by the authors' interest in fuzzy hypothesis testing [16, 17]. The ordering problem is approached here as a generalization of decision making under uncertainty. This means that the worth of a decision (payoff), instead of being a crisp number or interval over the real line, is now represented as a fuzzy set over the real line. Hence, given two fuzzy sets over the real line, it is assumed that a clearly defined decision objective is needed in order to decide which of the two precedes or ranks above the other. This also implies that the ordering of fuzzy sets involves the notion of 'preference' rather than the notion of 'greater than'. Furthermore, since preference between alternative courses of actions cannot be decided in an absolute sense, achieving a decision objective requires the use of an appropriate preference or decision criterion. Therefore, the approach developed here for dealing with the fuzzy set ordering problem is based on finding suitable generalizations of classical decision criteria for ordering crisp payoffs.

2. Comparison of sets on the real line

We begin by formulating the comparison of sets over the real line in terms of their characteristic functions. The comparison can be expressed in a number of alternative forms. Some of these relate more closely to traditional decision criteria, and turn out to be more useful for the purpose of comparing fuzzy sets. It is concluded that traditional decision criteria for the crisp (ordinary) case need to be redefined as a preliminary step toward applying them to the fuzzy case.

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2.1. Comparison of two intervals

Given two intervals, $A=[a_1, a_2]$ and $B=[b_1, b_2]$, let these be represented by their respective *characteristic functions*:

$$\mu_A(u)=\begin{cases}1, & a_1 \leqslant u \leqslant a_2,\\ 0, & \text{otherwise,}\end{cases} \tag{1a}$$

$$\mu_B(u)=\begin{cases}1, & b_1 \leqslant u \leqslant b_2,\\ 0, & \text{otherwise.}\end{cases} \tag{1b}$$

In order to arrive at a comparison of A and B in terms of $\mu_A(u)$ and $\mu_B(u)$, we first introduce the two accumulation mappings $f_r(u, k)$ and $f_\ell(u, k)$ from the u-axis to the x-axis. Let

$$x=f_r(u, k)=u+k \quad \text{for all } k \leqslant 0. \tag{2}$$

Thus, if x_0 is any given x-value then, through application of f_r, every u-value which is *greater* than or equal to x_0 gets mapped into x_0 (using a suitable choice of k). Similarly, let

$$x=f_\ell(u, k)=u+k \quad \text{for all } k \geqslant 0. \tag{3}$$

Thus, if x_0 is a given x-value then, through application of f_ℓ every u-value which is *less* than or equal to x_0 gets mapped into x_0.

The use of f_r and f_ℓ now allows straightforward application of the extension principle (cf. [7]) if the characteristic functions μ_A and μ_B are treated as if they were membership functions of fuzzy sets. That is, f_r transforms $\mu_A(u)$ and $\mu_B(u)$ into the characteristic functions of two semi-infinite intervals. These will be called the *right sets* of A and B, respectively, and denoted A_r and B_r. In other words,

$$\mu_{A_r}(x)=\sup_{u+k=x, k \leqslant 0} \mu_A(u) \iff A_r=(-\infty, a_2]. \tag{4}$$

At any x the function $\mu_{A_r}(x)$ represents a supremum of $\mu_A(u)$ over all points $u \geqslant x$. B_r is defined similarly.

f_ℓ also transforms $\mu_A(u)$ and $\mu_B(u)$ into the characteristic functions of two semi-infinite intervals, which will be called the *left sets* of A and B, respectively, and denoted A_ℓ and B_ℓ. Thus,

$$\mu_{A_\ell}(x)=\sup_{u+k=x, k \geqslant 0} \mu_A(u) \iff A_\ell=[a_1, \infty); \tag{5}$$

and similarly for B_ℓ.

It is now possible to select various pairs of indicators for comparing A and B in terms of their right and/or left sets [16]. We consider the following two indicators, each of which results in a comparison that agrees with a traditional decision criterion – the maximin or the maximax.

For the maximax criterion we choose the pair

$$(\mu_{A_r}(b_2), \mu_{B_r}(a_2))$$
$$=\begin{cases}(0,1) \Leftrightarrow b_2>a_2 \Leftrightarrow B>A & (B \text{ preferred to } A),\\ (1,0) \Leftrightarrow b_2<a_2 \Leftrightarrow A>B & (A \text{ preferred to } B),\\ (1,1) \Leftrightarrow a_2=b_2, \text{ equivalence under the maximax,} & \\ (0,0) \text{ is impossible.} & \end{cases} \tag{6}$$

For the maximin criterion, the following applies:

$$(\mu_{A_\ell}(b_1), \mu_{B_\ell}(a_1))$$
$$=\begin{cases}(0,1) \Leftrightarrow b_1<a_1 \Leftrightarrow A>B,\\ (1,0) \Leftrightarrow b_1>a_1 \Leftrightarrow B>A,\\ (1,1) \Leftrightarrow a_1=b_1, \text{ equivalence under the maximin,}\\ (0,0) \text{ is impossible.}\end{cases} \tag{7}$$

Thus, a comparison based either on the left sets only or on the right sets only leads to a decision that is consistent with a standard preference criterion. The same formulation applies also to the comparison of arbitrary closed sets. Unfortunately, this no longer works if A or B or both are fuzzy sets and the membership function is used in place of the characteristic function. This suggests that the standard decision criteria need to be redefined in a manner that permits their generalization to the fuzzy case.

3. Reformulation of some standard decision criteria for ordering intervals over the real line

3.1. Definitions and properties

In order to be able to extend various decision criteria to fuzzy sets, some additional definitions are introduced. First, note that from the definitions of the right and left sets of an interval A in (4) and (5) follows this property:

$$A=A_r \cap A_\ell \iff \mu_A(x)=\mu_{A_r}(x) \wedge \mu_{A_\ell}(x), \tag{8}$$

where $\wedge$ denotes the minimum.

Consider now two intervals, $A=[a_1, a_2]$ and $B=[b_1, b_2]$. Their *maximum*, which we write

$A \bigvee B$, is the interval $[a_1 \vee b_1, a_2 \vee b_2]$, where $\vee$ denotes the ordinary maximum. We define the *right maximum* of A and B, written $A \bigvee_r B$, as the maximum of A_r and B_r. Thus,

$$A \bigvee_r B = A_r \bigvee B_r = (-\infty, a_2] \bigvee (-\infty, b_2] = (-\infty, a_2 \vee b_2]. \tag{9}$$

Similarly, the *left maximum* of A and B, written $A \bigvee_\ell B$, is the maximum of A_ℓ and B_ℓ:

$$A \bigvee_\ell B = A_\ell \bigvee B_\ell = [a_1, \infty) \bigvee [b_1, \infty) = [a_1 \vee b_1, \infty). \tag{10}$$

The maximum of A and B can therefore also be expressed as

$$A \bigvee B = [a_1 \vee b_1, a_2 \vee b_2] = (A \bigvee_r B) \cap (A \bigvee_\ell B). \tag{11}$$

The following properties of the right and left maxima and right and left sets are worth noting:

$$(A_r)_r = A_r, \tag{12a}$$

$$(A_r)_\ell = (-\infty, \infty), \tag{12b}$$

$$(A \bigvee_r B)_r = A \bigvee_r B, \tag{12c}$$

$$(A \bigvee_r B)_\ell = (-\infty, \infty), \tag{12d}$$

$$A \bigvee_r B = (A \bigvee B)_r = A_r \cup B_r, \tag{12e}$$

$$(A_\ell)_\ell = A_\ell, \tag{12f}$$

$$(A_\ell)_r = (-\infty, \infty), \tag{12g}$$

$$(A \bigvee_\ell B)_\ell = A \bigvee_\ell B, \tag{12h}$$

$$(A \bigvee_\ell B)_r = (-\infty, \infty), \tag{12i}$$

$$A \bigvee_\ell B = (A \vee B)_\ell = A_\ell \cap B_\ell. \tag{12j}$$

We also define the *right distance* and *left distance*, respectively, between A and B as

$$d_r(A, B) = \int_{-\infty}^{\infty} |\mu_{A_r}(x) - \mu_{B_r}(x)| \, dx, \tag{13}$$

$$d_\ell(A, B) = \int_{-\infty}^{\infty} |\mu_{A_\ell}(x) - \mu_{B_\ell}(x)| \, dx. \tag{14}$$

The (total) *distance* between A and B (see [14]) can now be written

$$d(A, B) = d_r(A, B) + d_\ell(A, B). \tag{15}$$

It follows that

$$d_r(A, B) = d(A_r, B_r), \tag{16a}$$

and

$$d_\ell(A, B) = d(A_\ell, B_\ell), \tag{16b}$$

since $d_\ell(A_r, B_r) = 0$ and $d_r(A_\ell, B_\ell) = 0$.

3.2. Traditional decision criteria reformulated

We consider again the maximax and maximin principles, as well as one other ordering principle, namely, the Hurwicz criterion [15]. We express each principle, applied to two intervals A and B, in such a way that it can be extended in a meaningful way to fuzzy sets. These formulations call for the comparison of two distances, as follows:

(a) *Maximax principle*:

$$d(A_r, A \bigvee_r B) < d(B_r, A \bigvee_r B) \Leftrightarrow A > B. \tag{17}$$

By virtue of properties (12e) and (16), this can also be written

$$d_r(A, A \bigvee B) < d_r(B, A \bigvee B) \Leftrightarrow A > B. \tag{18}$$

(b) *Maximin principle*:

$$d(A_\ell, A \bigvee_\ell B) < d(B_\ell, A \bigvee_\ell B) \Leftrightarrow A > B. \tag{19}$$

By virtue of properties (12j) and (16), this can also be written

$$d_\ell(A, A \bigvee B) < d_\ell(B, A \bigvee B) \Leftrightarrow A > B. \tag{20}$$

The above formulations of the maximax and maximin principles are equivalent to (6) and (7), respectively.

(c) *Hurwicz criterion*: To apply this criterion, the quantities $\delta a_1 + (1-\delta)a_2$ and $\delta b_1 + (1-\delta)b_2$ are computed for a suitable choice of $\delta \in [0, 1]$ and compared. This can be expressed as

$$\delta d(A_\ell, A \bigvee_\ell B) + (1-\delta) d(A_r, A \bigvee_r B) < \delta d(B_\ell, A \bigvee_\ell B) + (1-\delta) d(B_r, A \bigvee_r B) \Leftrightarrow A > B. \tag{21}$$

Proof. Since $a_1 \le a_1 \vee b_1$, $a_2 \le a_2 \vee b_2$, and $b_1 \le a_1 \vee b_1$, $b_2 \le a_2 \vee b_2$, it follows that

$$\begin{aligned} &\delta d(A_\ell, A \bigvee_\ell B) + (1-\delta) d(A_r, A \bigvee_r B) \\ &\quad = \delta(a_1 \vee b_1 - a_1) + (1-\delta)(a_2 \vee b_2 - a_2) \\ &\quad < \delta(a_1 \vee b_1 - b_1) + (1-\delta)(a_2 \vee b_2 - b_2) \\ &\quad = \delta d(B_\ell, A \bigvee_\ell B) + (1-\delta) d(B_r, A \bigvee_r B) \\ &\quad \Leftrightarrow \delta a_1 + (1-\delta)a_2 > \delta b_1 + (1-\delta) b_2 \\ &\quad \Leftrightarrow A > B. \end{aligned}$$

By virtue of (12e), (12j), (15) and (16), the above criterion can be expressed in the following way for $\delta = \frac{1}{2}$:

$$d(A, A \bigvee B) < d(B, A \bigvee B) \Leftrightarrow A > B. \tag{22}$$

3.3. Transitivity of decision rules

It needs to be demonstrated that the preference criteria as formulated in the preceding section are consistent, i.e., transitivity holds.

Lemma 1. *Given any three intervals over* $\mathbb{R}$,

$$A=[a_1,a_2],\quad B=[b_1,b_2],\quad C=[c_1,c_2],$$

then there exists a non-negative constant k_1 *such that*

$$d(A, A\bigvee B)+k_1=d(A, A\bigvee B\bigvee C)$$

and

$$d(B, A\bigvee B)+k_1=d(B, A\bigvee B\bigvee C).$$

Proof. Since $a_1\leqslant a_1\vee b_1\leqslant a_1\vee b_1\vee c_1$, and $a_2\leqslant a_2\vee b_2\leqslant a_2\vee b_2\vee c_2$, it follows that

$$\begin{aligned}
&d(A, A\bigvee B\bigvee C)-d(A, A\bigvee B)\\
&\quad=[(a_1\vee b_1\vee c_1-a_1)-(a_1\vee b_1-a_1)]\\
&\qquad+[(a_2\vee b_2\vee c_2-a_2)-(a_2\vee b_2-a_2)]\\
&\quad=(a_1\vee b_1\vee c_1-a_1\vee b_1)\\
&\qquad+(a_2\vee b_2\vee c_2-a_2\vee b_2)\\
&\quad=k_1\geqslant 0.
\end{aligned}$$

Proceeding in a similar manner it follows that

$$d(B, A\bigvee B\bigvee C)-d(B, A\bigvee B)=k_1.$$

Corollary. *Given any three intervals* A, B, C, *over* $\mathbb{R}$, *then*

$$\begin{aligned}
&d(A, A\bigvee B)<d(B, A\bigvee B)\\
&\quad\Leftrightarrow d(A, A\bigvee B\bigvee C)<d(B, A\bigvee B\bigvee C).
\end{aligned}$$

Interpretation. Again, let A, B, C be intervals over $\mathbb{R}$ and assume that A is preferred to B and B is preferred to C according to the decision rule (22). Using Lemma 1 we can write

$$d(A, A\bigvee B\bigvee C)<d(B, A\bigvee B\bigvee C)$$

and

$$d(B, A\bigvee B\bigvee C)<d(C, A\bigvee B\bigvee C).$$

Therefore,

$$d(A, A\bigvee B\bigvee C)<d(C, A\bigvee B\bigvee C),$$

which implies that

$$d(A, A\bigvee C)<d(C, A\bigvee C).$$

Thus, A is preferred to C. For decision rules (17) and (19), the same argument can be applied to the right sets of A, B, C and to the left sets, respectively. Thus, transitivity holds for all three decision rules.

4. Extension of the criteria defined for intervals to fuzzy numbers

In this section we provide a generalization of the preference criteria defined in Section 3 to the fuzzy set ordering problem. To do so, we extend the notions of right and left sets and of the left and right maximum, as introduced in Section 3, to fuzzy numbers. Lemma 1 is extended here to fuzzy numbers in order to verify transitivity of the preference criteria as defined.

4.1. Definitions

Let $A=\{(x, \mu_A(x))\}$ and $B=\{(x, \mu_B(x))\}$ be fuzzy sets defined over $\mathbb{R}$. Analogous to (4) and (5), the left and right sets of A are defined in terms of their membership functions as follows:

$$\mu_{A_r}(x)=\sup_{u+k=x,k\leqslant 0}\mu_A(u), \tag{23a}$$

$$\mu_{A_\ell}(x)=\sup_{u+k=x,k\geqslant 0}\mu_A(u). \tag{23b}$$

Analogous definitions hold for B.

If A is a convex and normal fuzzy set, that is, a fuzzy number, and a_0 is a value for which $\mu_A(a_0)=1$, then

$$\mu_{A_r}(x)=\begin{cases}\mu_A(x), & x\geqslant a_0,\\ 1, & x\leqslant a_0,\end{cases} \tag{24a}$$

$$\mu_{A_\ell}(x)=\begin{cases}\mu_A(x), & x\leqslant a_0,\\ 1, & x\geqslant a_0.\end{cases} \tag{24b}$$

Now let the *right maximum* of the two fuzzy numbers A and B, denoted $A\bigvee_r B$, be defined by

$$A\bigvee\nolimits_r B=\{(z, \mu_{A\bigvee_r B}(z))\},\quad z\in\mathbb{R}, \tag{25}$$

where

$$\mu_{A\bigvee_r B}(z)=\sup_{\substack{(x,y):z=x\vee y\\ x,y\in\mathbb{R}}}(\mu_{A_r}(x)\wedge\mu_{B_r}(y)).$$

This can be defined in an equivalent manner using the concept of the α-level set of a fuzzy set [22], by specifying that for all $\alpha \in [0, 1]$,

$$(A \vee_r B)_\alpha = (A_r)_\alpha \vee (B_r)_\alpha = (-\infty, a_2(\alpha) \vee b_2(\alpha)]. \tag{26}$$

In a similar manner we define the *left maximum* of A and B,

$$A \vee_\ell B = \{(z, \mu_{A \vee_\ell B}(z))\}, \quad z \in \mathbb{R}, \tag{27}$$

where

$$\mu_{A \vee_\ell B}(z) = \sup_{\substack{(x,y): z = x \vee y \\ x,y \in \mathbb{R}}} (\mu_{A_\ell}(x) \wedge \mu_{B_\ell}(y)).$$

This is equivalent to specifying that for all $\alpha \in [0, 1]$,

$$(A \vee_\ell B)_\alpha = (A_\ell)_\alpha \vee (B_\ell)_\alpha = [a_1(\alpha) \vee b_1(\alpha), \infty). \tag{28}$$

The above definitions are straightforward extensions of (9) and (10), and it is easily seen that the *maximum* of A and B, as defined in [14], is given by the intersection of the right and left maxima, as in (11). The properties (12) also apply to fuzzy numbers.

These notions are illustrated in Figures 1–3. Consider two fuzzy numbers A and B as shown in Figure 1. The corresponding sets A_r, B_r, and $A \vee_r B$ are shown in Figure 2, and A_ℓ, B_ℓ, and $A \vee_\ell B$ are shown in Figure 3.

It is now possible to apply (13)–(16) to fuzzy numbers as well. The *right distance, left distance*, and *total distance* between two fuzzy numbers are thus defined. It can easily be seen that the definition of total distance between two fuzzy numbers turns out to be equivalent to the one introduced by Kaufmann and Gupta [14], which calls for integration along the membership axis, namely,

$$d(A, B) = \int_0^1 d(A_\alpha, B_\alpha)\, d\alpha. \tag{29}$$

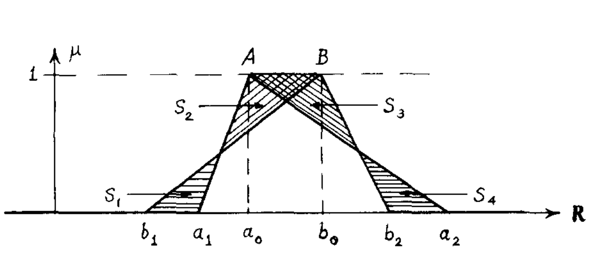

Fig. 1. Two overlapping fuzzy numbers.

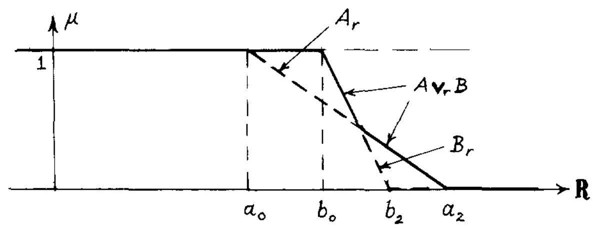

Fig. 2. Right sets and right maximum.

The right and left distances can be expressed similarly. The definition of total distance, however, is not equivalent to the definition introduced by Kaufmann [13], namely,

$$d(A, B) = \int_{-\infty}^{\infty} |\mu_A(x) - \mu_B(x)|\, dx,$$

called Hamming distance.

4.2. *Criteria for choosing between two fuzzy numbers*

The formulation of the decision criteria as presented in Section 3 can now be applied directly to fuzzy numbers.

(a) *Maximax principle*:

$$d_r(A, A \vee B) < d_r(B, A \vee B) \Leftrightarrow A > B. \tag{30}$$

Applied to the example in Figure 1, the distances in (30) express the areas identified by S_3 and S_4:

$$d_r(A, A \vee B) = \int_{-\infty}^{\infty} |\mu_{A_r}(x) - \mu_{A \vee_r B}(x)|\, dx = S_3,$$

$$d_r(B, A \vee B) = \int_{-\infty}^{\infty} |\mu_{B_r}(x) - \mu_{A \vee_r B}(x)|\, dx = S_4.$$

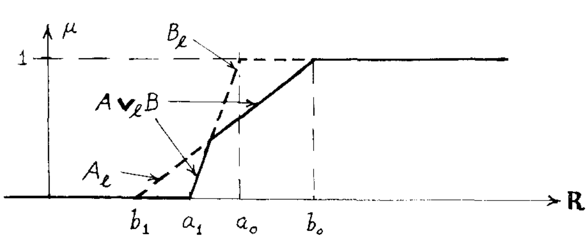

Fig. 3. Left sets and left maximum.

Therefore, based on the Maximax principle, if $S_3 < S_4$ then decide on A, otherwise on B.

(b) *Maximin principle*:

$$d_\ell(A, A \bigvee B) < d_\ell(B, A \bigvee B) \Leftrightarrow A > B. \quad (31)$$

When applying this to the example in Figure 1, we obtain

$$d_\ell(A, A \bigvee B) = \int_{-\infty}^{\infty} |\mu_{A_\ell}(x) - \mu_{A \vee_\ell B}(x)| \, \mathrm{d}x = S_2,$$

$$d_\ell(B, A \bigvee B) = \int_{-\infty}^{\infty} |\mu_{B_\ell}(x) - \mu_{A \vee_\ell B}(x)| \, \mathrm{d}x = S_1.$$

Thus, if $S_2 < S_1$, the decision is in favor of A, otherwise B.

(c) *Hurwicz principle*: Using the notion of total distance between two fuzzy sets we specify the Hurwicz criterion in the following form, which we call the *total distance criterion*:

$$d(A, A \bigvee B) < d(B, A \bigvee B) \Leftrightarrow A > B. \quad (32)$$

Applied to the example in Figure 1, we obtain

$$d(A, A \bigvee B) = S_2 + S_3$$

and

$$d(B, A \bigvee B) = S_1 + S_4.$$

The decision rule therefore becomes: If $S_2 + S_3 < S_1 + S_4$ then decide on A, otherwise decide on B.

Example. Consider the two fuzzy sets A and B defined as follows (see Figure 4):

$$\mu_A(x) = \begin{cases} \frac{1}{2}x - 2, & 4 \leq x \leq 6 \\ -\frac{1}{4}x + \frac{5}{2}, & 6 \leq x \leq 10, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mu_B(x) = \begin{cases} \frac{1}{3}x - \frac{4}{3}, & 4 \leq x \leq 7, \\ -x + 8, & 7 \leq x \leq 8, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$d(A, A \bigvee B) = S_2 + S_3 = \tfrac{2}{3},$$
$$d(B, A \bigvee B) = S_4 = \tfrac{2}{3}.$$

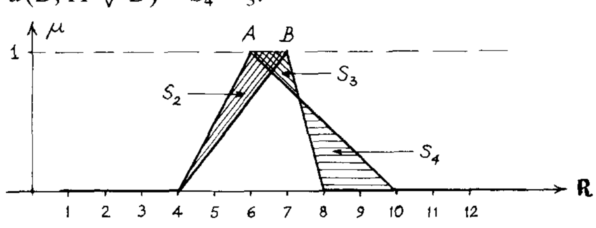

Fig. 4. Two fuzzy numbers (numerical example).

Thus, A is equivalent to B according to the total distance criterion (32). Also,

$$d(A_\ell, A \bigvee_\ell B) = S_2 = \tfrac{1}{2},$$
$$d(B_\ell, A \bigvee_\ell B) = S_1 = 0.$$

Thus, B is preferred to A in the maximin sense. Furthermore,

$$d(A_r, A_{A \vee_r B}B) = S_3 = \tfrac{1}{6},$$
$$d(B_r, A_{A \vee_r B}B) = S_4 = \tfrac{2}{3}.$$

Thus, A is preferred to B in the maximax sense.

4.3. *Transitivity of the decision criteria for ordering fuzzy numbers*

It will now be shown that the preference rules for fuzzy numbers (30), (31), and (32) are consistent; that is, transitivity holds.

Lemma 2. *Given any three convex and normal fuzzy sets over* $\mathbb{R}$,

$$A = \{(x, \mu_A(x))\}, \quad B = \{(x, \mu_B(x))\},$$
$$C = \{(x, \mu_C(x))\}$$

for all $x \in \mathbb{R}$, *then there exists a non-negative constant* k_2 *such that*

$$d(A, A \bigvee B) + k_2 = d(A, A \bigvee B \bigvee C)$$

and

$$d(B, A \bigvee B) + k_2 = d(B, A \bigvee B \bigvee C).$$

Proof. Denote the α-level intervals of the fuzzy numbers A, B, and C by

$$A_\alpha = [a_1(\alpha), a_2(\alpha)], \quad B_\alpha = [b_1(\alpha), b_2(\alpha)],$$
$$C_\alpha = [c_1(\alpha), c_2(\alpha)].$$

According to Lemma 1, for any given $\alpha \in [0, 1]$ there exists a non-negative constant $k(\alpha)$ such that

$$d[A_\alpha, (A \bigvee B)_\alpha] + k(\alpha) = d[A_\alpha, (A \bigvee B \bigvee C)_\alpha]$$

and

$$d[B_\alpha, (A \bigvee B)_\alpha] + k(\alpha) = d[B_\alpha, (A \bigvee B \bigvee C)_\alpha].$$

Integrating with respect to α and using (29), we obtain

$$d(A, A \bigvee B) + k_2 = d(A, A \bigvee B \bigvee C)$$

and

$$d(B, A \bigvee B) + k_2 = d(B, A \bigvee B \bigvee C). \qquad \square$$

This lemma's interpretation is analogous to the interpretation of Lemma 1, except that A, B and C are now fuzzy numbers instead of intervals. Therefore, the preference criterion (32) is transitive. It follows that the ordering of several fuzzy numbers, and determination of the optimum with respect to that criterion, can be carried out by a series of pairwise comparisons. In other words, there is no need to compute the distance from each fuzzy number to the overall fuzzy maximum as suggested in [12].

Lemma 2 also establishes transitivity for the decision rules (30) and (31). The sets A, B and C in the lemma must then be understood as representing right sets (in the case of (30)) or left sets (in the case of (31)).

4.4. *Relating the total distance criterion to Yager's ranking index*

We show now that, for convex and normal fuzzy sets, the total distance criterion is equivalent to the ranking index proposed by Yager [21]. This reduces the computational complexity of the ordering criterion and makes its use more convenient.

Lemma 3. *Given two convex and normal fuzzy sets A and B over $\mathbb{R}$, then*

$$2[F(A)-F(B)] = d(B, A \bigvee B) - d(A, A \bigvee B),$$

where, for a general fuzzy set over $\mathbb{R}$, $F(A)$ as defined by Yager is

$$F(A) = \int_0^{\alpha_{\max}} M(A_\alpha)\,\mathrm{d}\alpha.$$

$\alpha_{\max} = \max_x \mu_A(x)$ is the height of the membership function of the fuzzy set A, and $M(A_\alpha)$ is the center of gravity of the sets A_α. If A is convex, then $M(A_\alpha) = \frac{1}{2}(a_1(\alpha)+a_2(\alpha))$ for every $\alpha \in [0, \alpha_{\max}]$, and if A is normal then $\alpha_{\max} = 1$.

Proof.

$$\begin{aligned}
&d(B, A \bigvee B) - d(A, A \bigvee B)\\
&\quad = \int_0^1 [|b_1(\alpha) - a_1(\alpha) \vee b_1(\alpha)| \\
&\qquad\qquad + |b_2(\alpha) - a_2(\alpha) \vee b_2(\alpha)|]\,\mathrm{d}\alpha \\
&\qquad - \int_0^1 [|a_1(\alpha) - a_1(\alpha) \vee b_1(\alpha)| \\
&\qquad\qquad + |a_2(\alpha) - a_2(\alpha) \vee b_2(\alpha)|]\,\mathrm{d}\alpha \\
&\quad = \int_0^1 [(a_1(\alpha)+a_2(\alpha)) - (b_1(\alpha)+b_2(\alpha))]\,\mathrm{d}\alpha \\
&\quad = 2[F(A)-F(B)]. \qquad \square
\end{aligned}$$

It may be noted here that the ranking index proposed by Yager is not restricted to normal and convex fuzzy sets. In our case, normality is a requirement for the convergence of the integrals in (13) and (14). Also, if one or both of the fuzzy sets to be compared are not convex then the method based on the total distance criterion leads to results that are different from those obtained using Yager's index.

5. Discussion

The problem of ordering fuzzy sets over the real line has been addressed by several authors (see [1, 2, 4, 6, 8, 11, 12, 18, 19, 20, 21]). Some of the results in these studies were based on an intuitive notion as to which of two fuzzy sets or numbers is to be preferred over the other. This involved making existing ordering methods more sensitive to changes in the shape of the membership functions [2, 4]. Other results were derived by extending the notion of preference between crisp numbers to that of fuzzy numbers by using some form of the extension principle [1, 11]. Some of the methods have been criticized for the insensitivity to shift along the real line and a lack of transitivity in the preference ordering [3].

The ordering criteria developed in this paper do not have the above-mentioned disadvantages. Transitivity holds, and there is sensitivity to a shift along the real line. Besides, these criteria are not based on intuitive considerations. Instead, a fuzzy number is viewed as an extension of a crisp interval rather than a crisp number, and the standard decision criteria for preference between intervals are expressed in a manner that allows their generalization to the fuzzy case.

Although the proposed ordering criteria are, in certain ways, sensitive to changes in the shape of the membership functions, a lack of such sensitivity does not necessarily lead to inappropriate results. The maximin (maximax) criterion is only sensitive to increasing (decreasing) parts of the membership functions. Thus

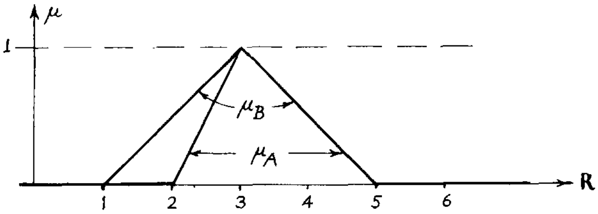

Fig. 5. Another example of two fuzzy numbers.

take, for instance, the example shown in Figure 5. The preference between A and B varies with the criterion that is applied. A is equivalent to B under the maximax. From that perspective, the claim by Chen [4] that the equivalence of A and B produced by Jain's method [11] in this example is counterintuitive, and that A has to be preferred to B, is not justified. The total distance criterion is less optimistic than the maximax and less pessimistic than the maximin, and thus can be regarded as intermediate to these two. When approaching an ordering problem in terms of the total distance criterion, Chen's observation may be appropriate.

Two aspects of the ordering criteria which have been developed here merit further comment:

(1) The preference ordering is based on the comparison of the areas by which two membership functions differ.

(2) Even though in most practical applications, membership functions are only approximately determined, the preference ordering rules presented here are sensitive to small modifications in the shape of a membership function—especially at low grades of membership.

The following remarks address these points.

(1) The expression of the standard decision criterion for preferences between intervals in terms of distance measures, and the extension to the fuzzy case, resulted in a distance expressed in two dimensions, in the form of areas. And this seems natural. For, it results from the fact that in comparing two fuzzy sets, both the membership axis and the real line ought to be taken into account. This is not the case in the comparison of intervals. It should also be noted that the preference criteria introduced here do not assign any significance to the area under a membership function, as would be the case in a probabilistic analysis, but rather to the areas of the non-overlapping portions of two membership functions.

(2) The extension of binary operations to fuzzy numbers (see [14]) turns out to be sensitive to small changes in the shape of the membership functions. Accordingly, the comparison of such numbers is also sensitive to such changes. Besides, in many fuzzy decision problems (see [1, 11, 17, 19]), the fuzzy sets to be compared are induced by a function of two variables, one of which is fuzzy. In that case, a change in the shape of the membership function of this variable results in a change in the shape of the membership functions of all the induced fuzzy sets, and thus the optimality decision may remain the same. Furthermore, when two fuzzy sets are to be compared, the way in which their membership functions were assigned should not be of concern. The membership grades, once assigned, are to be taken as exact, even though they might have been arrived at subjectively (Zadeh [22]).

6. A criterion for ordering fuzzy numbers based on Bernoulli's principle of diminishing marginal utility

We present now one additional ordering criterion for fuzzy sets which differs significantly from all the above. As discussed by Carnap [5], the concept of utility or 'moral gain' represents a measure of the satisfaction experienced by a person as a result of some material gain. Defined in this way, utility is a subjective measure that needs to be quantified before it can be treated analytically. This was achieved with Bernoulli's law of diminishing marginal utility:

$$\mathrm{d}u = k(\mathrm{d}f)/f.$$

where $\mathrm{d}u$ represents the utility associated with a person's small differential gain, $\mathrm{d}f$, and f is the person's initial fortune. k is a positive constant characterizing the person at the time at which the gain is experienced. Taking into account the additivity of this measure of utility, the utility of a change in fortune from level f_0 to level f_1 becomes:

$$u(f_1 - f_0) = \int_{f_0}^{f_1} k(1/f)\,\mathrm{d}f = k\ln(f_1/f_0).$$

Using Bernoulli's law, a criterion for the preference between two intervals $A=[a_1, a_2]$ and $B=[b_1, b_2]$ defined over $\mathbb{R}^+$ can be specified as follows:

$$u[d(A, A \bigvee B)] < u[d(B, A \bigvee B)] \Leftrightarrow A > B, \tag{33}$$

where the marginal utility function, $u[d(A, A \bigvee B)]$, is here defined as

$$\begin{aligned} u[d(A, A \bigvee B)] &= u[d(A_\ell, A \bigvee_\ell B)] + u[d(A_r, A \bigvee_r B)] \\ &= u\ |a_1 - a_1 \vee b_1| + u\ |a_2 - a_2 \vee b_2| \\ &= \ln[(a_1 \vee b_1)(a_2 \vee b_2)/a_1 a_2]. \end{aligned}$$

Similarly,

$$u[d(B, A \bigvee B)] = \ln[(a_1 \vee b_1)(a_2 \vee b_2)/b_1 b_2].$$

The decision criterion (33) therefore becomes

$$\ln(a_1 a_2) > \ln(b_1 b_2) \Leftrightarrow A > B.$$

The criterion (33) can now be extended in a straightforward manner to fuzzy numbers. Let A and B be two fuzzy numbers over $\mathbb{R}^+$. Then

$$u[d(A, A \bigvee B)] < u[d(B, A \bigvee B)] \Leftrightarrow A > B, \tag{34}$$

where now

$$u[d(A, A \bigvee B)] = \int_0^1 u[d(A_\alpha, (A \bigvee B)_\alpha)]\, \mathrm{d}\alpha,$$

$$u[d(B, A \bigvee B)] = \int_0^1 u[d(B_\alpha, (A \bigvee B)_\alpha)]\, \mathrm{d}\alpha.$$

Lemma 4. *The decision criterion (34) is equivalent to*

$$F_u(A) > F_u(B) \Leftrightarrow A > B, \tag{35}$$

where

$$F_u(A) = \int_0^1 \ln(a_1(\alpha) a_2(\alpha))\, \mathrm{d}\alpha,$$

$$F_u(B) = \int_0^1 \ln(b_1(\alpha) b_2(\alpha))\, \mathrm{d}\alpha.$$

$F_u(\cdot)$ *is called the utility ranking index.*

Proof. We observe the following equivalences:

$$\begin{aligned} & u[d(A, A \bigvee B)] < u[d(B, A \bigvee B)] \\ & \Leftrightarrow \int_0^1 u[d(A_\alpha, (A \bigvee B)_\alpha)]\, \mathrm{d}\alpha < \int_0^1 u[d(B_\alpha, (A \bigvee B)_\alpha)]\, \mathrm{d}\alpha \\ & \Leftrightarrow \int_0^1 \{\ln[(a_1(\alpha) \vee b_1(\alpha))(a_2(\alpha) \vee b_2(\alpha))] - \ln(a_1(\alpha)a_2(\alpha))\}\, \mathrm{d}\alpha \\ & \qquad < \int_0^1 \{\ln[(a_1(\alpha) \vee b_1(\alpha))(a_2(\alpha) \vee b_2(\alpha))] - \ln(b_1(\alpha)b_2(\alpha))\}\, \mathrm{d}\alpha \\ & \Leftrightarrow \int_0^1 \ln(a_1(\alpha)a_2(\alpha))\, \mathrm{d}\alpha > \int_0^1 \ln(b_1(\alpha)b_2(\alpha))\, \mathrm{d}\alpha \\ & \Leftrightarrow F_u(A) > F_u(B). \qquad \square \end{aligned}$$

Transitivity follows readily:

$$F_u(A) > F_u(B) > F_u(C) \Rightarrow F_u(A) > F_u(C).$$

Application to the numerical example in Figure 4 results in $F_u(A) = 3.6715$ and $F_u(B) = 3.7062$. Thus, B is preferred to A.

7. Concluding remarks

The ordering of fuzzy sets over the real line has been approached in this study from the point of view of ordering intervals rather than crisp numbers. Although a convex and normal fuzzy set over the real line is called a fuzzy number, it may be better thought of as a fuzzified interval rather than as a fuzzified crisp number. In Kaufmann and Gupta's terms [14], a fuzzy number is an interval of confidence at levels ranging from 0 to 1. Intervals cannot be ordered in an absolute sense, and neither can fuzzy sets over the real line.

Taking into account this interpretation of a fuzzy set over the real line, a systematic approach to the ordering problem has been presented. This approach consists of first redefining the standard decision criteria for intervals in such a way as to permit their generalization to fuzzy sets. The various decision criteria which were introduced were examined only for convex and normal fuzzy sets over the real line (fuzzy numbers). In all the analysis carried out here, it was assumed that a non-fuzzy (clear-cut) preference is required.

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