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CONTINUOUS $l_{n,p}$-SYMMETRIC DISTRIBUTIONS

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Abstract. For $p > 0$, the $l_{n,p}$-generalized surface measure on the $l_{n,p}$-unit sphere is studied and used for deriving a geometric measure representation for $l_{n,p}$-symmetric distributions having a density.

Keywords: geometric measure representation, indivisible method, intersection-percentage function, Minkowski geometry, $l_{n,p}$-generalized surface measure, generalized uniform distribution, cone measure, exact statistical distributions, p-generalized normal distribution, geometric probability, density-generating function, heavy tails, light tails, p-generalized Fisher distribution, (p, g)-generalized χ^2-distribution, $l_{n,p}$-ball numbers.

1 INTRODUCTION

In the literature, the main success in deriving exact distributions of functions of random vectors was in the case of the Gaussian sample law. For a solid basis of this research area, we refer to [1]. Many of the original results could be extended later to the class of spherically- or $l_{n,2}$-norm symmetric sample distributions. This view does not ignore the fact that a similar success was also achieved for more or less specific functions of random vectors for other sample distributions. Numerous results exist, e.g., if the sample distribution is a one- or two-sided exponential or an $l_{n,1}$-norm symmetric distribution. Considerably fewer explicit distributional results are known, however, for other types of sample distributions because such distributions are not yet studied as well as those mentioned before.

Much of the foundation of the classes of $l_{n,2}$- and $l_{n,1}$-symmetric distributions has been laid in [2, 5, 6, 12, 20, 38]. Statistical applications were discussed in [4, 9, 17, 43]. The separate but analogous theories of these distributions are well established nowadays and include stochastic representations of random vectors with respect to a suitably chosen so-called uniform basis and a certain positive random variable. Hence, when establishing these theories, the distribution ω_p of the n-dimensional basis vector ($n \geqslant 2$) plays a fundamental role. This distribution is not yet studied for arbitrary $p > 0$ as extensively as for the two special cases $p \in \{1, 2\}$. The study is closely connected with answering the question which geometry would be the most suitable for measuring the surface content of the $l_{n,p}$-sphere if p is not from $\{1, 2\}$. It will become obvious in this paper that studying the distribution ω_p for arbitrary $p > 0$ brings together measure theory and a certain type of non-Euclidean geometry that is a Minkowski geometry in the convex case $p \geqslant 1$.

Basic properties of $l_{n,p}$-norm symmetric distributions have been studied in [11, 25, 36, 39, 41]. Notice that the continuous $l_{n,p}$-symmetric distributions are natural generalizations of the p-generalized normal distri-

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bution N_p which was introduced in [40] and later on studied in [18, 10, 42]. The density of this distribution is

$$\varphi_p(x) = C_p^n \exp\Big\{-\frac{1}{p}|x|_p^p\Big\}, \quad |x|_p = \Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p},\ x = (x_1, \ldots, x_n) \in R^n,$$

where $C_p = p^{1-1/p}/(2\Gamma(1/p))$, $p > 0$, and, in accordance with the geometric properties of the level sets of this density, p will be called the form parameter. Obviously, this is the Gaussian density if $p = 2$ and the Laplace density if $p = 1$. The distribution N_p can be considered as an $l_{n,p}$-symmetric or $l_{n,p}$-spherical distribution, or even as an $l_{n,p}$-norm symmetric distribution if $p \geqslant 1$. It has zero expectation and covariance matrix $\sigma_p^2 I$ with $\sigma_p^2 = p^{2/p}\Gamma(3/p)/\Gamma(1/p)$, where I denotes the $n \times n$-unit matrix, and Γ denotes the Gamma function.

Let a random vector $X = (X_1, \ldots, X_n)$ be distributed according to this distribution, $X \sim N_p$, $p > 0$, and put

$$R_p = |X|_p \quad \text{and } U_p = X/R_p.$$

The random variable R_p can be considered as the p-radius variate of the random vector X. It is distributed according to the distribution function

$$F_p(x) = I_{(0,\infty)}(x) \int_0^x r^{n-1} \mathrm{e}^{-\frac{r^p}{p}}\,\mathrm{d}r \Big/ \int_0^\infty r^{n-1}\mathrm{e}^{-\frac{r^p}{p}}\,\mathrm{d}r,$$

and the statistic R_p^p follows the so-called χ^p- or p-generalized χ^2-distribution having the density

$$f_p(y) = I_{(0,\infty)}(y)\frac{1}{p^{\frac{n}{p}}\Gamma(\frac{n}{p})} y^{\frac{n}{p}-1}\mathrm{e}^{-\frac{y}{p}}.$$

Possibly without knowing yet the direct derivation of these formulas, the reader may check them immediately by induction. In accordance with [30], we write for short $R_p \sim \chi(p, n)$ and $R_p^p \sim \chi^p(n)$. This distribution will be further generalized in Section 4.

The vector U_p takes values in the unit sphere,

$$U_p|\Omega \rightarrow S_{n,p} = \{x \in R^n \colon |x|_p = 1\},$$

and is called the uniform basis of the random vector X. The distribution generated by U_p on the Borel σ-field $\mathfrak{B}_{S,p}$ over $S_{n,p}$ is singular w.r.t. the Lebesgue measure λ.

Moreover, for each $p > 0$, the random elements R_p and U_p are stochastically independent. The vector U_p has zero expectation and covariance matrix $(\sigma_p^2/ER_p^2)I$, and from the equation $ER_p^2 = p^{2/p}\Gamma((n + 2)/p)/\Gamma(n/p)$ it follows that

$$\sigma_p^2/ER_p^2 = \Gamma(3/p)\Gamma(n/p))/\big(\Gamma(1/p)\Gamma((n+2)/p)\big).$$

It is shown in [30] that U_p follows the $l_{n,p}$-generalized uniform probability distribution on $S_{n,p}$, which can be considered as a geometric probability distribution based upon the $l_{n,p}$-generalized surface measure also defined there. The latter notion was further studied for dimension $n = 2$ in [31, 32] and will be discussed for higher dimensions in Section 2. It will be shown there that if $p \geqslant 1$, then the notion of the $l_{n,p}$-generalized surface measure on the $l_{n,p}$-sphere relies on the Minkowski metric induced by the unit ball of the space l_{n,p^*} with the "dual" $p^* \geqslant 1$ satisfying $\frac{1}{p} + \frac{1}{p^*} = 1$. If, however, $p \in (0, 1)$, then this notion is based instead upon the Minkowski functional of the nonconvex star-shaped set $S(p^*) = \{x \in R^n \colon |x|_{p^*} \geqslant 1\}$ with $p^* = \frac{p}{p-1} < 0$, where $|x|_q$ is defined for $q < 0$ as for $q > 0$.

To be more specific, for arbitrary $p > 0$, let $K_{n,p}(\varrho) = \{x \in R^n \colon |x|_p \leqslant \varrho\}$ denote the $l_{n,p}$-ball of p-radius $\varrho > 0$,

$$Z_p(D) = \{x \in R^n \colon x/|x|_p \in D\}$$

the central projection cone corresponding to the set $D \in \mathfrak{B}_{S,p}$, and

$$\mathit{sector}_p(D, \varrho) = Z_p(D) \cap K_{n,p}(\varrho)$$

the D-based sector of the $l_{n,p}$-ball $K_{n,p}(\varrho)$. The $l_{n,p}$-generalized surface content of a set D from $\mathfrak{B}_{S,p}$ is defined according to [30] as

$$\mathfrak{O}_p(D) = \frac{\mathrm{d}}{\mathrm{d}\varrho}\lambda\big(\mathit{sector}_p(D, \varrho)\big)\big|_{\varrho=1},$$

i.e., it is the derivative of the sector's Lebesgue volume $\lambda(\mathit{sector}_p(D, \varrho))$ w.r.t. the p-radius ϱ taken at the point $\varrho = 1$. On using this measure, the distribution of U_p may be represented as

$$\omega_p(D) = \mathfrak{O}_p(D)/\mathfrak{O}_p(S_{n,p}), \quad D \in \mathfrak{B}_{S,p}.$$

The notion of the $l_{n,p}$-generalized surface content of a set D from $\mathfrak{B}_{S,p}$ may be equivalently expressed by means of the Minkowski functional d_M of an absorbing reference set M as follows. The functions

$$\varphi_{+(-)}(x_1, \dots, x_{n-1}) = +(-)\bigg(1 - \sum_{i=1}^{n-1} |x_i|^p\bigg)^{1/p}, \quad (x_1, \dots, x_{n-1}) \in K_{n-1,p},$$

with $K_{m,p} = K_{m,p}(1)$ describe in R^n the upper and lower half spheres $S^+_{n,p}$ and $S^-_{n,p}$, respectively. Let the standard unit vectors in R^n be denoted by $e_i, i = 1, \dots, n$, and the normal vector to $S_{n,p}$ at the point $(x_1, \dots, x_{n-1})$ by

$$N(x_1, \dots, x_{n-1}) = (-1)^n\Big(e_1\frac{\partial\varphi_{+(-)}}{\partial x_1} + \dots + e_{n-1}\frac{\partial\varphi_{+(-)}}{\partial x_{n-1}} - e_n\Big).$$

It will be shown in Section 2 that

$$\mathfrak{O}_p(D) = \int_{G(D)} d_{M(p)}\big(N(x_1, \dots, x_{n-1})\big)\,\mathrm{d}x, \quad D \in \mathfrak{B}_{S,p},$$

with $G(D) = \{(x_1, \dots, x_{n-1}) \in K_{n-1,p} \colon (x_1, \dots, x_n) \in D\}$ and the reference set being $M(p) = K_{n,p^*}$ if $p \geqslant 1$ and $M(p) = S(p^*)$ if $0 < p < 1$.

The $l_{n,p}$-generalized surface content of the whole $l_{n,p}$-sphere $S_{n,p}$ satisfies the equation $\mathfrak{O}_p(S_{n,p}) = \omega_{n,p}$, where

$$\omega_{n,p} = 2^n\Gamma\Big(\frac{1}{p}\Big)^n\Big/\Big(p^{n-1}\Gamma\Big(\frac{n}{p}\Big)\Big).$$

The constant $\pi_n(p) = \omega_{n,p}/n$ has the properties of the so-called $l_{n,p}$-ball number. The latter notion generalizes that of the circle number π to the case of an arbitrary form parameter $p > 0$ and an arbitrary dimension $n \geqslant 2$. This is outlined in [31, 32] for dimension $n = 2$ and will be further discussed in Section 5.

For the formal reason of defining a certain analogy to the case $p = 2$, the random vector U_p was said by several authors to be uniformly distributed on the sphere $S_{n,p}$. Notice that, e.g., the authors in [39] deal with this definition without explaining the geometric meaning of the word "uniform." Instead, they show that the Lebesgue density of an $(n-1)$-dimensional sub-vector of U_p satisfies a certain representation.

It was proved in [30] that, in general, ω_p does not coincide with the geometric probability distribution

$$\omega(D) := \mathfrak{O}(D)/\mathfrak{O}(S_{n,p}), \quad D \in \mathfrak{B}_{S,p},$$

which is based upon the usual (Euclidean) surface measure $\mathfrak{O}$ on $\mathfrak{B}_{S,p}$ over $S_{n,p}$. It is quite remarkable that even $\mathfrak{O}_1$ does not coincide with $\mathfrak{O}$. But, because $\mathfrak{O}_1(D)$ is a scalar multiple of $\mathfrak{O}(D)$, it holds nevertheless that

$$\omega_i(D) = \omega(D), \quad D \in \mathfrak{B}_{S,i}, \quad i \in \{1,2\}.$$

For other results characterizing the distribution of U_p, e.g., by the almost independence of its components, we refer to [25, 36].

In [41], a special system of generalized spherical coordinates is introduced for studying ω_p. It seems there, however, that the "uniformity" of this distribution does not refer to real, geometrical uniformity of the probability mass on the surface of the unit sphere.

Hence, if we call a random vector simply uniformly distributed on $S_{n,p}$ if it follows the distribution ω, then we can only say that U_p has this property if $p \in \{1,2\}$.

The $l_{n,p}$-generalized uniform distribution on the sphere $S_{n,p}$ can be represented alternatively as a geometric probability distribution which is based upon the volumes of p-generalized ball sectors, i.e.,

$$\omega_p(D) = \lambda\big(sector_p(D,1)\big)/\lambda\big(K_{n,p}(1)\big), \quad D \in \mathfrak{B}_{S,p}.$$

This is closely connected with the circumstance that ω_p is also called the cone measure. One might consider the cone measure representation of ω_p as the global and the $l_{n,p}$-generalized surface measure representation of ω_p as its local view. For a recent study comparing the Euclidean surface measure with the cone measure, we refer to [23].

Of the best of all the distributions $\omega_p, p > 0$, the two special cases ω_2 and ω_1 are studied. These play an important role in the literature for geometric measure representation formulas and their applications. A general foundation of geometric measure theory is given in [8] and [22]. Detailed studies were made in [5, 6] for several classes of symmetric multivariate distributions including those having l_2-norm or l_1-norm symmetric densities. Specific geometric measure representation formulas reflecting certain extensions of Cavalieri's and Torricelli's indivisible method were given for Gaussian measures in [26, 27], for more general $l_{n,2}$-spherical measures in [28, 29], and for $l_{n,1}$-symmetric measures in [13]. The spheres $S_{n,p}, p \in \{1,2\}$, may be interpreted thereby as generalized indivisibles, and measuring them may be based upon the usual (Euclidean) surface measure.

The key formulas in these papers are based upon the so-called $l_{n,p}$-sphere intersection-percentage functions for $p \in \{1,2\}$. In the present paper, we unify these two approaches and generalize them to the case of arbitrary $p > 0$. For all $p > 0$, the function

$$r \to \mathfrak{F}_p(A,r) := \omega_p\Big(r^{-1}[A \cap S_{n,p}(r)]\Big), \quad r > 0,$$

will be called the $l_{n,p}$-sphere intersection-percentage function of the set $A \in \mathfrak{B}^n$. The sets $A \cap S_{n,p}(r), r > 0$, may be considered now, for arbitrary $p > 0$, as generalized indivisibles, and measuring them is based upon $\mathfrak{O}_p$, which is a non-Euclidean surface measure, unless for $p = 2$. The corresponding geometric measure representation formula for $N_{g,p}$-measures will be proved in Section 3. On using suitably chosen density generating functions g, such measures allow modeling heavy or light distribution tails. Section 4 deals with some examples where the measure representation formula applies to generalizing classical exact statistical distributions.

2 THE $l_{n,p}$-GENERALIZED SURFACE MEASURE ON THE $l_{n,p}$-SPHERE

For simplicity, we restrict our consideration in this section partly to the half sphere $S^+_{n,p}$. Extension to the whole sphere $S_{n,p}$ will then be obvious.

2.1 The convex case

In this subsection, we are merely dealing with the case $1 \leqslant p < \infty$. It is well known that the usual Euclidean surface content of $S^+_{n,p}(r)$ can be represented as

$$\mathfrak{O}(S^+_{n,p}(r)) = \int_{S^+_{n,p}(r)} \mathrm{d}o = \int_{K_{n-1,p}(r)} \big\|N(x_1,\dots,x_{n-1})\big\|_2 \,\mathrm{d}x,$$

where $||.||_2$ denotes the Euclidean norm. For details, see, e.g., [37]. Notice that $\mathfrak{O}(S^+_{n,p}(r)) = r^{n-1}\mathfrak{O}(S^+_{n,p})$ and that it generally suffices to consider surface measures on the unit sphere. The given representation formula for $\mathfrak{O}(S^+_{n,p})$ is suitable for generalizing the notion of the $l_{2,q}$-arc-length $AL_{p,q}$ of the $l_{2,p}$-circle $S_{2,p}$ studied in [31] to the multivariate case $n \geqslant 2$. For doing this, we consider elements $(x_1,\dots,x_{n-1})$ from the Euclidean space $(R^{n-1}, ||.||_2)$ alternatively as elements from the Minkowski space $(R^{n-1}, ||.||_q)$, where $||.||_q$ denotes the $l_{n-1,q}$-norm, $q \geqslant 1$.

Let the $l_{n,q}$-surface content of a Borel subset A of a half-sphere $S^+_{n,p}$ be defined for $p \geqslant 1$ and $q \geqslant 1$ as

$$O_{p,q}(A) = \int_{G(A)} ||N(x_1,\dots,x_{n-1})||_q \,\mathrm{d}x.$$

Theorem 1. *If $(p,p^*) = (1,\infty)$ or $p > 1$ and $p^* = \frac{p}{p-1}$, then there holds*

$$O_{p,p^*}(A) = \mathfrak{O}_p(A), \quad A \in \mathfrak{B}_{S,p}.$$

Proof. It follows from the definition of $O_{p,q}(A)$ that it satisfies the representation formula

$$O_{p,q}(A) = \int_{G(A)} \left(1 + \sum_{i=1}^{n-1} \left|\frac{\partial \varphi_{+(-)}}{\partial x_i}\right|^q\right)^{1/q} \mathrm{d}x$$

with

$$\left|\frac{\partial \varphi_{+(-)}}{\partial x_i}\right|^q = |x_i|^{(p-1)q}\left(1 - \sum_{j=1}^{n-1} |x_j|^p\right)^{\frac{1-p}{p}q}.$$

In the case $p = 1$, there holds

$$O_{1,q}(A) = n^{1/q} \int_{G(A)} \mathrm{d}x \overset{q\to\infty}{\longrightarrow} \int_{G(A)} \mathrm{d}x = O_{1,\infty}(A).$$

Below, it will become obvious that $O_{1,\infty}(A)$ coincides with $\mathfrak{O}_1(A)$.

Now, let $p > 1$. On choosing $q = p^*$, it follows that

$$O_{p,p^*}(A) = \int_{G(A)} \left(1 - \sum_{i=1}^{n-1} |x_i|^p\right)^{\frac{1-p}{p}} \mathrm{d}x.$$

Let us assume for a while that the set A may be represented as

$$A = A(r_1, r_2, M^*) = \left\{\left(y_1,\dots,y_{n-1}, \left[1 - \sum_{i=1}^{n-1} |y_i|^p\right]^{1/p}\right) \colon (y_1,\dots,y_{n-1}) = SPH_p^{(n-1)}([r_1,r_2) \times M^*)\right\},$$

where $r_1 < r_2$ are real numbers from the interval $(0, 1)$,

$$M^* = \{(\varphi_1, \dots, \varphi_{n-2})\colon \varphi_{il} \leqslant \varphi_i < \varphi_{iu},\ i = 1, \dots, n-2\}$$

is a subset of $M^*_{n-1} = [0, \pi)^{\times(n-3)} \times [0, 2\pi)$, and $SPH_p^{(n-1)} | M^*_{n-1} \times [0, \infty) \to R^{n-1}$ denotes the $(n-1)$-dimensional p-generalized spherical coordinate transformation defined in [30]. The collection of all such sets $A(r_1, r_2, M^*)$ is a sub-semi-algebra of $\mathfrak{B}_{S,p}$, say $\mathfrak{S}_p$.

In the integral representation of $O_{p,p*}(A(r_1, r_2, M^*))$, changing the Cartesian coordinates with $(n-1)$-dimensional p-generalized spherical coordinates, we have

$$O_{p,p*}(A(r_1, r_2, M^*)) = \int_{r_1}^{r_2} \int_{M^*} J(SPH_p^{(n-1)})(\varphi, \varrho)(1 - \varrho^p)^{\frac{1-p}{p}} \,\mathrm{d}(\varphi, \varrho),$$

where $J(SPH_p^{(n-1)})(\varphi, \varrho)$ denotes the Jacobian of the p-generalized spherical coordinate transformation $SPH_p^{(n-1)}$. On exploiting the multiplicative structure of this Jacobian, $J(SPH_p^{(n-1)})(\varphi, \varrho) = \varrho^{n-2} J^*(SPH_p^{(n-1)})(\varphi)$, we get

$$O_{p,p*}(A(r_1, r_2, M^*)) = \int_{r_1}^{r_2} \int_{M^*} J^*(SPH_p^{(n-1)})(\varphi)\varrho^{n-2}(1 - \varrho^p)^{\frac{1-p}{p}} \,\mathrm{d}(\varphi, \varrho).$$

For describing now the set $sector_p((A(r_1, r_2, M^*)), \varrho)$, let us define the coordinate transformation T: $(R, r, \varphi) \to z[R, r, \varphi]$ as follows:

$$\begin{aligned}
&z_1 = Rr\cos_p(\varphi_1), \quad z_2 = Rr\sin_p(\varphi_1)\cos_p(\varphi_2), \dots, \\
&z_{n-2} = Rr\sin_p(\varphi_1)\cdots\sin_p(\varphi_{n-3})\cos_p(\varphi_{n-2}), \\
&z_{n-1} = Rr\sin_p(\varphi_1)\cdots\sin_p(\varphi_{n-3})\sin_p(\varphi_{n-2}), \quad z_n = R(1 - r^p)^{1/p},
\end{aligned}$$

where $\sin_p$ and $\cos_p$ are the p-generalized sine and cosine functions defined in [30]. The Jacobian of this coordinate transformation is

$$\begin{aligned}
J(T) &= \frac{D(z_1, \dots, z_n)}{D(R, r, \varphi_1, \dots, \varphi_{n-2})} = \left|\begin{vmatrix}
z_{1R} & z_{1r} & z_{1\varphi_1} & \cdots & z_{1\varphi_{n-2}} \\
 & \vdots & & & \\
z_{n-1R} & z_{n-1r} & z_{n-1\varphi_1} & \cdots & z_{n-1\varphi_{n-2}} \\
z_{nR} & z_{nr} & 0 & \cdots & 0
\end{vmatrix}\right| \\
&= R^{n-1}(1 - r^p)^{1/p-1} J(SPH_p^{(n-1)})(r, \varphi), \quad R > 0,\ r \in (0, 1),\ \varphi \in M^*_{n-1},
\end{aligned}$$

where $z_{i\theta}$ means the partial derivative $\frac{\partial z_i}{\partial \theta}$, and $J(SPH_p^{(n-1)})(r, \varphi)$ is still the Jacobian of the $(n-1)$-dimensional p-generalized spherical coordinate transformation $SPH_p^{(n-1)}$. On using the new coordinates, we get the representation formulas

$$A(r_1, r_2, M^*) = T(\{1\} \times [r_1, r_2) \times M^*) = \{z[R, r, \varphi] \in K_{n,p}\colon R = 1,\ r \in [r_1, r_2),\ \varphi \in M^*\}$$

and

$$sector_p((A(r_1, r_2, M^*), \varrho) = \{z[R, r, \varphi] \in K_{n,p}\colon 0 \leqslant R < \varrho,\ \varphi \in M^*,\ r_1 \leqslant r < r_2\}, \quad \varrho > 0.$$

For proving now that

$$O_{p,p^*}(A) = \frac{\mathrm{d}}{\mathrm{d}\varrho}\lambda\big(sector_p(A,\varrho)\big)\big|_{\varrho=1},$$

we start from

$$\begin{aligned}\lambda\Big(sector_p(A(r_1,r_2,M^*),\varrho)\Big) &= \int_{sector_p(A(r_1,r_2,M^*),\varrho)} \mathrm{d}z \\ &= \int_0^{\varrho}\int_{r_1}^{r_2}\int_{M^*} \frac{D(z_1,\dots,z_n)}{D(R,r,\varphi_1,\dots,\varphi_{n-2})}\big(R,r,\varphi_{(n-2)}\big)\,\mathrm{d}\varphi_{(n-2)}\,\mathrm{d}r\,\mathrm{d}R \\ &= \int_0^{\varrho}\int_{r_1}^{r_2}\int_{M^*} R^{n-1}(1-r^p)^{1/p-1}r^{n-2}J^*\big(SPH_p^{(n-1)}\big)(\varphi)\,\mathrm{d}\varphi_{(n-2)}\,\mathrm{d}r\,\mathrm{d}R.\end{aligned}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}\varrho}\lambda\Big(sector_p(A(r_1,r_2,M^*),\varrho)\Big)\Big|_{\varrho=1} = \int_{r_1}^{r_2}\int_{M^*}(1-r^p)^{1/p-1}r^{n-2}J^*\big(SPH_p^{(n-1)}\big)(\varphi)\,\mathrm{d}(\varphi,r).$$

This integral representation for $\mathfrak{O}_p(A(r_1,r_2,M^*))$ coincides with that for $O_{p,p*}(A(r_1,r_2,M^*))$ for both $p>1$ and $p=1$. Hence, the measures $O_{p,p*}$ and $\mathfrak{O}_p$ coincide on the semi-algebra $\mathfrak{S}_p$. They are countably additive on the smallest algebra $\mathfrak{A}_p$ including $\mathfrak{S}_p$. The smallest σ-algebra including $\mathfrak{A}_p$ includes at least the intersections of $l_{n,p}$-balls with the sphere $S_{n,p}$. Thus, by the measure extension theorem, the measures $O_{p,p*}$ and $\mathfrak{O}_p$ coincide on $\mathfrak{B}_{S,p}$.

Notice that by substituting $z=\varrho^{1/p}$ in the latter integral, we get

$$\mathfrak{O}_p(A(r_1,r_2,M^*)) = \frac{1}{p}\int_{r_1^{1/p}}^{r_2^{1/p}} z^{\frac{n-1}{p}-1}(1-z)^{\frac{1}{p}-1}\,\mathrm{d}z\int_{M^*}J^*\big(SPH_p^{(n-1)}\big)(\varphi)\,\mathrm{d}\varphi.$$

This shows that the measure $\mathfrak{O}_p = O_{p,p*}$ is closely connected with a certain type of incomplete Beta function because

$$\int_0^1 z^{\frac{n-1}{p}-1}(1-z)^{\frac{1}{p}-1}\,\mathrm{d}z = B\Big(\frac{n-1}{p},\frac{1}{p}\Big).$$

Notice further that there hold both

$$\int_{M^*_{n-1}} J^*\big(SPH_p^{(n-1)}\big)(\varphi)\,\mathrm{d}\varphi = \omega_{n-1,p} \quad \text{and } 2\cdot\frac{1}{p}\cdot B\Big(\frac{n-1}{p},\frac{1}{p}\Big)\cdot\omega_{n-1,p} = \omega_{n,p}.$$

Hence,

$$\mathfrak{O}_p(S_{n,p}) = \omega_{n,p}.$$

What we have done so far shows that the present approach is closely connected with a generalization of the indivisible method of Cavalieri and Torricelli in the sense that here the indivisibles are the boundaries of convex $l_{n,p}$-balls, and integrating their l_{n,p^*}-surface contents means measuring the $l_{n,p}$-ball volume.

The very special role played by the case $p=1$ is expressed in the following theorem. It says that the 1-generalized surface measure on the $l_{n,1}$-sphere is closely connected with, but nevertheless different from, the Euclidean one.

Theorem 2.

$$\mathfrak{O}_1(A) = \frac{1}{\sqrt{n}}\mathfrak{O}(A), \quad A\in\mathfrak{B}_{S,1}.$$

Proof. We start with considering the measure $\mathfrak{O}_1$:

$$O_{1,q}(A) = n^{1/q} \int_{G(A)} \mathrm{d}x \overset{q\to\infty}{\longrightarrow} \int_{G(A)} \mathrm{d}x = \mathfrak{O}_1(A).$$

On the other hand,

$$\mathfrak{O}(A) = \mathfrak{O}_2(A) = O_{1,2}(A) = n^{1/2} \int_{G(A)} \mathrm{d}x.$$

2.2 The nonconvex case

Using the notion of the Minkowski functional, one can rewrite the integral in the definition of the $l_{n,q}$-surface content of a Borel subset A of the half-sphere $S^+_{n,p}$ in the just considered case $p \geqslant 1$ as

$$O_{p,q}(A) = \int_{G(A)} d_{K_{n,q}(1)}(N(x_1, \dots, x_{n-1}))\,\mathrm{d}x.$$

We may pass from this representation formula to a suitable modification for the case $p \in (0, 1)$. In the remaining part of this subsection, we are merely dealing with this case.

For the announced purposes, we define for $q < 0$, the $d_{S(q)}$-surface content of a Borel subset A of the half-sphere $S^+_{n,p}$ with $p \in (0, 1)$ as

$$O_{p,q}(A) = \int_{G(A)} d_{S(q)}(N(x_1, \dots, x_{n-1}))\,\mathrm{d}x.$$

Notice that $S(q)$ is a nonconvex star-shaped set and that, because of

$$\lambda \cdot S(q) = \bigl\{(\lambda x_1, \dots, \lambda x_n) \in R^n\colon |x|_q \geqslant 1\bigr\} = \bigl\{y \in R^n\colon \lambda \geqslant |y|_q\bigr\}, \quad \lambda > 0,$$

there holds $z \in \lambda \cdot S(q)$ if and only if

$$\lambda \geqslant |z|_q \equiv \inf\bigl\{\lambda > 0\colon z \in \lambda \cdot S(q)\bigr\}.$$

Hence, the Minkowski functional $d_{S(q)}(z) = \inf\{\lambda > 0\colon z \in \lambda \cdot S(q)\}$ of the set $S(q)$ may be written as

$$d_{S(q)}(z) = I_{\prod_{i=1}^n z_i \neq 0}(z) \cdot |z|_q, \quad z \in R^n.$$

Consequently,

$$O_{p,q}(A) = \int_{G(A)} \biggl(1 + \sum_{i=1}^{n-1} \Bigl|\frac{\partial \varphi_{+(-)}(x)}{\partial x_i}\Bigr|^q\biggr)^{1/q} \mathrm{d}x.$$

This formula generalizes the notion of the $l_{2,q}$-arc-length $AL_{p,q}$ of the $l_{2,p}$-circle $S_{2,p}$ studied in [32] to the multivariate case $n \geqslant 2$.

Theorem 3. *If $p \in (0, 1)$ and $p^* = \frac{p}{p-1}$, then there holds*

$$O_{p,p^*}(A) = \mathfrak{O}_p(A), \quad A \in \mathfrak{B}_{S,p}.$$

Proof. Starting from the last equation before the theorem, we observe that

$$O_{p,q}(A) = \int_{G(A)} \left(1 + \sum_{i=1}^{n-1} |x_i|^{(p-1)q} \left(1 - \sum_{1}^{n-1} |x_j|^p\right)^{\frac{1-p}{p}q}\right)^{1/q} \mathrm{d}x.$$

On choosing $q = p^*$, it follows that

$$O_{p,p^*}(A) = \int_{G(A)} \left(1 + \sum_{i=1}^{n-1} \frac{|x_i|^p}{1 - \sum_1^{n-1} |x_j|^p}\right)^{\frac{p-1}{p}} \mathrm{d}x = \int_{G(A)} \left(1 - \sum_{1}^{n-1} |x_i|^p\right)^{\frac{1-p}{p}} \mathrm{d}x.$$

Proceeding now as in the proof of Theorem 1, we get the assertion.

This theorem shows that we are also in the present case confronted with a generalization of the indivisible method of Cavalieri and Torricelli in the sense that the indivisibles are now the boundaries of nonconvex $l_{n,p}$-balls, and integrating their $d_{S(p^*)}$-surface contents means measuring the $l_{n,p}$-ball's volume.

3 GEOMETRIC MEASURE REPRESENTATION

According to [5] and [13], we denote by $\mathfrak{R}$ the set of all nonnegative random variables defined on the same probability space where the random variable R_p and the random vector U_p are defined. Let F be any distribution function of a positive random variable and put

$$L_n(F) = \left\{X\colon\ X \overset{d}{=} RU_p,\ R \in \mathfrak{R} \text{ has distribution function } F,\ R \text{ and } U_p \text{ are stochastically independent}\right\}.$$

From now on, let X denote an arbitrary element of $L_n(F)$. The random vector X is called $l_{n,p}$-symmetric or $l_{n,p}$-spherical distributed, or even $l_{n,p}$-norm symmetric distributed if $p \geqslant 1$, and the corresponding random variable R is called its generating variate. The assumption $X \in L_n(F)$ implies that X has a density iff R has a density. In this case, the density of X is

$$\varphi_{g,p}(x) = C_p(n,g)g\big(|x|_p^p\big),$$

where $g|R^+ \to R^+$ is called the density-generating function. It is assumed that g satisfies the assumption $I_{n+2,g,p} < \infty$. Here, we use the notation $I_{k,g,p} = \int_0^\infty r^{k-1} g(r^p)\,\mathrm{d}r$, and $C_p(n,g) = 1/(n\pi_n(p)I_{n,\,g,\,p})$ is a suitably chosen normalizing constant.

Because of the independence of the random variables R and U_p, it holds that $P(R < r) = P(X \in K_{n,p}(r))$. Changing the Cartesian coordinates with p-generalized spherical coordinates in the integral representation of the latter probability, one can see immediately that the connection between the density-generating function g and the density f of the random variable R is given by the equation

$$f(r) = I_{n,g,p}^{-1} r^{n-1} g(r^p) I_{(0,\infty)}(r).$$

For an $l_{n,p}$-spherical distribution defined in this way, we shall use the notation $N_{g,p}$. This distribution is the p-generalized normal distribution if the density-generating function is $g(r) = \mathrm{e}^{-r/p} I_{(0,\infty)}(r)$. In this case, we have $1/I_{n,g,p} = p^{1-n/p}/\Gamma(n/p)$. For each form parameter $p > 0$, there exists a density-generating function g such that $N_{g,p}$ is a product measure.

Theorem 4. *The $l_{n,p}$-symmetric distribution with density-generating function g satisfies the representation*

$$N_{g,p}(A) = \frac{1}{I_{n,\,g,\,p}} \int_0^\infty \mathfrak{F}_p(A,r) r^{n-1} g(r^p)\,\mathrm{d}r, \quad A \in \mathfrak{B}^n.$$

Proof. Let us first consider Borel sets of the type

$$A_p(D, \varrho_1, \varrho_2) = sector_p(D, \varrho_2) \setminus sector_p(D, \varrho_1), \quad D \in \mathfrak{B}_{S,p},$$

for arbitrary $\varrho_1 < \varrho_2$ from $[0, \infty)$. The collection of all such sets is a semi-algebra on R^n, say $\mathfrak{S}_p$. The smallest algebra including $\mathfrak{S}_p$, i.e., the collection of finite unions of elements from $\mathfrak{S}_p$, will be denoted by $\mathfrak{A}_p$. Let us further consider the finitely additive function $N^*_{g,p}$ on $\mathfrak{A}_p$ which is already defined by its values for elements from $\mathfrak{S}_p$ as

$$N^*_{g,p}(A) = \frac{1}{I_{n,\,g,\,p}} \int_{\varrho_1}^{\varrho_2} \mathfrak{F}_p(A, r) r^{n-1} g(r^p)\,\mathrm{d}r, \quad A = A_p(D, \varrho_1, \varrho_2),\ 0 \leqslant \varrho_1 < \varrho_2 < \infty,\ D \in \mathfrak{B}_{S,p}.$$

Note that $N^*_{g,p}(A_n)$ tends to zero whenever $(A_n)_{n \in N}$ is a decreasing sequence of sets from $\mathfrak{A}_p$ satisfying $\bigcap_n A_n = \emptyset$ for the empty set $\emptyset$. This means that $N^*_{g,p}$ is continuous at $\emptyset$ and therefore is a countably additive function on $\mathfrak{A}_p$. Now, it remains to show that $N^*_{g,p}$ coincides with $N_{g,p}$ on $\mathfrak{A}_p$. Then, by the measure extension theorem, $N^*_{g,p}$ coincides with $N_{g,p}$ on the whole σ-algebra $\mathfrak{B}^n$.

For doing the latter, it suffices to show that $N^*_{g,p}(A)$ is the same as $N_{g,p}(A)$ for sets of the type $A = A_p(D, \varrho_1, \varrho_2)$, $0 \leqslant \varrho_1 < \varrho_2 < \infty$, $D \in \mathfrak{B}_{S,p}$. Due to the product structure of the set $A_p(D, \varrho_1, \varrho_2) = [\varrho_1, \varrho_2] \cdot D$, the function $r \to \mathfrak{F}_p(A_p(D, \varrho_1, \varrho_2), r)$ is constant with the constant being $\omega_p(D)$. Hence,

$$N^*_{g,p}(A_p(D, \varrho_1, \varrho_2)) = \frac{\omega_p(D)}{I_{n,\,g,\,p}} \int_{\varrho_1}^{\varrho_2} r^{n-1} g(r^p)\,\mathrm{d}r = \omega_p(D) \int_{\varrho_1}^{\varrho_2} f(r)\,\mathrm{d}r.$$

Let us consider now $N_{g,p}$. Because R and U_p are independent and $sector_p(D, \varrho) = [0, \varrho] \cdot D$, it follows that

$$N_{g,p}(sector_p(D, \varrho)) = P^R([0, \varrho]) P^{U_p}(D) = \omega_p(D) \int_0^{\varrho} f(r)\,\mathrm{d}r, \quad \varrho > 0,\ D \in \mathfrak{B}_{S,p}.$$

Hence,

$$N_{g,p}(A_p(D, \varrho_1, \varrho_2)) = \omega_p(D) \int_{\varrho_1}^{\varrho_2} f(r)\,\mathrm{d}r, \quad 0 \leqslant \varrho_1 < \varrho_2 < \infty,\ D \in \mathfrak{B}_{S,p}.$$

Geometric measure representation formulas apply to several problems from probability theory and mathematical statistics. Just to mention some of them, we refer to determining the asymptotic behavior of large-deviation probabilities as a function of the geometric structure of the multivariate domain of deviations in [26] (and succeeding papers), remainder-term estimation in the multivariate central limit theorem in [27], studying functionals of Gaussian vectors of fixed or increasing dimensions in [7, 24], proving a Hilbert space probability inequality in [35], constructing the coefficients in an asymptotic expansion for probabilities of multivariate large deviations in [3, 33], studying finite sample and large deviation properties in two-way ANOVA in [34], evaluating probabilities and large quantiles of noncentral statistical distributions in [14, 21], constructing exact tests and confidence regions in nonlinear regression in [15, 29], and evaluating probabilities of correct classification in [21]. Theorem 4 opens now the possibility for extending several of these results to the case of arbitrary $p > 0$ and to give exact representation formulas for distributions of functions not only of p-generalized normally distributed random vectors but also of $l_{n,p}$-symmetric distributed ones. Some first examples will be studied in the next section.

4 EXAMPLES

In the first example, we derive a further generalization of the χ^p- distribution w.r.t. the more general density-generating function g. To this end, let $A = A(\varrho) = K_{n,p}(\varrho), \varrho > 0$. Then, the $l_{n,p}$-sphere intersection-percentage function satisfies the representation

$$\mathfrak{F}_p(A(\varrho), r) = I_{[0,\varrho)}(r), \quad r > 0,$$

and from Theorem 4 it follows that

$$N_{g,p}(A(\varrho)) = \frac{1}{I_{n,g,p}} \int_0^{\varrho} r^{n-1} g(r^p)\,\mathrm{d}r.$$

Let us now consider the χ^p-statistic R^p. For all $y > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}y} P(R^p < y) = \frac{\mathrm{d}}{\mathrm{d}y} N_{g,p}(A(y^{1/p})) = \frac{y^{\frac{n}{p}-1} g(y)}{p \cdot I_{n,g,p}} =: f^{\chi}_{n,g,p}(y).$$

This motivates the following definition. A continuous random variable Z will be said to be distributed according to the χ^p_g-distribution with n d.f. if its density is $f^{\chi}_{n,g,p}$. Symbolically, in this case, we write $Z \sim \chi^p_g(n)$. The following theorem summarizes this example.

Theorem 5. *If X follows the $l_{n,p}$-symmetric sample distribution $N_{g,p}$ with form parameter p and density-generating function g, then $R^p \sim \chi^p_g(n)$.*

In the second example, let A satisfy the assumption

$$\mathfrak{F}_p(A, r) = C \quad \text{for all } r > 0 \text{ and a certain constant } C \in (0, 1).$$

It follows immediately from Theorem 1 that $N_{g,p}(A) = C$. Thus, the probability integral does not depend on whether the distribution has light or heavy tails. This motivates the following definition.

If the sets $A(t)$ are generated by a statistic T, $A(t) = \{T < t\}, t \in R$, and if, for each fixed t, the $l_{n,p}$-sphere intersection-percentage function of $A(t)$ does not depend on $r > 0$, then the statistic T is called robust with respect to the density-generating function g.

We now consider a situation where this applies. Let the p-generalized Fisher statistic be defined for $1 \leqslant m < n$ by

$$T_{m,n-m}(p) = \frac{\frac{1}{m}(|X_1|^p + \cdots + |X_m|^p)}{\frac{1}{n-m}(|X_{m+1}|^p + \cdots + |X_n|^p)}.$$

The random event $\{T_{m,n-m}(p) < t\}$ can be reformulated for arbitrary $p > 0$ and $t > 0$ as X falling into the set

$$C_m(p\,; t) = \left\{ x \in R^n \colon \frac{|x_1|^p + \cdots + |x_m|^p}{|x_{m+1}|^p + \cdots + |x_n|^p} < \frac{mt}{n-m} \right\}.$$

Notice that $C_m(p; t)$ is a cone with vertex in the origin. This ensures that if $x \in S_{n,p}$, then x belongs to $C_m(p; t)$ if and only if rx belongs to $C_m(p; t)$ for all $r > 0$. Hence, the $l_{n,p}$-sphere intersection-percentage function of the set $C_m(p; t)$ is constant w.r.t. the p-radius r. The p-generalized Fisher statistic is therefore robust w.r.t. the density-generating function, and its distribution function can be written as

$$P(T_{m,n-m}(p) < t) = \mathfrak{F}_p(C_m(p\,; t), 1) I_{(0,\infty)}(t).$$

We determine now the actual $\mathfrak{F}_p$-value.

Theorem 6. (a) *The $l_{n,p}$-sphere intersection-percentage function of the cone $C_m(p\,;t)$ allows the representation formula*

$$\mathfrak{F}_p(C_m(p\,;t),1) = \frac{p}{B\left(\frac{m}{p},\frac{n-m}{p}\right)} \int_{arccot((\frac{mt}{n-m})^{1/p})}^{\pi/2} \frac{(\cos\varphi)^{m-1}(\sin\varphi)^{n-m-1}}{(N_p(\varphi))^n}\,\mathrm{d}\varphi,$$

where $N_p(\varphi) = ((\sin\varphi)^p + (\cos\varphi)^p)^{1/p}$.

(b) *The density of the random variable $T_{m,n-m}(p)$ is*

$$\frac{\mathrm{d}}{\mathrm{d}t}P(T_{m,n-m}(p) < t) = \frac{\left(\frac{m}{n-m}\right)^{\frac{m}{p}} t^{\frac{m}{p}-1}}{B\left(\frac{m}{p},\frac{n-m}{p}\right)\left(1+\frac{mt}{n-m}\right)^{\frac{n}{p}}} I_{(0,\infty)}(t).$$

Proof. (a) In order to determine the function

$$t \to \mathfrak{F}_p(C_m(p\,;t),1) = \frac{1}{\omega_{n,p}} \int_{\{x\in S_{n,p}\cap C_m(p;t)\}} \mathfrak{O}_p(\,\mathrm{d}x),$$

we replace the Cartesian coordinates by p-generalized spherical coordinates. Following the line in [13, 29], we do this separately in two subspaces of R^n, $\mathfrak{L}(\{e_1,\ldots,e_m\})$ and $\mathfrak{L}(\{e_{m+1},\ldots,e_n\})$, which are spanned by the standard unit vectors $e_1,\ldots,e_m$ and $e_{m+1},\ldots,e_n$, respectively. If

$$SPH_{p,1}|(r_1,\varphi_1,\ldots,\varphi_{m-1}) \to (x_1,\ldots,x_m)$$

and

$$SPH_{p,2}|(r_2,\varphi_{m+1},\ldots,\varphi_{n-1}) \to (x_{m+1},\ldots,x_n)$$

denote the corresponding p-generalized coordinate transformations, then the cone $C_m(p;t)$ coincides with the $SPH_{p,2}\circ SPH_{p,1}$-image of the set

$$\Big\{(r_1,\varphi_1,\ldots,\varphi_{m-1},r_2,\varphi_{m+1},\ldots,\varphi_{n-1})\colon \frac{r_1^p}{r_2^p} < \frac{m}{n-m}\,t\Big\}.$$

Replacing now the coordinates r_1, r_2 by the usual polar coordinates r, φ_m,

$$POL|(r,\varphi_m) \to (r_1,r_2),$$

gives the final representation of the cone under consideration:

$$\begin{aligned} C_m(p;t) \;\; &= \;\; SPH_{p,2}\circ SPH_{p,1}\circ POL\bigg(\Big\{(r,\varphi_1,\ldots,\varphi_{n-1})\colon (\varphi_1,\ldots,\varphi_{m-1})\in M_m^*, \\ &\qquad (\varphi_{m+1},\ldots,\varphi_{n-1})\in M_{n-m}^*,\;\; r\in[0,\infty),\;\; 0\leqslant \cot\varphi_m < \Big(\frac{m}{n-m}\,t\Big)^{1/p}\Big\}\bigg). \end{aligned}$$

The Jacobian of the composite transformation $SPH_{p,2}\circ SPH_{p,1}\circ POL$ is the product of the three separate Jacobians, i.e.,

$$\begin{aligned} &J(SPH_{p,2}\circ SPH_{p,1}\circ POL)(r,\varphi_1,\ldots,\varphi_{n-1}) \\ &\quad = \frac{r^{n-1}(\cos\varphi_m)^{m-1}(\sin\varphi_m)^{n-m-1}}{(N_p(\varphi_m))^{2+(m-1)+(n-m-1)}} J^*(SPH_{p,1})(\varphi_1,\ldots,\varphi_{m-1}) J^*(SPH_{p,2})(\varphi_{m+1},\ldots,\varphi_{n-m-1}), \end{aligned}$$

where

$$J^*(SPH_{p,1})(\varphi_1, \dots, \varphi_{m-1}) = \prod_{i=1}^{m-1} (\sin \varphi_i)^{m-1-i} / N_p(\varphi_i)^{m+1-i},$$

and $J^*(SPH_{p,2})(\varphi_{m+1}, \dots, \varphi_{n-m-1})$ is defined similarly. Recalling the notation $\omega_{k,p}$ for the $l_{k,p}$-generalized surface content of the unit sphere $S_{k,p}$, we see that

$$\int_{M_m^*} J^*(SPH_{p,1})(\varphi_1, \dots, \varphi_{m-1}) \, \mathrm{d}\varphi_1 \dots \mathrm{d}\varphi_{m-1} = \omega_{m,p}$$

and

$$\int_{M_{n-m}^*} J^*(SPH_{p,2})(\varphi_{m+1}, \dots, \varphi_{n-m-1}) \, \mathrm{d}\varphi_{m+1} \dots \mathrm{d}\varphi_{n-m-1} = \omega_{n-m,p}.$$

It follows that

$$\omega_{n,p} \mathfrak{F}_p(C_m(p\,; t), 1) = \omega_{m,p} \omega_{n-m,p} \int_{arccot((\frac{mt}{n-m})^{1/p})}^{\pi/2} \frac{(\cos \varphi)^{m-1} (\sin \varphi)^{n-m-1}}{(N_p(\varphi))^n} \, \mathrm{d}\varphi.$$

(b) The probability density function corresponding to the distribution function of the p-generalized Fisher statistic $T_{m,n-m}(p)$ is therefore

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} P(T_{m,n-m}(p) < t) \quad &= \quad -\frac{p}{B(\frac{m}{p}, \frac{n-m}{p})} \Bigg(\frac{(\frac{mt}{n-m})^{1/p}}{\sqrt{1 + (\frac{mt}{n-m})^{2/p}}} \Bigg)^{m-1} \Bigg(\frac{1}{\sqrt{1 + (\frac{mt}{n-m})^{2/p}}} \Bigg)^{n-m-1} \\ &\quad \times \frac{(1 + (\frac{mt}{n-m})^{2/p})^{n/2}}{(1 + \frac{mt}{n-m})^{n/p}} \Bigg(-\frac{1}{1 + (\frac{mt}{n-m})^{2/p}} \Bigg) \frac{1}{p} \Big(\frac{mt}{n-m} \Big)^{1/p-1} \frac{m}{n-m}. \end{aligned}$$

Due to Theorem 6, we say that a continuous random variable Z is distributed according to the p-generalized Fisher- or F-distribution with m and n d.f. if its density is

$$f_{m,n;p}^F(t) = \frac{(\frac{m}{n})^{\frac{m}{p}} t^{\frac{m}{p}-1}}{B(\frac{m}{p}, \frac{n}{p})} \Big(1 + \frac{m}{n} t\Big)^{-\frac{m+n}{p}} I_{(0,\infty)}(t).$$

In this case, we write $Z \sim F_{m,n}(p)$. This density formula coincides with the corresponding well-known density formula of the (usual) $F_{m,n}$-distribution if $p = 2$ and with that of the $F_{2m,2n}$-distribution as derived in [13] for the case $p = 1$.

Notice that the analytical structure of $f_{m,n;p}^F$ is closely connected with that of the generalized F-density $f_{GF(m_1,m_2,\lambda,p)}$, which may be derived for the random variable T in the four-parameter (m_1, m_2, λ, p)-model considered in [19],

$$\ln T = -\ln \lambda + \frac{1}{p} W,$$

if it is assumed that $\ln W$ follows the Fisher distribution with $2m_1$ and $2m_2$ d.f. To be more concrete, it holds that

$$f_{m,n,p}^F\Big(\frac{x}{p}\Big) = f_{GF(\frac{m}{p}, \frac{n}{p}, \lambda(x), p)}(x),$$

where the natural numbers m_1, m_2 and the parameter λ have been formally replaced in $f_{GF(m_1,m_2,\lambda,p)}$ by the real numbers $\frac{m}{p}, \frac{n}{p}$ and the function $\lambda(x) = \frac{1}{p}(\frac{x}{p})^{1/p-1}$, respectively.

Quite another type of generalized F-distribution was considered in [16], where the ratio of (partly) weighted sums of squares is considered for Gaussian sample distribution, and the resulting density is given by a series representation.

The $l_{n,p}$-sphere intersection-percentage function with $p \in \{1,2\}$ has been determined in the literature for different types of Borel sets which occur in a natural way in different applications of geometric measure representation formulae. Just to mention some of them, we refer to Euclidean balls and $l_{n,2}$-cones in [28], half-spaces in [7, 34], shifted balls in [14, 35], $l_{n,1}$-cones in [13], parabolic cylinder type sets in [21], and curved transformed cone type sets in [15].

Just to extend the second example, let us define a system $\mathfrak{A}_m^{(p)} \subset \mathfrak{B}^n$ of Borel sets as follows. We say that a set A belongs to $\mathfrak{A}_m^{(p)}$ if there exists a function $\lambda^*|[0,\infty) \to [0,\infty]$ such that A satisfies the condition

$$A \cap S_{n,p}(r) = C_m(p;\lambda^*(r)) \cap S_{n,p}(r), \quad r > 0.$$

This means that, for fixed $r > 0$, the value $\mathfrak{F}_p(A;r)$ of the $l_{n,p}$-sphere intersection-percentage function of such a set A coincides with the constant value of the intersection-percentage function of the cone $C_m(p;\lambda^*(r))$. Hence, if $A \in \mathfrak{A}_m^{(p)}$, we have the geometric measure representation

$$N_{g,p}(A) = \frac{1}{I_{n,g,p}} \int_0^\infty \mathfrak{F}_p\Big(C_m(p;\lambda^*(r)),1\Big) r^{n-1} g(r^p)\,\mathrm{d}r,$$

where $\mathfrak{F}_p(C_m(p;t),1)$ has been defined in Theorem 6.

To additionally illustrate this example, let $N_n = \{1,\ldots,n\}$, $1 \leqslant m < n, 1 \leqslant i_1 < \ldots < i_m \leqslant n$, $I_m = \{i_1,\ldots,i_m\}$, $R_p^{(m)} = (\sum_{j\in I_m} |X_j|^p)^{1/p}$, $R_{p,m} = (\sum_{j\in N_n\setminus I_m} |X_j|^p)^{1/p}, A = \{R_p^{(m)} < R_{p,m}^2\}$, and $\lambda^*(r) = \frac{n-m}{n}(\sqrt{\frac{1}{4}+r^p} - \frac{1}{2})$. Then, $A \in \mathfrak{A}_m^{(p)}$.

5 ON $l_{n,p}$-BALL NUMBERS

It was recently shown in [31, 32] that Archimedes' or Ludolph's number π is not alone as a circle number. There exists a continuous increasing function $p \to \pi(p)$, $p > 0$, with $\lim_{p\to 0}\pi(p) = 0$, $\pi(1) = 2$, $\pi(2) = \pi$, $\lim_{p\to\infty}\pi(p) = 4$ and such that, for each $p > 0$, $\pi(p)$ reflects both the area content and the $l_{2,p}$-generalized perimeter properties of the $l_{2,p}$-circle. This means that if $\mathfrak{U}_p(r) = AL_{p,p^*}(r)$ and $A_p(r)$ denote the $l_{2,p}$-generalized circumference and the area content of an $l_{2,p}$-circle $S_{2,p}(r)$ of p-generalized radius r and its included disc $K_{2,p}(r)$, respectively, then the ratios $\mathfrak{U}_p(r)/(2r)$ and $A_p(r)/r^2$ attain the same values and do not depend on $r > 0$, and their common value is $\pi(p)$, $p > 0$. It turns out that, as in the case $p = 2$, there holds

$$\pi(p) = A_p(1) = \frac{2\Gamma^2(\frac{1}{p})}{p\Gamma(\frac{2}{p})}$$

in all cases $p > 0$.

Here, the notion of a circle number will be extended to that of a ball number of an arbitrary $l_{n,p}$-ball $K_{n,p}(r)$ of p-radius $r > 0$ for $p > 0$, $n \in \{2,3,\ldots\}$. To this end, let us denote the usual (Euclidean) volume of $K_{n,p}(r)$ by $V_{n,p}(r)$. The ratio $V_{n,p}(r)/r^n = V_{n,p}(1)$ does not depend on $r > 0$. It follows immediately from Sections 2 and 3 that this ratio is in fact equal to the ratio $\mathfrak{O}_p(r)/(nr^{n-1})$. We will say that the common value actually attained by these ratios reflects the volume and the p-generalized surface content properties of the $l_{n,p}$-balls. This value is, for each $p > 0$,

$$\pi_n(p) = V_{n,p}(1) = \frac{2^n \Gamma^n(\frac{1}{p})}{np^{n-1}\Gamma(\frac{n}{p})}$$

and is called the $l_{n,p}$-ball number.

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