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A method for generating uniformly scattered points on the L_p-norm unit sphere and its applications

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Abstract Tashiro (Ann Inst Stat Math 29:295–300, 1977) studied methods for generating unform points on the surface of the regular unit sphere. The L_p-norm unit sphere is a generalization of the regular unit sphere. In this paper we propose a method associated with an algorithm for generating uniformly scattered points on the L_p-norm unit sphere and discuss its applications in statistical simulation, representative points of a wide class of multivariate probability distributions and optimization problems. Some examples are illustrated for these applications.

Keywords L_p-norm spherical distribution · L_p-norm uniform distribution · Stochastic representation · Uniformly scattered points

1 Introduction

Sets of uniformly scattered points in some regular regions in the Euclidean space R^s ($s \geq 2$) have found many applications in the field of numerical integration, quasi Monte Carlo methods and statistical simulation (see, for instance, Fang and Wang 1994; Hua and Wang 1981; Niederreiter 1992; Sloan and Joe 1994). Some methods

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have been proposed to generate uniformly scattered points in the regular regions such as the square (in R^2), the cube (in R^3) or the hypercube (in R^s, $s > 3$), the simplex, and the unit sphere (Tashiro 1977) in the sense of the Euclidean norm in R^s ($s \geq 2$). These methods have been used to generate empirical samples or representative points (Fang and Wang 1994) from some multivariate probability distributions such as the spherical and elliptical distributions (Fang et al. 1990). In the literature of number-theoretic methods or quasi Monte Carlo methods, there are different criteria on the uniformity of a set of points in a given region (Fang and Wang 1994). For an arbitrarily given region $D \subset R^s$ ($s \geq 2$), it is often difficult to find a direct method for generating a set of uniformly scattered points in D. A transformation method is usually necessary. One of the commonly used approaches as pointed out by Fang and Wang (1994) (Sects. 1.5–1.6) is to generate a set of uniformly scattered points $\mathcal{P} = \{z_1, \ldots, z_n\}$ in the hypercube

$$C^s = [0, 1]^s = \{z = (z_1, \ldots, z_s)' \in R^s,\ 0 \leq z_i \leq 1,\ i = 1, \ldots, s\} \tag{1}$$

and then use the inverse transformation method (Fang and Wang 1994, p. 45) to project the set of points in $\mathcal{P} = \{z_1, \ldots, z_n\} \subset C^s$ into the given region $D \subset R^s$. The uniformity of the set of points in $\mathcal{P} \subset C^s$ is measured by discrepancy defined by (Fang and Wang 1994, p. 15)

$$D(n, \mathcal{P}) = \sup_{\boldsymbol{\gamma} \in C^s} \left| \frac{N(\boldsymbol{\gamma}, \mathcal{P})}{n} - v([\mathbf{0}, \boldsymbol{\gamma}]) \right|, \tag{2}$$

where $v([\mathbf{0}, \boldsymbol{\gamma}])$ denotes the volume of the rectangle $[\mathbf{0}, \boldsymbol{\gamma}]$, and $N(\boldsymbol{\gamma}, \mathcal{P})$ denotes the number of points in $\mathcal{P}$ satisfying $z_i \leq \boldsymbol{\gamma}$ ("$\leq$" means componentwise, $i = 1, \ldots, n$). After projecting the points in $\mathcal{P} \subset C^s$ into the region D by the inverse transformation method, the uniformity of the projected set of points $\mathcal{P}_F \subset D$ can be measured by the quasi F-discrepancy defined as follows.

Definition 1 (Fang and Wang 1994, p. 42) Let $\boldsymbol{x}$ be an $s \times 1$ random vector with c.d.f. (cumulative distribution function) $F(\boldsymbol{x})$. $\boldsymbol{x}$ has a stochastic representation $\boldsymbol{x} = \boldsymbol{h}(z)$, where $z \sim U(C^t)$ (the uniform distribution on the hypercube C^t defined by (1) with s replaced by t, $t \leq s$). Let $\{\boldsymbol{c}_k : k = 1, \ldots, n\}$ be a set of points that are uniformly scattered in C^t with discrepancy d in the sense of (2). Then the set of points $\mathcal{P}_F = \{\boldsymbol{h}(\boldsymbol{c}_k) : k = 1, \ldots, n\}$ is said to have a quasi F-discrepancy d with respect to $F(\boldsymbol{x})$.

Definition 2 (Fang and Wang 1994, p. 15) Let $F(\boldsymbol{x})$ be a c.d.f. in R^s and $\mathcal{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ a set of points in R^s. The empirical distribution function on $\mathcal{P}$ is defined by

$$F_n(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} I\{\boldsymbol{x}_i \leq \boldsymbol{x}\},$$

where $I(\cdot)$ stands for the indicator function on a set and "$\boldsymbol{x}_i \leq \boldsymbol{x}$" means componentwise. Then

$$D_F(n, \mathcal{P}) = \sup_{\boldsymbol{x} \in R^s} |F_n(\boldsymbol{x}) - F(\boldsymbol{x})| \tag{3}$$

is called the F-discrepancy of $\mathcal{P}$ with respect to $F(\boldsymbol{x})$.

(Fang and Wang 1994, pp. 42–44) gives a detailed discussion on using the quasi F-discrepancy as a reasonable measure of uniformity on the projected set of points $\mathcal{P}_F \subset D$ in Definition 1. When the random vector $\boldsymbol{z} = (z_1, \ldots, z_t)'$ in the stochastic representation $\boldsymbol{x} = \boldsymbol{h}(\boldsymbol{z})$ in Definition 1 has independent components $z_1, \ldots, z_t$ with a c.d.f. $H_i(z_i)$ for z_i $(i = 1, \ldots, t)$, (Fang and Wang 1994, pp. 42–44) prove that the quasi F-discrepancy d in Definition 1 with respect to c.d.f. $F(\boldsymbol{x})$ is actually the regular F-discrepancy (3) with respect to the c.d.f. $H(\boldsymbol{z}) = \prod_{i=1}^{t} H_i(z_i)$ of $\boldsymbol{z}$. It turns out that

$$d = \sup_{\boldsymbol{r} \in C^t} \left| \frac{N(\mathcal{P}_F, G_{\boldsymbol{r}})}{n} - H(\boldsymbol{r}) \right|, \tag{4}$$

where $\boldsymbol{r} = (r_1, \ldots, r_t)' \in C^t$, $H(\boldsymbol{r}) = \prod_{i=1}^{t} H_i(r_i)$, $\mathcal{P}_F$ is the set defined in Definition 1, $N(\mathcal{P}_F, G_{\boldsymbol{r}})$ stands for the number of points in $\mathcal{P}_F$ that fall in the set $G_{\boldsymbol{r}}$ defined by

$$G_{\boldsymbol{r}} = \{\boldsymbol{x} : \boldsymbol{x} = \boldsymbol{h}(\boldsymbol{z}),\ \boldsymbol{z} \leq \boldsymbol{r}\}. \tag{5}$$

Equation (4) is exactly the regular F-discrepancy of $\mathcal{P}_F$ with respect to the c.d.f. $H(\boldsymbol{z})$ of $\boldsymbol{z}$. Therefore, in order to obtain a set of uniformly scattered points $\mathcal{P}_F \subset D$ in a given region D, three steps are essential:

Step 1 Find a set of uniformly scattered points $\mathcal{P} = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\} \subset C^s$ with the smallest discrepancy d in the sense of (2);

Step 2 Find a suitable stochastic representation $\boldsymbol{x} = \boldsymbol{h}(\boldsymbol{z})$ so that $\boldsymbol{z} = (z_1, \ldots, z_s)'$ has independent components $z_1, \ldots, z_s$ with some c.d.f. $H_i(z_i)$ for z_i $(i = 1, \ldots, s)$;

Step 3 Perform the inverse transformations

$$\boldsymbol{x}_k = \boldsymbol{h}\Big(H_1^{-1}(c_{k1}), \ldots, H_s^{-1}(c_{ks})\Big), \quad k = 1, \ldots, n, \tag{6}$$

where $\boldsymbol{c}_k = (c_{k1}, \ldots, c_{ks}) \in \mathcal{P} = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\} \subset C^s$ in step (1). Then the set of points $\mathcal{P}_F = \{\boldsymbol{x}_k$ given by (6) : $k = 1, \ldots, n\} \subset D$ has the smallest quasi F-discrepancy (with respect to the c.d.f. of $\boldsymbol{x}$ in step (2)) d that is equal to (4), or $\mathcal{P}_F$ has the smallest F-discrepancy with respect to the c.d.f. $H(\boldsymbol{z}) = \prod_{i=1}^{s} H_i(z_i)$ of $\boldsymbol{z}$ in step (2).

Step (1) is realized by using the set of good lattice points (called the *glp* set by Fang and Wang (1994, p. 21)) in C^s as defined below. Step (2) will be realized by using the uniform distribution on the given region D.

Definition 3 (Fang and Wang 1994, p. 21) Let $(n; h_1, \ldots, h_s)$ be a vector with integer components satisfying $1 \le h_i \le n, h_i \ne h_j\ (i \ne j), s < n$, and the greatest common divisors $(n, h_i) = 1, i = 1, \ldots, s$. Let

$$\begin{cases} q_{ki} = kh_i \pmod n, \\ x_{ki} = (2q_{ki} - 1)/2n, \end{cases} \quad k = 1, \ldots, n;\ i = 1, \ldots, s, \tag{7}$$

where $1 \le q_{ki} < n$ and mod=module (e.g., $4 \times 2 = 3 \pmod 5$). Then the set $\mathcal{P}_n = \{\boldsymbol{x}_k = (x_{k1}, \ldots, x_{ks})',\ k = 1, \ldots, n\}$ is called the lattice point set with the generating vector $(n; h_1, \ldots, h_s)$. If the set $\mathcal{P}_n$ has the smallest discrepancy in the sense of (2) among all possible generating vectors, then the set $\mathcal{P}_n$ is called a *glp* set.

It can be verified that the points $\boldsymbol{x}_k = (x_{k1}, \ldots, x_{ks})'$ given by (7) can be calculated by

$$x_{ki} = \left\{ \frac{2kh_i - 1}{2n} \right\}, \quad k = 1, \ldots, n;\ i = 1, \ldots, s, \tag{8}$$

where $\{\cdot\}$ stands for the fractional part of a real number. Fang and Wang (1994, p. 286, Appendix A) tabulates the generating vectors for obtaining the *glp* sets for a wide range of n and the dimension $2 \le s \le 18$. In order to realize the above step (2), we need to construct the uniform distribution in the given region D. In this paper we consider the surface of the L_p-norm unit sphere as the region D and construct the inverse transformation from the uniform distribution on the L_p-norm unit sphere.

Definition 4 Let

$$S_s^p = \left\{ \boldsymbol{x} = (x_1, \ldots, x_s)' \in R^s,\ \|\boldsymbol{x}\|_p = (|x_1|^p + \cdots + |x_s|^p)^{1/p} = 1,\ p > 0,\ s \ge 2 \right\}. \tag{9}$$

S_s^p ($s \ge 2$ is a positive integer) is called the surface of the L_p-norm unit sphere or simply called the L_p-norm unit sphere.

Definition 5 (Song and Gupta 1997) A random vector $\boldsymbol{u}_s = (U_1, \ldots, U_s)'$ is said to have an L_p-norm uniform distribution ($p > 0$), denoted by $\boldsymbol{u}_s \sim \mathcal{U}(s, p)$, if $\sum_{i=1}^s |U_i|^p = 1$ and the joint p.d.f. (probability density function) of $U_1, \ldots, U_{s-1}$ is given by

$$g(u_1, \ldots, u_{s-1}) = \frac{p^{s-1}\Gamma(s/p)}{2^{s-1}\Gamma^s(1/p)} \left(1 - \sum_{i=1}^{s-1} |u_i|^p \right)^{(1-p)/p}, \tag{10}$$
$$-1 < u_i < 1, \quad i = 1, \ldots, s-1, \quad \textstyle\sum_{i=1}^{s-1} |u_i|^p < 1.$$

The L_p-norm uniform distribution is a generalization of the uniform distribution on the usual unit sphere S_s^2 ($p = 2$) (see Fang et al. 1990, Chap. 3). However, some algorithms, such as the algorithms given by Tashiro (1977) and Fang and Wang (1994, pp. 167–170), which have been used to generate uniformly scattered points on S_s^2, cannot

be directly generalized to generating uniformly scattered points on S_s^p for any $p > 0$ ($p \neq 2$). We will present the theoretical derivation for the inverse transformation in Sect. 2. Some applications of the proposed method are illustrated in Sect. 3.

2 Theoretical derivation

The following two theorems provide the basis for our inverse transformation method for generating uniformly scattered points in the L_p-norm unit sphere S_s^p defined by (9).

Theorem 1 *Let* $\boldsymbol{u}_s = (U_1, \ldots, U_s)' \sim \mathcal{U}(s, p)$. *Define the random variables* B_i ($i = 1, \ldots, s-1$) *by the following conditional distributions*:

$$\begin{aligned} B_1 &\stackrel{d}{=} |U_1|^p, \\ B_2 &\stackrel{d}{=} \{(1 - |U_1|^p)^{-1}|U_2|^p \Big| U_1\}, \\ &\vdots \\ B_m &\stackrel{d}{=} \{(1 - \textstyle\sum_{i=1}^{m-1} |U_i|^p)^{-1}|U_m|^p \Big| (U_1, \ldots, U_{m-1})\}, \end{aligned} \tag{11}$$

where $m = 2, \ldots, s-1$, *the sign* "$\stackrel{d}{=}$" *means that the two sides of the equality have the same probability distribution, and* $\{\cdot \Big| \cdot\}$ *stands for the conditional distribution given the part on the right hand side of* "$\Big|$". *Then* $B_1, \ldots, B_{s-1}$ *are mutually independent and* $B_k \sim Beta[1/p, (s-k)/p]$ *(the beta distribution,* $k = 1, \ldots, s-1$*).*

Proof Define the following conditional random variables

$$\begin{aligned} T_1 &\stackrel{d}{=} U_1, \\ T_2 &\stackrel{d}{=} \{U_2 \Big| U_1\} \\ &\vdots \\ T_{s-1} &\stackrel{d}{=} \{U_{s-1} \Big| U_1, \ldots, U_{s-2}\}. \end{aligned} \tag{12}$$

Because the random vector $(U_1, \ldots, U_{s-1})'$ has a joint p.d.f. given by (10), according to the conditional probability integral transformation (Rosenblatt 1952), the random variables $T_1, \ldots, T_{s-1}$ are mutually independent. Note that

$$\begin{aligned} &B_1 \text{ is a function of } T_1 \text{ and uniquely determined by } T_1 \\ &B_2 \text{ is a function of } T_2 \text{ and uniquely determined by } T_2 \\ &\quad\vdots \\ &B_{s-1} \text{ is a function of } T_{s-1} \text{ and uniquely determined by } T_{s-1}. \end{aligned}$$

So $B_1, \ldots, B_{s-1}$ are also mutually independent.

From Theorem 2.1 of Song and Gupta (1997), the random variables $|U_i|^p$ $(i = 1, \ldots, s)$ has a beta distribution $|U_i|^p \sim \text{Beta}[1/p, (s-1)/p]$, and the marginal density of $(U_1, \ldots, U_m)$ $(m = 2, \ldots, s-1)$ is given by

$$h(u_1, \ldots, u_m) = \frac{p^m \Gamma(s/p)}{2^m \Gamma^m(1/p)\Gamma[(s-m)/p]} \left(1 - \sum_{i=1}^{m} |u_i|^p\right)^{(s-m)/p-1}, \quad -1 < u_i < 1, \quad i = 1, \ldots, m, \quad \sum_{i=1}^{m} |u_i|^p < 1. \tag{13}$$

Therefore $B_1 \overset{d}{=} |U_1|^p \sim \text{Beta}[1/p, (s-1)/p]$, and the conditional density function of $(U_m | U_1, \ldots, U_{m-1})$ $(m = 2, \ldots, s-1)$ is given by

$$\begin{aligned} & f(u_m | u_1, \ldots, u_{m-1}) \\ &= \frac{p\Gamma[(s-m+1)/p]}{2\Gamma(1/p)\Gamma[(s-m)/p]} \cdot \left(1 - \sum_{i=1}^{m-1} |u_i|^p\right)^{-1/p} \\ &\quad \times \left(1 - \frac{|u_m|^p}{1 - \sum_{i=1}^{m-1} |u_i|^p}\right)^{(s-m)/p-1}, \\ &\text{for } |u_m| < \left(1 - \sum_{i=1}^{m-1} |u_i|^p\right)^{1/p}. \end{aligned} \tag{14}$$

Then from (14), we can obtain the density function of the random variable $Y = B_m$ defined in (11) as

$$b(y) = \frac{\Gamma[(s-m+1)/p]}{\Gamma(1/p)\Gamma[(s-m)/p]} y^{\frac{1}{p}-1}(1-y)^{\frac{s-m}{p}-1}, \qquad |y| < 1, \tag{15}$$

which is the density function of the beta distribution $B_m \sim \text{Beta}[1/p, (s-m)/p]$ $(m = 2, \ldots, s-1)$. This completes the proof. □

Theorem 2 *Assume that* $\boldsymbol{u} = (U_1, \ldots, U_s)' \sim \mathcal{U}(s, p)$. *Let* $V_1, \ldots, V_s$ *be i.i.d. and* $V_i \sim U(0, 1)$ (*the uniform distribution in* (0, 1), $1 \le i \le s$), *and* $B_1, \ldots, B_{s-1}$ *be independent such that* $B_k \sim Beta[1/p, (s-k)/p]$ $(k = 1, \ldots, s-1)$. *Denote by* $F_k(\cdot)$ *the c.d.f. of* B_k *and* $F_k^{-1}(\cdot)$ *the inverse function of* $F_k(\cdot)$. *Then the random vector* $\boldsymbol{u} = (U_1, \ldots, U_s)' \sim \mathcal{U}(s, p)$ *has a stochastic representation*

$$\boldsymbol{u} \overset{d}{=} \boldsymbol{x} = (X_1, \ldots, X_s)', \tag{16}$$

where the components $X_1, \ldots, X_s$ are given by

$$\begin{aligned} X_1 &= S_1 \left[F_1^{-1}(W_1) \right]^{1/p}, \\ X_2 &= S_2 \left[(1 - |X_1|^p) F_2^{-1}(W_2) \right]^{1/p}, \\ &\vdots \\ X_{s-1} &= S_{s-1} \left[(1 - \textstyle\sum_{i=1}^{s-2} |X_i|^p) F_{s-1}^{-1}(W_{s-1}) \right]^{1/p}, \\ X_s &= S_s \left(1 - \textstyle\sum_{i=1}^{s-1} |X_i|^p \right)^{1/p}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} S_i &= S_i(V_i) = sign(2V_i - 1), \quad i = 1, \ldots, s, \\ W_k &= W_k(V_k) = (2V_k - 1)S_k, \quad k = 1, \ldots, s-1, \end{aligned} \tag{18}$$

and $sign(\cdot)$ stands for the sign function.

Proof From the assumption that $V_1, \ldots, V_s$ are i.i.d. and $V_i \sim U(0, 1)$, we can verify that the random variables W_k's in (18) are i.i.d. with a uniform distribution $U(0, 1)$. Then

$$B_k = F_k^{-1}(W_k) \sim \text{Beta}\left(\frac{1}{p}, \frac{s-k}{p} \right) \tag{19}$$

for $k = 1, \ldots, s-1$. With probability 1, the equations given by (17) can be written as

$$\begin{aligned} X_1 &= \pm B_1^{1/p}, \\ X_2 &= \pm \left[(1 - |X_1|^p) B_2 \right]^{1/p}, \\ &\vdots \\ X_{s-1} &= \pm \left[\left(1 - \sum_{i=1}^{s-2} |X_i|^p \right) B_{s-1} \right]^{1/p}, \\ X_s &= \pm \left(1 - \sum_{i=1}^{s-1} |X_i|^p \right)^{1/p}, \end{aligned} \tag{20}$$

where $B_1, \ldots, B_{s-1}$ given by (19) are independent and $B_k \sim \text{Beta}[1/p, (s-k)/p]$ $(k = 1, \ldots, s-1)$.

Consider the transformation $(X_1, \ldots, X_{s-1}) \to (B_1, \ldots, B_{s-1})$ given by (20). In each of the 2^{s-1} quadrants of R^{s-1}, the transformation given by (20) is uniquely

determined with probability 1. The first-order differential form for the transformation (20) can be calculated as follows:

$$dx_1 \cdots dx_{s-1} = \pm p^{1-s} \left(1 - |x_1|^p\right)^{1/p} \cdots \left(1 - \sum_{i=1}^{s-2} |x_i|^p\right)^{1/p} \times (b_1 \cdots b_{s-1})^{1/p-1} db_1 \cdots db_{s-1}. \tag{21}$$

Then the Jacobian

$$\begin{aligned} &J\Big((X_1, \ldots, X_{s-1}) \to (B_1, \ldots, B_{s-1})\Big) \\ &= 2^{s-1} p^{1-s} \Big(1 - |x_1|^p\Big)^{1/p} \cdots \left(1 - \sum_{i=1}^{s-2} |x_i|^p\right)^{1/p} (b_1 \cdots b_{s-1})^{1/p-1}. \end{aligned} \tag{22}$$

The joint p.d.f. of $(B_1, \ldots, B_{s-1})$ given by (19) can be expressed as

$$\frac{\Gamma(s/p)}{\Gamma^s(1/p)} (b_1, \ldots, b_{s-1})^{\frac{1}{p}-1} (1 - b_1)^{\frac{s-1}{p}-1} (1 - b_2)^{\frac{s-2}{p}-1} \cdots (1 - b_{s-1})^{\frac{1}{p}-1}, \tag{23}$$

where $0 < b_i < 1, i = 1, \ldots, s-1$.

From (20), we obtain

$$\begin{aligned} 1 - b_1 &= 1 - |x_1|^p, \\ 1 - b_2 &= \left(1 - |x_1|^p\right)^{-1} \left(1 - |x_1|^p - |x_2|^p\right), \\ &\vdots \\ 1 - b_{s-1} &= \left(1 - \sum_{i=1}^{s-2} |x_i|^p\right)^{-1} \left(1 - \sum_{i=1}^{s-1} |x_i|^p\right). \end{aligned} \tag{24}$$

Using (22)–(24) and combining some common terms, we can obtain the joint p.d.f. of $(X_1, \ldots, X_{s-1})$:

$$g(x_1, \ldots, x_{s-1}) = \frac{p^{s-1}\Gamma(s/p)}{2^{s-1}\Gamma^s(1/p)} \left(1 - \sum_{i=1}^{s-1} |x_i|^p\right)^{(1-p)/p}, \tag{25}$$

where $-1 < x_i < 1, i = 1, \ldots, s-1, \sum_{i=1}^{s-1} |x_i|^p < 1$. Thus, the random vector $\boldsymbol{x} = (X_1, \ldots, X_s)'$ with the components $X_1, \ldots, X_s$ given by (20) has a uniform distribution $\mathcal{U}(s, p)$, i.e., $\boldsymbol{u} = (U_1, \ldots, U_s)' \stackrel{d}{=} \boldsymbol{x} = (X_1, \ldots, X_s)'$ as defined in (10). This completes the proof.

The stochastic representation given by (16)–(18) for $\boldsymbol{x} \sim \mathcal{U}(s, p)$ determines a transformation from C^s (defined by (1)) to S_s^p (defined by (9)). Let

$\mathcal{P}_z = \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n\} \subset C^s$ be a *glp* set with discrepancy d in the sense of (2). Let us obtain the set of points $\mathcal{P}_x = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subset S_s^p$ in the following way: denote by $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{is})'$ and $\boldsymbol{z}_i = (z_{i1}, \ldots, z_{is})'$ $(i = 1, \ldots, n)$, let

$$
\begin{aligned}
x_{i1} &= S_1(z_{i1}) \left\{ F_1^{-1}[W1(z_{i1})] \right\}^{1/p}, \\
x_{i2} &= S_2(z_{i2}) \left\{ (1 - |x_{i1}|^p) F_2^{-1}[W_2(z_{i2})] \right\}^{1/p}, \\
&\vdots \\
x_{i,s-1} &= S_{s-1}(z_{i,s-1}) \left\{ \left(1 - \sum_{j=1}^{s-2} |x_{ij}|^p\right) F_{s-1}^{-1}[W_{s-1}(z_{i,s-1})] \right\}^{1/p}, \qquad (26) \\
x_{is} &= \begin{cases} S_s(z_{is}) \left(1 - \sum_{j=1}^{s-1} |x_{ij}|^p\right)^{1/p}, & \text{if } z_{is} \neq 0.5, \\ \left(1 - \sum_{j=1}^{s-1} |x_{ij}|^p\right)^{1/p}, & \text{if } z_{is} = 0.5, \end{cases}
\end{aligned}
$$

where $S_l(z_{il})$ $(l = 1, \ldots, s)$ and $W_k(z_{ik})$ $(k = 1, \ldots, s-1)$ are computed through (18) by replacing V_l with z_{il} for each fixed i and $l = 1, \ldots, s$. Based on the steps (1)–(3) in the introduction, the set of points $\mathcal{P}_x = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subset S_s^p$ has a quasi F-discrepancy d with respect to the c.d.f. of $\boldsymbol{x} \sim \mathcal{U}(s, p)$, or $\mathcal{P}_x$ has an F-discrepancy d with respect to the c.d.f. $H(\boldsymbol{v}) = \prod_{i=1}^s v_i$ of the random vector $\boldsymbol{V} = (V_1, \ldots, V_s)'$ with independent components $V_i \sim U(0, 1)$ $(\boldsymbol{v} = (v_1, \ldots, v_s) \in C^s)$. According to Eq. (4), d is equal to:

$$
d = \sup_{\boldsymbol{r} \in C^s} \left| \frac{N(\mathcal{P}_x, G_{\boldsymbol{r}})}{n} - H(\boldsymbol{r}) \right|, \qquad (27)
$$

where $\boldsymbol{r} = (r_1, \ldots, r_s)' \in C^s$, $H(\boldsymbol{r}) = \prod_{i=1}^s r_i$, $N(\mathcal{P}_x, G_{\boldsymbol{r}})$ stands for number of points in $\mathcal{P}_x$ that fall in the set $G_{\boldsymbol{r}}$ defined by

$$
G_{\boldsymbol{r}} = \{\boldsymbol{x} : \boldsymbol{x} = \boldsymbol{h}(\boldsymbol{v}), \ \boldsymbol{v} \leq r\}. \qquad (28)
$$

where $\boldsymbol{h}(\boldsymbol{v}) = (h_1(\boldsymbol{v}), \ldots, h_s(\boldsymbol{v}))'$ $(\boldsymbol{v} = (v_1, \ldots, v_s)' \in C^s)$ with

$$
\begin{aligned}
h_1(\boldsymbol{v}) &= S_1(v_1) \left\{ F_1^{-1}[W_1(v_1)] \right\}^{1/p}, \\
h_2(\boldsymbol{v}) &= S_2(v_2) \left\{ \left[1 - F_1^{-1}(W_1(v_1))\right] F_2^{-1}(W_2(v_2)) \right\}^{1/p},
\end{aligned}
$$

$$h_m(\boldsymbol{v}) = S_m(v_m)\left\{\left[1 - \sum_{i=1}^{m-1} F_i^{-1}(W_i(v_i))\right] F_m^{-1}(W_m(v_m))\right\}^{1/p}, \quad m = 2, \ldots, s-1,$$
$$h_s(\boldsymbol{v}) = S_s(v_s)\left(1 - \sum_{i=1}^{s-1} |h_i(\boldsymbol{v})|^p\right)^{1/p}, \tag{29}$$

where $S_i(v_i)$ $(i = 1, \ldots, s)$ is computed through (18) by replacing V_i with v_i, and $W_k(v_k)$ $(k = 1, \ldots, s-1)$ computed through (18) by replacing V_k with v_k. Therefore, if we choose the set of points $\mathcal{P}_z = \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n\}$ in (26) as a *glp* set in C^s with the smallest discrepancy d in the sense of (2), then the set of points $\mathcal{P}_x = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ determined by (26) has the smallest quasi F-discrepancy d with respect to the c.d.f. of the uniform distribution $\boldsymbol{x} \sim \mathcal{U}(s, p)$ on S_s^p, or $\mathcal{P}_x$ has the smallest F-discrepancy d with respect to the uniform c.d.f. $H(\boldsymbol{r}) = \prod_{i=1}^{s} r_i$ with independent components, where $\boldsymbol{r} = (r_1, \ldots, r_s)' \in C^s$.

The algorithm given by (26) is easily implemented in practice. The computation of the inverse function $F_k^{-1}(\cdot)$ $(k = 1, \ldots, s-1)$ (i.e., the inverse beta function) in (26) is provided by many statistical packages such as SAS and MATLAB. Figure 1 illustrates the projection of some *glp* sets in C^2 onto S_2^p by the algorithm given by (26) for the 2-dimensional case and $p = 1/2, 1, 2$ and 3, respectively, where the generating vectors $glp_k(n; h_1, h_2) = glp_k(n; h_2) = \left(\frac{k-0.5}{n}, \left\{\frac{h_2 k - 0.5}{n}\right\}\right)$ with $h_1 = 1$ for the *glp* sets are given as follows (see also Appendix A of Fang and Wang (1994)), where $\{\cdot\}$ has the same meaning as in (8) (i.e., the fractional part of a real number). From Fig. 1, it can be seen that the projected points of the *glp* sets in C^2 are nearly uniformly scattered on S_2^p for each chosen p.

1. $s = 2$, $n = 8$, $p = 1/2,\ 1,\ 2,\ 3$, $h_2 = 5$, the *glp* set in C^2 is $glp_k(n; h_2)$ for $1 \le k \le n$ and $n = 8$;
2. $s = 2$, $n = 21$, $p = 1/2,\ 1,\ 2,\ 3$, $h_2 = 13$, the *glp* set in C^2 is $glp_k(n; h_2)$ for $1 \le k \le n$ and $n = 21$;
3. $s = 2$, $n = 55$, $p = 1/2,\ 1,\ 2,\ 3$, $h_2 = 34$, the *glp* set in C^2 is $glp_k(n; h_2)$ for $1 \le k \le n$ and $n = 55$;
4. $s = 2$, $n = 144$, $p = 1/2,\ 1,\ 2,\ 3$, $h_2 = 89$, the *glp* set in C^2 is $glp_k(n; h_2)$ for $1 \le k \le n$ and $n = 144$. □

3 Applications

Stochastic representation of a random variable (or vector) plays an important role in statistical simulation. Johnson (1987) comprehensively studied statistical simulation by the method of stochastic representation. Fang and Wang (1994, Chap. 4) used stochastic representation to generate the *representative points* of a multivariate probability distribution. The stochastic representation given by (16)–(18) provides a way to

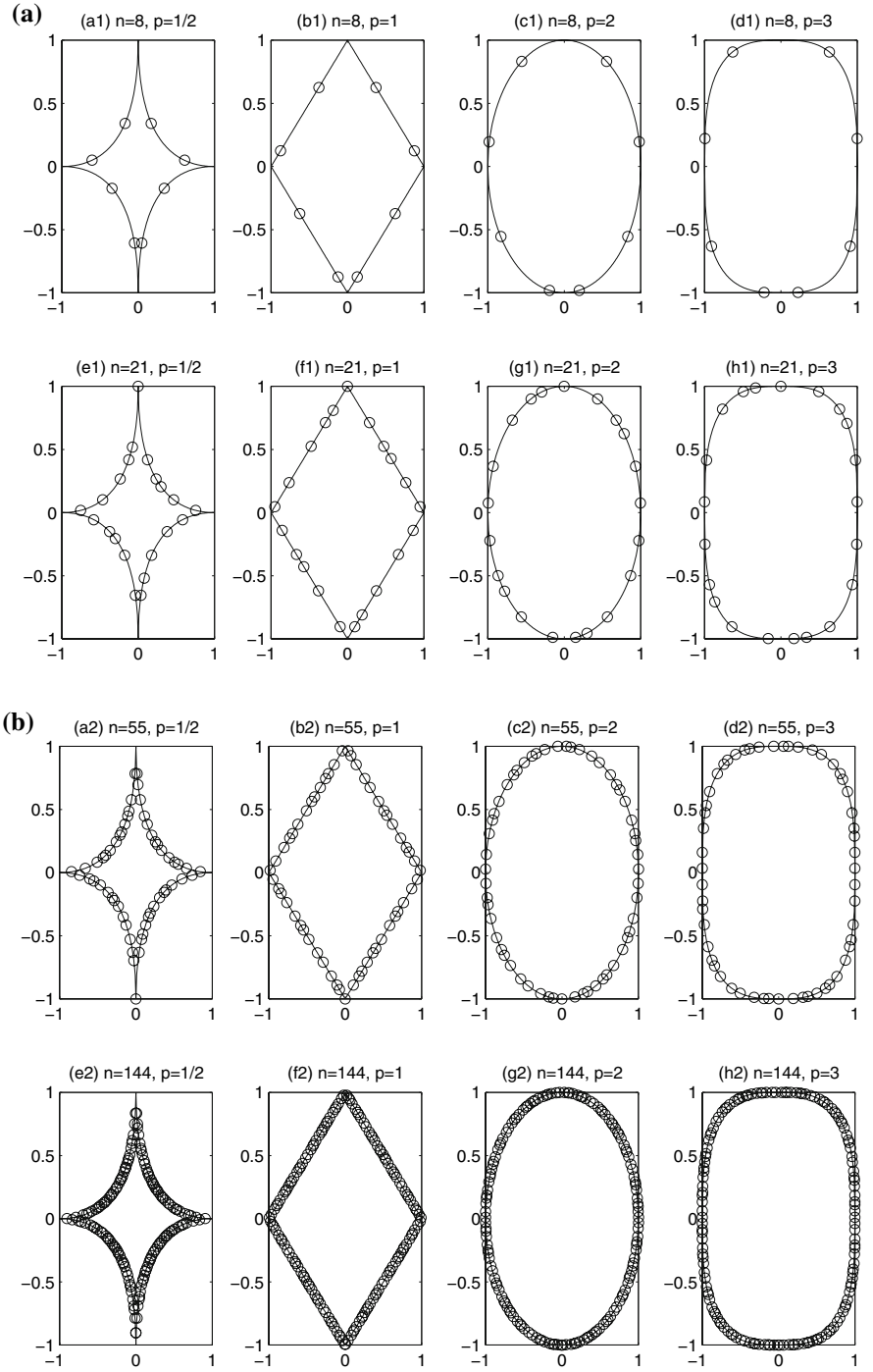

Fig. 1 Illustration of projecting some *glp* sets in C^2 onto the L_p-norm unit sphere S_2^p by the algorithm given by (26) for some selected values of p

generate the representative points or empirical samples from the uniform distribution $\mathcal{U}(s, p)$. Based on (16)–(18) and the properties of the L_p-norm spherical distributions, we can generate empirical samples from a given L_p-norm spherical distribution conveniently. The following definition was given by Gupta and Song (1997).

Definition 6 An s-variate random vector $\boldsymbol{x}$ is said to have an L_p-norm spherical distribution (denoted by $\boldsymbol{x} \sim SP(s, p)$) if

$$\boldsymbol{x} \stackrel{d}{=} R\boldsymbol{u}, \tag{30}$$

where $\boldsymbol{u} \sim \mathcal{U}(s, p)$, R is a nonnegative univariate random variable that is independent of $\boldsymbol{u}$.

By Definition 6, we can generate a random sample from a given L_p-norm spherical distribution as follows: generate a sample $\boldsymbol{u}_i = (U_{i1}, \ldots, U_{is})' \sim \mathcal{U}(s, p)$ by using a uniform sample $\boldsymbol{z}_i = (z_{i1}, \ldots, z_{is})' \in C^s$ with z_{ij} $(i = 1, \ldots, n;\ j = 1, \ldots, s)$ being i.i.d. $U(0, 1)$ and the algorithm given by (26), then generate a random sample r_i $(i = 1, \ldots, n)$ such that $r_i \stackrel{d}{=} R$ and r_i is independent of $\boldsymbol{u}$, where R in (30) is a given nonnegative random variable that is independent of $\boldsymbol{u}$. Then $\{\boldsymbol{x}_i = (X_{i1}, \ldots, X_{is})' = (r_i U_{i1}, \ldots, r_i U_{is})' :\ i = 1, \ldots, n\}$ is a random sample satisfying (30).

Example 1 Application in generating empirical samples from the class of p-generalized normal distributions. The p-generalized normal distribution was given by Goodman and Kotz (1973). Denote it by $N_s(\boldsymbol{0}, \boldsymbol{I}_s, p)$.$\boldsymbol{x} = (X_1, \ldots, X_s)' \sim N_s(\boldsymbol{0}, \boldsymbol{I}_s, p)$ has a p.d.f.

$$f(x_1, \ldots, x_s) = \frac{p^s r^{s/p}}{2^s \Gamma^s(1/p)} \cdot \exp\left\{-r \sum_{i=1}^{s} |x_i|^p\right\}, \quad (x_1, \ldots, x_s)' \in R^s,$$

where $r > 0$ is a parameter. It is easy to verify that $\boldsymbol{x} \sim N_s(\boldsymbol{0}, \boldsymbol{I}_s, p) \stackrel{d}{=} R\boldsymbol{u}$, $\boldsymbol{u} \sim \mathcal{U}(s, p)$, and R has a p.d.f. (Lemma 2.1 of Gupta and Song 1997)

$$g(t) = \frac{p r^{s/p}}{\Gamma(s/p)} \cdot t^{s-1} \exp(-r t^p), \quad t > 0.$$

Then the random variable $Y = r R^p$ has a gamma distribution with a p.d.f.

$$g(y) = \frac{1}{\Gamma(s/p)} y^{s/p-1} \exp(-y), \quad y > 0. \tag{31}$$

An i.i.d. sample $\{Y_1, \ldots, Y_n\}$ can be easily generated from the gamma distribution (31). Then an i.i.d. sample $\{R_1, \ldots, R_n\}$ can be obtained by

$$R_i = (Y_i/r)^{1/p}, \qquad i = 1, \ldots, n. \tag{32}$$

A random sample $\{\boldsymbol{x}_1, \ldots, x_n\}$ from $N_s(\boldsymbol{0}, \boldsymbol{I}_s, p)$ is obtained by

$$\boldsymbol{x}_i = R_i \boldsymbol{u}_i, \qquad i = 1, \ldots, n. \tag{33}$$

where $\{\boldsymbol{u}_i : i = 1, \ldots, n\}$ is a random sample from $\mathcal{U}(s, p)$, which is obtained through (26) by generating a uniform sample $\boldsymbol{z}_i = (z_{i1}, \ldots, z_{is})' \in C^s$ with z_{ij} $(j = 1, \ldots, s)$ i.i.d. $U(0, 1)$.

Example 2 Application in generating representative points (simply called rep-points) for the class of L_p-norm spherical distributions. We need the following definition.

Definition 7 (Fang and Wang 1994, p. 155) Let $F(\boldsymbol{x}) = F(x_1, \ldots, x_s)$ be a given s-dimensional continuous c.d.f. and $\mathcal{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ a set of points on R^s. The F-discrepancy $D_F(n, \mathcal{P})$ defined by (3) is a measure of the representation of $\mathcal{P}$ to $F(\boldsymbol{x})$. If we can find a set of points $\mathcal{P}^* = \{\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_n^*\}$ such that

$$D_F(n, \mathcal{P}^*) = \min_{\mathcal{P}} D_F(n, \mathcal{P}), \tag{34}$$

where $\mathcal{P}$ runs over all sets of n points in R^s, then $\mathcal{P}^*$ is called a set of cdf-rep-points of $F(\boldsymbol{x})$.

It can be seen that the empirical distribution function based on the set of cdf-rep-points of $F(\boldsymbol{x})$ is the best approximation of $F(\boldsymbol{x})$ in the sense of (34). For the one-dimensional case $s = 1$, it is easy to find the set of cdf-rep-points of any given continuous c.d.f. as stated in Theorem 4.1 of Fang and Wang (1994, p. 156). As pointed out by Fang and Wang (1994, p. 156), for the high-dimensional case $s > 1$, it is usually difficult to find the set of cdf-rep-points $\mathcal{P}^*$ of any given c.d.f. $F(\boldsymbol{x})$. We have to turn to looking for a sequence of rep-points $\mathcal{P}_n = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ of $F(\boldsymbol{x})$ such that

$$D_F(n, \mathcal{P}_n) = o(n^{-\frac{1}{2}}), \tag{35}$$

where $o(n^{-\frac{1}{2}})$ stands for a term satisfying $\lim_{n\to\infty} n^{\frac{1}{2}} o(n^{-\frac{1}{2}}) = 0$. Fang and Wang (1994, pp. 158–159) gave the NTSR algorithm for generating a sequence of sets of rep-points of a c.d.f. $F(\boldsymbol{x})$ $(\boldsymbol{x} \in R^s, s \geq 2)$ with the associated random vector $\boldsymbol{x}$ that has the stochastic representation of the type

$$\boldsymbol{x} \stackrel{d}{=} R\boldsymbol{y}, \tag{36}$$

where $\boldsymbol{x} \sim F(\boldsymbol{x})$, $R > 0$ is a positive random variable, and $\boldsymbol{y} \sim U(D)$ is the uniform distribution on an $(s-1)$-dimensional bounded domain $D \subset R^s$. The NTSR algorithm consists of the following steps:

Step 1. Generate a uniformly scattered set of points $\mathcal{P}_c = \{\boldsymbol{c}_k = (c_{k1}, \ldots, c_{ks})' : k = 1, \ldots, n\}$ in C^s defined by (1);

Step 2. Denote the c.d.f. of R by $F_R(r)$ and let F_R^{-1} be its inverse function. Compute $r_k = F_R^{-1}(c_{ks})$, $k = 1, \ldots, n$;

Step 3. Generate a uniformly scattered set of points $\mathcal{P}_y = \{\boldsymbol{y}_k : k = 1, \ldots, n\}$ on the bounded domain $D \subset R^s$ with the first $(s-1)$-components of the point $\boldsymbol{c}_k = (c_{k1}, \ldots, c_{ks})'$ in step 1 $(k = 1, \ldots, n)$;

Step 4. Then the set of points $\mathcal{P}_x = \{\boldsymbol{x}_k = r_k \boldsymbol{y}_k : k = 1, \ldots, n\}$ is a set of rep-points of $F(\boldsymbol{x})$.

By the above NTSR algorithm, to generate a set of rep-points $\mathcal{P}_x = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ for an L_p-norm spherical distribution defined by (30), we can carry out step 1 by choosing the $\mathcal{P}_c = \{\boldsymbol{c}_k = (c_{k1}, \ldots, c_{ks})' : k = 1, \ldots, n\} \subset C^s$ in step 1 as the *glp* set defined by (7). That is, let $(n; h_1, \ldots, h_s)$ be a generating vector for given n and $\{\boldsymbol{c}_{ki} = \boldsymbol{x}_{ki} : i = 1, \ldots, s;\ k = 1, \ldots, n\}$ as given by (8). In step 2, the c.d.f. $F_R(r)$ of R is known in the L_p-norm spherical distribution defined by (30). So the above step 2 can be carried out. In step 3, the bounded domain $D \subset R^s$ is the L_p-norm unit sphere S_s^p defined by (9). A uniformly scattered set $\mathcal{P}_y = \mathcal{P}_u = \{\boldsymbol{u}_k = (u_{k1}, \ldots, u_{ks})' : k = 1, \ldots, n\}$ on S_s^p in the above step 3 is realized by the algorithm defined by (26). That is, the components of $\boldsymbol{u}_k = (u_{k1}, \ldots, u_{ks})' \in \mathcal{P}_u \subset S_s^p$ are given by

$$
\begin{aligned}
u_{k1} &= S_1(c_{k1}) \left\{ F_1^{-1}[W_1(c_{k1})] \right\}^{1/p}, \\
u_{k2} &= S_2(c_{k2}) \left\{ (1 - |u_{k1}|^p) F_2^{-1}[W_2(c_{k2})] \right\}^{1/p}, \\
&\vdots \\
u_{k,s-1} &= S_{s-1}(c_{k,s-1}) \left\{ \left(1 - \sum_{j=1}^{s-2} |u_{kj}|^p \right) F_{s-1}^{-1}[W_{s-1}(c_{k,s-1})] \right\}^{1/p}, \qquad (37) \\
u_{ks} &= \begin{cases} S_s(c_{ks}) \left(1 - \sum_{j=1}^{s-1} |u_{kj}|^p \right)^{1/p}, & \text{if } c_{ks} \neq 0.5, \\ \left(1 - \sum_{j=1}^{s-1} |u_{kj}|^p \right)^{1/p}, & \text{if } c_{ks} = 0.5, \end{cases}
\end{aligned}
$$

where $S_i(c_{ki})$ $(i = 1, \ldots, s)$ is computed through (18) by replacing V_i with c_{ki}, $W_l(c_{kl})$ $(l = 1, \ldots, s-1)$ computed through (18) by replacing V_k with c_{kl}, and $\{c_{ki} : i = 1, \ldots, s;\ k = 1, \ldots, n\}$ are the components of $\{c_{ki} = x_{ki} : i = 1, \ldots, s;\ k = 1, \ldots, n\}$ as given by (8). The inverse functions $F_l^{-1}(\cdot)$ $(l = 1, \ldots, s-1)$ are the same as in (26). Then the set of points $\mathcal{P}_x = \{\boldsymbol{x}_k = r_k \boldsymbol{u}_k : k = 1, \ldots, n\}$ from the above step 4 is a set of rep-points of the L_p-norm spherical distribution defined by (30).

Example 3 Application in optimization problems. Let $f(\boldsymbol{x})$ $(\boldsymbol{x} = (x_1, \ldots, x_s)' \in R^s)$ be a continuous function. Suppose that we want to find the maximal point $\boldsymbol{x}^* \in S_s^p$ such that

$$
M = f(\boldsymbol{x}^*) = \max_{\boldsymbol{x} \in S_s^p} f(\boldsymbol{x}). \qquad (38)
$$

This is an optimization problem of $f(\boldsymbol{x}) = f(x_1, \ldots, x_s)$ subject to the restriction

$$|x_1|^p + \cdots + |x_s|^p = 1, \qquad p > 0.$$

Assume that $\{\mathcal{P}_{n_i} : i = 1, \ldots, \}$ $(n_1 < n_2 < \cdots, n_i \to \infty$ as $i \to \infty)$ is a sequence of sets such that the points in $\mathcal{P}_{n_i}$ are obtained by (37). So the points in $\mathcal{P}_{n_i}$ can be considered as uniformly scattered on S_s^p. From Example 1.2 of Fang and Wang (1994, p. 19), the discrepancy d of $\mathcal{P}_c = \{\boldsymbol{c}_k = (c_{k1}, \ldots, c_{ks})' : k = 1, \ldots, n\} \subset C^s$ with $c_{ki} = x_{ki}$ given by (8) satisfies

$$c_1(s)n^{-1/s} \le d \le c_2(s)n^{-1/s},$$

where $c_1(s)$ and $c_2(s)$ are two constants determined only by s. Let d_{n_i} be the discrepancy (in the sense of (2)) of $\mathcal{P}_c(n_i) = \{\boldsymbol{c}_k = (c_{k1}, \ldots, c_{ks})' : k = 1, \ldots, n_i\}$. According to Theorem 1.6 of Fang and Wang (1994, pp. 44–45), d_{n_i} is the quasi F-discrepancy of $\mathcal{P}_{n_i}$ with respect to the c.d.f. of $\boldsymbol{u}_s \sim \mathcal{U}(s, p)$, or the F-discrepancy of $\mathcal{P}_{n_i}$ with respect to the c.d.f. $H(z_1, \ldots, z_s) = \prod_{i=1}^s z_i$ $(0 < z_i < 1, i = 1, \ldots, s)$ by the discussion following equations (3)–(5). Then we have

$$c_1(s)n_i^{-1/s} \le d_{n_i} \le c_2(s)n_i^{-1/s}.$$

So $\lim_{n_i \to \infty} d_{n_i} = \lim_{i \to \infty} d_{n_i} = 0$. We write $d_{n_i} = o(1)$. Suppose that $\boldsymbol{x}_{n_i}^* \in \mathcal{P}_{n_i}$ be a point satisfying

$$M_{n_i} = f(\boldsymbol{x}_{n_i}^*) = \max\{f(\boldsymbol{x}_k^{(n_i)}) : \boldsymbol{x}_k^{(n_i)} \in \mathcal{P}_{n_i}\}. \tag{39}$$

By Theorem 3.1 of Fang and Wang (1994, p. 107), we have

$$M_{n_i} \to M, \qquad i \to \infty. \tag{40}$$

Equation (40) implies that the optimization problem (38) can be approximated by way of (39) through generating a suitable sequence of sets $\mathcal{P}_{n_i}$ such that the points in $\mathcal{P}_{n_i}$ are uniformly scattered on S_s^p.

Furthermore, if the function $f(\boldsymbol{x}) = f(x_1, \ldots, x_s)$ is differentiable such that $\bigtriangledown f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_s})$ is continuous and

$$\| \bigtriangledown f \|_2 = \left[\sum_{i=1}^s \left(\frac{\partial f}{\partial x_i} \right)^2 \right]^{1/2} < C, \tag{41}$$

where C is a constant, a more accurate approximation of the M_{n_i} given by (39) to the solution M defined by (38) is possible based on the following definition.

Definition 8 (Fang and Wang 1994, p. 34) Let D be a closed and bounded domain and $\mathcal{P} = \{\boldsymbol{x}_k : k = 1, \ldots, n\} \subset D$ a set of points on D. The term

$$DP(\mathcal{P}, D) = \max_{\boldsymbol{x} \in D} \min_{1 \le k \le n} d(\boldsymbol{x}, \boldsymbol{x}_k), \tag{42}$$

is called the dispersion of $\mathcal{P}$ on D, where $d(\boldsymbol{x}, \boldsymbol{x}_k)$ denotes the Euclidean distance between the two points $\boldsymbol{x}$ and $\boldsymbol{x}_k$. By the above suitably chosen sequence of sets $\mathcal{P}_{n_i}$ and using Theorem 3.2 of Fang and Wang (1994, p. 108), we have

$$M_{n_i} \leq M \leq M_{n_i} + C \cdot DP(\mathcal{P}_{n_i}, S_s^p), \tag{43}$$

where C is the constant given in (41) and $DP(\mathcal{P}_{n_i}, S_s^p)$ is the dispersion of $\mathcal{P}_{n_i}$ on S_s^p. When the points in $\mathcal{P}_{n_i}$ are fairly uniformly scattered on S_s^p, one can expect the dispersion $DP(\mathcal{P}_{n_i}, S_s^p)$ to be very small for large n_i. But unfortunately, obtaining the exact value or the accurate upper bound of the dispersion $DP(\mathcal{P}_{n_i}, S_s^p)$ can be extremely difficult, see Fang and Wang (1994, pp. 34–36) for more discussion on dispersion. Examples 1–3 above provide a few applications of the proposed method in this paper. By following the same discussions in Chapters 2, 5 and 6 of Fang and Wang (1994), it is possible to find more applications of the proposed method in this paper. We only present the above three examples as illustrations for applying the sets of uniformly scattered points on the L_p-norm unit sphere to various problems. The method in this paper sheds some additional light to the application of number-theoretic methods in statistics as comprehensively studied by Fang and Wang (1994), and it could be also considered as a generalization of the method in Tashiro (1977).

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