# Solutions to *Algebra* by Thomas W. Hungerford

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December 16, 2025

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**Exercise 1.** *Let $(A, \leq)$ be a partially ordered set and $B$ a nonempty subset. A* ***lower bound*** *of $B$ is an element $d \in A$ such that $d \leq b$ for* every *$b \in B$. A* ***greatest lower bound (g.l.b.)*** *of $B$ is a lower bound $d_0$ of $B$ such that $d \leq d_0$ for every other lower bound $d$ of $B$. A* ***least upper bound (l.u.b.)*** *of $B$ is an upper bound $t_0$ of $B$ such that $t_0 \leq t$ for every other upper bound $t$ of $B$. $(A, \leq)$ is a* ***lattice*** *if for all $a, b \in A$ the set $\{a, b\}$ has both a greatest lower bound and a least upper bound.*

*(a) If $S \neq \varnothing$, then the power set $P(S)$ ordered by set-theoretic inclusion is a lattice, which has a unique maximal element.*

*(b) Give an example of a partially ordered set which is* not *a lattice.*

*(c) Give an example of a lattice with no maximal element and an example of a partially ordered set with two maximal elements.*

*Solution.* (a) For $X, Y \subset S$ the greatest lower bound is

$$X \cap Y.$$

The least upper bound is

$$X \cup Y.$$

Thus every pair $X, Y$ has a g.l.b. and l.u.b., so $(P(S), \subset)$ is a lattice.

A maximal element in $P(S)$ is an element that is not properly contained in any other element. The whole set $S$ is an upper bound for every subset of $S$ and is not contained in any strictly larger subset of $S$, so $S$ is a maximal element. It is unique because if $T$ is any subset with $U \subset T$ for all $U \subset S$, then in particular $S \subset T$, so $T = S$.

(b) Take the set $A = \{a, b\}$ with the only order relations being reflexivity:

$$a \leq a, \qquad b \leq b,$$

For the pair $a, b$ there is no lower bound other than possibly elements $\leq a$ and $\leq b$; but the only candidates are $a$ and $b$ themselves, and neither is $\leq$ the other. Hence there is no greatest lower bound of $a, b$. (Similarly there is no least upper bound.) Therefore this poset is not a lattice.

(c) Take the integers $\mathbb{Z}$ with the usual order. For any $m, n \in \mathbb{Z}$ the least upper bound is $\max m, n$ and the greatest lower bound is $\min m, n$; thus $(\mathbb{Z}, \leq)$ is a lattice. But $\mathbb{Z}$ has no maximal element because for every $n \in \mathbb{Z}$ there exists $n + 1 > n$. So $\mathbb{Z}$ is a lattice with no maximal element.

Let $A = \{0, a, b\}$ and define the order by

$$0 \leq a, \qquad 0 \leq b.$$

**Exercise 2.** *A lattice $(A, \leq)$ (see Exercise 1) is said to be **complete** if every nonempty subset of $A$ has both a least upper bound and a greatest lower bound. A map of partially ordered sets $f : A \to B$ is said to preserve order if $a \leq a'$ in $A$ implies $f(a) \leq f(a')$ in $B$. Prove that an order-preserving map $f$ of a complete lattice $A$ into itself has at least one fixed element (that is, an $a \in A$ such that $f(a) = a$).*

*Solution.* Let $S = \{\, a \in A : f(a) \leq a \,\}$ be the set of all pre-fixed points of $f$. Since $A$ is complete, it has a greatest element, say 1. Because $f$ preserves order, $f(1) \leq 1$, so $1 \in S$. Thus $S \neq \varnothing$ and, since $A$ is complete, $S$ has a g.l.b; call it

$$m = \inf S.$$

*First*, we show that $f(m) \leq m$. For every $s \in S$ we have $m \leq s$, hence $f(m) \leq f(s)$ by order preservation. Since $s \in S$, $f(s) \leq s$, and thus $f(m) \leq s$ for all $s \in S$. Hence $f(m)$ is a lower bound of $S$, and by maximality of $m$ as greatest lower bound, $f(m) \leq m$.

*Second*, we show that $m \leq f(m)$. Since $m$ is a lower bound of $S$ and $f$ is order-preserving, the argument above shows that $f(m)$ is also a lower bound of $S$. Therefore $f(m) \leq s$ for all $s \in S$, so $f(m)$ is a lower bound of $S$. Because $m$ is the greatest lower bound, we must have $m \leq f(m)$.

Combining the inequalities $f(m) \leq m$ and $m \leq f(m)$, we conclude that $f(m) = m$. Thus $f$ has a fixed element.

**Exercise 3.** *Exhibit a well ordering of the set $\mathbb{Q}$ of rational numbers.*

*Solution.* Write each rational number in $\mathbb{Q}$ in its unique reduced form $a/b$ with $b > 0$ and $\gcd(a, b) = 1$. (Under this convention the rational 0 is represented uniquely as $0/1$.)

Define a binary relation $\trianglelefteq$ on $\mathbb{Q}$ by declaring

$$\frac{a}{b} \trianglelefteq \frac{c}{d}$$

iff either

1. $|a| + b < |c| + d$, or
2. $|a| + b = |c| + d$ and $a < c$, or
3. $|a| + b = |c| + d,\ a = c$, and $b \leq d$.

Since every rational is written in the unique reduced form specified above, the quantities $|a| + b$, $a$, and $b$ are well defined for each rational, so $\trianglelefteq$ is well defined.

It is immediate that $\trianglelefteq$ is a total order. To see that it is a well ordering, let $S \subseteq \mathbb{Q}$ be nonempty and for each $x = a/b \in S$ set $N(x) = |a| + b \in \mathbb{N}$. The set $\{N(x) : x \in S\}$ is a nonempty subset of $\mathbb{N}$, hence has a least element $n_0$. The subset $T = \{x \in S : N(x) = n_0\}$ is therefore nonempty. Among elements of $T$, the numerators form a finite (hence well-ordered) subset of $\mathbb{Z}$, so there is a least numerator $a_0$. Finally, among rationals in $T$ with numerator $a_0$ the denominator is minimal for the $\trianglelefteq$-least element. Thus $T$ (and hence $S$) has a least element with respect to $\trianglelefteq$. Therefore $\trianglelefteq$ is a well ordering of $\mathbb{Q}$.

**Exercise 4.** *Let $S$ be a set. A* ***choice function*** *for $S$ is a function $f$ from the set of all nonempty subsets of $S$ to $S$ such that $f(A) \in A$ for all $A \neq \varnothing$, $A \subset S$. Show that the Axiom of Choice is equivalent to the statement that every set $S$ has a choice function.*

*Solution.* We show the two statements are equivalent.

**(AC $\Rightarrow$ choice functions exist).** Let $S$ be any set and let $\mathcal{I}$ denote the collection of all nonempty subsets of $S$. If $\mathcal{I} = \varnothing$ then $S = \varnothing$, and the unique function $\varnothing \to \varnothing$ is a choice function for $S$. Thus assume $\mathcal{I} \neq \varnothing$. Consider the family $\{X_A\}_{A \in \mathcal{I}}$ where $X_A = A$ for each $A \in \mathcal{I}$. Every $X_A$ is nonempty by definition, and the family is indexed by the nonempty set $\mathcal{I}$. By the Axiom of Choice (the product of a family of nonempty sets indexed by a nonempty set is nonempty), the product $\prod_{A \in \mathcal{I}} X_A$ is nonempty. An element of this product is precisely a function $f \colon \mathcal{I} \to S$ with $f(A) \in X_A = A$ for each $A$; that is exactly a choice function for $S$. Hence every set $S$ admits a choice function.

**(Choice functions exist $\Rightarrow$ AC).** Assume every set $T$ admits a choice function $c_T$ defined on the collection of nonempty subsets of $T$. Let $\{X_i\}_{i \in I}$ be any family of nonempty sets indexed by a nonempty set $I$. Put $S = \bigcup_{i \in I} X_i$. Then each $X_i$ is a nonempty subset of $S$, so the hypothesis supplies a choice function $c_S$ for $S$. Define $g \colon I \to S$ by $g(i) := c_S(X_i)$. By construction $g(i) \in X_i$ for every $i \in I$, so $g \in \prod_{i \in I} X_i$. Hence the product is nonempty. This establishes the Axiom of Choice.

Therefore the two statements are equivalent.

**Exercise 5.** *Let $S$ be the set of all points $(x, y)$ in the plane with $y \leq 0$. Define an ordering by $(x_1, y_1) \leq (x_2, y_2) \iff x_1 = x_2$ and $y_1 \leq y_2$. Show that this is a partial ordering of $S$, and that $S$ has infinitely many maximal elements.*

*Solution.* Let $S = \{(x, y) \in \mathbb{R}^2 : y \leq 0\}$ and define

$$(x_1, y_1) \leq (x_2, y_2) \iff x_1 = x_2 \text{ and } y_1 \leq y_2.$$

**(i) This relation is a partial order.**

- *Reflexive:* For any $(x, y) \in S$ we have $x = x$ and $y \leq y$, so $(x, y) \leq (x, y)$.
- *Antisymmetric:* If $(x_1, y_1) \leq (x_2, y_2)$ and $(x_2, y_2) \leq (x_1, y_1)$, then $x_1 = x_2$ and $y_1 \leq y_2$, and also $x_2 = x_1$ and $y_2 \leq y_1$. Hence $y_1 = y_2$ and therefore $(x_1, y_1) = (x_2, y_2)$.
- *Transitive:* If $(x_1, y_1) \leq (x_2, y_2)$ and $(x_2, y_2) \leq (x_3, y_3)$, then $x_1 = x_2$ and $x_2 = x_3$, so $x_1 = x_3$, and $y_1 \leq y_2 \leq y_3$, hence $y_1 \leq y_3$. Thus $(x_1, y_1) \leq (x_3, y_3)$.

Therefore the relation is reflexive, antisymmetric, and transitive, i.e. a partial order.

**(ii) $S$ has infinitely many maximal elements.**

Fix any real number $x_0$. For that $x_0$ the point $(x_0, 0) \in S$ satisfies the following: if $(x_0, 0) \leq (x, y)$ then $x = x_0$ and $0 \leq y$. Since every element of $S$ has $y \leq 0$, the only possibility is $y = 0$, so $(x, y) = (x_0, 0)$. Thus there is no element of $S$ strictly greater than $(x_0, 0)$; i.e. $(x_0, 0)$ is maximal.

As $x_0$ ranges over $\mathbb{R}$ we obtain the family $\{(x, 0) : x \in \mathbb{R}\}$ of maximal elements,which is infinite (indeed uncountable). Hence $S$ has infinitely many maximal elements.

(Observe also that any point $(x, y)$ with $y < 0$ is not maximal because $(x, y) < (x, 0)$.)

**Exercise 6.** *Prove that if all the sets in the family $\{A_i \mid i \in I \neq \varnothing\}$ are nonempty, then each of the projections $\pi_k \colon \prod_{i \in I} A_i \to A_k$ is surjective.*

*Solution.* Let $\{A_i\}_{i \in I}$ be a family of sets with $A_i \neq \varnothing$ for each $i \in I$. Fix $k \in I$ and let $\pi_k : \prod_{i \in I} A_i \to A_k$ be the projection onto the $k$-th coordinate. We must show that $\pi_k$ is surjective, i.e. that for every $a \in A_k$ there exists $f \in \prod_{i \in I} A_i$ with $\pi_k(f) = f(k) = a$.

For a given $a \in A_k$ we need to define a function $f : I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$ and $f(k) = a$. To do this we must choose, for each $i \in I - \{k\}$, an element $f(i) \in A_i$. The existence of a choice function selecting one element from each $A_i$ (for $i \neq k$) is exactly an instance of the Axiom of Choice. Assuming Choice (or equivalently the hypothesis that the product $\prod_{i \in I} A_i$ is nonempty), pick such elements $f(i)$ for all $i \neq k$, and put $f(k) = a$. Then $f \in \prod_{i \in I} A_i$ and $\pi_k(f) = a$. Since $a$ was arbitrary, $\pi_k$ is surjective.

**Remark.** If the index set $I$ is finite, no form of the Axiom of Choice is needed: one can choose elements from the finitely many $A_i$ inductively (or by a finite product of nonempty sets being nonempty). The use of Choice becomes essential only when $I$ is infinite.

**Exercise 7.** *Let $(A, \leq)$ be a linearly ordered set. The **immediate successor** of $a \in A$ (if it exists) is the least element in the set $\{x \in A \mid a < x\}$. Prove that if $A$ is well ordered by $\leq$, then at most one element of $A$ has no immediate successor. Give an example of a linearly ordered set in which precisely two elements have no immediate successor.*

*Solution.* First remark: if $a \in A$ has no immediate successor, that means the set $\{x \in A : x > a\}$ either is empty (so $a$ is maximal) or is nonempty but has no least element.

**At most one element has no immediate successor.** Suppose for contradiction that $a$ and $b$ are two distinct elements of $A$ with no immediate successor. Since $A$ is linearly ordered, either $a < b$ or $b < a$. Without loss of generality assume $a < b$. Then $b \in \{x \in A : x > a\}$, so this set is nonempty. But $A$ is well ordered, hence every nonempty subset has a least element; therefore $\{x \in A : x > a\}$ has a least element $c$. By definition $c$ is the immediate successor of $a$, contradicting the assumption that $a$ has no immediate successor. Thus it is impossible for two distinct elements to both lack immediate successors; at most one element of $A$ can have no immediate successor. $\square$

**Example with exactly two elements having no immediate successor.** Let

$$B = \{0\} \cup \{1/n : n \in \mathbf{N}^*\} \subset \mathbb{R}$$

equipped with the usual order inherited from $\mathbb{R}$. Every element of $B$ except 0 is of the form $1/n$ for some $n \in \mathbf{N}^*$. For $n \geq 2$, the least element strictly greater than $1/n$ is $1/(n-1)$, so $1/n$ has an immediate successor. The element $1 = 1/1$ is maximal in $B$ (no larger element of $B$ exists), hence it has no immediate successor. The element 0 also has no immediate successor: the set $\{x \in B : x > 0\} = \{1/n : n \in \mathbf{N}^*\}$ has no least element because for each $1/n$ there is $1/(n+1) \in B$ with $0 < 1/(n+1) < 1/n$. Therefore 0 has no immediate successor. No other elements of $B$ lack immediate successors, so exactly two elements of $B$ (namely 0 and 1) have no immediate successor.

## 0.8 Cardinal Numbers

**Exercise 1.** *Let $I_0 = \varnothing$ and for each $n \in \mathbf{N}^*$ let $I_n = \{1, 2, 3, \ldots, n\}$.*

*(a) $I_n$ is not equipollent to any of its proper subsets [*Hint*: induction].*

*(b) $I_m$ and $I_n$ are equipollent if and only if $m = n$.*

*(c) $I_m$ is equipollent to a subset of $I_n$ but $I_n$ is not equipollent to any subset of $I_m$ if and only if $m < n$.*

*Solution.* Recall that $I_0 = \varnothing$ and $I_n = \{1, 2, \ldots, n\}$ for $n \geq 1$.

**Lemma.** For every $n \geq 0$, every injective map $g\colon I_n \to I_n$ is surjective (hence bijective).

*Proof.* We proceed by strong induction on $n$.

*Base cases.* For $n = 0$, the statement is trivial: the only map $\varnothing \to \varnothing$ is bijective. For $n = 1$, any injective map $g : \{1\} \to \{1\}$ must send 1 to 1, so it is surjective.

*Inductive step.* Fix $n \geq 2$ and assume the claim holds for all $k < n$. Let $g : I_n \to I_n$ be injective. Suppose, for a contradiction, that $g$ is not surjective. Then $g(I_n)$ is a proper subset of $I_n$, so there exists an element of $I_n$ not in the image of $g$; choose $m$ to be the largest such element. (A largest element exists since $I_n$ is finite and totally ordered.)

Because $m \notin g(I_n)$, the image of $g$ is contained in $I_n - \{m\}$. Define

$$\phi : I_n - \{m\} \longrightarrow I_{n-1}, \qquad \phi(k) = \begin{cases} k, & k < m, \\ k - 1, & k > m. \end{cases}$$

Define also

$$\phi^{-1} : I_{n-1} \longrightarrow I_n - \{m\}, \qquad \phi^{-1}(j) = \begin{cases} j, & j < m, \\ j+1, & j \geq m. \end{cases}$$

A direct check shows that $\phi$ and $\phi^{-1}$ are inverse bijections.

Now consider the composition

$$\psi = \phi \circ g \circ \phi^{-1} : I_{n-1} \to I_{n-1}.$$

The map $\psi$ is injective, since it is a composition of injective maps. By the induction hypothesis, $\psi$ is surjective, hence bijective. Since $\phi^{-1}$ is also bijective, the composition

$$\phi^{-1} \circ \psi = g \circ \phi^{-1}$$

is bijective. In particular, $g \circ \phi^{-1}$ is surjective onto $I_n - \{m\}$. This means that the restriction

$$g|_{I_n - \{m\}} : I_n - \{m\} \longrightarrow I_n - \{m\}$$

is surjective.

Now consider $g(m)$. Since $m \notin g(I_n)$ by assumption, we must have $g(m) \in I_n - \{m\}$. But because $g|_{I_n-\{m\}}$ is surjective, there exists some $j \in I_n - \{m\}$ with $g(j) = g(m)$, contradicting the injectivity of $g$. This contradiction shows that $g$ must be surjective.

This completes the induction and the proof of the lemma.

**(a) $I_n$ is not equipollent to any of its proper subsets.**

Assume, for a contradiction, that there exists a bijection $f : I_n \to S$ with $S \subsetneq I_n$. Let $i : S \hookrightarrow I_n$ denote the inclusion map. Then $i \circ f : I_n \to I_n$ is injective. By the Lemma, $i \circ f$ is surjective. But $(i \circ f)(I_n) = i(S) = S$, a proper subset of $I_n$, which is impossible. Hence $I_n$ is not equipollent to any of its proper subsets.

**(b) $I_m$ and $I_n$ are equipollent if and only if $m = n$.**

If $m = n$, the identity map is a bijection. Conversely, suppose $I_m$ and $I_n$ are equipollent and assume $m \neq n$. Without loss of generality, let $m < n$. Then a bijection $I_m \to I_n$ would make $I_n$ equipollent to a proper subset of itself, contradicting part (a). Thus $m = n$.

**(c) $I_m$ is equipollent to a subset of $I_n$ but $I_n$ is not equipollent to any subset of $I_m$ if and only if $m < n$.**

If $m < n$, the inclusion $I_m \hookrightarrow I_n$ is injective, so $I_m$ is equipollent to the subset $I_m \subset I_n$. If $I_n$ were equipollent to a subset of $I_m$, then $I_n$ would be equipollent to a proper subset of itself, contradicting part (a). Hence the stated asymmetry holds when $m < n$.

Conversely, suppose the asymmetry in the statement holds. The existence of an injection $I_m \to I_n$ implies $m \leq n$. If $m = n$, then the two sets are equipollent, contradicting the assumption. Therefore $m < n$. This completes the proof.

**Exercise 2.** *(a) Every infinite set is equipollent to one of its proper subsets.*

*(b) A set is finite if and only if it is not equipollent to one of its proper subsets [see Exercise 1].*

*Solution.* (a) **Every infinite set is equipollent to one of its proper subsets (assuming the Axiom of Choice).**

Assume the Axiom of Choice in the form that every set admits a choice function. Let $S$ be an infinite set. Using a choice function, we construct an infinite sequence of distinct elements of $S$.

Let $\mathcal{P}^*(S)$ denote the collection of all nonempty subsets of $S$, and let $c : \mathcal{P}^*(S) \to S$ be a choice function. Define inductively

$$S_1 = S, \qquad s_1 = c(S_1),$$

and, having chosen distinct elements $s_1, \ldots, s_n$, set

$$S_{n+1} = S - \{s_1, \ldots, s_n\}, \qquad s_{n+1} = c(S_{n+1}).$$

Since $S$ is infinite, each $S_{n+1}$ is nonempty, so the construction continues indefinitely. Thus we obtain an infinite sequence $(s_n)_{n \geq 1}$ of distinct elements of $S$.

Define a map $f : S \to S$ by

$$f(s_n) = s_{n+1} \quad (n \geq 1), \qquad f(x) = x \text{ for } x \notin \{s_n : n \geq 1\}.$$

Then $f$ is injective: it is the identity off $\{s_n\}$, and on $\{s_n\}$ it is a shift. Moreover, $f$ is not surjective, since $s_1$ is not in the image. Hence $f(S) \subsetneq S$, and since $f : S \to f(S)$ is a bijection, $S$ is equipollent to a proper subset of itself.

*Remark.* The statement proved here is not provable in ZF alone. Without the Axiom of Choice, there may exist infinite sets that are not equipollent to any proper subset (so-called *Dedekind-finite* infinite sets). Thus part (a) genuinely requires some form of Choice.

(b) **A set is finite if and only if it is not equipollent to one of its proper subsets (assuming the Axiom of Choice).**

If $S$ is finite, then $S$ is equipollent to $I_n$ for some $n$, and by Exercise 1(a) no finite set is equipollent to any proper subset of itself. Hence a finite set is not equipollent to a proper subset.

Conversely, suppose $S$ is not finite, i.e. $S$ is infinite. By part (a), assuming the Axiom of Choice, $S$ is equipollent to a proper subset of itself. Therefore, a set is finite if and only if it is not equipollent to one of its proper subsets.

**Exercise 3.** *(a) $\mathbb{Z}$ is a denumerable set.*

*(b) The set $\mathbb{Q}$ of rational numbers is denumerable. [Hint: show that $|\mathbb{Z}| \leq |\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{Z}| = |\mathbb{Z}|$.]*

*Solution.* (a) **$\mathbb{Z}$ is denumerable.**

Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(0) = 0, \qquad f(2n-1) = n, \qquad f(2n) = -n \quad (n \geq 1).$$

Then $f$ is bijective: every integer occurs exactly once (positive integers at odd inputs, negative integers at even inputs, and 0 at 0). Hence $\mathbb{Z}$ is denumerable.

(b) $\mathbb{Q}$ **is denumerable.**

We show that $|\mathbb{Z}| \le |\mathbb{Q}| \le |\mathbb{Z} \times \mathbb{Z}|$, and that $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{Z}|$.

First, $\mathbb{Z} \subset \mathbb{Q}$ via $n \mapsto n/1$, so the inclusion gives an injection $\mathbb{Z} \hookrightarrow \mathbb{Q}$; hence $|\mathbb{Z}| \le |\mathbb{Q}|$.

Next define $g : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$ by sending each rational $r$ to its reduced numerator–denominator pair: write $r = a/b$ with $a \in \mathbb{Z}$, $b \in \mathbb{Z} - \{0\}$, $\gcd(a, b) = 1$, and $b > 0$, and set $g(r) = (a, b)$. The representation $a/b$ with these conditions is unique, so $g$ is injective. Hence $|\mathbb{Q}| \le |\mathbb{Z} \times \mathbb{Z}|$.

Finally, $\mathbb{Z} \times \mathbb{Z}$ is denumerable. Since $\mathbb{Z}$ is denumerable by part (a), it suffices to exhibit a bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and then transport it to $\mathbb{Z} \times \mathbb{Z}$ using a bijection $\mathbb{N} \to \mathbb{Z}$. For example, the Cantor pairing function

$$\pi(m, n) = \frac{(m+n)(m+n+1)}{2} + n$$

is a bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Therefore $\mathbb{Z} \times \mathbb{Z}$ is denumerable, i.e. $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{Z}|$.

Combining the inequalities,

$$|\mathbb{Z}| \le |\mathbb{Q}| \le |\mathbb{Z} \times \mathbb{Z}| = |\mathbb{Z}|,$$

so $|\mathbb{Q}| = |\mathbb{Z}|$. Hence $\mathbb{Q}$ is denumerable.

**Exercise 4.** *If $A$, $A'$, $B$, $B'$ are sets such that $|A| = |A'|$ and $|B| = |B'|$, then $|A \times B| = |A' \times B'|$. If in addition $A \cap B = \varnothing = A' \cap B'$ then $|A \cup B| = |A' \cup B'|$. Therefore multiplication and addition of cardinals is well defined.*

*Solution.* Assume $|A| = |A'|$ and $|B| = |B'|$. Then there exist bijections $\alpha : A \to A'$ and $\beta : B \to B'$.

**Products.** Define

$$\Phi : A \times B \longrightarrow A' \times B', \qquad \Phi(a, b) = (\alpha(a), \beta(b)).$$

Then $\Phi$ is bijective. Indeed, its inverse is

$$\Psi : A' \times B' \longrightarrow A \times B, \qquad \Psi(a', b') = (\alpha^{-1}(a'), \beta^{-1}(b')).$$

Thus $|A \times B| = |A' \times B'|$.

**Unions (disjoint case).** Assume in addition that $A \cap B = \varnothing$ and $A' \cap B' = \varnothing$. Define $F : A \cup B \to A' \cup B'$ by

$$F(x) = \begin{cases} \alpha(x), & x \in A, \\ \beta(x), & x \in B. \end{cases}$$

This is well defined because $A \cap B = \varnothing$, so each $x \in A \cup B$ lies in exactly one of the two sets. Similarly, the map

$$G : A' \cup B' \to A \cup B, \qquad G(y) = \begin{cases} \alpha^{-1}(y), & y \in A', \\ \beta^{-1}(y), & y \in B', \end{cases}$$

is well defined because $A' \cap B' = \varnothing$. One checks immediately that $G \circ F = \mathrm{id}_{A \cup B}$ and $F \circ G = \mathrm{id}_{A' \cup B'}$, so $F$ is a bijection. Hence $|A \cup B| = |A' \cup B'|$.

Therefore, if we define cardinal multiplication by $|A| \cdot |B| := |A \times B|$ and cardinal addition (for disjoint sets) by $|A| + |B| := |A \cup B|$, these operations depend only on the cardinalities of $A$ and $B$, and not on the particular representatives chosen. In other words, addition and multiplication of cardinals are well defined.

**Exercise 5.** *For all cardinal numbers $\alpha$, $\beta$, $\gamma$:*

*(a)* $\alpha + \beta = \beta + \alpha$ *and* $\alpha\beta = \beta\alpha$ *(commutative laws).*

*(b)* $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ *and* $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ *(associative laws).*

*(c)* $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ *and* $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$ *(distributive laws).*

*(d)* $\alpha + 0 = \alpha$ *and* $\alpha 1 = \alpha$.

*(e) If $\alpha \neq 0$, then there is no $\beta$ such that $\alpha + \beta = 0$ and if $\alpha \neq 1$, then there is no $\beta$ such that $\alpha\beta = 1$. Therefore subtraction and division of cardinal numbers cannot be defined.*

*Solution.* Let $\alpha, \beta, \gamma$ be cardinals. Choose sets $A, B, C$ such that $|A| = \alpha$, $|B| = \beta$, $|C| = \gamma$, and assume (replacing by equipollent copies if necessary) that $A, B, C$ are pairwise disjoint. Recall that $\alpha + \beta := |A \cup B|$ (for disjoint representatives) and $\alpha\beta := |A \times B|$.

(a) **Commutativity.** Since $A \cup B = B \cup A$, we have $\alpha + \beta = |A \cup B| = |B \cup A| = \beta + \alpha$. Define $\tau : A \times B \to B \times A$ by $\tau(a, b) = (b, a)$. Then $\tau$ is a bijection, so $|A \times B| = |B \times A|$, i.e. $\alpha\beta = \beta\alpha$.

(b) **Associativity.** Because $A, B, C$ are disjoint,

$$(\alpha + \beta) + \gamma = |(A \cup B) \cup C| = |A \cup (B \cup C)| = \alpha + (\beta + \gamma).$$

For products, define $\Phi : (A \times B) \times C \to A \times (B \times C)$ by $\Phi((a, b), c) = (a, (b, c))$. This is a bijection with inverse $(a, (b, c)) \mapsto ((a, b), c)$. Hence $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

(c) **Distributivity.** Since $B$ and $C$ are disjoint, so are $A \times B$ and $A \times C$ if we identify them as subsets of $A \times (B \cup C)$ via the inclusions $B \hookrightarrow B \cup C$ and $C \hookrightarrow B \cup C$. Define

$$\Phi : A \times (B \cup C) \longrightarrow (A \times B) \cup (A \times C)$$

by

$$\Phi(a, x) = \begin{cases} (a, x), & x \in B, \\ (a, x), & x \in C. \end{cases}$$

This is well defined (each $x \in B \cup C$ lies in exactly one of $B, C$) and is clearly bijective, with inverse given by the inclusion of the union into $A \times (B \cup C)$. Therefore

$$|A \times (B \cup C)| = |(A \times B) \cup (A \times C)|,$$

i.e. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$. The identity $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$ follows similarly by swapping the roles of left and right factors.

(d) **Identities.** Let $0 = |\varnothing|$ and $1 = |\{*\}|$. If $A \cap \varnothing = \varnothing$, then $A \cup \varnothing = A$, so $\alpha + 0 = |A| = \alpha$. Also $A \times \{*\} \cong A$ via $a \mapsto (a, *)$, so $\alpha 1 = \alpha$.

(e) **No additive inverses and no multiplicative inverses in general.** If $\alpha \neq 0$, choose a nonempty set $A$ with $|A| = \alpha$. For any set $B$ disjoint from $A$, the union $A \cup B$ is nonempty, hence $|A \cup B| \neq 0$. Therefore there is no $\beta$ such that $\alpha + \beta = 0$.

If $\alpha \neq 1$, then either $\alpha = 0$ or $\alpha \geq 2$. In either case, there is no $\beta$ with $\alpha\beta = 1$. Indeed, if $\alpha = 0$ then $\alpha\beta = 0$ for all $\beta$. If $\alpha \geq 2$, let $A$ be a set of cardinality $\alpha$, so $A$ has distinct elements $a_1 \neq a_2$. For any nonempty $B$, the two subsets $\{a_1\} \times B$ and $\{a_2\} \times B$ are disjoint and nonempty, so $A \times B$ has at least two elements and hence cannot have cardinality 1. If $B = \varnothing$, then $A \times B = \varnothing$ has cardinality 0. Thus $|A \times B| \neq 1$ for all $B$, i.e. there is no $\beta$ with $\alpha\beta = 1$.

Therefore subtraction and division of cardinal numbers cannot be defined so as to make $(\text{Cardinals}, +, \cdot)$ into a ring or field in the usual way.

**Exercise 6.** *Let $I_n$ be as in Exercise 1. If $A \sim I_m$ and $B \sim I_n$ and $A \cap B = \varnothing$, then $(A \cup B) \sim I_{m+n}$ and $A \times B \sim I_{mn}$. Thus if we identify $|A|$ with $m$ and $|B|$ with $n$, then $|A| + |B| = m + n$ and $|A||B| = mn$.*

*Solution.* Let $A \sim I_m$ and $B \sim I_n$, and assume $A \cap B = \varnothing$. Choose bijections

$$f : A \longrightarrow I_m, \qquad g : B \longrightarrow I_n.$$

**Unions.** Define $h : A \cup B \to I_{m+n}$ by

$$h(x) = \begin{cases} f(x), & x \in A, \\ m + g(x), & x \in B. \end{cases}$$

This is well defined because $A \cap B = \varnothing$. It is injective: on $A$ it agrees with the injection $f$; on $B$ it agrees with the injection $x \mapsto m + g(x)$; and no value coming from $A$ (which lies in $\{1, \ldots, m\}$) can equal a value coming from $B$ (which lies in $\{m+1, \ldots, m+n\}$). It is surjective because every $t \in I_{m+n}$ satisfies either $1 \leq t \leq m$, in which case $t = f(a)$ for $a = f^{-1}(t) \in A$, or $m + 1 \leq t \leq m + n$, in which case $t = m + g(b)$ for $b = g^{-1}(t - m) \in B$. Hence $h$ is a bijection and $(A \cup B) \sim I_{m+n}$.

**Products.** Define $\Phi : A \times B \to I_{mn}$ by

$$\Phi(a, b) = (f(a) - 1)\, n + g(b).$$

Since $1 \leq f(a) \leq m$ and $1 \leq g(b) \leq n$, we have $0 \leq (f(a) - 1)n \leq (m - 1)n$, so $\Phi(a, b) \in \{1, 2, \ldots, mn\} = I_{mn}$.

To see that $\Phi$ is injective, suppose $\Phi(a, b) = \Phi(a', b')$. Then

$$(f(a) - 1)n + g(b) = (f(a') - 1)n + g(b'),$$

so

$$(f(a) - f(a'))n = g(b') - g(b).$$

The right-hand side lies in $\{-(n-1),\dots,n-1\}$, while the left-hand side is a multiple of $n$. Hence both sides must be 0, so $f(a)=f(a')$ and $g(b)=g(b')$, and therefore $a=a'$ and $b=b'$.

For surjectivity, let $t\in I_{mn}$. By the division algorithm there exist unique integers $q,r$ with

$$t-1=qn+r, \qquad 0\le r\le n-1, \qquad 0\le q\le m-1.$$

Set $i=q+1\in I_m$ and $j=r+1\in I_n$. Choose $a\in A$ with $f(a)=i$ and $b\in B$ with $g(b)=j$. Then

$$\Phi(a,b)=(i-1)n+j=qn+(r+1)=t.$$

Thus $\Phi$ is surjective, hence bijective, and $A\times B\sim I_{mn}$.

Therefore, identifying $|A|$ with $m$ and $|B|$ with $n$, we obtain

$$|A|+|B|=m+n, \qquad |A|\,|B|=mn,$$

i.e. cardinal addition and multiplication agree with the usual addition and multiplication on finite cardinalities.

**Exercise 7.** *If $A\sim A'$, $B\sim B'$ and $f:A\to B$ is injective, then there is an injective map $A'\to B'$. Therefore the relation $\le$ on cardinal numbers is well defined.*

*Solution.* Assume $A\sim A'$ and $B\sim B'$, and let $f:A\to B$ be injective. Choose bijections $\alpha:A'\to A$ and $\beta:B\to B'$. Define

$$f'=\beta\circ f\circ\alpha\;:\;A'\longrightarrow B'.$$

Then $f'$ is injective, since it is a composition of injective maps ($\alpha$ and $\beta$ are bijections, hence injective, and $f$ is injective). Thus there exists an injection $A'\to B'$, as required.

Consequently, if we define $|A|\le|B|$ to mean that there exists an injective map $A\to B$, then this relation depends only on the cardinalities of $A$ and $B$, and not on the particular representatives chosen. Hence $\le$ on cardinal numbers is well defined.

**Exercise 8.** *An infinite subset of a denumerable set is denumerable.*

*Solution.* Let $S$ be denumerable and let $T\subset S$ be an infinite subset. Choose a bijection $f:\mathbb{N}\to S$. Consider the set of indices

$$J=f^{-1}(T)=\{\,n\in\mathbb{N}:f(n)\in T\,\}\subset\mathbb{N}.$$

Since $T$ is infinite and $f$ is bijective, $J$ is infinite.

We now enumerate $J$ in increasing order. Define $j_0=\min J$, and having defined $j_0<\cdots<j_k$, set

$$j_{k+1}=\min\bigl(J-\{j_0,\dots,j_k\}\bigr).$$

This is well defined because $J$ is infinite, so after removing finitely many elements it is still nonempty, and $\mathbb{N}$ is well ordered.

Define $g:\mathbb{N}\to T$ by $g(k)=f(j_k)$. Then $g(k)\in T$ for all $k$, and $g$ is injective since the $j_k$ are distinct and $f$ is injective. Moreover $g$ is surjective onto $T$: if $t\in T$, then $t=f(n)$ for a unique $n\in\mathbb{N}$, and $n\in J$. Since $(j_k)$ lists all elements of $J$, we have $n=j_k$ for some $k$, hence $t=f(n)=f(j_k)=g(k)$.

Thus $g$ is a bijection $\mathbb{N}\to T$, so $T$ is denumerable.

**Exercise 9.** *The infinite set of real numbers $\mathbb{R}$ is not denumerable (that is, $\aleph_0 < |\mathbb{R}|$). [Hint: it suffices to show that the open interval $(0,1)$ is not denumerable by Exercise 8. You may assume each real number can be written as an infinite decimal. If $(0,1)$ is denumerable there is a bijection $f : \mathbf{N}^* \to (0,1)$. Construct an infinite decimal (real number) $.a_1a_2\dots$ in $(0,1)$ such that $a_n$ is not the nth digit in the decimal expansion of $f(n)$. This number cannot be in $\operatorname{Im} f$.]*

*Solution.* We prove that $(0,1)$ is not denumerable. Since $(0,1) \subset \mathbb{R}$, this implies $|\mathbb{R}| > \aleph_0$. (Equivalently, if $\mathbb{R}$ were denumerable then its infinite subset $(0,1)$ would be denumerable, contrary to what we prove below.)

Assume for contradiction that $(0,1)$ is denumerable. Then there exists a bijection $f : \mathbf{N}^* \to (0,1)$. For each $n \in \mathbf{N}^*$, write the decimal expansion of $f(n)$ as

$$f(n) = 0.d_{n1}d_{n2}d_{n3}\cdots,$$

where each $d_{nk} \in \{0,1,\dots,9\}$. We may (and do) choose the expansion so that it does *not* end in an infinite string of 9's; this makes the decimal representation unique.

Now define a new decimal

$$x = 0.a_1a_2a_3\cdots$$

by the rule

$$a_n = \begin{cases} 1, & d_{nn} \neq 1, \\ 2, & d_{nn} = 1. \end{cases}$$

Then each $a_n \in \{1,2\}$, so $x \in (0,1)$. Moreover, for every $n$ we have $a_n \neq d_{nn}$ by construction. Hence $x \neq f(n)$ for every $n$, since $x$ and $f(n)$ differ in the $n$-th decimal digit. Therefore $x \notin \operatorname{Im}(f)$, contradicting surjectivity of $f$.

Thus no bijection $\mathbf{N}^* \to (0,1)$ exists, so $(0,1)$ is not denumerable. Consequently $\mathbb{R}$ is not denumerable, i.e. $\aleph_0 < |\mathbb{R}|$.

**Exercise 10.** *If $\alpha$,$\beta$ are cardinals, define $\alpha^\beta$ to be the cardinal number of the set of all functions $B \to A$, where $A$, $B$ are sets such that $|A| = \alpha$, $|B| = \beta$.*

*(a) $\alpha^\beta$ is independent of the choice of $A$, $B$.*

*(b) $\alpha^{\beta+\gamma} = (\alpha^\beta)(\alpha^\gamma)$; $(\alpha\beta)^\gamma = (\alpha^\gamma)(\beta^\gamma)$; $\alpha^{\beta\gamma} = (\alpha^\beta)^\gamma$.*

*(c) If $\alpha \leq \beta$, then $\alpha^\gamma \leq \beta^\gamma$.*

*(d) If $\alpha$, $\beta$ are finite with $\alpha > 1$, $\beta > 1$ and $\gamma$ is infinite, then $\alpha^\gamma = \beta^\gamma$.*

*(e) For every finite cardinal $n$, $\alpha^n = \alpha\alpha\cdots\alpha$ ($n$ factors). Hence $\alpha^n = \alpha$ if $\alpha$ is infinite.*

*(f) If $P(A)$ is the power set of a set $A$, then $|P(A)| = 2^{|A|}$.*

*Solution.* Let $|A| = \alpha$ and $|B| = \beta$. Write $A^B$ for the set of all functions $B \to A$; by definition $\alpha^\beta = |A^B|$.

(a) $\alpha^\beta$ **is well defined.** Suppose $A, A', B, B'$ satisfy $|A| = |A'| = \alpha$ and $|B| = |B'| = \beta$. Choose bijections $\varphi : A \to A'$ and $\psi : B' \to B$. Define

$$T : A^B \longrightarrow (A')^{B'}, \qquad T(f) = \varphi \circ f \circ \psi.$$

Then $T$ is a bijection, with inverse $g \mapsto \varphi^{-1} \circ g \circ \psi^{-1}$. Hence $|A^B| = |(A')^{B'}|$, so $\alpha^\beta$ is independent of the choices of $A, B$.

(b) **Exponent laws.** Let $|A| = \alpha$, $|B| = \beta$, $|C| = \gamma$, and take $B \cap C = \varnothing$.

*(i)* $\alpha^{\beta+\gamma} = \alpha^\beta \, \alpha^\gamma$. A function $h : B \cup C \to A$ is uniquely determined by its restrictions $h|_B : B \to A$ and $h|_C : C \to A$. Conversely, any pair $(f, g) \in A^B \times A^C$ determines a unique $h \in A^{B \cup C}$ by $h|_B = f$, $h|_C = g$. Thus the map

$$A^{B\cup C} \longrightarrow A^B \times A^C, \qquad h \mapsto (h|_B, h|_C)$$

is a bijection, so $|A^{B\cup C}| = |A^B \times A^C|$, i.e. $\alpha^{\beta+\gamma} = (\alpha^\beta)(\alpha^\gamma)$.

*(ii)* $(\alpha\beta)^\gamma = (\alpha^\gamma)(\beta^\gamma)$. A function $u : C \to A \times B$ is equivalent to an ordered pair of functions $(f, g)$ with $f : C \to A$ and $g : C \to B$, via $u(c) = (f(c), g(c))$. Hence

$$(A \times B)^C \sim A^C \times B^C,$$

so $|(A \times B)^C| = |A^C \times B^C|$, i.e. $(\alpha\beta)^\gamma = (\alpha^\gamma)(\beta^\gamma)$.

*(iii)* $\alpha^{\beta\gamma} = (\alpha^\beta)^\gamma$. Identify $B \times C$ as the domain. A function $F : B \times C \to A$ is equivalent to a function $\widetilde{F} : C \to A^B$ given by

$$\widetilde{F}(c)(b) = F(b, c).$$

This correspondence is bijective (currying/uncurrying), so

$$A^{B\times C} \sim (A^B)^C,$$

hence $\alpha^{\beta\gamma} = (\alpha^\beta)^\gamma$.

(c) **Monotonicity in the base.** Assume $\alpha \le \beta$. Choose sets $A, B$ with $|A| = \alpha$, $|B| = \beta$, and an injection $i : A \hookrightarrow B$. For any set $C$ with $|C| = \gamma$, define

$$I : A^C \longrightarrow B^C, \qquad I(f) = i \circ f.$$

If $I(f) = I(g)$, then $i \circ f = i \circ g$, and since $i$ is injective we have $f = g$. Thus $I$ is injective, so $|A^C| \le |B^C|$, i.e. $\alpha^\gamma \le \beta^\gamma$.

(d) **If $\alpha, \beta$ are finite $> 1$ and $\gamma$ is infinite, then $\alpha^\gamma = \beta^\gamma$.**

Let $\gamma = |C|$ with $C$ infinite. Since $\alpha > 1$, there exists an injection $\{0, 1\} \hookrightarrow A$, hence $2^\gamma \le \alpha^\gamma$ by (c). Also $A$ is finite, so there is an injection $A \hookrightarrow \{0, 1\}^k$ for some $k \in \mathbb{N}$ (e.g. take $k$ with $2^k \ge \alpha$). Then by (c)

$$\alpha^\gamma \le (2^k)^\gamma.$$

Using (b)(iii) and (b)(v) below, $(2^k)^\gamma = 2^{k\gamma}$. Since $C$ is infinite and $k \ge 1$ is finite, $k\gamma = \gamma$ (there is a bijection $C \times I_k \cong C$), hence $(2^k)^\gamma = 2^\gamma$. Therefore $2^\gamma \le \alpha^\gamma \le 2^\gamma$, so $\alpha^\gamma = 2^\gamma$. The same argument gives $\beta^\gamma = 2^\gamma$, hence $\alpha^\gamma = \beta^\gamma$.

(e) **Finite exponents.** Let $n$ be a finite cardinal and choose $I_n = \{1, \dots, n\}$. A function $I_n \to A$ is the same as an $n$-tuple $(a_1, \dots, a_n) \in A^n$. Thus

$$A^{I_n} \cong \underbrace{A \times \cdots \times A}_{n \text{ factors}},$$

so $\alpha^n = \alpha \cdot \alpha \cdots \alpha$ ($n$ factors).

In particular, if $\alpha$ is infinite and $n \geq 1$ is finite, then $\alpha^n = \alpha$. (This uses the earlier result that $\alpha n = \alpha$ for infinite $\alpha$ and finite $n \geq 1$, proved by exhibiting a bijection $A \times I_n \sim A$ when $A$ is infinite.)

(f) **Power sets.** Let $P(A)$ denote the power set of $A$. Identify a subset $S \subset A$ with its characteristic function $\chi_S : A \to \{0,1\}$, where $\chi_S(a) = 1$ if $a \in S$ and $\chi_S(a) = 0$ otherwise. The map

$$P(A) \longrightarrow \{0,1\}^A, \qquad S \mapsto \chi_S$$

is a bijection, with inverse $f \mapsto f^{-1}(\{1\})$. Hence $|P(A)| = |\{0,1\}^A| = 2^{|A|}$.

**Exercise 11.** *If $I$ is an infinite set, and for each $i \in I$ $A_i$ is a finite set, then $|\bigcup_{i\in I} A_i| \leq |I|$.*

*Solution.* Let $I$ be infinite and suppose each $A_i$ is finite. For each $i \in I$, choose a bijection $f_i : A_i \to I_{n_i}$ for some $n_i \in \mathbb{N}$. Since $A_i$ is finite, there exists an injection $A_i \hookrightarrow \mathbb{N}$ (for instance, compose $f_i$ with the inclusion $I_{n_i} \hookrightarrow \mathbb{N}$). Fix such an injection and denote it by $\phi_i : A_i \hookrightarrow \mathbb{N}$.

Define a map

$$F : \bigcup_{i\in I} A_i \longrightarrow I \times \mathbb{N}$$

by

$$F(x) = (i, \phi_i(x)) \quad \text{where } i \text{ is any index with } x \in A_i.$$

To make $F$ well defined, replace $\bigcup_{i\in I} A_i$ by the disjoint union

$$\bigsqcup_{i\in I} A_i = \{(i,x) : i \in I,\ x \in A_i\},$$

which is equipollent to $\bigcup_{i\in I} A_i$ via $(i,x) \mapsto x$. On the disjoint union define

$$\widetilde{F} : \bigsqcup_{i\in I} A_i \longrightarrow I \times \mathbb{N}, \qquad \widetilde{F}(i,x) = (i, \phi_i(x)).$$

This map is injective: if $\widetilde{F}(i,x) = \widetilde{F}(j,y)$, then $(i, \phi_i(x)) = (j, \phi_j(y))$, hence $i = j$ and $\phi_i(x) = \phi_i(y)$. Since $\phi_i$ is injective, $x = y$. Thus $(i,x) = (j,y)$.

Therefore

$$\left|\bigsqcup_{i\in I} A_i\right| \leq |I \times \mathbb{N}|.$$

Because $I$ is infinite, we have $|I \times \mathbb{N}| = |I|$ (since $|\mathbb{N}| = \aleph_0 \leq |I|$ and for infinite cardinals $\kappa$, $\kappa \cdot \aleph_0 = \kappa$). Hence

$$\left|\bigsqcup_{i\in I} A_i\right| \leq |I|.$$

Finally, the canonical surjection $\bigsqcup_{i\in I} A_i \to \bigcup_{i\in I} A_i$, $(i,x) \mapsto x$, shows $\left|\bigcup_{i\in I} A_i\right| \leq \left|\bigsqcup_{i\in I} A_i\right|$. Combining, we obtain

$$\left|\bigcup_{i\in I} A_i\right| \leq |I|.$$

**Exercise 12.** *Let $\alpha$ be a fixed cardinal number and suppose that for every $i \in I$, $A_i$ is a set with $|A_i| = \alpha$. Then $|\bigcup_{i\in I} A_i| \le |I|\alpha$.*

*Solution.* Let $I$ be an index set and suppose $|A_i| = \alpha$ for all $i \in I$. Choose a set $A$ with $|A| = \alpha$. For each $i \in I$, choose a bijection $\varphi_i : A_i \to A$.

Consider the disjoint union

$$\bigsqcup_{i\in I} A_i = \{(i,x) : i \in I,\ x \in A_i\}.$$

Define

$$F : \bigsqcup_{i\in I} A_i \longrightarrow I \times A, \qquad F(i,x) = (i, \varphi_i(x)).$$

Then $F$ is injective: if $F(i,x) = F(j,y)$, then $(i,\varphi_i(x)) = (j,\varphi_j(y))$, hence $i = j$ and $\varphi_i(x) = \varphi_i(y)$, and since $\varphi_i$ is injective, $x = y$. Thus $(i,x) = (j,y)$.

Therefore

$$\left|\bigsqcup_{i\in I} A_i\right| \le |I \times A| = |I|\,|A| = |I|\,\alpha.$$

Finally, the canonical map $\bigsqcup_{i\in I} A_i \to \bigcup_{i\in I} A_i$, $(i,x) \mapsto x$, is surjective, so

$$\left|\bigcup_{i\in I} A_i\right| \le \left|\bigsqcup_{i\in I} A_i\right|.$$

Combining these inequalities gives

$$\left|\bigcup_{i\in I} A_i\right| \le |I|\,\alpha,$$

as required.