Solutions to *Algebra* by Thomas W. Hungerford

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Exercise 1. *Let $(A, \leq)$ be a partially ordered set and B a nonempty subset. A* ***lower bound*** *of B is an element $d \in A$ such that $d \leq b$ for* every *$b \in B$. A* ***greatest lower bound (g.l.b.)*** *of B is a lower bound d_0 of B such that $d \leq d_0$ for every other lower bound d of B. A* ***least upper bound (l.u.b.)*** *of B is an upper bound t_0 of B such that $t_0 \leq t$ for every other upper bound t of B. $(A, \leq)$ is a* ***lattice*** *if for all $a, b \in A$ the set $\{a, b\}$ has both a greatest lower bound and a least upper bound.*

(a) If $S \neq \varnothing$, then the power set $P(S)$ ordered by set-theoretic inclusion is a lattice, which has a unique maximal element.

(b) Give an example of a partially ordered set which is not *a lattice.*

(c) Give an example of a lattice with no maximal element and an example of a partially ordered set with two maximal elements.

Solution. (a) For $X, Y \subset S$ the greatest lower bound is

$$X \cap Y.$$

The least upper bound is

$$X \cup Y.$$

Thus every pair X, Y has a g.l.b. and l.u.b., so $(P(S), \subset)$ is a lattice.

A maximal element in $P(S)$ is an element that is not properly contained in any other element. The whole set S is an upper bound for every subset of S and is not contained in any strictly larger subset of S, so S is a maximal element. It is unique because if T is any subset with $U \subset T$ for all $U \subset S$, then in particular $S \subset T$, so $T = S$.

(b) Take the set $A = \{a, b\}$ with the only order relations being reflexivity:

$$a \leq a, \qquad b \leq b,$$

For the pair a, b there is no lower bound other than possibly elements $\leq a$ and $\leq b$; but the only candidates are a and b themselves, and neither is $\leq$ the other. Hence there is no greatest lower bound of a, b. (Similarly there is no least upper bound.) Therefore this poset is not a lattice.

(c) Take the integers $\mathbb{Z}$ with the usual order. For any $m, n \in \mathbb{Z}$ the least upper bound is $\max m, n$ and the greatest lower bound is $\min m, n$; thus $(\mathbb{Z}, \leq)$ is a lattice. But $\mathbb{Z}$ has no maximal element because for every $n \in \mathbb{Z}$ there exists $n + 1 > n$. So $\mathbb{Z}$ is a lattice with no maximal element.

Let $A = \{0, a, b\}$ and define the order by

$$0 \leq a, \qquad 0 \leq b.$$

Exercise 2. *A lattice $(A, \leq)$ (see Exercise 1) is said to be **complete** if every nonempty subset of A has both a least upper bound and a greatest lower bound. A map of partially ordered sets $f : A \to B$ is said to preserve order if $a \leq a'$ in A implies $f(a) \leq f(a')$ in B. Prove that an order-preserving map f of a complete lattice A into itself has at least one fixed element (that is, an $a \in A$ such that $f(a) = a$).*

Solution. Let $S = \{\, a \in A : f(a) \leq a \,\}$ be the set of all pre-fixed points of f. Since A is complete, it has a greatest element, say 1. Because f preserves order, $f(1) \leq 1$, so $1 \in S$. Thus $S \neq \varnothing$ and, since A is complete, S has a g.l.b; call it

$$m = \inf S.$$

First, we show that $f(m) \leq m$. For every $s \in S$ we have $m \leq s$, hence $f(m) \leq f(s)$ by order preservation. Since $s \in S$, $f(s) \leq s$, and thus $f(m) \leq s$ for all $s \in S$. Hence $f(m)$ is a lower bound of S, and by maximality of m as greatest lower bound, $f(m) \leq m$.

Second, we show that $m \leq f(m)$. Since m is a lower bound of S and f is order-preserving, the argument above shows that $f(m)$ is also a lower bound of S. Therefore $f(m) \leq s$ for all $s \in S$, so $f(m)$ is a lower bound of S. Because m is the greatest lower bound, we must have $m \leq f(m)$.

Combining the inequalities $f(m) \leq m$ and $m \leq f(m)$, we conclude that $f(m) = m$. Thus f has a fixed element.

Exercise 3. *Exhibit a well ordering of the set $\mathbb{Q}$ of rational numbers.*

Solution. Write each rational number in $\mathbb{Q}$ in its unique reduced form a/b with $b > 0$ and $\gcd(a, b) = 1$. (Under this convention the rational 0 is represented uniquely as $0/1$.)

Define a binary relation $\trianglelefteq$ on $\mathbb{Q}$ by declaring

$$\frac{a}{b} \trianglelefteq \frac{c}{d}$$

iff either

1. $|a| + b < |c| + d$, or
2. $|a| + b = |c| + d$ and $a < c$, or
3. $|a| + b = |c| + d,\ a = c$, and $b \leq d$.

Since every rational is written in the unique reduced form specified above, the quantities $|a|+b$, a, and b are well defined for each rational, so $\trianglelefteq$ is well defined.

It is immediate that $\trianglelefteq$ is a total order. To see that it is a well ordering, let $S \subseteq \mathbb{Q}$ be nonempty and for each $x = a/b \in S$ set $N(x) = |a| + b \in \mathbb{N}$. The set $\{N(x) : x \in S\}$ is a nonempty subset of $\mathbb{N}$, hence has a least element n_0. The subset $T = \{x \in S : N(x) = n_0\}$ is therefore nonempty. Among elements of T, the numerators form a finite (hence well-ordered) subset of $\mathbb{Z}$, so there is a least numerator a_0. Finally, among rationals in T with numerator a_0 the denominator is minimal for the $\trianglelefteq$-least element. Thus T (and hence S) has a least element with respect to $\trianglelefteq$. Therefore $\trianglelefteq$ is a well ordering of $\mathbb{Q}$.

Exercise 4. *Let S be a set. A* ***choice function*** *for S is a function f from the set of all nonempty subsets of S to S such that $f(A) \in A$ for all $A \neq \varnothing$, $A \subset S$. Show that the Axiom of Choice is equivalent to the statement that every set S has a choice function.*

Solution. We show the two statements are equivalent.

(AC $\Rightarrow$ choice functions exist). Let S be any set and let $\mathcal{I}$ denote the collection of all nonempty subsets of S. If $\mathcal{I} = \varnothing$ then $S = \varnothing$, and the unique function $\varnothing \to \varnothing$ is a choice function for S. Thus assume $\mathcal{I} \neq \varnothing$. Consider the family $\{X_A\}_{A \in \mathcal{I}}$ where $X_A = A$ for each $A \in \mathcal{I}$. Every X_A is nonempty by definition, and the family is indexed by the nonempty set $\mathcal{I}$. By the Axiom of Choice (the product of a family of nonempty sets indexed by a nonempty set is nonempty), the product $\prod_{A \in \mathcal{I}} X_A$ is nonempty. An element of this product is precisely a function $f \colon \mathcal{I} \to S$ with $f(A) \in X_A = A$ for each A; that is exactly a choice function for S. Hence every set S admits a choice function.

(Choice functions exist $\Rightarrow$ AC). Assume every set T admits a choice function c_T defined on the collection of nonempty subsets of T. Let $\{X_i\}_{i \in I}$ be any family of nonempty sets indexed by a nonempty set I. Put $S = \bigcup_{i \in I} X_i$. Then each X_i is a nonempty subset of S, so the hypothesis supplies a choice function c_S for S. Define $g \colon I \to S$ by $g(i) := c_S(X_i)$. By construction $g(i) \in X_i$ for every $i \in I$, so $g \in \prod_{i \in I} X_i$. Hence the product is nonempty. This establishes the Axiom of Choice.

Therefore the two statements are equivalent.

Exercise 5. *Let S be the set of all points (x, y) in the plane with $y \leq 0$. Define an ordering by $(x_1, y_1) \leq (x_2, y_2) \iff x_1 = x_2$ and $y_1 \leq y_2$. Show that this is a partial ordering of S, and that S has infinitely many maximal elements.*

Solution. Let $S = \{(x, y) \in \mathbb{R}^2 : y \leq 0\}$ and define

$$(x_1, y_1) \leq (x_2, y_2) \iff x_1 = x_2 \text{ and } y_1 \leq y_2.$$

(i) This relation is a partial order.

- *Reflexive:* For any $(x, y) \in S$ we have $x = x$ and $y \leq y$, so $(x, y) \leq (x, y)$.
- *Antisymmetric:* If $(x_1, y_1) \leq (x_2, y_2)$ and $(x_2, y_2) \leq (x_1, y_1)$, then $x_1 = x_2$ and $y_1 \leq y_2$, and also $x_2 = x_1$ and $y_2 \leq y_1$. Hence $y_1 = y_2$ and therefore $(x_1, y_1) = (x_2, y_2)$.
- *Transitive:* If $(x_1, y_1) \leq (x_2, y_2)$ and $(x_2, y_2) \leq (x_3, y_3)$, then $x_1 = x_2$ and $x_2 = x_3$, so $x_1 = x_3$, and $y_1 \leq y_2 \leq y_3$, hence $y_1 \leq y_3$. Thus $(x_1, y_1) \leq (x_3, y_3)$.

Therefore the relation is reflexive, antisymmetric, and transitive, i.e. a partial order.

(ii) S has infinitely many maximal elements.

Fix any real number x_0. For that x_0 the point $(x_0, 0) \in S$ satisfies the following: if $(x_0, 0) \leq (x, y)$ then $x = x_0$ and $0 \leq y$. Since every element of S has $y \leq 0$, the only possibility is $y = 0$, so $(x, y) = (x_0, 0)$. Thus there is no element of S strictly greater than $(x_0, 0)$; i.e. $(x_0, 0)$ is maximal.

As x_0 ranges over $\mathbb{R}$ we obtain the family $\{(x, 0) : x \in \mathbb{R}\}$ of maximal elements,which is infinite (indeed uncountable). Hence S has infinitely many maximal elements.

(Observe also that any point (x, y) with $y < 0$ is not maximal because $(x, y) < (x, 0)$.)

Exercise 6. *Prove that if all the sets in the family $\{A_i \mid i \in I \neq \varnothing\}$ are nonempty, then each of the projections $\pi_k \colon \prod_{i \in I} A_i \to A_k$ is surjective.*

Solution. Let $\{A_i\}_{i \in I}$ be a family of sets with $A_i \neq \varnothing$ for each $i \in I$. Fix $k \in I$ and let $\pi_k : \prod_{i \in I} A_i \to A_k$ be the projection onto the k-th coordinate. We must show that π_k is surjective, i.e. that for every $a \in A_k$ there exists $f \in \prod_{i \in I} A_i$ with $\pi_k(f) = f(k) = a$.

For a given $a \in A_k$ we need to define a function $f : I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$ and $f(k) = a$. To do this we must choose, for each $i \in I - \{k\}$, an element $f(i) \in A_i$. The existence of a choice function selecting one element from each A_i (for $i \neq k$) is exactly an instance of the Axiom of Choice. Assuming Choice (or equivalently the hypothesis that the product $\prod_{i \in I} A_i$ is nonempty), pick such elements $f(i)$ for all $i \neq k$, and put $f(k) = a$. Then $f \in \prod_{i \in I} A_i$ and $\pi_k(f) = a$. Since a was arbitrary, π_k is surjective.

Remark. If the index set I is finite, no form of the Axiom of Choice is needed: one can choose elements from the finitely many A_i inductively (or by a finite product of nonempty sets being nonempty). The use of Choice becomes essential only when I is infinite.

Exercise 7. *Let $(A, \leq)$ be a linearly ordered set. The **immediate successor** of $a \in A$ (if it exists) is the least element in the set $\{x \in A \mid a < x\}$. Prove that if A is well ordered by $\leq$, then at most one element of A has no immediate successor. Give an example of a linearly ordered set in which precisely two elements have no immediate successor.*

Solution. First remark: if $a \in A$ has no immediate successor, that means the set $\{x \in A : x > a\}$ either is empty (so a is maximal) or is nonempty but has no least element.

At most one element has no immediate successor. Suppose for contradiction that a and b are two distinct elements of A with no immediate successor. Since A is linearly ordered, either $a < b$ or $b < a$. Without loss of generality assume $a < b$. Then $b \in \{x \in A : x > a\}$, so this set is nonempty. But A is well ordered, hence every nonempty subset has a least element; therefore $\{x \in A : x > a\}$ has a least element c. By definition c is the immediate successor of a, contradicting the assumption that a has no immediate successor. Thus it is impossible for two distinct elements to both lack immediate successors; at most one element of A can have no immediate successor. $\square$

Example with exactly two elements having no immediate successor. Let

$$B = \{0\} \cup \{1/n : n \in \mathbf{N}^*\} \subset \mathbb{R}$$

equipped with the usual order inherited from $\mathbb{R}$. Every element of B except 0 is of the form $1/n$ for some $n \in \mathbf{N}^*$. For $n \geq 2$, the least element strictly greater than $1/n$ is $1/(n-1)$, so $1/n$ has an immediate successor. The element $1 = 1/1$ is maximal in B (no larger element of B exists), hence it has no immediate successor. The element 0 also has no immediate successor: the set $\{x \in B : x > 0\} = \{1/n : n \in \mathbf{N}^*\}$ has no least element because for each $1/n$ there is $1/(n+1) \in B$ with $0 < 1/(n+1) < 1/n$. Therefore 0 has no immediate successor. No other elements of B lack immediate successors, so exactly two elements of B (namely 0 and 1) have no immediate successor.