Machine Learning Exercise 3

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1. Prove this inequality:

$$\frac{1}{2}P(\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-E\phi(z)|\geq 2\varepsilon)\leq P(\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)|\geq\varepsilon)\leq 2P(\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-E\phi(z)|\geq\frac{1}{2}\varepsilon)$$

Proof:

Lemma:

$$\frac{1}{2}P(|\nu_1-p|\geq 2\varepsilon)\leq P(|\nu_1-\nu_2|\geq\varepsilon)\leq 2P(|\nu_1-p|\geq\frac{\varepsilon}{2})$$

where $X_1,\cdots,X_{2n}$ are i.i.d. r.v. and

$$\nu_1=\frac{1}{n}\sum_{i=1}^{n}X_i,\nu_2=\frac{1}{n}\sum_{i=n+1}^{2n}X_i,p=EX,n\geq\frac{ln2}{\varepsilon^2}$$

We have a conclusion that

$$\{\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)|\geq\varepsilon\}\subseteq\{\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-E\phi(z)|\geq\frac{1}{2}\varepsilon\}\cup\{\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)-E\phi(z)|\geq\frac{1}{2}\varepsilon\}$$

and $z_1,\cdots,z_{2n}$ are i.i.d. r.v.

$$\begin{aligned}P(\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)|\geq\varepsilon) &\leq P(\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-E\phi(z)|\geq\varepsilon)+P(\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)-E\phi(z)|\geq\varepsilon)\\ &= 2P(\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-E\phi(z)|\geq\varepsilon)\end{aligned}$$

Another conclusion

$$\{\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-E\phi(z)|\geq 2\varepsilon\}\cap\{|\frac{1}{n}\sum_{i=n+1}^{2n}\phi_0(z_i)-E\phi_0(z)|\leq\varepsilon\}\subseteq\{\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)|\geq\varepsilon\}$$

where ϕ_0 is selected in Φ such that $|\frac{1}{n}\sum_{i=1}^{n}\phi_0(z_i)-E\phi_0(z)|\geq 2\varepsilon$ holds true. Therefore

$$P(\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)|\geq\varepsilon)\geq P(\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-E\phi(z)|\geq 2\varepsilon)P(|\frac{1}{n}\sum_{i=n+1}^{2n}\phi_0(z_i)-E\phi_0(z)|\leq\varepsilon)$$

Given the condition

$$n\geq\frac{ln2}{\varepsilon^2}$$

then using Chernoff bound, we get

$$\begin{aligned} P(|\frac{1}{n}\sum_{i=n+1}^{2n}\phi_0(z_i)-E\phi_0(z)|\leq\varepsilon) &= 1-P(|\frac{1}{n}\sum_{i=n+1}^{2n}\phi_0(z_i)-E\phi_0(z)|>\varepsilon) \\ &= 1-2P(\frac{1}{n}\sum_{i=n+1}^{2n}\phi_0(z_i)-E\phi_0(z)>\varepsilon) \\ &\geq 1-2e^{-2n\varepsilon^2} \\ &\geq 1-2e^{-2ln2}=\frac{1}{2} \end{aligned}$$

Therefore

$$P(\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)|\geq\varepsilon)\geq\frac{1}{2}P(\sup_{\phi\in\Phi}|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-E\phi(z)|\geq 2\varepsilon)$$

2. Prove the bound

$$\sum_{i=0}^{d}\binom{n}{i}\leq(\frac{en}{d})^d$$

Proof:

Lemma:

$$dln(1-x)-ln(1-dx)\leq 0,\forall x,0\leq x\leq\frac{1}{d+1}$$

Lemma's proof:

Let $f(x)=dln(1-x)-ln(1-dx)$, then $f(0)=0$, and

$$\begin{aligned} f'(x) &= \frac{d}{x-1}-\frac{1}{dx-1} \\ &= \frac{(d-1)((d+1)x-1)}{(1-x)(1-dx)}\leq 0 \end{aligned}$$

Therefore $f(x)\leq 0$ holds true $\forall x,0\leq x\leq\frac{1}{d+1}$.

$n=d$

$$\sum_{i=0}^{d}\binom{n}{i} = 2^d \leq e^d$$

<u>$n = k, k \geq d$</u>

Suppose that the inequality holds true for all $n \leq k$, and when $n = k+1$, we desire to prove

$$\sum_{i=0}^{d}\binom{k+1}{i} \leq (\frac{e(k+1)}{d})^d$$

namely to prove

$$\begin{aligned}
\sum_{i=0}^{d}\binom{k}{i}\frac{k+1}{k-i+1} &\leq (\frac{e(k+1)}{d})^d && (1)\\
\frac{k+1}{k-d+1}\sum_{i=0}^{d}\binom{k}{i} &\leq (\frac{e(k+1)}{d})^d && (2)\\
\frac{k+1}{k-d+1}(\frac{ek}{d})^d &\leq (\frac{e(k+1)}{d})^d && (3)\\
k^d &\leq (k+1)^{d-1}(k-d+1) && (4)\\
(\frac{k}{k+1})^d &\leq \frac{k-d+1}{k+1} && (5)\\
(1-\frac{1}{k+1})^d &\leq 1-\frac{d}{k+1} && (6)\\
dln(1-\frac{1}{k+1}) &\leq ln(1-\frac{d}{k+1}) && (7)
\end{aligned}$$

The last inequality is true according to lemma. and in (1) to (2) we use

$$\frac{k+1}{k-i+1} \leq \frac{k+1}{k-d+1}$$

in (2) to (3) we use the suppose

$$\sum_{i=0}^{d}\binom{k}{i} \leq (\frac{ek}{d})^d$$

To conclude, the original inequality holds true.