Machine Learning Exercise 1

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1. $X \sim N(0,1)$(Guassian Distribution), $t > 0$, give an upper bound (UB) and a lower bound (LB) of $Pr[X > t]$ such that $UB(t) \sim LB(t)$ when $t \to +\infty$.

Answer:

the p.d.f of X is

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

and

$$\begin{aligned}
Pr[X > t] &= \int_t^\infty p(x)dx \\
&= \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{1}{2}x^2} dx \\
&\leq \frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{x}{t} e^{-\frac{1}{2}x^2} \\
&= \frac{1}{\sqrt{2\pi}t} \int_t^\infty x e^{-\frac{1}{2}x^2} dx \\
&= \frac{1}{\sqrt{2\pi}t} e^{-\frac{1}{2}t^2}
\end{aligned}$$

namely

$$UB(t) = \frac{1}{\sqrt{2\pi}t} e^{-\frac{1}{2}t^2}$$

assume that

$$LB(t) = \frac{t}{\sqrt{2\pi}(t^2+1)} e^{-\frac{1}{2}t^2}$$

let

$$
\begin{aligned}
h(t) &= Pr[X > t] - LB(t) \\
&= \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-\frac{1}{2}x^2} - \frac{t}{\sqrt{2\pi}(t^2+1)} e^{-\frac{1}{2}t^2}
\end{aligned}
$$

then

$$
\begin{aligned}
h'(t) &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} - \frac{1 - 2t^2 - t^4}{\sqrt{2\pi}(t^2+1)^2} e^{-\frac{1}{2}t^2} \\
&= -\frac{2}{\sqrt{2\pi}(t^2+1)^2} e^{-\frac{1}{2}t^2} < 0
\end{aligned}
$$

note that

$$\lim_{t\to\infty} h(t) = 0$$

therefore

$$h(t) > 0, \forall t > 0$$

equals to

$$Pr[X > t] > LB(t)$$

finally we note that

$$\lim_{t\to\infty} \frac{UB}{LB} = \lim_{t\to\infty} \frac{t^2+1}{t^2} = 1$$

2. $X_1, X_2, \cdots, X_n$ i.i.d, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, EX and $varX$ are known, give a bound to $\bar{X}$ using Chebyshev Ineq.

Answer:

$$E\bar{X} = E(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n} EX_i = EX$$

$$
\begin{aligned}
var\bar{X} &= E\bar{X}^2 - (EX)^2 \\
&= \frac{1}{n^2}E(\sum_{i=1}^{n} X_i^2 + \sum_{i\neq j} X_i X_j) - (EX)^2
\end{aligned}
$$

given that $X_1, X_2, \cdots, X_n$ i.i.d

$$EX_iX_j = EX_iEX_j, (i \neq j)$$

$$EX_i^2 = varX_i + (EX_i)^2$$

therefore

$$
\begin{aligned}
var\bar{X} &= \frac{1}{n^2}(n(varX + (EX)^2) + C_n^2(EX)^2) - (EX)^2 \\
&= \frac{1}{n}varX
\end{aligned}
$$

using Chebyshev Ineq.

$$Pr[(\bar{X} - E\bar{X})^2 \geq t^2] \leq \frac{var\bar{X}}{t^2} = \frac{varX}{nt^2}$$