STA623 - Bayesian Data Analysis - Practical 3 (Solutions)

Marc Henrion

2025-09-24

Practical 3

Notation

- X, Y, Z - random variables
- x, y, z - measured / observed values
- $\bar{X}$, $\bar{Y}, \bar{Z}$ - sample mean estimators for X, Y, Z
- $\bar{x}$, $\bar{y}, \bar{z}$ - sample mean estimates of X, Y, Z
- $\hat{T}$, $\hat{t}$ - given a statistic T, estimator and estimate of T
- $P(A)$ - probability of an event A occurring
- $f_X(.)$, $f_Y(.), f_Z(.)$ - probability mass / density functions of X, Y, Z; sometimes $p_X(.)$ etc. rather than $f_X(.)$
- p(.) - used as a shorthand notation for pmfs / pdfs if the use of this is unambiguous (i.e. it is clear which is the random variable)
- $X \sim F$ - X distributed according to distribution function F
- $E[X]$, $E[Y]$, $E[Z]$, $E[T]$ - the expectation of X, Y, Z, T respectively

Exercise 1

Show that the Bayes estimator $\widehat{\theta}_B$ for the quadratic loss function $\mathcal{C}(\theta - \widehat{\theta}) = (\theta - \widehat{\theta})^2$ is given by the posterior mean. In other words, show that:

$$E[\theta|y] = \arg\min_{\hat{\theta}} \int_y \int_\Theta \mathcal{C}(\theta - \widehat{\theta}) p(\theta, y) d\theta dy$$

Exercise 1 (Solution)

We saw in lectures that we can use the multiplication rule and then optimise the inner integral only:

$$\begin{aligned} \widehat{\theta}_B &= \arg\min_{\hat{\theta}} \int \mathcal{C}(\theta - \widehat{\theta}) p(\theta|y) d\theta && (1) \\ &= \arg\min_{\hat{\theta}} \int (\theta - \widehat{\theta})^2 p(\theta|y) d\theta && (2) \end{aligned}$$

To find the minimum, we solve

$$\begin{aligned} & \frac{d}{d\widehat{\theta}} \int (\theta - \widehat{\theta})^2 p(\theta|y) d\theta = 0 && (3) \\ \iff & \int 2(\theta - \widehat{\theta})(-1) p(\theta|y) d\theta = 0 && (4) \\ \iff & \int (\theta - \widehat{\theta}) p(\theta|y) d\theta = 0 && (5) \\ \iff & \int \theta p(\theta|y) d\theta = \int \widehat{\theta} p(\theta|y) d\theta && (6) \end{aligned}$$

Note that $\int \widehat{\theta} p(\theta|y) d\theta = \widehat{\theta} \int p(\theta|y) d\theta = \widehat{\theta}$ since the posterior distribution for θ is a probability distribution and needs to integrate to 1.

Therefore we see that

$$\widehat{\theta}_B = \int \theta p(\theta|y) d\theta$$

This is the posterior mean $E[\theta|y] = \int \theta p(\theta|y) d\theta$.

Exercise 2

Suppose $\pi \sim \text{Beta}(2,3)$ and $Y_1, \dots, Y_n \sim_{\text{iid}} \textit{Bernoulli}(\pi)$. Further suppose we observe data $y_1, \dots, y_n$ with $n = 25, k = \sum_i y_i = 16$.

Find the following:

- posterior distribution $p(\pi|k)$ and plot it, comparing it to the prior distribution
- posterior predictive distribution $p(\tilde{y}|y_1, \dots, y_n)$
- a Bayesian point estimate $\hat{\pi}$
- the 95% quantile-based Bayesian confidence interval for π
- the 95% HPD interval

Further, compute:

- $P(\pi > 0.5|k)$
- For the following 2 hypotheses: $H_1 : \pi \in [0.3, 0.5], H_2 : \pi \in [0.5, 0.7]$, compute the prior and posterior odds and calculate the Bayes factor.

Exercise 2 (Solution)

Posterior distribution

Recall:

$$\begin{cases} \text{prior } \Pi & \sim \text{Beta}(a,b) \\ \\ \text{likelihood } Y|\Pi & \sim \text{Bin}(n,\pi) \end{cases}$$

$$\Rightarrow \text{posterior } \Pi|Y = k \sim \text{Beta}(a + k, b + n - k)$$

Here:

- a = 2
- b = 3
- n = 25
- k = 16

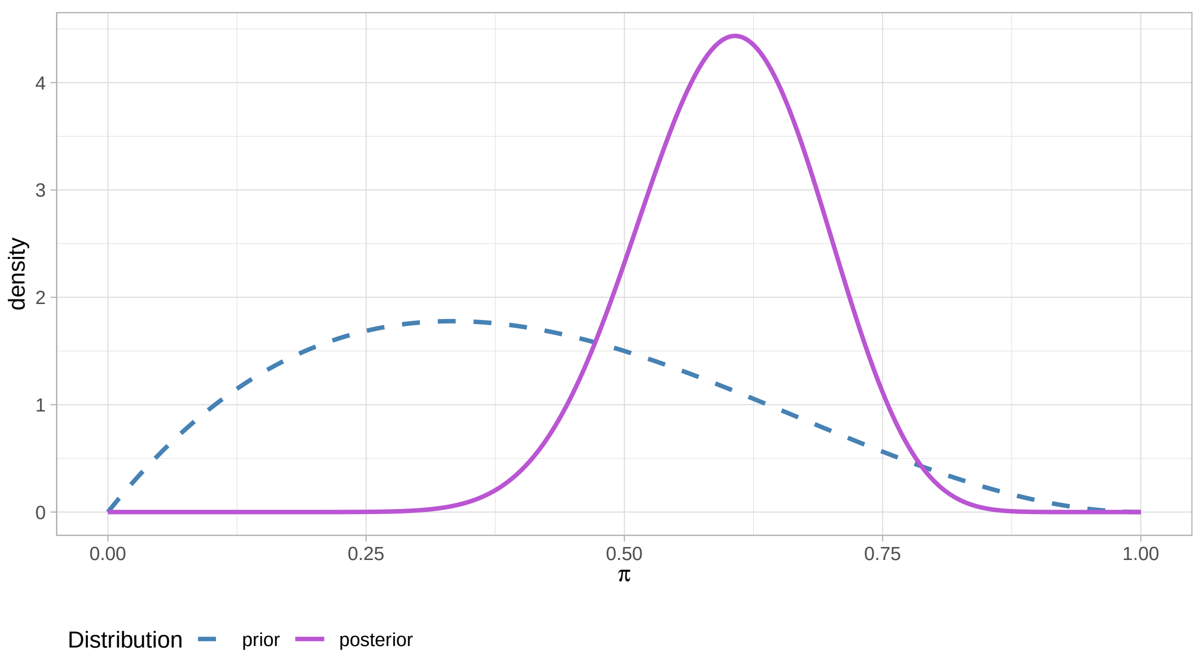
Hence, we have a Beta(2+16,3+9) = Beta(18,12) distribution.

Alternatively, you can also see that

$$p(\pi|k=16) \propto \pi^{2-1}(1-\pi)^{3-1}\pi^{16}(1-\pi)^{25-16} = \pi^{17}(1-\pi)^{11}$$
$$\Rightarrow \Pi|Y=k \sim \text{Beta}(18,12)$$

```r
df<-data.frame(
  type=factor(levels=c("prior","posterior"),c(rep("prior",1000),rep("posterior",1000))),
  xx=rep(seq(0,1,length=1000),2)
) %>%
  mutate(
    yy=case_when(
      type=="prior"~dbeta(xx,2,3),
      type=="posterior"~dbeta(xx,18,12))
  )

df %>%
  ggplot(mapping=aes(x=xx,y=yy,col=type,lty=type)) +
  geom_line(mapping=aes(y=yy),lwd=1) +
  scale_color_manual(name="Distribution",values=c("steelblue","mediumorchid")) +
  scale_linetype_manual(name="Distribution",values=c(2,1)) +
  theme_light() + theme(legend.position = "bottom",legend.justification = "left") +
  ylab("density") + xlab(expression(pi))
```



Posterior predictive distribution

$$
\begin{aligned}
p(\tilde{Y}=1|y_1,\dots,y_n) = & \int p(\tilde{y},\pi|y_1,\dots,y_n)d\pi & (7)\\
= & \int p(\tilde{y}|\pi)p(\pi|y_1,\dots,y_n)d\pi & (8)\\
= & \int \pi\frac{\Gamma(30)}{\Gamma(18)\Gamma(12)}\pi^{17}(1-\pi)^{11}d\pi & (9)\\
= & \frac{\Gamma(30)\Gamma(19)}{\Gamma(18)\Gamma(31)}\int \frac{\Gamma(31)}{\Gamma(19)\Gamma(12)}\pi^{18}(1-\pi)^{11}d\pi & (10)\\
= & \frac{\Gamma(30)\Gamma(19)}{\Gamma(18)\Gamma(31)} = \frac{18}{30} = 3/5 = 0.6 & (11)
\end{aligned}
$$

From this it follows that $p(\tilde{Y}=0|y_1,\dots,y_n) = 1-\frac{3}{5} = \frac{2}{5} = 0.4$.

Bayesian point estimate

As the question did not specify, you choose which Bayesian estimator to use - computing one is enough.

For example:

- Posterior mean

  With $\Pi|k \sim \text{Beta}(18,12)$, the posterior mean $\hat{\pi} = E[\Pi|k] = \frac{18}{18+12} = 0.6$.

- Posterior median

  For this we can use R: it is simply the 50th percentile: $\hat{\pi} =$ `qbeta(0.5,18,12)` $= 0.6023$.

- Posterior mode

  Since a>1 and b>1, the mode exists and is equal to $\hat{\pi} = \frac{a-1}{a+b-2}\frac{17}{28} = 0.6071$.

Credible intervals

We have:

- $q_{0.025;\text{Beta}(18,12)} = 0.4226$
- $q_{0.975;\text{Beta}(18,12)} = 0.7648$

Therefore the 95% quantile based Bayesian confidence interval for π is given by [`0.42`,`0.76`]

```
qbeta(c(0.025,0.975),18,12)
```

```
[1] 0.4226046 0.7647598
```

For the HPD we can use the function `hdi()` from the R package `HDInterval`:

```
library(HDInterval)
hdp<-hdi(qbeta,credMass=0.95,shape1=18,shape2=12)
print(hdp)
```

```
    lower     upper
0.4273464 0.7690367
attr(,"credMass")
[1] 0.95
```

From this we find that the 95% HDP interval for π is given by [`0.43`,`0.77`]

Alternatively, we can used the function `hdi()` from the R package `bayestestR` - but not that despite the same name, this function is used different: you have to provide a vector of values to it and cannot provide a parametric function as we did above. So this means, we first need to sample from the Beta(18,12) distribution.

```
library(bayestestR)
```

```
Attaching package: 'bayestestR'


The following object is masked from 'package:HDInterval':

    hdi
```

```
sbeta<-rbeta(n=1e5,shape1=18,shape2=12)
hdi(sbeta,ci=0.95)
```

```
95% HDI: [0.43, 0.77]
```

Probability that π is larger than 0.5

We have

$$
\begin{aligned}
P(\pi > 0.5|Y = k) &= \int_{0.5}^{1} p(\pi|k)d\pi & (12)\\
&= \int_{0.5}^{1} \frac{\Gamma(30)}{\Gamma(18)\Gamma(12)}\pi^{17}(1-\pi)^{11}d\pi & (13)\\
&= \approx 0.87 & (14)
\end{aligned}
$$

```
integrate(dbeta,lower=0.5,upper=1,shape1=18,shape2=12)
```

```
0.8675346 with absolute error < 2.8e-09
```

Alternatively we could have worked with the posterior cdf:

```
1-pbeta(0.5,18,12)
```

```
[1] 0.8675346
```

Bayes factor

Prior odds

$$\frac{P(H_1)}{P(H_2)} = \frac{\int_{0.3}^{0.5} p_{\beta(3,2)}(\pi)d\pi}{\int_{0.5}^{0.7} p_{\beta(3,2)}(\pi)d\pi} = 1.48$$

Prosterior odds

$$\frac{P(H_1|k)}{P(H_2|k)} = \frac{\int_{0.3}^{0.5} p_{\beta(18,12)}(\pi)d\pi}{\int_{0.5}^{0.7} p_{\beta(18,12)}(\pi)d\pi} = 0.18$$

```
priorOdds<-
  integrate(dbeta,0.3,0.5,shape1=2,shape2=3)$value /
    integrate(dbeta,0.5,0.7,shape1=2,shape2=3)$value
#priorOdds<-
#  (pbeta(0.5,2,3)-pbeta(0.3,2,3))/(pbeta(0.7,2,3)-pbeta(0.5,2,3))
#  alternative way of doing this
priorOdds
```

```
[1] 1.482517
```

```r
posteriorOdds<-
  integrate(dbeta,0.3,0.5,shape1=18,shape2=12)$value /
    integrate(dbeta,0.5,0.7,shape1=18,shape2=12)$value
posteriorOdds
```

```
[1] 0.1789838
```

Bayes Factor

$\text{BF} = \frac{\text{posterior odds}}{\text{prior odds}} = 0.18/1.48 = 0.12$

```r
bayesFactor<-posteriorOdds/priorOdds
bayesFactor
```

```
[1] 0.1207296
```

Exercise 3

Suppose $\lambda \sim \text{Gamma}(5,2)$ and $Y_1, \ldots, Y_n \sim_{\text{iid}} Poisson(\lambda)$. Further suppose we observe data $y_1, \ldots, y_n$ with $n = 18, k = \sum_i y_i = 40$.

Find the following:

- posterior distribution $p(\lambda | y_1, \ldots, y_n)$ and plot it, comparing it to the prior distribution
- posterior predictive distribution $p(\tilde{y} | y_1, \ldots, y_n)$
- the 95% quantile-based Bayesian confidence interval for λ
- the 95% HPD interval

Further, compute:

- $P(\lambda \leq 1 | y_1, \ldots, y_n)$
- For the following 2 hypotheses: $H_1 : \lambda \in [0.75, 1.25], H_2 : \lambda \in [1.75, 2.25]$, compute the prior and posterior odds and calculate the Bayes factor.

Exercise 3 (Solution)

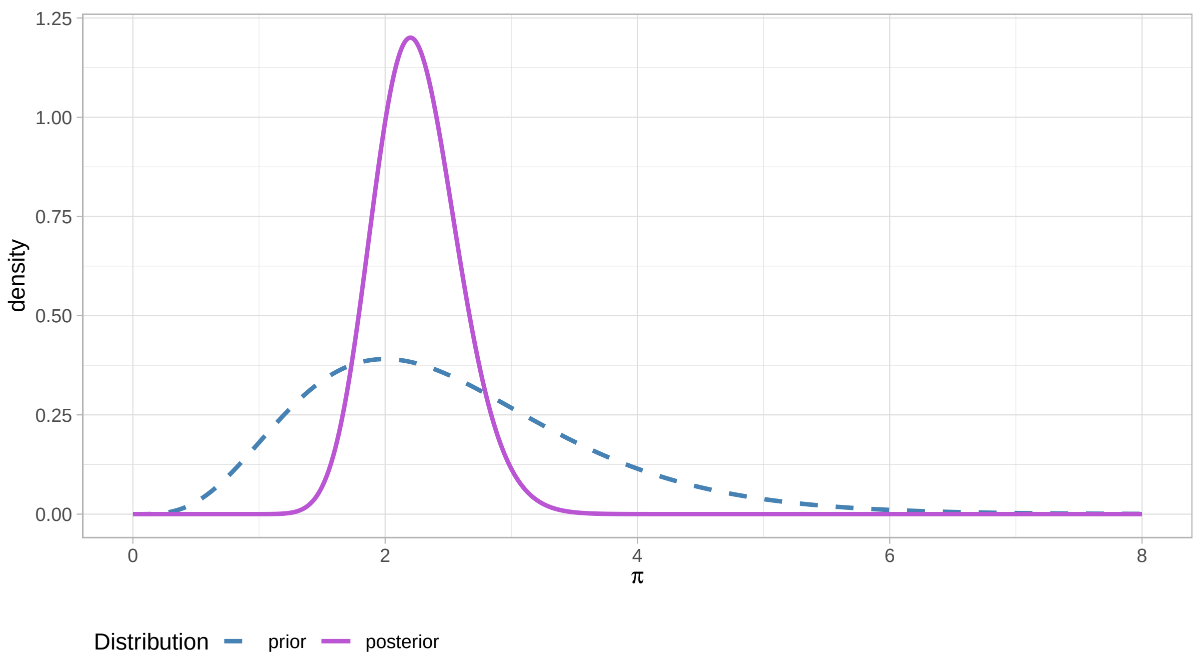
Posterior distribution

$$
\begin{aligned}
p(\lambda | y_1, \ldots, y_n) \propto & \quad \lambda^{5-1} e^{-2\lambda} \lambda^{\sum_i y_i} e^{-n\lambda} && (15) \\
\propto & \quad \lambda^{5+40-1} e^{-(2+18)\lambda} && (16) \\
\Rightarrow \Lambda | y_1, \ldots, y_n \sim & \quad \Gamma(45, 20) && (17)
\end{aligned}
$$

```
df<-data.frame(
  type=factor(levels=c("prior","posterior"),c(rep("prior",1000),rep("posterior",1000))),
  xx=rep(seq(0,8,length=1000),2)
) %>%
  mutate(
    yy=case_when(
      type=="prior"~dgamma(xx,shape=5,rate=2),
      type=="posterior"~dgamma(xx,shape=45,rate=20))
  )

df %>%
  ggplot(mapping=aes(x=xx,y=yy,col=type,lty=type)) +
```

```
  geom_line(mapping=aes(y=yy),lwd=1) +
  scale_color_manual(name="Distribution",values=c("steelblue","mediumorchid")) +
  scale_linetype_manual(name="Distribution",values=c(2,1)) +
  theme_light() + theme(legend.position = "bottom",legend.justification = "left") +
  ylab("density") + xlab(expression(pi))
```



Posterior predictive distribution

From Practical 1&2, Exercise 4, we know:

$$\tilde{Y}|y_1, \ldots, y_n \sim \text{NegBin}(45, 20/21)$$

Credible intervals

We have:

- $q_{0.025;\Gamma(45,20)} = 1.6412$
- $q_{0.975;\Gamma(45,20)} = 2.9534$

Therefore the 95% quantile based Bayesian confidence interval for π is given by [`1.64,2.95`]

```
qgamma(c(0.025,0.975),shape=45,rate=20)
```

```
[1] 1.641165 2.953397
```

For the HPD we are going to use the function `hdi()` from the R package `HDInterval` (you can also use the same function from the `bayestestR` package instead as we did for Exercise 2 - this will require sampling from the posterior gamma distribution first):

```
library(HDInterval)
hdp<-hdi(qgamma,credMass=0.95,shape=45,rate=20)
print(hdp)
```

```
   lower    upper
1.611342 2.917067
attr(,"credMass")
[1] 0.95
```

From this we find that the 95% HDP interval for λ is given by [`1.61`,`2.92`]

We have

$$
\begin{aligned}
P(\lambda \leq 1|y_1, \dots, y_n) &= \int_0^1 p(\lambda|y_1, \dots, y_n) d\lambda & (18)\\
&= \int_0^1 \frac{20^{45}}{\Gamma(45)} \lambda^{44} e^{-20\lambda} d\lambda & (19)\\
&= 1.06 \cdot 10^{-6} & (20)
\end{aligned}
$$

```
integrate(dgamma,lower=0,upper=1,shape=45,rate=20)
```

```
1.060263e-06 with absolute error < 4.4e-13
```

Bayes factor

Prior odds

$$
\frac{P(H_1)}{P(H_2)} = \frac{\int_{0.75}^{1.25} p_{\gamma(5,2)}(\lambda) d\lambda}{\int_{1.75}^{2.25} p_{\Gamma(5,2)}(\lambda) d\lambda} = 0.2513
$$

Prosterior odds

$$
\frac{P(H_1|y_1, \dots, y_n)}{P(H_2|y_1, \dots, y_n)} = \frac{\int_{0.75}^{1.25} p_{\gamma(45,20)}(\lambda) d\lambda}{\int_{1.75}^{2.25} p_{\Gamma(45,20)}(\lambda) d\lambda} = 0.0004
$$

```r
priorOdds<-
  integrate(dgamma,0.75,1.25,shape=5,rate=2)$value /
    integrate(dgamma,1.75,2.25,shape=5,rate=2)$value

# priorOdds<-(pgamma(1.25,shape=5,rate=2)-pgamma(0.75,shape=5,rate=2)) /
#                (pgamma(2.25,shape=5,rate=2)-pgamma(1.75,shape=5,rate=2))
# same result

priorOdds
```

```
[1] 0.4667705
```

```r
posteriorOdds<-
  integrate(dgamma,0.75,1.25,shape=45,rate=20)$value /
    integrate(dgamma,1.75,2.25,shape=45,rate=20)$value

# posteriorOdds<-(pgamma(1.25,shape=45,rate=20)-pgamma(0.75,shape=45,rate=20)) /
#                (pgamma(2.25,shape=45,rate=20)-pgamma(1.75,shape=45,rate=20))
# same result

posteriorOdds
```

```
[1] 0.0004338494
```

Bayes Factor

$$\text{BF} = \frac{\text{posterior odds}}{\text{prior odds}} = 0.004/0.2513 = 0.0015$$

```r
bayesFactor<-posteriorOdds/priorOdds
bayesFactor
```

```
[1] 0.0009294706
```

The Bayes factor is very low, which means that observing the data reduced the odds of hypothesis 1 over hypothesis 2 substantially (i.e. hypothesis 2 much more likely under the posterior compared to hypothesis 1 than it was under the prior).