TWIN-STAR HYPOTHESIS AND CYCLE-FREE d-PARTITIONS OF K_{2d}

MATTHEW J. FYFE, STEVEN R. LIPPOLD, MIHAI D. STAIC, AND ALIN STANCU

1. Summary

In this document we explain how one can use MATLAB to prove that the twin-star hypothesis $\mathcal{H}(4)$ holds true. This combined with a Theorem from the article "Twin-Star Hypothesis and Cycle-Free d-Partitions of K_{2d}" to show that a Conjecture from the same article is true for $d = 4$. These results prove the twin-star hypothesis in this case. For more information on the Theorem and Conjecture in question, as well as the statement of the twin-star hypothesis, see the article cited.

2. MATLAB Explanation

2.1. **Initial Observations.** First, notice that from the article we know that in order to show that the twin-star hypothesis $\mathcal{TS}(4)$ holds true it is enough to show that very cycle-free 4-partition of K_8 that contains the graph T_{19} as one of its components is involution equivalent with a cycle-free 4-partition of K_8 that contains the graph TS_4 as one of its components.

Let $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ be a cycle-free partition of K_8 such that one of the graphs Γ_i is isomorphic to T_{19}. Using the action of S_4 on $\mathcal{P}_4^{cf}(K_8)$ we may assume that the cycle-free 4-partition $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ has the property that Γ_1 is isomorphic to T_{19}. Up to the S_4 action, this is equivalent to Γ_4 being isomorphic to T_{19}, as given in the article. However, it is more straightforward in the MATLAB code to take Γ_1 to be isomorphic to T_{19}.

Next, using the action of the group S_8 on on $\mathcal{P}_4^{cf}(K_8)$) (via the action on the vertices of K_8), we may assume that $E(\Gamma_1) = \{(1,2), (1,4), (1,6), (1,8), (2,3), (2,5), (4,7)\}$. Finally, since $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ is cycle-free, the vertex 1 must be incident to at least one edge of Γ_i for all $1 \le i \le 4$, so we may assume that $(1,3) \in E(\Gamma_2)$, $(1,5) \in E(\Gamma_3)$, and $(1,7) \in E(\Gamma_4)$ (here we use again the action of S_4 on $\mathcal{P}_4^{cf}(K_8)$). A template for our partition is presented in Figure 1.

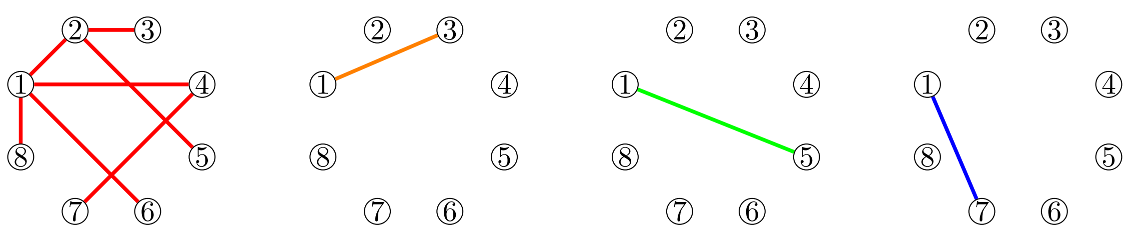

Figure 1. $\Gamma_1 = X_{19}$, $(1,3) \in E(\Gamma_2)$, $(1,5) \in E(\Gamma_3)$, and $(1,7) \in E(\Gamma_4)$

The plan is to analyze all cycle-free 4-partitions of K_8 that have these properties and show that each such partition is involution equivalent with a cycle-free 4-partition of K_8 that contains the graph TS_4 as one of its components.

2.2. **Step 1: Generate Partitions.** Next we explain the MATLAB code that we developed to check this property. First we enumerate the edges of K_8 as follows: $(1,2)$ is the first edge, $(1,3)$ is the second, and so on with the edge $(7,8)$ being the the 28-th (see the vector `AlphS2D4` in the file `Step1GeneratePartitions.m`). With this convention, a homogeneous 4-partition $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ of K_8 is described by a vector $v \in \mathbb{Z}^{28}$ whose entries are distinct values of the set $\{1, 2, \dots, 28\}$. The first seven entries of the vector v correspond to Γ_1, the next seven to Γ_2, the next seven to Γ_3, and the last seven to Γ_4. In order to get a unique description of our partition we add the assumption that $v(1) < v(2) < \cdots < v(7)$, $v(8) < v(9) < \cdots < v(14)$, $v(15) < v(16) < \cdots < v(21)$, and $v(22) < v(23) < \cdots < v(28)$. For example, the vector

$$v = [1, 2, 4, 6, 9, 11, 13, 3, 8, 14, 15, 17, 20, 22, 5, 10, 16, 19, 23, 24, 27, 7, 12, 18, 21, 25, 26, 28]$$

corresponds to the 4-partition in Figure 2.

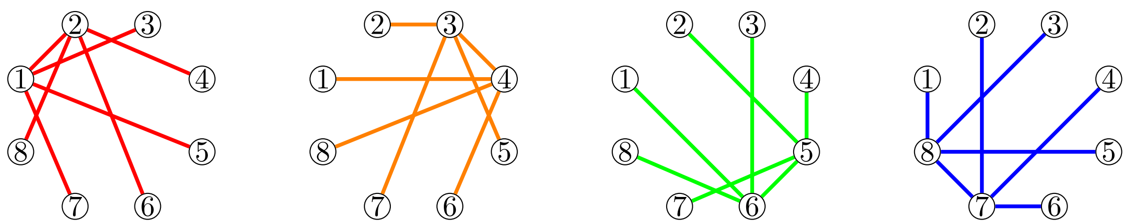

FIGURE 2. A homogeneous cycle-free 4-partition of K_8 associated with the given vector

With these conventions, we are looking for vectors $v \in \mathbb{Z}^{28}$ with the $v(i) \in \{1, 2, \dots, 28\}$, $v(i) \neq v(j)$ if $i \neq j$, and $v(1) < v(2) < \cdots < v(7)$, $v(8) < v(9) < \cdots < v(14)$, $v(15) < v(16) < \cdots < v(21)$, and $v(22) < v(23) < \cdots < v(28)$. Moreover, since we fixed Γ_1 and the edges $(1,3) \in E(\Gamma_2)$, $(1,5) \in E(\Gamma_3)$, and $(1,7) \in E(\Gamma_4)$, we want that $v(1) = 1$, $v(2) = 3$, $v(3) = 5$, $v(4) = 7$, $v(5) = 8$, $v(6) = 10$ and $v(7) = 21$ (edges for Γ_1), $v(8) = 2$ (because $(1,3) \in E(\Gamma_2)$), $v(15) = 4$ (because $(1,5) \in E(\Gamma_3)$), and $v(22) = 6$ (because $(1,7) \in E(\Gamma_4)$).

The first part of the code (file `Step1GeneratePartitions.m`) describes an algorithm that generates all cycle-free 4-partitions with the above properties. It turns out that there are 617 088 such partitions. The output of this search is the file `CycleFreePartitionsGamma1X19.txt`. This is primarily accomplished through empty arrays, the use of the nchoosek and setdiff functions in MATLAB, and with a user defined function that converts from the vector into a matrix that labels the edges.

2.3. **Step 2: Klein Group Action.** Next notice that the configuration in Figure 1 is invariant under the action of a group isomorphic to the Klein group

$$K = \{e_{S_8} \times e_{S_4}, (6,8) \times e_{S_4}, (3,5) \times (2,3), (3,5)(6,8) \times (2,3)\} \subseteq S_8 \times S_4$$

By use of this group action, we can reduce the number of partitions we need to examine by restricting ourselves to a representative of each of the equivalence classes. This is accomplished through the second part of code (file `Step2ActionKlein.m`), where we use this action to reduce the number of partitions we need to analyze. As expected, there are 154 272 equivalence classes. The input is the file `CycleFreePartitionsGamma1X19.txt` and the output file is `CycleFreePartitionsKlein.txt`.

2.4. **Step 3: Check for the twin-star graph.** The last part of the code (file `Step3CheckTS4.m`) shows that every partition from the file `CycleFreePartitionsKlein.txt` is at most three involutions away from a cycle-free 4-partition that has one of the graphs isomorphic to TS_4. More precisely, for every $\mathcal{P} = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ partition listed in `CycleFreePartitionsKlein.txt` there exist $1 \leq a_i < b_i < c_i \leq 8$, for $1 \leq i \leq 3$ such that if we denote

$$\begin{aligned} \mathcal{P}_1 &= \mathcal{P}^{(a_1,b_1,c_1)}, \\ \mathcal{P}_2 &= \mathcal{P}_1^{(a_2,b_2,c_2)}, \\ \mathcal{P}_3 &= \mathcal{P}_2^{(a_3,b_3,c_3)}, \end{aligned}$$

then at least one of the cycle-free 4-partition $\mathcal{P}_1$, $\mathcal{P}_2$, or $\mathcal{P}_3$ contains a graph isomorphic to TS_4.

Remark 2.1. To check these results we used a PC with Core i7 processor, 12GB (RAM). The run time was approximately 1.5 hours for the first file, 16 hours for the second, and 30 hours for the last. With minimal modifications, one can skip the second file (use only the first and the last file), but in that case the run time is about 120 hours.

2.5. **MATLAB Items.** Here is a short description of some of the main vectors, files, and functions used in our code. For the complete MATLAB code see the Github Repository at https://github.com/Steven-R-Lippold/Twin-Star-Hypothesis.

- the vector `AlphS2D4` lists all edges in K_8.
- the file `CycleFreePartitionsGamma1X19.txt` gives a list of all cycle-free partitions with $E(\Gamma_1) = \{(1,2),(1,4),(1,6),(1,8),(2,3),(2,5),(4,7)\}$, $(1,3) \in \Gamma_2$, $(1,5) \in \Gamma_3$ and $(1,7) \in \Gamma_4$.
- the function `Perm23(w28)` gives the action of the permutation $(2,3) \in S_4$ on the partition vector $w28 \in \mathbb{Z}^{28}$.
- the function `sxw(sigma8,w28,AlphS2D4)` gives the action of the permutation $\sigma_8 \in S_8$ on the partition vector $w28 \in \mathbb{Z}^{28}$.
- the file `CycleFreePartitionsKlein.txt` gives a list of representatives for the action of the Klein group $K \in S_8 \times S_4$ on the cycle-free partitions from the file `CycleFreePartitionsGamma1X19.txt`.
- the function `PartToVect(uu,AlphS2D4)`, associates to a 1×7 vector uu the corresponding matrix of edges.
- the vector `AlphS2ABC` lists all triples (a,b,c) with $1 \leq a < b < c \leq 8$.
- `vHH=[4,4,1,1,1,1,1,1]` is the incidence vector for the graph TS_4.
- the function `Step3DS(vxx,vHH,AlphS2D4,AlphS2ABC,rowsS2abc)` checks if the partition corresponding to vxx is at most three involutions away from a cycle-free 4-partition which contains a graph isomorphic to TS_4.
- the function `hasDS(txx,vHH,AlphS2D4)` decides if the cycle-free 4-partition associated to the vector txx contains a graph isomorphic to TS_4.
- the function `ActABC(a,b,c,w28,AlphS2D4)` gives the action of the involution (a,b,c) on the cycle-free 4-partition associated to the vector $w28$.
- the file `NoPartitions.txt` list all partitions from `CycleFreePartitionsKlein.txt` that are not three involutions away from a cycle-free 4-partition which contains a graph isomorphic to TS_4 (if our claim is correct, this file is empty).

Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403
Email address: mjfyfe@bgsu.edu

Department of Chemistry, Mathematics, and Physics, Geneva College, Beaver Falls, PA 15010
Email address: srlippol@geneva.edu

Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403

Institute of Mathematics of the Romanian Academy, PO.BOX 1-764, RO-70700 Bucharest, Romania.
Email address: mstaic@bgsu.edu

Department of Mathematics, Columbus State University, Columbus, GA 31907
Email address: stancu_alin1@columbusstate.edu