Option Pricing Beyond BSM

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Why BSM Fails After 1987

BS Model Setting

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{1}$$

Postulates:

- Constant r, μ, σ
- $S(t)$ follows geometric brownian motion

Method 1: Dynamic Hedging

Assume we can replicate a vanilla option, we set up a portfolio $\Pi(t, S_t) = \Delta_t S_t - c(t, S_t)$.

By no-arbitrage assumption, we have $\dfrac{d\Pi}{\Pi} = rdt$.
We arrive at

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} - rc = 0 \tag{2}$$

Method 2: Risk-Neutral Measure

Under the risk-neutral measure $\mathbb{Q}$ (changing the measure by 1d version of Girsanov Theorem), the BSM becomes:

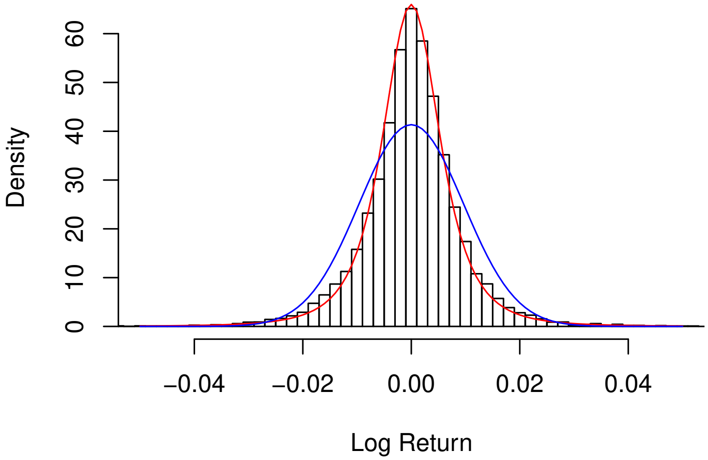
$$dS(t) = rS(t)dt + \sigma dW^{\mathbb{Q}}(t) \tag{3}$$

We can then show that $e^{-rt}S(t)$ is a martingale under $\mathbb{Q}$. Furthermore,

$$c(t, S(t)) = \mathbb{E}[e^{-r(T-t)}(S(t) - K)^{+}|S(t)] \tag{4}$$

This representation allow us to either do numerical integration or easily get the explicit BSM formula.

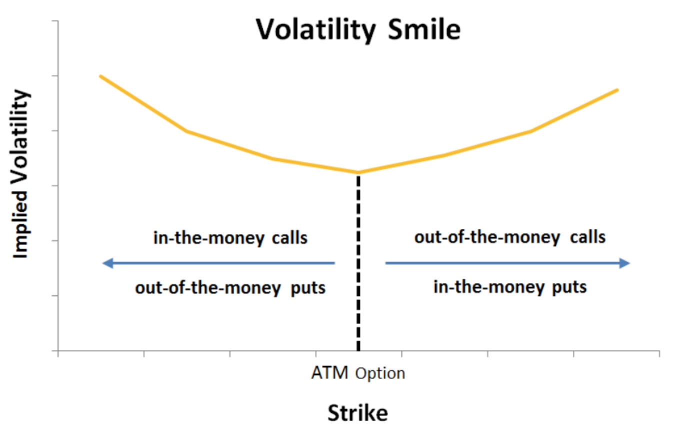
Empirical Fact: Not Normal



Implied Volatility

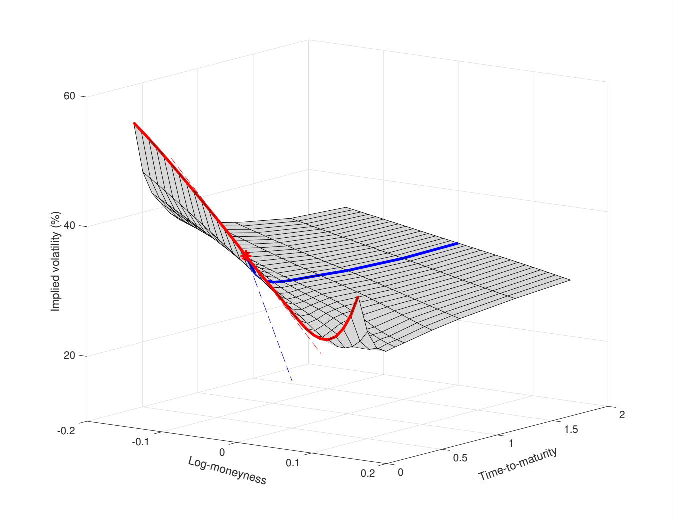
Generated by a 'wrong' pricing tool with a right market price

$$P_{BS}(r, \tau, S_t, K, \sigma_{BS}) = P_{market} \tag{5}$$



IV ctd.: SPX Data, Interpolation using cubic spline

On 2010 May 20th, SPX data, with $log(K/S)$ from -0.15 to 0.15, τ from 1 month to 2 years:



Stochastic Volatility Models

General Model

$$\frac{dS(t)}{S(t)} = \mu dt + \sqrt{V(t)}dW_1 \tag{6a}$$

$$dV(t) = a(V(t))dt + b(V(t))(\rho dW_1 + \sqrt{1-\rho^2}dW_2) \tag{6b}$$

We can solve this set of equations with dynamic hedging method, in which case three kind of assets will be used: risk-free bonds, stock and an arbitrary asset whose payoff only depends on volatility (ex. variance swap).

The process to obtain PDE is tedious, so not represented here. The interesting part is how to obtain a tractable solution.

Heston Model (1993)

Variance follows a mean-reverting square root process:

$$\frac{dS(t)}{S(t)} = rdt + \sqrt{V(t)}dW_1(t) \tag{7a}$$

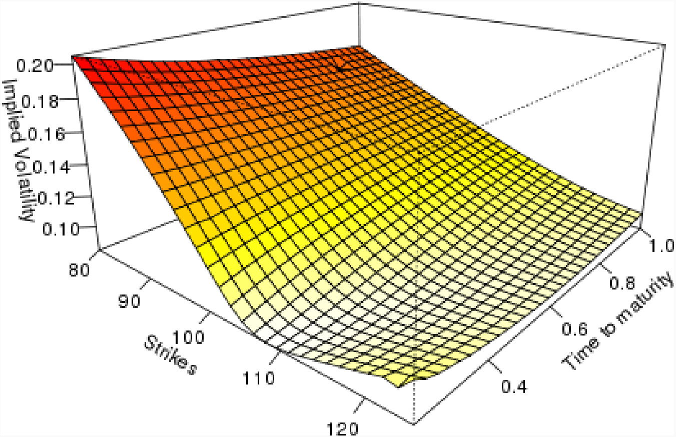
$$dV(t) = \kappa(\theta - V(t))dt + \xi\sqrt{V(t)}dW_2(t) \tag{7b}$$

$$dW_1(t)dW_2(t) = \rho dt \tag{7c}$$

Why Heston model is popular?

All SV model produces similar IVS, but Heston is rather simple and has some useful features.

Heston Model IV



Why not Monte Carlo?

Simulation: Euler Scheme

Change variable to $Y = lnS$ and decompose $dW_2(t)$.
The Euler-discretization version gives:

$$Y_{(i+1)\Delta} = Y_{i\Delta} + (\mu - \frac{1}{2} V_{i\Delta})\Delta + \sqrt{V_{i\Delta}\Delta}\epsilon_{1,i\Delta} \tag{8a}$$

$$V_{(i+1)\Delta} = V_{i\Delta} + \kappa(\theta - V_{i\Delta})\Delta + \xi\sqrt{V_{i\Delta}\Delta}[\rho\epsilon_{1,i\Delta} + \sqrt{1-\rho^2}\epsilon_{2,i\Delta}] \tag{8b}$$

Problem: V_i can become negative during simulation.
Fix: apply a truncation function at each step, for ex., $|x|$ or $(x)^+$, called absorbing and reflecting condition

Other Scheme: Milstein Scheme

Milstein:

$$v_{i+1} = v_i - \lambda\,(v_i - \bar{v})\,\Delta t + \eta\,\sqrt{v_i}\,\sqrt{\Delta t}\,Z + \frac{\eta^2}{4}\,\Delta t\,\left(Z^2 - 1\right)$$

This can be rewritten as

$$v_{i+1} = \left(\sqrt{v_i} + \frac{\eta}{2}\,\sqrt{\Delta t}\,Z\right)^2 - \lambda\,(v_i - \bar{v})\,\Delta t - \frac{\eta^2}{4}\,\Delta t$$

Even under second-order scheme, this problem still exists. But the frequency with which the process goes negative is substantially reduced relative to the Euler case.

Other Scheme: Implicit Scheme

An Implicit Scheme We follow Alfonsi (2005) and consider

$$\begin{aligned} v_{i+1} &= v_i - \lambda\,(v_i - \bar{v})\,\Delta t + \eta\,\sqrt{v_i}\,\sqrt{\Delta t}\,Z \\ &= v_i - \lambda\,(v_{i+1} - \bar{v})\,\Delta t + \eta\,\sqrt{v_{i+1}}\,\sqrt{\Delta t}\,Z \\ &\quad -\eta\,\left(\sqrt{v_{i+1}} - \sqrt{v_i}\right)\,\sqrt{\Delta t}\,Z + \text{higher order terms} \end{aligned}$$

We note that

$$\sqrt{v_{i+1}} - \sqrt{v_i} = \frac{\eta}{2}\,\sqrt{\Delta t}\,Z + \text{higher order terms}$$

$$v_{i+1} = v_i - \lambda\,(v_{i+1} - \bar{v})\,\Delta t + \eta\,\sqrt{v_{i+1}}\,\sqrt{\Delta t}\,Z - \frac{\eta}{2}\,\Delta t \tag{2.19}$$

Then $\sqrt{v_{i+1}}$ may be obtained as a root of the quadratic equation (2.19). Explicitly,

$$\sqrt{v_{i+1}} = \frac{\sqrt{4\,v_i + \Delta t\,\left[(\lambda\,\bar{v} - \eta^2/2)\,(1 + \lambda\,\Delta t) + \eta^2\,Z^2\right]} + \eta\,\sqrt{\Delta t}\,Z}{2\,(1 + \lambda\,\Delta t)}$$

Characteristic Function Method

Motivation

- Heston's paper used this method
- Semi-closed form solution
- Can be applied to all AJD process
- Can be implemented efficiently, for ex. by FFT

Risk-Neutral Measure

$$\frac{dS}{S} = \mu dt + \sigma dW_t \tag{9a}$$

$$\frac{dS}{S} = r dt + \sigma d^Q W_t \tag{9b}$$

According to Girsanov Theorem, the two measures are linked as $\frac{dQ}{dP}(t) = M(t) = exp(\int_0^t \theta(u) dW - \frac{1}{2}\int_0^t \theta(u)^2 du)$. We can easily verify that $M(t)$ is a martingale under $\mathbb{P}$ and $dW^Q = dW - \theta dt$ is a brownian motion under $\mathbb{Q}$. In the above case, $\theta = \frac{\mu - r}{\sigma}$, namely sharpe ratio in economic sense.

Characteristic Function

Consider the Fourier transform on $f(x)$:

$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} e^{iux} f(x) dx$

In case $f(x)$ represents the probability density of a random variable X, then $\Phi(u) = \mathcal{F}[f(x)] = \mathbb{E}[e^{iux}]$, is called the characteristic function.

If we know $\Phi(u)$, we can inverse it to obtain

$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \Phi(u) du.$

For sufficiently small ε, we can extend the integral to the complex plane: $f(x) = \frac{1}{2\pi} \int_{-\varepsilon i-\infty}^{-\varepsilon i+\infty} e^{-iux} \Phi(u) du.$

Inverse Trick

Consider

$$G(x) = \mathbb{P}(X > x) = \frac{1}{2\pi}\int_{-\varepsilon i-\infty}^{-\varepsilon i+\infty} \Phi(u)(\int_x^{\infty} e^{-iuy}dy)du$$
$$= \frac{1}{2\pi i}\int_{-\varepsilon i-\infty}^{-\varepsilon i+\infty} \Phi(u)\frac{e^{-iux}}{u}du$$

Note that on one hand,
$\frac{1}{2\pi i}\int_{-\varepsilon i-\infty}^{-\varepsilon i+\infty} \Phi(u)\frac{e^{-iux}}{u}du - \frac{1}{2\pi i}\int_{\varepsilon i-\infty}^{\varepsilon i+\infty} \Phi(u)\frac{e^{-iux}}{u}du = 1,$
on the other hand,
$\frac{1}{2\pi i}\int_{-\varepsilon i-\infty}^{-\varepsilon i+\infty} \Phi(u)\frac{e^{-iux}}{u}du + \frac{1}{2\pi i}\int_{\varepsilon i-\infty}^{\varepsilon i+\infty} \Phi(u)\frac{e^{-iux}}{u}du =$
$2lim_{\varepsilon\to 0}\int_{|u|>\epsilon} \Phi(u)\frac{e^{-iux}}{u}du.$

Inverse Trick ctd.

Combining the two results, we get (integral in P.V. sense):

$$G(x) = \frac{1}{2} + \frac{1}{2\pi} lim_{\varepsilon \to 0} \int_{|u|>\epsilon} \Phi(u) \frac{e^{-iux}}{u} du \tag{11}$$

To match what appears in literature, one last step is to note that $\frac{e^{iux}\Phi(-u)}{-iu}$ is the complex conjugate of $\frac{e^{-iux}\Phi(u)}{iu}$. So finally we have the graceful expression:

$$G(x) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \Re[\frac{e^{-iux}\Phi(u)}{iu}] du \tag{12}$$

Pricing European Call Option

Let $x_T = lnS_T, k = lnK$, call option price can be expressed as:

$$C = \mathbb{E}(e^{-rT}e^{x_T}\mathbb{1}_{\{x_T>k\}}) - e^k e^{-rT}\mathbb{P}(x_T > k) \tag{13}$$

Clearly the second term is just $G(k)$.

Pricing European Call Option ctd.

The first term can be evaluated by changing from risk neutral measure $\mathbb{P}$ to a new measure $\widetilde{\mathbb{P}}$ defined by $\dfrac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \dfrac{e^{x_T}}{\mathbb{E}[e^{x_T}]}$.

Now we have $\mathbb{E}(e^{-rT}e^{x_T}\mathbb{1}_{\{x_T>k\}}) = \mathbb{E}[e^{-rT}e^{x_T}]\widetilde{\mathbb{P}}(x_T > k)$.

Note that under the new measure,

$$\widetilde{\Phi}(u) = \widetilde{\mathbb{E}}[e^{iux_T}] = \frac{\mathbb{E}[e^{iux_T}e^{x_T}]}{\mathbb{E}[e^{x_T}]} = \frac{\Phi(u-i)}{\Phi(-i)} \tag{14}$$

Therefore, we obtain $\widetilde{\mathbb{P}}(x_T > k) = \dfrac{1}{2} + \dfrac{1}{\pi}\int_0^\infty \Re[\dfrac{e^{-iuk}\Phi(u-i)}{iu\Phi(-i)}]du$.

Pricing European Call Option Conclusion

- First, we need to find the CF $\Phi(u)$.
- Secondly, we calculate the two probability $\widetilde{\mathbb{P}}(x_T > k)$ and $\mathbb{P}(x_T > k)$.
- Finally, we have $C = S_0\widetilde{\mathbb{P}}(x_T > k) - Ke^{-rT}\mathbb{P}(x_T > k)$

Clearly the expression is very similar to BSM's explicit formula. CF is just a generalization of risk neutral measure method.

Carr-Madan FFT

The previous graceful formula, unfortunately, cannot be evaluated by FFT (singularity at 0).

Carr-Madan (1999) proposed a method to avoid it. Denote the call price as $C_T(k)$. Define $c_T(k) = e^{\alpha k} C_T(k)$, with $\alpha > 0$. Define $\psi_T(u) = \mathcal{F}[c_T(k)]$.

$\psi_T(u)$ can be obtained using the trick mentioned previously (changing measure). The result is

$$\psi_T(u) = \frac{e^{-rT}\Phi_T(u - (\alpha+1)i)}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} \tag{15}$$

Carr-Madan FFT ctd.

Call values can be calculated through inverse transformation (the integral is odd in *Im* and even in *Re*):

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \psi_T(u) du = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-iuk} \psi_T(u) du \quad (16)$$

Note that introducing α has remove the singularity at 0.

In practice, α is often chosen as 0.75 for call option and 1.75 for put option. (Rule-of-thumb)

Carr-Madan FFT ctd.

Change the upper limit to $N\eta$ (N is a power of 2), $u_j = \eta(j-1)$. Employ equal sizing of $k_l = \frac{N\lambda}{2} + \lambda(l-1)$. In order to apply FFT, we note that we should have $\lambda\eta = \frac{2\pi}{N}$. Now we write:

$$C_T(k_l) \approx \frac{e^{-\alpha k_l}}{\pi} \Sigma_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(l-1)} e^{iu_j\frac{N\lambda}{2}} \psi_T(u_j) \tag{17}$$

Note that k is actually the log of strike, so we would rather want smaller λ with larger η. For this purpose, we would further incorporate Simpson's rule weightings $\frac{\eta}{3}(3 + (-1)^j - \delta_{j-1})$.

Carr-Madan FFT ctd.

Writing FFT in matrix, let $g(u) = e^{-iuk}\Psi_T(u)$

$$\begin{pmatrix} c_T(k_1) \\ c_T(k_2) \\ \vdots \\ c_T(k_N) \end{pmatrix} \approx \Delta u \begin{pmatrix} e^{-iu_1k_1} & e^{-iu_2k_1} & \dots & e^{-iu_Nk_1} \\ e^{-iu_1k_2} & e^{-iu_2k_2} & \dots & e^{-iu_Nk_2} \\ \vdots & \vdots & \vdots & \vdots \\ e^{-iu_1k_N} & e^{-iu_2k_N} & \dots & e^{-iu_Nk_N} \end{pmatrix} \times \begin{pmatrix} \psi_T(u_1) \\ \psi_T(u_2) \\ \vdots \\ \psi_T(u_N) \end{pmatrix} - \frac{1}{2} \begin{pmatrix} g(u_1) \\ 0 \\ \vdots \\ g(u_N) \end{pmatrix}$$

$$= \Delta u \underbrace{\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & e^{-i1\Delta u1\Delta k} & \dots & e^{-i(N-1)\Delta u1\Delta k} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & e^{-i1\Delta u(N-1)\Delta k} & \dots & e^{-i(N-1)\Delta u(N-1)\Delta k} \end{pmatrix}}_{\text{This Fourier matrix is solved efficiently by the FFT}} \cdot \begin{pmatrix} e^{iu_1b}\psi_T(u_1) \\ e^{iu_2b}\psi_T(u_2) \\ \vdots \\ e^{iu_Nb}\psi_T(u_N) \end{pmatrix} - \frac{1}{2} \begin{pmatrix} g(u_1) \\ 0 \\ \vdots \\ g(u_N) \end{pmatrix}$$

Application of Carr-Madan: BSM

Let $x_0 = log(S_0)$, S_T is lognormal, we can easily get:

$$\Phi(u) = exp\{iu[x_0 + (r - \frac{\sigma^2}{2}T] - \frac{1}{2}\sigma^2 u^2 T\} \tag{18}$$

Remember that once we have $\Phi(u)$, we only need to calculate two integrals. Done!

Application of FFT: Heston

LEMMA A.1.1. *For the Heston model, the characteristic function $\phi_T(u)$ conditioned on the initial value x_0, v_0 of the underlying diffusion process $x_T = \log S_T$ is defined by:*

$$\phi_T(u) = \exp(\underbrace{iu(x_0 + rT)}_{C}) \tag{A.7}$$

$$\cdot \exp\left(\underbrace{\frac{v_0}{\eta^2}\left[\frac{1-e^{-DT}}{1-Ge^{-DT}}\right](\kappa - \rho\eta iu - D)}_{B(u,T)v_0}\right) \tag{A.8}$$

$$\cdot \exp\left(\underbrace{\frac{\kappa\theta}{\eta^2}\left[T(\kappa - \rho\eta iu - D) - 2\log\left(\frac{1-Ge^{-DT}}{1-G}\right)\right]}_{A(u,T)}\right) \tag{A.9}$$

where

$$D = \sqrt{(\kappa - \rho\eta iu)^2 + (u^2 + iu)\eta^2}, \quad G = \frac{\kappa - \rho\eta iu - D}{\kappa - \rho\eta iu + D}$$

and x_0, v_0 are the initial values for the log-price and the volatility process, respectively.

Conclusion

SV Model Pricing Method

BSM suffers from several problems. To model the IVS, stochastic models are proposed.

SV models are generally difficult to obtain simple formula. The most tractable one is Heston model, which may be the second popular model in pricing theory.

CF method is the standard way to deal with SV models. The advantages include:

- Compared to PDE, easier to arrive at closed-form
- CFs are always continuous, even in jump process
- Modular concept. Intuitively $\Phi(u) = \Phi_{SI}(u)\Phi_{SV}(u)\Phi_{Jump}(u)$

References

1. Jim Gatheral, *The Volatility Surface*, Wiley Finance, 2006
2. Jianwei Zhu, *Applications of Fourier Transform to Smile Modeling*, Springer Finance, 2009
3. Paul Glasserman, *Monte Carlo Methods in Financial Engineering*, 2003
4. Steven Heston, *A closed-form solution for options with stochastic volatility, with application to bond and currency options*, 1993
5. Peter Carr, D.B.Madan, *Option Valuation Using the Fast Fourier Transform*, 1999