

Black-Scholes Formula, a physicist's perspective

1 (a)

Rewritten Using *Brownian Motion*:

$ds(t) = \phi s(t)dt + \sigma s(t)dW(t)$, where $W(t)$ is a standard brownian motion.

To illustrate the relation between **Gaussian Noise** and **Brownian Motion**, consider when using $R(t)$, we're actually suggesting $s(t + \epsilon) = s(t) + \phi s(t)\epsilon + \sigma R\epsilon$. In this case, $R \sim \mathcal{N}(0, \frac{1}{\epsilon})$, therefore $R\epsilon \sim \mathcal{N}(0, \epsilon)$, which can be characterized as $W(t + \epsilon) - W(t)$. As $\epsilon \rightarrow 0$, $W(t + \epsilon) - W(t) \rightarrow dW(t)$. (It's really clearer to use *brownian motion* notation.) Brownian motion has the property that $dW(t)dW(t) = dt$, $dW(t)dt = 0$.

The original statement can be rewritten as:

$$df(t, s(t)) = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t)$$

According to *Taylor expansion formula*, we can write

$$df = f_t dt + f_s ds + \frac{1}{2}\{f_{tt}dt^2 + (f_{ts} + f_{st})dtds(t) + f_{ss}ds(t)ds(t)\} + o(dt^2) \quad (1)$$

$$\implies df = f_t dt + f_s ds + \frac{1}{2}f_{ss}ds(t)ds(t) \quad (2)$$

Considering $ds(t) = \phi s(t)dt + \sigma s(t)dW(t)$, $dW(t)dW(t) = dt$, $dW(t)dt = 0$, we have

$$df = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t) \quad (3)$$

2 (b)

$$c = c(t, s(t)), \quad d\Pi = c_t dt + c_s ds + \frac{1}{2}c_{ss}dsds - c_s ds.$$
$$d\Pi = c_t dt + \frac{1}{2}c_{ss}\sigma^2 s^2 dt$$

3 (c)

$$d\Pi = r(c - c_s s)dt = c_t dt + \frac{1}{2}c_{ss}\sigma^2 s^2 dt$$

$$c_t + \frac{1}{2}\sigma^2 s^2 c_{ss} + rsc_s - rc = 0$$

Another Way of Obtaining BS Formula

According to non-arbitrage postulate, if at time 0, $c(0, S(0)) = c_0$, then we should be able to construct a portfolio $X(t)$ (with $X(0) = c_0$) to replicate exactly this option. We should use the underlying stock $S(t)$ and the money market with interest rate r . Suppose we hold $\Delta(t)$ share of stock at time t , then it follows:

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt = D_t dt + D_w dW(t) \quad (4)$$

where $D_t = \Delta(t)S(t)(\phi - r) + rX(t)$, $D_w = \Delta(t)S(t)\sigma$.

At the same time,

$$dc(t, S(t)) = c_t dt + c_s dS(t) + \frac{1}{2}c_{ss}dS(t)dS(t) = D'_t dt + D'_w dW(t) \quad (5)$$

where $D'_t = c_t + c_s \phi S(t) + \frac{1}{2}\sigma^2 S(t)^2 c_{ss}$, $D'_w = c_s \sigma S(t)$.

Therefore, it should follow that $D_t = D'_t$, $D_w = D'_w$. The second relation yields instantly that $\Delta(t) = c_s$, while the first would amount to the Black-Scholes formula:

$$c_t + \frac{1}{2}\sigma^2 s^2 c_{ss} + rsc_s - rc = 0 \quad (6)$$

4 (d)

Change variable $s = e^x$, we have $c_x = e^{-x}c_s$.

$$c_t = rc - rsc_s - \frac{1}{2}\sigma^2 s^2 c_{ss} = (r - (r - \frac{1}{2}\sigma^2)\frac{\partial}{\partial x} - \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2})c.$$

$$\text{Therefore } H_{BS} = (1 - \frac{\partial}{\partial x})(r + \frac{1}{2}\sigma^2 \frac{\partial}{\partial x}).$$

5 (e)

$$p_{BS}(x, \tau; x') = \langle x | e^{-\tau H} | x' \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle x | e^{-\tau H} | p \rangle \langle p | x' \rangle \quad (7)$$

Taking $p = i \frac{\partial}{\partial x}$, using $\langle x | p \rangle = e^{ipx}$:

$$p_{BS}(x, \tau; x') = e^{-r\tau} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\{-\frac{1}{2}\sigma^2 p^2 \tau + ip(x - x') + ip\tau(r - \frac{\sigma^2}{2})\} \quad (8)$$

Finally, perform the Gaussian integration:

$$p_{BS}(x, \tau; x') = e^{-r\tau} \frac{1}{\sqrt{2\pi\tau\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2\tau} \left(x - x' + \tau\left(r - \frac{\sigma^2}{2}\right)\right)^2\right\} \quad (9)$$