Black-Scholes Formula, a physicist's perspective

1 (a)

Rewritten Using *Brownian Motion*:

$$ds(t) = \phi s(t)dt + \sigma s(t)dW(t). \tag{1}$$

where $W(t)$ is a standard brownian motion.

To illustrate the relation between **Gaussian Noise** and **Brownian Motion**, consider when using $R(t)$, we're actually suggesting $s(t+\epsilon) = s(t) + \phi s(t)\epsilon + \sigma R\epsilon$. In this case, $R \sim \mathcal{N}(0, \frac{1}{\epsilon})$, therefore $R\epsilon \sim \mathcal{N}(0, \epsilon)$, which can be characterized as $W(t+\epsilon) - W(t)$. As $\epsilon \to 0$, $W(t+\epsilon) - W(t) \to dW(t)$. (It's really clearer to use *brownian motion* notation.) Brownian motion has the property that $dW(t)dW(t) = dt$, $dW(t)dt = 0$.

The original statement can be rewritten as:

$$df(t, s(t)) = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t). \tag{2}$$

According to *Taylor expansion formula*, we can write

$$df = f_t dt + f_s ds + \frac{1}{2}\{f_{tt}dt^2 + (f_{ts} + f_{st})dtds(t) + f_{ss}ds(t)ds(t)\} + o(dt^2) \tag{3}$$

$$\Longrightarrow df = f_t dt + f_s ds + \frac{1}{2} f_{ss} ds(t)ds(t) \tag{4}$$

Considering $ds(t) = \phi s(t)dt + \sigma s(t)dW(t)$, $dW(t)dW(t) = dt$, $dW(t)dt = 0$, we have

$$df = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t) \tag{5}$$

Using the relationship between W(t) and R(t), we can change dW(t) and derive:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} + \frac{\partial f}{\partial s}(\phi s + \sigma s R). \tag{6}$$

2 (b)

We've already known that $c = c(t, s(t))$. Consider a portfolio $\Pi = c - \frac{\partial c}{\partial s} s$. To calculate the derivative of Π we can write down the differential of Π:

$$d\Pi = c_t dt + c_s ds + \frac{1}{2} c_{ss} ds ds - c_s ds. \tag{7}$$

Because we've got $dS^2 = \sigma^2 s^2 dt$, we can derive:

$$\frac{d\Pi}{dt} = c_t + \frac{1}{2} c_{ss} \sigma^2 s^2. \tag{8}$$

3 (c)

r means the short-term risk-free interest rate. The following equation must be true:

$$\frac{d\Pi}{dt} = r\Pi. \tag{9}$$

Otherwise arbitrage will exist, which contradicts our assumption. Therefore we can write $d\Pi = r(c - c_s s)dt$. It's equivalent to the previous result $c_t dt + \frac{1}{2} c_{ss} \sigma^2 s^2 dt$.

At last we have the **Black-Scholes formula**:

$$c_t + \frac{1}{2} \sigma^2 s^2 c_{ss} + rsc_s - rc = 0. \tag{10}$$

Another Way of Obtaining B-S Formula

According to non-arbitrage postulate, if at time 0, $c(0, S(0)) = c_0$, then we should should be able to construct a portfolio $X(t)$(with $X(0) = c_0$) to replicate exactly this option. We should use the underlying stock $S(t)$ and the money market with interest rate r. Suppose we hold $\Delta(t)$ share of stock at time t, then it follows:

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt = D_t dt + D_w dW(t) \tag{11}$$

where $D_t = \Delta(t)S(t)(\phi - r) + rX(t)$, $D_w = \Delta(t)S(t)\sigma$.

At the same time,

$$dc(t, S(t)) = c_t dt + c_s dS(t) + \frac{1}{2} c_{ss} dS(t) dS(t) = D'_t dt + D'_w dW(t) \tag{12}$$

where $D'_t = c_t + c_s \phi S(t) + \frac{1}{2}\sigma^2 S(t)^2 c_{ss}$, $D'_w = c_s \sigma S(t)$.

Therefore, it should follows that $D_t = D'_t, D_w = D'_w$. The second relation yields instantly that $\Delta(t) = c_s$, while the first would amount to the Black-Scholes formula:

$$c_t + \frac{1}{2} \sigma^2 s^2 c_{ss} + rsc_s - rc = 0 \tag{13}$$

4 (d)

Change variable $s = e^x$, we have:

$$c_x = e^{-x} c_s. \tag{14}$$

$$c_t = rc - rsc_s - \frac{1}{2}\sigma^2 s^2 c_{ss} = (r - (r - \frac{1}{2}\sigma^2)\frac{\partial}{\partial x} - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2})c. \tag{15}$$

Therefore we can easily prove that

$$H_{BS} = (1 - \frac{\partial}{\partial x})(r + \frac{1}{2}\sigma^2\frac{\partial}{\partial x}) = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (\frac{\sigma^2}{2} - r)\frac{\partial}{\partial x} + r. \tag{16}$$

which is the Hamiltonian for Black-Scholes model.

5 (e)

$$p_{BS}(x, \tau; x') = \langle x|e^{-\tau H}|x'\rangle = \int_{\infty}^{\infty} \frac{dp}{2\pi} \langle x|e^{-\tau H}|p\rangle \langle p|x'\rangle \tag{17}$$

Taking $p = i\frac{\partial}{\partial x}$, using $\langle x|p\rangle = e^{ipx}$:

$$p_{BS}(x, \tau; x') = e^{-r\tau}\int_{\infty}^{\infty} \frac{dp}{2\pi} exp\{-\frac{1}{2}\sigma^2 p^2\tau + ip(x - x') + ip\tau(r - \frac{\sigma^2}{2})\} \tag{18}$$

Finally, perform the Gaussian integration:

$$p_{BS}(x, \tau; x') = e^{-r\tau}\frac{1}{\sqrt{2\pi\tau\sigma^2}} exp\{-\frac{1}{2\sigma^2\tau}(x - x' + \tau(r - \frac{\sigma^2}{2}))^2\} \tag{19}$$

6 (f)

After obtaining the pricing kernel, we can get the price of call option at time t simply by integrating the final value with the kernel.

$$c(\tau, x) = \int_{-\infty}^{\infty} g(x')P_{BS}dx' \tag{20}$$

where in this case, $g(x') = (e^{x'} - K)^+$. $(x)^+$ takes x when $x > 0$ and takes 0 when $x \le 0$. Therefore, the integration only takes place in $x \in (lnK, \infty)$.

After noticing, the pricing kernel is actually $e^{-r\tau}$ mutilpying a normal distribution, we can define $d_-(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}[\frac{x}{lnK} + (r - \frac{1}{2}\sigma^2)\tau], d_+ = d_- + \sigma\sqrt{\tau}$, $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{x^2}{2}}$. We separate the integration into two parts: K and e^x.

$$\int_{lnK}^{\infty} KP_{BS}dx' = e^{-r\tau}KN(d_-(\tau, x)) \tag{21a}$$
$$\int_{lnK}^{\infty} e^x P_{BS}dx' = e^x N(d_+(\tau, x)) \tag{21b}$$

Remember that $s = e^x$, we have the pricing formula for $c(t, s)$:

$$c(t, s) = sN(d_+(\tau, s)) - e^{-r\tau}KN(d_-(\tau, s)) \tag{22}$$

With this formula, we can plot the time-evolution of stock prices and call option prices.

The following code does the job in R, results shown in $fig.1 \sim 3$

```
c_time_evolution <- function(sigma, x0, r, K)
{
    tau = seq(0, 1, 0.01)[-1]
    w = rnorm(100, 0, 0.01)
    w = cumsum(w)
    # calculate s(t)
    x = (r - 0.5*sigma^2) * tau + sigma * w
    x = exp(x)
    x = x0 * x
    #fi
    plot(tau, x, type='l')

    d1 = log(x/K) + (1 - tau) * (r + 0.5*sigma^2)
    d1 = d1/(sqrt(1 - tau) * sigma)
    d2 = d1 - sigma*sqrt(1 - tau)
    c = x * pnorm(d1) - exp(-r * (1-tau)) * K * pnorm(d2)
    plot(tau, c, 'l')
}
#fig.1
c_time_evolution(sigma = 0.5, x0 = 100, r = 0.20, K = 100)
#fig.2
c_time_evolution(sigma = 0.5, x0 = 80, r = 0.20, K = 100)
#fig.3
c_time_evolution(sigma = 0.5, x0 = 120, r = 0.20, K = 100)
```

Fig. 1: Time-evolution of call option price

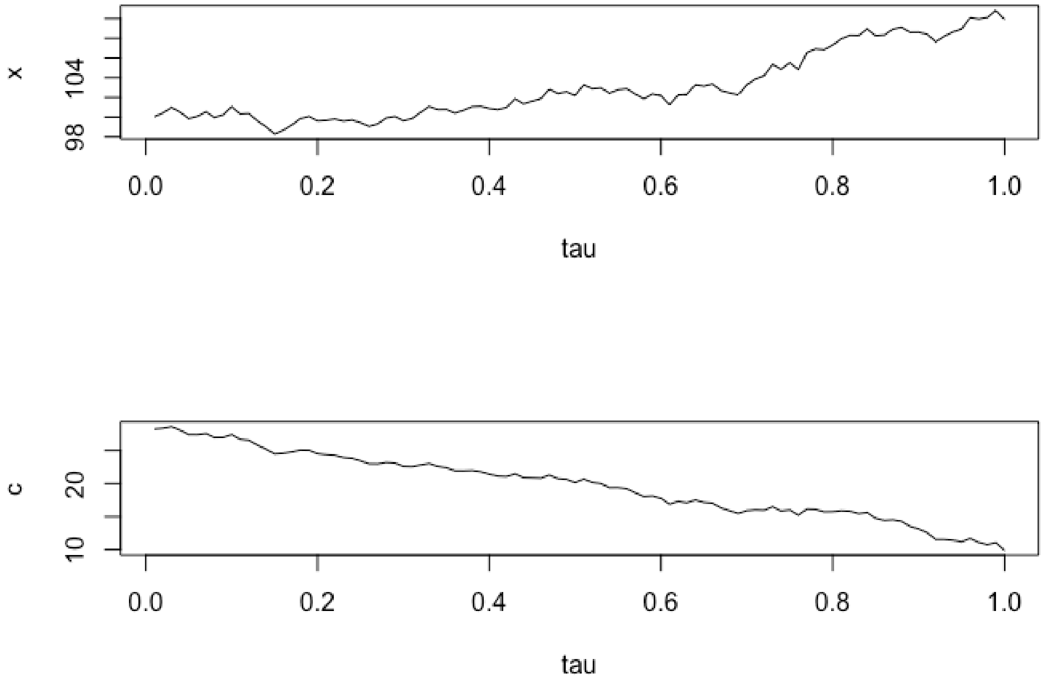


Fig. 2: Time-evolution of call option price

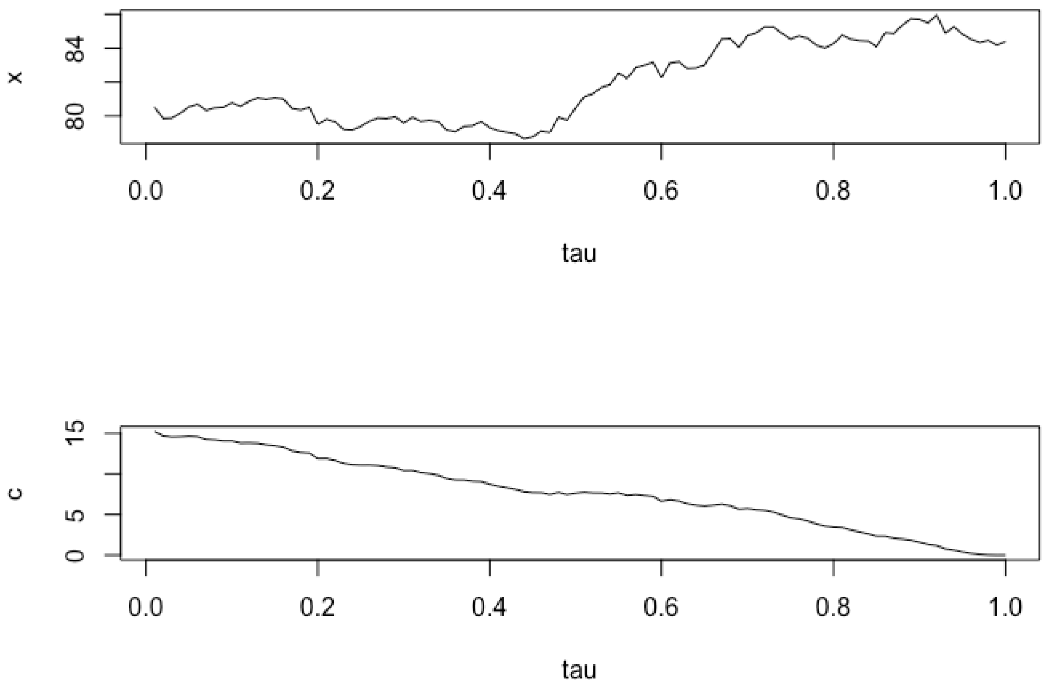
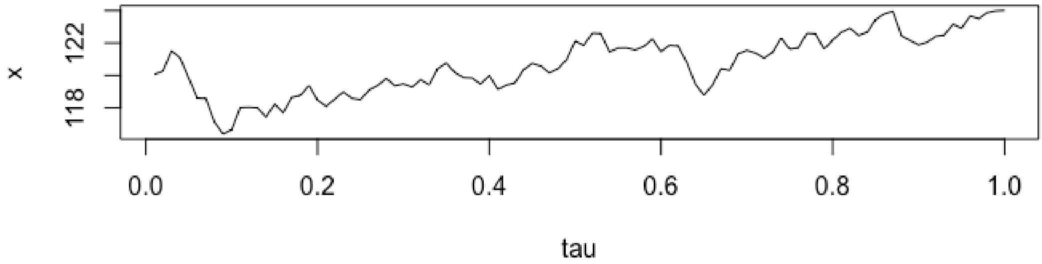
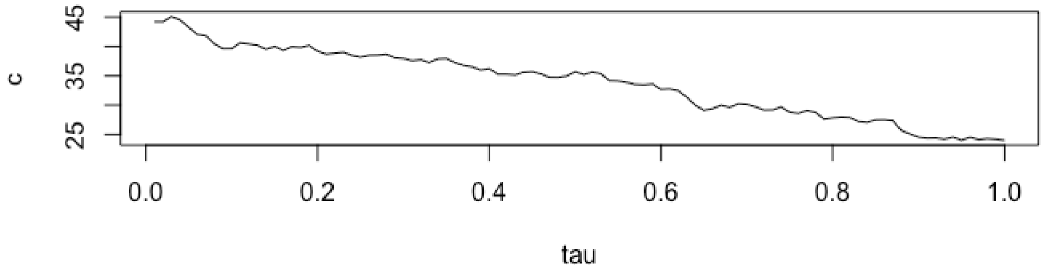


Fig. 3: Time-evolution of call option price





As shown in these figures, call option price drops as $(T-t)$ decreases.

7 (g)

Consider a down-and-out barrier European option. If $s \leq e^B(x \leq B)$ it will become worthless, which means $c = 0$.

For an arbitrary potential $v(x)$, we have: $\frac{\partial c}{\partial t} = Hc$, where H can be written as:

$$H = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (\frac{\sigma^2}{2} - r)\frac{\partial}{\partial x} + V(x). \tag{23}$$

Obviously, if we set $V(x) = \infty$ when $x \leq B$, then $c = 0$ is automaticlly satisfied. However, when $x > B$, the changing of c still conform to the B-S formula. comparing with H_{BS}, we have: V(x)= $\begin{cases} +\infty, x \leq B. \\ r, x > B. \end{cases}$

8 (h)

The Hamiltonian for the down and out option is given by:

$$H_{DO} = H_{BS} + V(x) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (\frac{\sigma^2}{2} - r)\frac{\partial}{\partial x} + V(x) \tag{24}$$

V(x) is defined in (g).

Now we can construct the eigenfunctions similar to the $Black - Scholes$ case except that these must satisfy $\Psi_E(B) = 0$.

Define the quantities:

$$\beta = \frac{(\frac{\sigma^2}{2} + r)^2}{\sigma^4}, p = \sqrt{\frac{2E}{\sigma^2} - \beta}, \alpha = \frac{\frac{\sigma^2}{2} - r}{\sigma^2}, i\lambda_{\pm} = \alpha \pm ip \tag{25}$$

The eigenfunctions are given by:

$$x > B : \langle x|\Psi_E\rangle = e^{i\lambda_+(x-B)} - e^{i\lambda_-(x-B)} = 2ie^{\alpha(x-B)}\sin\left[p(x-B)\right] \tag{26}$$

$$\langle\widetilde{\Psi_E}|x\rangle = e^{-i\lambda_+(x-B)} - e^{-i\lambda_-(x-B)} = -2ie^{-\alpha(x-B)}\sin\left[p(x-B)\right] \tag{27}$$

$$\left\langle\widetilde{\Psi_E}|\Psi_E{}'\right\rangle = [2\pi\sigma^2\sqrt{2E/\sigma^2 - \beta}]\delta(E - E'). \tag{28}$$

$$x \leq B : \langle x|\Psi_E\rangle = 0 = \left\langle\widetilde{\Psi_E}|x\right\rangle. \tag{29}$$

Now we can use the eigenfunctions to evaluate the pricing kernel. Again we work with the p variable:

$$\begin{aligned} p_{DO}(x,\tau;x') &= \left\langle x|e^{-\tau H_{DO}}|x'\right\rangle \\ &= e^{-\frac{\tau\beta\sigma^2}{2}+\alpha(x-x')}\int_0^\infty \frac{dp}{2\pi}e^{-\frac{1}{2}\tau\sigma^2p^2}\big[e^{ip(x-x')} + e^{-ip(x-x')} \\ &- e^{ip(x+x'-2B)} - e^{-ip(x+x'-2B)}\big]. \end{aligned}$$

Consider that we have already derived:

$$p_{BS}(x,\tau;x') = e^{-r\tau}\frac{1}{\sqrt{2\pi\tau\sigma^2}}e^{-\frac{1}{2\tau\sigma^2}[x-x'+\tau(r-\frac{\sigma^2}{2})]^2}. \tag{30}$$

We can simplify the pricing kernel using the previous results:

$$p_{DO}(x,\tau;x') = p_{BS}(x,\tau;x') - (\frac{e^x}{e^B})^{2\alpha}p_{BS}(2B - x,\tau;x'). \tag{31}$$

9 (i)

We know that $r(t)$ satisfied Langevin equation:

$$\frac{dr}{dt} = a(r,t) + \sigma(r,t)R(t). \tag{32}$$

Here $R(t)$ is still Gaussian noise.

Define the propagator $P(r,t;r_0)$: if $r(t_0) = r_0$, the probability of $r(t) = r$ equals $P(r,t;r_0)$.

From the Langevin equation we have:

$$r(t+\varepsilon) = r(t) + \varepsilon[a + \sigma R(t)]. \tag{33}$$

Change it into the following formula:

$$r = r' + \varepsilon[a(r') + \sigma(r')R(t)]. \tag{34}$$

Thus we can calculate the propagator:

$$\begin{aligned}
P(r,t+\varepsilon,r_0) &= P(r',t;r_0)|_{r'=r-\varepsilon[a(r')+\sigma(r')R(t)]} \\
&= \int P(r',t;r_0)\delta(r-r'-\varepsilon[a(r')+\sigma(r')R(t)])dr' \\
&\simeq \int P(r',t;r_0)dr'\{\delta(r-r') + \frac{\partial\delta(r-r')}{\partial r'}\varepsilon[a(r')+\sigma(r')R(t)] \\
&+ \frac{1}{2}\frac{\partial^2\delta(r-r')}{\partial r'^2}\varepsilon^2[a(r')+\sigma(r')R(t)]^2 + ...\}.
\end{aligned}$$

Because $< R^2(t) >= \frac{1}{\varepsilon}$ and $< R(t) >= 0$, the previous formula can be rewritten like this:

$$\begin{aligned}
P(r,t+\varepsilon,r_0) &= P(r',t;r_0) + \int dr'P(r',t;r_0)\{\frac{\partial\delta(r-r')}{\partial r'}\varepsilon a(r') + \frac{1}{2}\frac{\partial^2\delta(r-r')}{\partial r'^2}\varepsilon^2\sigma^2(r')\frac{1}{\varepsilon}\} \\
&= P(r',t;r_0) - \varepsilon\frac{\partial}{\partial r}[a(r)P(r,t;r_0)] + \frac{\varepsilon}{2}\frac{\partial^2}{\partial r^2}[\sigma(r)^2P(r,t;r_0)].
\end{aligned}$$

If a variable is o(ε), it is automatically neglected.

Thus, from the definition of derivative, we have:

$$\frac{\partial P(r,t;r_0)}{\partial t} = [\frac{1}{2}\frac{\partial^2}{\partial r^2}\sigma^2(r) - \frac{\partial}{\partial r}a(r)]P(r,t;r_0). \tag{35}$$

We've already known that:

$$\frac{\partial P(r,t;r_0)}{\partial t} = -H_F P(r,t;r_0). \tag{36}$$

Therefore we can prove:

$$H_F = -\frac{1}{2}\frac{\partial^2}{\partial r^2}\sigma^2(r) + \frac{\partial}{\partial r}a(r) = -\frac{1}{2}\frac{\partial^2}{\partial r^2}\sigma^2(r) + a(r)\frac{\partial}{\partial r} + \frac{\partial a(r)}{\partial r}. \tag{37}$$

From a different perspective, we define $P_B(R,t;r)$ as the back propagator. Similarly we have(since the time flows backwards this time):

$$\frac{\partial P_B(R,t;r)}{\partial t} = +H_B P_B(R,t;r). \tag{38}$$

Finally we can calculate:

$$H_B = -\frac{1}{2}\sigma^2(r)\frac{\partial^2}{\partial r^2} - a(r)\frac{\partial}{\partial r}. \tag{39}$$

$H_B = H_F^\dagger$ is obvious.

10 (j)

In this part we'll focus on the so-called **stochastic Quantization**.
The Langevin equation:

$$\frac{dr}{dt} = a(r,t) + \sigma(r.t)R(t) \tag{40}$$

is satisfied at any time between t_0 and T. We must consider that both r and R are stochastic variables. Therefore when calculating z_B(can be compared to partition function in statistical mechanics), we must integrate over all possible paths

$$Z_B = \int DRDr\prod_{t=t_0}^{T}\delta[\frac{dr}{dc} - a(r,t) - \sigma(r,t)R(t)]e^{-}\frac{1}{2}\int_{t_0}^{T}R^2(t)dt \tag{41}$$

Dr means integrating over all possible $r(t)$: $\int Dr = \int_{-\infty}^{+\infty}\prod_{t=t_0}^{T}dr(t)$.

We hope to write Z_B int the form of $z_B = \int Dre^{s_B}$. Because of this, we calculate the integral over R first:

$$Z_B = \int Drexp(-\frac{1}{2}\int_{t_0}^{T}\frac{[\frac{dr}{dt} - a(r,t)]^2}{\sigma^2(r,t)}dt). \tag{42}$$

Because the $Dirac-\delta$ function makes sure we only keep $R(t)$ that satisfies the Langevin equation: $R(t) = \frac{\frac{dr}{dt}-a(r,t)}{\sigma(r,t)}$.

It's easy to calculate S_B:

$$S_B = -\frac{1}{2}\int_{t_0}^{T}\frac{[\frac{dr}{dt} - a(r,t)]^2}{\sigma^2(r,t)}dt. \tag{43}$$

From the definition of $S_B : S_B = \int_{t_0}^{T} Ldt$, it's clear that:

$$L = -\frac{[\frac{dr}{dt} - a(r,t)]^2}{2\sigma^2(r,t)}dt. \tag{44}$$

11 (k)

The Vasicek model can be described using the following equation:

$$\frac{dr}{dt} = a(b-r) + \sigma R(t). \tag{45}$$

Note that if we set $\begin{cases} a(r,t) = a(b-r). \\ \sigma(r,t) = \sigma. \end{cases}$ in Langevin equation(a,b,σ are all constants), we have the Vasicek model naturally.

Thus we get:

$$L_V = -\frac{[\frac{dr}{dt} - a(b-r)]^2}{2\sigma^2}. \tag{46}$$

$$S_V = -\frac{1}{2\sigma^2}\int_{t_0}^{T}[\frac{dr}{dt} - a(b-r)]^2 dt. \tag{47}$$

In the next section we'll use S_V to calculate the propagator of Vasicek model.

12 (l)

For different paths(which all have different S_V), so that e^{S_V}/Z is the distribution of probability. $Z = \int Dre^{S_V}$. Given by the question itself, we know that the propagator equals:

$$P(t_0,T) = \frac{1}{Z}\int Dre^{S_V}e^{-\int_{t_0}^{T} r(t)dt}. \tag{48}$$

If we use a new variable $S = S_V - \int_{t_0}^{T} r(t)dt$, then we have:

$$P(t_0,T) = \frac{1}{Z}\int Dre^{S}. \tag{49}$$

Change the variable: $u = r - b$. Thus we can write:

$$\begin{aligned} S = &-\frac{1}{2\sigma^2}\int_{t_0}^{T} dt[\frac{du}{dt} + au]^2 - \int_{t_0}^{T}(u+b)dt \\ &= -\frac{1}{2\sigma^2}\int_{t_0}^{T} dt[\frac{dr}{dt} + ar]^2 - \int_{t_0}^{T}(r+b)dt. \end{aligned}$$

Next, we can define $v(t) = ar(t) + \frac{dr(t)}{dt}$. View this formula as a differential equation of $r(t)$:

$$\frac{dr}{dt} + ar = v(t). \tag{50}$$

The solution can be easily derived:

$$r(t) = e^{-a(t-t_0)}r_0 + e^{-at}\int_{t_0}^{t} e^{at'}v(t')dt'. \tag{51}$$

Since we want to calculate the integral of r(t), we now have:

$$\int_{t_0}^{T} r(t)dt = B(t_0,T)r_0 + \int_{t_0}^{T} B(t,T)v(t)dt. \tag{52}$$

where $B(t,T) = \frac{1-e^{-a(T-t)}}{a}$.

From the definition of $v(t)$ we can tell that $v(T)$ is free to take all possible values. Therefore we can use the following path integral to calculate propagator:

$$\begin{aligned} P(t_0,T) &= e^{-b(T-t_0)-B(t_0,T)r_0}\frac{1}{Z}\int Dve^{-\frac{1}{2\sigma^2}\int_{t_0}^{T} dt[v(t)^2+2\sigma^2 B(t,T)v(t)]} \\ &= e^{-b(T-t_0)-B(t_0,T)r_0}e^{\frac{\sigma^2}{2}\int_{t_0}^{T} dtB(t,T)^2}. \end{aligned}$$

The $v(t)$ integrations are decoupled Gaussian integrations, with the overall normalization being canceled by the factor Z.

Finally, assume $B(\theta) = \frac{1-e^{-a\theta}}{a}$, where $\theta = T - t_0$, and $A(\theta) = exp[(\frac{\sigma^2}{2a^2} - b)(\theta - B(\theta)) - \frac{\sigma^2}{4a}B(\theta)^2]$.

$$A(\theta)e^{-r_0 B(\theta)} = exp[-b\theta - r_0\frac{1-e^{-a\theta}}{a} + \frac{\sigma^2}{2a^2}\theta - \frac{\sigma^2(1-e^{-a\theta})}{2a^3} + \frac{b}{a}(1-e^{-a\theta}) - \frac{\sigma^2}{4a^3}(1-e^{-a\theta})^2]. \tag{53}$$

If we want to prove $P(t_0,T) = A(\theta)e^{-r_0 B(\theta)}$, then we must prove:

$$\begin{aligned} -b(T-t_0) - B(t_0,T)r_0 + \frac{\sigma^2}{2}\int_{t_0}^{T} dtB(t,T)^2 &= -b\theta - r_0\frac{1-e^{-a\theta}}{a} + \frac{\sigma^2}{2a^2}\theta \\ &- \frac{\sigma^2(1-e^{-a\theta})}{2a^3} + \frac{b}{a}(1-e^{-a\theta}) - \frac{\sigma^2}{4a^3}(1-e^{-a\theta})^2. \end{aligned}$$

Change all $T - t_0$ to θ and all $B(t_0,T)$ to $B(\theta)$, and calculate the integral of $B(\theta)$, we can simplify the equation:

$$\int_{t_0}^{T} \frac{(1-e^{-a\theta})^2}{a^2}dt = \frac{\theta}{a^2} - \frac{1-e^{-a\theta}}{a^3} + \frac{2b}{a\sigma^2}(1-e^{-a\theta}) - \frac{(1-e^{-a\theta})^2}{2a^3}. \tag{54}$$

the left side can be written as:

$$\int_{t_0}^{T} \frac{(1-e^{-a\theta})^2}{a^2} dt = \frac{\theta}{a^2} - \frac{2}{a^3} e^{-aT}(e^{aT} - e^{at_0}) + \frac{1}{2a^3} e^{-2aT}(e^{2aT} - e^{2at_0}). \tag{55}$$

After some comparison, the correctness is easy to prove.

13 Conclusion

14 Reference

[1]Belal E.Baaquie, Quantum Finance: Path integrals and Hamiltonians for Options and Interest Rates. Cambridge University Press (2004). [2]Zee.A, Quantum field theory in a nutshell. Princeton university press (2010). [3]J. C. Hull, Options, Futures and Other Derivatives. Fifth Edition, Prentice-Hall International (2003). [4]M. Otto, 'Using path integrals to price interest rate derivatives', http://xxx.lanl.gov/cond-mat/9812318. [5]O. Vasicek, 'An Equilibrium Characterization of the Term Structure'. Journal of Financial Economics, 5: 177.