Basic Maths

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1 Topological Space

The most fundamental property of topological space is the concept of continuity.
First we have a set X. We define **topological space** to be a structure on X that has the property of **connectivity** between elements of X. Any space that has the same connectivity is topological invariant. We can define connectivity by **Euler Number**: $v - e + f$.(But I can only imagine Euler number within a continuous space)

The point is to clarify what is the most basic property, i.e. invariant of equivalent topological spaces.

1.1 Formation 1: Starting from Neighborhood

Particular Example:
In a E^m space, define the neighborhood $N_p \subseteq E^m$ of any $p \in E^m$:
$\exists r \in R$, $r > 0$, a m-dimensional ball $B(p, r)$ centering at p with radius r, s.t. $B(r, p) \subseteq N_p$.
This is an example in a Euclidean Space where distance is well defined. But actually given a set, we can define it in whatever way we want it.

The General Definition of Neighborhood:
Given a set X, for all $x \in X$, we assign a family F_x of subsets of X. Every element in F_x is defined as the neighborhood of x, if F_x satisfies:

1. $x \in N_x$, for all $N_x \in F_x$
2. $\forall N_x1, N_x2 \in F_x$, $N_x1 \bigcap N_x2 \in F_x$
3. if $N_x \subseteq U \& \& U \subseteq X$, $U \in F_x$
4. $\forall N_x \in F_x$, define $\tilde{N_x} = \{z \in N_x \& \& N_x \in F_z\}$, then $\tilde{N_x} \in F_x$

Given set X, if we find a F_x for all $x \in X$, we then call the (X, F) a topological space.($T - space$ below for simplicity)

Given two $T - space$ X, Y and a mapping $f : X \mapsto Y$, we then can define f to be a **continuous mapping**, if:
$\forall x_0 \in X$, we have $y_0 = f(x_0)$. For all $N_{y_0} \in F_{y_0}$, $f^{-1}(N_{y_0}) \in F_{x_0}$.

Further we define **homomorphism** between $T - space$ X, Y: if $\exists h : X \mapsto Y$, s.t. h is one-to-one and continuous mapping, $h(X) = Y$, $\exists$ continuous $h^{-1} : Y \mapsto X$.

We can see homomorphism between two $T - space$ is equivalent to having a bijective continuous mapping.

1.1.1 Open sets on top of Neighborhood

After we have a topology on X, it is easy to define **Open sets**: In a $T - space(X, F)$, a subset of X O is called an open set if: $\forall x \in O$, $O \in F_x$.

It's straightforward to prove the following preoperty(derived from the definition of neighborhood family):

1. any union(finite, countable, uncountable) of open sets is an open set
2. finite open sets' joint is an open set
3. $X \& \emptyset$ are open sets

1.2 Formation 2: Starting from Open Sets

We've found out that it's possible to define the property of open sets from the definition of neighorhood. It's also possible vice versa.

Primitive Definition of Open Sets:

Given a set X, construct a family F ,which consists of subsets of X. (X, F) is called a topology on X, and the members of F are called open sets, if F satisfies:

1. any union(finite, countable, uncountable) of open sets is an open set
2. finite open sets' joint is an open set
3. $X \& \emptyset$ are open sets

A set X with F satisfying above conditions are called topological space(X, F).

We can prove this definition complies with Formation 1.

we can then define neighborhood:
N_x is called the neighborhood of $x \in X$ if $\exists O \in F$, s.t. $x \in O \subseteq N_x$.

Therefore, defining **Open Sets** is equivalent to defining **Neighborhood**, and both defines **topology** on set X.

There're many kinds of $T - space$, for instance:

Hausdorff Topological Space

Definition: Points x and y in a topological space (X, F) can be separated by neighbourhoods if there exists a neighbourhood N_x of x and a neighbourhood N_y of y such that N_x and N_y are disjoint. (X, F) is a Hausdorff space if all distinct points in X are pairwise neighborhood-separable.

1.3 Other Definition Given a Topological Space X

1.3.1 Closed Set

Limit Point: $A \subseteq X$, $p \in X$ is called a limit point if $\forall N_p$ includes at least one member of $A - \{p\}$.

Closed Set: A set is closed if and only if it includes all its limit points.

2 Measurability

2.1 $\sigma - algebra$

Definition: A collection $\mathcal{M}$ of subsets of X is called a $\sigma - algebra$ in X if:

1. $X \in \mathcal{M}$
2. $\forall A \in \mathcal{M}, A^c \in \mathcal{M}$
3. countable union of members of $\mathcal{M}$ still belongs to $\mathcal{M}$

It can be shown that there are many ways to select a $\sigma - algebra$ in X. For any collection of subsets in X, named $\mathcal{F}$, we can find a smallest $\sigma - algebra$ generated by $\mathcal{F}$ called $\mathcal{M}^*$, such that $\mathcal{F} \subseteq \mathcal{M}^*$.

It is intuitive to remember the definition by thinking about probability.

1. $P(\Omega) = 1$
2. $P(A^c) = 1 - P(A)$
3. if we know $P(A_i)$, we can calculate their countable unions' P

2.2 measurable space

With a set X equipped with $\sigma - algebra$, we can define measurability with respect to the selection of $\mathcal{M}$:

if $\mathcal{M}$ is a $\sigma - algebra$ in X, then $(X, \mathcal{M})$ is called a measurable space. The members of $\mathcal{M}$ is called measurable sets in X.

2.3 Borel Sets

Definition: Let (X, F) be a topological space(where F is the collection of all open sets). There exists a smallest $\sigma - algebra$ $\mathcal{B}$ in (X, F) such that for all $O \in F$, $O \in \mathcal{B}$. The members of $\mathcal{B}$ are called the Borel Sets in (X, F).

A **Borel Measure** on a topological space (X, F) is a measure that is defined on all open sets (and thus on all Borel sets).

3 Probability

3.1 Probability Space

A probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is defined by three part:

1. A sample space, the set of all possible outcomes(sample paths) Ω
2. A collection of all events(subsets of Ω) $\mathcal{F}$, which is a $\sigma - algebra$
3. A mapping $\mathcal{P} : A \mapsto \mathcal{R}, \forall A \in \mathcal{F}$