

Notes on Quantum Physics

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1 Experimental Facts

1.1 The Stern-Gerlach Experiment 1921 ~ 1922

For a heavy atom as a whole, magnetic moment μ is proportional to the electron spin \vec{S} , i.e. $\vec{\mu} = \frac{e\vec{S}}{m_e c}$. Place a magnetic field \vec{B} , the interaction energy yields $-\vec{\mu} \cdot \vec{B}$. Therefore, if B_z is not homogeneous, then $F_z = \mu_z \partial_z B_z$.

We have a beam of silver atoms (47 electrons in total, with 46 of them having spherical symmetry, no net angular momentum, therefore the atom's angular momentum is solely decided by the 47th electron), going through a \vec{B} inhomogeneous in B_z . Then atoms will split in z direction according to their spin. We only observe two distinct components on the other side, where $S_z = \pm \frac{\hbar}{2}$.

The experiment suggests quantization of the electron spin angular momentum.

1.2 Sequential SG Experiments

$Oven \Rightarrow SG\hat{z}$ (filtering S_z-) $\Rightarrow SG\hat{z} \Rightarrow S_z+$ only

$Oven \Rightarrow SG\hat{z}$ (filtering S_z-) $\Rightarrow SG\hat{x} \Rightarrow S_x+, S_x-$

$Oven \Rightarrow SG\hat{z}$ (filtering S_z-) $\Rightarrow SG\hat{z} \Rightarrow SG\hat{z} \Rightarrow S_z+, S_z-$

These results are similar to the polarization of light.

Suppose $x \perp y, x' = \frac{1}{\sqrt{2}}(x + y)$

$\Rightarrow x - filter \Rightarrow y - filter \Rightarrow$ No light

$\Rightarrow x - filter \Rightarrow x' - filter \Rightarrow y - filter \Rightarrow x_+, x_-$

The experiment suggests that we cannot determine S_x and S_z simultaneously. Previous information is destroyed by the new apparatus. It also suggests the superposition principle.

Further we can use abstract vectors to represent the states in SG experiment on the basis of $|S_z+\rangle$ and $|S_z-\rangle$.

$|S_x\rangle, |S_y\rangle, |S_z\rangle$'s relations:

$$|S_z+\rangle = \frac{1}{\sqrt{2}}(|S_z+\rangle + |S_z-\rangle), \quad (1a)$$

$$|S_z-\rangle = \frac{1}{\sqrt{2}}(-|S_z+\rangle + |S_z-\rangle), \quad (1b)$$

$$|S_y+\rangle = \frac{1}{\sqrt{2}}(|S_z+\rangle + i|S_z-\rangle), \quad (1c)$$

$$|S_y-\rangle = \frac{1}{\sqrt{2}}(|S_z+\rangle - i|S_z-\rangle) \quad (1d)$$

This example is clear to demonstrate the abstractness of vector space. Quantum-mechanical states are to be represented by vectors in an abstract complex vector space.

1.3 Feynman's Remarks on electron's split

See Quantum Field Theory in a Nutshell by A. Zee, Chap 1. Illustrate the motive of path integral.

2 Mathematics

2.1 Dirac, Ket & Bra

2.1.1 Ket $|\alpha\rangle$ & State Vector Space(Ket Space) \mathcal{H}

We want it clear in the beginning. When referring to ket vectors, we're speaking of functions in a functional vector space, ex. functions expanded in fourier forms. The dimension of the complex vector space is decided by the physics system's degree of freedom, ex. in the case of an electron's spin(upward & downward), $dim = 2$.

A physical state is represented by a **state vector** in a complex vector space, called **ket**, denoted by $|\alpha\rangle$. All information about that state is contained in the vector. Our postulation of the existence of vectors to represent states already suggests the superposition principle.

An observable, on the other hand, is represented by an operator which acts on the ket, on the left,

$$A \cdot (|\alpha\rangle) = A|\alpha\rangle \quad (2)$$

which yields another ket. If all kets of a system forms the space \mathcal{H} , an observable in that system is an operator on \mathcal{H} (operator defined as the same in linear algebra). Therefore, there should be eigenkets $|\alpha'\rangle$ of A ,

$$A|\alpha'\rangle = \alpha'|\alpha'\rangle \quad (3)$$

Here α' is just a number. It's convention to 'ket' an eigenvalue to stand for the corresponding eigenvector. The physical state corresponding to an eigenket is called an eigenstate:

$$S_z |S_z+\rangle = \frac{\hbar}{2} |S_z+\rangle, S_z |S_z-\rangle = -\frac{\hbar}{2} |S_z-\rangle \quad (4)$$

Next, we consider a $N - \dim$ vector space \mathcal{H} , spanned according to the N eigenkets of observable A . Then $\forall |\alpha\rangle \in \mathcal{H}$, $|\alpha\rangle = \sum_{\alpha'} c_{\alpha'} |\alpha'\rangle$.

2.1.2 Bra $\langle\alpha|$, Dual Space(Bra Space) \mathcal{H}^* and Inner Products

Bra Space is dual to Ket Space. Corresponding to every $|\alpha\rangle$ in \mathcal{H} , there exists a bra in dual space \mathcal{H}^* , denoted by $\langle\alpha|$.

Inner product is defined as a mapping $i(*, *) : \mathcal{H} \times \mathcal{H} \mapsto \mathbb{C}$. Think of $i(*, *)$ as a machine receiving two kets and returns a complex number. We all know how the inner product should be defined.

Dual space is by now unclear. Yet with $i(*, *)$ we can deduce a dual space with a particular $i(*, *)$ defined. With $i(*, *)$ and $\forall f \in \mathcal{H}$, we have $i(f, *) : \mathcal{H} \mapsto \mathbb{C}$, $i(f, *) \in \mathcal{H}^*$. Moreover, this is indeed a **one-to-one mapping**:

$$\forall \eta \in \mathcal{H}^*, \exists f_\eta \in \mathcal{H}, s.t. \forall g \in \mathcal{H}, \eta(g) = i(f_\eta, g) \quad (5)$$

Denote this mapping $v : \mathcal{H}^* \mapsto \mathcal{H}$, and denote $v^{-1}(\eta) = g_\eta$. (Right now I can only understand dual space this way, remained to be uncovered.)

Now we know a bra $\langle\eta|$ acts on a ket $|f\rangle$ yields the (g_η, f) , which is called the inner product of $\langle\eta|$ and $|f\rangle$ in physics, when mathematically this is inaccurate.

In physics, we often put a complex vector in \mathcal{H} into a bra, ex. $\langle\alpha|$, which actually means that $\langle\alpha| = v^{-1}(|\alpha\rangle)$. Though very confusing, we only need to remember finally $\langle\eta|$ acts on $|f\rangle$ yields $(|\eta\rangle, |f\rangle)$.

2.1.3 Probability & Measurement

Dirac:"A measurement always cause the system jump into an eigenstate of the dynamical variable that is being measured."

$$|\alpha\rangle = \sum_{\alpha'} c_{\alpha'} |\alpha'\rangle = \sum_{\alpha'} |\alpha'\rangle \langle\alpha'|\alpha\rangle \quad (6)$$

Postulate the probability for the system to jump from $|\alpha\rangle$ to $|\alpha'\rangle$ after measurement of \hat{A} to be $|\langle\alpha'|\alpha\rangle|^2$.

An observable's expectation value w.r.t. state $|\Psi\rangle$ is defined as:

$$\langle\hat{A}\rangle = \sum_{\alpha'} \sum_{\alpha''} \langle\alpha|\alpha''\rangle \langle\alpha''|\hat{A}|\alpha'\rangle \langle\alpha'|\alpha\rangle = \sum_{\alpha'} \alpha' |\langle\alpha'|\alpha\rangle|^2 \quad (7)$$

Continuous Version:

$$\langle Q \rangle_{\Psi} = \int \Psi^* \hat{Q} \Psi dx = \langle \Psi | \hat{Q} | \Psi \rangle \quad (8)$$

A selective measurement amounts to the projection operator defined as:

$$\Lambda_{\alpha'} |\alpha\rangle = |\alpha'\rangle \langle \alpha' | \alpha \rangle \quad (9)$$

2.1.4 Compatible Observables

Observables \hat{A} and \hat{B} are defined to be compatible when

$$[\hat{A}, \hat{B}] = 0 \quad (10)$$

Then there exists a set of simultaneous eigenkets of \hat{A}, \hat{B} : $|K'\rangle = |a', b'\rangle$.

If $\hat{A}, \hat{B}, \hat{C}$ are incompatible, consider a sequence of selective measurements. The result coming out of the C filter depends on whether or not B measurements have actually been carried out, i.e.

$$\sum_{b'} |\langle c' | b' \rangle|^2 |\langle b' | a' \rangle|^2 \neq |\langle c' | a' \rangle|^2 \quad (11)$$

2.1.5 The Uncertainty Relation

Define an operator:

$$\Delta \hat{A} = \hat{A} - \langle \hat{A} \rangle \quad (12)$$

Then the general uncertainty relation follows from Schwarz inequality:

$$\langle (\hat{A})^2 \rangle \langle (\hat{B})^2 \rangle \geq \frac{1}{4} |\langle [\hat{A}, \hat{B}] \rangle|^2 \quad (13)$$

3 Schrodinger's Equation

Define

$$\hat{H} = -\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} + \hat{V} \quad (14)$$

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi \quad (15)$$

By separation of variable, $\psi(x, t) = \psi(x)\varphi(t)$, we obtain:

$$\frac{d\varphi(t)}{dt} = -\frac{iE}{\hbar} \varphi \quad (16a)$$

$$\hat{H}\psi = E\psi \quad (16b)$$

which yields: time-independent solution $\Psi(t, x) = \psi(x)e^{-\frac{iEt}{\hbar}}$:

$$|\Psi(x, t)|^2 = |\psi(x)|^2 \quad (17a)$$

$$\langle \hat{H} \rangle = \int \psi^* \hat{H} \psi dx = E \int |\Psi|^2 dx = E \quad (17b)$$

$$\langle \hat{H}^2 \rangle = E^2 \quad (17c)$$

$$\sigma_H^2 = 0 \quad (17d)$$