# Black-Scholes Formula, a physicist's perspective

## 1 (a)

Rewritten Using *Brownian Motion*:
$ds(t) = \phi s(t)dt + \sigma s(t)dW(t)$, where $W(t)$ is a standard brownian motion.

To illustrate the relation between **Gaussian Noise** and **Brownian Motion**, consider when using $R(t)$, we're actually suggesting $s(t+\epsilon) = s(t) + \phi s(t)\epsilon + \sigma R\epsilon$. In this case, $R \sim \mathcal{N}(0, \frac{1}{\epsilon})$, therefore $R\epsilon \sim \mathcal{N}(0, \epsilon)$, which can be characterized as $W(t+\epsilon) - W(t)$. As $\epsilon \to 0$, $W(t+\epsilon) - W(t) \to dW(t)$. (It's really clearer to use *brownian motion* notation.) Brownian motion has the property that $dW(t)dW(t) = dt$, $dW(t)dt = 0$.

The original statement can be rewritten as:
$df(t, s(t)) = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t)$

According to *Taylor expansion formula*, we can write

$$df = f_t dt + f_s ds + \frac{1}{2}\{f_{tt}dt^2 + (f_{ts} + f_{st})dtds(t) + f_{ss}ds(t)ds(t)\} + o(dt^2) \tag{1}$$

$$\Longrightarrow df = f_t dt + f_s ds + \frac{1}{2} f_{ss} ds(t) ds(t) \tag{2}$$

Considering $ds(t) = \phi s(t)dt + \sigma s(t)dW(t)$, $dW(t)dW(t) = dt$, $dW(t)dt = 0$, we have

$$df = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t) \tag{3}$$

## 2 (b)

$c = c(t, s(t))$, $d\Pi = c_t dt + c_s ds + \frac{1}{2} c_{ss} ds ds - c_s ds$.
$d\Pi = c_t dt + \frac{1}{2} c_{ss} \sigma^2 s^2 dt$

# 3 (c)

$d\Pi = r(c - c_s s)dt = c_t dt + \frac{1}{2}c_{ss}\sigma^2 s^2 dt$
$c_t + \frac{1}{2}\sigma^2 s^2 c_{ss} + rsc_s - rc = 0$

# 4 Another Way of writing

According to non-arbitrage postulate, if at time 0, $c(0, S(0)) = c_0$, then we should should be able to construct a portfolio $X(t)$(with $X(0) = c_0$) to replicate exactly this option. We should use the underlying stock $S(t)$ and the money market with interest rate $r$. Suppose we hold $\Delta(t)$ share of stock at time $t$, then it follows:

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt = D_t dt + D_w dW(t) \tag{4}$$

where $D_t = \Delta(t)S(t)(\phi - r) + rX(t)$, $D_w = \Delta(t)S(t)\sigma$.

At the same time,

$$dc(t, S(t)) = c_t dt + c_s dS(t) + \frac{1}{2}c_{ss}dS(t)dS(t) = D'_t dt + D'_w dW(t) \tag{5}$$

where $D'_t = c_t + c_s\phi S(t) + \frac{1}{2}\sigma^2 S(t)^2 c_{ss}$, $D'_w = c_s\sigma S(t)$.

Therefore, it should follows that $D_t = D'_t, D_w = D'_w$. The second relation yields instantly that $\Delta(t) = c_s$, while the first would amount to the Black-Scholes formula:

$$c_t + \frac{1}{2}\sigma^2 s^2 c_{ss} + rsc_s - rc = 0 \tag{6}$$

# 5 (d)

Change variable $s = e^x$, we have $c_x = e^{-x}c_s$.
$c_t = rc - rsc_s - \frac{1}{2}\sigma^2 s^2 c_{ss} = (r - (r - \frac{1}{2}\sigma^2)\frac{\partial}{\partial x} - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2})c$.

Therefore $H_{BS} = (1 - \frac{\partial}{\partial x})(r + \frac{1}{2}\sigma^2\frac{\partial}{\partial x})$.

# 6 (e)

$$p_{BS}(x, \tau; x') = \langle x|e^{-\tau H}|x'\rangle = \int_{\infty}^{\infty}\frac{dp}{2\pi}\langle x|e^{-\tau H}|p\rangle\langle p|x'\rangle \tag{7}$$

Taking $p = i\frac{\partial}{\partial x}$, using $\langle x|p\rangle = e^{ipx}$:

$$p_{BS}(x, \tau; x') = e^{-r\tau}\int_{\infty}^{\infty}\frac{dp}{2\pi}exp\{-\frac{1}{2}\sigma^2 p^2\tau + ip(x - x') + ip\tau(r - \frac{\sigma^2}{2})\} \tag{8}$$

Finally, perform the Gaussian integration:

$$p_{BS}(x,\tau;x') = e^{-r\tau}\frac{1}{\sqrt{2\pi\tau\sigma^2}}exp\{-\frac{1}{2\sigma^2\tau}(x - x' + \tau(r - \frac{\sigma^2}{2}))^2\} \tag{9}$$