Black-Scholes Formula, a physicist's perspective

1 (a)

Rewritten Using $Brownian\ Motion$:
$ds(t) = \phi s(t)dt + \sigma s(t)dW(t)$, where $W(t)$ is a standard brownian motion.

To illustrate the relation between **Gaussian Noise** and **Brownian Motion**, consider when using $R(t)$, we're actually suggesting $s(t+\epsilon) = s(t) + \phi s(t)\epsilon + \sigma R\epsilon$. In this case, $R \sim \mathcal{N}(0, \frac{1}{\epsilon})$, therefore $R\epsilon \sim \mathcal{N}(0, \epsilon)$, which can be characterized as $W(t+\epsilon) - W(t)$. As $\epsilon \to 0$, $W(t+\epsilon) - W(t) \to dW(t)$. (It's really clearer to use *brownian motion* notation.) Brownian motion has the property that $dW(t)dW(t) = dt$, $dW(t)dt = 0$.

The original statement can be rewritten as:
$df(t, s(t)) = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t)$

According to $Taylor\ expansion\ formula$, we can write

$$df = f_t dt + f_s ds + \frac{1}{2}\{f_{tt}dt^2 + (f_{ts} + f_{st})dtds(t) + f_{ss}ds(t)ds(t)\} + o(dt^2) = f_t dt + f_s ds + \frac{1}{2}f_{ss}ds(t)ds(t) \tag{1}$$

Considering $ds(t) = \phi s(t)dt + \sigma s(t)dW(t)$ & $dW(t)dW(t) = dt$, $dW(t)dt = 0$, we have $df = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t)$.

2 (b)

$c = c(t, s(t))$, $d\Pi = c_t dt + c_s ds + \frac{1}{2}c_{ss} ds ds - c_s ds$.
$d\Pi = c_t dt + \frac{1}{2}c_{ss}\sigma^2 s^2 dt$

3 (c)

$d\Pi = r(c - c_s s)dt = c_t dt + \frac{1}{2}c_{ss}\sigma^2 s^2 dt$
$c_t + \frac{1}{2}\sigma^2 s^2 c_{ss} + rsc_s - rc = 0$

4 d

Change variable $s = e^x$, we have $c_x = e^{-x}c_s$.
$c_t = rc - rsc_s - \frac{1}{2}\sigma^2 s^2 c_{ss} = (r - (r - \frac{1}{2}\sigma^2)\frac{\partial}{\partial x} - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2})c$.

Therefore $H_{BS} = (1 - \frac{\partial}{\partial x})(r + \frac{1}{2}\sigma^2\frac{\partial}{\partial x})$.

5 Notes

Consider an electron which can only stays on a lattice if discrete points: $x = na$. The eigenvects should be:

$$|n\rangle = \begin{bmatrix} ... \\ 0 \\ 1 \\ 0 \\ ... \end{bmatrix}$$

Then $\langle m|n\rangle = \delta_{n,m}$