Notes on Quantum Physics

Steven Shen

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1 Experimental Facts

1.1 The Stern-Gerlach Experiment 1921 1922

For a heavy atom as a whole, magnetic moment μ is proportional to the electron spin $\overrightarrow{S}$, i.e. $\overrightarrow{\mu} = \frac{e\overrightarrow{S}}{m_e c}$. Place a magnetic field $\overrightarrow{B}$, the interaction energy yields $-\overrightarrow{\mu} \cdot \overrightarrow{B}$. Therefore, if B_z is not homogeneous, then $F_z = \mu_z \partial_z B_z$.

We have a beam of silver atoms(47 electrons in total, with 46 of them having spherical symmetry, no net angular momentum, therefore the atom's angular momentum is sorely decided by the $47th$ electron), going through a $\overrightarrow{B}$ inhomogeneous in B_z. Then atoms will split in z direction according to their spin. We only observe two distinct component on the other side, where $S_z = \pm\frac{\hbar}{2}$.

The experiment suggests quantization of the electron spin angular momentum.

1.2 Sequential SG Experiments

$Oven \Longrightarrow SG\hat{z}$ (filtering S_z-) $\Longrightarrow SG\hat{z} \Longrightarrow S_z+$ only
$Oven \Longrightarrow SG\hat{z}$ (filtering S_z-) $\Longrightarrow SG\hat{x} \Longrightarrow S_x+, S_x-$
$Oven \Longrightarrow SG\hat{z}$ (filtering S_z-) $\Longrightarrow SG\hat{z} \Longrightarrow SG\hat{z} \Longrightarrow S_z+, S_z-$

These results is similar to the polarization of light.

Suppose $x \perp y, x' = \frac{1}{\sqrt{2}}(x+y)$
$\Longrightarrow x - filter \Longrightarrow y - filter \Longrightarrow$ No light
$\Longrightarrow x - filter \Longrightarrow x' - filter \Longrightarrow y - filter \Longrightarrow x_+, x_-$

The experiment suggests that we cannot determine S_z and S_z simultaneously. Previous information is destroyed by the new apparatus. It also suggests the superposition principle.

Further we can use abstact vectors to represent the states in SG experiment on the basis of $|S_z+\rangle$ and $|S_z-\rangle$.

$|S_x\rangle, |S_y\rangle, |S_z\rangle$'s relations:

$$|S_z+\rangle = \frac{1}{\sqrt{2}}(|S_z+\rangle + |S_z-\rangle), \tag{1a}$$

$$|S_z-\rangle = \frac{1}{\sqrt{2}}(-|S_z+\rangle + |S_z-\rangle), \tag{1b}$$

$$|S_y+\rangle = \frac{1}{\sqrt{2}}(|S_z+\rangle + i\,|S_z-\rangle), \tag{1c}$$

$$|S_y-\rangle = \frac{1}{\sqrt{2}}(|S_z+\rangle - i\,|S_z-\rangle) \tag{1d}$$

This example is clear to demonstrate the abstractness of vector space. Quantum-mechanical states are to be represented by vectors in an abstract complex vector space.

2 Mathematics

2.1 Dirac, Ket & Bra

2.1.1 Ket $|\alpha\rangle$ & State Vector Space(Ket Space) $\mathcal{H}$

We want it clear in the beginning. When referring to ket vectors, we're speaking of functions in a functional vector space, ex. functions expanded in fourier forms. The dimension of the complex vector space is decided by the physics system's degree of freedom, ex. in the case of an electron's spin(upward & downward), $dim = 2$.

A physical state is represented by a **state vector** in a complex vector space, called **ket**, denoted by $|\alpha\rangle$. All information about that state is contained in the vector. Our postulation of the existence of vectors to represent states already suggests the superposition principle.
An observable, on the other hand, is represented by an operator which acts on the ket, on the left,

$$A \cdot (|\alpha\rangle) = A\,|\alpha\rangle \tag{2}$$

which yields another ket. If all kets of a system forms the space $\mathcal{H}$, an observable in that system is an operator on $\mathcal{H}$(operator defined as the same in linear algebra). Therefore, there should be eigenkets $|\alpha'\rangle$ of A,

$$A\,|\alpha'\rangle = \alpha'\,|\alpha'\rangle \tag{3}$$

Here α' is just a number. It's convention to 'ket' an eigenvalue to stand for the corresponding eigenvector. The physical state corresponding to an eigenket is called an eigenstate:

$$S_z\,|S_z+\rangle = \frac{\hbar}{2}\,|S_z+\rangle, S_z\,|S_z-\rangle = -\frac{\hbar}{2}\,|S_z-\rangle \tag{4}$$

Next, we consider a $N - dim$ vector space $\mathcal{H}$, spanned according to the N eigenkets of observable A. Then $\forall \left|\alpha\right\rangle \in \mathcal{H}$, $\left|\alpha\right\rangle = \sum_{\alpha'} c_{\alpha'} \left|\alpha'\right\rangle$.

2.1.2 Bra $\langle\alpha|$, Dual Space(Bra Space) $\mathcal{H}^*$ and Inner Products

Bra Space is dual to Ket Space. Corresponding to every $\left|\alpha\right\rangle$ in $\mathcal{H}$, there exists a bra in dual space $\mathcal{H}^*$, denoted by $\langle\alpha|$.

Inner product is defined as a mapping $i(*,*) : \mathcal{H} \times \mathcal{H} \mapsto \mathbb{C}$. Think of $i(*,*)$ as a machine receiving two kets and returns a complex number. We all know how the inner product should be defined.

Dual space is by now unclear. Yet with $i(*,*)$ we can deduce a dual space with a particular $i(*,*)$ defined. With $i(*,*)$ and $\forall f \in \mathcal{H}$, we have $i(f,*) : \mathcal{H} \mapsto \mathbb{C}$, $i(f,*) \in \mathcal{H}^*$. Moreover, this is indeed a **one-to-one mapping**:

$$\forall \eta \in \mathcal{H}^*, \exists f_\eta \in \mathcal{H}, s.t. \forall g \in \mathcal{H}, \eta(g) = i(f_\eta, g) \tag{5}$$

Denote this mapping $v : \mathcal{H}^* \mapsto \mathcal{H}$, and denote $v^{-1}(\eta) = g_\eta$.
(Right now I can only understand dual space this way, remained to be uncovered.)

Now we know a bra $\langle\eta|$ acts on a ket $\left|f\right\rangle$ yields the (g_η, f), which is called the inner product of $\langle\eta|$ and $\left|f\right\rangle$ in physics, when mathematically this is inacurate.

In physics, we often put a complex vector in $\mathcal{H}$ into a bra, ex. $\langle\alpha|$, which actually means that $\langle\alpha| = v^{-1}(\left|\alpha\right\rangle)$. Though very confusing, we only need to remember finally $\langle\eta|$ acts on $\left|f\right\rangle$ yields $(\left|\eta\right\rangle, \left|f\right\rangle)$.