Black-Scholes Formula, a physicist's perspective

1 (a)

Rewritten Using *Brownian Motion*:

$$ds(t) = \phi s(t)dt + \sigma s(t)dW(t). \tag{1}$$

where $W(t)$ is a standard brownian motion.

To illustrate the relation between **Gaussian Noise** and **Brownian Motion**, consider when using $R(t)$, we're actually suggesting $s(t+\epsilon) = s(t) + \phi s(t)\epsilon + \sigma R\epsilon$. In this case, $R \sim \mathcal{N}(0, \frac{1}{\epsilon})$, therefore $R\epsilon \sim \mathcal{N}(0, \epsilon)$, which can be characterized as $W(t+\epsilon) - W(t)$. As $\epsilon \to 0$, $W(t+\epsilon) - W(t) \to dW(t)$. (It's really clearer to use *brownian motion* notation.) Brownian motion has the property that $dW(t)dW(t) = dt$, $dW(t)dt = 0$.

The original statement can be rewritten as:

$$df(t, s(t)) = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t). \tag{2}$$

According to *Taylor expansion formula*, we can write

$$df = f_t dt + f_s ds + \frac{1}{2}\{f_{tt} dt^2 + (f_{ts} + f_{st}) dt ds(t) + f_{ss} ds(t) ds(t)\} + o(dt^2) \tag{3}$$

$$\Longrightarrow df = f_t dt + f_s ds + \frac{1}{2} f_{ss} ds(t) ds(t) \tag{4}$$

Considering $ds(t) = \phi s(t)dt + \sigma s(t)dW(t)$, $dW(t)dW(t) = dt$, $dW(t)dt = 0$, we have

$$df = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t) \tag{5}$$

Using the relationship between W(t) and R(t), we can change dW(t) and derive:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} + \frac{\partial f}{\partial s}(\phi s + \sigma s R). \tag{6}$$

2 (b)

We've already known that $c = c(t, s(t))$. Consider a portfolio $\Pi = c - \frac{\partial c}{\partial s} s$. To calculate the derivative of Π we can write down the differential of Π:

$$d\Pi = c_t dt + c_s ds + \frac{1}{2} c_{ss} ds ds - c_s ds. \tag{7}$$

Because we've got $dS^2 = \sigma^2 s^2 dt$, we can derive:

$$\frac{d\Pi}{dt} = c_t + \frac{1}{2} c_{ss} \sigma^2 s^2. \tag{8}$$

3 (c)

r means the short-term risk-free interest rate. The following equation must be true:

$$\frac{d\Pi}{dt} = r\Pi. \tag{9}$$

Otherwise arbitrage will exist, which contradicts our assumption. Therefore we can write $d\Pi = r(c - c_s s)dt$. It's equivalent to the previous result $c_t dt + \frac{1}{2} c_{ss} \sigma^2 s^2 dt$.

At last we have the **Black-Scholes formula**:

$$c_t + \frac{1}{2} \sigma^2 s^2 c_{ss} + rsc_s - rc = 0. \tag{10}$$

Another Way of Obtaining B-S Formula

According to non-arbitrage postulate, if at time 0, $c(0, S(0)) = c_0$, then we should should be able to construct a portfolio $X(t)$(with $X(0) = c_0$) to replicate exactly this option. We should use the underlying stock $S(t)$ and the money market with interest rate r. Suppose we hold $\Delta(t)$ share of stock at time t, then it follows:

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt = D_t dt + D_w dW(t) \tag{11}$$

where $D_t = \Delta(t)S(t)(\phi - r) + rX(t)$, $D_w = \Delta(t)S(t)\sigma$.

At the same time,

$$dc(t, S(t)) = c_t dt + c_s dS(t) + \frac{1}{2} c_{ss} dS(t) dS(t) = D'_t dt + D'_w dW(t) \tag{12}$$

where $D'_t = c_t + c_s \phi S(t) + \frac{1}{2} \sigma^2 S(t)^2 c_{ss}$, $D'_w = c_s \sigma S(t)$.

Therefore, it should follows that $D_t = D'_t, D_w = D'_w$. The second relation yields instantly that $\Delta(t) = c_s$, while the first would amount to the Black-Scholes formula:

$$c_t + \frac{1}{2} \sigma^2 s^2 c_{ss} + rsc_s - rc = 0 \tag{13}$$

4 (d)

Change variable $s = e^x$, we have:

$$c_x = e^{-x} c_s. \tag{14}$$

$$c_t = rc - rsc_s - \frac{1}{2}\sigma^2 s^2 c_{ss} = (r - (r - \frac{1}{2}\sigma^2)\frac{\partial}{\partial x} - \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2})c. \tag{15}$$

Therefore we can easily prove that

$$H_{BS} = (1 - \frac{\partial}{\partial x})(r + \frac{1}{2}\sigma^2 \frac{\partial}{\partial x}) = -\frac{\sigma^2}{2}\frac{\partial^2 c}{\partial x^2} + (\frac{\sigma^2}{2} - r)\frac{\partial c}{\partial x} + rc. \tag{16}$$

which is the Hamiltonian for Black-Scholes model.

5 (e)

$$p_{BS}(x, \tau; x') = \langle x|e^{-\tau H}|x'\rangle = \int_{\infty}^{\infty} \frac{dp}{2\pi} \langle x|e^{-\tau H}|p\rangle \langle p|x'\rangle \tag{17}$$

Taking $p = i\frac{\partial}{\partial x}$, using $\langle x|p\rangle = e^{ipx}$:

$$p_{BS}(x, \tau; x') = e^{-r\tau} \int_{\infty}^{\infty} \frac{dp}{2\pi} exp\{-\frac{1}{2}\sigma^2 p^2 \tau + ip(x - x') + ip\tau(r - \frac{\sigma^2}{2})\} \tag{18}$$

Finally, perform the Gaussian integration:

$$p_{BS}(x, \tau; x') = e^{-r\tau} \frac{1}{\sqrt{2\pi\tau\sigma^2}} exp\{-\frac{1}{2\sigma^2\tau}(x - x' + \tau(r - \frac{\sigma^2}{2}))^2\} \tag{19}$$

6 (f)

We've known that P(x,T-t,x') is the conditional probability density that, given security price x at time t, it will have a value of x' at time T.

The expectation of x' at time τ can therefore be calculated(it's a simple convolution):

$$< x(\tau) >= \int_{-\infty}^{+\infty} x' P_{BS}(x, \tau; x') dx' = \frac{e^{-r\tau}}{\sqrt{2\pi\tau\sigma^2}} \int_{-\infty}^{+\infty} x' e^{-\frac{1}{2\tau\sigma^2}[x - x' + \tau(r - \frac{\sigma^2}{2})]^2} dx'. \tag{20}$$

Finally we can derive:

$$< x(\tau) >= [x + \tau(r - \frac{\sigma^2}{2})]e^{-r\tau}. \tag{21}$$

The result tells us the evolution of x(which equals ln(S)) over time.

7 (i)

We know that r(t) satisfied Langevin equation:

$$\frac{dr}{dt} = a(r,t) + \sigma(r,t)R(t). \tag{22}$$

Here R(t) is still Gaussian noise.

Define the propagator $P(r,t;r_0)$: if $r(t_0) = r_0$, the probability of r(t)=r equals $P(r,t;r_0)$.

From the Langevin equation we have:

$$r(t+\varepsilon) = r(t) + \varepsilon[a + \sigma R(t)]. \tag{23}$$

Change it into the following formula:

$$r = r' + \varepsilon[a(r') + \sigma(r')R(t)]. \tag{24}$$

Thus we can calculate the propagator:

$$P(r,t+\varepsilon,r_0) = P(r',t;r_0)|_{r'=r-\varepsilon[a(r')+\sigma(r')R(t)]} = \int P(r',t;r_0)\delta(r-r'-\varepsilon[a(r')+\sigma(r')R(t)])dr' \simeq \int P(r \tag{25}$$

Because $< R^2(t) >= \frac{1}{\varepsilon}$ and $< R(t) >= 0$, the previous formula can be rewritten like this:

$$P(r,t+\varepsilon,r_0) = P(r',t;r_0) + \int dr' P(r',t;r_0)\{\frac{\partial\delta(r-r')}{\partial r'}\varepsilon a(r') + \frac{1}{2}\frac{\partial^2\delta(r-r')}{\partial r'^2}\varepsilon^2\sigma^2(r')\frac{1}{\varepsilon}\} = P(r',t;r_0)- \tag{26}$$

If a variable is o(ε), it is automatically neglected.

Thus, from the definition of derivative, we have:

$$\frac{\partial P(r,t;r_0)}{\partial t} = [\frac{1}{2}\frac{\partial^2}{\partial r^2}\sigma^2(r) - \frac{\partial}{\partial r}a(r)]P(r,t;r_0). \tag{27}$$

We've already known that:

$$\frac{\partial P(r,t;r_0)}{\partial t} = -H_F P(r,t;r_0). \tag{28}$$

Therefore we can prove:

$$H_F = -\frac{1}{2}\frac{\partial^2}{\partial r^2}\sigma^2(r) + \frac{\partial}{\partial r}a(r) = -\frac{1}{2}\frac{\partial^2}{\partial r^2}\sigma^2(r) + a(r)\frac{\partial}{\partial r} + \frac{\partial a(r)}{\partial r}. \tag{29}$$

From a different perspective, we define $P_B(R,t;r)$ as the back propagator. Similarly we have(since the time flows backwards this time):

$$\frac{\partial P_B(R,t;r)}{\partial t} = +H_B P_B(R,t;r). \tag{30}$$

Finally we can calculate:

$$H_B = -\frac{1}{2}\sigma^2(r)\frac{\partial^2}{\partial r^2} - a(r)\frac{\partial}{\partial r}. \tag{31}$$

$H_B = H_F^\dagger$. is obvious.

8 (k)

The Vasicek model can be described using the following equation:

$$\frac{dr}{dt} = a(b-r) + \sigma R(t). \tag{32}$$

Note that if we set $\begin{cases} a(r,t) = a(b-r). \\ \sigma(r,t) = \sigma. \end{cases}$ in Langevin equation(a,b,σ are all constants), we have the Vasicek model naturally.

Thus we get:

$$L_V = -\frac{[\frac{dr}{dt} - a(b-r)]^2}{2\sigma^2}. \tag{33}$$

$$S_V = -\frac{1}{2\sigma^2}\int_{t_0}^{T}[\frac{dr}{dt} - a(b-r)]^2 dt. \tag{34}$$

In the next section we'll use S_V to calculate the propagator of Vasicek model.