Black-Scholes Formula, a physicist's perspective

1 (a)

Rewritten Using *Brownian Motion*:

$$ds(t) = \phi s(t)dt + \sigma s(t)dW(t). \tag{1}$$

where $W(t)$ is a standard brownian motion.

To illustrate the relation between **Gaussian Noise** and **Brownian Motion**, consider when using $R(t)$, we're actually suggesting $s(t+\epsilon) = s(t) + \phi s(t)\epsilon + \sigma R\epsilon$. In this case, $R \sim \mathcal{N}(0, \frac{1}{\epsilon})$, therefore $R\epsilon \sim \mathcal{N}(0, \epsilon)$, which can be characterized as $W(t+\epsilon) - W(t)$. As $\epsilon \to 0$, $W(t+\epsilon) - W(t) \to dW(t)$. (It's really clearer to use *brownian motion* notation.) Brownian motion has the property that $dW(t)dW(t) = dt$, $dW(t)dt = 0$.

The original statement can be rewritten as:

$$df(t, s(t)) = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t). \tag{2}$$

According to *Taylor expansion formula*, we can write

$$df = f_t dt + f_s ds + \frac{1}{2}\{f_{tt}dt^2 + (f_{ts} + f_{st})dtds(t) + f_{ss}ds(t)ds(t)\} + o(dt^2) \tag{3}$$

$$\Longrightarrow df = f_t dt + f_s ds + \frac{1}{2} f_{ss} ds(t)ds(t) \tag{4}$$

Considering $ds(t) = \phi s(t)dt + \sigma s(t)dW(t)$, $dW(t)dW(t) = dt$, $dW(t)dt = 0$, we have

$$df = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t) \tag{5}$$

Using the relationship between W(t) and R(t), we can change dW(t) and derive:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} + \frac{\partial f}{\partial s}(\phi s + \sigma s R). \tag{6}$$

2 (b)

We've already known that $c = c(t, s(t))$. Consider a portfolio $\Pi = c - \frac{\partial c}{\partial s}s$. To calculate the derivative of Π we can write down the differential of Π:

$$d\Pi = c_t dt + c_s ds + \frac{1}{2}c_{ss} ds ds - c_s ds. \tag{7}$$

Because we've got $dS^2 = \sigma^2 s^2 dt$, we can derive:

$$\frac{d\Pi}{dt} = c_t + \frac{1}{2}c_{ss}\sigma^2 s^2. \tag{8}$$

3 (c)

r means the short-term risk-free interest rate. The following equation must be true:

$$\frac{d\Pi}{dt} = r\Pi. \tag{9}$$

Otherwise arbitrage will exist, which contradicts our assumption. Therefore we can write $d\Pi = r(c - c_s s)dt$. It's equivalent to the previous result $c_t dt + \frac{1}{2}c_{ss}\sigma^2 s^2 dt$.

At last we have the **Black-Scholes formula**:

$$c_t + \frac{1}{2}\sigma^2 s^2 c_{ss} + rsc_s - rc = 0. \tag{10}$$

Why the equation does not contain ϕ ?

Intuitively, it means that the expected return of an stock has nothing to do with the derivatives pricing. In other words, if there are two stocks at the same price today, which are expected to have different rate of returns in the future, will have the same the price for their derivatives, whether it be options or something else. It might seem weird at first sight because a call option should have a higher price if the the underlying stock is expected to rise up. But actually it is not. Derivatives pricing is relative pricing instead of absolute pricing. If the return is high, it might have a higher volatility. And the price of the stock today reflects every piece of information at present. So when pricing a Derivative, we just consider the price S, and do not include the expected return, ϕ. It's just derived from the underlying assets.

Another Way of Obtaining B-S Formula

According to non-arbitrage postulate, if at time 0, $c(0, S(0)) = c_0$, then we should should be able to construct a portfolio $X(t)$(with $X(0) = c_0$) to replicate exactly this option. We should use the underlying stock $S(t)$ and the money market with interest rate r. Suppose we hold $\Delta(t)$ share of stock at time t, then it follows:

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt = D_t dt + D_w dW(t) \tag{11}$$

where $D_t = \Delta(t)S(t)(\phi - r) + rX(t)$, $D_w = \Delta(t)S(t)\sigma$.

At the same time,

$$dc(t, S(t)) = c_t dt + c_s dS(t) + \frac{1}{2}c_{ss} dS(t)dS(t) = D'_t dt + D'_w dW(t) \tag{12}$$

where $D'_t = c_t + c_s\phi S(t) + \frac{1}{2}\sigma^2 S(t)^2 c_{ss}$, $D'_w = c_s\sigma S(t)$.

Therefore, it should follows that $D_t = D'_t, D_w = D'_w$. The second relation yields instantly that $\Delta(t) = c_s$, while the first would amount to the Black-Scholes formula:

$$c_t + \frac{1}{2}\sigma^2 s^2 c_{ss} + rsc_s - rc = 0 \tag{13}$$

4 (d)

Change variable $s = e^x$, we have:

$$c_x = e^{-x} c_s. \tag{14}$$

$$c_t = rc - rsc_s - \frac{1}{2}\sigma^2 s^2 c_{ss} = (r - (r - \frac{1}{2}\sigma^2)\frac{\partial}{\partial x} - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2})c. \tag{15}$$

Therefore we can easily prove that

$$H_{BS} = (1 - \frac{\partial}{\partial x})(r + \frac{1}{2}\sigma^2\frac{\partial}{\partial x}) = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (\frac{\sigma^2}{2} - r)\frac{\partial}{\partial x} + r. \tag{16}$$

which is the Hamiltonian for Black-Scholes model.

5 (e)

$$p_{BS}(x, \tau; x') = \langle x|e^{-\tau H}|x'\rangle = \int_{\infty}^{\infty}\frac{dp}{2\pi}\langle x|e^{-\tau H}|p\rangle\langle p|x'\rangle \tag{17}$$

Taking $p = i\frac{\partial}{\partial x}$, using $\langle x|p\rangle = e^{ipx}$:

$$p_{BS}(x, \tau; x') = e^{-r\tau}\int_{\infty}^{\infty}\frac{dp}{2\pi}exp\{-\frac{1}{2}\sigma^2 p^2\tau + ip(x - x') + ip\tau(r - \frac{\sigma^2}{2})\} \tag{18}$$

Finally, perform the Gaussian integration:

$$p_{BS}(x, \tau; x') = e^{-r\tau}\frac{1}{\sqrt{2\pi\tau\sigma^2}}exp\{-\frac{1}{2\sigma^2\tau}(x - x' + \tau(r - \frac{\sigma^2}{2}))^2\} \tag{19}$$

6 (f)

After obtaining the pricing kernel, we can get the price of call option at time t simply by integrating the final value with the kernel.

$$c(\tau, x) = \int_{-\infty}^{\infty} g(x')P_{BS}dx' \tag{20}$$

where in this case, $g(x') = (e^{x'} - K)^+$. $(x)^+$ takes x when $x > 0$ and takes 0 when $x \leq 0$. Therefore, the integration only takes place in $x \in (lnK, \infty)$.

After noticing, the pricing kernel is actually $e^{-r\tau}$ mutilpying a normal distribution, we can define $d_-(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}[\frac{x}{lnK} + (r - \frac{1}{2}\sigma^2)\tau], d_+ = d_- + \sigma\sqrt{\tau}$, $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{x^2}{2}}$. We separate the integration into two parts: K and e^x.

$$\int_{lnK}^{\infty} KP_{BS}dx' = e^{-r\tau}KN(d_-(\tau, x)) \tag{21a}$$

$$\int_{lnK}^{\infty} e^x P_{BS}dx' = e^x N(d_+(\tau, x)) \tag{21b}$$

Remember that $s = e^x$, we have the pricing formula for $c(t, s)$:

$$c(t, s) = sN(d_+(\tau, s)) - e^{-r\tau}KN(d_-(\tau, s)) \tag{22}$$

With this formula, we can plot the time-evolution of stock prices and call option prices.

The following code does the job in R, results shown in $fig.1 \sim 3$

```
c_time_evolution <- function(sigma, x0, r, K)
{
    tau = seq(0, 1, 0.01)[-1]
    w = rnorm(100, 0, 0.01)
    w = cumsum(w)
    # calculate s(t)
    x = (r - 0.5*sigma^2) * tau + sigma * w
    x = exp(x)
    x = x0 * x
    #fi
    plot(tau, x, type='l')

    d1 = log(x/K) + (1 - tau) * (r + 0.5*sigma^2)
    d1 = d1/(sqrt(1 - tau) * sigma)
```

```
    d2 = d1 - sigma*sqrt(1 - tau)
    c = x * pnorm(d1) - exp(-r * (1-tau)) * K * pnorm(d2)
    plot(tau, c, 'l')
}
#fig.1
c_time_evolution(sigma = 0.5, x0 = 100, r = 0.20, K = 100)
#fig.2
c_time_evolution(sigma = 0.5, x0 = 80, r = 0.20, K = 100)
#fig.3
c_time_evolution(sigma = 0.5, x0 = 120, r = 0.20, K = 100)
```

Fig. 1: Time-evolution of call option price

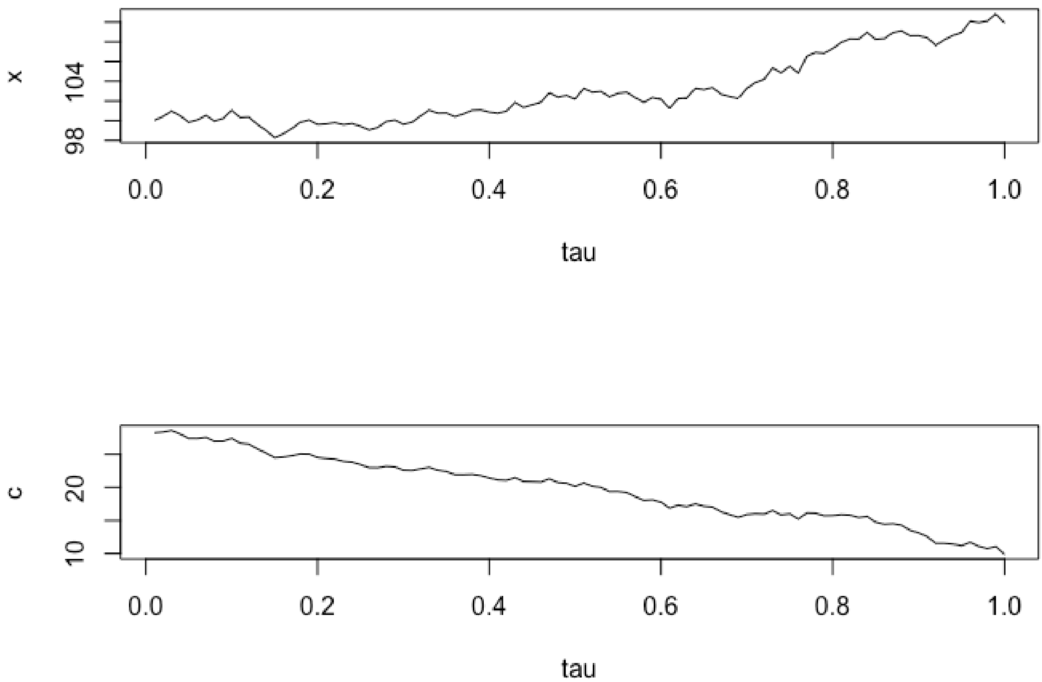


Fig. 2: Time-evolution of call option price

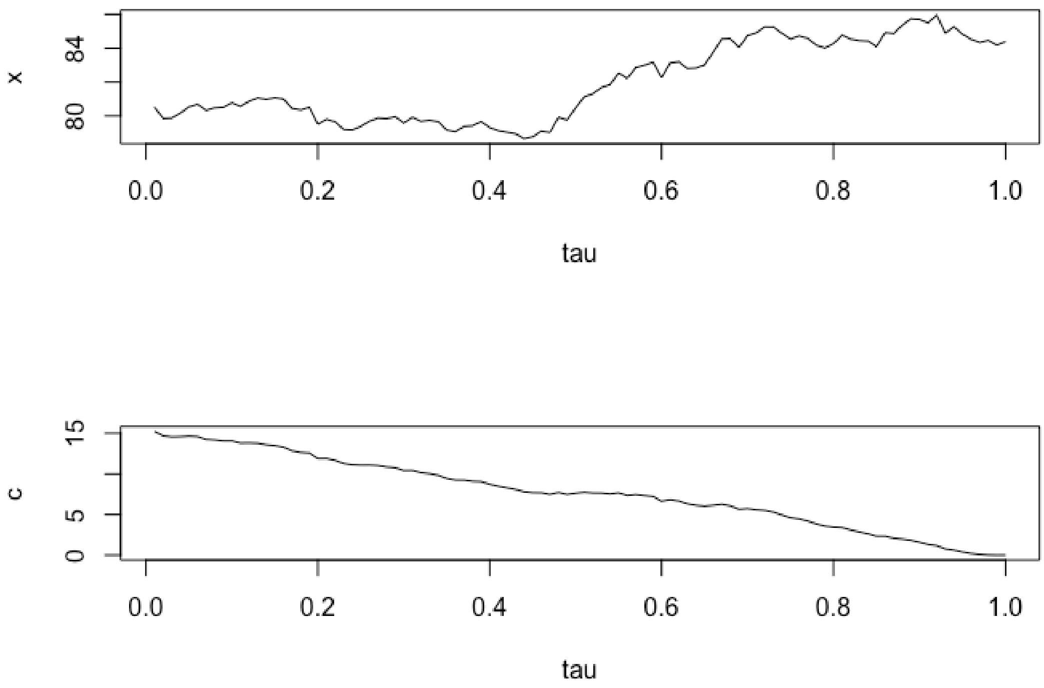
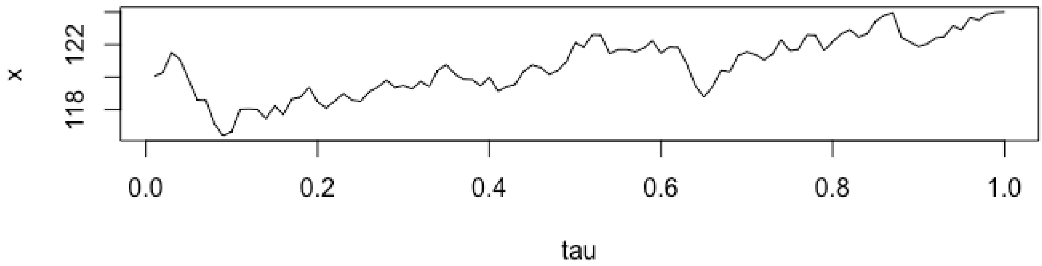
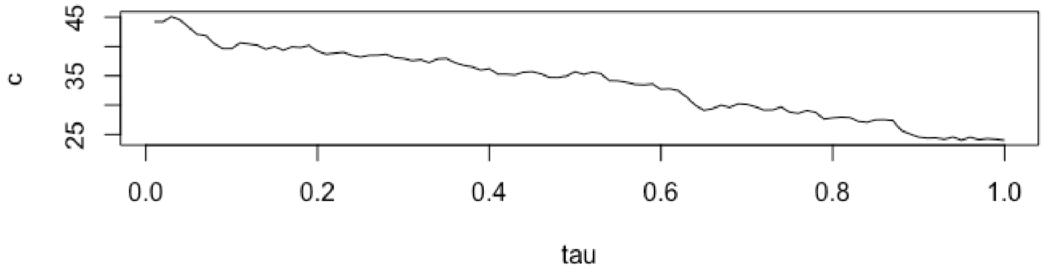


Fig. 3: Time-evolution of call option price





As shown in these figures, call option price drops as $(T-t)$ decreases.

7 (g)

Consider a down-and-out barrier European option. If $s \leq e^B (x \leq B)$ it will become worthless, which means $c = 0$.

For an arbitrary potential $v(x)$, we have: $\frac{\partial c}{\partial t} = Hc$, where H can be written as:

$$H = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (\frac{\sigma^2}{2} - r)\frac{\partial}{\partial x} + V(x). \tag{23}$$

Obviously, if we set $V(x) = \infty$ when $x \leq B$, then $c = 0$ is automaticlly satisfied. However, when $x > B$, the changing of c still conform to the B-S formula. comparing with H_{BS}, we have: V(x)= $\begin{cases} +\infty, x \leq B. \\ r, x > B. \end{cases}$

8 (h)

The Hamiltonian for the down and out option is given by:

$$H_{DO} = H_{BS} + V(x) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (\frac{\sigma^2}{2} - r)\frac{\partial}{\partial x} + V(x) \tag{24}$$

V(x) is defined in (g).

Now we can construct the eigenfunctions similar to the $Black - Scholes$ case except that these must satisfy $\Psi_E(B) = 0$.

Define the quantities:

$$\beta = \frac{(\frac{\sigma^2}{2} + r)^2}{\sigma^4}, p = \sqrt{\frac{2E}{\sigma^2} - \beta}, \alpha = \frac{\frac{\sigma^2}{2} - r}{\sigma^2}, i\lambda_\pm = \alpha \pm ip \tag{25}$$

The eigenfunctions are given by:

$$x > B : \langle x|\Psi_E\rangle = e^{i\lambda_+(x-B)} - e^{i\lambda_-(x-B)} = 2ie^{\alpha(x-B)}\sin\left[p(x-B)\right] \tag{26}$$

$$\langle\widetilde{\Psi_E}|x\rangle = e^{-i\lambda_+(x-B)} - e^{-i\lambda_-(x-B)} = -2ie^{-\alpha(x-B)}\sin\left[p(x-B)\right] \tag{27}$$

$$\left\langle\widetilde{\Psi_E}|\Psi_E{}'\right\rangle = [2\pi\sigma^2\sqrt{2E/\sigma^2 - \beta}]\delta(E - E'). \tag{28}$$

$$x \le B : \langle x|\Psi_E\rangle = 0 = \left\langle\widetilde{\Psi_E}|x\right\rangle. \tag{29}$$

Now we can use the eigenfunctions to evaluate the pricing kernel. Again we work with the p variable:

$$\begin{aligned} p_{DO}(x,\tau;x') &= \left\langle x|e^{-\tau H_{DO}}|x'\right\rangle \\ &= e^{-\frac{\tau\beta\sigma^2}{2}+\alpha(x-x')}\int_0^\infty \frac{dp}{2\pi}e^{-\frac{1}{2}\tau\sigma^2p^2}\big[e^{ip(x-x')} + e^{-ip(x-x')} \\ &- e^{ip(x+x'-2B)} - e^{-ip(x+x'-2B)}\big]. \end{aligned}$$

Consider that we have already derived:

$$p_{BS}(x,\tau;x') = e^{-r\tau}\frac{1}{\sqrt{2\pi\tau\sigma^2}}e^{-\frac{1}{2\tau\sigma^2}[x-x'+\tau(r-\frac{\sigma^2}{2})]^2}. \tag{30}$$

We can simplify the pricing kernel using the previous results:

$$p_{DO}(x,\tau;x') = p_{BS}(x,\tau;x') - (\frac{e^x}{e^B})^{2\alpha}p_{BS}(2B - x,\tau;x'). \tag{31}$$

9 (i)

We know that $r(t)$ satisfied Langevin equation:

$$\frac{dr}{dt} = a(r,t) + \sigma(r,t)R(t). \tag{32}$$

Here $R(t)$ is still Gaussian noise.

Define the propagator $P(r,t;r_0)$: if $r(t_0) = r_0$, the probability of $r(t) = r$ equals $P(r,t;r_0)$.

From the Langevin equation we have:

$$r(t+\varepsilon) = r(t) + \varepsilon[a + \sigma R(t)]. \tag{33}$$

Change it into the following formula:

$$r = r' + \varepsilon[a(r') + \sigma(r')R(t)]. \tag{34}$$

Thus we can calculate the propagator:

$$\begin{aligned} P(r,t+\varepsilon,r_0) &= P(r',t;r_0)|_{r'=r-\varepsilon[a(r')+\sigma(r')R(t)]} \\ &= \int P(r',t;r_0)\delta(r - r' - \varepsilon[a(r') + \sigma(r')R(t)])dr' \\ &\simeq \int P(r',t;r_0)dr'\{\delta(r-r') + \frac{\partial\delta(r-r')}{\partial r'}\varepsilon[a(r') + \sigma(r')R(t)] \\ &+ \frac{1}{2}\frac{\partial^2\delta(r-r')}{\partial r'^2}\varepsilon^2[a(r') + \sigma(r')R(t)]^2 + ...\}. \end{aligned}$$

Because $< R^2(t) >= \frac{1}{\varepsilon}$ and $< R(t) >= 0$, the previous formula can be rewritten like this:

$$\begin{aligned} P(r,t+\varepsilon,r_0) &= P(r',t;r_0) + \int dr' P(r',t;r_0)\{\frac{\partial\delta(r-r')}{\partial r'}\varepsilon a(r') + \frac{1}{2}\frac{\partial^2\delta(r-r')}{\partial r'^2}\varepsilon^2\sigma^2(r')\frac{1}{\varepsilon}\} \\ &= P(r',t;r_0) - \varepsilon\frac{\partial}{\partial r}[a(r)P(r,t;r_0)] + \frac{\varepsilon}{2}\frac{\partial^2}{\partial r^2}[\sigma(r)^2P(r,t;r_0)]. \end{aligned}$$

If a variable is o(ε), it is automatically neglected.

Thus, from the definition of derivative, we have:

$$\frac{\partial P(r,t;r_0)}{\partial t} = [\frac{1}{2}\frac{\partial^2}{\partial r^2}\sigma^2(r) - \frac{\partial}{\partial r}a(r)]P(r,t;r_0). \tag{35}$$

We've already known that:

$$\frac{\partial P(r,t;r_0)}{\partial t} = -H_F P(r,t;r_0). \tag{36}$$

Therefore we can prove:

$$H_F = -\frac{1}{2}\frac{\partial^2}{\partial r^2}\sigma^2(r) + \frac{\partial}{\partial r}a(r) = -\frac{1}{2}\frac{\partial^2}{\partial r^2}\sigma^2(r) + a(r)\frac{\partial}{\partial r} + \frac{\partial a(r)}{\partial r}. \tag{37}$$

From a different perspective, we define $P_B(R,t;r)$ as the back propagator.
Similarly we have(since the time flows backwards this time):

$$\frac{\partial P_B(R,t;r)}{\partial t} = +H_B P_B(R,t;r). \tag{38}$$

Finally we can calculate:

$$H_B = -\frac{1}{2}\sigma^2(r)\frac{\partial^2}{\partial r^2} - a(r)\frac{\partial}{\partial r}. \tag{39}$$

$H_B = H_F^\dagger$ is obvious.

10 (j)

In this part we'll focus on the so-called **stochastic Quantization**.
The Langevin equation:

$$\frac{dr}{dt} = a(r,t) + \sigma(r.t)R(t) \tag{40}$$

is satisfied at any time between t_0 and T. We must consider that both r and R are stochastic variables. Therefore when calculating z_B(can be compared to partition function in statistical mechanics), we must integrate over all possible paths

$$Z_B = \int DRDr\prod_{t=t_0}^{T}\delta[\frac{dr}{dc} - a(r,t) - \sigma(r,t)R(t)]e^-\frac{1}{2}\int_{t_0}^{T}R^2(t)dt \tag{41}$$

Dr means integrating over all possible $r(t)$: $\int Dr = \int_{-\infty}^{+\infty}\prod_{t=t_0}^{T} dr(t)$.
We hope to write Z_B int the form of $z_B = \int Dre^{s_B}$. Because of this, we calculate the integral over R first:

$$Z_B = \int Drexp(-\frac{1}{2}\int_{t_0}^{T}\frac{[\frac{dr}{dt} - a(r,t)]^2}{\sigma^2(r,t)}dt). \tag{42}$$

Because the $Dirac-\delta$ function makes sure we only keep $R(t)$ that satisfies the Langevin equation: $R(t) = \frac{\frac{dr}{dt}-a(r,t)}{\sigma(r,t)}$.
It's easy to calculate S_B:

$$S_B = -\frac{1}{2}\int_{t_0}^{T}\frac{[\frac{dr}{dt} - a(r,t)]^2}{\sigma^2(r,t)}dt. \tag{43}$$

From the definition of $S_B : S_B = \int_{t_0}^{T} Ldt$, it's clear that:

$$L = -\frac{[\frac{dr}{dt} - a(r,t)]^2}{2\sigma^2(r,t)}dt. \tag{44}$$

11 (k)

The Vasicek model can be described using the following equation:

$$\frac{dr}{dt} = a(b-r) + \sigma R(t). \tag{45}$$

Note that if we set $\begin{cases} a(r,t) = a(b-r). \\ \sigma(r,t) = \sigma. \end{cases}$ in Langevin equation(a,b,σ are all constants), we have the Vasicek model naturally.

Thus we get:

$$L_V = -\frac{[\frac{dr}{dt} - a(b-r)]^2}{2\sigma^2}. \tag{46}$$

$$S_V = -\frac{1}{2\sigma^2}\int_{t_0}^{T}[\frac{dr}{dt} - a(b-r)]^2 dt. \tag{47}$$

In the next section we'll use S_V to calculate the propagator of Vasicek model.

12 (l)

For different paths(which all have different S_V), so that e^{S_V}/Z is the distribution of probability. $Z = \int Dre^{S_V}$. Given by the question itself, we know that the propagator equals:

$$P(t_0,T) = \frac{1}{Z}\int Dre^{S_V}e^{-\int_{t_0}^{T} r(t)dt}. \tag{48}$$

If we use a new variable $S = S_V - \int_{t_0}^{T} r(t)dt$, then we have:

$$P(t_0,T) = \frac{1}{Z}\int Dre^{S}. \tag{49}$$

Change the variable: $u = r - b$. Thus we can write:

$$\begin{aligned} S = &-\frac{1}{2\sigma^2}\int_{t_0}^{T} dt[\frac{du}{dt} + au]^2 - \int_{t_0}^{T}(u+b)dt \\ &= -\frac{1}{2\sigma^2}\int_{t_0}^{T} dt[\frac{dr}{dt} + ar]^2 - \int_{t_0}^{T}(r+b)dt. \end{aligned}$$

Next, we can define $v(t) = ar(t) + \frac{dr(t)}{dt}$. View this formula as a differential equation of $r(t)$:

$$\frac{dr}{dt} + ar = v(t). \tag{50}$$

The solution can be easily derived:

$$r(t) = e^{-a(t-t_0)} r_0 + e^{-at} \int_{t_0}^{t} e^{at'} v(t') dt'. \tag{51}$$

Since we want to calculate the integral of r(t), we now have:

$$\int_{t_0}^{T} r(t) dt = B(t_0, T) r_0 + \int_{t_0}^{T} B(t, T) v(t) dt. \tag{52}$$

where $B(t,T) = \frac{1-e^{-a(T-t)}}{a}$.

From the definition of $v(t)$ we can tell that $v(T)$ is free to take all possible values. Therefore we can use the following path integral to calculate propagator:

$$\begin{aligned} P(t_0, T) &= e^{-b(T-t_0) - B(t_0,T)r_0} \frac{1}{Z} \int Dv e^{-\frac{1}{2\sigma^2} \int_{t_0}^{T} dt [v(t)^2 + 2\sigma^2 B(t,T) v(t)]} \\ &= e^{-b(T-t_0) - B(t_0,T)r_0} e^{\frac{\sigma^2}{2} \int_{t_0}^{T} dt B(t,T)^2}. \end{aligned}$$

The $v(t)$ integrations are decoupled Gaussian integrations, with the overall normalization being canceled by the factor Z.

Finally, assume $B(\theta) = \frac{1-e^{-a\theta}}{a}$, where $\theta = T - t_0$, and $A(\theta) = exp[(\frac{\sigma^2}{2a^2} - b)(\theta - B(\theta)) - \frac{\sigma^2}{4a} B(\theta)^2]$.

$$A(\theta) e^{-r_0 B(\theta)} = exp[-b\theta - r_0 \frac{1-e^{-a\theta}}{a} + \frac{\sigma^2}{2a^2}\theta - \frac{\sigma^2(1-e^{-a\theta})}{2a^3} + \frac{b}{a}(1-e^{-a\theta}) - \frac{\sigma^2}{4a^3}(1-e^{-a\theta})^2]. \tag{53}$$

If we want to prove $P(t_0, T) = A(\theta) e^{-r_0 B(\theta)}$, then we must prove:

$$\begin{aligned} -b(T-t_0) - B(t_0,T) r_0 + \frac{\sigma^2}{2} \int_{t_0}^{T} dt B(t,T)^2 = &-b\theta - r_0 \frac{1-e^{-a\theta}}{a} + \frac{\sigma^2}{2a^2}\theta \\ &- \frac{\sigma^2(1-e^{-a\theta})}{2a^3} + \frac{b}{a}(1-e^{-a\theta}) - \frac{\sigma^2}{4a^3}(1-e^{-a\theta})^2. \end{aligned}$$

Change all $T - t_0$ to θ and all $B(t_0, T)$ to $B(\theta)$, and calculate the integral of $B(\theta)$, we can simplify the equation:

$$\int_{t_0}^{T} \frac{(1-e^{-a\theta})^2}{a^2} dt = \frac{\theta}{a^2} - \frac{1-e^{-a\theta}}{a^3} + \frac{2b}{a\sigma^2}(1-e^{-a\theta}) - \frac{(1-e^{-a\theta})^2}{2a^3}. \tag{54}$$

the left side can be written as:

$$\int_{t_0}^{T} \frac{(1-e^{-a\theta})^2}{a^2} dt = \frac{\theta}{a^2} - \frac{2}{a^3} e^{-aT}(e^{aT} - e^{at_0}) + \frac{1}{2a^3} e^{-2aT}(e^{2aT} - e^{2at_0}). \tag{55}$$

After some comparison, the correctness is easy to prove.

13 Conclusion

14 Reference

[1]Belal E.Baaquie, Quantum Finance: Path integrals and Hamiltonians for Options and Interest Rates. Cambridge University Press (2004). [2]Zee.A, Quantum field theory in a nutshell. Princeton university press (2010). [3]J. C. Hull, Options, Futures and Other Derivatives. Fifth Edition, Prentice-Hall International (2003). [4]M. Otto, 'Using path integrals to price interest rate derivatives', http://xxx.lanl.gov/cond-mat/9812318. [5]O. Vasicek, 'An Equilibrium Characterization of the Term Structure'. Journal of Financial Economics, 5: 177.