Stochastic Analysis

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1 Foundation

A stochastic process is a collection of $R.V.$: $X = \{X_t; 0 \leq t < \infty\}$ on sample space $(\Omega, \mathcal{F})$, which take values in a second measurable state space $(\mathcal{R}^d, \mathcal{B}(\mathcal{R}^d))$.

2 Understanding $\sigma - algebra$

2.1 Conditional Expectation

$\mathbb{E}[X|\mathcal{G}]$ is the unique random variable that satisfies:

1. $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G} - measurable$
2. $\int_A \mathbb{E}[X|\mathcal{G}](w)d\mathbb{P}(w) = \int_A X(w)d\mathbb{P}(w)$, for all $A \in \mathcal{G}$

3 Ito Integral

Property of $I(t)$:

1. Continuity
2. $\mathcal{F}(t) - measurable$
3. Linearity
4. Martingale
5. $Isometry : \mathbb{E}I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)du$
6. $QV(t) = [I, I](t) = \int_0^t \Delta^2(u)du$

There is a useful exercise on Shreve $P_{151} - 4.4.11$.

4 Risk-Neutral Measure

4.1 Change of Measure

In $(\Omega, \mathcal{F}, \mathbb{P})$, *R.V.* Z is a.s. nonnegative, $\mathbb{E}Z = 1$.
Then for all $A \in \mathcal{F}$, we can define $\widetilde{\mathbb{P}}(A) = \int_A Z(w)d\mathbb{P}(w)$.

4.2 Radon-Nikodym Derivative Process

We have $(\Omega, \mathcal{F}, \mathbb{P})$ and *filtration* $\mathcal{F}(t)$ defined on $0 \leq t \leq T(T$ fixed). *R.V.* Z is a.s. nonnegative, $\mathbb{E}Z = 1$. Define $\widetilde{\mathbb{P}}$ as in previous subsection.

Define the Radon-Nikodym Derivative Process to be $Z(t) = \mathbb{E}[Z|\mathcal{F}(t)]$, $Z(t)$ is a *martingale* with respect to $\mathcal{F}(t)$.

Property of $Z(t)$:

1. if Y is a $\mathcal{F}(t) - measurable$ *R.V.*, then $\widetilde{\mathbb{E}}Y = \mathbb{E}[YZ(t)]$
2. if $0 \leq s \leq t \leq T$, Y is a $\mathcal{F}(t) - measurable$ *R.V.*, then $Z(s)\widetilde{\mathbb{E}}[Y|\mathcal{F}(s)] = \mathbb{E}[YZ(t)|\mathcal{F}(s)]$

4.3 Girsanov Theorem, one dimension

4.4 Martingale Representation Theorem, one dimension

4.5 Application of Risk-Neutral

5 Stochastic Differentiation Equations

5.1 Markov Property

5.2 Feynmann-Kac Theorem, one dimension

5.3 Kolmogorov Backward & Forward Equation

Shreve P_{291}

5.4 Volatility Smile & Surface