Notes on Quantum Physics

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1 Experimental Facts

1.1 The Stern-Gerlach Experiment 1921 $\sim$ 1922

For a heavy atom as a whole, magnetic moment μ is proportional to the electron spin $\overrightarrow{S}$, i.e. $\overrightarrow{\mu} = \frac{e\overrightarrow{S}}{m_e c}$. Place a magnetic field $\overrightarrow{B}$, the interaction energy yields $-\overrightarrow{\mu} \cdot \overrightarrow{B}$. Therefore, if B_z is not homogeneous, then $F_z = \mu_z \partial_z B_z$.

We have a beam of silver atoms(47 electrons in total, with 46 of them having spherical symmetry, no net angular momentum, therefore the atom's angular momentum is sorely decided by the $47th$ electron), going through a $\overrightarrow{B}$ inhomogeneous in B_z. Then atoms will split in z direction according to their spin. We only observe two distinct component on the other side, where $S_z = \pm\frac{\hbar}{2}$.

The experiment suggests quantization of the electron spin angular momentum.

1.2 Sequential SG Experiments

$Oven \Longrightarrow SG\hat{z}$ (filtering S_z-) $\Longrightarrow SG\hat{z} \Longrightarrow S_z+$ only
$Oven \Longrightarrow SG\hat{z}$ (filtering S_z-) $\Longrightarrow SG\hat{x} \Longrightarrow S_x+, S_x-$
$Oven \Longrightarrow SG\hat{z}$ (filtering S_z-) $\Longrightarrow SG\hat{z} \Longrightarrow SG\hat{z} \Longrightarrow S_z+, S_z-$

These results is similar to the polarization of light.

Suppose $x \perp y, x' = \frac{1}{\sqrt{2}}(x+y)$
$\Longrightarrow x - filter \Longrightarrow y - filter \Longrightarrow$ No light
$\Longrightarrow x - filter \Longrightarrow x' - filter \Longrightarrow y - filter \Longrightarrow x_+, x_-$

The experiment suggests that we cannot determine S_z and S_z simultaneously. Previous information is destroyed by the new apparatus. It also suggests the superposition principle.

Further we can use abstact vectors to represent the states in SG experiment on the basis of $|S_z+\rangle$ and $|S_z-\rangle$.

$|S_x\rangle, |S_y\rangle, |S_z\rangle$'s relations:

$$|S_z+\rangle = \frac{1}{\sqrt{2}}(|S_z+\rangle + |S_z-\rangle), \tag{1a}$$

$$|S_z-\rangle = \frac{1}{\sqrt{2}}(-|S_z+\rangle + |S_z-\rangle), \tag{1b}$$

$$|S_y+\rangle = \frac{1}{\sqrt{2}}(|S_z+\rangle + i|S_z-\rangle), \tag{1c}$$

$$|S_y-\rangle = \frac{1}{\sqrt{2}}(|S_z+\rangle - i|S_z-\rangle) \tag{1d}$$

This example is clear to demonstrate the abstractness of vector space. Quantum-mechanical states are to be represented by vectors in an abstract complex vector space.

1.3 Feynman's Remarks on electron's split

See Quantum Field Theory in a Nutshell by A. Zee, Chap 1. Illustrate the motive of path integral.

2 Mathematics

2.1 Dirac, Ket & Bra

2.1.1 Ket $|\alpha\rangle$ & State Vector Space(Ket Space) $\mathcal{H}$

We want it clear in the beginning. When referring to ket vectors, we're speaking of functions in a functional vector space, ex. functions expanded in fourier forms. The dimension of the complex vector space is decided by the physics system's degree of freedom, ex. in the case of an electron's spin(upward & downward), $dim = 2$.

A physical state is represented by a **state vector** in a complex vector space, called **ket**, denoted by $|\alpha\rangle$. All information about that state is contained in the vector. Our postulation of the existence of vectors to represent states already suggests the superposition principle.

An observable, on the other hand, is represented by an operator which acts on the ket, on the left,

$$A \cdot (|\alpha\rangle) = A\,|\alpha\rangle \tag{2}$$

which yields another ket. If all kets of a system forms the space $\mathcal{H}$, an observable in that system is an operator on $\mathcal{H}$(operator defined as the same in linear algebra). Therefore, there should be eigenkets $|\alpha'\rangle$ of A,

$$A\,|\alpha'\rangle = \alpha'\,|\alpha'\rangle \tag{3}$$

Here α' is just a number. It's convention to 'ket' an eigenvalue to stand for the corresponding eigenvector. The physical state corresponding to an eigenket is called an eigenstate:

$$S_z\,|S_z+\rangle = \frac{\hbar}{2}\,|S_z+\rangle\,, S_z\,|S_z-\rangle = -\frac{\hbar}{2}\,|S_z-\rangle \tag{4}$$

Next, we consider a $N - dim$ vector space $\mathcal{H}$, spanned according to the N eigenkets of observable A. Then $\forall\,|\alpha\rangle \in \mathcal{H}$, $|\alpha\rangle = \sum_{\alpha'} c_{\alpha'}\,|\alpha'\rangle$.

2.1.2 Bra $\langle\alpha|$, Dual Space(Bra Space) $\mathcal{H}^*$ and Inner Products

Bra Space is dual to Ket Space. Corresponding to every $|\alpha\rangle$ in $\mathcal{H}$, there exists a bra in dual space $\mathcal{H}^*$, denoted by $\langle\alpha|$.

Inner product is defined as a mapping $i(*,*) : \mathcal{H} \times \mathcal{H} \mapsto \mathbb{C}$. Think of $i(*,*)$ as a machine receiving two kets and returns a complex number. We all know how the inner product should be defined.

Dual space is by now unclear. Yet with $i(*,*)$ we can deduce a dual space with a particular $i(*,*)$ defined. With $i(*,*)$ and $\forall f \in \mathcal{H}$, we have $i(f,*) : \mathcal{H} \mapsto \mathbb{C}$, $i(f,*) \in \mathcal{H}^*$. Moreover, this is indeed a **one-to-one mapping**:

$$\forall \eta \in \mathcal{H}^*, \exists f_\eta \in \mathcal{H}, s.t. \forall g \in \mathcal{H}, \eta(g) = i(f_\eta, g) \tag{5}$$

Denote this mapping $v : \mathcal{H}^* \mapsto \mathcal{H}$, and denote $v^{-1}(\eta) = g_\eta$.

(Right now I can only understand dual space this way, remained to be uncovered.)

Now we know a bra $\langle\eta|$ acts on a ket $|f\rangle$ yields the (g_η, f), which is called the inner product of $\langle\eta|$ and $|f\rangle$ in physics, when mathematically this is inacurate.

In physics, we often put a complex vector in $\mathcal{H}$ into a bra, ex. $\langle\alpha|$, which actually means that $\langle\alpha| = v^{-1}(|\alpha\rangle)$. Though very confusing, we only need to remember finally $\langle\eta|$ acts on $|f\rangle$ yields $(|\eta\rangle, |f\rangle)$.

2.1.3 More Notes about Vector Space

Suppose we take $\mathcal{F}$ the subspace of *Hilbert Space* that we concern.
Discrete orthonormal basis $\{u_i(r)\}$:

1. orthonormalization relation: $\langle u_i|u_j\rangle = \delta_{ij} \Longrightarrow$ normalized and orthogonal
2. closure relation: $\sum_i u_i(r)u_i^*(r') = \delta(r-r') \Longrightarrow$ constitute a basis

Basis outside $\mathcal{F}$:

Consider $v_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$. It is not integrable on $(-\infty, \infty)$. We label members of set $\{v_p(x)\}$ by the continuous index p. So for any state $|\psi\rangle$, $\{v_p(x)\}$ is a set of eigenfunctions for basis $\{|x\rangle\}$. We have:

$$\langle x|\psi\rangle = \psi(x) = \int_{-\infty}^{\infty} dp\widetilde{\psi(p)}v_p(x) \tag{6}$$

Obviously here $\widetilde{\psi(p)} = \langle v_p|\psi\rangle$, which is exactly the fourier transform of $\psi(x)$, i.e. $\widetilde{\psi(p)} = \int_{-\infty}^{\infty} dxv_p^*(x)\psi(x)$. The fourier frequency spectra is equivalent to the projector operator on $|x_i\rangle$ in discrete case.

In a continuous case, the two relations become:

1. orthonormalization relation: $\langle v_p|v_{p'}\rangle = \int_{-\infty}^{\infty} dxv_{p'}(x')v_p^*(x) = \delta(p-p')$
2. closure relation: $\int_{-\infty}^{\infty} dpv_p(x)v_p^*(x') = \delta(x-x')$

2.1.4 Riesz representation theorem

$\varphi_y(x) = \langle x|y\rangle$, when often $\varphi_y(x)$ is simply denoted as $y(x)$.

2.2 Probability & Measurement

Dirac:"A measurement always cause the system jump into an eigenstate of the dynamical variable that is being measured."

$$|\alpha\rangle = \sum_{a'} c_{a'}|\alpha'\rangle = \sum_{a'} |\alpha'\rangle\langle\alpha'|\alpha\rangle \tag{7}$$

Postulate the probability for the system to jump from $|\alpha\rangle$ to $|\alpha'\rangle$ after measurement of $\widehat{A}$ to be $|\langle\alpha'|\alpha\rangle|^2$.

An observable's expectation value w.r.t. state $|\Psi\rangle$ is defined as:

$$\langle\widehat{A}\rangle = \sum_{a'}\sum_{a''} \langle\alpha|\alpha''\rangle\langle\alpha''|\widehat{A}|\alpha'\rangle\langle\alpha'|\alpha\rangle = \sum_{a'} a'|\langle\alpha'|\alpha\rangle|^2 \tag{8}$$

Continuous Version:

$$\langle Q\rangle_\Psi = \int \Psi^*\widehat{Q}\Psi dx = \langle\Psi|\widehat{Q}|\Psi\rangle \tag{9}$$

A selective measurement amounts to the projection operator defined as:

$$\Lambda_{a'}|\alpha\rangle = |\alpha'\rangle\langle\alpha'|\alpha\rangle \tag{10}$$

2.2.1 Compatible Observables

Observables $\widehat{A}$ and $\widehat{B}$ are defined to be compatible when

$$[\widehat{A}, \widehat{B}] = 0 \tag{11}$$

Then there exists a set of simultaneous eigenkets of $\widehat{A}, \widehat{B}$: $|K'\rangle = |a', b'\rangle$.

If $\widehat{A}, \widehat{B}, \widehat{C}$ are incompatible, consider a sequence of selective measurements. The result coming out of the C filter depends on whether or not B measurements have actually been carried out, i.e.

$$\sum_{b'} |\langle c'|b'\rangle|^2 |\langle b'|a'\rangle|^2 \neq |\langle c'|a'\rangle|^2 \tag{12}$$

2.2.2 The Uncertainty Relation

Define an operator:

$$\Delta\widehat{A} = \widehat{A} - \langle\widehat{A}\rangle \tag{13}$$

Then the general uncertainty relation follows from Schwarz inequality:

$$\langle(\widehat{A})^2\rangle \langle(\widehat{B})^2\rangle \geq \frac{1}{4} |\langle[\widehat{A}, \widehat{B}]]\rangle|^2 \tag{14}$$

2.2.3 Translation and Momentum

Define **infinitesimal translation operator** $\mathcal{T}(dx')|x'\rangle = |x' + dx'\rangle$, where the x' is $3 - dim$ by convention.

Expand arbitratry state $|\alpha\rangle$ to examine the effect of $\mathcal{T}(dx')$:

$$\mathcal{T}(dx')|\alpha\rangle = \int d^3x' |x' + dx'\rangle \langle x'|\alpha\rangle = \int d^3x' |x'\rangle \langle x' - dx'|\alpha\rangle \tag{15}$$

If we take $\mathcal{T}(dx') = 1 - iK \cdot dx'$, where K's component K_x, K_y, K_z are all hermitians. Then $\mathcal{T}(dx')$ has the following properties:

1. $\mathcal{T}(dx')\mathcal{T}(dx')^\dagger = 1$
2. $\mathcal{T}(dx')\mathcal{T}(dx'') = \mathcal{T}(dx' + dx'')$
3. $\mathcal{T}(-dx')\mathcal{T}(dx') = 1$
4. $lim_{dx' \to 0}\mathcal{T}(dx') = 1$

We can also derive the following relation:

1. $[x, \mathcal{T}(dx')] = dx'$
2. $[x_i, K_j] = i\delta_{ij}$

Dirac:"Momentum is the generator of an infinitesimal translation."

From the generating function of an infinitesimal translation $F_2(x, P) = x \cdot P + p \cdot dx$, we have

$X = \partial_p F_2 = x + dx, \quad p = \partial_x F_2 = P$

We find that $K = \frac{p}{\hbar}$, i.e. $\frac{2\pi}{\lambda} = \frac{p}{\hbar}$. Therefore, we obtain the final expression for infinitesimal translation operator with exact physical meaning:

$$\mathcal{T}(dx') = 1 - \frac{i}{\hbar} p \cdot dx' \tag{16}$$

The commutation relation now becomes $[x_i, p_j] = i\hbar\delta ij$, together with $[x_i, x_j] = 0, [p_i, p_j] = 0$.

3 Time Evolution

Define the time-evolution operator $\widehat{U(t)}$ for state vector $|\psi(t)\rangle$ to be the operator such that:

$$|\psi(t)\rangle = \widehat{U(t)}\,|\psi(0)\rangle \tag{17}$$

Intuitively, time-evolution operator is the coefficient matrix to express $\psi(t)$ in terms of eigenfunctions.

Define the propagator $K(x'', t; x', t_0)$ such that $\psi(x'', t) = \int d^3x' K(x'', t; x', t_0)\psi(x', t_0)$, where it is given by

$$K(x'', t; x', t_0) = \sum_{a'} \langle x''|a'\rangle \langle a'|x'\rangle\, e^{-iE_{a'}(t-t_0)/\hbar} \tag{18}$$

4 Schrodinger's Equation

Define

$$\widehat{H} = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} + \widehat{V} \tag{19}$$

$$i\hbar\frac{\partial \Psi}{\partial t} = \widehat{H}\Psi \tag{20}$$

By separation of variable, $\psi(x, t) = \psi(x)\varphi(t)$, we obtain:

$$\frac{d\varphi(t)}{dt} = -\frac{iE}{\hbar}\varphi \tag{21a}$$

$$\widehat{H}\psi = E\psi \tag{21b}$$

which yields: time-independent solution $\Psi(t, x) = \psi(x)e^{-\frac{iEt}{\hbar}}$:

$$|\Psi(x, t)|^2 = |\psi(x)|^2 \tag{22a}$$

$$\langle\widehat{H}\rangle = \int \psi^* \widehat{H}\psi dx = E\int |\Psi|^2 dx = E \tag{22b}$$

$$\langle\widehat{H}^2\rangle = E^2 \tag{22c}$$

$$\sigma_H^2 = 0 \tag{22d}$$

5 Appendix

Useful formulas:
(x_is are the zeros of $f(x)$, $f(a)$ is the minimum of $f(x)$)

$$\delta(x) = \int_{\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tag{23a}$$

$$\int_{\infty}^{\infty} dx\delta(f(x))s(x) = \sum_i \frac{s(x_i)}{|f'(x_i)|} \tag{23b}$$

$$\int_{\infty}^{\infty} dq e^{-f(q)/\hbar} = e^{-f(a)/\hbar}(\frac{2\pi\hbar}{f''(a)})^{\frac{1}{2}} e^{-O(\hbar^{\frac{1}{2}})} \tag{23c}$$