

Black-Scholes Formula, a physicist's perspective

1 (a)

Rewritten Using *Brownian Motion*:

$$ds(t) = \phi s(t)dt + \sigma s(t)dW(t). \quad (1)$$

where $W(t)$ is a standard brownian motion.

To illustrate the relation between **Gaussian Noise** and **Brownian Motion**, consider when using $R(t)$, we're actually suggesting $s(t + \epsilon) = s(t) + \phi s(t)\epsilon + \sigma R\epsilon$. In this case, $R \sim \mathcal{N}(0, \frac{1}{\epsilon})$, therefore $R\epsilon \sim \mathcal{N}(0, \epsilon)$, which can be characterized as $W(t + \epsilon) - W(t)$. As $\epsilon \rightarrow 0$, $W(t + \epsilon) - W(t) \rightarrow dW(t)$. (It's really clearer to use *brownian motion* notation.) Brownian motion has the property that $dW(t)dW(t) = dt$, $dW(t)dt = 0$.

The original statement can be rewritten as:

$$df(t, s(t)) = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t). \quad (2)$$

According to *Taylor expansion formula*, we can write

$$df = f_t dt + f_s ds + \frac{1}{2}\{f_{tt} dt^2 + (f_{ts} + f_{st}) dt ds(t) + f_{ss} ds(t) ds(t)\} + o(dt^2) \quad (3)$$

$$\implies df = f_t dt + f_s ds + \frac{1}{2} f_{ss} ds(t) ds(t) \quad (4)$$

Considering $ds(t) = \phi s(t)dt + \sigma s(t)dW(t)$, $dW(t)dW(t) = dt$, $dW(t)dt = 0$, we have

$$df = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t) \quad (5)$$

Using the relationship between $W(t)$ and $R(t)$, we can change $dW(t)$ and derive:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} + \frac{\partial f}{\partial s}(\phi s + \sigma s R). \quad (6)$$

2 (b)

We've already known that $c = c(t, s(t))$. Consider a portfolio $\Pi = c - \frac{\partial c}{\partial s}s$. To calculate the derivative of Π we can write down the differential of Π :

$$d\Pi = c_t dt + c_s ds + \frac{1}{2}c_{ss}dsds - c_s ds. \quad (7)$$

Because we've got $dS^2 = \sigma^2 s^2 dt$, we can derive:

$$\frac{d\Pi}{dt} = c_t + \frac{1}{2}c_{ss}\sigma^2 s^2. \quad (8)$$

3 (c)

r means the short-term risk-free interest rate. The following equation must be true:

$$\frac{d\Pi}{dt} = r\Pi. \quad (9)$$

Otherwise arbitrage will exist, which contradicts our assumption. Therefore we can write $d\Pi = r(c - c_s s)dt$. It's equivalent to the previous result $c_t dt + \frac{1}{2}c_{ss}\sigma^2 s^2 dt$.

At last we have the **Black-Scholes formula**:

$$c_t + \frac{1}{2}\sigma^2 s^2 c_{ss} + rsc_s - rc = 0. \quad (10)$$

Another Way of Obtaining B-S Formula

According to non-arbitrage postulate, if at time 0, $c(0, S(0)) = c_0$, then we should be able to construct a portfolio $X(t)$ (with $X(0) = c_0$) to replicate exactly this option. We should use the underlying stock $S(t)$ and the money market with interest rate r . Suppose we hold $\Delta(t)$ share of stock at time t , then it follows:

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt = D_t dt + D_w dW(t) \quad (11)$$

where $D_t = \Delta(t)S(t)(\phi - r) + rX(t)$, $D_w = \Delta(t)S(t)\sigma$.

At the same time,

$$dc(t, S(t)) = c_t dt + c_s dS(t) + \frac{1}{2}c_{ss}dS(t)dS(t) = D'_t dt + D'_w dW(t) \quad (12)$$

where $D'_t = c_t + c_s \phi S(t) + \frac{1}{2}\sigma^2 S(t)^2 c_{ss}$, $D'_w = c_s \sigma S(t)$.

Therefore, it should follow that $D_t = D'_t$, $D_w = D'_w$. The second relation yields instantly that $\Delta(t) = c_s$, while the first would amount to the Black-Scholes formula:

$$c_t + \frac{1}{2}\sigma^2 s^2 c_{ss} + rsc_s - rc = 0 \quad (13)$$

4 (d)

Change variable $s = e^x$, we have:

$$c_x = e^{-x} c_s. \quad (14)$$

$$c_t = rc - rsc_s - \frac{1}{2}\sigma^2 s^2 c_{ss} = \left(r - \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x} - \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\right)c. \quad (15)$$

Therefore we can easily prove that

$$H_{BS} = \left(1 - \frac{\partial}{\partial x}\right)\left(r + \frac{1}{2}\sigma^2 \frac{\partial}{\partial x}\right) = -\frac{\sigma^2}{2} \frac{\partial^2 c}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right) \frac{\partial c}{\partial x} + rc. \quad (16)$$

which is the Hamiltonian for Black-Scholes model.

5 (e)

$$p_{BS}(x, \tau; x') = \langle x | e^{-\tau H} | x' \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle x | e^{-\tau H} | p \rangle \langle p | x' \rangle \quad (17)$$

Taking $p = i \frac{\partial}{\partial x}$, using $\langle x | p \rangle = e^{ipx}$:

$$p_{BS}(x, \tau; x') = e^{-r\tau} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left\{-\frac{1}{2}\sigma^2 p^2 \tau + ip(x - x') + ip\tau\left(r - \frac{\sigma^2}{2}\right)\right\} \quad (18)$$

Finally, perform the Gaussian integration:

$$p_{BS}(x, \tau; x') = e^{-r\tau} \frac{1}{\sqrt{2\pi\tau\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2\tau}(x - x' + \tau(r - \frac{\sigma^2}{2}))^2\right\} \quad (19)$$

6 (f)

We've known that $P(x, T-t, x')$ is the conditional probability density that, given security price x at time t , it will have a value of x' at time T .

The expectation of x' at time τ can therefore be calculated (it's a simple convolution):

$$\langle x(\tau) \rangle = \int_{-\infty}^{+\infty} x' P_{BS}(x, \tau; x') dx' = \frac{e^{-r\tau}}{\sqrt{2\pi\tau\sigma^2}} \int_{-\infty}^{+\infty} x' e^{-\frac{1}{2\tau\sigma^2}[x-x'+\tau(r-\frac{\sigma^2}{2})]^2} dx'. \quad (20)$$

Finally we can derive:

$$\langle x(\tau) \rangle = \left[x + \tau\left(r - \frac{\sigma^2}{2}\right)\right] e^{-r\tau}. \quad (21)$$

The result tells us the evolution of x (which equals $\ln(S)$) over time.

7 (g)

Consider a down-and-out barrier European option. If $s \leq e^B (x \leq B)$ it will become worthless, which means $c = 0$.

For an arbitrary potential $v(x)$, we have: $\frac{\partial c}{\partial t} = Hc$, where H can be written as:

$$H = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right) \frac{\partial}{\partial x} + V(x). \quad (22)$$

Obviously, if we set $V(x) = \infty$ when $x \leq B$, then $c = 0$ is automatically satisfied. However, when $x > B$, the changing of c still conform to the B-S formula. comparing with H_{BS} , we have: $V(x) = \begin{cases} +\infty, & x \leq B. \\ r, & x > B. \end{cases}$

8 (i)

We know that $r(t)$ satisfied Langevin equation:

$$\frac{dr}{dt} = a(r, t) + \sigma(r, t)R(t). \quad (23)$$

Here $R(t)$ is still Gaussian noise.

Define the propagator $P(r, t; r_0)$: if $r(t_0) = r_0$, the probability of $r(t)=r$ equals $P(r, t; r_0)$.

From the Langevin equation we have:

$$r(t + \varepsilon) = r(t) + \varepsilon[a + \sigma R(t)]. \quad (24)$$

Change it into the following formula:

$$r = r' + \varepsilon[a(r') + \sigma(r')R(t)]. \quad (25)$$

Thus we can calculate the propagator:

$$P(r, t + \varepsilon, r_0) = P(r', t; r_0)|_{r'=r-\varepsilon[a(r')+\sigma(r')R(t)]} = \int P(r', t; r_0) \delta(r - r' - \varepsilon[a(r') + \sigma(r')R(t)]) dr' \simeq \int P(r', t; r_0) \delta(r - r' - \varepsilon a(r')) dr' \quad (26)$$

Because $\langle R^2(t) \rangle = \frac{1}{\varepsilon}$ and $\langle R(t) \rangle = 0$, the previous formula can be re-written like this:

$$P(r, t + \varepsilon, r_0) = P(r', t; r_0) + \int dr' P(r', t; r_0) \left\{ \frac{\partial \delta(r - r')}{\partial r'} \varepsilon a(r') + \frac{1}{2} \frac{\partial^2 \delta(r - r')}{\partial r'^2} \varepsilon^2 \sigma^2(r') \frac{1}{\varepsilon} \right\} = P(r', t; r_0) + \dots \quad (27)$$

If a variable is $o(\varepsilon)$, it is automatically neglected.

Thus, from the definition of derivative, we have:

$$\frac{\partial P(r, t; r_0)}{\partial t} = \left[\frac{1}{2} \frac{\partial^2}{\partial r^2} \sigma^2(r) - \frac{\partial}{\partial r} a(r) \right] P(r, t; r_0). \quad (28)$$

We've already known that:

$$\frac{\partial P(r, t; r_0)}{\partial t} = -H_F P(r, t; r_0). \quad (29)$$

Therefore we can prove:

$$H_F = -\frac{1}{2} \frac{\partial^2}{\partial r^2} \sigma^2(r) + \frac{\partial}{\partial r} a(r) = -\frac{1}{2} \frac{\partial^2}{\partial r^2} \sigma^2(r) + a(r) \frac{\partial}{\partial r} + \frac{\partial a(r)}{\partial r}. \quad (30)$$

From a different perspective, we define $P_B(R, t; r)$ as the back propagator. Similarly we have(since the time flows backwards this time):

$$\frac{\partial P_B(R, t; r)}{\partial t} = +H_B P_B(R, t; r). \quad (31)$$

Finally we can calculate:

$$H_B = -\frac{1}{2} \sigma^2(r) \frac{\partial^2}{\partial r^2} - a(r) \frac{\partial}{\partial r}. \quad (32)$$

$H_B = H_F^\dagger$. is obvious.

9 (j)

In this part we'll focus on the so-called "**stochastic Quantization**".

The Langevin equation:

$$\frac{dr}{dt} = a(r, t) + \sigma(r, t) R(t) \quad (33)$$

is satisfied at any time between t_0 and T . We must consider that both r and R are stochastic variables. Therefore when calculating z_B (can be compared to partition function in statistical mechanics), we must integrate over all possible paths

$$Z_B = \int DR Dr \prod_{t=t_0}^T \delta\left[\frac{dr}{dc} - a(r, t) - \sigma(r, t) R(t)\right] e^{-\frac{1}{2} \int_{t_0}^T R^2(t) dt} \quad (34)$$

Dr means integrating over all possible $r(t) : \int Dr = \int_{-\infty}^{+\infty} \prod_{t=t_0}^T dr(t)$.

We hope to write Z_B in the form of $z_B = \int Dr e^{S_B}$. Because of this, we calculate the integral over R first:

$$Z_B = \int Dr \exp\left(-\frac{1}{2} \int_{t_0}^T \frac{\left[\frac{dr}{dt} - a(r, t)\right]^2}{\sigma^2(r, t)} dt\right). \quad (35)$$

Because the *Dirac* δ function makes sure we only keep $R(t)$ that satisfies the Langevin equation: $R(t) = \frac{\frac{dr}{dt} - a(r, t)}{\sigma(r, t)}$.

It's easy to calculate S_B :

$$S_B = -\frac{1}{2} \int_{t_0}^T \frac{\left[\frac{dr}{dt} - a(r, t)\right]^2}{\sigma^2(r, t)} dt. \quad (36)$$

From the definition of $S_B : S_B = \int_{t_0}^T L dt$, it's clear that:

$$L = -\frac{\left[\frac{dr}{dt} - a(r, t)\right]^2}{2\sigma^2(r, t)} dt. \quad (37)$$

10 (k)

The Vasicek model can be described using the following equation:

$$\frac{dr}{dt} = a(b - r) + \sigma R(t). \quad (38)$$

Note that if we set $\begin{cases} a(r, t) = a(b - r). \\ \sigma(r, t) = \sigma. \end{cases}$ in Langevin equation (a, b, σ are all constants), we have the Vasicek model naturally.

Thus we get:

$$L_V = -\frac{\left[\frac{dr}{dt} - a(b - r)\right]^2}{2\sigma^2}. \quad (39)$$

$$S_V = -\frac{1}{2\sigma^2} \int_{t_0}^T \left[\frac{dr}{dt} - a(b - r)\right]^2 dt. \quad (40)$$

In the next section we'll use S_V to calculate the propagator of Vasicek model.