

Black-Scholes Formula, a physicist's perspective

1 (a)

Rewritten Using *Brownian Motion*:

$$ds(t) = \phi s(t)dt + \sigma s(t)dW(t). \quad (1)$$

where $W(t)$ is a standard brownian motion.

To illustrate the relation between **Gaussian Noise** and **Brownian Motion**, consider when using $R(t)$, we're actually suggesting $s(t + \epsilon) = s(t) + \phi s(t)\epsilon + \sigma R\epsilon$. In this case, $R \sim \mathcal{N}(0, \frac{1}{\epsilon})$, therefore $R\epsilon \sim \mathcal{N}(0, \epsilon)$, which can be characterized as $W(t + \epsilon) - W(t)$. As $\epsilon \rightarrow 0$, $W(t + \epsilon) - W(t) \rightarrow dW(t)$. (It's really clearer to use *brownian motion* notation.) Brownian motion has the property that $dW(t)dW(t) = dt$, $dW(t)dt = 0$.

The original statement can be rewritten as:

$$df(t, s(t)) = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t). \quad (2)$$

According to *Taylor expansion formula*, we can write

$$df = f_t dt + f_s ds + \frac{1}{2}\{f_{tt} dt^2 + (f_{ts} + f_{st}) dt ds(t) + f_{ss} ds(t) ds(t)\} + o(dt^2) \quad (3)$$

$$\implies df = f_t dt + f_s ds + \frac{1}{2}f_{ss} ds(t) ds(t) \quad (4)$$

Considering $ds(t) = \phi s(t)dt + \sigma s(t)dW(t)$, $dW(t)dW(t) = dt$, $dW(t)dt = 0$, we have

$$df = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t) \quad (5)$$

Using the relationship between $W(t)$ and $R(t)$, we can change $dW(t)$ and derive:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} + \frac{\partial f}{\partial s}(\phi s + \sigma s R). \quad (6)$$

2 (b)

We've already known that $c = c(t, s(t))$. Consider a portfolio $\Pi = c - \frac{\partial c}{\partial s}s$. To calculate the derivative of Π we can write down the differential of Π :

$$d\Pi = c_t dt + c_s ds + \frac{1}{2}c_{ss}dsds - c_s ds. \quad (7)$$

Because we've got $dS^2 = \sigma^2 s^2 dt$, we can derive:

$$\frac{d\Pi}{dt} = c_t + \frac{1}{2}c_{ss}\sigma^2 s^2. \quad (8)$$

3 (c)

r means the short-term risk-free interest rate. The following equation must be true:

$$\frac{d\Pi}{dt} = r\Pi. \quad (9)$$

Otherwise arbitrage will exist, which contradicts our assumption. Therefore we can write $d\Pi = r(c - c_s s)dt$. It's equivalent to the previous result $c_t dt + \frac{1}{2}c_{ss}\sigma^2 s^2 dt$.

At last we have the **Black-Scholes formula**:

$$c_t + \frac{1}{2}\sigma^2 s^2 c_{ss} + rsc_s - rc = 0. \quad (10)$$

Another Way of Obtaining B-S Formula

According to non-arbitrage postulate, if at time 0, $c(0, S(0)) = c_0$, then we should be able to construct a portfolio $X(t)$ (with $X(0) = c_0$) to replicate exactly this option. We should use the underlying stock $S(t)$ and the money market with interest rate r . Suppose we hold $\Delta(t)$ share of stock at time t , then it follows:

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt = D_t dt + D_w dW(t) \quad (11)$$

where $D_t = \Delta(t)S(t)(\phi - r) + rX(t)$, $D_w = \Delta(t)S(t)\sigma$.

At the same time,

$$dc(t, S(t)) = c_t dt + c_s dS(t) + \frac{1}{2}c_{ss}dS(t)dS(t) = D'_t dt + D'_w dW(t) \quad (12)$$

where $D'_t = c_t + c_s \phi S(t) + \frac{1}{2}\sigma^2 S(t)^2 c_{ss}$, $D'_w = c_s \sigma S(t)$.

Therefore, it should follow that $D_t = D'_t$, $D_w = D'_w$. The second relation yields instantly that $\Delta(t) = c_s$, while the first would amount to the Black-Scholes formula:

$$c_t + \frac{1}{2}\sigma^2 s^2 c_{ss} + rsc_s - rc = 0 \quad (13)$$

4 (d)

Change variable $s = e^x$, we have:

$$c_x = e^{-x} c_s. \quad (14)$$

$$c_t = rc - rsc_s - \frac{1}{2}\sigma^2 s^2 c_{ss} = (r - (r - \frac{1}{2}\sigma^2) \frac{\partial}{\partial x} - \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2})c. \quad (15)$$

Therefore we can easily prove that

$$H_{BS} = (1 - \frac{\partial}{\partial x})(r + \frac{1}{2}\sigma^2 \frac{\partial}{\partial x}) = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + (\frac{\sigma^2}{2} - r) \frac{\partial}{\partial x} + r. \quad (16)$$

which is the Hamiltonian for Black-Scholes model.

5 (e)

$$p_{BS}(x, \tau; x') = \langle x | e^{-\tau H} | x' \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle x | e^{-\tau H} | p \rangle \langle p | x' \rangle \quad (17)$$

Taking $p = i \frac{\partial}{\partial x}$, using $\langle x | p \rangle = e^{ipx}$:

$$p_{BS}(x, \tau; x') = e^{-r\tau} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\{-\frac{1}{2}\sigma^2 p^2 \tau + ip(x - x') + ip\tau(r - \frac{\sigma^2}{2})\} \quad (18)$$

Finally, perform the Gaussian integration:

$$p_{BS}(x, \tau; x') = e^{-r\tau} \frac{1}{\sqrt{2\pi\tau\sigma^2}} \exp\{-\frac{1}{2\sigma^2\tau}(x - x' + \tau(r - \frac{\sigma^2}{2}))^2\} \quad (19)$$

6 (f)

After obtaining the pricing kernel, we can get the price of call option at time t simply by integrating the final value with the kernel.

$$c(\tau, x) = \int_{-\infty}^{\infty} g(x') P_{BS} dx' \quad (20)$$

where in this case, $g(x') = (e^{x'} - K)^+$. $(x)^+$ takes x when $x > 0$ and takes 0 when $x \leq 0$. Therefore, the integration only takes place in $x \in (\ln K, \infty)$.

After noticing, the pricing kernel is actually $e^{-r\tau}$ mutlipying a normal distribution, we can define $d_-(\tau, x) = \frac{1}{\sigma\tau}[\frac{x}{\ln K} + (r - \frac{1}{2}\sigma^2)\tau]$, $d_+ = d_- + \sigma\sqrt{\tau}$, $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{x^2}{2}}$. We separate the integration into two parts: K and e^x .

$$\int_{\ln K}^{\infty} K P_{BS} dx' = e^{-r\tau} K N(d_-(\tau, x)) \quad (21a)$$

$$\int_{\ln K}^{\infty} e^x P_{BS} dx' = e^x N(d_+(\tau, x)) \quad (21b)$$

Remember that $s = e^x$, we have the pricing formula for $c(t, s)$:

$$c(t, s) = sN(d_+(\tau, s)) - e^{-r\tau} K N(d_-(\tau, s)) \quad (22)$$

7 (g)

Consider a down-and-out barrier European option. If $s \leq e^B$ ($x \leq B$) it will become worthless, which means $c = 0$.

For an arbitrary potential $v(x)$, we have: $\frac{\partial c}{\partial t} = Hc$, where H can be written as:

$$H = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right) \frac{\partial}{\partial x} + V(x). \quad (23)$$

Obviously, if we set $V(x) = \infty$ when $x \leq B$, then $c = 0$ is automatically satisfied. However, when $x > B$, the changing of c still conform to the B-S formula. comparing with H_{BS} , we have: $V(x) = \begin{cases} +\infty, & x \leq B. \\ r, & x > B. \end{cases}$

8 (h)

The Hamiltonian for the down and out option is given by:

$$H_{DO} = H_{BS} + V(x) = \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right) \frac{\partial}{\partial x} + V(x) \quad (24)$$

$V(x)$ is defined in (g).

Now we can construct the eigenfunctions similar to the *Black – Scholes* case except that these must satisfy $\Psi_E(B) = 0$.

Define the quantities:

$$\beta = \frac{(\frac{\sigma^2}{2} + r)^2}{\sigma^4}, p = \sqrt{\frac{2E}{\sigma^2} - \beta}, \alpha = \frac{\frac{\sigma^2}{2} - r}{\sigma^2}, i\lambda_{\pm} = \alpha \pm ip \quad (25)$$

The eigenfunctions are given by:

$$x > B : \langle x | \Psi_E \rangle = e^{i\lambda_+(x-B)} - e^{i\lambda_-(x-B)} = 2ie^{\alpha(x-B)} \sin[p(x-B)] \quad (26)$$

$$\langle \widetilde{\Psi}_E | x \rangle = e^{-i\lambda_+(x-B)} - e^{-i\lambda_-(x-B)} = -2ie^{-\alpha(x-B)} \sin[p(x-B)] \quad (27)$$

$$\langle \widetilde{\Psi}_E | \Psi_{E'} \rangle = [2\pi\sigma^2 \sqrt{2E/\sigma^2 - \beta}] \delta(E - E'). \quad (28)$$

$$x \leq B : \langle x | \Psi_E \rangle = 0 = \langle \widetilde{\Psi}_E | x \rangle. \quad (29)$$

Now we can use the eigenfunctions to evaluate the pricing kernel. Again we work with the p variable:

$$\begin{aligned} p_{DO}(x, \tau; x') &= \langle x | e^{-\tau H_{DO}} | x' \rangle \\ &= e^{-\frac{\tau\beta\sigma^2}{2} + \alpha(x-x')} \int_0^\infty \frac{dp}{2\pi} e^{-\frac{1}{2}\tau\sigma^2 p^2} [e^{ip(x-x')} + e^{-ip(x-x')} \\ &\quad - e^{ip(x+x'-2B)} - e^{-ip(x+x'-2B)}]. \end{aligned}$$

Consider that we have already derived:

$$p_{BS}(x, \tau; x') = e^{-r\tau} \frac{1}{\sqrt{2\pi\tau\sigma^2}} e^{-\frac{1}{2\tau\sigma^2} [x-x'+\tau(r-\frac{\sigma^2}{2})]^2}. \quad (30)$$

We can simplify the pricing kernel using the previous results:

$$p_{DO}(x, \tau; x') = p_{BS}(x, \tau; x') - \left(\frac{e^x}{e^B}\right)^{2\alpha} p_{BS}(2B - x, \tau; x'). \quad (31)$$

9 (i)

We know that $r(t)$ satisfied Langevin equation:

$$\frac{dr}{dt} = a(r, t) + \sigma(r, t)R(t). \quad (32)$$

Here $R(t)$ is still Gaussian noise.

Define the propagator $P(r, t; r_0)$: if $r(t_0) = r_0$, the probability of $r(t) = r$ equals $P(r, t; r_0)$.

From the Langevin equation we have:

$$r(t + \varepsilon) = r(t) + \varepsilon[a + \sigma R(t)]. \quad (33)$$

Change it into the following formula:

$$r = r' + \varepsilon[a(r') + \sigma(r')R(t)]. \quad (34)$$

Thus we can calculate the propagator:

$$\begin{aligned}
P(r, t + \varepsilon, r_0) &= P(r', t; r_0)|_{r'=r-\varepsilon[a(r')+\sigma(r')R(t)]} \\
&= \int P(r', t; r_0) \delta(r - r' - \varepsilon[a(r') + \sigma(r')R(t)]) dr' \\
&\simeq \int P(r', t; r_0) dr' \left\{ \delta(r - r') + \frac{\partial \delta(r - r')}{\partial r'} \varepsilon[a(r') + \sigma(r')R(t)] \right. \\
&\quad \left. + \frac{1}{2} \frac{\partial^2 \delta(r - r')}{\partial r'^2} \varepsilon^2[a(r') + \sigma(r')R(t)]^2 + \dots \right\}.
\end{aligned}$$

Because $\langle R^2(t) \rangle = \frac{1}{\varepsilon}$ and $\langle R(t) \rangle = 0$, the previous formula can be re-written like this:

$$\begin{aligned}
P(r, t + \varepsilon, r_0) &= P(r', t; r_0) + \int dr' P(r', t; r_0) \left\{ \frac{\partial \delta(r - r')}{\partial r'} \varepsilon a(r') + \frac{1}{2} \frac{\partial^2 \delta(r - r')}{\partial r'^2} \varepsilon^2 \sigma^2(r') \frac{1}{\varepsilon} \right\} \\
&= P(r', t; r_0) - \varepsilon \frac{\partial}{\partial r} [a(r) P(r, t; r_0)] + \frac{\varepsilon}{2} \frac{\partial^2}{\partial r^2} [\sigma(r)^2 P(r, t; r_0)].
\end{aligned}$$

If a variable is $o(\varepsilon)$, it is automatically neglected.

Thus, from the definition of derivative, we have:

$$\frac{\partial P(r, t; r_0)}{\partial t} = \left[\frac{1}{2} \frac{\partial^2}{\partial r^2} \sigma^2(r) - \frac{\partial}{\partial r} a(r) \right] P(r, t; r_0). \quad (35)$$

We've already known that:

$$\frac{\partial P(r, t; r_0)}{\partial t} = -H_F P(r, t; r_0). \quad (36)$$

Therefore we can prove:

$$H_F = -\frac{1}{2} \frac{\partial^2}{\partial r^2} \sigma^2(r) + \frac{\partial}{\partial r} a(r) = -\frac{1}{2} \frac{\partial^2}{\partial r^2} \sigma^2(r) + a(r) \frac{\partial}{\partial r} + \frac{\partial a(r)}{\partial r}. \quad (37)$$

From a different perspective, we define $P_B(R, t; r)$ as the back propagator.

Similarly we have(since the time flows backwards this time):

$$\frac{\partial P_B(R, t; r)}{\partial t} = +H_B P_B(R, t; r). \quad (38)$$

Finally we can calculate:

$$H_B = -\frac{1}{2} \sigma^2(r) \frac{\partial^2}{\partial r^2} - a(r) \frac{\partial}{\partial r}. \quad (39)$$

$H_B = H_F^\dagger$ is obvious.

10 (j)

In this part we'll focus on the so-called **stochastic Quantization**.

The Langevin equation:

$$\frac{dr}{dt} = a(r, t) + \sigma(r, t)R(t) \quad (40)$$

is satisfied at any time between t_0 and T . We must consider that both r and R are stochastic variables. Therefore when calculating z_B (can be compared to partition function in statistical mechanics), we must integrate over all possible paths

$$Z_B = \int DRDr \prod_{t=t_0}^T \delta\left[\frac{dr}{dt} - a(r, t) - \sigma(r, t)R(t)\right] e^{-\frac{1}{2} \int_{t_0}^T R^2(t) dt} \quad (41)$$

Dr means integrating over all possible $r(t) : \int Dr = \int_{-\infty}^{+\infty} \prod_{t=t_0}^T dr(t)$.

We hope to write Z_B in the form of $z_B = \int Dre^{S_B}$. Because of this, we calculate the integral over R first:

$$Z_B = \int Dr \exp\left(-\frac{1}{2} \int_{t_0}^T \frac{\left[\frac{dr}{dt} - a(r, t)\right]^2}{\sigma^2(r, t)} dt\right). \quad (42)$$

Because the *Dirac* δ function makes sure we only keep $R(t)$ that satisfies the Langevin equation: $R(t) = \frac{\frac{dr}{dt} - a(r, t)}{\sigma(r, t)}$.

It's easy to calculate S_B :

$$S_B = -\frac{1}{2} \int_{t_0}^T \frac{\left[\frac{dr}{dt} - a(r, t)\right]^2}{\sigma^2(r, t)} dt. \quad (43)$$

From the definition of $S_B : S_B = \int_{t_0}^T L dt$, it's clear that:

$$L = -\frac{\left[\frac{dr}{dt} - a(r, t)\right]^2}{2\sigma^2(r, t)} dt. \quad (44)$$

11 (k)

The Vasicek model can be described using the following equation:

$$\frac{dr}{dt} = a(b - r) + \sigma R(t). \quad (45)$$

Note that if we set $\begin{cases} a(r, t) = a(b - r). \\ \sigma(r, t) = \sigma. \end{cases}$ in Langevin equation(a, b, σ are all constants), we have the Vasicek model naturally.

Thus we get:

$$L_V = -\frac{[\frac{dr}{dt} - a(b - r)]^2}{2\sigma^2}. \quad (46)$$

$$S_V = -\frac{1}{2\sigma^2} \int_{t_0}^T [\frac{dr}{dt} - a(b - r)]^2 dt. \quad (47)$$

In the next section we'll use S_V to calculate the propagator of Vasicek model.

12 (I)

For different paths(which all have different S_V), so that e^{S_V}/Z is the distribution of probability. $Z = \int D r e^{S_V}$. Given by the question itself, we know that the propagator equals:

$$P(t_0, T) = \frac{1}{Z} \int D r e^{S_V} e^{-\int_{t_0}^T r(t) dt}. \quad (48)$$

If we use a new variable $S = S_V - \int_{t_0}^T r(t) dt$, then we have:

$$P(t_0, T) = \frac{1}{Z} \int D r e^S. \quad (49)$$

Change the variable: $u = r - b$. Thus we can write:

$$\begin{aligned} S &= -\frac{1}{2\sigma^2} \int_{t_0}^T dt \left[\frac{du}{dt} + au \right]^2 - \int_{t_0}^T (u + b) dt \\ &= -\frac{1}{2\sigma^2} \int_{t_0}^T dt \left[\frac{dr}{dt} + ar \right]^2 - \int_{t_0}^T (r + b) dt. \end{aligned}$$

Next, we can define $v(t) = ar(t) + \frac{dr(t)}{dt}$. View this formula as a differential equation of $r(t)$:

$$\frac{dr}{dt} + ar = v(t). \quad (50)$$

The solution can be easily derived:

$$r(t) = e^{-a(t-t_0)} r_0 + e^{-at} \int_{t_0}^t e^{at'} v(t') dt'. \quad (51)$$

Since we want to calculate the integral of $r(t)$, we now have:

$$\int_{t_0}^T r(t)dt = B(t_0, T)r_0 + \int_{t_0}^T B(t, T)v(t)dt. \quad (52)$$

where $B(t, T) = \frac{1-e^{-a(T-t)}}{a}$.

From the definition of $v(t)$ we can tell that $v(T)$ is free to take all possible values. Therefore we can use the following path integral to calculate propagator:

$$\begin{aligned} P(t_0, T) &= e^{-b(T-t_0)-B(t_0, T)r_0} \frac{1}{Z} \int Dv e^{-\frac{1}{2\sigma^2} \int_{t_0}^T dt [v(t)^2 + 2\sigma^2 B(t, T)v(t)]} \\ &= e^{-b(T-t_0)-B(t_0, T)r_0} e^{\frac{\sigma^2}{2} \int_{t_0}^T dt B(t, T)^2}. \end{aligned}$$

The $v(t)$ integrations are decoupled Gaussian integrations, with the overall normalization being canceled by the factor Z .

Finally, assume $B(\theta) = \frac{1-e^{-a\theta}}{a}$, where $\theta = T - t_0$, and $A(\theta) = \exp[(\frac{\sigma^2}{2a^2} - b)(\theta - B(\theta)) - \frac{\sigma^2}{4a} B(\theta)^2]$.

$$A(\theta)e^{-r_0 B(\theta)} = \exp[-b\theta - r_0 \frac{1-e^{-a\theta}}{a} + \frac{\sigma^2}{2a^2} \theta - \frac{\sigma^2(1-e^{-a\theta})}{2a^3} + \frac{b}{a}(1-e^{-a\theta}) - \frac{\sigma^2}{4a^3}(1-e^{-a\theta})^2]. \quad (53)$$

If we want to prove $P(t_0, T) = A(\theta)e^{-r_0 B(\theta)}$, then we must prove:

$$\begin{aligned} -b(T-t_0) - B(t_0, T)r_0 + \frac{\sigma^2}{2} \int_{t_0}^T dt B(t, T)^2 &= -b\theta - r_0 \frac{1-e^{-a\theta}}{a} + \frac{\sigma^2}{2a^2} \theta \\ &\quad - \frac{\sigma^2(1-e^{-a\theta})}{2a^3} + \frac{b}{a}(1-e^{-a\theta}) - \frac{\sigma^2}{4a^3}(1-e^{-a\theta})^2. \end{aligned}$$

Change all $T - t_0$ to θ and all $B(t_0, T)$ to $B(\theta)$, and calculate the integral of $B(\theta)$, we can simplify the equation:

$$\int_{t_0}^T \frac{(1-e^{-a\theta})^2}{a^2} dt = \frac{\theta}{a^2} - \frac{1-e^{-a\theta}}{a^3} + \frac{2b}{a\sigma^2}(1-e^{-a\theta}) - \frac{(1-e^{-a\theta})^2}{2a^3}. \quad (54)$$

the left side can be written as:

$$\int_{t_0}^T \frac{(1-e^{-a\theta})^2}{a^2} dt = \frac{\theta}{a^2} - \frac{2}{a^3} e^{-aT} (e^{aT} - e^{at_0}) + \frac{1}{2a^3} e^{-2aT} (e^{2aT} - e^{2at_0}). \quad (55)$$

After some comparison, the correctness is easy to prove.

13 Conclusion

14 Reference

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