Black-Scholes Formula, a physicist's perspective

1 (a)

Rewritten Using $Brownian\ Motion$:
$ds(t) = \phi s(t)dt + \sigma s(t)dW(t)$, where $W(t)$ is a standard brownian motion.

To illustrate the relation between **Gaussian Noise** and **Brownian Motion**, consider when using $R(t)$, we're actually suggesting $s(t+\epsilon) = s(t) + \phi s(t)\epsilon + \sigma R\epsilon$. In this case, $R \sim \mathcal{N}(0, \frac{1}{\epsilon})$, therefore $R\epsilon \sim \mathcal{N}(0, \epsilon)$, which can be characterized as $W(t+\epsilon) - W(t)$. As $\epsilon \to 0$, $W(t+\epsilon) - W(t) \to dW(t)$. (It's really clearer to use *brownian motion* notation.) Brownian motion has the property that $dW(t)dW(t) = dt$, $dW(t)dt = 0$.

The original statement can be rewritten as:
$df(t, s(t)) = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t)$

According to $Taylor\ expansion\ formula$, we can write

$$df = f_t dt + f_s ds + \frac{1}{2}\{f_{tt}dt^2 + (f_{ts} + f_{st})dtds(t) + f_{ss}ds(t)ds(t)\} + o(dt^2) \tag{1}$$

$$\Longrightarrow df = f_t dt + f_s ds + \frac{1}{2} f_{ss} ds(t)ds(t) \tag{2}$$

Considering $ds(t) = \phi s(t)dt + \sigma s(t)dW(t)$, $dW(t)dW(t) = dt$, $dW(t)dt = 0$, we have

$$df = f_t dt + \frac{1}{2}\sigma^2 s^2 f_{ss} dt + \phi s f_s dt + \sigma s dW(t) \tag{3}$$

2 (b)

$c = c(t, s(t))$, $d\Pi = c_t dt + c_s ds + \frac{1}{2} c_{ss} ds ds - c_s ds$.
$d\Pi = c_t dt + \frac{1}{2} c_{ss} \sigma^2 s^2 dt$

3 (c)

$d\Pi = r(c - c_s s)dt = c_t dt + \frac{1}{2}c_{ss}\sigma^2 s^2 dt$
$c_t + \frac{1}{2}\sigma^2 s^2 c_{ss} + rsc_s - rc = 0$

Another Way of Obtaining BS Formula

According to non-arbitrage postulate, if at time 0, $c(0, S(0)) = c_0$, then we should should be able to construct a portfolio $X(t)$(with $X(0) = c_0$) to replicate exactly this option. We should use the underlying stock $S(t)$ and the money market with interest rate r. Suppose we hold $\Delta(t)$ share of stock at time t, then it follows:

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt = D_t dt + D_w dW(t) \tag{4}$$

where $D_t = \Delta(t)S(t)(\phi - r) + rX(t)$, $D_w = \Delta(t)S(t)\sigma$.

At the same time,

$$dc(t, S(t)) = c_t dt + c_s dS(t) + \frac{1}{2}c_{ss}dS(t)dS(t) = D'_t dt + D'_w dW(t) \tag{5}$$

where $D'_t = c_t + c_s\phi S(t) + \frac{1}{2}\sigma^2 S(t)^2 c_{ss}$, $D'_w = c_s\sigma S(t)$.

Therefore, it should follows that $D_t = D'_t, D_w = D'_w$. The second relation yields instantly that $\Delta(t) = c_s$, while the first would amount to the Black-Scholes formula:

$$c_t + \frac{1}{2}\sigma^2 s^2 c_{ss} + rsc_s - rc = 0 \tag{6}$$

4 (d)

Change variable $s = e^x$, we have $c_x = e^{-x}c_s$.
$c_t = rc - rsc_s - \frac{1}{2}\sigma^2 s^2 c_{ss} = (r - (r - \frac{1}{2}\sigma^2)\frac{\partial}{\partial x} - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2})c$.

Therefore $H_{BS} = (1 - \frac{\partial}{\partial x})(r + \frac{1}{2}\sigma^2\frac{\partial}{\partial x})$.

5 (e)

$$p_{BS}(x, \tau; x') = \langle x|e^{-\tau H}|x'\rangle = \int_\infty^\infty \frac{dp}{2\pi}\langle x|e^{-\tau H}|p\rangle\langle p|x'\rangle \tag{7}$$

Taking $p = i\frac{\partial}{\partial x}$, using $\langle x|p\rangle = e^{ipx}$:

$$p_{BS}(x, \tau; x') = e^{-r\tau}\int_\infty^\infty \frac{dp}{2\pi}exp\{-\frac{1}{2}\sigma^2 p^2\tau + ip(x - x') + ip\tau(r - \frac{\sigma^2}{2})\} \tag{8}$$

Finally, perform the Gaussian integration:

$$p_{BS}(x,\tau;x') = e^{-r\tau}\frac{1}{\sqrt{2\pi\tau\sigma^2}}exp\{-\frac{1}{2\sigma^2\tau}(x - x' + \tau(r - \frac{\sigma^2}{2}))^2\} \tag{9}$$

section(f) We've known that P(x,T-t,x') is the conditional probability density that, given security price x at time t, it will have a value of x' at time T.

The expectation of x' at time τ can therefore be calculated(it's a simple convolution):

$$< x(\tau) >= \int_{-\infty}^{+\infty} x' P_{BS}(x,\tau;x')dx' = \frac{e^{-r\tau}}{\sqrt{2\pi\tau\sigma^2}}\int_{-\infty}^{+\infty} x' e^{-\frac{1}{2\tau\sigma^2}[x-x'+\tau(r-\frac{\sigma^2}{2})]^2}dx'. \tag{10}$$

Finally we can derive:

$$< x(\tau) >= [x + \tau(r - \frac{\sigma^2}{2})]e^{-r\tau}. \tag{11}$$

The result tells us the evolution of x(which equals ln(S)) over time.

6 (g)

Consider a down-and-out barrier European option. If $s \leq e^B (x \leq B)$ it will become worthless, which means $c = 0$.

For an arbitrary potential $v(x)$, we have:$\frac{\partial c}{\partial t} = Hc$, where H can be written as:

$$H = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (\frac{\sigma^2}{2} - r)\frac{\partial}{\partial x} + V(x). \tag{12}$$

Obviously, if we set $V(x) = \infty$ when $x \leq B$, then $c = 0$ is automaticlly satisfied. However, when $x > B$, the changing of c still conform to the B-S formula. comparing with H_{BS}, we have: V(x)= $\begin{cases} +\infty, x \leq B. \\ r, x > B. \end{cases}$

7 (i)

We know that r(t) satisfied Langevin equation:

$$\frac{dr}{dt} = a(r,t) + \sigma(r,t)R(t). \tag{13}$$

Here R(t) is still Gaussian noise.

Define the propagator $P(r,t;r_0)$: if $r(t_0) = r_0$, the probability of r(t)=r equals

$P(r, t; r_0)$.

From the Langevin equation we have:

$$r(t+\varepsilon) = r(t) + \varepsilon[a + \sigma R(t)]. \tag{14}$$

Change it into the following formula:

$$r = r' + \varepsilon[a(r') + \sigma(r')R(t)]. \tag{15}$$

Thus we can calculate the propagator:

$$P(r, t+\varepsilon, r_0) = P(r', t; r_0)|_{r'=r-\varepsilon[a(r')+\sigma(r')R(t)]} = \int P(r', t; r_0)\delta(r-r'-\varepsilon[a(r')+\sigma(r')R(t)])dr' \simeq \int P(r \tag{16}$$

Because $< R^2(t) >= \frac{1}{\varepsilon}$ and $< R(t) >= 0$, the previous formula can be rewritten like this:

$$P(r, t+\varepsilon, r_0) = P(r', t; r_0) + \int dr' P(r', t; r_0)\{\frac{\partial\delta(r-r')}{\partial r'}\varepsilon a(r') + \frac{1}{2}\frac{\partial^2\delta(r-r')}{\partial r'^2}\varepsilon^2\sigma^2(r')\frac{1}{\varepsilon}\} = P(r', t; r_0) \tag{17}$$

If a variable is o(ε), it is automatically neglected.

Thus, from the definition of derivative, we have:

$$\frac{\partial P(r, t; r_0)}{\partial t} = [\frac{1}{2}\frac{\partial^2}{\partial r^2}\sigma^2(r) - \frac{\partial}{\partial r}a(r)]P(r, t; r_0). \tag{18}$$

We've already known that:

$$\frac{\partial P(r, t; r_0)}{\partial t} = -H_F P(r, t; r_0). \tag{19}$$

Therefore we can prove:

$$H_F = -\frac{1}{2}\frac{\partial^2}{\partial r^2}\sigma^2(r) + \frac{\partial}{\partial r}a(r) = -\frac{1}{2}\frac{\partial^2}{\partial r^2}\sigma^2(r) + a(r)\frac{\partial}{\partial r} + \frac{\partial a(r)}{\partial r}. \tag{20}$$

From a different perspective, we define $P_B(R, t; r)$ as the back propagator.

Similarly we have(since the time flows backwards this time):

$$\frac{\partial P_B(R, t; r)}{\partial t} = +H_B P_B(R, t; r). \tag{21}$$

Finally we can calculate:

$$H_B = -\frac{1}{2}\sigma^2(r)\frac{\partial^2}{\partial r^2} - a(r)\frac{\partial}{\partial r}. \tag{22}$$

$H_B = H_F^\dagger$. is obvious.

8 (j)

In this part we'll focus on the so-called "**stochastic Quantization**".
The Langevin equation:

$$\frac{dr}{dt} = a(r,t) + \sigma(r.t)R(t) \tag{23}$$

is satisfied at any time between t_0 and T. We must consider that both r and R are stochastic variables. Therefore when calculating z_B(can be compared to partition function in statistical mechanics), we must integrate over all possible paths

$$Z_B = \int DRDr \prod_{t=t_0}^{T} \delta[\frac{dr}{dc} - a(r,t) - \sigma(r,t)R(t)]e^{-}\frac{1}{2}\int_{t_0}^{T} R^2(t)dt \tag{24}$$

Dr means integrating over all possible $r(t) : \int Dr = \int_{-\infty}^{+\infty} \prod_{t=t_0}^{T} dr(t)$.

We hope to write Z_B int the form of $z_B = \int Dre^{s_B}$. Because of this, we calculate the integral over R first:

$$Z_B = \int Drexp(-\frac{1}{2}\int_{t_0}^{T} \frac{[\frac{dr}{dt} - a(r,t)]^2}{\sigma^2(r,t)}dt). \tag{25}$$

Because the $Dirac - \delta$ function makes sure we only keep $R(t)$ that satisfies the Langevin equation: $R(t) = \frac{\frac{dr}{dt} - a(r,t)}{\sigma(r,t)}$.

It's easy to calculate S_B:

$$S_B = -\frac{1}{2}\int_{t_0}^{T} \frac{[\frac{dr}{dt} - a(r,t)]^2}{\sigma^2(r,t)}dt. \tag{26}$$

From the definition of $S_B : S_B = \int_{t_0}^{T} Ldt$, it's clear that:

$$L = -\frac{[\frac{dr}{dt} - a(r,t)]^2}{2\sigma^2(r,t)}dt. \tag{27}$$

9 (k)

The Vasicek model can be described using the following equation:

$$\frac{dr}{dt} = a(b-r) + \sigma R(t). \tag{28}$$

Note that if we set $\begin{cases} a(r,t) = a(b-r). \\ \sigma(r,t) = \sigma. \end{cases}$ in Langevin equation(a,b,σ are all constants), we have the Vasicek model naturally.

Thus we get:

$$L_V = -\frac{[\frac{dr}{dt} - a(b-r)]^2}{2\sigma^2}. \tag{29}$$

$$S_V = -\frac{1}{2\sigma^2}\int_{t_0}^{T}[\frac{dr}{dt} - a(b-r)]^2 dt. \tag{30}$$

In the next section we'll use S_V to calculate the propagator of Vasicek model.