Stochastic Analysis

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1 Foundation

A stochastic process is a collection of $R.V.$: $X = \{X_t; 0 \leq t < \infty\}$ on sample space $(\Omega, \mathcal{F})$, which take values in a second measurable state space $(\mathcal{R}^d, \mathcal{B}(\mathcal{R}^d))$.

1.1 Understanding $\sigma - algebra$

1.2 Filtration

A non-decreasing family $\{\mathcal{F}_t; t \geq 0\}$ of $sub - \sigma - field$ of $\mathcal{F}$: $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s < t < \infty$. Set $\mathcal{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$.

Given a process $X(t)$, the simplest choice of a filtration is $\mathcal{F}_t^X = \sigma(X_s; 0 \leq s \leq t)$.

1.3 Conditional Expectation

$\mathbb{E}[X|\mathcal{G}]$ is the unique random variable that satisfies:

1. $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G} - measurable$
2. $\int_A \mathbb{E}[X|\mathcal{G}](w) d\mathbb{P}(w) = \int_A X(w) d\mathbb{P}(w)$, for all $A \in \mathcal{G}$

1.4 Stopping Times

Consider a measurable space $(\Omega, \mathcal{F})$ equipped with a filtration $\{\mathcal{F}_t\}$. A random time T is a stopping time w.r.t. that filtration, if the event $\{T \leq t\}$ belongs to $\mathcal{F}_t$, $\forall t \geq 0$.

Let T, S be stopping times and Z an integrable $R.V.$. We have:

1. $\mathbb{E}[Z|\mathcal{F}_T] = \mathbb{E}[Z|\mathcal{F}_{S \bigwedge T}]$, P-a.s. on $\{T \leq S\}$
2. $\mathbb{E}[\mathbb{E}[Z|\mathcal{F}_T]|\mathcal{F}_S] = \mathbb{E}[Z|\mathcal{F}_{S \bigwedge T}]$, P-a.s.

2 Brownian Motion

2.1 Construction

2.2 Levy Theorem

2.3 First Passage Time

2.4 Maximum of Brownian Motion with Drift

3 Ito Integral

Property of $I(t)$:

1. Continuity
2. $\mathcal{F}(t) - measurable$
3. Linearity
4. Martingale
5. $Isometry : \mathbb{E}I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)du$
6. $QV(t) = [I, I](t) = \int_0^t \Delta^2(u)du$

There is a useful exercise on Shreve $P_{151} - 4.4.11$.

4 Risk-Neutral Measure

4.1 Change of Measure

In $(\Omega, \mathcal{F}, \mathbb{P})$, $R.V.$ Z is a.s. nonnegative, $\mathbb{E}Z = 1$.
Then for all $A \in \mathcal{F}$, we can define $\widetilde{\mathbb{P}}(A) = \int_A Z(w)d\mathbb{P}(w)$.

4.2 Radon-Nikodym Derivative Process

We have $(\Omega, \mathcal{F}, \mathbb{P})$ and $filtration$ $\mathcal{F}(t)$ defined on $0 \le t \le T(T$ fixed). $R.V.$ Z is a.s. nonnegative, $\mathbb{E}Z = 1$. Define $\widetilde{\mathbb{P}}$ as in previous subsection.

Define the Radon-Nikodym Derivative Process to be $Z(t) = \mathbb{E}[Z|\mathcal{F}(t)]$, $Z(t)$ is a $martingale$ with respect to $\mathcal{F}(t)$.

Property of $Z(t)$:

1. if Y is a $\mathcal{F}(t) - measurable$ $R.V.$, then $\widetilde{\mathbb{E}}Y = \mathbb{E}[YZ(t)]$

2. if $0 \leq s \leq t \leq T$, Y is a $\mathcal{F}(t) - measurable$ *R.V.*, then $Z(s)\widetilde{\mathbb{E}}[Y|\mathcal{F}(s)] = \mathbb{E}[YZ(t)|\mathcal{F}(s)]$

4.3 Girsanov Theorem, one dimension

4.4 Martingale Representation Theorem, one dimension

4.5 Application of Risk-Neutral

5 Stochastic Differentiation Equations

5.1 Markov Property

5.2 Feynmann-Kac Theorem, one dimension

5.3 Kolmogorov Backward & Forward Equation

Shreve P_{291}

5.4 Volatility Smile & Surface