Stochastic Analysis

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1 Foundation

A stochastic process is a collection of $R.V.$: $X = \{X_t; 0 \leq t < \infty\}$ on sample space $(\Omega, \mathcal{F})$, which take values in a second measurable state space $(\mathcal{R}^d, \mathcal{B}(\mathcal{R}^d))$.

1.1 Understanding $\sigma - algebra$

1.2 Filtration

A non-decreasing family $\{\mathcal{F}_t; t \geq 0\}$ of $sub - \sigma - field$ of $\mathcal{F}$: $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s < t < \infty$. Set $\mathcal{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$.

Given a process $X(t)$, the simplest choice of a filtration is $\mathcal{F}_t^X = \sigma(X_s; 0 \leq s \leq t)$.

1.3 Conditional Expectation

$\mathbb{E}[X|\mathcal{G}]$ is the unique random variable that satisfies:

1. $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G} - measurable$
2. $\int_A \mathbb{E}[X|\mathcal{G}](w) d\mathbb{P}(w) = \int_A X(w) d\mathbb{P}(w)$, for all $A \in \mathcal{G}$
   (alternative expression: $\forall A \in \mathcal{G}, \mathbb{E}[\mathbb{E}[\mathbb{1}_A X|\mathcal{G}]] = \mathbb{E}[\mathbb{1}_A X]$)

P.S.: a very useful thing to remember: $\mathbb{P}(A) = \mathbb{E}\mathbb{1}_A$.

1.4 Stopping Times

Consider a measurable space $(\Omega, \mathcal{F})$ equipped with a filtration $\{\mathcal{F}_t\}$. A random time T is a stopping time w.r.t. that filtration, if the event $\{T \leq t\}$ belongs to $\mathcal{F}_t$, $\forall t \geq 0$.

Let T, S be stopping times and Z an integrable $R.V.$. We have:

1. $\mathbb{E}[Z|\mathcal{F}_T] = \mathbb{E}[Z|\mathcal{F}_{S \bigwedge T}]$, P-a.s. on $\{T \leq S\}$
2. $\mathbb{E}[\mathbb{E}[Z|\mathcal{F}_T]|\mathcal{F}_S] = \mathbb{E}[Z|\mathcal{F}_{S \bigwedge T}]$, P-a.s.

2 Brownian Motion

2.1 Construction

2.2 Levy Theorem

Let $M(t), t \geq 0$, be a martingale w.r.t. $\mathcal{F}(t)$. We have $M(0) = 0$, $M(t)$ has continuous sample paths, $\langle M, M \rangle (t) = t, \forall t \geq 0$.
$\Longrightarrow M(t)$ is a Brownian motion.

Sketch of Proof:

For any function $f(t,x)$, we have:

$$f(t, M(t)) = f_0 + \int_0^t [f_t + \frac{1}{2} f_{xx}] ds + \int_0^t f_x dM(s) \tag{1}$$

where we've used $\langle M, M\rangle(t) = t \to dM(t)dM(t) = dt$. Taking expectation on both sides, due to the martingale property of $M(t)$, the expectation of the integral w.r.t. $dM(t)$ disappears. Due to the arbitrarity of $f(t,x)$, we can select $f(t,x) = e^{ux - \frac{1}{2}u^2 t}$. Thus we obtain:

$$f_t + \frac{1}{2} f_{xx} = 0, \tag{2a}$$

$$\mathbb{E} exp\{uM(t) - \frac{1}{2}u^2 t\} = e^{0-0} = 1, \tag{2b}$$

$$\Longrightarrow \mathbb{E} e^{uM(t)} = e^{\frac{1}{2}u^2 t} \tag{2c}$$

We believe two $R.V.$ who have the same moment fenerating function should have the same distribution. Therefore we prove the normality of $M(t)$.

2.3 First Passage Time

2.4 Maximum of Brownian Motion with Drift

3 Ito Integral

Property of $I(t)$:

1. Continuity
2. $\mathcal{F}(t) - measurable$
3. Linearity
4. Martingale
5. $Isometry : \mathbb{E} I^2(t) = \mathbb{E} \int_0^t \Delta^2(u) du$
6. $QV(t) = [I, I](t) = \int_0^t \Delta^2(u) du$

There is a useful exercise on Shreve $P_{151} - 4.4.11$.

4 Risk-Neutral Measure

4.1 Change of Measure

In $(\Omega, \mathcal{F}, \mathbb{P})$, $R.V.$ Z is a.s. nonnegative, $\mathbb{E}Z = 1$.
Then for all $A \in \mathcal{F}$, we can define $\widetilde{\mathbb{P}}(A) = \int_A Z(w) d\mathbb{P}(w)$.

4.2 Radon-Nikodym Derivative Process

We have $(\Omega, \mathcal{F}, \mathbb{P})$ and $filtration$ $\mathcal{F}(t)$ defined on $0 \leq t \leq T$(T fixed). $R.V.$ Z is a.s. nonnegative, $\mathbb{E}Z = 1$. Define $\widetilde{\mathbb{P}}$ as in previous subsection.

Define the Radon-Nikodym Derivative Process to be $Z(t) = \mathbb{E}[Z|\mathcal{F}(t)]$, $Z(t)$ is a *martingale* with respect to $\mathcal{F}(t)$.
Property of $Z(t)$:

1. if Y is a $\mathcal{F}(t)-measurable$ *R.V.*, then $\widetilde{\mathbb{E}}Y = \mathbb{E}[YZ(t)]$
2. if $0 \leq s \leq t \leq T$, Y is a $\mathcal{F}(t)-measurable$ *R.V.*, then $Z(s)\widetilde{\mathbb{E}}[Y|\mathcal{F}(s)] = \mathbb{E}[YZ(t)|\mathcal{F}(s)]$

4.3 Girsanov Theorem, one dimension

We have $W(t), 0 \leq t \leq T$ on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \leq t \leq T$ be the filtration for $W(t)$ and $\Theta(t)$ be an adapted process to it. Define

$$Z(t) = exp\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\}, \tag{3a}$$

$$\widetilde{W(t)} = W(t) + \int_0^t \Theta(u)du, \tag{3b}$$

$$assume \quad \mathbb{E}\int_0^T \Theta^2(u)Z^2(u)du < \infty \tag{3c}$$

Set $Z = Z(T)$, it follows:

$$\mathbb{E}Z = 1 \tag{4}$$

Define a new probability measure by:

$$d\widetilde{\mathbb{P}} = Zd\mathbb{P} \tag{5}$$

Then under $\widetilde{\mathbb{P}}$ measure, $\widetilde{W(t)}$ is a Brownian motion.

4.4 Martingale Representation Theorem, one dimension

We have $W(t), 0 \leq t \leq T$ on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \leq t \leq T$ be the filtration for $W(t)$. Let $M(t)$ be a martingale w.r.t. $\mathcal{F}(t)$. $\Longrightarrow \exists$ an $\mathcal{F}(t)$ adapted process $\Gamma(u), 0 \leq u \leq T$, such that:

$$M(t) = M(0) + \int_0^t \Gamma(u)dW(u), 0 \leq t \leq T. \tag{6}$$

Using the Martingale Representation Theorem & Girsanov Theorem, it can be proved that:

Let $M(t)$ be a martingale under $\widetilde{\mathbb{P}}$. Then there exists an adapted process $\widetilde{\Gamma(u)}$ w.r.t $\mathcal{F}(t)$, such that:

$$\widetilde{M(t)} = \widetilde{M(0)} + \int_0^t \widetilde{\Gamma(u)}d\widetilde{W(u)}, 0 \leq t \leq T. \tag{7}$$

4.5 Application of Risk-Neutral

5 Stochastic Differentiation Equations

5.1 Markov Property

Solutions to *S.D.E.* are *Markov processes.*

5.2 Feynmann-Kac Theorem, one dimension

Consider the following *S.D.E.*:

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u) \tag{8}$$

Fix $T > 0$, let $0 \le t \le T$. Let $h(y)$ be *Borel-measurable*. Define the function

$$g(t, x) = \mathbb{E}^{t,x} h(X(T)) \tag{9}$$

$\implies g(t, x)$ satisfies *P.D.E.*:

$$g_t + \beta g_x + \frac{1}{2}\gamma^2 g_{xx} = 0 \tag{10a}$$

$$g(T, x) = h(x) \tag{10b}$$

Sketch of Proof:

It follows immediately that the process $g(t, X(t))$ is a martingale.

On the other hand, we have

$$dg(t, X(t)) = [g_t + \beta g_x + \frac{1}{2}\gamma^2 g_x x]dt + \gamma g_x dW(t) \tag{11}$$

Therefore, the dt term must be 0.

5.3 General Version of Feynman-Kac

For $0 \le t \le T$, $x \in \mathbb{R}^d$, $d - dim\ W(t)$, σ is a $d \times d$ diffusion matrix,

$$X(\theta) = x + \int_t^\theta \mu(s, X(s))ds + \int_t^\theta \sigma(s, X(s))dW(s) \tag{12}$$

We define operator $\mathcal{A}_t = \sum_i \mu_i \partial_{x_i} + \frac{1}{2}\sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$, where $a_{ij} = \sum_k \sigma_{ik}\sigma jk$.

Under some technical conditions which often hold, we have

$$V(t, x) = \mathbb{E}[\int_t^T e^{-\int_t^\theta k(u,X(u))du} f(\theta, X(\theta))d\theta + g(X(T))e^{-\int_t^T k(u,X(u))du}] \tag{13}$$

solves the *P.D.E.* with terminal condition:

$$\frac{\partial V}{\partial t} + \mathcal{A}_t V + f = kV \tag{14a}$$

$$V(T, y) = g(y) \tag{14b}$$

We can verify $V(t, x)$ solves the equation. *Define* $\beta(t) = e^{-\int_0^t k(u,X(u))du}$, then it follows

$$V(t, X(t)) = \mathbb{E}[\int_t^T \frac{\beta(s)}{\beta(t)} f(s, X(s))ds + \frac{\beta(T)}{\beta(t)} g(X(T))|\mathcal{F}(t)] \tag{15a}$$

$$M(t) = \beta(t)V(t, X(t)) + \int_0^t \beta(s)f(s, X(s))ds \tag{15b}$$

$$\implies M(t) = \mathbb{E}[\int_0^T \beta(s)f(s, X(s))ds + \beta(T)g(X(T))|\mathcal{F}(t)] \tag{15c}$$

Obviously $M(t)$ is a *Levy martingale.* What is left to show is calculate $dM(t)$ and set dt term to 0 to obtain the *P.D.E.* $V(t, X(t))$ solves.

Notes:

1. $V(t, X(t))$ is the asset price at time t. In practice, we often change to risk-neutral measure first.

2. $g(y)$ is the final payoff function w.r.t. stock price y

3. $k(u, X(u))$ is the interest rate, then $\beta(t)$ is the discount factor

4. Setting $f = 0, k(u, X(u)) = r = const$, and change to risk-neutral measure w.r.t. an underlying stock for a call option, we have the common form of *Discounted Feynman-Kac*:

$$c(t, X(t)) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}(S(T) - K)^+|\mathcal{F}(t)] \tag{16}$$

5.4 Transitional Density

Definition:

$$\mathbb{P}(X(T) \in A | X(t) = x) = \int_A p(t, T, x, y) dy \tag{17}$$

$$\mathbb{E}^{t,x} h(X(T)) = \int h(y) p(t, T, x, y) dy \tag{18}$$

$$p^{t,x}(T, y) = \int p(t, T, x, y) p(t, x) dx \tag{19}$$

5.5 Kolmogorov Backward & Forward Equation

Shreve P_{291}
For process $X(t)$:

$$dX(t) = \beta(t, X(t))dt + \gamma(t, X(t))dW(t) \tag{20}$$

Let $\mathcal{A}_t = \beta(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\gamma^2(t,x)\frac{\partial^2}{\partial x^2}$, $\mathcal{A}_t^\dagger = \beta(t,x)\frac{\partial}{\partial x} - \frac{1}{2}\gamma^2(t,x)\frac{\partial^2}{\partial x^2}$.
(Note that $t \sim x, T \sim y$)
Let the transitional density be $p(t, T, x, y)$.

$$(\frac{\partial}{\partial t} + \mathcal{A}_t) p(t, T, x, y) = 0 \tag{21a}$$

$$(\frac{\partial}{\partial T} - \mathcal{A}_T^\dagger) p(t, T, x, y) = 0 \tag{21b}$$

5.6 Volatility Smile & Surface

6 Excellent Exercise on Courseware

6.1 Introduction to SA

1. P.156 the property of infinitismal generator
2. P.160 Prove the Komogorov supplem P.55
3. P.162 Show $V(t,x)$ is the solution to $\partial_t V(t,x) + \mathcal{A}_t V(t,x) + f(t,x) = k(t,x)V(t,x)$ on the previous page.
4. P.165 top
5. P.172
6. P.178

6.2 Supplementary Notes on Introduction

1. P.26 bottom
2. P.31
3. P.50 51
4. P.53
5. P.59

6.3 Application of SA in Financial Engineering

1. P.12 bottom
2. P.14 prove (3)
3. P.15 top
4. P.23 top
5. P.26 $dX = \sum \Delta_i dSi + r(X - \sum \Delta_i S_i)dt$, prove $d(e^{-rt}X(t)) = \sum \Delta_i d(e^{-rt}S_i(t))$
6.