Pricing Options Under Stochastic Volatility with Fourier-Cosine Series Expansions

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Summary. An option pricing method for European options based on the Fourier-cosine series, called the COS method, is presented. It can cover underlying asset processes for which the characteristic function is known, and in this paper, in particular, we consider stochastic volatility dynamics.

1 Introduction: The COS Method

Efficient numerical methods are required to rapidly price complex contracts and calibrate financial models. During calibration, i.e., when fitting model parameters of the stochastic asset processes to market data, we typically need to price European options at a single spot price, with many different strike prices, very quickly. Particular examples of where this is important would be processes with several parameters, like the Heston model [4] or the infinite activity Lévy processes, since there the pricing problem (for many strikes) is used inside an optimization method.

The integration methods are used for calibration purposes whenever the characteristic function of the asset price process is known analytically. State-of-the-art numerical integration techniques have in common that they rely on a transformation to the Fourier domain. The Carr-Madan method [1] is one of the best known examples of this class.

In this paper we will focus on *Fourier-cosine expansions* in the context of numerical integration as an alternative for methods based on the FFT. We will show that this method, called the COS method [2, 3], can improve the speed of pricing plain vanilla options.

The point-of-departure for pricing European options with numerical integration techniques is the risk-neutral valuation formula:

$$v(x, t_0) = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[v(y, T)|x\right] = e^{-r\Delta t}\int_{\mathbb{R}} v(y, T)f(y|x)dy, \tag{1}$$

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where v denotes the option value, Δt is the difference between the maturity, T, and the initial date, t_0, and $\mathbb{E}^{\mathbb{Q}}[\cdot]$ is the expectation operator under risk-neutral measure $\mathbb{Q}$. x and y are state variables at time t_0 and T, respectively; $f(y|x)$ is the probability density of y given x, and r is the risk-neutral interest rate.

Since the density rapidly decays to zero as $y \to \pm\infty$ in (1), we truncate the infinite integration range without loosing significant accuracy to $[a, b] \subset \mathbb{R}$, and we obtain approximation v_1:

$$v_1(x, t_0) = e^{-r\Delta t} \int_a^b v(y, T) f(y|x) dy. \tag{2}$$

Since $f(y|x)$ is usually not known whereas the characteristic function is, we replace the density by its cosine expansion in y,

$$f(y|x) = {\sum_{k=0}^{+\infty}}' A_k(x) \cos\left(k\pi \frac{y-a}{b-a}\right) \tag{3}$$

with

$$A_k(x) := \frac{2}{b-a} \int_a^b f(y|x) \cos\left(k\pi \frac{y-a}{b-a}\right) dy, \tag{4}$$

so that

$$v_1(x, t_0) = e^{-r\Delta t} \int_a^b v(y, T) {\sum_{k=0}^{+\infty}}' A_k(x) \cos\left(k\pi \frac{y-a}{b-a}\right) dy. \tag{5}$$

$\sum'$ indicates that the first term in the summation is weighted by one-half.

We interchange the summation and integration, and insert the definition

$$V_k := \frac{2}{b-a} \int_a^b v(y, T) \cos\left(k\pi \frac{y-a}{b-a}\right) dy, \tag{6}$$

resulting in

$$v_1(x, t_0) = \frac{1}{2}(b-a) e^{-r\Delta t} \cdot {\sum_{k=0}^{+\infty}}' A_k(x) V_k. \tag{7}$$

The V_k are the cosine series coefficients of payoff function $v(y, T)$ in y.

We have analytic solutions for V_k for several contracts. As we assume the characteristic function of the log-asset price to be known, we represent the payoff as a function of the log-asset price, $x := \ln(S_0/K)$ and $y := \ln(S_T/K)$, with S_t the underlying price at time t and K the strike price. Focusing on a put option, we obtain

$$V_k^{put} = \frac{2}{b-a} K \left(-\chi_k(a, 0) + \psi_k(a, 0)\right). \tag{8}$$

where χ_k and ψ_k are given by

$$\chi_k(c,d) := \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\left[\cos\left(k\pi\frac{d-a}{b-a}\right)e^d - \cos\left(k\pi\frac{c-a}{b-a}\right)e^c \right.$$
$$\left. + \frac{k\pi}{b-a}\sin\left(k\pi\frac{d-a}{b-a}\right)e^d - \frac{k\pi}{b-a}\sin\left(k\pi\frac{c-a}{b-a}\right)e^c\right] \quad (9)$$

and

$$\psi_k(c,d) := \begin{cases} \left[\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right]\frac{b-a}{k\pi} & k \neq 0, \\ (d-c) & k = 0. \end{cases} \quad (10)$$

For a call we find a similar expression.

Due to the rapid decay rate of the V_k, we further truncate the series summation in (7) to obtain approximation v_2:

$$v_2(x,t_0) = \frac{1}{2}(b-a)e^{-r\Delta t} \cdot {\sum_{k=0}^{N-1}}' A_k(x)V_k. \quad (11)$$

Coefficients $A_k(x)$, defined in (4), can be approximated by $F_k(x)$ defined as

$$F_k(x) := \frac{2}{(b-a)}\mathrm{Re}\left\{\phi\left(\frac{k\pi}{b-a};x\right)\cdot e^{-i\frac{ka\pi}{b-a}}\right\} \quad (12)$$

with $\phi(\omega;x)$ the characteristic function:

$$\phi(\omega;x) := \int_{\mathbb{R}} e^{i\omega y} f(y|x)dy.$$

This gives

$$v(x,t_0) \approx v_3(x,t_0) = e^{-r\Delta t}{\sum_{k=0}^{N-1}}' \mathrm{Re}\left\{\phi\left(\frac{k\pi}{b-a};x\right)e^{-ik\pi\frac{a}{b-a}}\right\}V_k, \quad (13)$$

where $\mathrm{Re}\{\cdot\}$ denotes taking the real part of the argument.

Equation (13) can be improved for the Lévy and the Heston models, so that options *for many strike prices* can be computed simultaneously. In the Heston model [4], the volatility, denoted by $\sqrt{u_t}$, is modeled by an additional stochastic differential equation,

$$\begin{aligned} dx_t &= \left(\mu - \tfrac{1}{2}u_t\right)dt + \sqrt{u_t}dW_{1t}, \\ du_t &= \lambda(\bar{u} - u_t)dt + \eta\sqrt{u_t}dW_{2t} \end{aligned} \quad (14)$$

where x_t denotes the log-asset price variable and u_t the variance of the asset price process. Parameters $\lambda \geq 0, \bar{u} \geq 0$ and $\eta \geq 0$ are the speed of mean reversion, the mean level of variance and the volatility of volatility, respectively. Furthermore, the Brownian motions W_{1t} and W_{2t} are assumed to be correlated with correlation coefficient ρ.

For the Heston model, we have $\phi(\omega;\mathbf{x},u_0)=\varphi_{hes}(\omega;u_0)\cdot e^{i\omega\mathbf{x}}$, with u_0 the volatility of the underlying at the initial time and $\varphi_{hes}(\omega;u_0):=\phi(\omega;0,u_0)$. The characteristic function of the log-asset price, $\varphi_{hes}(\omega;u_0)$, reads

$$\varphi_{hes}(\omega;u_0)=\exp\left(i\omega\mu\Delta t+\frac{u_0}{\eta^2}\left(\frac{1-e^{-D\Delta t}}{1-Ge^{-D\Delta t}}\right)(\lambda-i\rho\eta\omega-D)\right)\cdot$$
$$\exp\left(\frac{\lambda\bar{u}}{\eta^2}\left(\Delta t(\lambda-i\rho\eta\omega-D)-2\log(\frac{1-Ge^{-D\Delta t}}{1-G})\right)\right),$$

with

$$D=\sqrt{(\lambda-i\rho\eta\omega)^2+(\omega^2+i\omega)\eta^2}\quad\text{and}\quad G=\frac{\lambda-i\rho\eta\omega-D}{\lambda-i\rho\eta\omega+D}.$$

Recalling the V_k-formula for a European options, like (8), we now define them as a vector multiplied by a scalar, $\mathbf{V}_k=U_k\mathbf{K}$, where

$$U_k=\begin{cases}\frac{2}{b-a}\left(-\chi_k(a,0)+\psi_k(a,0)\right) & \text{for a put}\\ \frac{2}{b-a}\left(\chi_k(0,b)-\psi_k(0,b)\right) & \text{for a call.}\end{cases} \tag{15}$$

We then find

$$v(\mathbf{x},t_0,u_0)\approx\mathbf{K}e^{-r\Delta t}\cdot\mathrm{Re}\left\{{\sum_{k=0}^{N-1}}'\varphi_{hes}\left(\frac{k\pi}{b-a};u_0\right)U_k\cdot e^{ik\pi\frac{\mathbf{x}-a}{b-a}}\right\}. \tag{16}$$

This is the COS formula, pricing European options under Heston dynamics very efficiently. The convergence rate of the Fourier-cosine series depends on the properties of the functions on the interval $[a,b]$. From the error analysis in [2], we can summarize that, with a properly chosen truncation of the integration range, the overall error converges either exponentially for density functions, with nonzero derivatives, belonging to $\mathbb{C}^\infty([a,b]\subset\mathbb{R})$.

We define the truncation range by

$$[a,b]:=[c_1-12\sqrt{|c_2|},c_1+12\sqrt{|c_2|}].$$

in which the cumulants, c_n, are given by the derivatives, at zero, of $g(t)=\log(E(e^{t\cdot X}))$,

$$c_1=\mu T+(1-e^{-\lambda T})\frac{\bar{u}-u_0}{2\lambda}-\frac{1}{2}\bar{u}T,$$

$$\begin{aligned}c_2=\frac{1}{8\lambda^3}\big(&\eta T\lambda e^{-\lambda T}(u_0-\bar{u})(8\lambda\rho-4\eta)+\lambda\rho\eta(1-e^{-\lambda T})(16\bar{u}-8u_0)+\\&2\bar{u}\lambda T(-4\lambda\rho\eta+\eta^2+4\lambda^2)+\eta^2((\bar{u}-2u_0)e^{-2\lambda T}+\bar{u}(6e^{-\lambda T}-7)+2u_0)+\\&8\lambda^2(u_0-\bar{u})(1-e^{-\lambda T})\big)\end{aligned}$$

Cumulant c_2 may become negative for sets of Heston parameters that do not satisfy the Feller condition, i.e., $2\bar{u}\lambda>\eta^2$. We therefore use the absolute value of c_2.

The Greeks, like Delta, Gamma and also Vega can be obtained, basically at no cost, by differentiating the COS formula (16).

2 Numerical Results

We perform a numerical test on European options under the Heston process to evaluate the efficiency and accuracy of the COS method. We compare our results to the Carr-Madan method [1], in which the FFT has been used. Parameter N, in the experiments to follow, denotes for the COS method the number of terms in the Fourier-cosine expansion, and the number of grid points for the Carr-Madan method. Some experience is helpful when choosing the correct truncation range and damping factor in Carr-Madan's method. A suitable choice appears to be $\alpha = 0.75$ for the Heston experiments.

The computer used has an Intel Pentium 4 CPU, 2.80 GHz with cache size 1,024 KB; The code is written in Matlab 7-4.

2.1 The Heston Model

We choose the Heston model and price puts with the following parameters:

$$\begin{aligned} S_0 = 100, K = 100, r = 0, q = 0, \lambda = 1.5768, \eta = 0.5751, \\ \bar{u} = 0.0398, u_0 = 0.0175, \rho = -0.5711, T = 1. \end{aligned} \tag{17}$$

In this test, we compare the COS method with the Carr-Madan method. The option price reference values are obtained by the Carr-Madan method using $N = 2^{17}$ points, and the truncated Fourier domain is set to [0, 1,200] for the experiment with $T = 1$.

We mimic the calibration situation and price several strikes simultaneously. We choose $T = 1$ and 21 consecutive strikes, $K = 50, 55, 60, \ldots, 150$, see the results in Table 1. The maximum error over all strike prices is presented. Note the very different values of N, that the two methods require for satisfactory convergence. With $N = 160$, the COS method can price all options for 21 strikes highly accurately, within 3 ms. The COS method appears to be approximately a factor 20 faster than the Carr-Madan method for the same level of accuracy.

Table 1. Error convergence and cpu time for puts under the Heston model by the COS and Carr-Madan method, pricing 21 strikes, with $T = 1$, parameters as in (17)

	N	32	64	96	128	160
COS	cpu time (ms)	0.85	1.45	2.04	2.64	3.22
	max. abs. err.	1.43×10^{-1}	6.75×10^{-3}	4.52×10^{-4}	2.61×10^{-5}	4.40×10^{-6}
	N	512	1,024	2,048	4,096	8,192
Carr-Madan	cpu time (ms)	7.44	12.84	20.36	37.69	76.02
	max. error	4.70×10^{6}	6.69×10^{1}	2.61×10^{-1}	2.15×10^{-3}	2.08×10^{-7}

3 Conclusions and Discussion

In this paper we have discussed an option pricing method based on Fourier-cosine series expansions, the COS method, for European-style options. The method can be used as long as a characteristic function for the underlying price process is available. The COS method is based on the insight that the series coefficients of many density functions can be accurately retrieved from their characteristic functions. The computational complexity of the COS method is linear in the number of terms, N, chosen in the Fourier-cosine series expansion. Very fast computing times were reported here for the Heston model.

References

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