1 Time change and Heston

The diffusion time change model:

$$dS = rSdt + \sigma S dW^1_\tau \tag{1}$$
$$\tau = \int_0^t \bar{v} ds \tag{2}$$
$$d\bar{v} = k(1-\bar{v})dt + \eta\sqrt{\bar{v}}dW^2_t \tag{3}$$
$$dW^1 dW^2 = \rho dt \tag{4}$$

and the traditional Heston model:

$$dS = rSdt + \sqrt{v}SdW^1_t \tag{5}$$
$$dv = a(b-v)dt + c\sqrt{v}dW^2_t \tag{6}$$
$$dW^1 dW^2 = \rho dt \tag{7}$$

are equivalent. To get from the Heston to time change, use the following change of variables:

$$k = a \tag{8}$$
$$\eta = \frac{c}{\sqrt{b}} \tag{9}$$
$$\sigma = \sqrt{b} \tag{10}$$
$$\bar{v}_0 = \frac{v_0}{b} \tag{11}$$

2 Time change and Black Scholes

The diffusion time change model reduces to the Black Scholes model if $\bar{v}_0 = 1$ and $\eta = 0$. This can be demonstrated either through a convoluted and unenlightening application of l'hospital's rule on the moment generating function of a CIR process or, more simply, by solving the integral of a non-stochastic mean-reverting process:

$$\int_0^T \bar{v}_s ds$$

Where

$$d\bar{v}_s = k(1-\bar{v})dt$$

Since there is no stochastic component, this is just an ODE with solution

$$\bar{v}_t = e^{-kt}(\bar{v}_0 - 1) + 1$$

The characteristic function in the Black Scholes world is thus

$$e^{-\psi\left(\frac{1-e^{-kT}}{k}(\bar{v}_0-1)+T\right)}$$

When $\bar{v}_0 = 1$, this becomes (as expected)

$$e^{-\psi T}$$