Math Assignment

MTH1098_01E
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Solutions:

1.

Proof:
$\forall \epsilon > 0$, assume $0 < |x-2| < \delta$, set $f(x) = 4x - 3$ then:

$$|f(x) - 5| = |4x - 8| = 4|x - 2| < 4\delta$$

Choose $\delta = \frac{\epsilon}{4}$. This implies:

$$4|x - 2| < \epsilon$$

That is:

$$|x - 2| < \frac{\epsilon}{4}$$

Which is equivalent to:

$$|x - 2| < \epsilon$$

Therefore, $\forall \epsilon > 0$, we choose $\delta = \frac{\epsilon}{4}$, which satisfies the condition that if $0 < |x-2| < \delta$, then $|f(x) - 5| < \epsilon$. This proves that $\lim_{x\to 2}(4x - 3) = 5$.

2.

- (a) The domain of $g(x)$ is $\{x : x \neq 1\}$.
- (b) $\lim\limits_{x\to 1} g(x) = 3 \cdot x^2 + 2 \cdot x = 5$.
- (c) $f(x)$ is continuous at $x = 1$ because the limit exists and equals the function value.

3.

- (a) We know that the sine function is bounded between -1 and 1 for all real numbers. Therefore:

  $$-1 \leq \sin\left(\frac{2}{x^2}\right) \leq 1$$

  Multiply by x^2:

  $$-x^2 \leq x^2 \sin\left(\frac{2}{x^2}\right) \leq x^2$$

  Apply the Squeeze Theorem:

  $$\lim_{x\to 0} -x^2 = 0$$
  $$\lim_{x\to 0} x^2 = 0$$

  Therefore, we can conclude:

  $$\lim_{x\to 0} x^2 \sin\left(\frac{2}{x^2}\right) = 0$$

- (b) Transfer the question to:

$$\lim_{x\to\infty} \frac{(\sqrt{x^2+2x}-\sqrt{x^2-2x})(\sqrt{x^2+2x}+\sqrt{x^2-2x})}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$$

This can be simplified to:

$$\lim_{x\to\infty} \frac{(x^2+2x)-(x^2-2x)}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$$
$$= \lim_{x\to\infty} \frac{4x}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$$
$$= \lim_{x\to\infty} \frac{4x}{x\left(\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{2}{x}}\right)}$$
$$= \lim_{x\to\infty} \frac{4}{\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{2}{x}}}$$

As x approaches infinity, $\frac{2}{x}$ approaches 0. That is :

$$\frac{4}{\sqrt{1+0}+\sqrt{1-0}}$$
$$= \frac{4}{1+1}$$
$$= 2$$

So, the limit is:

$$\lim_{x\to\infty}\left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right) = 2$$

- (c) As t approaches infinity, $\frac{1}{t}$ approaches 0, so:

The Nominator:

$$1-\frac{t}{t+1} = 1-\frac{t}{t(1+\frac{1}{t})} = 1-\frac{1}{1+\frac{1}{t}} = 1-\frac{1}{1+0} = 1-1 = 0$$

The Denominator:

$$1-\sqrt{\frac{t}{t+1}} = 1-\sqrt{\frac{t}{t(1+\frac{1}{t})}} = 1-\sqrt{\frac{1}{1+\frac{1}{t}}} = 1-\sqrt{\frac{1}{1+0}} = 1-\sqrt{1} = 1-1 = 0$$

Giving the indeterminate form:

$$\lim_{t\to\infty} \frac{\left(1-\frac{t}{t+1}\right)}{\left(1-\sqrt{\frac{t}{t+1}}\right)} = \frac{0}{0}$$

Clearly, applying L'Hôpital's rule:

$$\lim_{t\to\infty} \frac{\frac{d}{dt}\left(1-\frac{t}{t+1}\right)}{\frac{d}{dt}\left(1-\sqrt{\frac{t}{t+1}}\right)}$$

Differentiate the Numerator:

$$\frac{d}{dt}\left(1-\frac{t}{t+1}\right) = \frac{d}{dt}\left(\frac{t+1-t}{t+1}\right) = \frac{d}{dt}\left(\frac{1}{t+1}\right) = -\frac{1}{(t+1)^2}$$

Differentiate the Denominator:

$$\frac{d}{dt}\left(1-\sqrt{\frac{t}{t+1}}\right) = \frac{d}{dt}\left(\sqrt{\frac{t}{t+1}}\right)$$

Using the chain rule:

$$\frac{d}{dt}\left(\sqrt{\frac{t}{t+1}}\right) = \frac{1}{2\sqrt{\frac{t}{t+1}}}\cdot\frac{d}{dt}\left(\frac{t}{t+1}\right)$$

$$\frac{d}{dt}\left(\frac{t}{t+1}\right) = \frac{(t+1)\cdot 1 - t\cdot 1}{(t+1)^2} = \frac{1}{(t+1)^2}$$

So:

$$\frac{d}{dt}\left(\sqrt{\frac{t}{t+1}}\right) = \frac{1}{2\sqrt{\frac{t}{t+1}}} \cdot \frac{1}{(t+1)^2}$$

Combine the Results:

$$\lim_{t\to\infty} \frac{1-\frac{1}{t+1}}{1-\sqrt{\frac{1}{t+1}}} = \lim_{t\to\infty} \frac{-\frac{1}{(t+1)^2}}{\frac{1}{2\sqrt{\frac{t}{t+1}}}\cdot\frac{1}{(t+1)^2}} = \lim_{t\to\infty} -2\sqrt{\frac{t}{t+1}} = \lim_{t\to\infty} -2\sqrt{\frac{1}{1+\frac{1}{t}}} = -2$$

Therefore, the limit is

$$-2$$

- (d)
  Calculate $(\sin\left(\frac{3\pi}{4}\right))and(cos\left(\frac{3\pi}{4}\right))$:

$$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

  Thus,

$$\sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 0.$$

  Calculate $(\cos(2x))at(x = \frac{3\pi}{4})$:

$$\cos(2x) = \cos\left(2\cdot\frac{3\pi}{4}\right) = \cos\left(\frac{3\pi}{2}\right) = 0.$$

  We have an indeterminate form $\frac{0}{0}$.
  Thus, apply L'Hôpital's Rule:

$$\lim_{x\to\frac{3\pi}{4}} \frac{\cos x - \sin x}{-2\sin(2x)}.$$

  Evaluate at $x = \frac{3\pi}{4}$:

$$\cos\left(\frac{3\pi}{4}\right) - \sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$$

$$\sin(2x) = \sin\left(\frac{3\pi}{2}\right) = -1$$

  Thus:

$$\lim_{x\to\frac{3\pi}{4}} \frac{-\sqrt{2}}{-2(-1)} = \frac{-\sqrt{2}}{2} = \frac{\sqrt{2}}{2}.$$

  Therefore, the limit is

$$\frac{\sqrt{2}}{2}.$$

- (e) As $x \to 1$:
  The numerator: $x\ln x \to 1\cdot\ln(1) = 0$.
  The denominator: $x^3 - 1 \to 1^3 - 1 = 0$.
  This gives us the indeterminate form

$$\frac{0}{0}.$$

  Thus, apply L'Hôpital's Rule

$$\lim_{x\to 1} \frac{x\ln x}{x^3-1} = \frac{\frac{d}{dx}(x\ln x) = \ln x + 1 \quad ,}{\frac{d}{dx}(x^3-1) = 3x^2} = \lim_{x\to 1} \frac{\ln x + 1}{3x^2}.$$

Now, as $x \to 1$:

The numerator

$$\ln(1) + 1 \to 0 + 1 = 1.$$

The denominator

$$3(1^2) \to 3.$$

Thus, we have:

$$\lim_{x \to 1} \frac{\ln x + 1}{3x^2} = \frac{1}{3}.$$

Therefore, the limit is

$$\frac{1}{3}$$

4.

- (a) $\forall x \in (0,4)$,

  $$\because \frac{d}{dx}(\sqrt[3]{2x} + 3x - 4) > 0 \implies f(x) \nearrow,$$

  and

  $$\begin{cases} f(0) = -4 < 0 \\ f(4) = 10 > 0 \end{cases}$$

  $\therefore$

  $$\exists x \in (0,4), \text{that } \ f(x) = 0$$

- (b) The fuction $f(x)$ is continuous on [0,4] as all its terms are differentiable. Thus we calculate $f(0)$ and $f(4)$:

  $$\begin{cases} f(0) = -4 \\ f(4) = 10 \end{cases}$$

  Accroding to MVT, $\exists x \in (0,4)$, such that

  $$f'(x) = \frac{f(4) - f(0)}{4 - 0} = \frac{7}{2}$$

  $\therefore$ There exist $c \in (0,4)$ such that $f'(c) = 3\frac{1}{2}$

5.

Let

$$f(x) = \begin{cases} 11 + c^2 x & \text{if } x < 2 \\ 1 - 6cx & \text{if } x \geq 2 \end{cases}$$

- **(a)** Find $\lim_{x \to 2^-} f(x)$:

  $$\lim_{x \to 2^-} f(x) = 11 + 2c^2$$

- **(b)** Find $\lim_{x \to 2^+} f(x)$:

  $$\lim_{x \to 2^+} f(x) = 1 - 12c$$

To ensure $f(x)$ is continuous at $x = 2$, set the limits equal:

$$11 + 2c^2 = 1 - 12c$$

Thus:

$$c = -1 \quad \text{or} \quad c = -5$$

6.

- (a) Rewrite the function:
$$f(x) = (4x+3)^{1/2}$$
Using the Chain Rule:
$$f'(x) = \frac{1}{2}(4x+3)^{-1/2} \cdot \frac{d}{dx}(4x+3) = \frac{1}{2}(4x+3)^{-1/2} \cdot 4 = \frac{2}{\sqrt{4x+3}}$$

- (b)
The definition of the derivative is:
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
Calculating $f(x+h)$:
$$f(x+h) = \sqrt{4(x+h)+3} = \sqrt{4x+4h+3}$$
Setting up the limit:
$$f'(x) = \lim_{h \to 0} \frac{\sqrt{4x+4h+3} - \sqrt{4x+3}}{h} = \lim_{h \to 0} \frac{(4x+4h+3) - (4x+3)}{h\left(\sqrt{4x+4h+3} + \sqrt{4x+3}\right)}$$
$$= \lim_{h \to 0} \frac{4h}{h\left(\sqrt{4x+4h+3} + \sqrt{4x+3}\right)} = \lim_{h \to 0} \frac{4}{\sqrt{4x+4h+3} + \sqrt{4x+3}} = \frac{4}{2\sqrt{4x+3}} = \frac{2}{\sqrt{4x+3}}$$
Therefore:
$$f'(x) = \frac{2}{\sqrt{4x+3}}$$

7.

- (a)
$$(\frac{fg}{h})' = \frac{f'g' \cdot h - fg \cdot h'}{h^2}$$

- (b) Acoording to (a), we can calculate:
$$(\frac{x^2 \sin x}{e^x})' = \frac{(2x \sin x + x^2 \cos x)e^x - (x^2 \sin x \cdot e^x)}{e^{2x}} = \frac{2x \sin x + x^2 \cos x - x^2 \sin x}{e^x}$$

8.

- (a)
$$\frac{d}{dx}\sqrt{5^x} = \frac{d}{dx}5^{\frac{x}{2}} = \frac{1}{2}5^{\frac{x}{2}} \ln(5) = \frac{\ln(5)}{2} \cdot 5^{\frac{x}{2}}$$

- (b)
$$\frac{d}{dx}e^{\tan(2x)} \ln(\sin x) = (e^{\tan(2x)} \sec^2 2x \cdot 2) \cdot (\ln(\sin x)) + (e^{\tan(2x)}) \cdot (\frac{1}{\sin x} \cdot \cos x)$$
$$= (2e^{\tan(2x)} sec^2(2x)) \cdot (\ln(\sin x)) + e^{\tan(2x)} \cot x$$
$$= e^{\tan(2x)}(2sec^2(2x) \ln(\sin x) + \cot x)$$

- (c) Differentiate the euqation:
$$\frac{d}{dx}(xy^2) + \frac{d}{dx}(y \ln x) + \frac{d}{dx}(e^x) = 0$$
Simplify each element:
$$\frac{d}{dx}(xy^2) = x \cdot \frac{d}{dx}(y^2) + y^2 \cdot \frac{d}{dx}(x) = x(2y\frac{dy}{dx}) + y^2$$
$$\frac{d}{dx}(y \ln x) = \frac{dy}{dx} \ln x + y \cdot \frac{1}{x}$$

$$\frac{d}{dx}(e^x) = e^x$$

Thus

$$x(2y\frac{dy}{dx}) + y^2 + \left(\frac{dy}{dx}\ln x + \frac{y}{x}\right) + e^x = 0$$

This equals to:

$$\frac{dy}{dx}(2xy + \ln x) = -\left(y^2 + \frac{y}{x} + e^x\right)$$

Therefore:

$$\frac{dy}{dx} = \frac{-(y^2 + \frac{y}{x} + e^x)}{2xy + \ln x}$$

- (d) The derivative of $\sin^{-1}(u)$ is given by:

$$\frac{d}{du}[\sin^{-1}(u)] = \frac{1}{\sqrt{1-u^2}}$$

Here, $u = 4x$, so we apply the chain rule:

$$\frac{dy}{dx} = \frac{d}{du}[\sin^{-1}(u)] \cdot \frac{du}{dx}$$

Next, we find $\frac{du}{dx}$:

$$u = 4x \implies \frac{du}{dx} = 4$$

Substituting back into the derivative, we have:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(4x)^2}} \cdot 4$$

Thus, the derivative is:

$$\frac{dy}{dx} = \frac{4}{\sqrt{1-16x^2}}$$

9.

Given:

- Ladder length $(L) = 17$ m
- Rate at which the foot is pulled away $(\frac{dx}{dt}) = 0.8$ m/s
- Distance from the wall $(x) = 8$ m

Using the Pythagorean theorem:

$$x^2 + y^2 = L^2 \implies y^2 = L^2 - x^2$$

Substituting $x = 8$:

$$y^2 = 17^2 - 8^2 \implies y = 15\,\text{m}$$

Differentiating:

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

Substituting values:

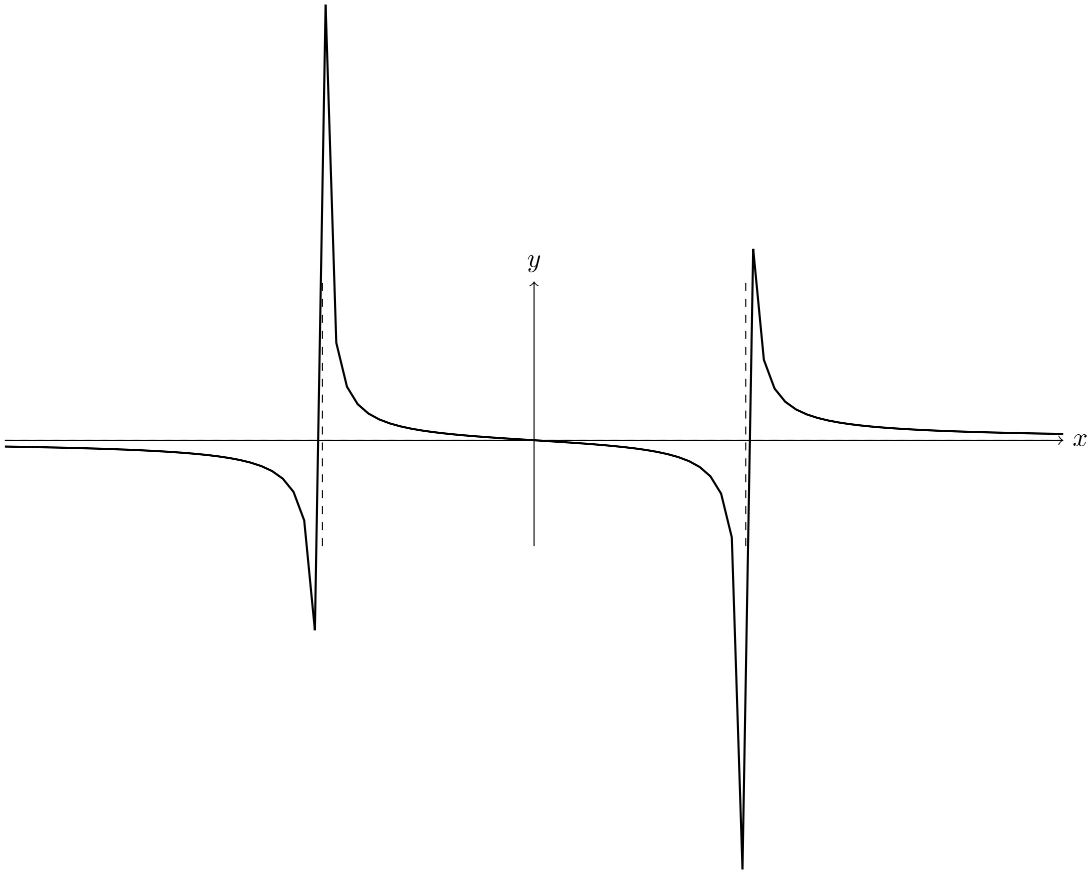
$$\frac{dy}{dt} = -\frac{8}{15} \cdot 0.8 = -\frac{6.4}{15} \approx -0.427\,\text{m/s}$$

Conclusion: The top of the ladder slides down at approximately $0.427\,\text{m/s}$.

10.

- **Domain:** $x \in (-\infty, -4) \cup (-4, 4) \cup (4, \infty)$
- **Intercepts:**
  - x-intercept: $(0, 0)$
  - y-intercept: $(0, 0)$
- **Asymptotes:**
  - Vertical: $x = -4,\ 4$
  - Horizontal: $y = 0$
- **Local Extrema:** None in the domain.

Graph:



11.

A piece of cardboard measures 2 m by 3 m. A square with side length x is cut from each corner. The volume V of the resulting box is given by:

$$V = x(2 - 2x)(3 - 2x) = 6x - 10x^2 + 4x^3$$

To find the maximum volume, take the derivative:

$$\frac{dV}{dx} = 6 - 20x + 12x^2$$

Setting the derivative to zero gives:

$$12x^2 - 20x + 6 = 0 \implies x = \frac{1}{2}$$

Using the quadratic formula:

$$x = \frac{20 \pm \sqrt{(-20)^2 - 4 \cdot 12 \cdot 6}}{2 \cdot 12} = \frac{20 \pm \sqrt{400 - 288}}{24} = \frac{20 \pm \sqrt{112}}{24} = \frac{20 \pm 4\sqrt{7}}{24} = \frac{5 \pm \sqrt{7}}{6}$$

The valid solution for x must be:

$$x = \frac{5 - \sqrt{7}}{6} \quad \text{(valid)}$$

Calculating the approximate value:

$$\sqrt{7} \approx 2.64575 \implies x \approx \frac{5 - 2.64575}{6} \approx 0.3923$$

The dimensions of the box become:

$$\begin{aligned} \text{Length} &\approx 3 - 2(0.3923) \approx 2.2154\,\text{m} \\ \text{Width} &\approx 2 - 2(0.3923) \approx 1.2154\,\text{m} \\ \text{Height} &\approx 0.3923\,\text{m} \end{aligned}$$

The maximum volume is calculated as:

$$V \approx 4(0.0605) - 10(0.154) + 2.3538 \approx 0.242 - 1.54 + 2.3538 \approx 1.06\,\text{m}^3$$

Thus, the dimensions of the box are $2\,\text{m} \times 1\,\text{m} \times 0.5\,\text{m}$ and the maximum possible volume is about $1.06\,\text{m}^3$.

After all, I would like to express my gratitude for your guidance throughout this course. I used LaTeX to format my assignment, which greatly helped me in presenting the mathematical concepts clearly.