Improvement Of the Economic Scenarios Generator

Rapport de Stage Milliman

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Abstract

Ce document offre une exploration approfondie de la Génération de Scénarios Économiques (ESG) dans le secteur de l'assurance. Il commence par une introduction au secteur de l'assurance, mettant en évidence le cadre Solvabilité II et les complexités des actifs et des passifs. Une attention particulière est accordée à la nécessité d'un modèle de valorisation du bilan cohérent avec le marché, ainsi qu'aux défis posés par un générateur de scénarios économiques. La section suivante approfondit la construction d'un ESG, en mettant l'accent sur la modélisation du risque de taux d'intérêt. Elle propose un examen approfondi des composants de la théorie des "Fixed incomes" tels que les obligations zéro-coupon, les swaps de taux d'intérêt et le modèle de diffusion décalée du Libor Market (DDLMM). La modélisation du risque lié aux actions est également discutée, en se concentrant spécifiquement sur le Modèle de Volatilité Constante Par Étapes (SCVM). Une partie importante est consacrée à la réduction du coût computationnel de l'ESG à travers diverses techniques, comme la réduction de scénarios. Le document évoque les avantages pour les petites compagnies d'assurance et présente le Cadre d'Apprentissage Statistique. Cela inclut des aperçus de la structure du cadre, du défi de la dimensionnalité et de l'échantillonnage Quasi Monte Carlo pour la génération de sensibilités de base. La dernière partie aborde la construction de sensibilités stressées, explorant des techniques d'ajustement de la volatilité du marché, et l'Algorithme Monte Carlo Pondéré (WMC). Le contenu se termine par une analyse des résultats numériques, évaluant les résultats d'apprentissage, l'impact du RNG et les prédictions sur les sensibilités choquées.

Keywords: Machine Learning, Fixed incomes, Insurance Industry, Finance, Stochastic Processes,Libor Market model

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Part I

Introduction

Chapter 1

Introduction

1.1 The Insurance Industry

Life insurance corporations and pension funds play a pivotal role as key institutional investors in the economy. They stand out prominently in nations such as France and the UK, where the volume of deposits from policyholders in life insurance savings accounts exceeds that in banks [8]. In 2018, the value of life insurance funds in France soared to a remarkable 1,700 billion, with approximately 80% of these funds funneled into Euro-denominated savings contracts [12]. The amassed funds provide insurance firms with the capacity to perform large-scale investments across various sectors, encompassing the debt market (both corporate and sovereign debt), the stock market, and real estate. As of 2014, insurance companies and pension funds were in possession of nearly 41% of the total outstanding Eurozone sovereign debt [9]. Conventional savings accounts constitute the structural core of the life insurance industry, accounting for over 80% of the total premiums. The risks of these accounts are borne by the insurance firms, while the risks pertaining to unit-linked securities are shouldered by the policyholders [2]. Despite their inherent complexity and the legal restrictions imposed on them, these traditional savings accounts continue to hold substantial importance in the life insurance industry. The high complexity of these products, along with differences from banking rules (Basel settlement), requires changes to the rules that govern financial safety. This is done to consider the company's risk and to safeguard the customers from the chance of the insurance company going bankrupt, therefore policyholders are shielded from the potential insolvency of insurance firms. The Solvency II regulatory framework, used across Europe for the insurance sector, shares certain characteristics with banking regulations, including a pillar approach and risk-based capital. Its core emphasis, however, is placed on protecting policyholders from the insolvency risk of a single insurance company by enhancing their financial stability. It aims to synchronize insurance laws across the European Union, thereby promoting competition and transparency [13]. This is distinct from the Basel III agreements which prioritize systemic risk and the spread of financial contagion effects [10]. Solvency II gives lesser importance to these factors, primarily because insurance companies are relatively less interdependent compared to banks, and also due to the high liquidity demands of the banking sector [14]. It's important to remember that banks and insurance companies function within differing economic frameworks and serve distinct roles in the economy.

1.2 The Solvency II framework

The Solvency II (SII) directive, enacted on January 1, 2016, marked a comprehensive transformation of the regulatory framework that governs the European insurance industry, with an emphasis on enhanced transparency and competitiveness. This restructuring reflects the dynamic changes within insurance and risk management sectors, aiming to standardize the oversight of insurance conglomerates and promote further synergy among regulatory bodies. The SII directive replaced the prior flat-rate Solvency I regulation, adopting a nuanced methodology to capital obligations and control mechanisms that resonate with the individual risk profiles of insurers. It also requires the fair value assessment of assets and liabilities in adherence to market consistency, alongside enforcing more rigorous governance and risk management protocols. Implementation of Solvency II directive sought to amend the inadequacies of the previous solvency system, including its insensitivity to risk, deviation from banking norms (such as Basel settlement), and global standards like IASB/IFRS. Furthermore, it was designed to foster coherence in regulation across EU jurisdictions. With the launch of Solvency II, a transition occurred from a static, rule-based capital requirement to a system sensitive to varying risk factors. This renewed framework requires substantial economic capital, known as Solvency Capital Requirement (SCR), to validate the insurer's ability to meet its financial obligations. Within this structure, the SCR is a vital solvency regulation metric, whereas the Minimal Capital Requirement serves as an indicator for regulatory intervention should capital concentrations descend beneath this threshold. Echoing the banking industry's Basel agreement, Solvency II's tripartite structure contributes significant enhancements to the actuarial field. Pillar one focuses on quantitative prerequisites, encompassing market-consistent valuation of the insurer's financial statement and risk-sensitive capital obligations. The second pillar encompasses qualitative measures, including a governance system with provisions for Enterprise Risk Management, internal control, and compliance, alongside mandating an Own Risk and Solvency Assessment (ORSA) as a core component of internal Risk Management protocols. The third pillar is related to supervisory reporting and public transparency, as detailed by Sandstrom (2016) [17]. Regulators permit two methodologies for the computation of capital requirements: a universal formula based on diverse stress-test modules, and an internal model strategy centered on a quantile of the insurer's one-year loss distribution at a 99.5% confidence interval. Though the standard formula is generally preferred for its accessibility, some insurers choose a partial internal model that substitutes the standard formula for a particular risk component. The primary concentration of our research is directed towards the first pillar of Solvency II. Consequently, a more comprehensive elucidation of this dimension will be provided in the ensuing section.

1.3 Assets & Liabilities lines of an Insurer

1.3.1 The need of a market consistent Balance Sheet Valuation model

In the insurance domain, the valuation of liabilities, which are neither traded nor possess a liquid market, presents unique challenges. For the technical reserves, a "best estimate" is required, utilizing available information through a marked-to-model approach. This corresponds to the expected present value of future liability cash flows, intricately dealing with financial guarantees, discretionary rules, bonus mechanisms, and investment strategies. Asset and Liability Management Models (ALM) and Economic Scenario Generators (ESG) play a critical role in valuing these balance-sheet items.The complexity is further intensified in Solvency Capital assessment, where integration with real-world and risk-neutral pricing scenarios is vital. Challenges include calibrating the risk-neutral ESG to market prices, reproducing interest rate term structures, and discounting cash flows over long maturities without available market data. Methods such as Smith-Wilson interpolation and regulatory curves like the Ultimate Forward Rate (UFR) are used, along with adjustments like Volatility Adjustment (VA) and Credit Risk Adjustments (CRA). ESG , capable of generating both risk-neutral and real-world trajectories, emerges as a critical tool for market-consistent balance sheet valuation. Compliance with Solvency 2 regulatory norms, requiring risk-neutral valuation, finds an effective solution in the ESG methodology. It substantiates the stochastic nature of asset/liability interactions within insurance, justifying the valuation via a risk-neutral approach.Within this framework, our study meticulously discusses the major risk factors R_f that an ESG should elicit, including interest rate models, equity indices, and real estate indices. Key criteria for model selection include:

- No arbitrage-free assumptions
- Ability to reproduce initial rate structures
- Spot price of vanillas at the Money (ATM)
- Capability to generate negative rates
- Simplicity, comprehensive documentation, and ready calibration with market data

1.3.2 A simplified Model of Balance sheet Model Under Solvency II Framework

Pillar 1 is designed to guarantee that an insurer's equity is adequate to resist an economic downfall in 99.5% of potential scenarios within a one-year span. Economic downfall is defined as the juncture when the market value of assets dips below the best estimated liabilities. As a result, it's imperative to ensure that the insurer's balance sheet can maintain non-zero economic equity across a wide spectrum of circumstances, even those that are highly adverse.

We will now delve deeper into a streamlined balance sheet model under the Solvency II framework and describe our principal metrics in a more quantitative manner. These metrics will persistently feature throughout our case study.

Assets	Liabilities
MV_t	NAV_t
	RM_t
	BEL_t

Table 1.1: SII Market value Balance sheet

On the left-hand side, the variable MV_t represents the assets of the company. These encompass all income accrued by the firm, such as premiums from policyholders, which are invested in financial instruments like bonds, stocks, and real estate, for which market values, or quoted prices, are available. Holding the customer's funds now translates into a liability for the insurance company. Under the Solvency II (SII) regime, it is presupposed that the company is in a run-off state. This means that when computing the company's total liabilities, no new business is incorporated into the future projections; potential new policyholders are overlooked. The company stops the sale of new contracts and concentrates on calculating the liabilities towards its existing clientele in the insurer's portfolio.

The most substantial component of liability is the **Best Estimate of Liabilities** (BEL). This is determined as the present value of the company's future cash outflows. It is conceivable that the company's operations are evaluated annually at $t-1,\dots,T$, where T denotes the projection horizon. For a savings portfolio, T could be considerably extended (up to 60 years). Let $(L_t)_{t\leq T}$ denote the liability cash outflows that occur at each year t (like insurers' claim payments, for instance). We consider a filtered probability space $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q}\right)$, where $\mathbb{Q}$ is the risk-neutral pricing measure. Assuming the company utilizes a market consistent, short-rate model $(r_t)_{t>0}$ (calibrated to both the EIOPA regulatory zero-coupon curve and market prices), the mathematical formula for the Best Estimate is:

$$BEL_t = \mathbb{E}^{\mathbb{Q}}\left[\sum_{u=t}^{T} e^{\int_t^u r_x\,ds} L_u \mid \mathcal{F}_t\right]. \tag{1.1}$$

often L_u can be decomposed in the following form (under SII assumptions) :

$$L_u = (\text{Flow}_u - \ \text{Contribution}_u + \ \text{Fee}\ _u - \text{Provisions}_u) \tag{1.2}$$

In the context of traditional insurance, most risks are not replicable, which consequently means that only a small number of insurance liabilities can be replicated. This leads to challenges in accurately assessing these liabilities. To address this issue, a **Value**

of Inforce is added to the Best Estimate of the insurance liability. This Value of Inforce acts as a safety buffer in the calculation of technical provisions, providing an extra layer of protection against unforeseen events. According to the European Directive 2009/138/EC, the Value of Inforce must be calculated in such a way that the value of technical provisions equals the total amount that insurance and reinsurance companies would require in order to assume and fulfill the relevant insurance and reinsurance obligations. Essentially, the Value of Inforce quantifies the additional cost that these firms would demand to take on the risks associated with the obligations. The process of calculating the Value of Inforce involves determining the cost of a suitable amount of the company's own funds that would be necessary to support the assumed obligations. This calculation is complex and is grounded in specific regulatory requirements and financial principles (for further details, see also [15]).

$$VIF_t = CoC \times \mathbf{E}\left[\sum_{u=t}^{T} e^{-\int_u^t r_u du} SCR_u \mid \mathcal{F}_t\right] \tag{1.3}$$

where:

- CoC represents the cost of capital rate, which is set to 6%,
- SCR_u signifies the capital requirement for the period $[t, t+1)$.
- $e^{-\int_u^t r_x ds}$ is the risk-free rate of maturity t.

Predicting future SCR values is a considerable challenge, and even computing SCR_0 is demanding due to the requirement for nested simulations. The calculation of future SCR values is not the main objective of our study case (the SCR can be computed using the standard formula, for more information see [6]). Today, forecasting future SCR values remains an unresolved issue in terms of computational time. At present, the regulator allows simplifications and presumes that future SCR values are proportional to the future Best Estimate. Essentially, this supposition suggests that the amount to be retained for solvency is a fraction of the insurer's best estimate of its current obligations towards its clients.

The sum $VIF + BEL$ represents the overall insurer's debt and is typically referred to as the Technical Provisions. The difference

$$NAV_t = MV_t - (VIF_t + BEL_t) \tag{1.4}$$

is indicative of the company's own funds. The company is declared insolvent if the **Net Asset Value** NAV_t becomes negative because the value of its assets is lower than its technical provisions. A comprehensive introduction to the Solvency II balance sheet is provided in [4].

Finally, we define **TVOG** or **Leakage** as the ratio:

$$TVOG_{\text{BEL}} = \frac{NAV_t}{BEL_{\text{Real}}}, \quad TVOG_{\text{VIF}} = \frac{NAV_t}{VIF_{\text{Real}}}. \tag{1.5}$$

which corresponds to the real error between the Value of Inforce/BE which could be estimated from a model against the actual realized Value of Inforce/BE of an insurer. However, since the real Value of Inforce/BE is only available at the present date, this measure can only be used in the past to backtest a model, thereby assessing the model's relevance.

1.3.3 Description Of an Asset Liabilities Management (ALM) Framework

In the landscape of insurance companies, Asset and Liability Management (ALM) stands out as an indispensable framework. At its core, ALM models the interactions between assets and liabilities, considering factors like revaluation strategies, investment strategies, and policyholder behaviors. These interactions necessitate the implementation of stochastic scenarios to determine the fair value of insurance contract options and guarantees. Using the Euro fund as an example, where policyholders are guaranteed both principal and a minimum return primarily through bond investments, the ALM model becomes vital. As interest rate fluctuations represent a primary risk, an understanding of how changes in the interest rate environment can significantly affect insurers is crucial, especially those offering surrender options in their policies. Literature has shown that factors such as surrender rates, mortality effects, and bonus allocation schemes play critical roles in the risk exposure and solvency of insurance companies. For instance, when bond portfolio returns inch closer to the guaranteed rate, a company's own funds could approach zero, posing solvency threats. The real strength of ALM lies in its capability to value insurance contract options and guarantees in a market-consistent manner, thereby giving a true reflection of an insurer's commitments to its policyholders. This valuation is especially significant when considering the complexities of non-homogeneous contracts and the dynamic interplay between book and market values, as seen in models from countries like Germany. Notably, the consistency of these models, intertwined with the uniqueness of each insurer's portfolio and the regulatory landscape, determines the long-term viability and solvency of the firm. Building upon the principles of Asset and

Liability Management (ALM), insurance firms can derive crucial metrics like the Best Estimate Liability (BEL) and the Solvency Capital Requirement (SCR). BEL represents the present value of future cash flows from insurance obligations, while SCR indicates the capital required to cover unforeseen risks, ensuring that the firm can meet its obligations over a one-year period with a 99.5% confidence level. When utilizing an ALM model, BEL can be calculated by projecting the future cash flows of both assets and liabilities under various stochastic scenarios. This projection takes into account policyholder behavior, market conditions, and the intricate interactions between assets and liabilities. The SCR, on the other hand, is determined by simulating the impact of predefined stress scenarios on the ALM model, focusing on key risks such as market, credit, operational, and underwriting risks. By examining the difference in the balance sheet under these stress scenarios compared to a baseline scenario, insurers can gauge the capital buffer necessary to absorb severe unexpected losses. Essentially, the ALM model acts as a dynamic tool, offering a holistic view of an insurance firm's financial health, ensuring that it remains resilient even in adverse market conditions. Economic Scenario Generators

(ESGs) are vital for Asset and Liability Management (ALM) systems. By producing varied future economic scenarios, ESGs provide essential inputs for ALM models, enabling projections of assets and liabilities across diverse economic conditions. This integration allows firms to stress-test portfolios and assess their resilience, making ESGs indispensable for a comprehensive ALM approach.

1.4 Role and Challenges of an Economic Scenario Generator (ESG)

Economic scenarios embody potential changes in economic and financial magnitudes, including interest rates, inflation rates, equity returns, and real estate returns. An Economic Scenario Generator (ESG) functions as a projection tool for these economic and financial risk factors, providing input parameters for models used to calculate the economic value of assets and liabilities [16]. ESGs should be tailored to the insurer's risk profile, based on robust mathematical and economic theories, and calibrated with reliable data. In life insurance, when evaluating the insurer's obligations, simulation techniques are crucial for considering options and guarantees resulting from interactions between the insurer's assets and liabilities. Insurance companies are particularly susceptible to interest rate risk. A decrease in rates boosts the value of bonds and immediate latent gains, but it also escalifies medium and long-term obligations Conversely, an increase in rates lowers the market value of bonds and thereby reduces latent wealth. An ESG helps to anticipate changes in market conditions and quantify the impact of deviations in interest rates. Modelling insured behaviour is also part of an ESG's function. By generating numerous economic scenarios, an ESG facilitates the estimation of circumstantial surrenders based on the gap between the served rate and a representative market benchmark rate. The Market-Consistent Embedded Value (MCEV), which discounts flows at the risk-free rate and **requires risk-neutral scenarios**, is another standard that necessitates an ESG. This standard, among other things, requires the determination of the Time Value of Financial Options and Guarantees (TVFOG), estimated as the difference between the Best Estimate Liability (BEL) derived from stochastic scenarios generated by an ESG, and the BEL obtained from a central deterministic scenario.

1.4.1 Risk neutral vs Real World for Valuation

In assessing solvency, the risk manager employs a two-stage simulation process, leveraging both real-world and risk-neutral probability measures. The outer simulations model the 'real' evolution of risk factors under the real-world probability measure, $\mathbb{P}$. These simulations represent the most probable trajectory of financial markets, aiming to pinpoint the least likely scenarios that could jeopardize the firm's solvency. On the other hand, the inner simulations, which are tasked with re-pricing assets and liabilities, operate under the risk-neutral measure $\mathbb{Q}$. These scenarios value the portfolio conditional on the state of the economy, aligning with market-based valuations as mandated by Solvency II. This dualistic approach effectively bridges the need to depict realistic financial behavior with the imperative for market-consistent valuation of the portfolio.

Part II

Building of an ESG

Chapter 2

Interest rate Risk Modelling

2.1 Basics Elements of Fixed Incomes Theory

We introduce Basics interest rates Theory and instruments quoted on market. We then turn our attention on Interest rates model and more specifically the LIBOR Market Model (LMM) to better understand the Displaced Diffusion LIBOR Market Model (DD LMM), underscores its pertinence in the pursuit of a market-consistent balance sheet valuation model.The following section is greatly inspired from [5]

2.1.1 Zero Coupon Bonds and Forward rate

We first introduce our first notion of Time value of money named Discount Factor ,with is an essential risk factors R_f directly given by a standard GSE model.

Definition 2.1.1: Discount Factor/Deflator

The (stochastic) discount factor $D(t,T)$ between two time instants t and T is the amount at time t that is "equivalent" to one unit of currency payable at time T, and is given by :

$$D(t,T) = \exp\left(-\int_t^T r_s ds\right) \tag{2.1}$$

Remark By convention, we will denote $D(0,t) = D(t)$ in future

One can see from the very Definition of the Deflator that to be able to replicate a large a wide range of financial instrument one has to provide a dynamic (could be stochastic or determinist) for the short rate r_s.Indeed, the primary interest of the Deflator is to demonstrate that, under a proper measure namely risk neutral measure, it is possible to replicate any options of a financial contract. This is the fundamental theorem of replication.

Theorem 2.1.1: Fundamental Theorem Of Valuation

Consider a financial market with an interest rate r_s, and let H_T denote the value of a certain payoff at time T. The discounted value of this payoff at an earlier time t can be represented as a conditional expectation under the risk-neutral measure $\mathbb{Q}$:

$$H_t = \mathbb{E}^{\mathbb{Q}} \left\{ D(t,T) H_T \mid \mathcal{F}_t \right\} = \mathbb{E}^{\mathbb{Q}} \left\{ e^{-\int_t^T r(s)ds} H_T \mid \mathcal{F}_t \right\} \tag{2.2}$$

We now turn to other basic definitions concerning the interest-rate world :

Definition 2.1.2: Zero-coupon-bond

A T-maturity zero-coupon bond (pure discount bond) is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $t < T$ is denoted by $P(t,T)$. Clearly, $P(T,T) = 1$ for all T.

The distinction between the previously defined deflator and the price of a zero-coupon bond lies in the possible stochastic nature of the short rate. Indeed, if r is deterministic, then for every pair (t,T), we have $D(t,T) = P(t,T)$. If r is a stochastic process adapted to $(\mathcal{F}_t)$ and integrable with respect to time, we then have:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}} \left\{ e^{-\int_t^T r(s)ds} \mid \mathcal{F}_t \right\} \tag{2.3}$$

It is also noteworthy that the measurement of time between the moments t and T plays a significant role in the valuation of a financial product. The choice of this measure is a key element in the quotations on the financial market.

In the following, by slight abuse of notation, t and T will denote both times, as measured by a real number from an instant chosen as time origin 0 , and dates expressed as days/months/years.

Definition 2.1.3: Time to Maturity

The time to maturity $T - t$ is the amount of time (in years) from the present time t to the maturity time $T > t$.

Definition 2.1.4: Day-count convention

denote by $\tau(t,T)$ the chosen time measure between t and T, which is usually referred to as year fraction between the dates t and T. When t and T are less than one-day distant (typically when dealing with limit quantities involving time to maturities tending to zero), $\tau(t,T)$ is to be interpreted as the time difference $T-t$ (in years). The particular choice that is made to measure the time between two dates reflects what is known as the day-count convention.

On the bond markets, several conventions are employed:

- Actual/365: in this system, years comprise 365 days
- Actual/360: this convention corresponds to years consisting of 360 days
- 30/360: the months contain 30 days, and the years 360 days
- ...

The following definitions introduce the various zero-coupon interest rates associated with a zero-coupon bond.

Definition 2.1.5: Simply-compounded spot interest rate/Libor Rate

The simply-compounded spot interest rate prevailing at time t for the maturity T is denoted by $L(t,T)$ and is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from $P(t,T)$ units of currency at time t, when accruing occurs proportionally to the investment time. In formulas:

$$L(t,T) := \frac{1-P(t,T)}{\tau(t,T)P(t,T)} \tag{2.4}$$

In the context of market analysis, LIBOR (London Interbank Offered Rate) rates are recognized as simply-compounded rates. This particular characteristic leads us to represent such rates by the symbol L. Typically, LIBOR rates are connected to zero-coupon bond prices through the use of the Actual/360 day-count convention, employed to calculate the value of $\tau(t,T)$.

Let us define forward rates, characterized by three time instants: the time t, expiry T, and maturity S, with $t \leq T \leq S$. These rates can be explained using a forward-rate agreement (FRA), a contract that allows one to lock-in an interest rate for the period between T and S. Formally, a FRA involves:

- Three time instants: current time t, expiry time $T > t$, and maturity time $S > T$.

- A fixed payment based on a fixed rate K, exchanged against a floating payment based on the spot rate $L(T, S)$, resetting in T and maturing in S.

The contract value at S of a **FRA**, given by:

$$N\tau(T, S)(K - L(T, S)) \tag{2.5}$$

where N is the contract's nominal value, and the rates are simply compounded.

Starting from the definition of $L(T, S)$ and noting that an amount of $1/P(T, S)$ units at time S is equivalent to $P(t, T)$ units at time t, the value of a forward contract evaluated at time t is given by:

$$FRA(t, T, S, \tau(T, S), N, K) = N[P(t, S)\tau(T, S)K - P(t, T) + P(t, S)] \tag{2.6}$$

There exists a unique value K that nullifies the contract at date t. It is referred to as the forward rate and justify the following definition :

Definition 2.1.6: The simply-compounded forward interest rate

The simply-compounded forward interest rate prevailing at time t for the expiry $T > t$ and maturity $S > T$ is denoted by $F(t; T, S)$ and is defined by

$$F(t; T, S) := \frac{1}{\tau(T, S)} \left(\frac{P(t, T)}{P(t, S)} - 1 \right). \tag{2.7}$$

It is that value of the fixed rate in a prototypical FRA with expiry T and maturity S that renders the FRA a fair contract at time t.

To value a Forward Rate Agreement (FRA), one simply needs to replace the LIBOR rate $L(T, S)$ in the payoff (2.5) with the corresponding forward rate $F(t; T, S)$, and then compute the present value of the resulting (deterministic) quantity. The forward rate $F(t; T, S)$ can thus be considered as an estimation of the future spot rate $L(T, S)$, which is stochastic at time t, based on the prevailing market conditions at time t.

2.1.2 Interest rate swap

Before seeing How the EIOPA curve can be obtained , we give a definition of a classical instruments in Fixed incomes products.

A forward swap contract involves a schedule of dates T_i where i ranges from 0 to M, and $T_0 = t \leq T_1 \leq \ldots \leq T_M$. For given α and β within the range $0 \leq \alpha < \beta \leq M$, we consider the time span $\mathcal{T} = \{T_\alpha, \ldots, T_\beta\}$ and intervals $\tau_i = T_i - T_{i-1}$, $i \in \{\alpha + 1, \beta\}$.

Definition 2.1.7: Swap contract

A receiver forward swap contract, with notional N, maturity T_α, and tenor $T_\beta - T_\alpha$, evaluated at time t, pays a fixed flow $\tau_i K$ (the fixed leg) and receives a variable flow $\tau_i L(T_{i-1}, T_i)$ (the variable leg) at each date within the range $T_{\alpha+1}, \ldots, T_\beta$. The discounted value of this contract is:

$$\begin{aligned} \mathrm{RFS}(t, T, N, K) &= N \sum_{i=\alpha+1}^{\beta} P(t, T_i) \tau_i (K - F(t, T_{i-1}, T_i)) \\ &= NP(t, T_\beta) - NP(t, T_\alpha) + N \sum_{i=\alpha+1}^{\beta} K \tau_i P(t, T_i) \end{aligned} \tag{2.8}$$

A payer forward swap contract, conversely, pays the fixed leg and receives the variable leg, with a discounted value given by:

$$PFS(t, T, N, K) = N \sum_{i=\alpha+1}^{\beta} P(t, T_i) \tau_i (F(t, T_{i-1}, T_i) - K) \tag{2.9}$$

Definition 2.1.8: Forward rate swap

The forward swap rate, evaluated at time t, denoted $S_{\alpha,\beta}(t)$, for the date schedule $\mathcal{T}$, is the fixed rate that neutralizes the value of the swap contract, resulting in:

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)} \tag{2.10}$$

The swap rate, evaluated at time t, denoted $S_\beta(t)$, is the forward swap rate defined for $T_\alpha = t$. Formally, it is given as:

$$S_\beta(t) = \frac{1 - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)} \tag{2.11}$$

These definitions encapsulate essential concepts in the financial derivatives space, outlining the structure, valuation, and rates associated with forward swap contracts.

2.1.3 EIOPA yield Curve

Forward LIBOR rates are not directly traded in the market, nor does one typically observe a continuum of discount bonds $P(\cdot)$ for every conceivable maturity. It is possible to observe the prices of actual zero-coupon bonds issued by governmental authorities, but it is neither advisable nor standard practice to interpolate these observed prices and use them to compute forward rates. Instead, market quotes from liquidly traded fixed-income securities are utilized as benchmarks, and a LIBOR curve is extracted from these.

Following conventional terminology in the literature, this curve will simply be referred to as the yield curve. Benchmark securities may include forward rate agreements (FRAs) and swaps. As delineated in Section 2.1.2 on swap pricing, swap rates can be dissected into a set of discount bonds, and thus, given market quotes of swaps for varying maturities (e.g., 2Y, 5Y, ..., 30Y), one can infer discount bonds from these swap market quotations.

A core construct that is derived from the market data of interest rates is the zero-coupon curve at a specific date t. This curve represents the function that maps maturities into rates at times t. The details are as follows:

Definition 2.1.9: Zero-coupon curve

he zero-coupon curve (also commonly referred to as the "yield curve") at time t is the graph of the function

$$T \mapsto \begin{cases} L(t,T) & t < T \leq t+1 \text{ (years)}, \\ Y(t,T) & T > t+1 \text{ (years)}. \end{cases} \tag{2.12}$$

where $Y(t,T)$ is a constant rate and $L(t,T)$ is the Libor rate

Such a curve is alternatively known as the term structure of interest rates at time t. It portrays, at time t, simply-compounded interest rates for all maturities T up to one year, and annually compounded rates for maturities T greater than one year.

The zero-coupon rate curve defined by EIOPA is constructed based on the swap rate curve, to which the following adjustments are applied:

- **Credit Risk Adjustment** (CRA): This aims to adjust the credit spread between overnight interbank swap rates and IBOR-based swap rates.
- **Volatility Adjustment** (VA): This replaces the counter-cyclical premium and corresponds to a premium applied to the entire rate curve to smooth the effects of spread movements on the bond assets of the prudential balance sheet.

The 20-year maturity for the swap rate is considered the Last Liquid Point (LLP) of the curve. Beyond the LLP, the rate curve is extrapolated over a period of 40 years using the Smith-Wilson method, with convergence to the Ultimate Forward Rate (UFR) fixed at 4.2%. The convergence point is set at 40 years after the LLP for the Eurozone.

2.1.4 Change Of Numeraire

The change of numeraire is a fundamental method used to derive closed-form solutions for the pricing of derivative options. In mathematical terms, the choice of numeraire and

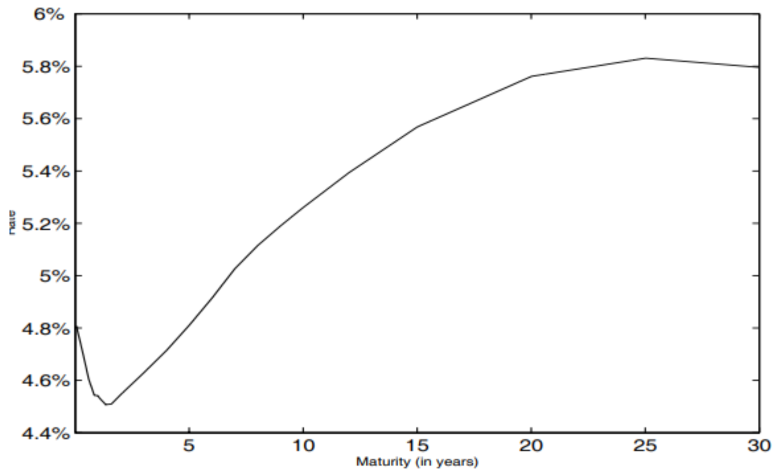

Figure 2.1: Zero-coupon curve stripped from market EURO rates on February 13, 2001 , at 5 p.m. [5]

corresponding measure shift allows for simplifying the pricing formulas and is employed multiple times throughout this study.

Definition 2.1.10: Numeraire

A numeraire is any strictly positive financial instrument that does not pay intermediate dividends. For example, a zero-coupon bond $P(t, T)$, evaluated at time t with maturity T, is a numeraire.

Theorem 2.1.2: Change of Numeraire

N and a probability $\mathbb{Q}^N$ equivalent to the historical probability $\mathbb{P}$, such that the price of any financial instrument without intermediate payment X, normalized by N, is a martingale under $\mathbb{Q}^N$:

$$\frac{X_t}{N_t} = \mathbb{E}^{\mathbb{Q}^N}\left[\frac{X_T}{N_T} \mid \mathcal{F}_t\right], \quad 0 \leq t \leq T \tag{2.13}$$

Let U be an arbitrarily chosen numeraire. Then there exists a probability $\mathbb{Q}^U$ equivalent to $\mathbb{P}$ such that the price of any financial instrument Y, normalized by U, is a martingale under $\mathbb{Q}^U$:

$$\frac{Y_t}{U_t} = \mathbb{E}^{\mathbb{Q}^U}\left[\frac{Y_T}{U_T} \mid \mathcal{F}_t\right], \quad 0 \leq t \leq T \tag{2.14}$$

Furthermore, the Radon-Nikodym density defining the probability $\mathbb{Q}^U$ is given by:

$$\frac{d\mathbb{Q}^U}{d\mathbb{Q}^N} = \frac{U_T N_0}{U_0 N_T} \tag{2.15}$$

By the definition of Radon-Nykodym density, for any financial instrument price Z, we have:

$$\mathbb{E}^{\mathbb{Q}^N}\left[\frac{Z_T}{N_T}\right] = \mathbb{E}^{\mathbb{Q}^U}\left[\frac{Z_T}{N_T}\frac{d\mathbb{Q}^N}{d\mathbb{Q}^U}\right] \tag{2.16}$$

We also have:

$$\mathbb{E}^{\mathbb{Q}^N}\left[\frac{Z_T}{N_T}\right] = \mathbb{E}^{\mathbb{Q}^U}\left[\frac{U_0 Z_T}{N_0 U_T}\right] \tag{2.17}$$

Consequently, the Radon-Nykodym density facilitates the transition from one universe to another (change of numeraire).

The fundamental implication of this theorem is that the price of any financial instrument divided by a numeraire is a martingale (absence of drift of the underlying dynamic) under the probability associated with that numeraire. The choice of numeraire enables the definition of the two most commonly used pricing universes, detailed below.Many applications of this theorem will be thefore shown in the next parts

2.1.5 Derivatives : Cap,Floor and Swaptions

We begin by examining the cap case. Consider an Actor that is indebted at the LIBOR rate and obligated to make payments at specified moments $T_{\alpha+1}, \ldots, T_\beta$ according to the LIBOR rates resetting at times $T_\alpha, \ldots, T_{\beta-1}$, with corresponding year fractions denoted

by $\tau = \{\tau_{\alpha+1}, \ldots, \tau_\beta\}$. For the purpose of this investigation, we assume that $N = 1$, and we define $\mathcal{T} = \{T_\alpha, \ldots, T_\beta\}$.
The Actor, aware of the potential increase in LIBOR rates in the future, may decide to safeguard its interests by setting the payment at a maximum cap rate K.The organization discharges its debt using the LIBOR rate L and in return receives $(L-K)^+$ from the cap contract. Thus, the net payment when considering both agreements can be expressed as:

$$L - (L-K)^+ = \min(L, K). \tag{2.18}$$

A cap can be viewed as a derivatives instrument employed to mitigate the risks of substantial fluctuations in interest rates when indebted at a variable (LIBOR) rate.Further analysis reveals that a cap contract payoff can be represented as a summation of terms such as:

$$\sum_{i=\alpha}^{\beta} D(t, T_i)\, N \tau_i \left(L(T_{i-1}, T_i) - K\right)^+. \tag{2.19}$$

In accordance with market conventions, the pricing of a cap can be modeled through the cumulative sum of Black's formulas at time zero, as given by:

$$\mathrm{Cap}^{\mathrm{Black}}\left(0, \mathcal{T}, \tau, N, K, \sigma_{\alpha,\beta}\right) = N \sum_{i-\alpha+1}^{\beta} P(0, T_i)\, \tau_i \mathrm{Bl}\left(K, F(0, T_1 - 1, T_i), v_i, 1\right) \tag{2.20}$$

Where $v_i = \sigma_{\alpha,\beta}\sqrt{T_{i-1}}$

We Remind the Black formula :

The core of Black's formula:

$$\mathrm{Bl}(K, F, v, \omega) = F\omega\Phi\left(\omega d_1(K, F, v)\right) - K\omega\Phi\left(\omega d_2(K, F, v)\right),$$

with,

$$d_1(K, F, v) = \frac{\ln(F/K) + v^2/2}{v}, d_2(K, F, v) = \frac{\ln(F/K) - v^2/2}{v},$$

where ω is either -1 or 1 and is meant to be 1 when omitted. The arguments of d_1 and d_2 may be omitted if clear from the context.

where Φ is the standard Gaussian cumulative distribution function, and the volatility parameter $\sigma_{\alpha,\beta}$ is obtained from market quotes. Analogously, the corresponding floor is priced in a similar fashion:

$$\mathrm{Flr}^{\mathrm{Black}}\left(0,\mathcal{T},\tau,N,K,\sigma_{\alpha,\beta}\right)=N\sum_{i=\alpha+1}^{\beta}P\left(0,T_i\right)\tau_i\,\mathrm{Bl}\left(K,F\left(0,T_{i-1},T_i\right),v_i,-1\right). \quad (2.21)$$

In the marketplace, there exists a variety of caps and floors, each with unique implied volatilities as quoted by market participants. For instance, certain quotes may have $\alpha = 0, T_0$ equal to three months, with subsequent T_i values spaced at three-month intervals, while other examples include quotes where T_0 is three months, the following T_i values up to one year are spaced equally at three-month intervals, and all remaining T_i values are spaced at six-month intervals.

Definition 2.1.11: Cap and Floor

a cap (or floor) with payment times $T_{\alpha+1},\ldots,T_\beta$, corresponding year fractions $\tau_{\alpha+1},\ldots,\tau_\beta$, and a strike K. A cap (or floor) is deemed to be at-the-money (ATM) if and only if:

$$K=K_{ATM}:=S_{\alpha,\beta}(0)=\frac{P\left(0,T_\alpha\right)-P\left(0,T_\beta\right)}{\sum_{i=\alpha+1}^{\beta}\tau_i P\left(0,T_i\right)}. \quad (2.22)$$

Furthermore, a cap is classified as in-the-money (ITM) if $K < K_{ATM}$, and out-of-the-money (OTM) if $K > K_{ATM}$, with the inverse criteria holding for a floor.

Definition 2.1.12: Swaption

A payer swaption is a contract that grants the holder the right to enter into a payer swap with strike K, at the date T_α, called the swaption's maturity. The duration of the swap, $T_\beta - T_\alpha$, is referred to as the tenor. The discounted payoff of a payer swaption is given by:

$$D\left(t,T_\alpha\right)\left(\sum_{i=\alpha+1}^{\beta}P\left(T_\alpha,T_i\right)\tau_i\left(F\left(T_\alpha,T_{i-1},T_i\right)-K\right)\right)^{+} \quad (2.23)$$

Analogously, a swaption is receiver if the holder, upon exercising the option, receives the fixed rate, and payer if they pay it. Its discounted payoff is then:

$$D\left(t,T_\alpha\right)\left(\sum_{i=\alpha+1}^{\beta}P\left(T_\alpha,T_i\right)\tau_i\left(K-F\left(T_\alpha,T_{i-1},T_i\right)\right)\right)^{+} \quad (2.24)$$

It is a standard practice in the market to value swaptions using a formula that resembles the Black model. Specifically, the price of the aforementioned payer swaption (at time zero) is given by:

$$\mathrm{PS}^{\mathrm{Black}}\left(0,\mathcal{T},\tau,N,K,\sigma_{\alpha,\beta}\right)=N\,\mathrm{Bl}\left(K,S_{\alpha,\beta}(0),\sigma_{\alpha,\beta}\sqrt{T_\alpha},1\right)\sum_{i=\alpha+1}^{\beta}\tau_i P\left(0,T_i\right), \tag{2.25}$$

where $\sigma_{\alpha,\beta}$ now denotes a volatility parameter quoted in the market, which differs from the corresponding $\sigma_{\alpha,\beta}$ in the caps/floors case.

This formal description presents a mathematical way to model the pricing of swaptions according to the market's established practice, incorporating specific variables and parameters.An important remark is that, if we consider a swaption at the money, is it possible to obtain a closed formula of the forward swap rate $S_{\alpha,\beta}$ like in definition 2.22 In the practical application of our study case (Part II), we will use this approach to perform a complete repricing of swaptions to obtain rate volatilities surface.

2.1.6 Review of Rate Models for Risk neutral Valuations

To review the most popular interest rate models used in the industry, we provide a summarized table based on [5], detailing the advantages and drawbacks of each model. It's worth noting that under the SII assumptions, all models presented here are risk-neutral; real-world cases are not addressed in this contex.

Table 2.1: Summary of Rates Models from [5]

Model	AOA	Closed formula	rate ≤ 0	Smile	Calibration	Diffusion
Vasicek	**N**	**Y**	**Y**	-	-	-
HW	**Y**	**Y**	**Y**	-	-	-
CIR	**N**	**Y**	**N**	-	-	-
BK	**Y**	**N**	**N**	-	++	-
CIR++	**Y**	**Y**	**Y** (bounded)	-	-	-
Vasicek 2F	**N**	**Y**	**Y**	-	-	-
G2++	**Y**	**Y**	**Y**	-	+	-
LMM	**Y**	**Y**	**N**	-	+	+
DD LMM	**Y**	**Y**	**Y** (bounded)	+	+	+
LMM+	**Y**	**Y****	**Y** (bounded)	++	+++	++

Abbreviations: Y: Yes / N: No / -: None or weak / +: Moderate / ++: Strong / +++: Very strong / *: Semi-closed on swaptions / **: Semi-closed
Criteria for choice:

- AOA: No arbitrage opportunities
- CF: Existence of closed-form solutions on swaptions

- Rate ≤ 0: Generation of negative rates by the model
- Smile: Replication of the volatility smile
- Calibration: Complexity of model calibration
- Diffusion: Complexity of model diffusion

2.2 The Displaced Diffusion Libor Market Model (DDLMM)

In the domain of market-consistent balance sheet valuation, particularly within the constraints specified for the Economic Scenario Generator (ESG), the selection of an appropriate model becomes vital. Our examination encompasses models categorized into one-factor short-rate models, two-factor short-rate models, and market models. The one-factor short-rate models are characterized by their limited adaptability, stemming from their dependence on a singular source of randomness. Although two-factor short-rate models address some of these constraints by introducing an additional random factor, they remain limited in scope. The market models, comprising the basic LIBOR Market Model (LMM) and its derivatives, emerge as notably applicable in the current context, modeling the evolution of LIBOR forward rates. However, several models within this category are precluded from consideration due to their inability to comply with critical market characteristics or the criteria previously outlined of market consistency. The basic LMM, while prevalent, reveals significant deficiencies, such as its exclusive generation of positive forward rates and failure to replicate the volatility smile. These limitations are strategically overcome by the Displaced Diffusion LIBOR Market Model (DDLMM), which not only introduces a shift but also integrates stochastic volatility to enhance the replication of the volatility smile. Despite its inherent complexity in calibration and diffusion, the DDLMM stands out as the preferred choice. Its alignment with the essential criteria, including the ability to generate negative rates and its aptitude for reproducing the term structure of initial rates, renders it particularly relevant for our study.

2.2.1 Relationship Between The Libor Market Model and Displaced Diffusion Libor Market Model (DDLMM)

The LIBOR Market Model, also known as the Brace-Gatarek-Musiela (BGM) model, is based on modeling the forward LIBOR rates using stochastic differential equations. The model allows each forward LIBOR rate to have its own volatility structure. The LMM is designed to be consistent with the current term structure of interest rates, which makes it useful for pricing interest rate derivatives. In the LMM, the forward LIBOR rates are modeled as lognormal random variables, which implies that interest rates are strictly positive. This lognormal assumption may lead to issues when dealing with negative interest rates or other market anomalies. The Displaced-Diffusion LIBOR Market Model is an extension of the basic LMM that attempts to address some of the limitations of the lognormal distribution assumption. In the Displaced-Diffusion model, the forward

LIBOR rates are transformed by a displacement factor, which shifts the distribution. This shift allows the model to accommodate a wider range of rate behaviors, including negative rates. After applying the displacement, a lognormal diffusion process is applied to the shifted rates. The key advantage of the Displaced-Diffusion model over the standard LMM is its flexibility. By adding the displacement factor, the model can better handle extreme market conditions and offers a more realistic representation of the distribution of interest rates.The following section will Introduce the Displaced Libor market model.

2.2.2 Dynamic of the DDLMM

Consider, for all $k \in \mathbb{N}^*$, for all $t \in [0, k]$, $\tilde{L}_i(t)$ the displaced forward Libor rate (for any displacement factor $\delta > 0$). This variable is built from the definition of $L_i(t)$ the forward Libor rate, of expiration date T_k and of maturity T_{k+1} by

$$\tilde{L}_i(t) = L_i(t) + \delta \tag{2.26}$$

for the following parts , to not abuse Notations we will simply denote $\tilde{L}_k(t)$ as $L_k(t)$. We are given a tenor structure,

$$0 \leq T_0 < T_1 < \ldots < T_N.$$

The Displaced Diffusion LIBOR Market Model assumes a system of stochastic differential equations for the joint evolution of N forward LIBOR rates under a $\mathbb{P}$ such that

$$dL_i(t) = \mu_i(t)L_i(t)dt + \sigma_i(t)L_i(t)dW_i(t) \quad i = 0, \ldots, N-1, \tag{2.27}$$

where $W_i(t)$ denotes instantaneously correlated Brownian motions with

$$dW_i(t)dW_j(t) = \rho_{ij}(t)dt. \tag{2.28}$$

We let $\rho = (\rho_{ij}(t))_{i,j=0,\ldots,N-1}$ be the instantaneous correlation matrix.Therefore the LMM can be seen as a collection of N Black models which are simultaneously evolved under a unified measure.

We now prove an important theorem that will be useful for subsequent parts to justify that the Libor rate is a martingale under the right measure.

Theorem 2.2.1:

Let $N(t)$ be a Numeraire and $\Pi(t)$ an underlying. Assume that the normalized price process $\Pi(t)/N(t)$ is a Q^N-martingale. If the density of the Radon-Nikodym derivative (likelihood process) is given by

$$\zeta(t) = \frac{M(t)/M(0)}{N(t)/N(0)}, \tag{2.29}$$

then $M(t)$ is a martingale measure for $\Pi(t)$.

Proof.

from Change of Numeraire theorem we have

$$\begin{aligned}
\mathbb{E}^{Q_M}\left(\frac{\Pi(t)}{M(t)} \mid \mathcal{F}(s)\right) &= \zeta(s)^{-1}\mathbb{E}^{Q_N}\left(\zeta(t)\frac{\Pi(t)}{M(t)} \mid \mathcal{F}(s)\right) \\
&= \zeta(s)^{-1}\mathbb{E}^{Q_N}\left(\frac{M(t)/M(0)}{N(t)/N(0)}\frac{\Pi(t)}{M(t)}\zeta(t) \mid \mathcal{F}(s)\right) \\
&= \zeta(s)^{-1}\frac{N(0)}{M(0)}\mathbb{E}^{Q_N}\left(\frac{\Pi(t)}{N(t)} \mid \mathcal{F}(s)\right) \\
&= \zeta(s)^{-1}\frac{N(0)}{M(0)}\frac{\Pi(s)}{N(s)} \\
&= \frac{N(s)/N(0)}{M(s)/M(0)}\frac{N(0)}{M(0)}\frac{\Pi(s)}{N(s)} \\
&= \frac{\Pi(s)}{M(s)}
\end{aligned} \tag{2.30}$$

□ □

Returning to theorem 1.5.1 , we know that by the risk-neutral pricing formula

$$\frac{P(t,T)}{\beta(t)} = \mathbb{E}^{Q}\left(\frac{1}{\beta(T)} \mid \mathcal{F}(t)\right), \tag{2.31}$$

so $P(t,T)$ is a Q-martingale. Applying this to the normalized Bond, then $\frac{P(t,T)}{P(t,T+\tau)}$ is a $Q^{T+\tau}$-martingale. Since the forward LIBOR rate is defined by :

$$L(t,T,T+\tau) = \tau^{-1}\left(\frac{P(t,T)}{P(t,T+\tau)} - 1\right) \tag{2.32}$$

It is a $Q^{T+\tau}$-martingale as well.

Under this T_{i+1}-forward measure, it holds that

$$dL_i(t) = \sigma_i(t)L_i(t)dW^{i+1}(t), \tag{2.33}$$

where $W^{i+1} \stackrel{\text{def}}{=} W^{Q_{T_{i+1}}}$ is a Brownian motion under $Q^{T_{i+1}}$.
Importantly, only one LIBOR rate can be a martingale once we opt for a specific numeraire, while the others are in general not martingales. In order to establish an arbitrage-free framework, we would like all rates to be martingales under a single common measure. We conclude that once a choice for a numeraire is made, the other rates need a drift adjustment to obey the martingale property.

A convenient choice to start with is to choose the discount bond $P(t, T_N)$ which induces the terminal measure Q^{T_N}. We then have the following theorem:

Theorem 2.2.2: Dynamic of the Libor Under the Terminal Measure

The arbitrage-free dynamics for the system of LIBOR forward rates under Q^{T_N}

$$dL_i(t) = L_i(t)\left(-\sum_{j=i+1}^{N-1} \frac{\tau_j L_j(t)}{1+\tau_j L_j(t)} \sigma_i(t)\sigma_j(t)\rho_{ij}(t)\right) dt + \sigma_i(t) L_i(t) dW_i^{Q_{T_N}}(t) \tag{2.34}$$

with $\quad i = 0, \ldots, N-1$

a proof of this fact can be found in Appendix [5]

We note that for $i = N-1$, the sum $\sum_{j=i+1}^{N-1}(\cdot)$ is empty so that the forward rate $L_{N-1}(t)$ has no drift adjustment, thus following the SDE

$$dL_{N-1}(t) = \sigma_{N-1}(t) L_{N-1}(t) dW^N(t)$$

The description of the Libor Market Model (LMM) dynamics has remained broad and general up until now. To implement a concrete model within this framework, specific functional forms must be chosen for the volatility function, $\sigma_i(t)$, and the correlation function, $\rho_{ij}(t)$. Given the model's high dimensionality pragmatic choices must be made. It is improbable to obtain reliable results for all $N(N-1)/2$ correlation parameters simply by calibrating to a limited set of market prices for actively traded derivatives. Reasonable specifications are therefore required to manage the complexity and provide meaningful insights.

2.2.3 Pricing in a DDLMM Model

before going further in the calibration problem, we recall the price of a Caplet in the DDLMM. By AOA, the caplet price is given by the expected discounted payoff under the risk neutral probability $\mathbb{Q}$:

$$\text{Caplet}\left(0, T_i, T_{i+1}\right) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{T_{i+1}} r_s ds} \delta_i \left(L_i\left(T_i\right) - K\right)_+\right] \tag{2.35}$$

We change the probability to the $\mathbb{Q}^{T_{i+1}}$-forward probability by taking $N_t = P(t, T_i)$ as numéraire.

$$\frac{d\mathbb{Q}^{T_{i+1}}}{d\mathbb{Q}} = \frac{N_{T_{i+1}}}{N_0} e^{-\int_0^{T_{i+1}} r_s} = \frac{e^{-\int_0^{T_{i+1}} r_s}}{P\left(0, T_{i+1}\right)} \tag{2.36}$$

Therefore:

$$\begin{aligned}\text{Caplet}\,(0,T_i,T_{i+1}) &= \mathbb{E}^{\mathbb{Q}^{T_{i+1}}}\left[\frac{d\mathbb{Q}}{d\mathbb{Q}^{T_{i+1}}}e^{-\int_0^{T_{i+1}} r_s ds}\delta_i\left(L_i\left(T_i\right)-K\right)_+\right] \\ &= \delta_i \mathbb{E}^{\mathbb{Q}^{T_{i+1}}}\left[\left(L_i\left(T_i\right)-K\right)_+\right]\end{aligned} \tag{2.37}$$

Recall that under the forward measure $\mathbb{Q}^{T_{i+1}} L_i$ follows a log-normal diffusion:

$$\frac{dL_i\left(T_i\right)}{L_i\left(T_i\right)} = \gamma_i(t) dW_t^{T_{i+1}} \tag{2.38}$$

Therefore, the problem amounts to value a call option with underlying given by the forward rate $L_i\left(t,T_i,T_{i+1}\right)$ in a Black model with null interest rate $r=0$, and *determinist* volatility $\sigma_i(t)=\sqrt{\frac{1}{T_i}\int_0^{T_i}\gamma_i(s)^2 ds}$, strike K, and maturity T_i:

$$\text{Cap}^{\text{LMM}} = \text{Bl}\left(S_0 = L_i\left(0,T_i,T_{i+1}\right), r=0, K=K, \sigma=\sigma_i(t), T=T_i\right) \tag{2.39}$$

We now Introduce the Case of the swaption which is the main purpose of this part , Indeed the whole calibration of a DDLMM Model is done through a repricing of Swaption,Hence is it crucial to obtain a formula under the DDLM model. Before introducing the case of swaption we give the following Definition :

Definition 2.2.1: Annuity Process

We define the Annuity process $A_{n,N}$ by :

$$A_{n,N}(t) = \sum_{i=n}^{N-1} P\left(t,T_{i+1}\right) \tag{2.40}$$

The annuity Process is a Numeraire.

in a Similar way as 2.28 , it is possible to prove that the swap rate $S(t,T_i,T_j)$ is a martingale under it's associated measure, Hence the price of a Swaptions admit a semi closed formula of the form:

$$\text{Swaption}\,(0,T_n,T_N) = P\left(0,T_n,T_N\right)\mathbb{E}^{\mathbb{Q}^{n,N}}\left[\left(S\left(T_n,T_n,T_N\right)-K\right)_+\right] \tag{2.41}$$

here $\mathbb{Q}^{n,N}$ is the probability under which the annuity process N_t is taken as numéraire.

However One has to find the Dynamic of the Swap rate :

$$S_{n,N}(t) := \frac{P\left(t,T_n\right)-P\left(t,T_N\right)}{A_{n,N}(t)} \quad \text{under} \quad \mathbb{Q}_{n,N} \tag{2.42}$$

we use the Following theorem (proof in Appendix):

Theorem 2.2.3: Swap Rate dynamic Under DDLMM model

Denote by $\mathbb{Q}_{n,N}$ the probability measure with numeraire $A_{n,N}(t)$ under which the swap rate $S_{n,N}(t)$ is a martingale. Let's denote $\left(Z^{\mathbb{Q}}_{n,N}\right)$ the associated Brownian motion, the swap rate dynamic under this probability measure is:

$$dS_{n,N}(t) = \left(S_{n,N}(t) + \delta\right)\left(\sum_{j=n}^{N-1} \frac{w_j^{n,N}(0)\left(L_j(t) + \delta\right)}{\left(S_{n,N}(t) + \delta\right)}\sigma_j(t)\right) dZ^{\mathbb{Q}}_{n,N}(t) \tag{2.43}$$

where:

$$w_j^{n,N}(0) = \frac{P\left(0, T_{j+1}\right)}{A_{n,N}(0)} \tag{2.44}$$

remark the proof of this theorem involve the important Rebonato hypothesis which consist of assuming that $w_j^{n,N}(t) \approx w_j^{n,N}(0)$, for all

given the dynamic of the swap rate we see that $S_{n,N}(t)$ follow a log-normal dynamic, However since the volatility term is stochastic we cannot derive price of swaption yet and a freezing technique is applied , that is :

$$\begin{aligned}\sigma^{\text{det}}_{N,n}(t) &= \left(S_{n,N}(t) + \delta\right)\left(\sum_{j=n}^{N-1} \frac{w_j^{n,N}(0)\left(L_j(t) + \delta\right)}{\left(S_{n,N}(t) + \delta\right)}\sigma_j(t)\right) \\ &\approx \sum_{j=n}^{N-1} w_j^{n,N}(0)\frac{\left(L_j(0) + \delta\right)}{\left(S_{n,N}(0) + \delta\right)}\sigma_j(t)\end{aligned} \tag{2.45}$$

therefore $S_{n,N}(t)$ is a shifted exponential martingale and we obtain:

$$\begin{aligned}\text{Swaption}\left(0, T_n, T_N\right) = \text{Bl}\,\Big(&S = S_{n,N}(0) + \delta, r = 0, K = K, \\ &\sigma = \sqrt{\frac{1}{T_n}\int_0^{T_n} \sigma^{\text{det}}_{n,N}(t)^2 dt}, T = T_n\Big)\end{aligned} \tag{2.46}$$

2.2.4 Calibration of the DDLMM

The calibration of a DDLMM is now straightforward using result 2.44 and consists of solving the following minimization problem. In this section, we focus only on the calibration of the volatility function. The correlation matrix is a more complex topic, but

details can be found in [?].
We define the volatility $\sigma_i(t)$ for $i = 1, \ldots, N$:

$$\sigma_i(t), \quad i = 1, \ldots, N \tag{2.47}$$

These volatilities can be derived from the market data of caps and swaptions. By analyzing the market price of swaptions, we can determine the root-mean-square volatility of the underlying forward, leading to the goal of recovering the instantaneous volatility function $\gamma_i(t)$ as:

$$\left(\sigma_i^{\text{swaption mkt}}\right)^2 = \frac{1}{T_i}\int_0^{T_i} \gamma_i(s)^2 ds \tag{2.48}$$

Many possible choices for the functions $\gamma_i(t)$ yield the same value $\sigma_i^{\text{caplet mkt}}$ as in equation (2.46). A common choice is a time-homogeneous function of the form $\sigma(T_i - t)$. The swaption market often shows that swaption volatility behaves as a "humped shaped" function of time to expiry. A straightforward parametrization to represent this is:

$$g_i(s) = (a + bs)e^{-cs} + d \tag{2.49}$$

Fitting all caplets simultaneously with $\sigma_i(s) = g_i(s)$ is infeasible, so we must rescale the functions g_i by a different factor ϕ_i for each Libor forward rate $L(0, T_i, T_{i+1})$ to price all swaptions correctly. We define ϕ as:

$$\left(\sigma_i^{\text{swaption mkt}}\right)^2 = \phi_i^2 \frac{1}{T_i}\int_0^{T_i} g_i(s)^2 ds \tag{2.50}$$

The calibration process for the instantaneous volatility involves finding parameters $(a, b, c, d) \in \mathbb{R}^4$ that minimize the Mean-Square Error (MSE) between the model volatility and market caplet volatility. We do this using a classic LSMC method:

$$(a^\star, b^\star, c^\star, d^\star) = \operatorname{argmin}_{(a,b,c,d)\in\mathbb{R}^4} \sum_{i=1}^{N} \left(\sigma_i^{\text{DDLMM}}(a, b, c, d)^2 - \sigma_i^{\text{swaptions mkt}}\right)^2 \tag{2.51}$$

where the DDLMM model volatility is given by:

$$\sigma_i^{\text{DDLMM}}(a, b, c, d)^2 = \frac{1}{T_i}\int_0^{T_i} \left((a + bs)e^{-cs} + d\right)^2 ds \tag{2.52}$$

Once the optimal parameters $(a^\star, b^\star, c^\star, d^\star)$ are found, the adjusted shifts $\phi_i, i = 1, \ldots, N$ are determined using equation (2.48):

$$\phi_i = \frac{\sigma_i^{\text{swaptions mkt}}}{\sqrt{\frac{1}{T_i}\int_0^{T_i} \left((a^\star + b^\star s)\, e^{-c^\star s} + d^\star\right)^2 ds}} \tag{2.53}$$

2.2.5 Simulation and Validation of the DDLMM

Before introducing the model difusion of the DDLMM observe we observe the following relationship on Zero coupons price :

$$P(T_k, T_{k+m}) = \prod_{i=0}^{m-1} \frac{P(T_k, T_{k+i+1})}{P(T_k, T_{k+i})} = \prod_{i=0}^{m-1} \frac{1}{1+\Delta F_{k+i}(T_k)} \tag{2.54}$$

This relationship will be useful to compute the forward libor Rate introduced in the DDLMM dynamic.

Let H be the table diffusion horizon (in years), and M be the maximal maturity (also in years) of the yield curve we want to project. We define $p = \frac{1}{\Delta} \in \mathbb{N}^*$. For all $k \in [\![0, (H+M)\times p - 1]\!]$, and for all $j \in [\![0, \min(k, p\times H) - 1]\!]$, the following relation holds:

$$\tilde{L}_k(T_{j+1}) = \tilde{L}_k(T_j)\exp\left[\Delta\left(\sum_{i=j+1}^{k}\left(\frac{\tilde{L}_i(T_j)\sum_{q=1}^{N_f}\sigma_i^q(T_j)\,\sigma_k^q(T_j)}{1+\Delta L_i(T_j)}\right)\right.\right.$$
$$\left. -\frac{1}{2}\|\sigma_k(T_j)\|^2 + \sigma_k(T_j)\times\varepsilon_j\sqrt{\Delta}\right]. \tag{2.55}$$

where, for all $j \in [\![0, p\times H-1]\!]$, $\varepsilon_j := \frac{1}{\sqrt{\Delta}}\left(Z(T_{j+1}) - Z(T_j)\right)$ follows a standard Gaussian distribution under the spot LIBOR measure (where Z is a Brownian motion under this measure). The intuiton behind this scheme is simply that from from (2.33) L follows a Lognormal dynamic where its closed formula is an Exponential martingale.
We can now use two techniques to check if our diffusion scheme is accurately generated and calibrated. The first involves a martingale test, as shown below:

Discount factor: The tests check for every simulation date T_j the equality:

$$P(0, T_j) = \mathbb{E}^*\left[D(T_j)\right] \tag{2.56}$$

where $\mathbb{E}^*$ denotes the expectation under the spot LIBOR measure and where we recall that the discount factor writes:

$$D(T_j) = \prod_{k\leq j-1} P(T_k, T_{k+1}). \tag{2.57}$$

where The expectation is estimated through a Monte-Carlo approach over N_s simulations;

$$\widetilde{\mathbb{E}^*}\left[D(T_j)\right] = \frac{1}{N_s}\sum_{s=1}^{N_s} D^{(s)}(T_j). \tag{2.58}$$

Zeros coupons : we check the equality $P(0,T_j) = \mathbb{E}^*[D(T_i)P(T_i,T_j)]$ for every simulation date $T_i > 0$ and maturity $T_j \geq T_i$. The expectation is estimated through a Monte-Carlo approach:

$$\widetilde{\mathbb{E}^*}[D(T_i)P(T_i,T_j)] = \frac{1}{N_s}\sum_{s=1}^{N_s} D^{(s)}(T_i)P^{(s)}(T_i,T_j). \tag{2.59}$$

Regarding the Repricing test , Similarly , we compute swaptions/caplet price using Monte Carlo simulation with each path and compare it to the theorical value with according calibrated volatility.

Chapter 3

Equity like risk Modelling

This chapter delves into the intricacies of modeling such equity-like risks, drawing contrasts between traditional and contemporary approaches and elucidating the rationale behind our modeling choices.

3.1 Chose of Equity like model

To model an Equity-Like index, our constraints were significantly less stringent.As shown on figure 3.1 The asset allocation of an insurer predominantly focuses on bonds and liabilities. Consequently, an insurer's sensitivity to interest rate fluctuations far surpasses that to equities or real estate. This heightened sensitivity underscores the need for advanced, refined models capable of aptly reflecting the intricacies of interest rate movements and their implications for the insurer's portfolio. As a result, we have chosen a Stepwise Constant volatility model over the traditional Black-Scholes model which add more flexibility to control the volatility parameter.

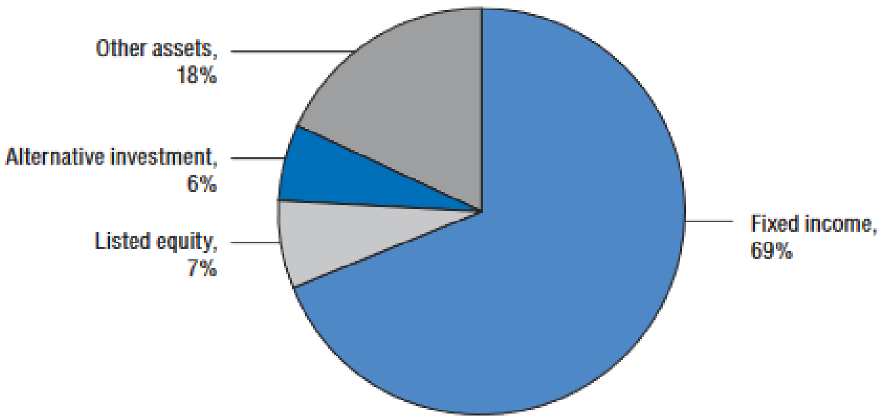

Figure 3.1: Target asset allocation for the longer term in the EU,EEA and Switzerland
Source : *OECD Large Insurer Survey.*

3.2 The Stepwise constant Volatility model SCVM

In a risk-neutral framework, the Stepwise constant Volatility model (SCVM) governs equity total returns. The deterministic local volatility is time-dependent and is chosen to be piecewise constant This adaptability enables the model to calibrate to an entire term structure of option implied volatilities, rather than to just a single point as seen in the constant volatility Black-Scholes model. The following parts provides an in-depth overview of the SCVM model's technical specifications, encompassing its modeling structure, calibration, simulation, and validation processes.

3.2.1 Dynamic of the SCVM

the price at time t of a zero-coupon bond maturing at T is denoted $P(t,T)$. In addition, we set D the discount factor process associated to the risk neutral measure.we consider a fixed horizon T and a tenor structure with annual dates

$$T_0 = 0 < T_1 < \ldots < T_{N-1} < T_N = T \quad \text{with} \quad \Delta T_k = T_{k+1} - T_k \quad \text{for} \quad k = 0, 1, \ldots, N-1. \tag{3.1}$$

Let us denote by $S(t)$ the equity total return index at time $t \in [0,T]$. Under the risk neutral probability $\mathbb{P}$ the Dynamic Of S is given by

$$\boxed{\frac{\mathrm{d}SP(t)}{SP(t)} = (r(t) - y(t))\mathrm{d}t + \sigma_{loc}(t)\mathrm{d}W_t} \tag{3.2}$$

Where r is a stochastic short rate determined by the nominal interest rate model (See 2.2) and W is a standard Brownian motion under $\mathbb{P}$. The dividends payment $y(t)$ is assumed to be constant, denoted as $y(t) = q$

for simplicity in our study case. The $\sigma_{loc}(t)$ is a deterministic step-wise constant function of the volatility satisfying:

$$\text{For } t \in [T_k, T_{k+1}), \quad \sigma_{loc}(t) = \sigma_{loc}(T_k). \tag{3.3}$$

3.2.2 Calibration under the SCVM

Using the fact that implied volatility can be viewed as the average of realized squared volatilities, One has:

$$\sigma_{\text{imp}}(T_N)^2 = \frac{1}{T_N}\int_0^{T_N} \sigma_{\text{loc}}(t)^2 dt = \frac{1}{T_N}\sum_{i=1}^{N}\int_{T_{i-1}}^{T_i} \sigma_{\text{loc}}(s)^2 ds = \frac{1}{T_N}\sum_{i=1}^{N} \Delta T_{i-1}\sigma_{\text{loc}}^2(T_{i-1}) \tag{3.4}$$

A term structure of implied volatilities can be derived from market data for a set of maturities $\mathcal{M} = \{T_1, \ldots, T_M\}$; these are denoted by:

$$\left\{\sigma_{\text{imp}}^{\text{Mkt}}\left(T_1\right), \ldots, \sigma_{\text{imp}}^{\text{Mkt}}\left(T_M\right)\right\}. \tag{3.5}$$

From these market values, one can directly determine the piecewise constant local volatilities as:

$$\sigma_{\text{loc}}\left(T_0\right) = \sigma_{\text{imp}}^{\text{Mkt}}\left(T_1\right) \tag{3.6}$$

Then recursively for $i \in [1, M-1]$:

$$\boxed{\sigma_{\text{loc}}\left(T_i\right) = \sqrt{\frac{T_{i+1}\sigma_{\text{imp}}^{\text{Mkt}}\left(T_{i+1}\right)^2 - T_i\sigma_{\text{imp}}^{\text{Mkt}}\left(T_i\right)^2}{\Delta T_i}}} \tag{3.7}$$

3.2.3 Simulation and Validation of the SCVM

A straightforward application of Itô's formula on (3.2) to $\ln(S(t))$ yields the classic result:

$$S\left(T_k\right) = \frac{S\left(T_{k-1}\right)}{P\left(T_{k-1}, T_k\right)} \exp\left(-q(\Delta T_k) + \int_{T_{k-1}}^{T_k} \sigma_{\text{loc}}(t)\mathrm{d}W_t^S - \frac{1}{2}\int_{T_{k-1}}^{T_k} \sigma_{\text{loc}}(t)^2\mathrm{d}t\right) \tag{3.8}$$

Using the properties of stochastic integrals and Itô's isometry, we have:

$$\int_{T_{k-1}}^{T_k} \sigma_{\text{loc}}(t)\mathrm{d}W_t^S \sim \sqrt{\int_{T_{k-1}}^{T_k} \sigma_{\text{loc}}(t)^2\mathrm{d}t} \cdot \mathcal{N}(0,1) \tag{3.9}$$

From which we obtain the following diffusion scheme where $X_k \sim \mathcal{N}(0,1)$:

$$\boxed{S\left(T_k\right) = \frac{S\left(T_{k-1}\right)}{P\left(T_{k-1}, T_k\right)} \exp\left(X_k\sqrt{\int_{T_{k-1}}^{T_k} \sigma_{\text{loc}}(t)^2\mathrm{d}t} - \frac{1}{2}\int_{T_{k-1}}^{T_k} \sigma_{\text{loc}}(t)^2\mathrm{d}t\right)} \tag{3.10}$$

Where:

$$\int_{T_{k-1}}^{T_k} \sigma_{\text{loc}}(t)^2\mathrm{d}t = \Delta T_{k-1}\sigma_{\text{loc}}\left(T_{k-1}\right)^2 \tag{3.11}$$

It is now Possible to check Validity of our model generated, The first procedure is to check the risk neutrality of our Index, that is applying the martingale test procedure :

The martingale test procedure consists checking the following identity for every simulation date $T_j, 1 \leq j \leq N$:

$$S(0) = \widehat{\mathbb{E}}\left[D\left(T_j\right) S\left(T_j\right)\right] = \frac{1}{N_s} \sum_{s=1}^{N_s} D^{(s)}\left(T_j\right) S^{(s)}\left(T_j\right), \tag{3.12}$$

a centered confidence interval at level α for the value $\frac{\widetilde{\mathbb{E}^*}[D(T_j)S(T_j)]-S(0)}{S(0)}$ is built as follows:

$$\left[-\frac{\widehat{sd\left(T_j\right)}}{S(0)\sqrt{N_s}} \Phi^{-1}\left(\frac{1+\alpha}{2}\right); +\frac{\widehat{sd\left(T_j\right)}}{S(0)\sqrt{N_s}} \Phi^{-1}\left(\frac{1+\alpha}{2}\right)\right], \tag{3.13}$$

Where the estimated standard deviation is:

$$\widehat{sd\left(T_j\right)}^2 = \frac{1}{N_s - 1} \sum_{s=1}^{N_s} \left(D^{(s)}\left(T_j\right) S^{(s)}\left(T_j\right) - \widehat{\mathbb{E}}\left[D\left(T_j\right) S\left(T_j\right)\right]\right)^2 \tag{3.14}$$

And where we recall that Φ is the normal cumulative distribution function.

Concerning Market consistency test, The computation of local volatilities using discounted log-returns ensures that the produced economic scenarios align with the market's implied volatility quotations.

We have a sample of the yearly discounted log-returns of the equity index, denoted as $\left(r_s^{(s)}\left(T_j\right)\right)_{j\in[1,N]}$, spanning the interval $[T_{j-1}, T_j]$ for $1 \leq j \leq N$. This is given by:

$$r_S^{(s)}\left(T_j\right) = \ln\left(\frac{D^{(s)}\left(T_j\right) S^{(s)}\left(T_j\right)}{D^{(s)}\left(T_{j-1}\right) S^{(s)}\left(T_{j-1}\right)}\right), \tag{3.15}$$

The set of yearly discounted log-returns, represented as $\left(r_S^{(1)}\left(T_j\right), \ldots, r_S^{(N_s)}\left(T_j\right)\right)$, follows a normal distribution and are considered independent and identically distributed. The Monte Carlo volatility at a given date T_j for $1 \leq j \leq N$ is derived from:

$$\widehat{\sigma_S}\left(T_{j-1}\right) = \sqrt{\frac{1}{N_s - 1} \sum_{s=1}^{N_s} \left(r_s^{(s)}\left(T_j\right) - \widehat{\mu_S}\left(T_j\right)\right)^2}, \tag{3.16}$$

with :

$$\widehat{\mu_S}\left(T_j\right) = \frac{1}{N_s} \sum_{s=1}^{N_s} r_S^{(s)}\left(T_j\right). \tag{3.17}$$

Aggregating these Monte Carlo volatilities gives us the estimate:

$$\widehat{\sigma_{S,Imp}}\left(T_j\right) = \frac{1}{\sqrt{T_j}}\sqrt{\sum_{i=0}^{j-1} \Delta T_i \widehat{\sigma}_S\left(T_i\right)^2}. \tag{3.18}$$

This value can be juxtaposed with the market's implied volatility, denoted by $\sigma_{imp}^{Mkt}\left(T_j\right)$

RemarkWhile in some scenarios this methodology enhances the implied volatility's estimation, it doesn't offer a means to construct confidence intervals around implied volatility due to the aggregation of non-independent and non-uniformly distributed variables.

Part III

Reduce the computational cost of ESG using Scenarios Reduction

Chapter 4

Introduction

In risk management and insurance valuation,computational complexity presents significant challenges. The tasks involved in economic scenario generation and asset and liability management necessitate extensive data handling and detailed simulations. Given the constraints imposed by regulatory frameworks and the demand for accuracy and efficiency, ongoing research is focused on developing new techniques to address these computational demands.This introductory chapter sets the stage for a detailed exploration of two principal areas

4.1 Computational cost improvement

Economic Scenario Generators (ESGs) are vital tools in risk management today. Their use can lead to significant computational costs, especially when generating a large number of scenarios. This problem is highlighted by the slow convergence rate of Monte Carlo approximation, which converges given M simulations in $O(\frac{1}{\sqrt{M}})$. This slow rate can make the process unmanageable, especially when there is a need to compute many sensitivities through recalibration of models introduced previously and simulations. To address this challenge, this chapter explores techniques for reducing the number of scenarios without losing the accuracy of risk predictions. We focus on methods that allow us to select the most relevant scenarios without needing to compute them all. The chapter discusses various approaches and how they can be applied in the context of ESGs under Solvency II regulations. The aim is to offer practical insights that can help the reader efficiently manage the computational demands of ESGs, given the slow convergence of Monte Carlo approximation and the need to simulate many scenarios.

4.2 Provide stochastic ALM valuation for small insurers

In the context of insurance balance sheet valuation, the execution of numerous simulations related to risk-neutral economic scenarios is a routine yet challenging task. Practical constraints on computational resources prevent the infinite scaling of scenarios, requiring innovative techniques to ensure convergence of values like the Best Estimate of Liabilities (BEL) and the Present Value of Future Profits (PVFP).With regulatory frameworks such as IFRS 17 and Solvency II, insurance entities are focused on reducing computation time for stochastic valuation in Asset and Liability Management (ALM), especially in the Least Square Monte Carlo (LSMC) calibration process. This research aims to explore methods to balance valuation accuracy with computational efficiency, with growing consideration for cost and energy implications.

4.3 Summary of the comprehensive approach

The central idea of the approach would be to completely reproduce the ALM model with a machine learning approach for several reasons. The primary one being that the ALM model in practice can happen to be modeled differently from one actor to another. Moreover, it is extremely computation-intensive and must be used with every sensitivity shift. As regulatory calculations are conducted multiple times a month, it becomes meaningful to adopt a technique that significantly reduces computation time. In practice, this can be achieved as follows, as detailed in the diagram: we train the ALM

model function f_{ML} on a central sensitivity generated beforehand; in our case, a DDLMM for rates and an SCVM for equity factors, using a vast number of scenarios (approximately 20,000 possible world states). The training of this model will be based on Target Y values of Best estimate BE pre-computed by a genuine ALM model initially. Our observations are the scenarios (in rows), and our columns then represent the trajectories for each timestamp of each of our risk factors.

After training this model, we implement a scenario reduction approach, in practice, an insurer will typically use sensitivities from a significantly reduced number of scenarios (e.g., 3000) due to regulatory standards and computation time Trade off. We then predict on this reduced set using the model trained by our machine learning approach, verifying its performance against a genuine ALM model to measure the model's efficacy using MSE metric. Subsequently, we attempt the Weighted approach without recalibration.

In practice, from our reduced sensitivity, we will construct using the WMC algorithm discussed later, a stressed sensitivity simulating various rate contexts and unfavorable volatility scenarios. We then predict the ALM metrics using our model and compare the results against a genuine ALM model to challenge our approach.

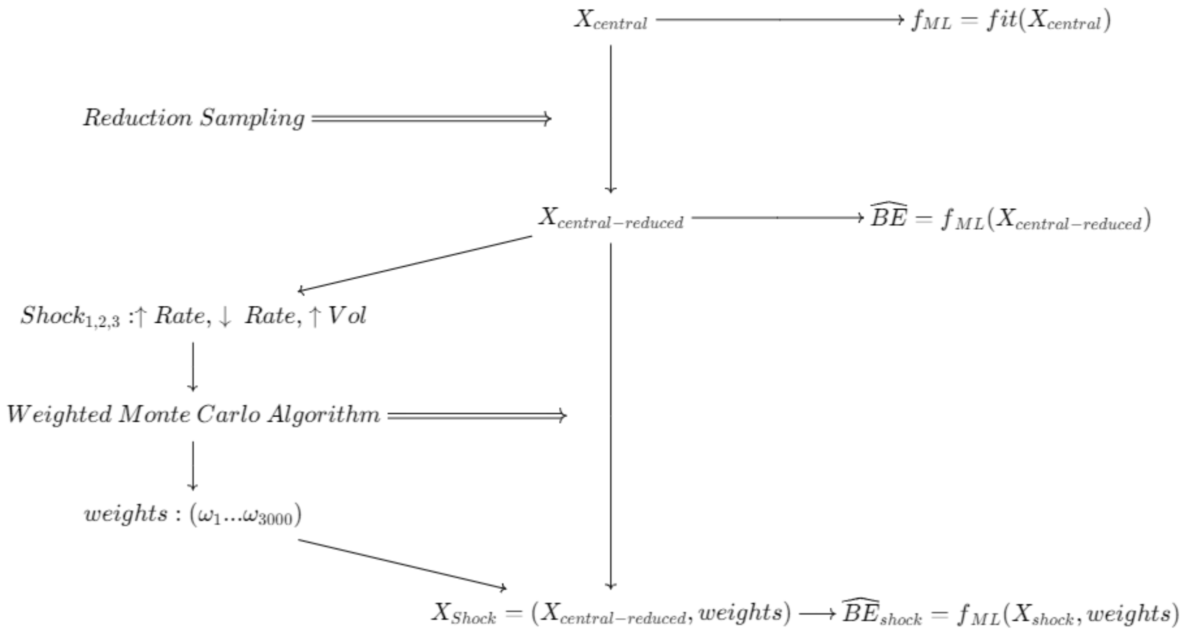

Figure 4.1: Representation of the Global Approach Employed

Chapter 5

The Statistical Learning Framework

The objective of the subsequent section is to outline the machine learning framework used for our case study. This covers the format of the input data, the target variables, and the choice of models. While one might consider a "black-box" model, like a deep neural network, such a choice could be challenging to justify from a regulatory or sell-side business perspective. Our foremost goal was to achieve optimal results with a clear and interpretable model, while also meeting Solvency II requirements.

5.1 Framework: Inputs and Observations

We're dealing with a collection of scenarios generated by our ESG. Rates are modeled using the DDLMM model, while equity-like instruments follow the local volatility model (SVCM). A Sobol sequence is utilized as the RNG to ensure broad distributions of each risk factor. Each scenario provides trajectories covering a 50-year maturity for every risk factor. The entire yield curve, $Y(t,T)_{t,T}$, is derived from zero coupon bond prices over a range of maturities. This can be ascertained by inversely applying the standard formula for annually compounded bonds. Additionally, the dataset encompasses trajectories for real estate and equity indexes, as well as the inflation curve.

To streamline the notation and maintain consistency with our previously established standards, we introduce the following notation:

- Let $m \in [0, 21000]$ represent our observations. In simpler terms, this corresponds to each scenario case.
- For $i \in [0, 50]$, $S_{i,m}^{EQ/RE}$ denotes our Equity/Real Estate indexes at each timestamp.
- For $i \in [0, 50]$ and $j \in [0, 40]$, $P_{i,j,m}$ are the trajectories of our zero-coupon bonds for 40 different time-to-maturities.
- For $i \in [0, 50]$, $I_{i,m}$ represents our Inflation Index.

In summary, our initial features matrix X is of the form:

$$\begin{aligned}\text{Rates} &= \begin{bmatrix} P_{1,1} & P_{2,1} & \cdots & P_{50,1} & P_{1,2} & \cdots & P_{49,40} & P_{50,40} \end{bmatrix}, \\ \text{Equity Indices} &= \begin{bmatrix} S_1^{EQ} & \cdots & S_{50}^{EQ} & \cdots & S_1^{RE} & \cdots & S_{50}^{RE} \end{bmatrix}, \\ \text{Inflation} &= \begin{bmatrix} I_1 & \cdots & I_{50} \end{bmatrix}.\end{aligned}$$

Thus, the columns of X are simply the concatenations of:

$$Col(X) = \begin{bmatrix} \text{Rates} & \text{Equity Indices} & \text{Inflation} \end{bmatrix},$$

and the rows of X represent the different values of Rates, Equity Indices, and Inflation, determined by the Economic Scenario Generator output for each sample from 1 to 2000.

Regarding the target variable, we have:

$$Y = \begin{bmatrix} BE_1 \\ \vdots \\ BE_{21000} \end{bmatrix}, \tag{5.1}$$

where the real BE values for each scenario sample case are obtained through a real Asset Liability Management Model and will be used to train our model.

5.2 Choice of the Model

Our primary objective is to determine an optimal estimate $\hat{Y}$ that minimizes the error given by:

$$\text{Error} = \left\| \frac{Y - \hat{Y}}{Y} \right\|_2 \tag{5.2}$$

For the purposes of this chapter, we define the aforementioned error as the Relative Mean Squared Error (RMSRE). This serves as the primary metric for evaluating the quality of our computational results.

To achieve our objective, the central challenge is selecting an appropriate function f_{ML} such that:

$$\hat{Y} = f_{ML}(X)$$

The choice of f_{ML} can vary significantly; it might be as straightforward as a linear regression or as complex as a deep neural network. Alternatively, a non-parametric approach can be adopted. For instance, we can determine the optimal solution by finding the saddle point of RMSRE through the Gradient Boosting Method.

In the upcoming sections, we'll base our choice of model on a combination of computational efficacy and financial intuition.

5.2.1 The idea Behind Polynomial Regressor

A first Observation is that the portfolio of an insurer is a linear combinaison of instruments derived from our risk factors , that is, for any insurer there must exist weights $(w_1, ...w_n)$ such that the Present value of a Portfolio is of the form :

$$PV = \sum_{i=1}^{n} Rf_i w_i = \underbrace{\text{Bonds} \cdot W_1}_{\text{liabilities}} + \underbrace{\text{Assets} \cdot W_2}_{\text{investments}} \tag{5.3}$$

An insurer's portfolio encompasses assets, which are investments, and liabilities, mainly future expected insurance obligations. The best estimate pertains to the projected liabilities. These assets, invested in securities like bonds and stocks, should yield returns sufficient to cover the projected best estimate liabilities. The financial health or solvency of an insurance company can be gauged by the difference between its total assets and this best estimate of liabilities. Notably, if the assets are insufficient, the insurer faces solvency risks. Regulatory frameworks, such as the Solvency II in the European Union, require insurance companies to have assets surpassing the sum of their best estimate and an additional capital buffer, ensuring they can handle substantial losses.

To conclude, predicting the "best estimate" using the linear relationship derived from its portfolio is intuitively meaningful. Additionally, assuming a nonlinear relationship between each risk factor (one can think of this as discounting asset values with rates), we employ a linear regression with polynomial features as our initial model.

that is

$$\hat{Y} = f_{ML}(X) \tag{5.4}$$

$$f_{ML}(X) = X\beta \tag{5.5}$$

$$\begin{cases} f_{ML}(X) = X\beta \\ \text{with} \\ Col(X) = \begin{bmatrix} x_1 & x_2 & \cdots & x_i & \cdots & x_1x_1 & x_1x_2 & \cdots & x_ix_j \end{bmatrix} \end{cases}$$

The principal advantage of this method is is its interpretability but also it admits a closed formula for β given by the Gauss Markov projection theorem :

Given a matrix of independent variables X and a vector of dependent variables Y, the objective is to find coefficients β that minimize the L2 norm :

$$\arg\min_{\beta} \|X\beta - Y\|_2 = (X^T X)^{-1} X^T Y = \hat{\beta} \tag{5.6}$$

Under the assumptions that the errors $Y - \hat{Y}$ have expectation zero, are uncorrelated, and have equal variances, the OLS estimator is the Best Linear Unbiased Estimator of the coefficients.

Remark : A residual analysis is conducted in Chapter 7 to check whether the assumptions of the theorem are rejected in our case study.
From the simplicity of the Linear Model and given closed formulas, is it possible to compute many test hypothesis on ours model including the nullity of certains coefficients of β but also the distributions of the Error (which ideally should follow a centered Gaussian law with constant variance)

From 5.6, it is clear that adding polynomial features to our initial data X results in impractical computing times. Indeed, starting with potentially $(40+2+1)\times 50=2250$ initial features could lead to a combinatorial explosion when considering all possible column combinations to compute the augmented matrix. We will address this challenge in Section 5.3. For now, we'll proceed under the assumption that this issue is resolved and discuss the Lasso regression approach in the subsequent section.

5.2.2 Lasso

Lasso Regression, which stands for *Least Absolute Shrinkage and Selection Operator*, is a type of linear regression that includes a regularization term. The Lagrangian formulation of Lasso regression is:

$$\min_{\beta} \left\{ \frac{1}{2} ||Y - X\beta||_2^2 + \lambda ||\beta||_1 \right\}$$

Lasso uses L1 regularization to reduce some model coefficients to zero, simplifying the model and aiding feature selection. This not only enhances interpretability but also combats overfitting, especially with many predictors. Additionally, Lasso addresses multicollinearity by selecting just one of the highly correlated features. However, there's no closed formula for β in this framework.

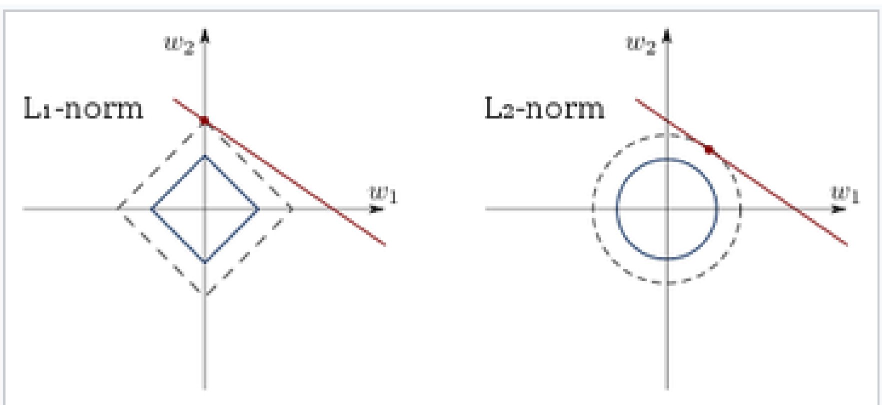

Figure 5.1: Geometric interpretation of Lasso Feature Selections , Source : Wikipedia

The decision to utilize Lasso in our approach largely stems from the assumption that only crucial timestamps can capture the overall trajectory of a risk factor. This leads to enhanced interpretability and selection in the model, especially when faced with a large number of features derived from the Augmented Polynomial Matrix X.

the Parameters λ is thus choosen Over a Randomsearch interval generated by a Sobol sequence to accelerate the research of an optimum.

5.3 Feature Selection : The curse of Dimensionnality

We revisit the dimensionality problem caused by the polynomial augmentation of our basic feature X. Given a matrix with n features, the number of coefficients resulting from polynomial augmentation is $O\left(\frac{n(n-1)}{2}\right)$ this would induce first long time computation and therefore a non tractable model but also introduce high overfitting if one tries to fit a model specifically to specific Dataset and increase the Generalisation error. To address this technical challenge, we tested two approaches,we first introduce the PCA (Principal Component Analysis) algorithm as used in our framework.We then discuss Sequential Feature Selection model, which is widely adopted in the insurance industry.

5.3.1 Principal Component Analysis

Principal Component Analysis (PCA) is a dimensionality reduction method that transforms a set of correlated variables into a set of orthogonal variables, called principal components. These components capture the maximum variance present in the original variables, in decreasing order, One can found a complete demistification of PCA in [3].

the Key point of PCA for dimensionnality reduction is to ensure that

$$X - \bar{X} \xrightarrow{\overset{PCA}{\Downarrow}} X_{PCA} = (X - \bar{X})W$$

Given the matrix X of size $n \times p$, and selecting the top k eigenvectors, we form the transformation matrix W with dimensions $p \times k$.
The transformation can be represented as:

$$X \times W = (n \times p) \times (p \times k)$$

This results in:

$$X \times W = (n \times k)$$

Therefore, the resulting matrix X_{PCA} after projection has the dimensions:

$$X_{PCA}\, is\, of\, dimension\, (n \times k)$$

The PCA Algorithm can thus significantly reduce the dimension of our features, especially when k is much smaller than the number of features.we thus keep 95 % of variance explained in our applications to find a good trade off between the numbers of features and total variance explained.

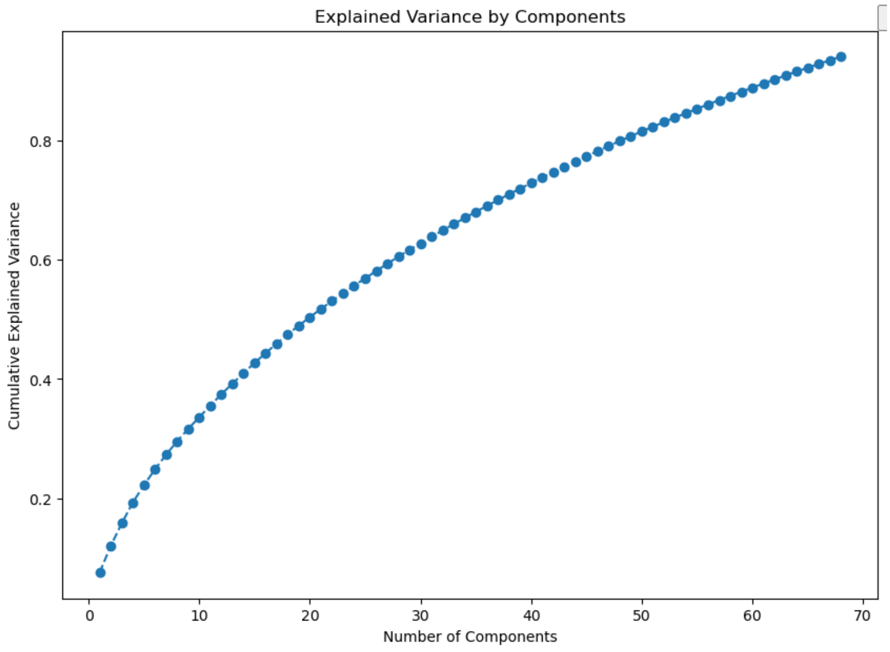

Figure 5.2: Figures shows that from a initial Dataframe with a numbers of features of 10 000 less than 90 principal components is able to contain more than 95% of explained variance

However the principal Drawback of PCA is its lack of interpretability, Indeed one can see that our features are now a melting pot of linear combinations of others features, going further by introducing a regression would not yield any interpretable results wether wich maturity or assets is predominant in our model. This approach is then alone hard to justify to a Sell side point of view.

5.3.2 The Stepwise Constant feature Selector algorithm

The Sequential Forward Selection (SFS) is a greedy search algorithm employed for feature selection. Its primary aim is to identify a subset of features from a larger set, where these features optimize a predefined criterion.

The process of SFS can be described as follows:

1. Start with an empty set of features.
2. At each iteration, evaluate all features not yet selected.
3. Select the feature which optimizes the criterion the most when combined with the previously selected features.

4. Repeat the process until a predefined stopping condition is met, such as a degradation of Score after the next iteration.

SFS can be advantageous because of its simplicity and effectiveness, especially in scenarios where the dimensionality of the dataset is high. However, it may sometimes fail to identify the global optimum subset of features as it does not reconsider previously selected features.

Algorithm 1 Sequential Forward Selection (SFS)

Require: $Y = \{y_1, y_2, \ldots, y_d\} \quad J = Score\ function$
Ensure: $X_k = \{x_j \mid j = 1, 2, \ldots, k; x_j \in Y\}$, where $k \in \{0, 1, 2, \ldots, d\}$

1: Initialize: $X_0 = \emptyset, k = 0$
2: **while** True **do**
3: $\quad x^+ = \arg\max J\left(X_k + x\right)$, where $x \in Y - X_k$
4: $\quad X_{k+1} = X_k + x^+$
5: $\quad k = k + 1$
6: **end while**
7: Termination when $J(X_{k+1}) < J(X_k)$

However finding x^+ in step 3 can be time heavily time computing considering the high amount of columns and thus make the SFS not usable in practice ,we thus introduce another way to use the power of SFS using taylor approximation

Recall that for a function f of multiple variables $x_1, x_2, \ldots, x_n$, its Taylor expansion up to the second order around a point $(a_1, a_2, \ldots, a_n)$ is:

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \underbrace{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{a})(x_i - a_i)}_{\mathrm{SFS}_{\mathrm{Linear}}} + \underbrace{\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})(x_i - a_i)(x_j - a_j)}_{\mathrm{SFS}_{\mathrm{Polynomial}}} \quad (5.7)$$

Our adopted approach would then be the following:

1. Apply the SFS algorithm first only on linear terms. This would be computationally super-efficient since the number of features would be greatly limited. Stock the k_{linear} first set of features selected.

2. Starting with the k_{linear} set of features, apply SFS on the polynomial terms. This part of the algorithm would be more greedy, but the score being already low from the first step. One can intuit this part as a tuning of our initial approach to increase sharply the final score. From this step, we obtain a new set of features k_{poly}.

3. Apply a regression model (Linear or Lasso) on the selected features $k_{\text{linear}} + k_{\text{poly}}$.

To conclude this chapter, given this machine learning framework, is it possible to learn f_{ML} efficiently while simultaneously preventing overfitting and maintaining the interpretability of our model? Chapter 8 provides numerical results and prediction outcomes from our chosen approach. Moreover, for further investigations, techniques such as the kernel trick and gradient boosting have been explored, but they are beyond the scope of this paper

Chapter 6

Base Sensivity Generation

We now delve into the second main step of our approach, which is to reduce our central sensitivity and gain the ability to generate shocked sensitivities. In this section, we present the theoretical aspects of the techniques used.

6.1 Quasi Monte carlo Sampling QMC

A preliminary adjustment considered consists of modifying the random number generator used in the creation of the reference set. In fact, the role of RNG is critical in the generation of Economic Scenario Generators (ESGs) and the quality of pseudo-RNG and its seed value can significantly impact validation tests and subsequently insurers' balance sheet valuations and leakage. Quasi-RNG as the Sobol generator is briefly discussed in the following to introduce the Milliman ESG hybrid-RNG. Moreover To be able to Fine tune our Machine Learning model hyperparameters, we employ a RandomsearchGrid based on uniform randomized Sobol sequence to improve computational cost due to high dimension of the Learning problem.

6.1.1 Why do we use QMC ?

The idea behind using QMC in simulations, such as those required for Economic Scenario Generators (ESGs) and machine learning hyperparameters tuning, is that the low-discrepancy sequences can provide a more accurate approximation with fewer simulation runs. The more evenly distributed samples lead to a reduction in the variance of the estimate, allowing for quicker convergence to the true value. In the context of simulation cost reduction, this is particularly valuable. By using QMC methods, one can achieve the same level of accuracy with fewer simulations, which translates to less computational time and resources. This efficiency makes QMC a preferred choice in scenarios where large-scale simulations are needed but computational budgets are constrained.In the Following Parts we presents it's main approach and results.

6.1.2 The Quasi Monte Carlo Approach to Estimate Sensitivities

Suppose that The aim is to compute the expectation

$$\mathbf{E}\left[F\left(U_1,\ldots,U_d\right)\right]=\int_{[0,1)^d} F(x)dx. \tag{6.1}$$

think of F as an unknown function form such as the realized payoff for each simulation from a base sensivity , that is using the Bel formula for instance can be seen as:

$$BEL_t=\mathbb{E}^{\mathbb{Q}}\left[\underbrace{\sum_{u=t}^{T} e^{\int_t^u r_x\,ds} L_u \mid \mathcal{F}_t}_{=F}\right]. \tag{6.2}$$

Quasi-Monte Carlo (QMC) provides an approximation using

$$\boxed{\int_{[0,1)^d} F(x)dx \approx \frac{1}{M}\sum_{i=1}^{M} F\left(x_i\right),} \tag{6.3}$$

where the points $x_1,\ldots,x_M$ are chosen within the unit hypercube $[0,1)^d$. Note the

following:

- The function F need not have an explicit form; it is sufficient to evaluate F, which is the purpose of a simulation algorithm , (One can think about ALM metrics whos cash flows functions does not explicity provide open formulas as shown below , but still can be computed through an ALM model)

- The specification of the boundary of the unit hypercube is insignificant for the integral's value and is irrelevant in standard Monte Carlo. However, in QMC, it matters due to certain definitions and results. Thus, we use $[0,1)^d$ as the unit hypercube, indicating an interval closed on the left and open on the right.

The objective of low-discrepancy methods is to formulate points x_i that minimize the error in equation across a wide range of integrands F. It is intuitively, and mathematically demonstrated in Glasserman that, selecting the points x_i that uniformly populate the hypercube is an optimal solution.

6.1.3 The Koksma Inequality

To quantify deviation from uniformity, we employ discrepancy measures.We introduce the following definition

Definition 6.1.1: Discrepancy

Given a collection $\mathcal{A}$ of Lebesgue measurable subsets of $[0,1)^d$, the discrepancy of the set $\{x_1, \ldots, x_M\}$ relative to $\mathcal{A}$ is given by :

$$D\left(x_1, \ldots, x_M ; \mathcal{A}\right)=\sup _{A \in \mathcal{A}}\left|\frac{\#\left\{x_i \in A\right\}}{M}-\operatorname{vol}(A)\right| . \tag{6.4}$$

Discrepancy counts the number of x_i within A, represented as $\#\{x_i \in A\}$, and the volume of A is denoted by $\operatorname{vol}(A)$. The discrepancy represents the supremum of errors when integrating the indicator function of A using the points $x_1, \ldots, x_M$.

if some demonstrated results are shown in dimension d=1 for a lower bounds wich states that $D\left(x_1, \ldots, x_M\right) \geq \frac{c \log M}{M}$ it is still an open question in higher dimensions even tho is it widely supposed of order error being in order of $O(\frac{logM^d}{M})$

from the very definition of Discrepancy (5.1.1) introduced previously , we give an alternative definition wich constest of taking subsets $\mathcal{A}$ of sub rectangle in $\mathcal{R}^d$

Definition 6.1.2: Star Discrepancy

the Star Discrepancy of the set $\{x_1, \ldots, x_M\}$ relative to $\mathcal{A}$ is given by :

$$D^*\left(x_1, \ldots, x_M ; \mathcal{A}\right)=\sup _{A \in \mathcal{A}}\left|\frac{\#\left\{x_i \in A\right\}}{n}-\operatorname{vol}(A)\right| . \tag{6.5}$$

Given a collection $\mathcal{A}$ the form $\prod_{j=1}^d\left[u_j, v_j\right), \quad 0 \leq u_j<v_j \leq 1$ the set of all sub rectangle of $[0,1]^d$,

We can now introduce the Main theorem of QMC sequences about errors bounds :

Theorem 6.1.1: Koksma Inequality $(d=1)$

et $(\xi_1, \ldots, \xi_n)$ be an n-tuple of $[0,1]^d$-valued vectors and let $f:[0,1]^d \rightarrow \mathbf{R}$ be a function with finite variation[a] Then

$$\left|\frac{1}{M} \sum_{k=1}^M f\left(\xi_k\right)-\int_{[0,1]^d} f(x) d x\right| \leq V(f) D^*\left(\left(\xi_1, \ldots, \xi_M\right)\right) . \tag{6.6}$$

where $V(f)$ is the measure variation of f

[a] A function $f:[a, b] \rightarrow \mathbf{R}$ is said to be of bounded variation if and only if there exists a real number V such that for any finite set of points $a=x_0<x_1<\ldots<x_M=b$ in $[a, b]$, we have $\sum_{i=1}^M\left|f\left(x_i\right)-f\left(x_{i-1}\right)\right| \leq V$.

Despite its theoretical significance, the Koksma-Hlawka inequality often lacks practical applicability as an error bound mostly for severals reasons :

- While the Koksma-Hlawka inequality provides a strict bound on integration error, probabilistic bounds from Central limitt Theorem (CLT) allow for adjustable error probabilities.
- Calculating the terms in the Koksma-Hlawka inequality can be more complex than the integral of f, unlike the easily estimated parameter
- When known, the Koksma-Hlawka bound often significantly overestimates the true integration error, whereas the central limit theorem provides a more reliable error estimate.

6.1.4 Low Discrepancy Sequences : Sobol sequences

given intuiton from high dimensional lower bounds of star discrepancy with Theorem 5.1.1 we now define low discrepancy sequence :

Definition 6.1.3: Low Discrepancy Sequence

$[0,1]^d$-valued sequence $(\xi_M)_{n\geq 1}$ is a sequence with low discrepancy if

$$D^*(\xi) = O\left(\frac{(\log M)^d}{M}\right) \text{ as } n \to +\infty. \tag{6.7}$$

Therefore , for a small value of d as in our case with only rates and equity like model to generate, it is easy to see that the rate of Convergence of Sobol Sequences is $O(\frac{1}{M})$ which clearly beat the CLT with $O(\frac{1}{\sqrt{M}})$ and justify our approach.

we can now introduce sobol sequences and their numerical advantages. the Sobol sequences we are interested in come into a larger class of sequences called $(t,d)-$sequences (more details about (t,d)-sequences in [11]) its main advantage is it's computational cost since it operate through bit-level operations by working in base 2.We present in appendix how to generate a sobol sequence in one direction of $[0,1]^d$ this approach is thus generalized in every directions for each of our risk factors and projections.

Pros and cons of Sobol Sequences can be summarized in this table :

Advantages	**Disadvantages**
•Highly uniform distribution.	•Deterministic nature could introduce artificial regularities, causing erroneous results in some cases.
•Deterministic method, allows for reproducible results for our reducted sets scenarios analysis	•May not cover the whole space efficiently when the number of points is not a power of two.
	•Sobol sequences are highly sensitive to dimension curse

Table 6.1: Advantages and Disadvantages of Sobol Sequences

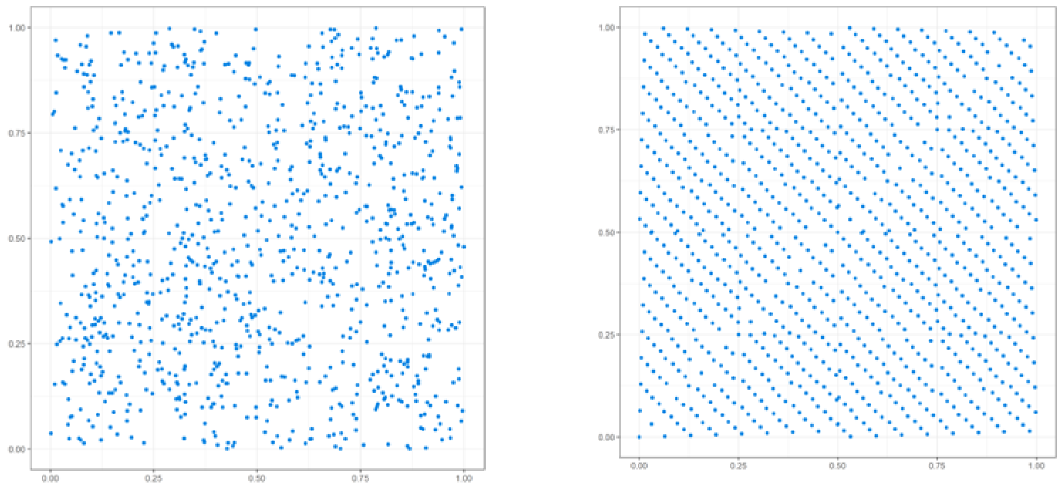

Figure 6.1: representation of the 9,999th (x-axis) and 10,000th (y-axis) coordinates of the first 1,000 points of the mersenne (left), and sobol (right) sequences in dimension 10,000

Figure 6.1 Show a clear improvement of space filling properties compared to a classic Random number Generated induced here by a Mersenne twitter
However, in order to reduce the main drawbacks of Classics Sobol Sequences, we need to add another adjustment : The randomization.

6.1.5 Randomization : Hybrid Sequences approach

To mitigate the disadvantages associated with classic Sobol sequences, hybrid or randomized versions have been introduced. They add a random "jitter" to the points in the Sobol sequence. This maintains the high uniformity of the sequence while mitigating the deterministic nature that could cause artificial regularities and also preserve the two

followings points :
Consider an original set of points $Q_n = \{y_1, \dots, y_M\}$, which consists of M elements from a low-discrepancy sequence. We apply a randomization process, resulting in a new point set $R_M = \{\widetilde{y}_1, \dots, \widetilde{y}_M\}$. This new set has the following two important properties:

- Every point $\widetilde{y}_i$ in the set R_M is uniformly distributed over the d-dimensional unit cube $[0,1)^d$.
- Despite the randomization, the set R_M maintains the low-discrepancy property of the original set Q_M.

The first property ensures that any estimator based on the randomized point set R_M is unbiased. This can be mathematically expressed as follows:

$$\mathbf{E}\left[\frac{1}{M}\sum_{i=1}^{M} F(\widetilde{y}_i)\right] = \frac{1}{M}\sum_{i=1}^{M}\mathbf{E}\left[F(\widetilde{y}_i)\right] = \frac{1}{M}\sum_{i=1}^{M}\int_{[0,1)^d} F(u)\,du = I(F), \tag{6.8}$$

where $I(F)$ is the integral of the function F over the unit cube $[0,1)^d$.

Various randomization algorithms are available, but this note will only present the digital shift randomization.low-discrepancy sequences and Sobol sequences it also preserve the class of $(t,d)-$ low discrepancy sequences of our hybrid randoms numbers.

Given a point set $Q_M = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_M\}$ in base b (typically our classical sobol sequence generated) and a uniform random variable $\boldsymbol{U} = (U_1, \dots, U_d)$ in $[0,1)^d$, the base-b expansion of its coordinates is expressed as:

$$U_i = \sum_{j=1}^{r} U_{i,j} b^{-j}, \quad i = 1, \dots, d \tag{6.9}$$

The digitally shifted version of Q_n, denoted $R_M = \{\widetilde{\boldsymbol{x}}_1, \dots, \widetilde{\boldsymbol{x}}_M\}$, is then defined with the j-th coordinate of the i-th point as:

$$\tilde{x}_i^{(j)} = \sum_{k=1}^{r}\left(x_{i,k}^{(j)} + U_{i,k} \bmod b\right) b^{-k} \tag{6.10}$$

where $x_{i,k}^{(j)}$ are the coefficients in the base-b expansion of $x_i^{(j)}$.

There exists an additional transformation that can be applied to Sobol sequences, which leverages the properties of Brownian bridges. This transformation aims to mitigate the curse of dimensionality. The idea is to utilize the characteristics of Brownian bridges to modify the trajectories of the hybrid sequence step by step, thereby improving

the uniformity of the distribution across the space in high dimension . Although this approach is available in the Milliman Economic Scenario Generator, it was not employed in our reduction scenario case. The reason is that the size of the dimension space in our case remained within manageable limits, thus making the utilization of this advanced technique unnecessary.
This two-stepped approach (QMC + Randomization) not only preserves the advantages of Sobol sequences, such as uniform distribution and space-filling, but also enhances the rate of convergence for high-dimensional problems.

6.2 Scenarios Reduction

the selection of representative scenarios is crucial for accurate modeling and prediction. Various approaches have been developed to select scenarios that capture the essential dynamics and statistical properties of financial markets. These methods aim to represent the underlying distribution of possible future states in a way that is both computationally efficient and consistent with observed market behavior. This section outlines 4 distinct approaches to scenario selection. The following discussion explores these approaches in detail, providing insights into their underlying principles and practical implications. Numerical Results of the application on two out of those 4 Methodologies can be found in chapter 7.

6.2.1 Stochastic Sampling

This method is straightforward and is based on the Central Limit Theorem to randomly select scenarios from the reference set. The method is easy to implement. However, its main limitation is its high sensitivity to the specifics of the random number generator and its seed. The Central Limit Theorem assumes strong independence of our trajectories. This assumption could be challenged if a random number generator is not robust enough and fails to generate sufficiently diversified samples. The second limitation is the error rate in $O\left(\frac{1}{\sqrt{\nu}}\right)$ where ν is the number of scenarios used for our stressed sensitivity. In the EIOPA case, where $\nu = 9$, the derived error rate (assuming the RNG is robust enough to apply the CLT) is of the order $O\left(\frac{\sigma}{\sqrt{9}}\right)$. This could be dramatically high and lead to significant instabilities.

6.2.2 the society Of Actuaries sampling Approach

Firstly introduced by Yvonne Chueh [7] and later adopted by the Society of Actuaries (SOA). This technique is theoretically designed to maintain a high level of homogeneity in the selection of scenarios.The primary principle of Yvonne Chueh's paper is the idea that the present value of the insurance portfolio can be expressed as a continuous function of the discount factor, specifically:

$$PV(t) = f(D(t)) \tag{6.11}$$

where f represents a continuous function and D is the discount factor observed at time t.
Under this assumption, if trajectories are in close proximity in terms of the distance on the discount factor, a negligible fluctuation in the present value of the insurance portfolio is expected.We proof this fact using a discounted Portfolio index we built in methodology 3 (only composed of w_{bonds}).In mathematical terms, we can express the Present value of our Portfolio as:

$$\mathrm{PV} = \sum_{t=1}^{30} \left(MV_t \cdot \prod_{k=1}^{t} \frac{1}{1+r_k} \right) \tag{6.12}$$

Here, we project for a duration of 30 years.

Proof. It is clear that the portfolio index is a function of the rate scenario:

$$\mathrm{PV} = f\left(r_1, r_2, \ldots, r_{30}\right) \tag{6.13}$$

where

$$\mathrm{PV} = f : X \rightarrow \boldsymbol{Y}, \boldsymbol{X} \subset \boldsymbol{R}^{30}, \boldsymbol{Y} \subset \boldsymbol{R} \tag{6.14}$$

The distance between two points in the space of scenarios x and s in the domain $\boldsymbol{X}, d_X(x,s)$ can be computed as

$$d_X(x,s) = \sqrt{\sum_{t=1}^{30} \left(\prod_{k=1}^{t} \frac{1}{1+r_k} - \prod_{k=1}^{t} \frac{1}{1+r'_k} \right)^2}. \tag{6.15}$$

The PV function f is a real function that exists on a 30-dimensional domain $\boldsymbol{X}$. It is said to be continuous on $\boldsymbol{X}$ if f is continuous at every point (scenario) of $\boldsymbol{X}$.
Given an arbitrary rate scenario s, for every $\varepsilon > 0$, there is a $\delta = \varepsilon/2M > 0$ such that $d_Y(f(x), f(s)) < \varepsilon$ for all points $\boldsymbol{x} \in \boldsymbol{X}$ that satisfy $d_X(x,s) < \delta$:

$$\begin{aligned}
d_Y^2(f(x), f(s)) =& \left| \sum_{t=1}^{30} \left(MV_t \cdot \prod_{k=1}^{t} \frac{1}{1+r_k} \right) \right. \\
& \left. - \sum_{t=1}^{30} \left(MV'_t \cdot \prod_{k=1}^{t} \frac{1}{1+r'_k} \right) \right|^2 \\
=& \left| \sum_{t=1}^{30} \left(MV_t \cdot \prod_{k=1}^{t} \frac{1}{1+r_k} - MV'_t \cdot \prod_{k=1}^{t} \frac{1}{1+r'_k} \right) \right|^2 \\
\leq& (2M)^2 \left| \sum_{t=1}^{30} \left(\prod_{k=1}^{t} \frac{1}{1+r_k} - \prod_{k=1}^{t} \frac{1}{1+r'_k} \right) \right|^2 \\
\leq& (2M)^2 \cdot \sum_{t=1}^{30} \left(\prod_{k=1}^{t} \frac{1}{1+r_k} - \prod_{k=1}^{t} \frac{1}{1+r'_k} \right)^2,
\end{aligned}$$

$$\leq (2M)^2 \cdot d_X^2(x, s) \leq \varepsilon^2.$$

Therefore, $d_Y \leq 2M \cdot d_X \leq \varepsilon$.

□ □

The goal would then be to select trajectories primarily based on the discount factor, which impacts the present values of all other risk factors through future cash flow discounting. Various methods exist to apply the principles of the SOA paper. However, the Second Sampling Algorithm is selected as it is most commonly utilized in practice by multiple entities and can be outlined as follows:

1. **Define Trajectory Significance:** Evaluate the significance of each trajectory by calculating the L^2 norm of the discount factor vector for that specific trajectory.
2. **Arrange Scenarios:** Order the scenarios from the reference set according to the significance computed in the previous step.
3. **Select Quantile Levels:** Choose specific quantile levels for analysis. Typically, the number of quantiles selected will match the number of scenarios we want to extract ν
4. **Select Trajectories:** Based on the quantiles and significance computed earlier, select the corresponding trajectories. These selected trajectories represent the most significant scenarios according to the defined criteria.

Aside from its theoretical robustness, a key feature of this methodology is its independence from a reference portfolio. As a result, this approach avoids any potential statistical inference that might be introduced by incorporating weights $w_{Bonds}, w_{EQ}, w_{RE}$ from differing insurers' portfolios. We now turn our focus towards the results related to ESG (including martingale and market-consistency tests) and ALM impacts of the various selection techniques and scenario adjustments introduced in this paper.

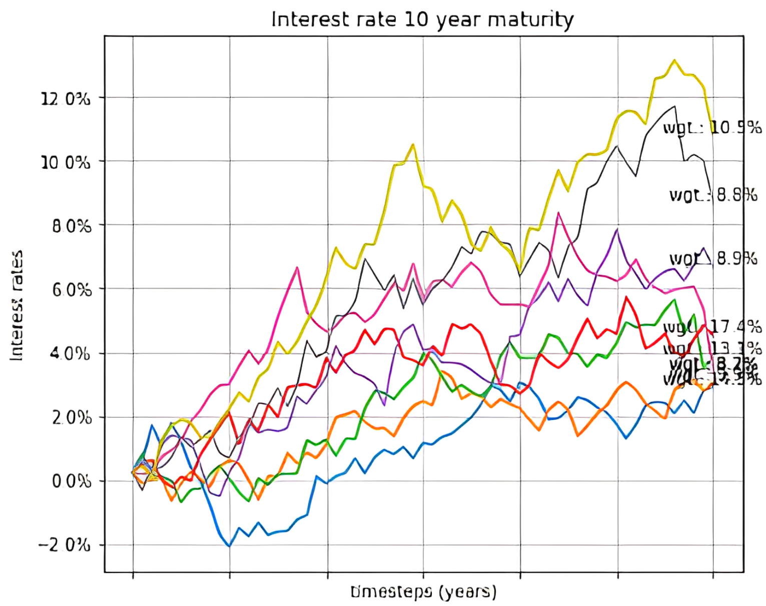

Figure 6.2: llustration of the method SOA Sampling

Chapter 7

Stressed Sensitivity Construction

7.1 Why do we transform our table ? : The sampling error problem

It is important to recognize that extracting the base sensitivity might require certain modifications to the trajectories retained during the creation of a reduced set of scenarios. These adjustments, although not immediately apparent after transformation, are essential in maintaining stability in Asset and Liability Management (ALM) impacts at the portfolio level and effectively preventing model leakages in real-world situations. Additionally, care must be taken in the extraction process, as a small number of trajectories can lead to significant sampling errors. This in turn may cause a noticeable change in the risk neutrality and market consistency of the scenarios, including aspects such as volatility We proceed to outline three specific adjustments performed on our reduced sensitivity to address the aforementioned issues. These modifications not only rectify the problems but also pave the way for introducing the weighted Monte Carlo algorithm. This algorithm primarily involves solving a non-parametric problem to identify the optimal measure that ensures our scenarios fit market data with precision.

7.2 Matching Market volatilities : The Rescaling

The objective of the rescaling adjustment method is to amend the equity-like trajectories so that the volatility levels of equity-like indices align with new target levels. In real-world applications, the WMC algorithm presented in next section, is frequently paired with this rescaling adjustment to enhance its applicability domain when executing equity volatility stresses, such as in internal models.The underlying principle of this method is outlined as follows.

With the Assumptions that such as S^{EQ} or S^{RE} indices follows to a log-normal dynamic: denote $S_{\text{ref}}(t)$ an Equity/RE like indice generated from a sensitivity.

$$S_{\text{ref}}(t) = D(t)S(0)\exp\left(-\frac{\sigma^2_{\text{reference}}(t)}{2} + \sigma_{\text{reference}}(t)W_t\right) \tag{7.1}$$

Here, W_t stands for the standard Brownian Motion, and $D(t)$ denotes the discount

factor. The aim is to change the volatility value to a novel target volatility σ_{target}. The rescaling factors can be evaluated as follows:

$$\lambda(t) = \frac{\sigma_{\text{target}}(t)}{\sigma_{\text{reference}}(t)} \tag{7.2}$$

Subsequently, the value of the rescaled index can be derived via the subsequent methodology:

$$S^i_{\text{adjust}}(t) = S^i_{\text{ref}}(0)\frac{\exp\left(\left(LR^i_{\text{ref}}(t) - \overline{LR_{\text{ref}}}(t)\right) \times \lambda(t) + \overline{LR_{\text{ref}}}(t) \times \lambda^2(t)\right)}{D^i_{\text{ref}}(t)} \tag{7.3}$$

Where:

- $\overline{LR_{\text{ref}}}(t) = \frac{1}{v}\sum_{i=1}^{v} LR^i_{\text{ref}}(t)$.
- $LR^i_{\text{ref}}(t) = \ln\left(\frac{s^i_{\text{ref}}(t)}{s^i_{\text{ref}}(0)} \times D^i_{\text{ref}}(t)\right)$.

Furthermore, it is important to note that $\overline{LR_{ref}}(t)$ estimates $\mathbf{E}\left[LR_{ref}\right]$ without weighting, as rescaling generally precedes the weighted Monte Carlo (WMC) application. This sequence helps avoid the severe distortion of index volatility, especially when simulations are restricted, as such distortion might not be corrected by an extreme choice of weights. Although this approach minimizes volatility distortion, it is not without warnings. The weighted estimator's effectiveness depends largely on the scenario selection method, and any marked asymmetry in the chosen volatilities can significantly bias the estimator.

7.3 The weighted Monte Carlo Algorithm (WMC)

We introduce here the Paper by Marco Avellaneda[1]: Weighted Monte Carlo-A new technique for calibrating Asset-Pricing Models, which discusses a conceptual approach to calibrate stochastic models for option pricing. We divert the practical use of this reference for our needs, which is the construction of a stressed sensitivity , the idea being to approximate $\mathbf{E}[R_f]$ by $\mathbf{E}^{\overset{*}{p}}[R_f]$ where $\overset{*}{p}$ would be the optimal measure of a minimization problem defined later or X would be all our risk factors generated from a diffusion model.At same time one wants also to keep this measure as a new reference measure with the hope that the estimators derived from this measure would accurately reflect the shock done on our risk factors.

7.3.1 Original Idea behind WMC

Suppose a Risk Factor R_f is now modeled by a general stochastic process X_t, starting with a Risk Factor Diffusion scheme Dynamic:

$$X_{n+1} = X_n + \sigma\left(X_n, n\right) \cdot \xi_{n+1}\sqrt{\Delta T} + \mu\left(X_n, n\right)\Delta T, \quad n = 1, 2, \ldots, M \tag{7.4}$$

where μ and σ are functions to define and calibrate according to the problem. One might consider calibrating the DDLMM for rates models. However, this approach can be very complex in some practical applications for reasons that are outside the scope of our study.

Avellaneda's original approach is to use a more general non-parametric method to calibrate our models. In short, given the dynamics of X, instead of looking for the best parameters of Equation (6.4), we directly look for the "risk-neutral" measure that would best approximate our market data.

We decided to revisit this approach for our own case of stressed sensitivity construction. Indeed, one might think that if changing our market instruments with for instance a shocked swaptions surface with shocked upward rates, finding an optimal measure to fit market data would mechanically improve the calibration from a central sensitivity without the need to recalibrate the DDLMM according to new rates changes and generate a shocked sensitivity mainly. In the next section, we introduce how this can be achieved in practice.

7.3.2 Fitting Market datas

Let Denote our shocked market prices of each instrument by $C_1, ..., C_N$ as well as the cash flows on each simulation for the instrument N:

$$g_{1j}, g_{2j}, \ldots g_{\nu j} \quad (j = 1, \ldots N) \tag{7.5}$$

where ν would be simulation budget (in practice in our case, number of scenarios we want to use) by the risk-neutral valuation theorem,we know that there exist optimal weights such that

$$\mathbf{E}^p\left\{g_j(\omega)\right\} = \sum_{i=1}^{v} p_i g_{ij} = C_j, \quad j = 1, \ldots, N \tag{7.6}$$

Before introducing the complete minimization problem, we define the sum of the squared weighted residuals $\chi(w)$ by

$$\boxed{\chi(w)^2 = \frac{1}{2}\sum_{j=1}^{N}\frac{1}{w_j}\left(\mathbf{E}^p\left\{g_j(\omega)\right\} - C_j\right)^2 \tag{7.7}}$$

Intuitively, we will seek to minimize this function over all possible weights $p = (p_1, \ldots, p_\nu)$ using the penalty parameter w_j. If $w_j \ll 1$, then we will obtain a perfect fitting of the shocked market datas. Furthermore, the error term is controlled by the central limit theorem and is of the order of $\frac{1}{\sqrt{w_j}} \forall j \in [1, N]$.

However, we now need to add constraints over weights $p = (p_1, \ldots, p_\nu)$ in order to solve this problem. The first one could obviously be that $\sum_{i=1}^{\nu}, p_i = 1$ to ensure that we are effectively finding a distribution. However, with only this constraint we also want to like our weights $(p_1, .., p_\nu)$ being close of our initial distributions of scenarios following by convention an uniform law $(\frac{1}{\nu}, \ldots. \frac{1}{\nu})$ For this reason, we introduce the Kullback-Leibler Entropy Function in the next section, which is a good metric to constrain weights in practical uses.

7.3.3 Conserve Prior Distribution Properties : The Kullback Leibler Entropy function

The relative entropy or Kullback-Leibler divergence quantifies the deviation of a probability distribution from a prior. In the context of Monte Carlo simulations, we use it to measure the deviation of a calibrated model from a uniform distribution u.

Definition 7.3.1: Kullback-Leibler Entropy function

Let p be the probabilities of the paths in a simulation with ν paths and u a uniform distribution. The Kullback-Leibler (KL) Entropy function is defined as:

$$D(p \mid u) = \log \nu + \sum_{i=1}^{\nu} p_i \log p_i \tag{7.8}$$

We are investigating how probability is distributed across different paths, each with a certain size, denoted by μ. This size is uniformly distributed and is related to the number of paths, ν, by $\mu = \nu^\alpha$, where α is a value between 0 and 1.When we use this relationship in our equations, we find that the relative entropy, $D(p \mid u)$, is equal to $(1 - \alpha) \log \nu$. This relationship helps us understand that the relative entropy can be seen as a way to measure the size of the support (or basis) of a distribution on a logarithmic scale.

Now, let's consider a more complex situation where the probability distribution is given by:

$$p_i = \frac{1}{\nu^{\alpha_i}}, \quad i = 1, 2, \ldots, \nu \tag{7.9}$$

Here, N_α represents the number of paths where $\alpha_i = \alpha$, and we have two conditions to satisfy:

$$\sum_\alpha N_\alpha = \nu, \quad \sum_\alpha \frac{N_\alpha}{\nu^\alpha} = 1 \tag{7.10}$$

Using these conditions in our calculations, we arrive at the following expression for the relative entropy:

$$D(p \mid u) = \log \nu \, (1 - \mathbf{E}^p(\alpha)) \tag{7.11}$$

This shows that if the expected value of α is high, the relative entropy will also be higher, meaning that there is a wider variation in α values. If there's a big difference between the measured probabilities and the original assumptions (prior), it implies that the measurement will focus on paths with smaller α values.

we conclude that Low Relative Entropy is desirable as it means the model is closely aligned with the original assumptions or prior model.Hence this metric is Used to solve the WMC Optimization Problem.

The figure depicts the evolution of relative errors on constraints as a function of the chosen weight value for the arbitrary weight family $w \in \mathbb{R}^N$. $\rightarrow$ Starting from the value $\log(w) = (-10, -10, \ldots, -10)$, that is, from $w = (e^{-10}, e^{-10}, \ldots, e^{-10})$, the computed relative differences become tolerable from a regulation point of view. We set this value for the rest of our study case as the mean value falls below a regulatory treshold.

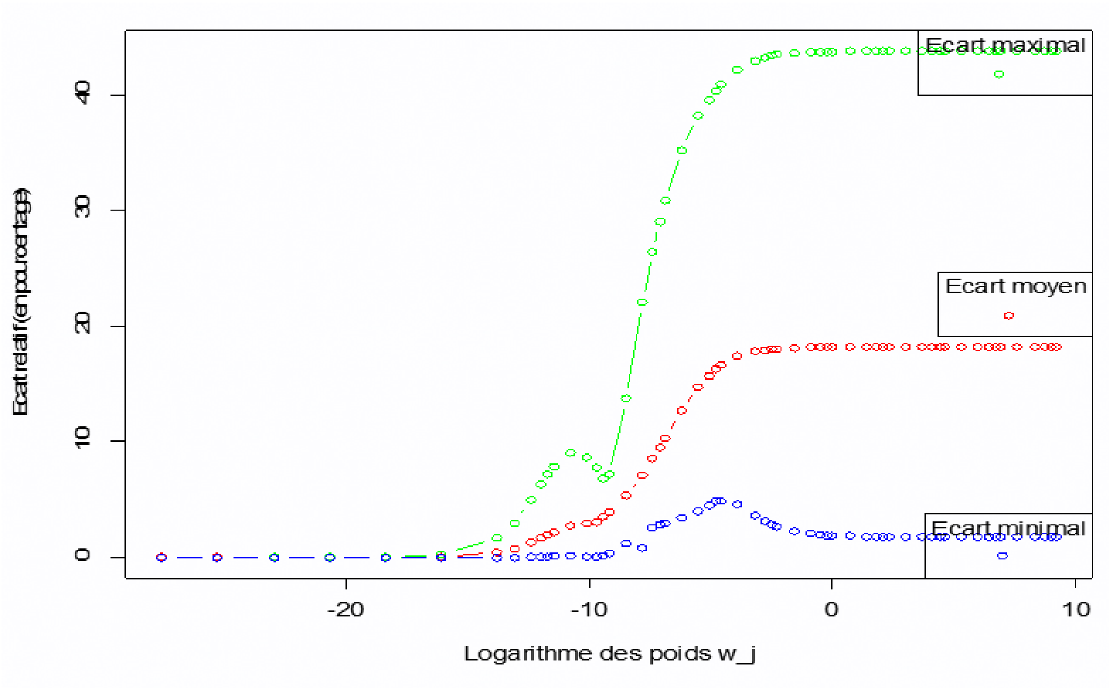

Figure 7.1: Relative error Between Simulated Prices and Real Prices

7.3.4 The Optimisation Problem

One approach in Avellaneda's paper consists of minimizing not only the relative entropy but also the sum of the squared residuals on the constraints, especially when exact matching on price constraints is not desired.

We introduce $w = (w_1, w_2, \ldots, w_N)$, a vector of positive weights, chosen to give more or less importance to the constraints as mentionned earlier.

The optimization problem is defined as:

$$(\mathcal{P}_2) : \min_p \left\{\chi_w^2(p) + D(p \mid u)\right\} \text{ with } \chi_w^2(p) = \frac{1}{2}\sum_{j=1}^{N} \frac{1}{w_j} \left(\mathbb{E}^p\left[g_j(w)\right] - C_j\right)^2 \tag{7.12}$$

We can show that:

$$\inf_p \left\{\chi_w^2(p) + D(p \mid u)\right\} = -\inf_\lambda \left\{W(\lambda) + \frac{1}{2}\sum_{j=1}^{N} w_j \lambda_j^2\right\}. \tag{7.13}$$

with :

$$W(\lambda) = -\log v + \log\left[\sum_{i=1}^{v} \exp\left(\sum_{j=1}^{N} \lambda_j g_{ij}\right)\right] - \sum_{j=1}^{N} \lambda_j C_j \tag{7.14}$$

This leads to an equivalence with the following problem:

$$(\mathcal{P}_2) \Leftrightarrow \min_\lambda \left\{U(\lambda) := \log\left[\sum_{i=1}^{v} \exp\left(\sum_{j=1}^{N} \lambda_j g_{ij}\right)\right] - \sum_{j=1}^{N} \lambda_j C_j + \frac{1}{2}\sum_{j=1}^{N} w_j \lambda_j^2\right\} \tag{7.15}$$

The solution to the minimization problem of this function U allows us to determine the optimal coefficients $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*)$, and then obtain the optimal weight family $p^* = (p_1^*, p_2^*, \ldots, p_v^*)$.

This algorithm is a minor modification of the initial algorithm provided by Avellaneda paper corresponding to the exact fitting of prices, and can be implemented similarly, for instance, using the L-BFGS method.

7.3.5 Final Algorithm

For a clear explanation , One can find how to apply the Weighted Monte Carlo Algorithm to compute the risk neutrals probabilities $p_i, i = 1, 2, \dots \nu$.

Algorithm 2 Apply WMC

1: **procedure** COMPUTE PROBABILITIES
2: **Construct** a set of reduced sensibility with ν scenarios from the Base central case Sensivity
3: **Compute** the shocked cashflow matrix $\{g_{ij}, i = 1, \dots \nu, j = 1, 2, \dots, N, \}$.
4: **Minimize** the problem (6.15) using a gradient-based optimization routine to find values of $\lambda_1 \dots \lambda_N$
5: **Compute** inject the values of $\lambda_1 \dots \lambda_N$ in 6.14 to obtain the risk-neutral probabilities $p_i, i = 1, 2, \dots \nu$.
6: **end procedure**

In practice, after the application of the WMC in Rebase, all our estimators will now be distorted. Indeed, given $p = (p_1; p_2; ...p_\nu)$ we now estimate:

$$\mathbf{E}[X] \text{ by } \mathbf{E}^p[X] = \sum_{i=1}^{\nu} p_i X_i \tag{7.16}$$

$$\mathbf{VAR}[X] \text{ by } \mathbf{VAR}^p[X] = \sum_{i=1}^{\nu} p_i (X_i - \mathbf{E}^p[X])^2 \tag{7.17}$$

Chapter 8

Numerical Results

introduction

While the preceding sections introduced the practical and theoretical foundations of the insurance world, as well as the main transformations and methods used to optimize computation time, this final chapter encompasses the core of our results. We detail our numerical outcomes from various applications. After presenting our primary comparison metrics, we begin by examining the learning results from our initial learning phase. We then delve into scenario sampling methods for reduction and explore adjustments such as the weighted algorithm in practice. Lastly, we analyze the predictions from our machine learning model in relation to the shocked sensitivities.

8.1 Learning Results

For this first section, we delve into the results of our Statistical Learning Framework. We recall our scoring metrics used for our algorithm: the RMSRE, which is defined in Chapter 5. Specifically, the RMSRE and the Relative Residual Formula are given by

$$\text{RMSRE} = \left\| \frac{Y - \hat{Y}}{Y} \right\|_2, \qquad \epsilon = \frac{Y - \hat{Y}}{Y}, \tag{8.1}$$

where Y represents our validation observations, and $\hat{Y}$ denotes the predictions of our model on the training set. The latter equation facilitates a deeper residual analysis in our Linear model.

8.1.1 Linear Models

We first conduct a residual analysis on a simple polynomial regression to assess the quality of our model's assumptions. we recall that under the assumption of gaussian

Linear model we should have.

$$\epsilon \sim \mathcal{N}(0, \sigma^2)$$

with σ a constant , therefore plotting ϵ with respect to $\hat{Y}$ should be centered around a straight line = 0 and with no spread.

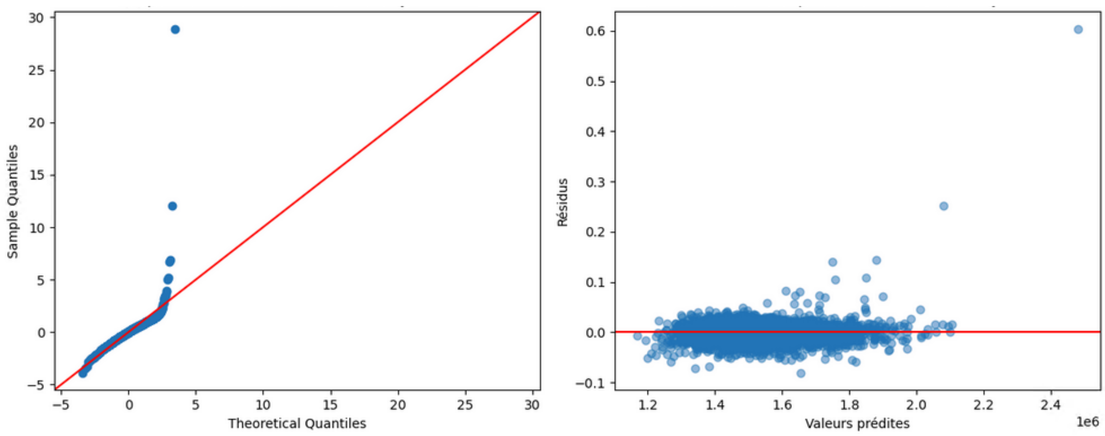

Figure 8.1: Left : QQplot of Linear regression , Right : Y - $\hat{Y}$ with respect to $\hat{Y}$

the residuals quantiles follows a straight line and centered, also no evident spread is seen on the linear basis.This suggest that the polynomial regression is a good fit for our model the distribution of coefficient on linear Model also doesn't follow any skew

	Linear (No regularization)	Linear (L1-Regularization)	Polynomial (No regularization)	Polynomial (L1-Regularization)
RMSRE	1.81 %	1.93 %	5.91 %	1.19 %
Relative Std: $\frac{\text{RMSRE}}{\text{Std}}$	2.78 %	1.86 %	2.35 %	1.81 %

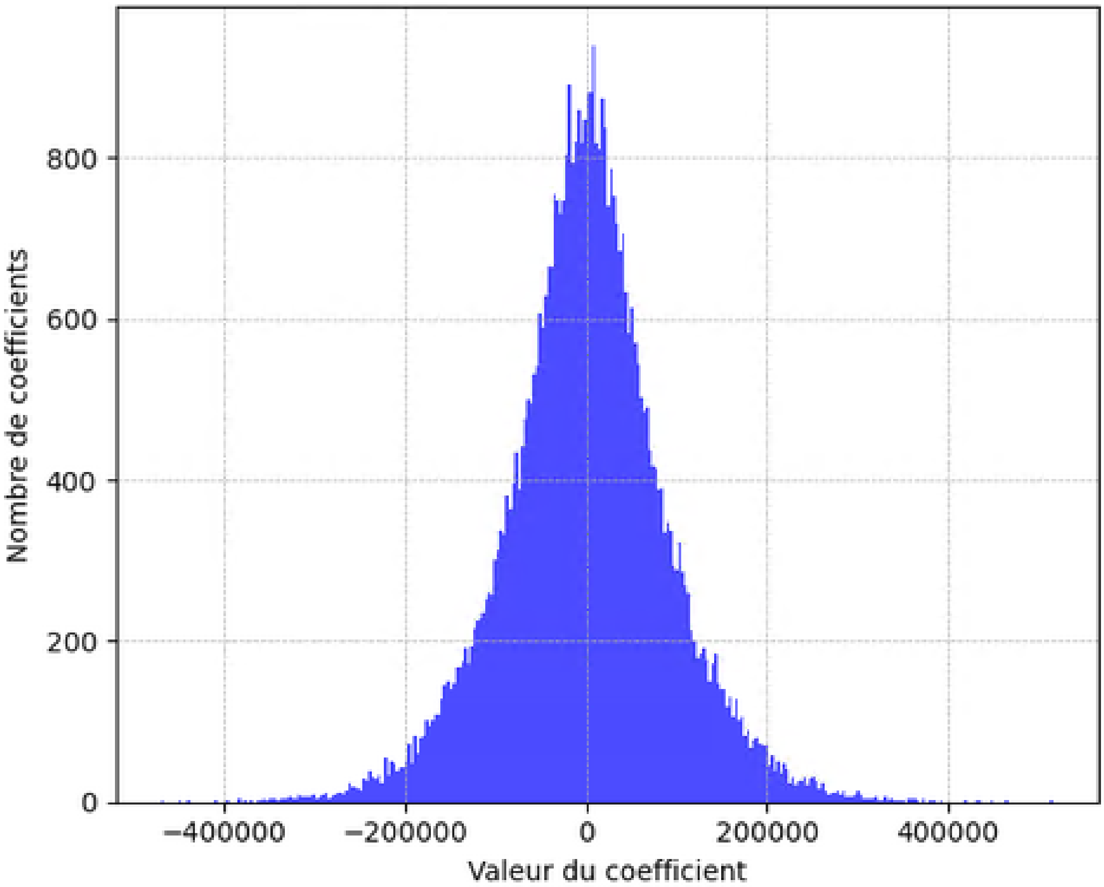

Figure 8.2: Distribution of Linear coefficient

we now inspect the cross validation score on the polynomial regression with and without penalisation (LASSO) , results can be submitted into this table :
If the results seem promising, we can verify that our model is not overfitting by examining the learning curves.

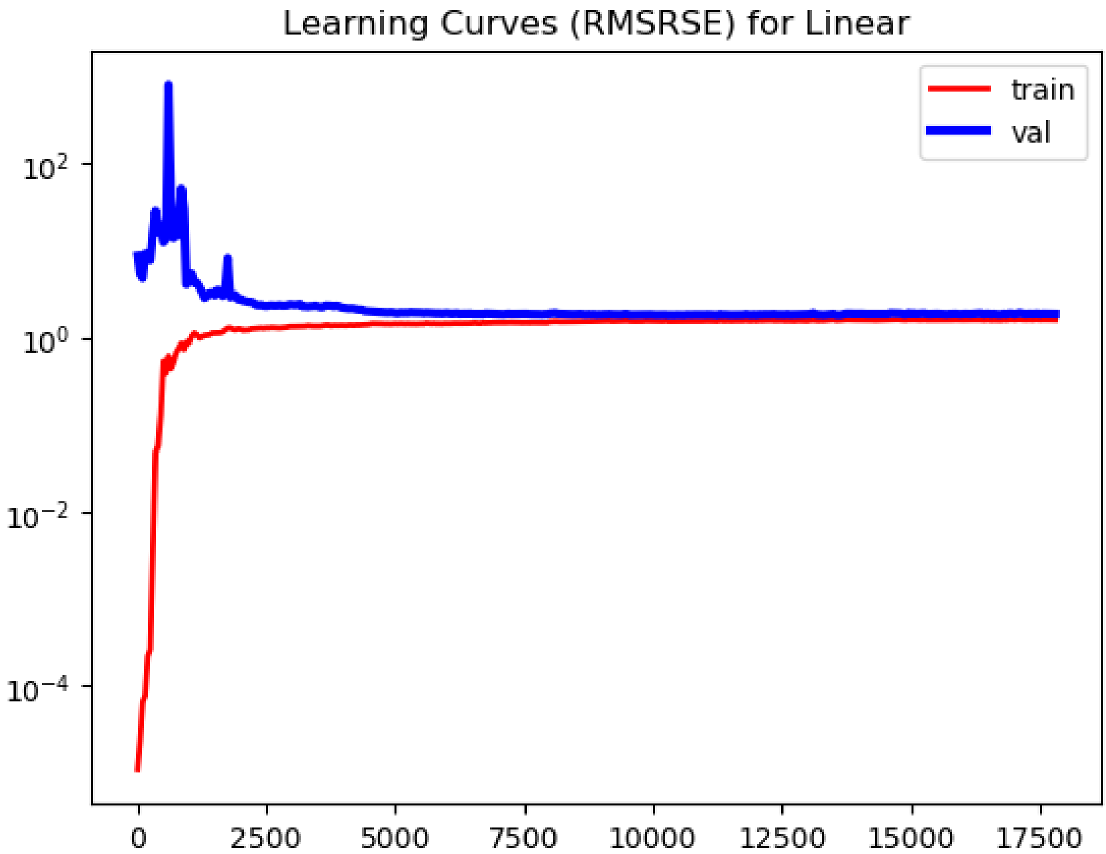

Figure 8.3: Learning curve of Linear Regression

To conclude this section, our conclusion is that an error of about 1% on the Best Estimate could actually be catastrophic (with the total balance being expressed in billions). On the other hand, the linear model, based on its analysis, seems promising and suggests potential improvements to the model in the long run. However, it could prove to be robust in assessing the magnitude of certain sensitivities during subsequent stress tests.

8.1.2 Features Tuning : PCA & Sequential Feature Algorithm

We now delve into the PCA et SFS algoritmh results for feature selection , we perform as previously with the selected features a classic Regression and a Lasso.
first the PCA seems to efficiently reduce the dimensionnality of problem by keeping a reasonnable numbers of explained components while keeping 95 % of the total variance :

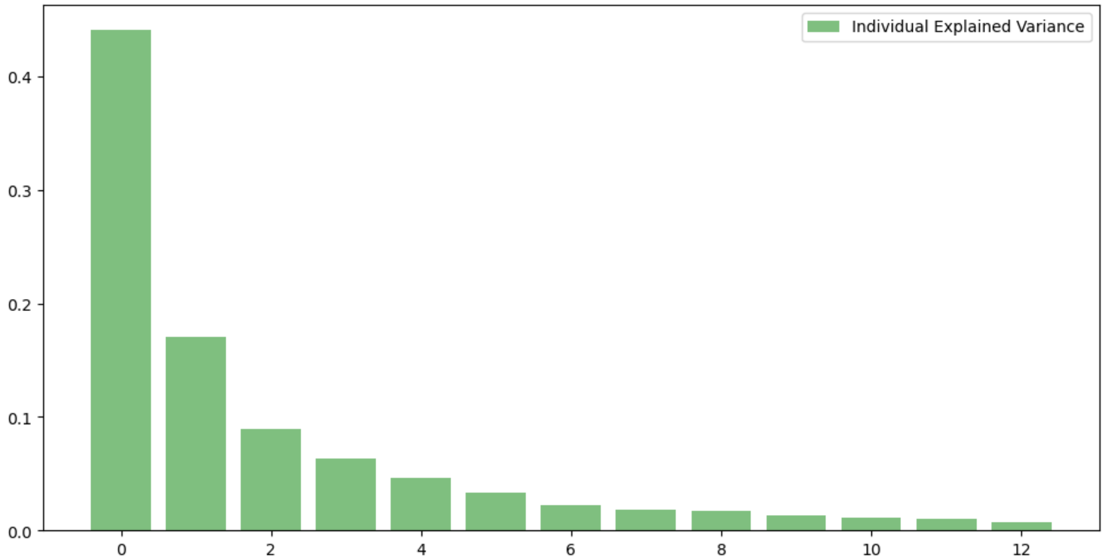

Figure 8.4: Explained variance of Component for a PCA with 95% variance kept

We now delve on the results of regression using Sequential feature algorithm and PCA for feature tuning before regression

	Stepwise Algorithm (No regularization)	Stepwise Algorithm (L1-Regularization)	PCA regression (No regularization)
RMSRE	2.39 %	1.64 %	2.72 %
Relative Std: $\frac{\text{RMSRE}}{\text{Std}}$	1.85 %	1.21 %	12,40 %

Even though the stepwise algorithm, when applied to polynomial features, is more time-consuming, a significant improvement is evident. Subsequently, the residuals analysis of both algorithms are displayed to ensure we aren't overfitting with these models

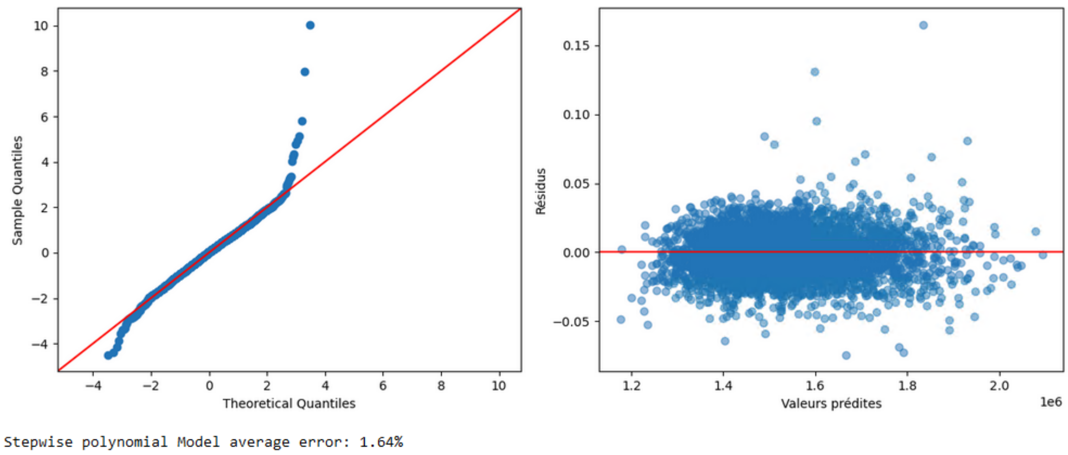

Figure 8.5: Residual Analysis for Stepwise Model

for the next parts we chose the SFS algorithm coupled with a polynomial regression with the reason of it's trade off between simplicity and results.

8.2 Reduction and Shocked Sensitivity Creation

We now focus on the Algorithm of Reduced sensitivity creation , we first review the Impact of Sobol sequence on accuracy compared to a classic Mersenne twister RNG, we then examine our two reduction sampling methods on results and to end we check the results on Stressed sensitivity construction with the use of WMC algorithm , Our benchmark is the Central sensitivity of 21k scenarios.

8.2.1 The Impact of RNG on accuracy

A comparison between hybrid-Sobol and Mersenne-Twister RNGs is demonstrated in the ensuing discussion. Initially, we introduce the formula for the root mean square relative error (RMSRE) as follows:

$$\text{RMSRE} = \sqrt{\frac{1}{|\mathcal{T}|}\sum_{i\in\mathcal{J}}\left(\frac{\hat{E}_i - E_i}{E_i}\right)^2},$$

where $\mathcal{T}$ is the collection of all martingale and/or market-consistency tests, $\hat{E}_i$ is the estimated value over the simulations for the test indexed by i and E_i is the corresponding expected target value.
Note that the WMC algorithm incorporates only a small subset of the RMSRE components to avoid issues with convergence. Therefore, the RMSRE also serves as a significant metric to ensure, a posteriori, that the WMC algorithm effectively enhances the martingale and market-consistency properties of the scenarios. This metric will be extensively

employed in the forthcoming sections to evaluate the performance of the scenarios from a purely ESG perspective.
The RMSREs and the convergence gaps of the cash-flow model are presented below for 3000 simulations sampled from both hybrid-Sobol and Mersenne-Twister RNGs, utilizing the capabilities of the Milliman ESG.

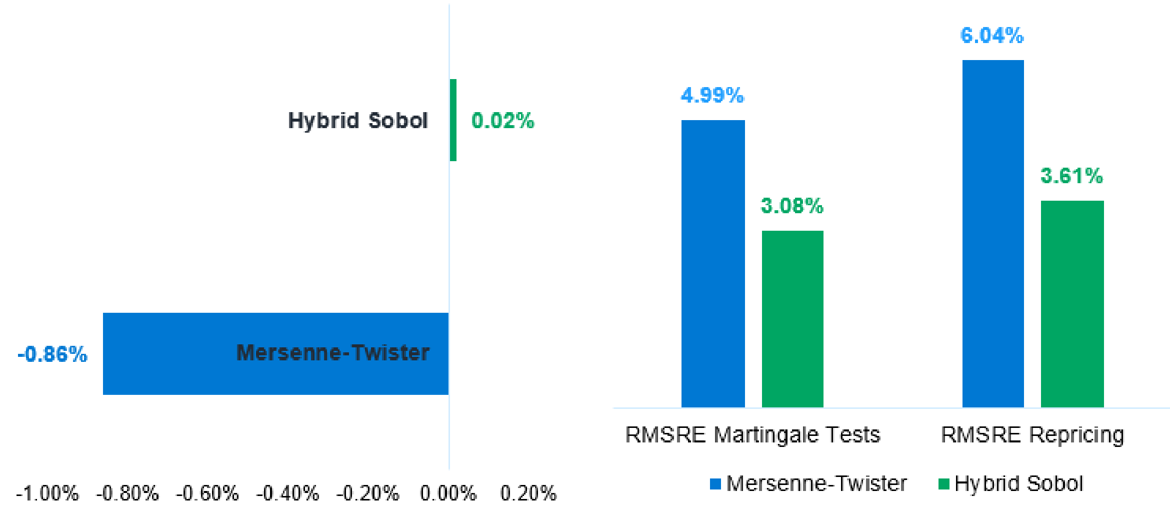

Figure 8.6: and impact of RNG on Leakage (right),impact of rng on RMSRES levels (left)

The application of the hybrid-Sobol RNG exhibits a marked enhancement in the overall metrics. This is evidenced by the substantial decrease in both martingale and market-consistency tests observed. In addition, when utilizing the hybrid-Sobol RNG, there is a significant reduction in the convergence leakage related to the Best estimate model. Remarkably, these improvements align with the findings obtained from the scenario selection methodologies introduced in this analysis.

8.2.2 Reduction Sampling

In this section we delve into the potential reduction adjustment on the Best estimate impact,compared to the Benchmark (our central sensivity with 21k samples)

	Method	
	3K STO	**3K SOA**
Relative Error Best Estimate	$0,01\%$	$0,16\%$
[IC up ; IC down]	[-0,7% ; 0,10%]	[-0,03% ; 0,36%]

Table 8.1: Comparison of Methods

As seen, the SOA approach does not give successful results compared to a classic stochastic sample methodology. This could be explained by the aim of the methodology to compulsively take edge case scenarios, which skews the final mean estimate of the best estimate. However, this approach has been shown to be more efficient with a very reduced set of scenarios, as demonstrated in the whitepaper on LSC reductions.

8.2.3 Stressed Sensitivity

We now investigate how the WMC algorithm affects our sensitivity to shocks on the best estimate and check whether the projected results from the WMC algorithm are close to the real best estimate. This involves determining if the WMC is indeed an efficient method to measure the adjustments made to our table. We will investigate various shocks involving rates and implied ATM volatilities.

Our benchmark is the initial chess inferred by our economic scenario generator :

Best Estimate Leakage	Rate up	Rate Down	Vol up	Rate & Vol up
SOA Central Sensivity (WMC)	0.09%	0.15%	-0.73%	0.69%
STO Central Sensivity (WMC)	0.12%	0.13%	0.73%	0.71%

Table 8.2: Comparison of Central Sensitivities using WMC

from table , no clear advantages of SOA vs Classic sampling methods can be seen , however WMC seems efficient to accuratly reproduce best estimate over a classic method using base case sensivity

8.3 Prediction of Best Estimate on Shocked Sensitivity

The final objective of this part now is the completly replace the ALM model computation with the use of your machine learning model. we remind the fundamental idea that under the Weighted Measure the RMSRE is therefore weighted as in formula (7.16) , (7.17).

8.3.1 Data Visualisation

The first step to determine if the method is appropriate involves data visualization using PCA to compare shocked sensitivity with central sensitivity. When observing the barycenters calculated using K-means, we find that the distance is relatively short. This observation supports the rationale that our learned central sensitivity should primarily be used for interpolation, as opposed to extrapolation, when predicting the best estimate of shocked sensitivity.

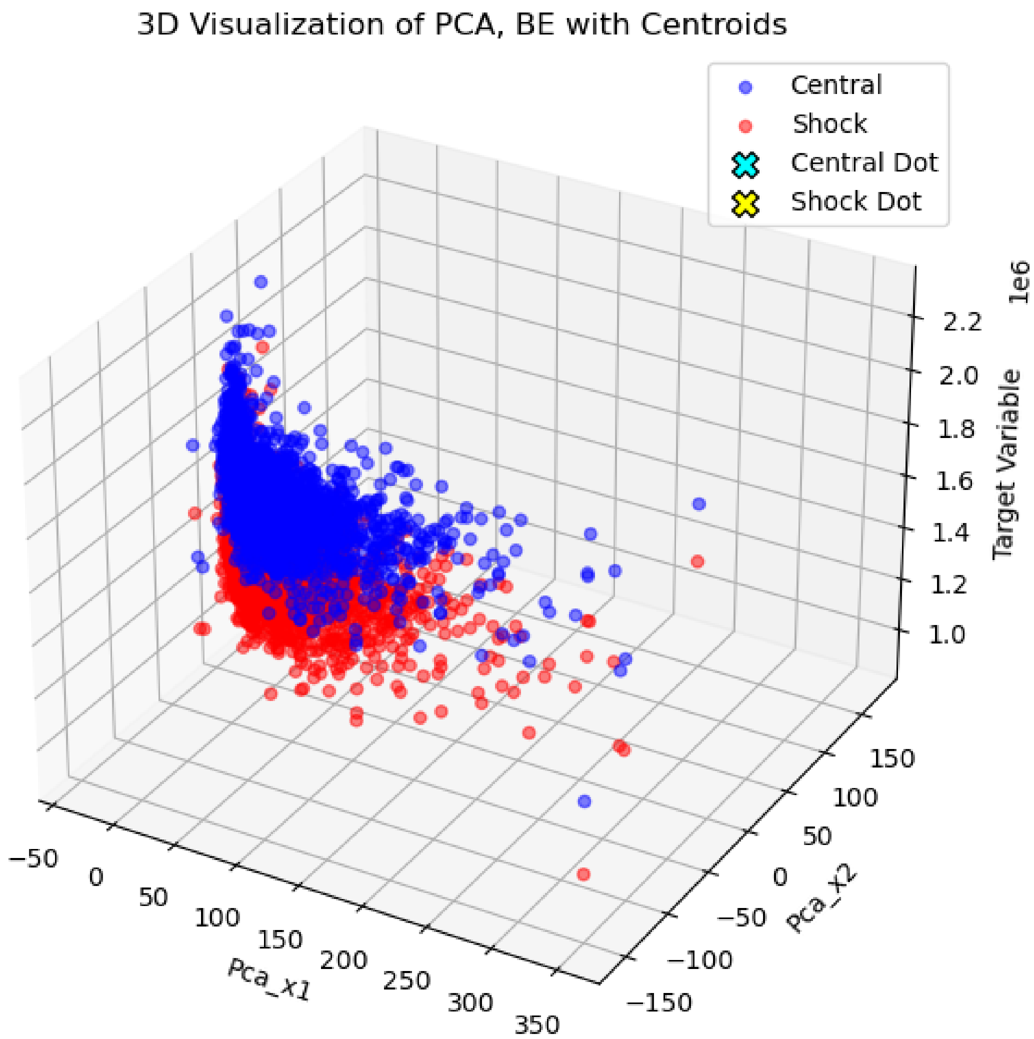

Figure 8.7: Visualisation of Shocked Best estimate (red) against Central best estimate (Blue)

8.3.2 Shocked Sensitivity Prediction Results

To conclude this section of the study, we examined the results of the machine learning model in predicting the best estimate based on a sensitivity shock (previously generated using a WMC). Initially, we used the WMC approach and derived the best estimate from a real ALM model. Subsequently, we compared our machine learning model against an additional approach, which involved a sensitivity generated from both our GSE and ALM models.

Best Estimate Leakage	Rate up	Rate Down	Vol up	Rate & Vol up
ML Model STO	2.20	2.12	1.72	2.20 %
ML Model SOA	2.26	2.15	1.77	2.27

We now compare the model leakage compared to a classic Complete Approach with Chess coupled with ALM model

Best Estimate Leakage	Rate up	Rate Down	Vol up	Rate & Vol up
ML Model STO	1.76	0.95	0.76	2.03%
ML Model SOA	1.50	0.89	0.81	2.09

8.4 Conclusion

In this comprehensive study, we embarked on an in-depth exploration of the insurance industry, focusing on its intricate financial components, from the Solvency II framework to the dynamics of Asset and Liabilities Management (ALM). A significant portion of our research has been dedicated to the construction and understanding of an Economic Scenario Generator (ESG). By diving into various modeling paradigms such as the Displaced Diffusion Libor Market Model (DDLMM) for interest rate risk and the Stepwise constant Volatility model (SCVM) for equity-like risk, we laid the groundwork for understanding their dynamics, calibration, and validation.

One of the critical innovations we introduced was the quest to reduce the computational cost of the ESG. This endeavor saw us exploring various statistical learning frameworks, delving into the nuances of Quasi Monte Carlo Sampling and introducing methodologies to generate base and stressed sensitivities. Our numerical results section stands testament to the robustness of our approach, detailing our learnings and our efforts at predicting the best estimates on shocked sensitivity.
Our approach offers dual benefits. First, we save computational time by avoiding the need to compute a new sensitivity through an ESG; the WMC eliminates the need for recalibration. Secondly, we don't need to use an ALM model more than once, as we utilize a proxy for the Best Estimate.

In essence, our research provides a holistic view of financial modeling within the insurance sector, presenting new methodologies, validating them, and offering insights into their practical implications. This work paves the way for future research, aiming to refine these methods further and ensuring the stability and sustainability of the insurance industry in an ever-evolving financial landscape

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