The primal linear program can be converted to its dual form:

$$\max_{\mu \geq 0, \lambda} \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \mu, \lambda) = \max_{\mu \geq 0, \lambda} \min_{\mathbf{x}} \mathbf{c}^\top \mathbf{x} - \mu^\top \mathbf{x} - \lambda^\top (\mathbf{A}\mathbf{x} - \mathbf{b}) \tag{11.31}$$

$$= \max_{\mu \geq 0, \lambda} \min_{\mathbf{x}} (\mathbf{c} - \mu - \mathbf{A}^\top \lambda)^\top \mathbf{x} + \lambda^\top \mathbf{b} \tag{11.32}$$

From the FONC, we know $\mathbf{c} - \mu - \mathbf{A}^\top \lambda = \mathbf{0}$, which allows us to drop the first term in the objective above. In addition, we know $\mu = \mathbf{c} - \mathbf{A}^\top \lambda \geq 0$, which implies $\mathbf{A}^\top \lambda \leq \mathbf{c}$. In summary, we have:

Primal Form (equality)	Dual Form
$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$	$\begin{aligned} \underset{\lambda}{\text{maximize}} \quad & \mathbf{b}^\top \lambda \\ \text{subject to} \quad & \mathbf{A}^\top \lambda \leq \mathbf{c} \end{aligned}$

If the primal problem has n variables and m equality constraints, then the dual problem has m variables and n constraints.[16] Furthermore, the dual of the dual is the primal problem.

[16] An alternative to the simplex algorithm, the *self-dual simplex algorithm*, tends to be faster in practice. It does not require that the matrix $\mathbf{A}_\mathcal{B}$ satisfy $\mathbf{x}_\mathcal{B} = \mathbf{A}_\mathcal{B}^{-1}\mathbf{b} \geq \mathbf{0}$. The self-dual simplex algorithm is a modification of the simplex algorithm for the dual of the linear programming problem in standard form.

Optimality can be assessed by verifying three properties. If someone claims $(\mathbf{x}^*, \lambda^*)$ is optimal, we can quickly verify the claim by checking whether all three of the following conditions are satisfied:

1. $\mathbf{x}^*$ is feasible in the primal problem.
2. λ^* is feasible in the dual problem.
3. $p^* = \mathbf{c}^\top \mathbf{x}^* = \mathbf{b}^\top \lambda^* = d^*$.

Dual certificates are used in example 11.9 to verify the solution to a linear program.

```julia
function dual_certificate(LP, x, λ, ϵ=1e-6)
    A, b, c = LP.A, LP.b, LP.c
    primal_feasible = all(x .≥ 0) && A*x ≈ b
    dual_feasible = all(A'*λ .≤ c)
    return primal_feasible && dual_feasible &&
           isapprox(c⋅x, b⋅λ, atol=ϵ)
end
```

Algorithm 11.6. A method for checking whether a candidate solution given by design point `x` and dual point `λ` for the linear program `LP` in equality form is optimal. The parameter `ϵ` controls the tolerance for the equality constraint.

Consider the standard-form linear program with

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

We would like to determine whether $\mathbf{x}^* = [2, 0, 1]$ and $\boldsymbol{\lambda}^* = [1, 0, 0]$ are an optimal solution pair. We first verify that $\mathbf{x}^*$ is feasible:

$$\mathbf{A}\mathbf{x}^* = [1, -2, 5] = \mathbf{b}, \qquad \mathbf{x}^* \geq \mathbf{0}$$

We then verify that $\boldsymbol{\lambda}^*$ is dual-feasible:

$$\mathbf{A}^\top \boldsymbol{\lambda}^* = [1, 1, -1] \leq \mathbf{c}$$

Finally, we verify that p^* and d^* are the same:

$$p^* = \mathbf{c}^\top \mathbf{x}^* = 1 = \mathbf{b}^\top \boldsymbol{\lambda}^* = d^*$$

We conclude that $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ are optimal.

Example 11.9. Verifying a solution using dual certificates.