Alg. Liniară

Seminar III

$M = \{v_1, v_K\} \subset V$

M SLI $\qquad \forall a_1, \dots, a_K \in K \quad a_1 v_1 + \dots + a_K v_K = 0 \Rightarrow a_1 = \dots = a_K = 0$

M este bază în $V \iff M$ SLI și $\langle M \rangle = V$.

$|M| = k = \dim V$.

1. Fie $L = \langle (1,1,1,1), (3,2,4,1), (2,1,3,0) \rangle$

$\dim L = ?$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 1 \\ 1 & 4 & 3 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & +1 & +1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \{(1,1,1,1), (3,2,4,1), (2,1,3,0)\} \text{ nu e SLI}$$

$\Rightarrow x_1 = x_3 \quad \leftarrow x_3$ este scris în funcție de x_1 și x_2

$x_2 = -x_3$

$(2,1,3,0) = (3,2,4,1) - (1,1,1,1)$

$\Rightarrow L = \langle (1,1,1,1), (3,2,4,1) \rangle$

$\{(1,1,1,1), (3,2,4,1)\}$ SLI. $\Rightarrow \dim L = 2$.

2. Fie $v_1 = (1,1,0,0)$, $v_2 = (1,0,1,0)$

a) ~~(Completați)~~ Ar că $\{v_1, v_2\}$ SLI

b) Completați $\{v_1, v_2\}$ la o bază a lui $\mathbb{R}^4$

a) $$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \{v_1, v_2\} \text{ SLI}$$

$\leftarrow$ FE + pivot pe fiecare col.

b) $v_3 = (0,0,1,0)$

$v_4 = (0,0,0,1)$

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \{v_1, v_2, v_3, v_4\} \text{ bază în } \mathbb{R}^4$$

$v_1 = (a_{11}, \dots, a_{n1})^t$

$v_m = (a_{1m}, \dots, a_{nm})^t$

$A = \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{1m} & & a_{nm} \end{pmatrix}$ $\quad M = \{v_1, v_m\}$

$\left.\begin{array}{l} M \text{ SLI} \iff rg\, A = m \\ M \text{ SG} \iff rg\, A = n \end{array}\right\} \Rightarrow [M \text{ bază} \iff m = n \text{ și } rg\, A = n]$.

$\forall (b_1, \dots, b_n)^t \ \exists (x_1, \dots, x_n)^t$ a.î.

$A \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \iff rg\, A = rg \left(A \,\middle|\, \begin{matrix} b_1 \\ b_n \end{matrix}\right) \in \mathbb{R}^n$

③ Fie $v_1 = (1,2,1,1)$, $v_2 = (2,1,3,3)$, $v_3 = (0,1,0,1)$, $v_4 = (1,1,2,1)$

a) verificați dacă $\{v_1, v_2, v_3, v_4\} = B$ este bază în $\mathbb{R}^4$

$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 2 \\ 1 & 3 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$

$\Rightarrow rg\, A = 4 \Rightarrow B$ bază
(pivot pe fiecare col).

b) scrieți coord vectorului $(1,1,1,1)$ în rap cu B

$A = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 2 & 1 \\ 1 & 3 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & -3 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -3 & 1 & -2 & -1 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \to$

$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -2 & -1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & \frac{1}{3} \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$

$(1,1,1,1) = \frac{2}{3}(1,2,1,1) + \frac{1}{3}(2,1,3,3) - \frac{1}{3}(0,1,0,1) + \frac{1}{3}(1,1,2,1)$

$= (1,1,1,1)$

$= \left(\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$ coord lui $(1,1,1,1)$ în rap cu B

⑤ Fie S spațiul soluțiilor sistemului

$$\begin{cases} 14x_1+35x_2-7x_3+63x_4=0 \\ -10x_1-25x_2+5x_3-45x_4=0 \\ 26x_1+65x_2-13x_3+117x_4=0 \end{cases} \iff \begin{cases} 2x_1+5x_2-x_3+9x_4=0 \\ 2x_1+5x_2-x_3+9x_4=0 \\ 2x_1+5x_2-x_3+9x_4=0 \end{cases}$$

Să se determine dim S.

$$\iff 2x_1+5x_2-x_3+9x_4=0 \Rightarrow x_1 = -\frac{5}{2}x_2+\frac{1}{2}x_3-\frac{9}{2}x_4$$

$$S = \left\{ \left(-\frac{5\alpha}{2}+\frac{\beta}{2}-\frac{9\gamma}{2}, \alpha, \beta, \gamma\right) \middle| \alpha,\beta,\gamma \in \mathbb{R} \right\}$$

$$v \in S \quad v = \left(-\frac{5\alpha}{2}+\frac{\beta}{2}-\frac{9\gamma}{2}, \alpha, \beta, \gamma\right) = \left(-\frac{5\alpha}{2}, \alpha, 0, 0\right) + \left(\frac{\beta}{2}, 0, \beta, 0\right) +$$

$$\left(-\frac{9\gamma}{2}, 0, 0, \gamma\right) = \alpha\underbrace{\left(-\frac{5}{2}, 1, 0, 0\right)}_{v_1} + \beta\underbrace{\left(\frac{1}{2}, 0, 1, 0\right)}_{v_2} + \gamma\underbrace{\left(-\frac{9}{2}, 0, 0, 1\right)}_{v_3}$$

$$A = \begin{pmatrix} -\frac{5}{2} & \frac{1}{2} & -\frac{9}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow rg\,A = 3 = \text{nr de coloane}$$

$\Rightarrow \{v_1, v_2, v_3\}$ bază în S.

$\Rightarrow \dim S = 3$

$v_1 \leftarrow x_2 = 1,\ x_3 = 0,\ x_4 = 0$

$v_2 \leftarrow x_2 = 0,\ x_3 = 1,\ x_4 = 0$

$v_3 \leftarrow x_2 = 0,\ x_3 = 0,\ x_4 = 1$

⑥
$$\begin{cases} 2x_1+x_2+4x_3+x_4=0 \\ 3x_1+2x_2-x_3-6x_4=0 \\ 7x_1+4x_2+6x_3-5x_4=0 \\ x_1+8x_3+7x_4=0 \end{cases}$$

$$\begin{pmatrix} 1 & 0 & 8 & 7 \\ 2 & 1 & 4 & 1 \\ 3 & 2 & -1 & -6 \\ 7 & 4 & 6 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 8 & 7 \\ 0 & 1 & -12 & -13 \\ 0 & 2 & -25 & -27 \\ 0 & 4 & -50 & -54 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 8 & 7 \\ 0 & 1 & -12 & -13 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 = x_4$
$x_2 = x_4$
$x_3 = -x_4$

$$S = \{(\alpha, \alpha, -\alpha, \alpha) \mid \alpha \in \mathbb{R}\} = \langle(1,1,-1,1)\rangle \quad [\dim S = 1]$$

⑦. Să se det. dimensiunea spațiului de sol în funcție de λ.

$$\begin{cases} (1-\lambda)x_1+\lambda x_2+2\lambda x_3+2\lambda x_4=0 \\ (-1+\lambda)x_1+(2-2\lambda)x_2-2\lambda x_3-2\lambda x_4=0 \\ (1-\lambda)x_1+\lambda x_2+(2+\lambda)x_3+(1+2\lambda)x_4=0 \\ (-1+\lambda)x_1-\lambda x_2-2\lambda x_3+(2-3\lambda)x_4=0 \end{cases}$$

Pp $\lambda \neq 1$

$$\begin{pmatrix} 1-\lambda & \lambda & 2\lambda & 2\lambda \\ -1+\lambda & 2-2\lambda & 2\lambda & -2\lambda \\ 1-\lambda & \lambda & 2+\lambda & 1+2\lambda \\ 1+\lambda & -\lambda & -2\lambda & 2-3\lambda \end{pmatrix} \to \begin{pmatrix} 1-\lambda & \lambda & 2\lambda & 2\lambda \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{pmatrix}$$

$\lambda \neq 1, \neq 2 \Rightarrow \boxed{\dim S_\lambda = 0}$

$$\xrightarrow{\lambda = 2} \begin{pmatrix} -1 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Leftrightarrow \begin{cases} -x_1 + 2x_2 + 4x_3 + 4x_4 = 0 \\ x_4 = 0 \end{cases}$$

x_3, x_4 principale $\Rightarrow \begin{cases} x_3 = -\frac{1}{4}x_1 - \frac{1}{2}x_2 \\ x_4 = 0 \end{cases}$

$S_2 = \{ (x_1, x_2, \frac{1}{4}x_1, -\frac{1}{2}x_2, 0) \mid x_1, x_2 \in \mathbb{R} \}$

$\Rightarrow \boxed{\dim S_2 = 2}$

$$\xrightarrow{\lambda = 1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Leftrightarrow \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 - x_4 \\ x_4 = 0 \end{matrix} \Rightarrow S_1 = \{ (0,0,0,0) \}$$

$\Rightarrow \boxed{\dim S_1 = 0}$