Spațiu vectorial peste un corp K - [Seminar 2]

$(V, +)$ grup abelian

$K \times V \to V \quad \forall a \in K, v_1, v_2 \in V \quad a(v_1 + v_2) = av_1 + av_2$

$\forall a, b \in K, v \in V \quad (a+b)v = av + bv$

$(ab)v = a(bv) \quad \forall a, b \in K, v \in V$

$1v = v \quad \forall v \in V$

1. Care din urm. mulțimi sunt spații vectoriale K.

a) $V_1 = \{P(x) \in K[x] \mid P(0) = 0\}$

b) $V_2 = \{P(x) \in K[x] \mid P(0) = 1\}$

c) $V_3 = \{P(x) \in K[x] \mid \text{grad } P(x) = n\}$.

d). $V_4 = \{P(x) \in K[x] \mid \text{grad } P(x) \leq n\}$.

a) $0 \in V_1$ adev.

$P(x) \in V_1 \Rightarrow (P(x)) \in V_1$

$-P(0) = 0 \Rightarrow -P(x) \in V_1$

$P(x), Q(x) \in V_1, \; R(x) = P(x) + Q(x)$

$R(0) = P(0) + Q(0) = 0 \Rightarrow R(x) \in V_1$

$(V_1, +)$ grup abelian

$\left.\begin{array}{l} \alpha \in K \\ P(x) \in V_1 \end{array}\right| \Rightarrow \alpha \cdot P(x) \in V_1$

$\alpha P(0) = \alpha 0 = 0 \Rightarrow (\alpha P(x)) \in V_1$

$\Rightarrow V_1$ spațiu vectorial

$\leftarrow$ relatie principală

$V_1 \subset K[x]$

$W \subset V$, V sp. vectorial

W sp. vectorial $\Rightarrow \forall a, b \in K, u, v \in W \quad au + bv \in W$

b) $P(x), Q(x) \in V_2$

$P(0) + Q(0) = 2 \Rightarrow P(x) + Q(x) \notin V_2$

$\Rightarrow V_2$ nu este sp. vectorial

c). $0 \notin V_3 \Rightarrow V_3$ nu este sp. vectorial

$V_3' = V_3 \cup \{0\}$. , $n \geq 1$

$P(x), Q(x) \in V_3$

grad $(P(x) + Q(x)) \leq n \Rightarrow (V_3, +)$ nu este parte stabilă

$a \in K$, $P(x) \in V_3' \Rightarrow a P(x) \in V_3'$

d) V_4 este sp vectorial (grad $(P(x) + Q(x)) \leq n \rightarrow (V_4, +)$ parte stabilă).

$\forall a \neq 0$ grad $(a P(x)) \leq n$

$V_4 = K_4[x]. \simeq \mathbb{R}^{n+1}$

2 $V = K[x]$.

$V_1 = \{P(x) \in K[x] \mid P(-x) = P(x)\}$

$V_2 = \{P(x) \in K[x] \mid P(-x) = -P(x)\}$

a) Arătați că V_1, V_2 subsp. vectoriale în V

b) $V = V_1 \oplus V_2$

a). • $a, b \in K$, $P(x), Q(x) \in V_1$ pt V_1

$R(x) = a P(x) + b Q(x)$

$R(-x) = a P(-x) + b Q(-x) = a P(x) + b Q(x) = R(x)$

$\Rightarrow R(x) \in V_1$

• pt V_2

$a, b \in K$, $P(x), Q(x) \in V_2$

$R(-x) = a P(-x) + b Q(-x)$

$-R(x) = -a P(x) - b Q(x) = a P(-x) + b Q(-x) = R(-x)$

$\Rightarrow R(-x) \in V_2$

b). $\forall P(x) \in V$

$\exists!\ P_1(x) \in V_1,\ P_2(x) \in V_2$ a.î.

$P(x) = P_1(x) + P_2(x)$

· $\exists! \Leftrightarrow V_1 \cap V_2 = \{0\}$

$P(x) \in V_1 \cap V_2$

$P(x) = P(-x) = -P(x) \Rightarrow P(x) = 0.$

$P_1'(x) = P(x) + P(-x)$

$P_1'(-x) = P(-x) + P(x) = P_1'(x)$

$P_2'(x) = P(x) - P(-x)$

$P_1'(x) + P_2'(x) = 2P(x)$

$P_1(x) = \frac{1}{2}(P(x) + P(-x)) \in V_1$

$P_2(x) = \frac{1}{2}(P(x) - P(-x)) \in V_2$

$P(x) = P_1(x) + P_2(x).$

Pp că $\exists Q_1(x) \in V_1,\ Q_2(x) \in V_2$ a.î. $P(x) = Q_1(x) + Q_2(x)$

$P_1(x) + P_2(x) = Q_1(x) + Q_2(x)$

$P_1(x) - Q_1(x) = Q_2(x) - P_2(x) \in V_1 \cap V_2 = \{0\} \Rightarrow Q_1(x) = P_1(x)$

$Q_2(x) = P_2(x).$

3. $V = M_n(K)$

$V_1 = \{A \in V \mid A = A^t\}.$

$V_2 = \{A \in V \mid A = -A^t\}.$

$A \in V \quad A_1 = \frac{1}{2}(A + A^t) \quad A_1 \in V_1$

$A_2 = \frac{1}{2}(A - A^t)$

$A_2 = A - A_1 = A - \frac{1}{2}(A + A^t) = \frac{1}{2}(A - A^t) \in V_2$

$A = A_1 + A_2$

$V_1 \cap V_2 = \{0\}?$

$A \in V_1 \cap V_2 \Rightarrow A = A^t = -A \Rightarrow A = -A \Rightarrow A = 0$

$\Rightarrow V = V_1 \oplus V_2$

4. Fie $v_1 = (1,0,1,0)$

$v_2 = (1,1,0,2)$

$V = \langle v_1, v_2 \rangle$

Verificați dacă $(1,0,0,0)$, $(1,1,0,0)$, $(1,-1,2,2) \in V$

$$V = \{ a_1 v_1 + a_2 v_2 \mid a_1 a_2 \in \mathbb{R} \}$$

$v = (1,0,0,0)$

$x_1 v_1 + x_2 V_2 = v$

$x_1 (1,0,1,0) + x_2 (1,1,0,2) = (1,0,0,0)$

$$\begin{cases} x_1 + x_2 = 1 \\ x_2 = 0 \\ x_1 = 0 \\ 2x_2 = 0 \end{cases} \iff \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{array}\right)$$ sistem incompatibil

$\Rightarrow v \notin V$

$u = (1,1,0,0)$

$x_1 u_1 + x_2 u_2 = u$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{array}\right)$$

$\Rightarrow u \notin V$

$w = (1,-1,2,-2)$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \\ 0 & 2 & -2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right)$$

$x_1 = 2$
$x_2 = -1$

$\iff (1,-1,2,-2)$

$= 2(1,0,1,0) - (1,1,0,2).$

$w \in V$

b) $V_1 = \langle v_1 \rangle$, $V_2 = \langle v_2 \rangle$

Atunci $V = V_1 \oplus V_2$

Știm $V = V_1 + V_2$

$V_1 \cap V_2 = ?$

$v \in V_1 \cap V_2 \Leftrightarrow v = a v_1 = b v_2$

$a v_1 - b v_2 = 0$

$V_1 \cap V_2 = \{0\} \Leftrightarrow a = b = 0 \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ sol. unică are

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\left.\begin{array}{l} \text{rang } A = 2 \\ n = 2 \end{array}\right| \Rightarrow x_1 = x_2 = 0$ sol unică

$\Rightarrow V_1 \cap V_2 = \{0\}$.

5) $V_1 = \langle (1,2,1,0), (0,1,1,2) \rangle$

$V_2 = \langle (1,3,2,2), (0,1,1,0) \rangle$.

$V = V_1 + V_2$ Este $V = V_1 \oplus V_2$?

$\Leftrightarrow V_1 \cap V_2 = \{0\}$

$v \in V_1 \cap V_2 \Leftrightarrow v = x_1 (1,2,1,0) + x_2 (0,1,1,2)$
$= y_1 (1,3,2,2) + y_2 (0,1,1,0)$

$\Leftrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & -3 & -1 \\ 1 & 1 & -2 & -1 \\ 0 & 2 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Leftrightarrow \begin{cases} x_1 - y_1 = 0 \\ x_2 - y_1 = 0 \\ y_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = y_1 \\ x_2 = y_1 \\ y_2 = 0 \end{cases}$

$v = \alpha (1,2,1,0) + \alpha (0,1,1,2) = \alpha (1,3,2,2)$

$V_1 \cap V_2 = \langle (1,3,2,2) \rangle \Rightarrow V = V_1 \oplus V_2$ (Fals)