Quantum State Discrimination

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Abstract

There are fundamental limits on how accurately one can determine the state of a quantum system due to the existence of non-orthogonal quantum states. The indistinguishability of quantum states poses a series of challenges in quantum communication and quantum information processing. In this report, we give an overview of various strategies for quantum state discrimination and discuss their connections.

1. Introduction

The state of a quantum system is appealing but mysterious. It contains all the information about a quantum system and provides a way to calculate the statistical properties of a desired observable. However, the state itself is not observable, which means that we might not determine the state by observation. Despite the difficulty, we can understand the state to some extent if some prior information is given. Consider, for example, a game involving two parties, typically called Alice and Bob. Alice first prepares a quantum state among a set of possible states $\{\rho_i\}_{i=1}^N$, each having been prepared with respect to a distribution $\{p_i\}_{i=1}^N$, and sends it to Bob. Suppose that Bob knows the set of possible states $\{\rho_i\}_{i=1}^N$ and the distribution $\{p_i\}_{i=1}^N$. When the elements in $\{\rho_i\}_{i=1}^N$ are orthogonal, Bob can construct a perfect measurement to determine which state has been sent by Alice with certainty. However, if the elements in $\{\rho_i\}_{i=1}^N$ are not orthogonal, it is impossible for Bob to distinguish each state from the others with certainty. The fact that non-orthogonal quantum states cannot be discriminated perfectly imposes fundamental limits to quantum state discrimination. Based on different figures of merit used to describe the performance of a measurement, we can systematically explore those limits and the best strategies.

2. Minimum Error Discrimination

Many figures of merit can be used to describe the performance of a measurement. Probably the simplest one is the success probability, which corresponds to minimum error discrimination. In minimum error discrimination, the criteria is to minimize the average error or equivalently to maximize the success probability of identifying a state. We seek for a measurement $\{M_i\}_{i=1}^N$ such that the detection event on each M_i leads to the state ρ_i with the minimum average error or equivalently with the optimal success probability. We can formulate minimum error discrimination as follows:

$$\begin{aligned} &\max \quad \sum_{i=1}^{N} p_i \operatorname{Tr}[M_i \rho_i] \\ &\text{subject to } \sum_{i=1}^{N} M_i = I \text{ and } M_i \geq 0 \quad \forall i = 1, \ldots, N. \end{aligned} \tag{1}$$

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2.1. Two-state Discrimination

When $N = 2$, the problem of minimum error discrimination can be simplified as

$$\begin{array}{rl} \max & p_2 + \mathrm{Tr}\,[M_1(p_1\rho_1 - p_2\rho_2)] \\ \text{subject to} & 0 \leq M_1 \leq I. \end{array} \tag{2}$$

According to Holevo-Helström theorem, the optimal success probability is given by

$$P_s^\star = \frac{1}{2} + \frac{1}{2}\|p_1\rho_1 - p_2\rho_2\|_1, \tag{3}$$

which can be achieved by

$$M_1 = \{p_1\rho_1 - p_2\rho_2 > 0\} \text{ and } M_2 = \{p_1\rho_1 - p_2\rho_2 \leq 0\}. \tag{4}$$

2.2. Minimum Error Conditions

Although minimum error discrimination for identifying two states has been fully understood, the optimal strategies for most of the other cases have remained unknown. We need to characterize minimum error discrimination more detailedly. The minimum error conditions can shed some light on the general problem.

Theorem 1. *If the POVM elements $\{M_i\}_{i=1}^N$ satisfy*

$$\sum_{i=1}^N p_i\rho_i M_i - p_j\rho_j \geq 0 \quad \forall j = 1, \ldots, N, \tag{5}$$

$$M_j\left(p_j\rho_j - p_i\rho_i\right)M_i = 0 \quad \forall i, j = 1, \ldots, N, \tag{6}$$

then they achieve the minimum error in distinguishing between the states $\{\rho_i\}_{i=1}^N$, occurring with probabilities $\{p_i\}_{i=1}^N$.

Remark 1. Note that (5) and (6) are not independent. (6) can be derived from (5).

Proof. Let P_s denote the success probability of the measurement satisfying the above conditions and $\tilde{P}_s$ denote the success probability of any other measurement $\{\tilde{M}_j\}_{j=1}^N$.

$$\begin{aligned} P_s - \tilde{P}_s &= \mathrm{Tr}\left[\sum_{i=1}^N p_i\rho_i M_i\right] - \mathrm{Tr}\left[\sum_{j=1}^N p_j\rho_j\tilde{M}_j\right] \\ &= \mathrm{Tr}\left[\sum_{i=1}^N p_i\rho_i M_i(\sum_{j=1}^N \tilde{M}_j)\right] - \mathrm{Tr}\left[\sum_{j=1}^N p_j\rho_j\tilde{M}_j\right] \\ &= \sum_{j=1}^N \mathrm{Tr}\left[(\sum_{i=1}^N p_i\rho_i M_i - p_j\rho_j)\tilde{M}_j\right] \geq 0 \end{aligned} \tag{7}$$

Hence, P_s is the optimal success probability. In other words, the POVM elements $\{M_i\}_{i=1}^N$ achieve the minimum error. Next, we prove that (5) implies (6). From (5), we see that $\sum_{i=1}^N p_i\rho_i M_i$ is positive and self-adjoint. Thus,

$$\sum_{i=1}^N p_i\rho_i M_i = \sum_{j=1}^N p_j M_j\rho_j, \tag{8}$$

$$\sum_{i=1}^N(\sum_{j=1}^N p_j M_j\rho_j - p_i\rho_i)M_i = \sum_{j=1}^N M_j(\sum_{i=1}^N p_i\rho_i M_i - p_j\rho_j) = 0. \tag{9}$$

Since both $\sum_{j=1}^{N} p_j M_j \rho_j - p_i\rho_i$ and M_i are positive,

$$(\sum_{j=1}^{N} p_j M_j \rho_j - p_i\rho_i) M_i = 0 \quad \forall i = 1, \ldots, N. \tag{10}$$

Similarly,

$$M_j(\sum_{i=1}^{N} p_i\rho_i M_i - p_j\rho_j) = 0 \quad \forall j = 1, \ldots, N, \tag{11}$$

$$M_j(\sum_{j=1}^{N} p_j M_j \rho_j - p_i\rho_i) M_i = M_j(\sum_{i=1}^{N} p_i\rho_i M_i - p_j\rho_j) M_i = 0 \quad \forall i, j = 1, \ldots N. \tag{12}$$

Thus,

$$M_j(p_j\rho_j - p_i\rho_i) M_i = 0 \quad \forall i, j = 1, \ldots N. \tag{13}$$

□

The above theorem indicates the sufficient conditions for a measurement to achieve the minimum error. To our surprise, those conditions are also necessary for an optimal measurement in minimum error discrimination. To see why those conditions hold for an optimal measurement, we use the following two lemmas concerning the properties of an optimal measurement to enhance our understanding of an optimal measurement.

Lemma 1. If the POVM elements $\{M_i\}_{i=1}^{N}$ achieve the minimum error in distinguishing between the states $\{\rho_i\}_{i=1}^{N}$ that occur with probabilities $\{p_i\}_{i=1}^{N}$, then

$$G_j = \frac{1}{2} \sum_{i=1}^{N} p_i(\rho_i M_i + M_i\rho_i) - p_j\rho_j \geq 0 \quad \forall j = 1, \ldots, N. \tag{14}$$

Proof. Suppose to the contrary that some G_j is not positive. We might assume G_1 is not positive. Then there exists a unit eigenvector $|\psi\rangle$ such that $G_1 |\psi\rangle = -\lambda |\psi\rangle$ for some positive λ. Consider a measurement $\{\tilde{M}_i\}_{i=1}^{N}$ defined by

$$\tilde{M}_1 = (I - \epsilon |\psi\rangle \langle\psi|) M_1 (I - \epsilon |\psi\rangle \langle\psi|) + \epsilon(2 - \epsilon) |\psi\rangle \langle\psi|, \tag{15}$$

$$\tilde{M}_i = (I - \epsilon |\psi\rangle \langle\psi|) M_i (I - \epsilon |\psi\rangle \langle\psi|) \quad \forall i = 2, \ldots, N. \tag{16}$$

Then

$$\begin{aligned} \tilde{P}_s &= \sum_{i=1}^{N} p_i \operatorname{Tr}\left[\rho_i \tilde{M}_i\right] \\ &= \sum_{i=1}^{N} p_i \operatorname{Tr}\left[\rho_i M_i\right] - \epsilon \sum_{i=1}^{N} \langle\psi| p_i(\rho_i M_i + M_i\rho_i) |\psi\rangle + 2\epsilon p_1 \langle\psi| \rho_1 |\psi\rangle + O(\epsilon^2) \\ &= P_s^\star - 2\epsilon |\psi\rangle G_1 \langle\psi| + O(\epsilon^2) \\ &= P_s^\star + 2\epsilon\lambda + O(\epsilon^2) \end{aligned} \tag{17}$$

Thus,

$$\lim_{\epsilon \to 0^+} \frac{\tilde{P}_s - P_s^\star}{\epsilon} = 2\lambda > 0, \tag{18}$$

which contradicts that $P_s^\star$ is the optimal success probability. Hence, G_j is positive. □

Lemma 2. If the POVM elements $\{M_i\}_{i=1}^{N}$ achieve the minimum error in distinguishing between the states $\{\rho_i\}_{i=1}^{N}$ that occur with probabilities $\{p_i\}_{i=1}^{N}$, then $\sum_{i=1}^{N} p_i\rho_i M_i$ is Hermitian.

Proof. Note that

$$\sum_{j=1}^{N} \mathrm{Tr}[G_j M_j] = \frac{1}{2}\,\mathrm{Tr}\left[\sum_{i=1}^{N} p_i(\rho_i M_i + M_i\rho_i)\right] - \mathrm{Tr}\left[\sum_{j=1}^{N} p_j\rho_j M_j\right] = 0. \tag{19}$$

Since G_j and M_j are positive, $G_j M_j = 0 \quad \forall j = 1, \ldots, N$.

$$\sum_{j=1}^{N} G_j M_j = \frac{1}{2}\sum_{i=1}^{n} p_i(M_i\rho_i - \rho_i M_i) = 0 \implies \sum_{i=1}^{n} p_i M_i \rho_i = \sum_{i=1}^{n} p_i\rho_i M_i \tag{20}$$

□

Accordingly, we can derive the necessary conditions for an optimal measurement from the two lemmas.

Theorem 2. *If the POVM elements $\{M_i\}_{i=1}^N$ achieve the minimum error in distinguishing between the states $\{\rho_i\}_{i=1}^N$ that occur with probabilities $\{p_i\}_{i=1}^N$, then they must satisfy*

$$\sum_{i=1}^{N} p_i\rho_i M_i - p_j\rho_j \geq 0 \quad \forall j = 1, \ldots, N, \tag{21}$$

$$M_j\left(p_j\rho_j - p_i\rho_i\right)M_i = 0 \quad \forall i, j = 1, \ldots, N. \tag{22}$$

Proof. By *Lemma* 2, we have

$$\frac{1}{2}\sum_{i=1}^{N} p_i(\rho_i M_i + M_i\rho_i) = \sum_{i=1}^{N} p_i\rho_i M_i. \tag{23}$$

Thus, *Lemma* 1 leads to

$$G_j = \sum_{i=1}^{N} p_i\rho_i M_i - p_j\rho_j \geq 0 \quad \forall j = 1, \ldots, N. \tag{24}$$

Clearly, (22) follows from (21) as mentioned in *Remark* 1. □

2.3. *Square-Root Measurement*

For a given set of states $\{\rho_i\}_{i=1}^N$ that occur with probabilities $\{p_i\}_{i=1}^N$, we can construct its square-root measurement as follows:

$$M_i = p_i\rho^{-1/2}\rho_i\rho^{-1/2}, \text{ where } \rho = \sum_{i=1}^{N} p_i\rho_i. \tag{25}$$

For many of the cases in which the optimal minimum error measurement is known, it happens to be the square-root measurement. Here, we give an example and apply **Theorem 1** to verify it.

Consider N symmetric pure states with equal probabilities $p_i = \frac{1}{N}$, given by

$$|\phi_i\rangle = U^{i-1}|\phi_1\rangle \quad \forall i = 1, \ldots, N, \tag{26}$$

where U is a unitary operator satisfying $U^N = I$.
For this set of states,

$$\rho = \frac{1}{N}\sum_{i=1}^{N} U^{i-1}|\phi_1\rangle\langle\phi_1| U^{\dagger i-1}, \tag{27}$$

$$U\rho U^\dagger = \frac{1}{N}\sum_{i=1}^{N} U^i|\phi_1\rangle\langle\phi_1| U^{\dagger i} = \rho, \tag{28}$$

$$M_i = \frac{1}{N}\rho^{-1/2}|\phi_i\rangle\langle\phi_i|\rho^{-1/2}. \tag{29}$$

From (28), we know U and ρ commute. We next check that the associated square-root measurement satisfies the sufficient conditions for minimum error discrimination.

$$\begin{aligned}
\sum_{i=1}^{N} p_i \rho_i M_i &= \frac{1}{N} \sum_{i=1}^{N} |\phi_i\rangle \langle\phi_i| \frac{1}{N}\rho^{-1/2} |\phi_i\rangle \langle\phi_i| \rho^{-1/2} \\
&= \frac{1}{N} \sum_{i=1}^{N} |\phi_i\rangle \langle\phi_1| U^{\dagger i-1} \frac{1}{N}\rho^{-1/2} U^{i-1} |\phi_1\rangle \langle\phi_i| \rho^{-1/2} \\
&= \frac{1}{N} \langle\phi_1| \rho^{-1/2} |\phi_1\rangle \sum_{i=1}^{N} \frac{1}{N} |\phi_i\rangle \langle\phi_i| \rho^{-1/2} \\
&= \frac{1}{N} \langle\phi_1| \rho^{-1/2} |\phi_1\rangle \rho^{1/2}
\end{aligned} \tag{30}$$

The third equality holds since U and ρ commute. It remains to show that

$$\frac{1}{N} \langle\phi_1| \rho^{-1/2} |\phi_1\rangle \rho^{1/2} - \frac{1}{N} |\phi_j\rangle \langle\phi_j| \geq 0 \quad \forall j = 1, \ldots, N. \tag{31}$$

For any j and $|\psi\rangle$,

$$\begin{aligned}
\langle\psi| \frac{1}{N} \langle\phi_1| \rho^{-1/2} |\phi_1\rangle \rho^{1/2} |\psi\rangle &= \frac{1}{N} \langle\phi_1| \rho^{-1/2} |\phi_1\rangle \langle\psi| \rho^{1/2} |\psi\rangle \\
&= \frac{1}{N} \langle\phi_j| \rho^{-1/2} |\phi_j\rangle \langle\psi| \rho^{1/2} |\psi\rangle \\
&= \frac{1}{N} ||\rho^{-1/4} |\phi_j\rangle||^2 ||\rho^{1/4} |\psi\rangle||^2 \\
&\geq \frac{1}{N} ||\langle\phi_j| \rho^{-1/4} \rho^{1/4} |\psi\rangle||^2 \\
&= \langle\psi| \frac{1}{N} |\phi_j\rangle \langle\phi_j| |\psi\rangle
\end{aligned} \tag{32}$$

The second equality holds since U and ρ commute. Hence,

$$\frac{1}{N} \langle\phi_1| \rho^{-1/2} |\phi_1\rangle \rho^{1/2} - \frac{1}{N} |\phi_j\rangle \langle\phi_j| \geq 0 \quad \forall j = 1, \ldots, N. \tag{33}$$

By **Theorem 1**, the square-root measurement is optimal for minimum error discrimination.

3. Unambiguous Discrimination

In minimum error discrimination, the observed outcome does not necessarily correspond to the prepared state. However, in some situation, the observer is not allowed to make an error of identifying the state. To avoid making any error, we further require that $\mathrm{Tr}[M_i \rho_j] = 0$ for $i \neq j = 1, \ldots, N$ so that the detection event on M_k corresponds only to the state ρ_k. In this case, we might need to incorporate an additional outcome which doesn't lead us to identify any state. The criteria that we cannot mistake the prepared state for the other underlies unambiguous discrimination. In addition, we usually impose a further requirement to minimize the probability of obtaining an inconclusive result in unambiguous discrimination. We can formulate unambiguous discrimination as follows:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{N} p_i \,\mathrm{Tr}\,[M_{N+1} \rho_i] \\
\text{subject to} \quad & \mathrm{Tr}[M_i \rho_j] = 0 \text{ for } i \neq j = 1, \ldots, N \\
& \sum_{i=1}^{N+1} M_i = I, \text{ and } M_i \geq 0 \quad \forall i = 1, \ldots, N+1.
\end{aligned} \tag{34}$$

3.1. Pure States

To unambiguously discriminate between N pure states $\{|\phi_i\rangle\}_{i=1}^N$, we need to construct a measurement $\{M_i\}_{i=1}^{N+1}$ satisfying

$$\langle\phi_j| M_i |\phi_j\rangle = P_i\delta_{ij}, \text{ where } 0 \le P_i \le 1. \tag{35}$$

The unambiguous measurement is feasible if and only if $\{|\phi_i\rangle\}_{i=1}^N$ are linearly independent. Moreover, we also want to minimize the probability of obtaining an inconclusive result. However, analytical solutions to the minimum achievable probability of the inconclusive result are not known in the general case. We present some special case in which the minimum probability of the inconclusive result is known. Consider a set of two pure states $|\phi_1\rangle$ and $|\phi_2\rangle$ occurring with prior probabilities p_1 and p_2. Suppose that $\langle\phi_1, \phi_2\rangle$ is real. Since the two pure states define a 2-dimensional space, we can choose an orthogonal basis $\{|0\rangle, |1\rangle\}$ such that

$$\begin{aligned} |\phi_1\rangle &= \cos\theta|0\rangle + \sin\theta|1\rangle, \\ |\phi_2\rangle &= \cos\theta|0\rangle - \sin\theta|1\rangle. \end{aligned} \tag{36}$$

To satisfy $M_1|\phi_2\rangle\langle\phi_2| = M_2|\phi_1\rangle\langle\phi_1| = 0$, the measurement should be taken as

$$\begin{aligned} M_1 &= a_1(\sin\theta|0\rangle + \cos\theta|1\rangle)(\sin\theta\langle 0| + \cos\theta\langle 1|), \\ M_2 &= a_2(\sin\theta|0\rangle - \cos\theta|1\rangle)(\sin\theta\langle 0| - \cos\theta\langle 1|), \end{aligned} \tag{37}$$

where $0 \le a_1, a_2$. Without loss of generality, we might assume $p_1 \ge p_2$ and $0 < \theta < \pi/4$. The probability of the inconclusive result is given by

$$p_1\langle\phi_1|(I - M_1 - M_2)|\phi_1\rangle + p_2\langle\phi_2|(I - M_1 - M_2)|\phi_2\rangle = 1 - (p_1a_1 + p_2a_2)\sin^2 2\theta. \tag{38}$$

Write down the matrix form of $I - M_1 - M_2$. From its determinant and trace, we can know that $I - M_1 - M_2 \ge 0$ if and only if $a_1 + a_2 \le 2$ and $1 - (a_1 + a_2) + a_1a_2\sin^2 2\theta \ge 0$. The problem can be simplified as

$$\begin{aligned} \min \quad & 1 - (p_1a_1 + p_2a_2)\sin^2 2\theta \\ \text{subject to} \quad & a_1 \ge 0 \\ & a_2 \ge 0 \\ & a_1 + a_2 \le 2 \\ & 1 - (a_1 + a_2) + a_1a_2\sin^2 2\theta \ge 0 \end{aligned} \tag{39}$$

Apply the KKT conditions to solve the optimization problem. Then we have [1]

$$\begin{aligned} a_1^\star &= \frac{1 - \sqrt{p_2/p_1}\cos 2\theta}{\sin^2 2\theta}, \\ a_2^\star &= \frac{1 - \sqrt{p_1/p_2}\cos 2\theta}{\sin^2 2\theta}. \end{aligned} \tag{40}$$

The minimum probability of the inconclusive result is given by $P_{\min} = 2\sqrt{p_1p_2}\cos 2\theta$ and the constraint $1 - (a_1 + a_2) + a_1a_2\sin^2 2\theta \ge 0$ is active.

Remark 2. (40) is valid for $\sqrt{p_1/p_2}\cos 2\theta \le 1$. If $\sqrt{p_1/p_2}\cos 2\theta$ exceeds 1, $a_1^\star$ and $a_2^\star$ will become 1 an 0 respectively. Hence, if p_1 is much larger than p_2, the optimal strategy in unambiguous discrimination is to rule out the less probable state ρ_2, which is much different from the optimal strategy in minimum error discrimination. Moreover, we can see that for $p_1 = p_2 = 1/2$, the minimum probability of the inconclusive result is $\cos 2\theta = |\langle\phi_1, \phi_2\rangle|$ and the optimal success probability for unambiguous discrimination is $1 - |\langle\phi_1, \phi_2\rangle|$, which is the Ivanovic-Dieks-Pere (IDP) limit.

[1] It is one of the solutions to the system: $p_1(1 - a_1\sin^2 2\theta) = p_2(1 - a_2\sin^2 2\theta)$ and $1 - (a_1 + a_2) + a_1a_2\sin^2\theta = 0$.

3.2. Mixed State

To unambiguously discriminate between two mixed states ρ_1 and ρ_2, a necessary and sufficient condition is that they have different kernels. If ρ_1 and ρ_2 have the same kernel, $M_1\rho_2 = M_2\rho_1 = 0$ will imply $M_1\rho_1 = M_2\rho_2 = 0$. We cannot distinguish between ρ_1 and ρ_2. If ρ_1 and ρ_2 have different kernels, we can define M_2 (or M_1) to lie in the kernel of ρ_1 (or ρ_2) such that $M_2\rho_1 = 0$ (or $M_1\rho_2 = 0$) and $\mathrm{Tr}[M_2\rho_2] \neq 0$ (or $\mathrm{Tr}[M_1\rho_1] \neq 0$). Thus, we can unambiguously discriminate between ρ_1 and ρ_2. The problem of finding the strategy that minimizes the probability of the inconclusive result is again difficult. The solutions are known only in some special cases. In general, semi-definite programming is applied to numerically solve the problem.

4. Maximum Confidence Measurement

Although in unambiguous discrimination, the guessed state always corresponds to the prepared state, it is feasible only when the states are linearly independent. In a more general case, we can take the confidence in a detection event as the figure of merit to describe the performance of a measurement. The criteria here is to construct a measurement that allows us to be as confident as possible to claim the prepared state is ρ_i when the detection event M_i happens. It is concerned with optimizing the posterior probability of a state given a particular measurement outcome. We can formulate the maximum confidence measurement as follows:

$$\begin{aligned} &\max \quad P(\rho_i \mid M_i) \quad \forall i = 1, \ldots, N \\ &\text{subject to } \sum_{i=1}^{N+1} M_i = I, \text{ and } M_i \geq 0 \quad \forall i = 1, \ldots, N+1. \end{aligned} \tag{41}$$

The next theorem tells us the maximum confidence in each detection event M_i and how we can construct the corresponding measurement.

Theorem 3. *For a given set of states $\{\rho_i\}_{i=1}^N$ that occur with probabilities $\{p_i\}_{i=1}^N$, the maximum confidence in the detection event M_i is given by*

$$P_{\max}(\rho_i \mid M_i) = \gamma_{\max}(\rho^{-1/2} p_i \rho_i \rho^{-1/2}), \tag{42}$$

where $\rho = \sum_{i=1}^N p_i\rho_i$ and $\gamma_{\max}(A)$ denote the largest eigenvalue of A.

Proof.

$$P(\rho_i \mid M_i) = \frac{p_i \,\mathrm{Tr}[\rho_i M_i]}{\mathrm{Tr}[\rho M_i]}, \text{ where } \rho = \sum_{i=1}^N p_i\rho_i. \tag{43}$$

We note that M_i appears in both the numerator and the denominator of this expression and thus can be determined only up to a multiplicative constant. Thus, it is always possible to choose M_i such that $\sum_{i=1}^N M_i \leq I$. An inconclusive result might be added to form a complete measurement. Write M_i as $M_i = \rho^{-1/2} Q_i \rho^{-1/2}$, where $Q_i \geq 0$. Then

$$P(\rho_i \mid M_i) = \mathrm{Tr}\left[\rho^{-1/2} p_i\rho_i\rho^{-1/2} \frac{Q_i}{\mathrm{Tr}\,[Q_i]}\right] \leq \gamma_{\max}(\rho^{-1/2} p_i\rho_i\rho^{-1/2}). \tag{44}$$

The upper bound can be achieved by choosing Q_i such that $\frac{Q_i}{\mathrm{Tr}[Q_i]}$ is a 1-dimensional projection in the eigenspace of $\rho^{-1/2} p_i\rho_i\rho^{-1/2}$ corresponding to the largest eigenvalue. □

Remark 3. When $\rho_i = |\phi_i\rangle\langle\phi_i|$, we should choose Q_i such that

$$\frac{Q_i}{\mathrm{Tr}\,[Q_i]} = \frac{\rho^{-1/2}\,|\phi_i\rangle\langle\phi_i|\,\rho^{-1/2}}{\|\rho^{-1/2}\,|\phi_i\rangle\|^2}. \tag{45}$$

Hence,

$$P_{\max}(\rho_i \mid M_i) = p_i \langle\phi_i|\,\rho^{-1}\,|\phi_i\rangle\,, \tag{46}$$

$$M_i \propto \rho^{-1}\,|\phi_i\rangle\langle\phi_i|\,\rho^{-1}. \tag{47}$$

To illustrate the strategy of the maximum confidence measurement, we consider the problem of discriminating between three pure states in a 2-dimensional space as follows:

$$
\begin{aligned}
|\phi_1\rangle &= \cos\theta\,|0\rangle + \sin\theta\,|1\rangle\,,\\
|\phi_2\rangle &= \cos\theta\,|0\rangle + e^{2\pi i/3}\sin\theta\,|1\rangle\,,\\
|\phi_3\rangle &= \cos\theta\,|0\rangle + e^{4\pi i/3}\sin\theta\,|1\rangle\,,
\end{aligned}
\tag{48}
$$

where $0 \le \theta \le \pi/4$ and $p_i = 1/3$ for $i = 1, 2, 3$.

$$
\rho = \cos^2\theta\,|0\rangle\langle 0| + \sin^2\theta\,|1\rangle\langle 1|
\tag{49}
$$

Calculating by using (46), we have

$$
P_{\max}(\rho_i \mid M_i) = \frac{2}{3} \text{ for } i = 1, 2, 3.
\tag{50}
$$

Calculating by using (47), we have the form $M_i = a_i\,|\psi_i\rangle\langle\psi_i|$ for $i = 1, 2, 3$, where $0 \le a_1, a_2, a_3$ and

$$
\begin{aligned}
|\psi_1\rangle &= \sin\theta\,|0\rangle + \cos\theta\,|1\rangle\,,\\
|\psi_2\rangle &= \sin\theta\,|0\rangle + e^{2\pi i/3}\cos\theta\,|1\rangle\,,\\
|\psi_3\rangle &= \sin\theta\,|0\rangle + e^{4\pi i/3}\cos\theta\,|1\rangle\,.
\end{aligned}
\tag{51}
$$

Generally speaking, it is impossible to choose a_1, a_2 and a_3 such that M_1, M_2, and M_3 form a complete measurement. We introduce an additional operator M_4, which is associated with the inconclusive result. Then we can complete the measurement by minimizing the probability of the inconclusive result. The probability of the inconclusive result is given by

$$
\mathrm{Tr}\,[\rho M_4] = \mathrm{Tr}\,[\rho(I - M_1 - M_2 - M_3)] = 1 - 2(a_1 + a_2 + a_3)\cos^2\theta\sin^2\theta.
\tag{52}
$$

Write down the matrix form of $I - M_1 - M_2 - M_3$. From its determinant and trace, we can know that $I - M_1 - M_2 - M_3 \ge 0$ if and only if $a_1 + a_2 + a_3 \le 2$ and $1 - (a_1 + a_2 + a_3) + \frac{3}{4}(a_1a_2 + a_1a_3 + a_2a_3)\sin^2 2\theta \ge 0$. The problem can be simplified as

$$
\begin{aligned}
\min \quad & 1 - 2(a_1 + a_2 + a_3)\cos^2\theta\sin^2\theta\\
\text{subject to} \quad & a_1 \ge 0\\
& a_2 \ge 0\\
& a_3 \ge 0\\
& a_1 + a_2 + a_3 \le 2\\
& 1 - (a_1 + a_2 + a_3) + \frac{3}{4}(a_1a_2 + a_1a_3 + a_2a_3)\sin^2 2\theta \ge 0
\end{aligned}
\tag{53}
$$

We can apply the KKT conditions to solve the optimization problem. Notice that when $0 \le \theta < \pi/4$, $a_1 + a_2 + a_3$ cannot equal to 2; otherwise,

$$
1 - (a_1 + a_2 + a_3) + \frac{3}{4}(a_1a_2 + a_1a_3 + a_2a_3)\sin^2 2\theta < 1 - 2 + \frac{3}{4}\left(\frac{4}{9} + \frac{4}{9} + \frac{4}{9}\right) = 0.
\tag{54}
$$

Hence, only the constraint $1 - (a_1 + a_2 + a_3) + \frac{3}{4}(a_1a_2 + a_1a_3 + a_2a_3)\sin^2 2\theta \ge 0$ is active for $0 \le \theta < \pi/4$. When $\theta = \pi/4$, the constraint $a_1 + a_2 + a_3 \le 2$ and the constraint $1 - (a_1 + a_2 + a_3) + \frac{3}{4}(a_1a_2 + a_1a_3 + a_2a_3)\sin^2 2\theta \ge 0$ are both active. In either case, we have[2]

$$
a_1^\star = a_2^\star = a_3^\star = \frac{1}{3\cos^2\theta}.
\tag{55}
$$

The minimum probability of the inconclusive result is given by $P_{\min} = 1 - 2\sin^2\theta$ and $M_4 = (1 - \tan^2\theta)\,|0\rangle\langle 0|$.

[2]It is one of the solutions to $1 - 3a + \frac{9}{4}a^2\sin^2 2\theta = 0$.

Remark 4. In this example, the pure states also satisfy (26). Thus, its square-root measurement is the minimum error measurement. The minimum error measurement is given by

$$M_i^{ME} = \frac{1}{3}\rho^{-1/2} |\phi_i\rangle \langle\phi_i| \rho^{-1/2} = \frac{2}{3} |\psi_i^{ME}\rangle \langle\psi_i^{ME}| \quad \text{for } i = 1, 2, 3, \tag{56}$$

where

$$\begin{aligned} |\psi_1^{ME}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \\ |\psi_2^{ME}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i/3} |1\rangle), \\ |\psi_3^{ME}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + e^{4\pi i/3} |1\rangle). \end{aligned} \tag{57}$$

The optimal success probability is given by

$$P_s^{ME} = \frac{1}{3}(1 + \sin 2\theta) \leq \max_{i=1,2,3} P_{\max}(\rho_i \mid M_i) = \frac{2}{3} \text{ for } 0 \leq \theta \leq \pi/4. \tag{58}$$

In contrast, the success probability in the maximum confidence measurement is given by

$$P_s^{MC} = \frac{4}{3}\sin^2\theta \leq P_s^{ME} = \frac{1}{3}(1 + \sin 2\theta) \text{ for } 0 \leq \theta \leq \pi/4. \tag{59}$$

Notice that when $\theta = \pi/4$, the maximum confidence measurement is the same as the minimum error measurement.

5. Relation between Strategies

The maximum confidence measurement can be viewed as a generalization of unambiguous discrimination. In fact, unambiguous discrimination is a special case of the maximum confidence measurement with $P(\rho_i \mid M_i) = 1$ for all $i = 1, \ldots, N$ and we can deduce the feasible conditions for unambiguous discrimination from the maximum confidence measurement. In addition, the maximum confidence measurement is less restrictive than unambiguous discrimination. It allows us to explore the extent to which one can unambiguously distinguish one state from the others. That is, it is possible to unambiguously discriminate some but not all states in a set.

In the maximum confidence measurement, we aim to maximize the confidence in each detection event. If we instead try to maximize the average confidence, then the criteria is to maximize

$$P(\rho_i \mid M_i)_{avg} = \sum_{i=1}^{N} P(M_i)P(\rho_i \mid M_i) = \sum_{i=1}^{N} p_i \operatorname{Tr}[M_i\rho_i], \tag{60}$$

which is precisely the success probability. It is tantamount to minimum error discrimination. Furthermore, we can deduce from (59) an upper bound for the optimal success probability, which is given by

$$\max_{i=1,\ldots,N} P_{\max}(\rho_i \mid M_i). \tag{61}$$

Actually, the upper bound has been seen in (58).

6. Conclusion

We have discussed a variety of optimal measurements based on three different figures of merit for quantum state discrimination: the success probability of identifying a state, the unambiguity of the discrimination, and the confidence in each detection event.

For minimum error discrimination, we have pointed out the minimum error conditions useful for numerically solving the optimal measurement. For unambiguous discrimination, we have explained the feasible conditions of unambiguous discrimination and given an example in which the optimal measurement can be solved through the KKT conditions. For the maximum confidence measurement, we have figured out the maximum confidence in each detection event along with its corresponding measurement up to a multiplicative constant. We have also illustrated a simple case in which the confidence in each detection event is maximized while the probability of the inconclusive result is minimized through the KKT conditions.

Although different strategies, for the most part, correspond to distinct optimal measurements, one can be transformed into another by applying different optimality conditions to the same quantity such as the confidence in a detection event. Those connections not only increase our understanding of different strategies but also help us interpret them from different perspectives.

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