Problem Solving Homework (Week 6)

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JH Chapter 3

3.7.2.1

(i) minimize

$$\sum_{i=1}^{n} c_i x_i$$

under the constraints

$$\sum_{i=1}^{n} a_{ji} x_i = b_j \text{ for } j \in M$$
$$\sum_{i=1}^{n} a_{ri} x_i + s_r = b_r \text{ for } r \in \{1, \ldots, m\} - M$$
$$x_i \geq 0 \text{ for } i \in Q$$
$$x_i + q_i \geq 0 \text{ for } i \in \{1, \ldots, n\} - Q$$
$$s_i \geq 0 \; \forall s_i$$

(ii) minimize

$$\sum_{i=1}^{n} c_i x_i$$

under the constrains

$$\sum_{i=1}^{n} a_{ji} x_i \geq b_j + s_j \text{ for } j \in M$$
$$\sum_{i=1}^{n} a_{ri} x_i \geq b_r \text{ for } r \in \{1, \ldots, m\} - M$$
$$x_i \geq 0 \text{ for } i \in Q$$
$$x_i + q_i \geq 0 \text{ for } i \in \{1, \ldots, n\} - Q$$
$$s_i \geq 0 \; \forall s_i$$

(iii) minimize

$$c^T \cdots X$$

under the constraints

$$\sum_{i=1}^{n} a_{ji} x_i \geq b_j \text{ for } j \in M$$
$$\sum_{i=1}^{n} -a_{ji} x_i \geq -b_j \text{ for } j \in M$$
$$\sum_{i=1}^{n} a_{ri} x_i \geq b_r \text{ for } r \in \{1, \ldots, m\} - M$$
$$x_i \geq 0 \text{ for } i \in Q$$

3.7.2.4

maximize

$$\sum_{e \in E} x_e$$

under the $|V|$ constrains

$$\sum_{e \in E} x_e = 1 \text{for every } v \in V$$

and the following $|E|$ constrains

$$x_e \in \{0, 1\} \text{for every } e \in E$$

3.7.2.5

Let x_{ij} be 1 if edge ij is in the tree T. Then we know that:

- Exactly $n - 1$ edges are in T;
- There is no cycle in T.

Suppose the weight function is w, i.e., $w(i, j)$ denotes the weight of edge ij. The formulation is:
minimize

$$\sum_{ij \in T.E} w_{ij}$$

under the first constraint

$$\sum_{ij \in E} x_{ij} = n - 1$$

and the second constraint

$$\sum_{ij \in E; i \in S, j \in S} x_{ij} \leq |S| - 1, \; \forall S \subseteq V$$

3.7.4.4

Proof. For step 1-3 in Algorithm 3.7.4.2, we changed the objective function to weighted sum $\sum_{i=1}^{n} c_i \alpha_i$. The inequality $\sum_{h \in Index(a_j)} \alpha_h \geq 1$ still holds. Therefore, $\exists t \in Index(a_j)$, where $\alpha_r \geq 1/k$. The rounding is that $\beta_i = 1 \Leftrightarrow a_i \geq 1/k, i = 1, \ldots, m$. Consequently, $\beta_i \leq k\alpha_i$, and $\sum_{i=1}^{n} c_i \beta_i \leq k \sum_{i=1}^{n} c_i \alpha_i$.
Based on what is stated above, the assertion still holds. □

3.7.4.12

(i) It can be proved by the following procedure. First, all feasible solutions in LP consists of the convex hull. Second, the optimal solution must be on the vertexes of it, and the boolean solutions make up the vertexes. In the end, non-boolean solutions must be on the lines (except vertexes). Therefore, $Opt_{LP}(I(G)) = Opt_{MMP}(G)$.

(ii) I'm sorry that I can't solve this problem.

3.7.4.16

The linear relaxation for SCP(k) is:
minimize

$$\sum_{i=1}^{m} x_i$$

under the constraints

$$\sum_{h \in Index(a_j)} x_h \geq 1 \text{ for } j = 1, \ldots, n$$

$$x_i \geq 0 \text{ for } i = 1, \ldots, m$$

The dual problem is:
maximize

$$\sum_{i=1}^{n} y_i$$

under the constraints

$$\sum_{j \in S_i} y_j \leq 1 \text{ for } i = 1, \ldots, m$$

$$y_i \geq 0 \text{ for } i = 1, \ldots, n$$

With the primal-dual scheme, we can solve the above problem. Denote their solutions by $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$, respectively. Then we have $\beta_i = 1 \Leftrightarrow \sum_{j \in S_i} \alpha_j = 1$.