Partial Differential Equations - Class Notes

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Project Information

If you want to mess with an image, you have to think about how computers view images. The image is stored. When we see it, we see an image, but what does the computer see?

Computers stores images as a point of pixels akin to a grid. Pixels contain RGB value information and location. Pictures are generally large.

The first thing is: An image is a grid of numbers (Array, matrix). This is what the computer sees.

Color: Saturation, hue, not the focus.

Focus: Grayscale, same functions, can be extended into color. Add extra dimensions.

Idea: Have a grid of values with pixels and a value at each point (Tells us the gray scale, point of scale between black and white).

Image Processing

Let's say we have a black and white digital image.

Here, we have:

- $u(x, y)$: u (int) is the darkness (grayscale) of the image at point (x, y).
- $u = 0$ black, $u = 255$ = white

Steps to our process:

1. Remove Noise. In the case of our project, we will remove Salt and Pepper noise.
2. Remove Blur.

We know Heat, Laplace, Wave, and Transport.

We want to use Heat.

Here, we want to remove the 'spikes' in our image. The idea is if we apply heat to our point, the peak of the spike reduces in magnitude and spreads out.

When we smoothen our picture, we also blur our ideal picture.

1. First step: Take an image and blur it.

   The heat equation blurs things. This will cause the salt and pepper noise fade.

   Here, let us consider an initial image with noise. Our initial image has $t = 0$. Then, we apply

   $$u_t = u_{xx} + u_{yy} \tag{1}$$
   $$u(x, y, 0) = f(x, y) \tag{2}$$

   Here, $f(x, y)$ is our image, the initial condition.

   However, when we apply our heat function, we have a pro and a con:

   - Pro: Salt and pepper noise are pretty much gone.
   - Con: Whole image is blurred.

   If we just have one point of noise, Let's say we have an $m \times n$ white grid and black dot in the center, then let us consider the following:

   $$f(x, y) = \delta\left(x - \frac{L}{2}\right)\delta\left(y - \frac{M}{2}\right) \tag{3}$$

   Here, we have the following conditions:

   (a) $u(x, y, 0) = f(x, y)$

   and

   (a) $u(0, y, t) = f(0, y)$

   (b) $u(L, y, t) = f(L, y)$

   (c) $u(x, 0, t) = f(x, 0)$

   (d) $u(x, M, t) = f(x, M)$

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How do you get rid of Salt and Pepper noise without blurring the image at the same time?

The heat equation removes the noise, but blurs as well.

Selective blur: Blur in some direction. It all depends on the boundary.

If we are parallel to the boundary, we could care less.

We want to blur in the direction perpendicular to the gradient.

We are going to try to modify the heat equation so that it does not blur the edge.

Here, we define:

- η : Direction of gradient
- ξ : Direction normal to gradient

To blur only in the direction perpendicular to ∇u, use

$$u_t = u_{xx} + u_{yy} - u_{yy} \tag{4}$$
$$u_t = u_{xx} \tag{5}$$

Here, Δu is $u_{xx} + u_{yy}$ and the component in the direction of ∇u is u_{yy}.

The equation that blurs only in the direction normal to the gradient is

$$\begin{aligned} u_t &= \Delta u - u_{\eta\eta} \quad (6)\\ &= u_{\xi\xi} + u_{\eta\eta} - u_{\eta\eta} \quad (7)\\ &= u_{\xi\xi} \quad (8) \end{aligned}$$

<u>Note:</u> $\Delta u = u_{xx} + u_{yy} = u_{\xi\xi} + u_{\eta\eta}$ since $\xi \perp \eta$ and they are unit vectors.

How do we express $u_{\eta\eta}$ in terms of $u_x, u_y, u_{xx}, u_{yy}, u_{xy}$?

Let $\vec{n}$ be a unit vector.

$$u_n = \nabla u \cdot \vec{n} \tag{9}$$

This is called the directional derivative (Calc III)

$$\begin{aligned} u_{nn} &= (u_n)_n \quad (10)\\ &= \nabla u_n \cdot \vec{n} \quad (11)\\ &= \nabla(\nabla u \cdot \vec{n}) \cdot \vec{n} \quad (12)\\ &= \nabla\nabla u \vec{n} \cdot \vec{n} \quad (13) \end{aligned}$$

Here, $\nabla\nabla u$ is the tensor. The tensor is also called the Hessian Matrix.

$\vec{u} \cdot \vec{v} = \vec{v}^T \vec{u}$

$\vec{n} = n_1 \hat{i} + n_2 \hat{j}$

$$\begin{aligned} &= \begin{bmatrix} n_1 & n_2 \end{bmatrix} \begin{bmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \quad (14)\\ &= \begin{bmatrix} n_1 & n_2 \end{bmatrix} \begin{bmatrix} u_{xx} n_1 + u_{xy} n_2 \\ u_{xy} n_1 + u_{yy} n_2 \end{bmatrix} \quad (15)\\ &= n_1(u_{xx} n_1 + u_{xy} n_2) + n_2(u_{xy} n_1 + u_{yy} n_2) \quad (16)\\ &= n_1^2 u_{xx} + 2 n_1 n_2 u_{xy} + n_2^2 u_{yy} \quad (17) \end{aligned}$$

We know the following:

$$\vec{\nabla} = \frac{\nabla}{||\nabla u||} \tag{18}$$
$$= \frac{\langle u_x, u_y \rangle}{\sqrt{u_x^2 + u_y^2}} \tag{19}$$
$$= \left\langle \frac{u_x}{\sqrt{u_x^2 + u_y^2}}, \frac{u_y}{\sqrt{u_x^2 + u_y^2}} \right\rangle \tag{20}$$
$$\vec{\xi} = \left\langle -\frac{u_y}{\sqrt{u_x^2 + u_y^2}}, \frac{u_x}{\sqrt{u_x^2 + u_y^2}} \right\rangle \tag{21}$$

Here, let us call the first vector n_1 and the second n_2. Now:

$$u_{\xi\xi} = \frac{u_y^2}{u_x^2 + u_y^2} u_{xx} - \frac{2u_x u_y}{u_x^2 + u_y^2} u_{xy} + \frac{u_x^2}{u_x^2 + u_y^2} u_{yy} \tag{22}$$
$$= \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{u_x^2 + u_y^2} \tag{23}$$

Here, let us write:

$$u_t = u_{\xi\xi} \tag{24}$$
$$\Downarrow \tag{25}$$
$$u_t = \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{u_x^2 + u_y^2} \tag{26}$$