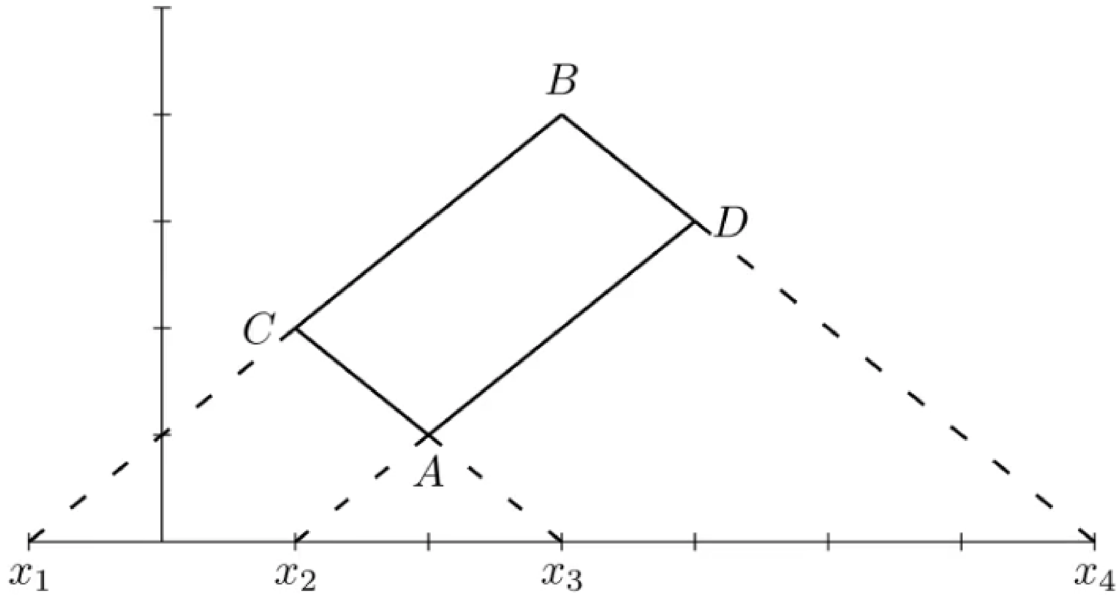
1. Use D'Alembert's formula to show the parallelogram property of the wave equation mentioned in class.



$$u(A) + u(B) = u(C) + u(D) \tag{1}$$

Note that our slope depends on c. Now, let us consider D'Alembert's Formula:

$$\frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(y)\mathrm{d}y \tag{2}$$

Now, let us consider using D'Alembert's Formula to generate the following equations:

$$u(A) = \frac{1}{2}[f(x_2) + f(x_3)] + \frac{1}{2c}\int_{x_2}^{x_3} g(y)\ dy \tag{3}$$
$$u(B) = \frac{1}{2}[f(x_1) + f(x_4)] + \frac{1}{2c}\int_{x_1}^{x_4} g(y)\ dy \tag{4}$$
$$u(C) = \frac{1}{2}[f(x_1) + f(x_3)] + \frac{1}{2c}\int_{x_1}^{x_3} g(y)\ dy \tag{5}$$
$$u(D) = \frac{1}{2}[f(x_2) + f(x_4)] + \frac{1}{2c}\int_{x_2}^{x_4} g(y)\ dy \tag{6}$$

From here, let us evaluate $u(A) + u(B)$ and $u(C) + u(D)$

$$u(A) + u(B) = \frac{1}{2}[f(x_2) + f(x_3)] + \frac{1}{2c}\int_{x_2}^{x_3} g(y)\ dy + \frac{1}{2}[f(x_1) + f(x_4)] + \frac{1}{2c}\int_{x_1}^{x_4} g(y)\ dy \tag{7}$$
$$= \frac{1}{2}\left(f(x_1) + f(x_4) + f(x_2) + f(x_3) + \frac{1}{c}\left[\int_{x_1}^{x_4} g(y)\mathrm{d}y + \int_{x_2}^{x_3} g(y)\mathrm{d}y\right]\right) \tag{8}$$

Next, evaluate $u(C) + u(D)$:

$$u(C) + u(D) = \frac{1}{2}[f(x_1) + f(x_3)] + \frac{1}{2c}\int_{x_1}^{x_3} g(y)\ dy + \frac{1}{2}[f(x_2) + f(x_4)] + \frac{1}{2c}\int_{x_2}^{x_4} g(y)\ dy \tag{9}$$
$$= \frac{1}{2}\left(f(x_1) + f(x_3) + f(x_2) + f(x_4) + \frac{1}{c}\left[\int_{x_1}^{x_3} g(y)\ dy + \int_{x_2}^{x_4} g(y)\ dy\right]\right) \tag{10}$$

If we analyze the regions of our integral, we can observe the interval length of the integral for $u(A) + u(B)$ spans over 10 units. In addition, $u(C) + u(D)$ also spans over 10 intervals once again. Here, both intervals are equal. Therefore,

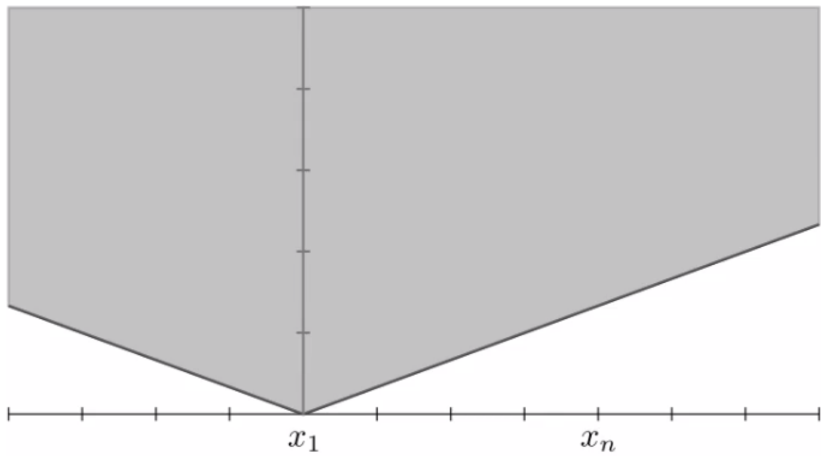
$$u(A) + u(B) = u(C) + u(D) \tag{11}$$

2. If $f(x)$ and $g(x)$ are changed on the region $x \in [0,4]$, on which region in the $(x,t)-$plane will the solutions of $u_{tt} = 9u_{xx}$ be altered?

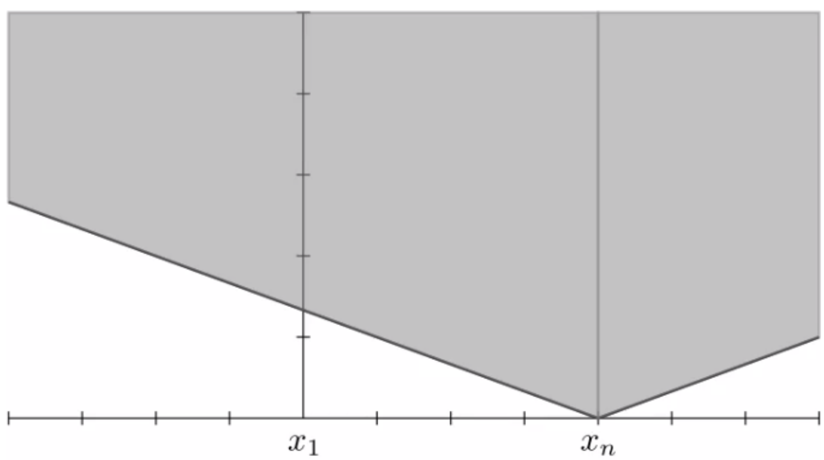
   Here, we are given a wave equation on the x boundary $[0,4]$ and a constant 3^2.

   Here, let us consider our wave equation, $u_{tt} = 9u_{xx}$, where $\sqrt{c} = 3$.

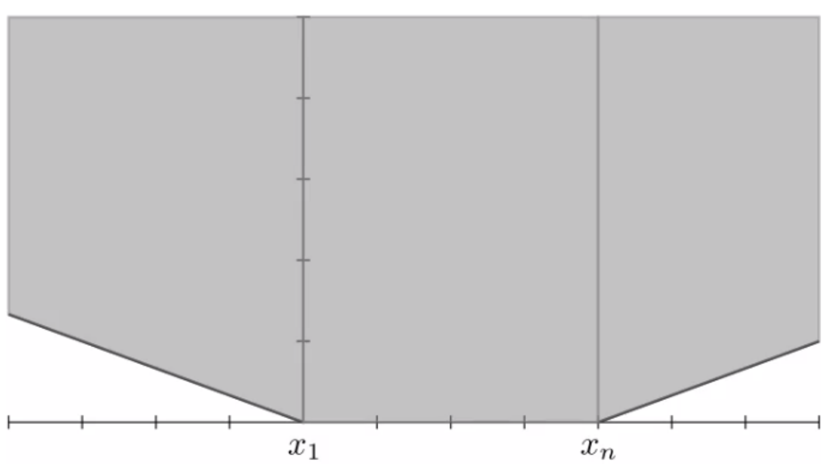
   Here, the slope of our characteristic line is $\frac{1}{3}$. If we alter $f(x)$ and $g(x)$, we are changing the range of influence of our equation. For instance, let us pick $x_1 = 1$ for $f(x)$, we would obtain a range of influence such as the following:



   Next, let us consider another point within our boundary, such as x_n, let us consider how our graph appears at $f(x_n)$:



   In essence, our range of influence as we tweak $f(x)$ and $g(x)$ lies between the points x_1 and x_n:



3. The solution to the non-homogeneous Laplace equation $\Delta u = f(x,y)$ on $x \in (-\infty, \infty), y \in (-\infty, \infty)$ is:

$$u(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x-\xi, y-\eta) f(\xi,\eta) \mathrm{d}\xi \mathrm{d}\eta \tag{1}$$

where

$$k(x,y) = -\frac{1}{2\pi} \ln\left(\sqrt{x^2+y^2}\right) \tag{2}$$

Show that if $f(\xi,\eta) = \delta(\xi)\delta(\eta)$, then $\Delta u = 0$ for $(x,y) \neq (0,0)$.

Let us consider the given equation 1). Here, let us use given assumption, $f(\xi,\eta) = \delta(\xi)\delta(\eta)$ and substitute it into 1)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x-\xi, y-\eta) f(\xi,\eta) \mathrm{d}\xi \mathrm{d}\eta = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x-\xi, y-\eta) \delta(\xi)\delta(\eta) \mathrm{d}\xi \mathrm{d}\eta \tag{3}$$

Let us consider our function, k. Equation 2) defines the function of k. Let us evaluate our function with the given parameters, $x-\xi$ and $y-\eta$:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x-\xi, y-\eta) \delta(\xi)\delta(\eta) \mathrm{d}\xi \mathrm{d}\eta = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{1}{2\pi} \left(\ln\sqrt{(x-\xi)^2+(y-\eta)^2}\right) \delta(\xi)\delta(\eta) \mathrm{d}\xi \mathrm{d}\eta \tag{4}$$

Here, let us consider our δ function and ways to manipulate the function. Here, we have the property:

$$\int_{-\infty}^{\infty} \delta(x-y) f(y) \mathrm{d}y = f(x) \tag{5}$$

If we apply it to equation 4, we get:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{1}{2\pi} \left(\ln\sqrt{(x-\xi)^2+(y-\eta)^2}\right) \delta(\xi)\delta(\eta) \mathrm{d}\xi \mathrm{d}\eta = -\frac{1}{2\pi} \ln\left(\sqrt{x^2+y^2}\right) \tag{6}$$

Now, let us take the x and y partial of line 6)

$$\begin{aligned}
u(x,y) &= -\frac{1}{2\pi} \ln\left(\sqrt{x^2+y^2}\right) && (7)\\
u_{xx}(x,y) + u_{yy}(x,y) &= \left(-\frac{1}{2\pi}\frac{x}{x^2+y^2}\right)_x + \left(-\frac{1}{2\pi}\frac{y}{x^2+y^2})\right)_y && (8)\\
&= \left(-\frac{1}{2\pi}\frac{y^2-x^2}{(x^2+y^2)^2}\right)\left(-\frac{1}{2\pi}\frac{x^2-y^2}{(x^2+y^2)^2}\right) && (9)\\
&= \left(\frac{1}{2\pi}\frac{x^2-y^2}{(x^2+y^2)^2}\right)\left(\frac{1}{2\pi}\frac{y^2-x^2}{(x^2+y^2)^2}\right) && (10)\\
&= \frac{1}{2\pi}\left(\frac{x^2-y^2+y^2-x^2}{(x^2+y^2)^2}\right) && (11)\\
&= 0 && (12)
\end{aligned}$$

4. Show the following:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \mathrm{d}x = 1$$

We want to find the integral of the following:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \mathrm{d}x \tag{1}$$

First, let us move the constant out of our integral:

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-x^2/4t} \mathrm{d}x \tag{2}$$

From here, let us rename our constant on the outside of our integral as ζ:

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-x^2/4t} \mathrm{d}x = \zeta \int_{-\infty}^{\infty} e^{-x^2/4t} \mathrm{d}x \tag{3}$$

Here, let us focus on our integral. First, let us square our integral and change our variables in the second integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2/4t} \mathrm{d}x \tag{4}$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2/4t} \mathrm{d}x \int_{-\infty}^{\infty} e^{-y^2/4t} \mathrm{d}y \tag{5}$$

From here, let us find the product of our integrals then combine our powers:

$$I^2 = \int_{-\infty}^{\infty} \int_{\infty}^{\infty} e^{-x^2/4t} e^{-y^2/4t} \mathrm{d}x\mathrm{d}y \tag{6}$$

$$= \int_{-\infty}^{\infty} \int_{\infty}^{\infty} e^{-(x^2+y^2)/4t} \mathrm{d}x\mathrm{d}y \tag{7}$$

$$\tag{8}$$

Here, let us write our integral and variables in terms of polar coordinates:

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2/4t} r \ \mathrm{d}r\mathrm{d}\theta \tag{9}$$

$$= \int_0^{\infty} \int_0^{2\pi} e^{-r^2/4t} r \ \mathrm{d}\theta\mathrm{d}r \tag{10}$$

$$= 2\pi \int_0^{\infty} r e^{-r^2/4t} \ \mathrm{d}r \tag{11}$$

$$\tag{12}$$

Here, let us perform u-substitution, where we write $u = \frac{r^2}{4t}$ and $\mathrm{d}u = \frac{r}{2t}\mathrm{d}r$

$$I^2 = 4\pi t \int_0^{\infty} e^{-u} \ \mathrm{d}r \tag{13}$$

$$I^2 = 4\pi t \tag{14}$$

$$I = \sqrt{4\pi t} \tag{15}$$

Here, let us plug our evaluation back to line 3 to find the solution:

$$\frac{1}{\sqrt{4\pi t}}\sqrt{4\pi t} = 1 \tag{16}$$

5. We know that the solution to the 2−D heat equation $u_t = u_{xx} + u_{yy}$, with $u(x, y, 0) = f(x, y)$ is

$$u(x,y,t) = \frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4t}}\mathrm{d}\xi\mathrm{d}\eta \tag{1}$$

If

$$f(x,y) = \begin{cases} 1 & 2 \leq r \leq 4, r = \sqrt{x^2+y^2} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Sketch $u(x, y, t)$ for different t values, say $t = 0, 5, 100, \infty$

Note, as t increases, the sharp edges near the top of the cylinder tend to smooth out. The drawings do not accurately represent this detail.