1. Plot the given functions and find their Fourier Transforms

 (a) $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

 (b) $f(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Here, let us consider both problems individually.

(a) First, let us consider our first given equation:

$$f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, let us plot our given function:

First image here

Now, let us find the Fourier Transform of this problem.

Here, let us consider our domain of integration, where f_i is the i^{th} position. Let us informally write:

$$f_1(x) = \int_{-1}^{0} -1 \text{ dx} \tag{1}$$

$$f_2(x) = \int_{0}^{1} 1 \text{ dx} \tag{2}$$

$$f_3(x) = \int_{-\infty}^{-1} 0 \text{ dx} + \int_{1}^{\infty} 0 \text{ dx} \tag{3}$$

Here, let us write our definition of the Fourier Transform:

$$F[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \text{ dx} \tag{4}$$

From here, let us split our integral:

$$\begin{aligned} F[f] &= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{0} -e^{-i\xi x} \text{ dx} + \int_{0}^{1} e^{-i\xi x} \text{ dx} + \int_{-\infty}^{-1} 0 e^{-i\xi x} \text{ dx} + \int_{1}^{\infty} 0 e^{-i\xi x} \text{ dx} \right] && (5) \\ &= \frac{1}{\sqrt{2\pi}} \left[-\int_{-1}^{0} e^{-i\xi x} \text{ dx} + \int_{0}^{1} e^{-i\xi x} \text{ dx} \right] && (6) \end{aligned}$$

Here, let us integrate our integrals:

$$\begin{aligned} F[f] &= \frac{1}{\sqrt{2\pi}} \left[-\int_{-1}^{0} e^{-i\xi x} \text{ dx} + \int_{0}^{1} e^{-i\xi x} \text{ dx} \right] && (7) \\ &= \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{-i\xi} e^{-i\xi x} \Big|_{-1}^{0} + \frac{1}{-i\xi} e^{-i\xi x} \Big|_{0}^{1} \right] && (8) \\ &= \frac{1}{i\xi\sqrt{2\pi}} \left[e^{-i\xi x} \Big|_{-1}^{0} - e^{-i\xi x} \Big|_{0}^{1} \right] && (9) \\ &= \frac{1}{i\xi\sqrt{2\pi}} \left[\left(e^{-i\xi 0} - e^{-i\xi(-1)} \right) - \left(e^{-i\xi 1} - e^{-i\xi 0} \right) \right] && (10) \\ & && (11) \end{aligned}$$

Here, let us evaluate our expressions and simplify:

$$= \frac{1}{i\xi\sqrt{2\pi}}\left[\left(1-e^{i\xi}\right)-\left(e^{-i\xi}-1\right)\right] \tag{12}$$
$$= \frac{1}{i\xi\sqrt{2\pi}}\left[1-e^{i\xi}-e^{-i\xi}+1\right] \tag{13}$$
$$= \frac{2}{i\xi\sqrt{2}\sqrt{\pi}}\left[1-e^{i\xi}\right] \tag{14}$$
$$= \frac{\sqrt{2}}{\xi\sqrt{\pi}}\frac{1-e^{i\xi}}{i} \tag{15}$$

(b) Now, let us consider our second given equation:

$$f(x) = \begin{cases} 1-|x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, let us plot our given function:

Now, let us find the Fourier Transform of this problem.

Here, let us write our equation and further divide our function:

$$f_1(x) = \int_0^1 1-x \text{ dx} \tag{1}$$
$$f_2(x) = \int_{-1}^0 1+x \text{ dx} \tag{2}$$
$$f_3(x) = \int_1^\infty 0 \text{ dx } + \int_{-\infty}^{-1} 0 \text{ dx} \tag{3}$$

Here, let us use the definition of the Fourier Transform from 4) is the previous part. First, let us split our integral:

$$F[f] = \frac{1}{\sqrt{2\pi}}\left[\int_0^1 (1-x)e^{-i\xi x} \text{ dx} + \int_{-1}^0 (1+x)e^{-i\xi x} \text{ dx} + \int_1^\infty 0e^{-i\xi x} \text{ dx } + \int_{-\infty}^{-1} 0e^{-i\xi x} \text{ dx}\right] \tag{4}$$
$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^1 (1-x)e^{-i\xi x} \text{ dx} + \int_{-1}^0 (1+x)e^{-i\xi x} \text{ dx}\right] \tag{5}$$
$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^1 e^{-i\xi x} - xe^{-i\xi x} \text{ dx} + \int_{-1}^0 e^{-i\xi x} + xe^{-i\xi x} \text{ dx}\right] \tag{6}$$

Before proceeding, let us create a table of integration:

x	$e^{-i\xi x}$
1	$\frac{1}{-i\xi}e^{-i\xi x}$
0	$\frac{1}{i^2\xi^2}e^{-i\xi x}$

Here, we have our integration by parts. Now, let us proceed with our integrals:

$$F[f] = \frac{1}{\sqrt{2\pi}}\left[\left[\left(\frac{1}{-i\xi}e^{-i\xi x}\right) - \left(\frac{x}{i\xi}e^{-i\xi x} + \frac{1}{-\xi^2}e^{-i\xi x}\right)\right]_0^1 + \left[\left(\frac{1}{-i\xi}e^{-i\xi x}\right) + \left(\frac{x}{i\xi}e^{-i\xi x} + \frac{1}{-\xi^2}e^{-i\xi x}\right)\right]_{-1}^0\right] \tag{7}$$
$$= \frac{1}{\sqrt{2\pi}}\left[\left[-\frac{1}{i\xi}e^{-i\xi x} - \frac{x}{i\xi}e^{-i\xi x} + \frac{1}{\xi^2}e^{-i\xi x}\right]_0^1 + \left[-\frac{1}{i\xi}e^{-i\xi x} + \frac{x}{i\xi}e^{-i\xi x} - \frac{1}{\xi^2}e^{-i\xi x}\right]_{-1}^0\right] \tag{8}$$

Here, let us evaluate both integrals side-by-side:

Let us consider the integral on the left:

$$\left[\frac{1}{\xi^2}e^{-i\xi x} - \frac{1}{i\xi}e^{-i\xi x} - \frac{x}{i\xi}e^{-i\xi x}\right]_0^1 \tag{9}$$

$$\frac{1}{\xi^2}e^{-i\xi} - \frac{2}{i\xi}e^{-i\xi} - \frac{1}{\xi^2} + \frac{1}{i\xi} \tag{10}$$

Now, let us consider the integral on the right:

$$\left[-\frac{1}{\xi^2}e^{-i\xi x} - \frac{1}{i\xi}e^{-i\xi x} + \frac{x}{i\xi}e^{-i\xi x}\right]_{-1}^0 \tag{11}$$

$$-\frac{1}{\xi^2} - \frac{1}{i\xi} + \frac{1}{\xi^2}e^{i\xi} + \frac{2}{i\xi}e^{i\xi} \tag{12}$$

Now, let us plug in our parts back into our integral:

$$\begin{aligned}
F[f] &= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\xi^2}e^{-i\xi} - \frac{2}{i\xi}e^{-i\xi} - \frac{1}{\xi^2} + \frac{1}{i\xi} - \frac{1}{\xi^2} - \frac{1}{i\xi} + \frac{1}{\xi^2}e^{i\xi} + \frac{2}{i\xi}e^{i\xi}\right] && (13)\\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\xi^2}e^{-i\xi} + \frac{1}{\xi^2}e^{i\xi} - \frac{2}{\xi^2} + \frac{2}{i\xi}e^{i\xi} - \frac{2}{i\xi}e^{-i\xi}\right] && (14)\\
&= \frac{1}{\xi^2\sqrt{2\pi}}\left[e^{-i\xi} + e^{i\xi}\right] + \frac{\sqrt{2}}{\xi\sqrt{\pi}}\left[\frac{e^{i\xi} - e^{-i\xi}}{i}\right] - \frac{2}{\xi^2\sqrt{2\pi}} && (15)
\end{aligned}$$