

1. Solve $\Delta u = 0$ on $x \in [0, 4]$, $y \in [0, 3]$, with $u(x, 0) = u(x, 3) = 0$ where

(a) $u_x(0, y) = 0$ and $u_x(4, y) = \cos(\pi y)$

(b) $u(0, y) = 1$ and $u(4, y) = 0$ (Hint: Translate the x coordinate so that $u(0, y) = 0$. This means that x goes between -4 and 0 . Now go back to 0 to 4 with a y -axis flip.)

Here, let us first solve part a) of the problem.

(a) First, let us assume our equation is separable:

$$u(x, y) = X(x)Y(y) \quad (1)$$

Let us take note of our u :

$$u_{xx} + u_{yy} = 0 \quad (2)$$

$$X''Y + XY'' = 0 \quad (3)$$

$$XY'' = -X''Y \quad (4)$$

$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda \quad (5)$$

Here, let us consider our boundary conditions:

$$u(x, 0) = 0 = X(0)Y(y) \quad (6)$$

Here, we either have $X(0) = 0$ or $Y(y) = 0$. We do not want to assume $Y(y) = 0$ because that would make $u(x, y) = 0 \forall y \in [0, 3]$. Therefore, we consider $X(0) = 0$. We will follow this assumption for the upcoming boundary conditions.

$$u(x, y) = X(x)Y(y) \quad (7)$$

$$u(x, 0) = X(x)Y(0) = 0 \Rightarrow Y(0) = 0 \quad (8)$$

$$u(x, 3) = X(x)Y(3) = 0 \Rightarrow Y(3) = 0 \quad (9)$$

Now, let us consider $u_x(x, y)$ as $X'(x)Y(y)$:

$$u_x(x, y) = X'(x)Y(y) \quad (10)$$

$$u_x(0, y) = X'(0)Y(y) = 0 \Rightarrow X'(0) = 0 \quad (11)$$

Now, let us write:

$$-\frac{X''}{X} = \frac{Y''}{Y} = -\lambda \quad (12)$$

(b) Next we solve for $Y(y)$ since we have more information on $Y(y)$.

$$\frac{Y''}{Y} = -\lambda \quad (13)$$

$$Y'' = -\lambda Y \quad (14)$$

Here, let us write out our general equation:

$$Y_n(y) = A \sin(\sqrt{\lambda}x) + B \cos(\sqrt{\lambda}x) \quad (15)$$

Here, we know $Y(0) = Y(3) = 0$. Let us input our y :

$$Y_n(0) = 0 = B \quad (16)$$

$$Y_n(3) = 0 = A \sin(\sqrt{\lambda}3) \quad (17)$$

Here, let us not consider A as 0, instead, the inside of our sine function to be 0.

$$\sqrt{\lambda}3 = n\pi \quad (18)$$

$$\sqrt{\lambda} = \frac{n\pi}{3} \quad (19)$$

$$\lambda_n = \left(\frac{n\pi}{3}\right)^2 \quad (20)$$

Now we have the following:

$$Y_n(y) = \sin\left(\frac{n\pi y}{3}\right) \quad (21)$$

(c) Now, let us solve for X :

$$\frac{X''}{X} = \lambda \quad (22)$$

$$X'' = \lambda X \quad (23)$$

Here, we solved for λ in the previous step:

$$X_n'' = \left(\frac{n\pi}{3}\right)^2 X_n \quad (24)$$

$$X_n(x) = C \sinh\left(\frac{n\pi x}{3}\right) + D \cosh\left(\frac{n\pi x}{3}\right) \quad (25)$$

(d)

(e)

2. Solve $u_{tt} = 4u_{xx}$ on $x \in [0, 3]$, $t \in [0, \infty)$, with $u(0, t) = u(3, t) = 0$ where

(a) $u(x, 0) = 4 \sin(2\pi x) + 7 \sin(6\pi x) - 2 \sin(\pi x)$, $u_t(x, 0) = 0$

(b) $u(x, 0) = x(3 - x)$, $u_t(x, 0) = \sin(\pi x)$

Let us begin with our boundary conditions.

(a) Let us assume our equation is separable.

$$u_{tt} = 2^2 u_{xx} \quad (26)$$

$$XT'' = 2^2 X''T \quad (27)$$

$$\frac{T''}{2^2 T} = \frac{X''}{X} = -\lambda \quad (28)$$

(b) Here, since we have more information regarding X , let us solve for X :

$$\frac{X''}{X} = -\lambda \quad (29)$$

$$X'' = -\lambda X \quad (30)$$

In this equation, we want to consider \cos and \sin for our general equation.

$$X(x) = A \sin(\sqrt{\lambda}x) + B \cos(\sqrt{\lambda}x) \quad (31)$$

Here, we know $X(0) = X(3) = 0$, so let us write:

$$X(0) = B = 0 \quad (32)$$

$$X(3) = A \sin(\sqrt{\lambda}3) = 0 \quad (33)$$

Here, let us find when our sine function is 0.

$$\sqrt{\lambda}3 = n\pi \quad (34)$$

$$\sqrt{\lambda} = \frac{n\pi}{3} \quad (35)$$

$$\lambda = \left(\frac{n\pi}{3}\right)^2 \quad (36)$$

Now, let us write our general equation for X :

$$X_n(x) = \sin\left(\frac{n\pi x}{3}\right) \quad (37)$$

(c) Next, let us solve for T :

$$\frac{T''}{2^2 T} = -\lambda \quad (38)$$

$$T'' = -2^2 T \lambda \quad (39)$$

Here, let us write the general equation:

$$T_n(t) = C_n \sin\left(\frac{2n\pi t}{L}\right) + D_n \cos\left(\frac{2n\pi t}{L}\right) \quad (40)$$

(d) Here, let us combine to find u_n and u :

$$u_n(x, t) = \sin\left(\frac{n\pi x}{3}\right) \left[C_n \sin\left(\frac{2n\pi t}{L}\right) + D_n \cos\left(\frac{2n\pi t}{L}\right) \right] \quad (41)$$

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{3}\right) \sin\left(\frac{2n\pi t}{L}\right) + D_n \sin\left(\frac{n\pi x}{3}\right) \cos\left(\frac{2n\pi t}{L}\right) \quad (42)$$

(e) Here, let us find the coefficients using the Initial Conditions:

- $u(x, 0) = 4 \sin(2\pi x) + 7 \sin(6\pi x) - 2 \sin(\pi x)$
- $u_t(x, 0) = 0$

Using this information, find our t partial from our summation series:

$$u_t(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{3}\right) \frac{2n\pi}{L} \cos\left(\frac{2n\pi t}{L}\right) - D_n \sin\left(\frac{n\pi x}{3}\right) \frac{2n\pi}{L} \sin\left(\frac{2n\pi t}{L}\right) \quad (43)$$

From here, let us plug in our initial condition, $u_t(x, 0) = 0$:

$$u_t(x, 0) = \sum_{n=1}^{\infty} C_n \frac{2n\pi}{L} \sin\left(\frac{n\pi x}{3}\right) = 0 \quad (44)$$