Partial Differential Equations - Class Notes

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1 Chapter 1

Sidenotes

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What is a PDE?
A PDE is an equation which contains partial derivatives of an unknown function and we want to find that unknown function.
Example: $F(t, x, y, z, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x \partial y}, \ldots) = 0$.
Note, the first partial derivatives are considered 1^{st} ordered partials
whereas the second ordered partials are considered 2^{nd} ordered partials.
The variables that are not u are considered independent variables and u is considered a dependent variable.
What PDEs do we study?
Generally, we restrict our attention to equations that model some phenomenom from physics, engineering, economics, geology, ... etc. We can use physical intuition to help guide the math.
Classification of PDEs

1. Order of PDE: Highest derivative.
   Example: $\frac{\partial^3 u}{\partial x^3} - \sin(y)u^7 = 3$ is a third order PDE.
   Example: $(\frac{\partial y}{\partial t})^5 - \frac{\partial^2 y}{\partial x \partial t} = e^x$ is a second order PDE.
2. Number of independent variables.
   Example: $\frac{du}{dt} = \frac{\partial^2 u}{\partial x^2}$ has two independent variables: t, x.
   This is the $1-D$ heat equation.
   Example: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \Delta u$ has 4 independent variables.
   This is the $3-D$ heat equation. Δu is Laplacian of u.
   Notation
   $\Delta u = \nabla^2 u = \nabla \cdot \nabla u = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$
   $\Delta u = 0$ is considered Laplace's equation.
3. Linear vs non-linear
   A linear PDE is any equation of the form $L[u(x)] = f(x)$ where $f(x)$ is a known function is a linear partial differential operator.

Definition: A differential operator is any rule that takes a function as its input and returns an expression that involves the derivatives of that function.
Example:

$$\begin{aligned} u(x,t) \qquad & v(x,t) && (1)\\ O[u] &= \frac{\partial^2 u}{\partial x^2} + \sin x + \pi - 7e^{tu} && (2)\\ O[u+3v] &= \frac{\partial^2}{\partial x^2}(u+3v) + \sin x + \pi - 7e^{tu+3tv} && (3)\\ &= \frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 v}{\partial x^2} + \sin x + \pi - 7e^{tu+3tv} && (4) \end{aligned}$$

Definition: A linear operator, L, is an operator that has the property:

$$L[au + bv] = aL[u] + bL[v] \qquad (5)$$

Where a and b are constants.
Theorem: If u and v are vectors and L is linear, then L can be represented by a matrix.
Theorem: If L is linear ordinary operator, it must take the form:

$$L[u] = f_0(x)u + f_1(x)u' + f_2(x)u'' + \ldots + f_n(x)u^{(n)} \qquad (6)$$

Where the f_i's are known functions.
Definition: A linear ODE is any ODE of the form where $f(x)$ is known is the following:

$$L[u] = f(x) \qquad (7)$$

If $f(x) = 0$, then the equation is homogeneous. Otherwise, the equation is non-homogeneous.
Ex: $(u')^2 = 0 \Rightarrow u' = 0 \to$ linear, homogeneous.
Theorem: If L is a linear partial differential operator, it must take the form (x is a vector with n unknowns)

$$L[u(x)] = f_0(x)u + \sum_{i=1}^{n} f_i(x)\frac{\partial u}{\partial x_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \ldots \qquad (8)$$

Definition: A linear PDE is any PDE of the form

$$L[u(x)] = f(x) \qquad (9)$$

If $f(x) = 0$, the equation is homogeneous, else it is non-homogeneous.
Ex: $u_t = 4u_x$ - Linear, homogeneous.

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Example:

$$u_{tt} = u_{xx} + uyy \quad \text{Linear, homogeneous} \tag{10}$$
$$\cos(xt) = u + u_t + u_{xyz} \quad \text{Linear, non-homogeneous} \tag{11}$$
$$u_t u_{xt} = 0 \quad \text{non-linear} \tag{12}$$
$$u_{xt} + e^x \cos t \; u_t = 0 \quad \text{linear, homogeneous} \tag{13}$$
$$u_t + u_{xx} + ue^u = 0 \quad \text{non-linear} \tag{14}$$

Note: You can add linear combinations of solutions to linear homogeneous equations and still get a solution. Example: $u_x = u_t$. Some solutions to this are:

1. $u_1(x,t) = 3$
2. $u_2(x,t) = x + t$
3. $u_3(x,t) = e^{x+t}\cos(x+t)$
4. $\vdots$
   $Au_1 + Bu_2 + Cu_3$ is also a solution.

How do we solve an ODE?

1. Use some technique to find an explicit solution.
2. Use power series and determine the coefficients

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{15}$$

3. Laplace Transforms

How do we solve PDEs?

1. Try to locate an explicit solution
2. We don't use power series, instead, we use a trigonometric series $\Rightarrow$ Fourier Series.

$$y(x) = \sum_{n=0}^{\infty} a_n \sin(nx) + b_n \cos(nx) \tag{16}$$

3. Laplace Transforms are good if the domain is $[0,\infty)$.
   Fourier Transforms are good if the domain is $(-\infty,\infty)$.
4. Reduce the PDE to a system of ODEs.

Initial Condiction

1. For ODEs, to solve a 1^{st} order equation, you need $y(0)$.
   2^{nd} order $\to y(0), y'(0)$
   3^{rd} order $\to y(0), y'(0), y''(0)$
   $\vdots$
   n^{th} order $\to y(0), y'(0), y''(0), \ldots, y^{(n-1)}(0)$
2. For PDEs, it's more complicated $\Rightarrow$ it depends on the PDE.
   Example: $u(x,t), x \in [a,b], t \in [0,\infty)$
   If $u_t = u_{xx}$
3. Boundary conditions:

$$u(a,t) = g_1(t) \tag{17}$$
$$u(b,t) = g_2(t) \tag{18}$$

   If $u_{tt} = u_{xx}$, we must specify:

   (a) Initial Conditions

$$u(x,0) = f_1(x) \tag{19}$$
$$u_t(x,0) = f_2(x) \tag{20}$$

   (b) Boundary Conditions

$$u(a,t) = g_1(t) \tag{21}$$
$$u(b,t) = g_2(t) \tag{22}$$

1-D Heat Equation
Assume cross sections are uniform Imagine a cross section:

O o==========o L

Assume cross sections are uniform and the lateral sides are well insulated $\Rightarrow$ heat only flows in the x-direction.
We need the following:

- $u(x,t)$: Temperature of rod at position x and time t.
- $u(x,0)$: Initial temperature

- $u(0,t)$ and $u(L,t)$: Boundary Conditions

Definition:

- $g(x,t)$: heat flux (energy / area time)
- $Q(x,t)$: heat energy density (energy / volume)
- A : Cross sectional area
- C_P : Heat capacity or specific heat
- ρ : Density
- K : Thermal conductivity

We want to find an equation for the temperature evolution. We will use conservation of energy : Look at a little Δx section of the rod starting at x_0.

Δx

o=====|o|=====o

x_0 $x_0 \Delta x$

Conservation of energy : heat in - heat out = heat accumulated

Heat in $=' qA\Delta t' = A \int_{t_0}^{t_0+\Delta t} q(x_0,t)$ dt

Heat out $= A \int_{t_0}^{t_0+\Delta t} q(x_0+\Delta x,t)$ dt

Heat Accumulated $= QA\Delta x|_{t_0+\Delta t} - QA\Delta x|_{t_0}$

$$= A \int_{x_0}^{x_0+\Delta x} Q(x,t_0+\Delta t)\ \mathrm{dx} - A \int_{x_0}^{x_0+\Delta x} Q(x,t_0)\ \mathrm{dx} \tag{23}$$

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Heat Equation
Conservation of energy
Heat in - heat out = heat accumulated

$$A\int_{t_0}^{t_0\to\Delta t} g(x_0,t)\ \mathrm{dt}\ - A\int_{t_0}^{t_0\to\Delta t} q(x_0+\Delta x,t)\ \mathrm{dt}\ = A\int_{t_0}^{t_0\to\Delta t} Q(x,t_0+\Delta t)\ \mathrm{dx}\ - A\int_{t_0}^{t_0\to\Delta t} Q(x,t_0)\ \mathrm{dx} \tag{24}$$

Let us simplify and divide by A. Then, let us combine the integrals:

$$\int_{t_0}^{t_0\to\Delta t} [q(x_0,t)-q(x_0+\Delta x,t)]\ \mathrm{dt}\ = \int_{t_0}^{t_0\to\Delta t} [Q(x,t_0+\Delta t)-Q(x,t_0)]\ \mathrm{dx} \tag{25}$$

Divide by $\Delta x\Delta t$ and take limit as $\Delta x,\Delta t\to 0$

$$\lim_{\Delta t,\Delta x\to 0}\frac{1}{\Delta x\Delta t}\int_{t_0}^{t_0\to\Delta t} [q(x_0,t)-q(x_0+\Delta x,t)]\ \mathrm{dt}\ = \lim_{\Delta t,\Delta x\to 0}\frac{1}{\Delta x\Delta t}\int_{t_0}^{t_0\to\Delta t} [Q(x,t_0+\Delta t)-Q(x,t_0)]\ \mathrm{dx} \tag{26}$$

$$\lim_{\Delta t}\frac{1}{\Delta t}\int_{t_0}^{t_0\to\Delta t} [\lim_{\Delta x\to 0}\frac{q(x_0,t)-q(x_0+\Delta x,t)}{\Delta x}]\ \mathrm{dt}\ = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_{t_0}^{t_0\to\Delta t}\lim_{\Delta t\to 0}\frac{Q(x,t_0+\Delta t)-Q(x,t_0)}{\Delta t}\ \mathrm{dx} \tag{27}$$

On the left side, we see the order is a bit difference. We want the delta to come first, such as in the difference quotient. The eft is now $-q_x(x_0,t)$ and the right is $Q_t(x,t_0)$.

$$\lim_{\Delta t\to}\frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} -q_x(x_0 t)\ \mathrm{dt}\ = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x} Q_t(x,t_0)\ \mathrm{dx} \tag{28}$$

$$\lim_{\Delta t\to 0} -q_x(x_0,t_0+\Delta t) = \lim_{\Delta x\to 0} Q_t(x_0+\Delta x,t_0) \tag{29}$$

At step 28, we used the fundamental theorem of calculus and derived both sides.

$$-q_x(x_0,t_0) = Q_t(x_0,t_0) \tag{30}$$

Since x_0 and t_0 are arbitrary, $-q_x(x,t) = Q_t(x,t)$
q and Q are related to u:

$$Q = \rho c_p u \qquad\qquad q = -Ku_x \tag{31}$$

$$-q_x = Q_t \Rightarrow Ku_{xx} = \rho c_p u_t \tag{32}$$

$$\Rightarrow u_t = \frac{k}{\rho c_p}u_{xx} \tag{33}$$

$$\Rightarrow u_t = \alpha^2 u_{xx} \tag{34}$$

$$\alpha = \sqrt{\frac{K}{\rho c_p}} \tag{35}$$

α is thermal diffusivity
$u_t = \alpha^2 u_{xx} \leftarrow$ 1-D heat equation (diffusivity equation)
We have a steady-state: $(t\to\infty)$, $u_t = 0 \Rightarrow u_{xx} = 0 \Rightarrow$ straight line
1-D: $-q_x = Q_t \Rightarrow -\nabla\cdot\vec{q} = Q_t$, $\quad \vec{q}$ is a vector.

$$q = -K\nabla u \Rightarrow -\nabla\cdot(-K\nabla u) = \rho c_p u_t \tag{36}$$

$$\Rightarrow K\Delta u = \rho c_p u_t \tag{37}$$

$$\Rightarrow u_t = \alpha^2\Delta u \tag{38}$$

What about a steady-state? $u_t = 0$

$$\Delta u = 0 \tag{39}$$

Here, we have Laplace's equation.
Note: It is not dependent on time.
The Wave Equation $u(x,t)$ is the height of the rope. We use Newton's 2^{nd} law on small segments of rope.

- $\rho =$ density of rope.
- dm $= \rho$ dx

$$F = ma \tag{40}$$

$$T\sin(\theta(x+\Delta x)) - T\sin(\theta(x)) = \int_x^{x+\Delta x} u_{tt}\ \mathrm{dm} \tag{41}$$

$$T[\sin(\theta(x+\Delta x)) - \sin(\theta(x))] = \rho\int_x^{x+\Delta x} u_{tt}\ \mathrm{dx} \tag{42}$$

Let us assume θ is small, $\sin\theta\approx\tan\theta$

$$T[\tan(\theta(x+\Delta x)) - \tan(\theta(x))] = \rho\int_x^{x+\Delta x} u_{tt}\ \mathrm{dx} \tag{43}$$

Also, $\tan(\theta(x)) = u_x(x,t)$.

$$T[u_x(x+\Delta x,t) - u_x(x,t)] = \rho\int_x^{x+\Delta x} u_{tt}\ \mathrm{dx} \tag{44}$$

Now, let us divide both sides by Δx and take the limit as $\Delta x\to 0$

$$\lim_{\Delta x\to 0} T\Big[\frac{u_x(x+\Delta x,t)-u_x(x,t)}{\Delta x}\Big] = \rho\lim_{\Delta x\to 0}\frac{\int_x^{x+\Delta x} u_{tt}\ \mathrm{dx}}{\Delta x} \tag{45}$$

On the left side, we have $u_x x$ and the right side we have $u_{tt}(x + \Delta x, t)$.

$$Tu_{xx}(x,t) = \rho u_{tt}(x,t) \tag{46}$$

$$u_{tt} = \frac{T}{\rho} u_{xx} = c^2 u_{xx} \tag{47}$$

$$c = \sqrt{\frac{T}{\rho}} = \text{ wave speed} \tag{48}$$

On the left, we have the $1 - D$ wave equation which is used for light, sound, rope, etc.
In 2-D, it corresponds to a vibrating membrane (drum)

$$u_{tt} = c^2 \Delta u \tag{49}$$

Remark:

$$u_t = u_{xx} \quad \text{Heat Equation} \tag{50}$$

$$u_{xx} + u_{yy} = 0 \quad \text{Laplace Equation} \tag{51}$$

$$u_{tt} = u_{xx} \quad \text{wave} \tag{52}$$

Here, we can replace:
u_t with t
u_x with x
u_{xx} with x^2

1. $t = x^2$ parabola
2. $x^2 + y^2 = 0$ ellipse
3. $t^2 = x^2$ hyperbolas

So, the equations behave like the following:

1. The Heat Equation is called parabolic
2. The Laplace Equation is called elliptic
3. The Wave Equation is called hyperbolic

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Approximating Functions with other Functions

1. Prove Series

$$f(x) = \sum_{n=0}^{M} a_n x^n \quad \text{Finite Power Series} \tag{53}$$

This is not the best way to approximate a function.
We choose the a_n's so that the power series is "close" to $f(x)$ which means we want to minimize the error.
We increase M to get a better approximation.
The problem begins when you change M, the values of a_n's change as well. Therefore, recalculating is a lot of work.
If we let $M \to \infty$ and if $f \in C^\infty$, so then $a_n = \frac{f^{(n)}(0)}{n!}$ and we get the Taylor series.
Note: C^∞: C means Continuous and the ∞ indicates the number of derivatives that are continuous.
Problem: This is only good inside the radius of convergence.

A Fourier Series is a trigonometric polynomial

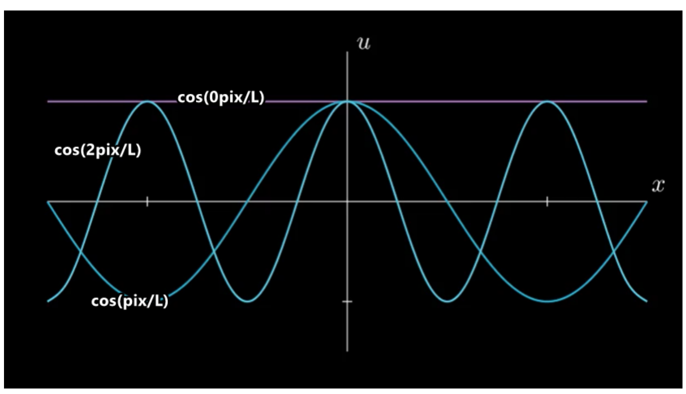
$$\sum_{n=0}^{M} a_n \sin\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right) \longleftarrow \text{period} = 2L \tag{54}$$

We use Fourier Series for a function on a bounded interval and we will use $x \in [-L, L]$

Advantages of Fourier Series

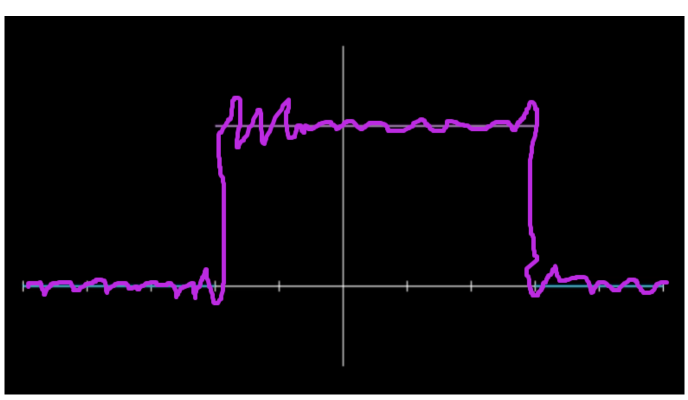
1. If M increases, we only need to calculate the new a_n's and b_n's. This property is due to the fact that the basis functions are orthogonal.
2. If $M = \infty$ and f is continuous, then the Fourier Series $= f(x) \forall x \in (-L, L)$. Our interval must be open for the case that $f(-L) \neq f(L)$.

Basis Functions : $\sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)$

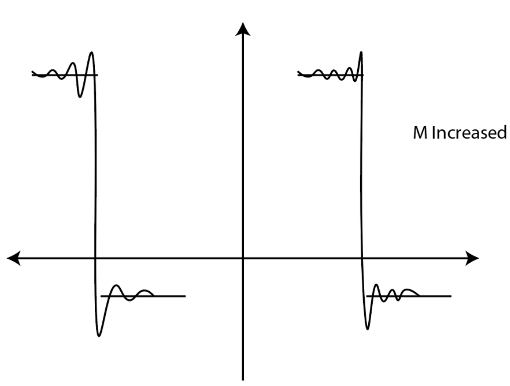


What happens if you use a Fourier Series on a discontinuous function?

$$f(x) = \begin{cases} 1 & x \in (-4, 6) \\ 0 & x \in [-10, -4] \bigcup [6, 10] \end{cases} \tag{55}$$

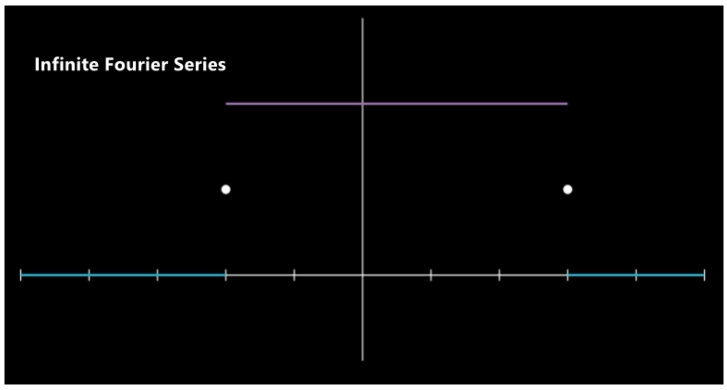


The Oscillations around the discontinuities are called Gibbs phenomenon. As M increases, the oscillation's amplitude does not change. However, the oscillations do get progressively closer to the discontinuities.



If $M = \infty$, then we have:

$$\text{Fourier Series } = \begin{cases} f(x) & = \text{ if } x \text{ is a point of continuity} \\ \lim_{c\to 0^+} \frac{f(x+c)+f(x-c)}{2} & \text{ if x is a point of discontinuity} \end{cases} \tag{56}$$



Orthogonality

Recall: The vectors

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \text{and} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{57}$$

are orthogonal if the dot product is zero.

$$u \circ v = \sum_{i=1}^{n} u_i v_i = 0 \tag{58}$$

We want to generalize this to function $x \in [-L, L]$.

Definition: Two functions $f(x)$ and $g(x)$ are orthogonal on $[a, b]$ if

$$\int_a^b f(x)g(x) \text{ dx } = 0 \tag{59}$$

Theorem: All basis functions in the Fourier Series are mutually orthogonal

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \text{ dx } = 0 \quad n \neq m \tag{60}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \text{ dx } = 0 \quad n \neq m \tag{61}$$

What happens if $m = n$?

$$\int_{-L}^{L} \sin^2\left(\frac{m\pi x}{L}\right) \text{ dx} \tag{62}$$

Here, we want to use the double angle formula: $\cos(2\theta) = 1 - 2\sin^2\theta$.

$$\begin{aligned} \int_{-L}^{L} \sin^2\left(\frac{m\pi x}{L}\right) \text{ dx } &= \frac{1}{2}\int_{-L}^{L} 1 - \cos\left(\frac{2m\pi x}{L}\right) \text{ dx} && (63) \\ &= \frac{1}{2}\Big[x - \frac{L}{2m\pi} \sin\left(\frac{2m\pi x}{L}\right)\Big]_{-L}^{L} && (64) \\ &= \frac{1}{2}\Big[L - \frac{L}{2m\pi}\sin(2m\pi) - \Big(-L - \frac{2}{2m\pi}\sin(-2m\pi)\Big)\Big] && (65) \\ &= L && (66) \end{aligned}$$