Partial Differential Equations - Class Notes

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March 7, 2022

Wave Equation Solutions

- $u_t = u_x x$
- $x \in (-\infty, \infty)$, $t \in [0, \infty)$
- $u(x,0) = f(x)$
- $u(x,t) = \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{4t}}\ \mathrm{dy}$

Let the initial condition be a "delta function," $\delta(x)$.

What is a delta function, $\delta(x)$?

It has two main properties:

1. $\delta(x) = 0$, $x \neq 0$.
2. $\displaystyle\int_{-\infty}^{\infty} \delta(x)\ \mathrm{dx}\ = 1$

The "mass" is centered at $x = 0$. The delta function is not a function because $\delta(0) =?$. Actually, the delta function is a measure.

Calculations with Delta Functions

$$\int_{-\infty}^{\infty} \delta(y) g(x-y)\ \mathrm{dy} = \int_{-\infty}^{\infty} \delta(x-y) g(y)\ \mathrm{dy} = g(x) \tag{1}$$

Here, $\delta(y)$ is zero except when $y = 0$ and $\delta(x-y) = 0$ except when $x = y$.

Here, we have a convolution $\delta * g$, where our variables can switch.

What do we expect when $f(x) = \delta(x)$?

When $t = 0$, our area is the t axis: $|$, however, as $t \to \infty$, then the area slowly flattens, akin to a candle.

Mathematically, what do we expect?

$$\begin{align} u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \delta(y) e^{-\frac{(x-y)^2}{4t}}\ \mathrm{dy} \tag{2} \\ &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \tag{3} \end{align}$$

The t's impact in the fraction reduces the amplitude and the t in the exponent flattens out the curve.

This is the Gaussian Normal Distributions

What if $f(x) = 7\delta(x) + 5\delta(x-3)$?

$$\begin{align} u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} [7\delta(y) + 5\delta(y-3)] e^{-\frac{(x-y)^2}{4t}}\ \mathrm{dy} \tag{4} \\ &= \frac{1}{\sqrt{4\pi t}} \left[\int_{-\infty}^{\infty} 7\delta(y) e^{-\frac{(x-y)^2}{4t}}\ \mathrm{dy} + \int_{-\infty}^{\infty} 5\delta(y-3) e^{-\frac{(x-y)^2}{4t}}\ \mathrm{dy} \right] \tag{5} \\ &= \frac{1}{\sqrt{4\pi t}} \left[7e^{-\frac{x^2}{4t}} + 5e^{-\frac{(x-3)^2}{4t}} \right] \tag{6} \end{align}$$

So for a general $f(x)$, think of $f(x)$ as a bunch of delta functions.