Partial Differential Equations - Class Notes

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Conservation Laws

Recall how we mentioned heat is conserved, accumulated heat is heat in - heat our.

1-D Conservation:

- $u(x,t)$: Quantity that is conserved: energy, mass, momentum, ...

$$g(x,t) = \text{ flux} = f(u(x,t)) \tag{1}$$

Here, flux is dependent on the gradient. Before, we have:

$$g(x,t) = g(u_x(x,t)) \tag{2}$$

This was our gradient.

onservation Law

$$\text{Accumulation} = \text{in - out}$$

$$\int_{x_0}^{x_1} u(x,t_1)\mathrm{d}x - \int_{x_0}^{x_1} u(x,t_0)\mathrm{d}x = u(x_0,t) \tag{3}$$

$$\int_{x_0}^{x} [u(x,t_1) - u(x,t_0)]\mathrm{d}x = \int_{t_0}^{t_1} [q(x_0,t) - q(x_1,t)]\mathrm{d}t \tag{4}$$

$$\int_{x_0}^{x_1}\int_{t_0}^{t_1} q_t(x,t)\mathrm{d}t\ \mathrm{d}x = -\int_{t_0}^{t_1}\int_{x_0}^{x_1} q_x(x,t)\mathrm{d}x\ \mathrm{d}t \tag{5}$$

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} [u_t(x,t) + u_x(x,t)]\mathrm{d}x\ \mathrm{d}t = -\int_{t_0}^{t_1}\int_{x_0}^{x_1} q_x(x,t)\mathrm{d}x\ \mathrm{d}t \tag{6}$$

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} [u_t(x,t) + q_x(x,t)]\mathrm{d}x\ \mathrm{d}t = 0 \tag{7}$$

Here, $u_t + q_x = 0$ if u_t and q_x are continuous.

Since $q = f(u)$, we get

$$u_t + [f(u)]_x = 0 \tag{8}$$

This is considered the conservation law. This is non-linear, first order.

Ex: Burger's Equation: For gas flow down a pipe.

$$u_t + \left(\frac{u^2}{2}\right)_x = \epsilon u_{xx} \tag{9}$$

Here, ϵ is the viscosity and u is the momentum. The viscosity of gases tend towards zero, therefore let us consider $\epsilon = 0$.

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{10}$$

Here, let $f(u) = \frac{u^2}{2}$:

$$u_t + uu_x = 0 \tag{11}$$

Domain: $x \in (-\infty,\infty), t \in [0,\infty)$

Initial condition: $u(x,0) = g(x)$

Recall: Transport equation: $u_t = cu_x \Rightarrow u_t - cu_x = 0$, $u(x,0) = g(x)$.