1 Wave Equation on Semi-Infinite Domain

- $x \in [0, \infty), t \in [0, \infty)$
- $u_{tt} = c^2 u_{xx}$
- $u(x, 0) = f(x)$
- $u_t(x, 0) = g(x)$
- $u(0, t) = 0$

Recall: If $x \in (-\infty, \infty)$, we use d'Alembert's Formula:

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\ \mathrm{d}y \tag{1}$$

We would like to use the solution to the wave equation for $x \in (-\infty, \infty)$ to help solve the wave equation when $x \in [0, \infty)$.

To do this, we use the odd extension of the initial conditions:

$$\widetilde{f}(x) = \begin{cases} f(x) & x > 0 \\ 0 & x = 0 \\ -f(-x) & x < 0 \end{cases} \tag{2}$$

$$\widetilde{g}(x) = \begin{cases} f(x) & x > 0 \\ 0 & x = 0 \\ -f(-x) & x < 0 \end{cases} \tag{3}$$

This system can be solved using d'Alembert's Formula:

$$u(x,t) = \frac{1}{2}\left[\widetilde{f}(x+ct) + \widetilde{f}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \widetilde{g}(y)\ \mathrm{d}y \tag{4}$$

Note: This solves some PDE on $[0, \infty)$, since it solves it on $(-\infty, \infty)$.

Note: $u(0,t) = \frac{1}{2}\left[\widetilde{f}(ct) + \widetilde{f}(-ct)\right] + \frac{1}{2}\int_{-ct}^{ct} \widetilde{g}(y)\ \mathrm{d}y$, but our integral will zero out since it is odd. In addition, since our functions are odd, the $\widetilde{f}$ will cancel out as well. **Case 1**: $x - ct > 0$

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\widetilde{f}(x+ct) + \widetilde{f}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \widetilde{g}(y)\ \mathrm{d}y && (5) \\ &= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\ \mathrm{d}y && (6) \end{aligned}$$

Staying on the right, we do not hit a wall and nothing changes.

Case 2: $x - ct < 0$

$$u(x,t) = \frac{1}{2}\left[\tilde{f}(x+ct) + \tilde{f}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{g}(y)\ \mathrm{d}y \tag{7}$$

$$= \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\left[\int_{x-ct}^{0} \tilde{g}(y)\ \mathrm{d}y + \int_{0}^{x+ct} \tilde{g}(y)\ \mathrm{d}y\right] \tag{8}$$

$$= \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\left[-\int_{x-ct}^{0} g(-y)\ \mathrm{d}y + \int_{0}^{x+ct} g(y)\ \mathrm{d}y\right] \tag{9}$$

Here, let us perform substitution with $w = -y$,

$$= \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\left[\int_{ct-x}^{0} g(w)\ \mathrm{d}w + \int_{0}^{x+ct} g(y)\ \mathrm{d}y\right] \tag{10}$$

$$= \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\left[\int_{ct-x}^{x+ct} g(y)\ \mathrm{d}y\right] \tag{11}$$

If we look at the domain of dependence, the left line reflect back to our domain and the line is represented as $ct - x$.

<u>Ex:</u> $u_{tt} = u_{xx}, x \in [0, \infty)$

$$u(x,0) = \begin{cases} 1 & 4 < x < 5 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

$$u_t(x,0) = 0 \tag{13}$$