1. Determine the solution to the heat equation $u_t = \alpha^2 u_{xx}$ with $\alpha \neq 1$ (We did the $\alpha = 1$ case in class).

Now, let us consider the following steps:

(a) Solve for $u_t = u_{xx}$

$$u_t = \alpha^2 u_{xx} \tag{1}$$
$$F[u_t] = F[\alpha^2 u_{xx}] \tag{2}$$
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t e^{-ix\xi} \, \mathrm{dx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \alpha^2 u_{xx} e^{-ix\xi} \, \mathrm{dx} \tag{3}$$
$$\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-ix\xi} \, \mathrm{dx}\right] = (i\xi)^2 \frac{\alpha^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-ix\xi} \, \mathrm{dx} \tag{4}$$
$$\frac{\partial}{\partial t}\hat{u}(\xi, t) = -\xi^2 \alpha^2 \hat{u}(\xi, t) \tag{5}$$

Our initial condition becomes:

$$F[u(x,0)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{-ix\xi} \, \mathrm{dx} \tag{6}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} \, \mathrm{dx} \tag{7}$$
$$= \hat{f}(\xi) \tag{8}$$

(b) Solve $\hat{u}_t = -\xi^2 \alpha^2 \hat{u}$,

$\hat{u}(\xi, 0) = \hat{f}(\xi)$. Here, let us write the general form of $\hat{u}$:

$$\hat{u}(\xi, t) = A(\xi) e^{-\xi^2 \alpha^2 t} \tag{9}$$

Here, let us use our initial condition to find $A(\xi)$

$$\hat{u}(\xi, 0) = \hat{f}(\xi) = A(\xi) \tag{10}$$
$$\hat{u}(\xi, t) = \hat{f}(\xi) e^{-\xi^2 \alpha^2 t} \tag{11}$$

(c) Retransform

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi, t) e^{-ix\xi} \, \mathrm{d}\xi \tag{12}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-\xi^2 t} e^{ix\xi} \, \mathrm{d}\xi \tag{13}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iy\xi} \, \mathrm{dy} \; e^{-\xi^2 t} e^{ix\xi} \, \mathrm{d}\xi \tag{14}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) e^{-iy\xi} e^{-\xi^2 t} e^{ix\xi} \, \mathrm{dy} \, \mathrm{d}\xi \tag{15}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{-iy\xi} e^{-\xi^2 t} e^{ix\xi} \, \mathrm{d}\xi \, \mathrm{dy} \tag{16}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{-\xi^2 t + i\xi(x-y)} \, \mathrm{d}\xi \, \mathrm{dy} \tag{17}$$

Here, let us focus our attention on the inner integral, where we rewrite it as Q.

$$Q = \int_{-\infty}^{\infty} e^{-\xi^2 t + i\xi(x-y)} \, \mathrm{d}\xi \tag{18}$$

Let's look at the following:

$$-\xi^2 t + i\xi(x-y) = -t\left[\xi^2 - \frac{i\xi(x-y)}{t}\right] \tag{19}$$

$$= -t\left[\left(\xi - \frac{i(x-y)}{2t}\right)^2 - \frac{i^2(x-y)^2}{4t^2}\right] \tag{20}$$

$$= -t\left[\left(\xi - \frac{i(x-y)}{2t}\right)^2 + \frac{(x-y)^2}{4t^2}\right] \tag{21}$$

Now, we have the following for Q:

$$Q = \int_{-\infty}^{\infty} e^{-t\left[\left(\xi - \frac{i(x-y)}{2t}\right)^2 + \frac{(x-y)^2}{4t^2}\right]}\ \mathrm{d}\xi \tag{22}$$

$$= \int_{-\infty}^{\infty} e^{-t\left(\xi - \frac{i(x-y)}{2t}\right)^2 - \frac{(x-y)^2}{4t}}\ \mathrm{d}\xi \tag{23}$$

$$= \int_{-\infty}^{\infty} e^{-t\left(\xi - \frac{i(x-y)}{2t}\right)^2} e^{-\frac{(x-y)^2}{4t}}\ \mathrm{d}\xi \tag{24}$$

$$= e^{-\frac{(x-y)^2}{4t}} \int_{-\infty}^{\infty} e^{-t\left(\xi - \frac{i(x-y)}{2t}\right)^2}\ \mathrm{d}\xi \tag{25}$$

Here, let us consider the following substitution:

$$w = \sqrt{t}\left(\xi - \frac{i(x-y)}{2t}\right) \tag{26}$$

$$dw = \sqrt{t}\ \mathrm{d}\xi \tag{27}$$

Now, let us write:

$$e^{-\frac{(x-y)^2}{4t}} \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-w^2}\ \mathrm{dw} = \sqrt{\frac{\pi}{t}} e^{-\frac{(x-y)^2}{4t}} \tag{28}$$

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \sqrt{\frac{\pi}{t}} e^{-\frac{(x-y)^2}{4t}}\ \mathrm{dy} \tag{29}$$

$$= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{ty}}\ \mathrm{dy} \tag{30}$$

2. Use Fourier Transforms to determine the solution to $u_t = cu_x$ is $u(x,t) = f(x+ct)$ where $u(x,0) = f(x)$.

Here, let us consider the given equation:

$$u_t = cu_x \tag{1}$$

Here, let us use Fourier Transform to transform the equation:

$$F[u_t] = F[cu_x] \tag{2}$$

Here, we are deriving the right side once. Since we're doing one derivative, we pick up one $ci\xi$.

$$\hat{u}_t = ci\xi\hat{u} \tag{3}$$

Here, let us consider the following equation:

$$u(x,0) = f(x) \tag{4}$$
$$\hat{u}(\xi,0) = \hat{f}(\xi) \tag{5}$$

Now, let us consider our equation $u_t = cu_x$. Here, recall we are picking up $ci\xi$ on the right. Let us attempt to write a general form for our equation:

$$\hat{u}_t = ci\xi\hat{u} \tag{6}$$
$$\hat{u}(\xi,t) = A(\xi)e^{ci\xi} \tag{7}$$

Here, we want to find a way to solve for our general solution. If we consider what we wrote for $hatu(\xi,0)$, we want to find:

$$\hat{(}\xi,0) = \hat{f}(\xi) = A(\xi) \tag{8}$$

So, our exponential must disappear at $t = 0$, which means we have the following:

$$\hat{u}(\xi,t) = A(\xi)e^{ic\xi t} \tag{9}$$

Here, let us rewrite $A(\xi)$ as $f(\xi)$

$$\hat{u}(\xi,t) = f(\xi)e^{ic\xi t} \tag{10}$$

Now, let us take our equation and retransform back.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(\xi,t)e^{ix\xi}\,\mathrm{d}\xi \tag{11}$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ic\xi t}e^{ix\xi}\,\mathrm{d}\xi \tag{12}$$

Here, let us combine our exponents.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)e^{i\xi(x+ct)}\,\mathrm{d}\xi \tag{13}$$

Here, we found that $u(x+t)$ is $f(x+ct) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)e^{i\xi(x+ct)}\,\mathrm{d}\xi$

3. Determine d'Alembert's solution for $u_{tt} = c^2 u_{xx}$

Here, let us consider the fourier transform on both sides of the equation:

$$F[u_{tt}] = F[c^2 u_{xx}] \tag{1}$$

Here, we are deriving u_{xx} twice, therefore:

$$\begin{align} \hat{u}_{tt} &= c^2 (i\xi)^2 \hat{u} \tag{2} \\ &= -c^2 \xi^2 \hat{u} \tag{3} \end{align}$$

Here, let us consider $\hat{u}(\xi, 0)$ and $\hat{u}_t(\xi, 0)$. Here, let us write:

$$\hat{u}(\xi, 0) = \hat{f}(\xi) \tag{4}$$

Next, let us consider $F[u_(x, 0)]$:

$$\begin{align} \hat{u}_t(\xi, 0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x, 0) e^{-ix\xi} \mathrm{d}x \tag{5} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ix\xi} \mathrm{d}x \tag{6} \\ &= \hat{g}(\xi) \tag{7} \end{align}$$

Next, let us solve for $\hat{u}_{tt} = -c^2\xi^2\hat{u}$. Recall we have $\hat{f}(\xi)$ and $\hat{g}(\xi)$. Here, we want to find the general form for our equations. Here, let us consider finding the general form for $\hat{u}$:

$$\hat{u}(\xi, t) = A(\xi) e^{ci\xi t} + B(\xi) e^{-ci\xi t} \tag{8}$$

Now, let us find $\hat{u}(\xi, 0)$

$$\hat{u}(\xi, 0) = A(\xi) + B(\xi) = \hat{f}(\xi) \tag{9}$$

Here, we found an equation for $f(\xi)$. Next, let us find the t partial and find $\hat{g}$:

$$\begin{align} \hat{u}_t(\xi, t) &= ci\xi A(\xi) e^{ci\xi t} - ci\xi B(\xi) e^{-ci\xi t} \tag{10} \\ \hat{u}_t(\xi, 0) &= ci\xi A(\xi) - ci\xi B(\xi) = \hat{g}(\xi) \tag{11} \end{align}$$

Here, let us rewrite $\hat{f}$ and $\hat{g}$:

$$\begin{align} ci\xi \hat{f}(\xi) &= ci\xi A(\xi) + ci\xi B(\xi) \tag{12} \\ \hat{g}(\xi) &= ci\xi A(\xi) - ci\xi B(\xi) \tag{13} \end{align}$$

Here, let us add these terms together,

$$\begin{align} ci\xi \hat{f}(\xi) + \hat{g}(\xi) &= 2ci\xi A(\xi) \tag{14} \\ \frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2ci\xi} &= A(\xi) \tag{15} \end{align}$$

Similarly for B:

$$ci\xi\hat{f}(\xi) - \hat{g}(\xi) = 2ci\xi B(\xi) \tag{16}$$

$$\frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2ci\xi} = B(\xi) \tag{17}$$

Next, let us substitute in for both $A(\xi)$ and $B(\xi)$ using our found equation, $\hat{u}(\xi,t)$.

$$\hat{u}(\xi,t) = \left(\frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2ci\xi}\right)e^{ci\xi t} + \left(\frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2ci\xi}\right)e^{-ci\xi t} \tag{18}$$

Now, let us retransform our equation back.

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(\xi,t)e^{ix\xi}\mathrm{d}\xi && (19)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\left(\frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2ci\xi}\right)e^{ci\xi t} + \left(\frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2ci\xi}\right)e^{-ci\xi t}\right)e^{ix\xi} && (20)\\
&= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\hat{f}(\xi)\left(e^{ci\xi t} + e^{-ci\xi t}\right)e^{ix\xi} + \frac{\hat{g}(\xi)}{ci\xi}\left(e^{ci\xi t} - e^{-ci\xi t}\right)e^{ix\xi}\right)\mathrm{d}\xi && (21)\\
&= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\hat{f}(\xi)\left(e^{i\xi(ct+x)} + e^{i\xi(x-ct)}\right) + \frac{\hat{g}(\xi)}{ci\xi}\left(e^{i\xi(ct+x)} - e^{i\xi(x-ct)}\right)\right)\mathrm{d}\xi && (22)
\end{aligned}$$

Here, note that we have the transforms for $f(x+ct)$ and $f(x-ct)$ present in our equation. Let us write:

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{g}(\xi)}{ci\xi}\left(e^{i\xi(ct+x)} - e^{i\xi(x-ct)}\right) \tag{23}$$

4. Derive d'Alembert's formula for $u_{tt} = u_{xx}$ by assuming that $u(x,t) = v(x+t, x-t) = v(y,z)$. Next show that the wave equation yields $v_{yz} = 0$ and hence $v = A(y) + B(z)$ and solve for A and B using the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

Here, let us take a look at our assumption:

$$u(x,t) = v(x+t, x-t) = v(y,z) \tag{1}$$

Here, we have the following relationship:

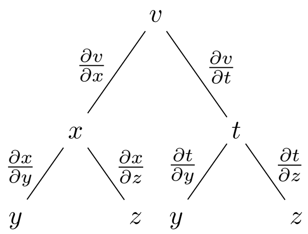
$$\begin{cases} y & = x + t \\ z & = x - t \end{cases} \tag{2}$$

Next, we want to show that the wave equation yields $v_{yz} = 0$. In orderfor the equation to be zero, that means v_y does not contain any $z's$, so when you differentiate v_y once more, then the non-z terms zero out. The same can be argued for v_z, where v_z does not contain any $y's$ and will zero out.

Therefore, v is a function of $y+z$ and we can write v as $v = A(y) + B(z)$.

Here, let us find the partial of v with respect to y and z.

Let us consider the following tree:



Here, let us consider our tree and find v_y:

$$v_y = \frac{\partial v}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y} \tag{3}$$

Earlier, we found the following system of equations:

$$\begin{cases} y & = x + t \\ z & = x - t \end{cases} \tag{4}$$

Here, notice we can isolate x or t by adding or subtracting the two equations together. First, let us add the equations to obtain the following:

$$y + z = 2x \tag{5}$$
$$x = \frac{1}{2}(y+z) \tag{6}$$

Using this information, we can find $\frac{\partial x}{\partial y}$:

$$\frac{\partial x}{\partial y} = \frac{1}{2} \tag{7}$$

Now, to find t, we subtract the equations:

$$y - z = 2t \tag{8}$$
$$t = \frac{1}{2}(y-z) \tag{9}$$

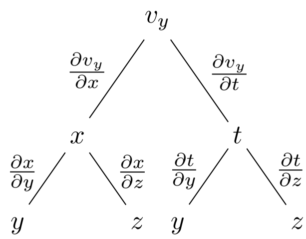
Similarly, let us find $\frac{\partial t}{\partial y}$

$$\frac{\partial t}{\partial y} = \frac{1}{2} \tag{10}$$

Here, now that we know $\frac{\partial x}{\partial y}$ and $\frac{\partial t}{\partial y}$, let us substitute this into line 3:

$$v_y = \frac{1}{2}v_x + \frac{1}{2}v_t \tag{11}$$

Now, let us find v_{yz}. Let us rewrite our tree:



Here, let us find v_{yz} using our tree.

$$\begin{aligned} v_{yz} &= \frac{\partial v_y}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial v_y}{\partial t}\frac{\partial t}{\partial z} && (12) \\ &= \frac{1}{2}v_{xx} + \frac{1}{2}v_{tx} && (13) \end{aligned}$$

Now, we still have the following terms for x and t:

$$\begin{aligned} x &= \frac{1}{2}(y+z) && (14) \\ t &= \frac{1}{2}(y-z) && (15) \end{aligned}$$

So let us substitute these back into the equation:

$$\begin{aligned} v_{yz} &= \frac{1}{2}\left(v_{xx} + v_{tx}\right) - \frac{1}{2}\left(v_{xt} + v_{tt}\right) && (16) \\ &= \frac{1}{2}v_{xx} - \frac{1}{2}v_{tt} && (17) \end{aligned}$$

Recall from our initial conditions, we have $v_{tt} = u_{xx}$. Now, we can say $v = A(y) + B(z)$ since $v_{yz} = v_{xx} - v_{tt} = 0$.

Let us reconsider what we know:

- $u(x, 0) = f(x)$
- $u_t(x, 0) = g(x)$
- $V = A(y) + B(z)$
- $u(x, t) = v(x + t, x - t)$

Here, let us plug in for $u(x, 0)$ and substitute for y and z:

$$\begin{aligned} v &= A(y) + B(z) && (18) \\ &= A(x+t) + B(x-t) && (19) \\ u(x,0) &= A(x) + B(x) = f(x) && (20) \end{aligned}$$

Now, let us find u_t, which requests for the t-partial, $u_t = v_t$. Let us use $A(x+t) + B(x-t)$ and find the derivative in terms of A and B:

$$u_t = A'(x+t) \cdot 1 - B'(x-t) \cdot 1 \tag{21}$$
$$u_t(x,0) = A'(x) - B'(x) = g(x) \tag{22}$$

Here, we found both $g(x)$ and $f(x)$:

$$f(x) = A(x) + B(x) \tag{23}$$
$$g(x) = A'(x) - B'(x) \tag{24}$$

Here, let us integrate the second line:

$$h(x) = \int g(x) \mathrm{d}x \tag{25}$$

Here, we want to integrate over an area while keeping g spanning over the entire number line.

$$h(x) = \int_{-\infty}^{x} g(y) \mathrm{d}y \tag{26}$$

Here, let us consider $g(y)$'s behavior as $-\infty$ tends to infinity. As g tends to $-\infty$, then g will head towards 0. Now, let us consider integrating $A'(x) - B'(x)$

$$A(x) - B(x) = \int_{-\infty}^{x} g(y) \mathrm{d}y \tag{27}$$

Here, recall $A(x) + B(x) = f(x)$. Here, let this function and the previous function together:

$$A(x) + B(x) = f(x) \tag{28}$$
$$A(x) - B(x) = \int_{-\infty}^{x} g(y) \mathrm{d}y \tag{29}$$

$$2A(x) = f(x) + \int_{-\infty}^{x} g(y) \mathrm{d}y \tag{30}$$
$$2B(x) = f(x) - \int_{-\infty}^{x} g(y) \mathrm{d}y \tag{31}$$

Here, we want to plug our previous two functions into v. See how $A(x+t) + B(x-t)$ has $x+t$ and $x-t$. Let us swap these into our new functions:

$$v = \frac{1}{2}\left(f(x+t) + \int_{-\infty}^{x+t} g(y)\right) \mathrm{d}y + \frac{1}{2}\left(f(x-t) - \int_{-\infty}^{x-t} g(y) \mathrm{d}y\right) \tag{32}$$
$$v = \frac{1}{2}\left(f(x+t) + \int_{-\infty}^{x+t} g(y)\right) \mathrm{d}y + \frac{1}{2}\left(f(x-t) + \int_{x-t}^{-\infty} g(y) \mathrm{d}y\right) \tag{33}$$

Here, notice how our integrations are similar. The bottom and top intervals are the same respectively, we can combine them:

$$v = \frac{1}{2}\left(f(x-t) + f(x+t) + \int_{x-t}^{x+t} g(y) \mathrm{d}y\right) \tag{34}$$

Here, we have derive D'Alembert's Formula