1. Determine the solution to the heat equation $u_t = \alpha^2 u_{xx}$ with $\alpha \neq 1$ (We did the $\alpha = 1$ case in class).

Now, let us consider the following steps:

(a) Solve for $u_t = u_{xx}$

$$u_t = \alpha^2 u_{xx} \tag{1}$$
$$F[u_t] = F[\alpha^2 u_{xx}] \tag{2}$$
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t e^{-ix\xi} \text{ dx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \alpha^2 u_{xx} e^{-ix\xi} \text{ dx} \tag{3}$$
$$\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-ix\xi} \text{ dx}\right] = (i\xi)^2 \frac{\alpha^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-ix\xi} \text{ dx} \tag{4}$$
$$\frac{\partial}{\partial t}\hat{u}(\xi, t) = -\xi^2\alpha^2\hat{u}(\xi, t) \tag{5}$$

Our initial condition becomes:

$$F[u(x,0)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{-ix\xi} \text{ dx} \tag{6}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} \text{ dx} \tag{7}$$
$$= \hat{f}(\xi) \tag{8}$$

(b) Solve $\hat{u}_t = -\xi^2\alpha^2\hat{u}$,

$\hat{u}(\xi, 0) = \hat{f}(\xi)$. Here, let us write the general form of $\hat{u}$:

$$\hat{u}(\xi, t) = A(\xi)e^{-\xi^2 t} \tag{9}$$

Here, let us use our initial condition to find $A(\xi)$

$$\hat{u}(\xi, 0)\hat{f}(\xi) = A(\xi) \tag{10}$$
$$\hat{u}(\xi, t) = \hat{f}(\xi)e^{-\xi^2 t} \tag{11}$$

(c) Retransform

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi, t) e^{-ix\xi} \text{ d}\xi \tag{12}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-\xi^2 t} e^{ix\xi} \text{ d}\xi \tag{13}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iy\xi} \text{ dy } e^{-\xi^2 t} e^{ix\xi} \text{ d}\xi \tag{14}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) e^{-iy\xi} e^{-\xi^2 t} e^{ix\xi} \text{ dy d}\xi \tag{15}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{-iy\xi} e^{-\xi^2 t} e^{ix\xi} \text{ d}\xi \text{ dy} \tag{16}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{-\xi^2 t + i\xi(x-y)} \text{ d}\xi \text{ dy} \tag{17}$$

Here, let us focus our attention on the inner integral, where we rewrite it as Q.

$$Q = \int_{-\infty}^{\infty} e^{-\xi^2 t + i\xi(x-y)} \text{ d}\xi \tag{18}$$

Let's look at the following:

$$
\begin{aligned}
-\xi^2 t + i\xi(x-y) &= -t\left[\xi^2 - \frac{i\xi(x-y)}{t}\right] && (19)\\
&= -t\left[\left(\xi - \frac{i(x-y)}{2t}\right)^2 - \frac{i^2(x-y)^2}{4t^2}\right] && (20)\\
&= -t\left[\left(\xi - \frac{i(x-y)}{2t}\right)^2 + \frac{(x-y)^2}{4t^2}\right] && (21)
\end{aligned}
$$

Now, we have the following for Q:

$$
\begin{aligned}
Q &= \int_{-\infty}^{\infty} e^{-t\left[\left(\xi-\frac{i(x-y)}{2t}\right)^2+\frac{(x-y)^2}{4t^2}\right]}\,\mathrm{d}\xi && (22)\\
&= \int_{-\infty}^{\infty} e^{-t\left(\xi-\frac{i(x-y)}{2t}\right)^2-\frac{(x-y)^2}{4t}}\,\mathrm{d}\xi && (23)\\
&= \int_{-\infty}^{\infty} e^{-t\left(\xi-\frac{i(x-y)}{2t}\right)^2} e^{-\frac{(x-y)^2}{4t}}\,\mathrm{d}\xi && (24)\\
&= e^{-\frac{(x-y)^2}{4t}} \int_{-\infty}^{\infty} e^{-t\left(\xi-\frac{i(x-y)}{2t}\right)^2}\,\mathrm{d}\xi && (25)
\end{aligned}
$$

Here, let us consider the following substitution:

$$
\begin{aligned}
w &= \sqrt{t}\left(\xi - \frac{i(x-y)}{2t}\right) && (26)\\
dw &= \sqrt{t}\,\mathrm{d}\xi && (27)
\end{aligned}
$$

Now, let us write:

$$
\begin{aligned}
e^{-\frac{(x-y)^2}{4t}}\frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-w^2}\,\mathrm{dw} &= \sqrt{\frac{\pi}{t}}e^{-\frac{(x-y)^2}{4t}} && (28)\\
u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\sqrt{\frac{\pi}{t}}e^{-\frac{(x-y)^2}{4t}}\,\mathrm{dy} && (29)\\
&= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(y)e^{-\frac{(x-y)^2}{ty}}\,\mathrm{dy} && (30)
\end{aligned}
$$

2. Use Fourier Transforms to determine the solution to $u_t = cu_x$ is $u(x,t) = f(x+ct)$ where $u(x,0) = f(x)$.

Here, let us consider the given equation:

$$u_t = cu_x \tag{1}$$

Here, let us use Fourier Transform to transform the equation:

$$F[u_t] = F[cu_x] \tag{2}$$

Here, we are deriving the right side once. Since we're doing one derivative, we pick up one $ci\xi$.

$$\hat{u}_t = ci\xi\hat{u} \tag{3}$$

Here, let us consider the following equation:

$$u(x,0) = f(x) \tag{4}$$
$$\hat{u}(\xi,0) = \hat{f}(\xi) \tag{5}$$

Now, let us consider our equation $u_t = cu_x$. Here, recall we are picking up $ci\xi$ on the right. Let us attempt to write a general form for our equation:

$$\hat{u}_t = ci\xi\hat{u} \tag{6}$$
$$\hat{u}(\xi,t) = A(\xi)e^{ci\xi} \tag{7}$$

Here, we want to find a way to solve for our general solution. If we consider what we wrote for $hatu(\xi,0)$, we want to find:

$$\hat{}(\xi,0) = \hat{f}(\xi) = A(\xi) \tag{8}$$

So, our exponential must disappear at $t = 0$, which means we have the following:

$$\hat{u}(\xi,t) = A(\xi)e^{ic\xi t} \tag{9}$$

Here, let us rewrite $A(\xi)$ as $f(\xi)$

$$\hat{u}(\xi,t) = f(\xi)e^{ic\xi t} \tag{10}$$

Now, let us take our equation and retransform back.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(\xi,t)e^{ix\xi}\ \mathrm{d}\xi \tag{11}$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ic\xi t}e^{ix\xi}\ \mathrm{d}\xi \tag{12}$$

Here, let us combine our exponents.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)e^{i\xi(x+ct)}\ \mathrm{d}\xi \tag{13}$$

3. Determine d'Alembert's solution for $u_{tt} = c^2 u_{xx}$

4. Derive d'Alembert's formula for $u_{tt} = u_{xx}$ by assuming that $u(x,t) = v(x+t, x-t) = v(y,z)$. Next show that the wave equation yields $v_{yz} = 0$ and hence $v = A(y) + B(z)$ and solve for A and B using the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

Here, let us take a look at our assumption:

$$u(x,t) = v(x+t, x-t) = v(y,z) \tag{1}$$

Here, we have the following relationship:

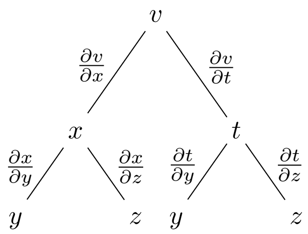
$$\begin{cases} y &= x+t \\ z &= x-t \end{cases} \tag{2}$$

Next, we want to show that the wave equation yields $v_{yz} = 0$. In orderfor the equation to be zero, that means v_y does not contain any $z's$, so when you differentiate v_y once more, then the non-z terms zero out. The same can be argued for v_z, where v_z does not contain any $y's$ and will zero out.

Therefore, v is a function of $y+z$ and we can write v as $v = A(y) + B(z)$.

Here, let us find the partial of v with respect to y and z.

Let us consider the following tree:



Here, let us consider our tree and find v_y:

$$v_y = \frac{\partial v}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y} \tag{3}$$

Earlier, we found the following system of equations:

$$\begin{cases} y &= x+t \\ z &= x-t \end{cases} \tag{4}$$

Here, notice we can isolate x or t by adding or subtracting the two equations together. First, let us add the equations to obtain the following:

$$y + z = 2x \tag{5}$$
$$x = \frac{1}{2}(y+z) \tag{6}$$

Using this information, we can find $\frac{\partial x}{\partial y}$:

$$\frac{\partial x}{\partial y} = \frac{1}{2} \tag{7}$$

Now, to find t, we subtract the equations:

$$y - z = 2t \tag{8}$$
$$t = \frac{1}{2}(y-z) \tag{9}$$

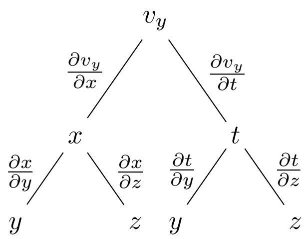
Similarly, let us find $\frac{\partial t}{\partial y}$

$$\frac{\partial t}{\partial y} = \frac{1}{2} \tag{10}$$

Here, now that we know $\frac{\partial x}{\partial y}$ and $\frac{\partial t}{\partial y}$, let us substitute this into line 3:

$$v_y = \frac{1}{2}v_x + \frac{1}{2}v_t \tag{11}$$

Now, let us find v_{yz}. Let us rewrite our tree:



Here, let us find v_{yz} using our tree.

$$\begin{align} v_{yz} &= \frac{\partial v_y}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial v_y}{\partial t}\frac{\partial t}{\partial z} \tag{12} \\ &= \frac{1}{2}v_{xx} + \frac{1}{2}v_{tx} \tag{13} \end{align}$$

Now, we still have the following terms for x and t:

$$\begin{align} x &= \frac{1}{2}(y+z) \tag{14} \\ t &= \frac{1}{2}(y-z) \tag{15} \end{align}$$

So let us substitute these back into the equation:

$$\begin{align} v_{yz} &= \frac{1}{2}\left(v_{xx} + v_{tx}\right) - \frac{1}{2}\left(v_{xt} + v_{tt}\right) \tag{16} \\ &= \frac{1}{2}v_{xx} - \frac{1}{2}v_{tt} \tag{17} \end{align}$$

Recall from our initial conditions, we have $v_{tt} = u_{xx}$. Now, we can say $v = A(y) + B(z)$ since $v_{yz} = v_{xx} - v_{tt} = 0$.

Let us reconsider what we know:

- $u(x, 0) = f(x)$
- $u_t(x, 0) = g(x)$
- $V = A(y) + B(z)$
- $u(x, t) = v(x + t, x - t)$

Here, let us plug in for $u(x, 0)$ and substitute for y and z:

$$\begin{align} v &= A(y) + B(z) \tag{18} \\ &= A(x+t) + B(x-t)u(x,0) & &= A(x) + B(x) = f(x) \tag{19} \end{align}$$

Now, let us find u_t, which requests for the t-partial, $u_t = v_t$. Let us use $A(x+t) + B(x-t)$ and find the derivative in terms of A and B:

$$u_t = A'(x+t) \cdot 1 - B'(x-t) \cdot 1 \tag{20}$$

$$u_t(x,0) = A'(x) - B'(x) = g(x) \tag{21}$$

Here, we found both $g(x)$ and $f(x)$:

$$f(x) = A(x) + B(x) \tag{22}$$

$$g(x) = A'(x) - B'(x) \tag{23}$$