1. Prove the maximum principle using the Mean Value Theorem(s). If $\Delta u = 0$ on a bounded domain Ω, show that

$$\max_{x \in \Omega} u(x) = \max_{x \in \partial\Omega} u(x) \tag{1}$$

In other words, the max of a harmonic function is attained on its boundary. Hint: Use proof by contradiction.

Here, let us consider our given statement from line 1. We are given the assumption that for a point in a given area, Ω, there is a point that gives us the max $u(\text{point})$.

Here, let us consider the area Ω, where Ω is any closed area. For the sake of coding these graphs, let us assume our boundary assumes the shape of a circle

$$+\ \Omega$$

Now, let us consider a point within this interval, x_0. Here, let us consider point x_0 to be an arbitrary point within our area, Ω:

To begin, let us consider a neighborhood, or ball, around our point, x_0. Here, our ball around x_0, has a radius ϵ, where $\epsilon > 0$, and the area of our ball is a subset of Ω.

Now, let us consider what is $u(x)$. Here, $u(x)$ is a harmonic function, allowing us to make the assumption that the average value of the function within its neighborhood is the center point of our neighborhood. In this case, our neighborhood is centered about x_0.

Here, let us consider a function where the average value of its neighborhood is also the maximum value of our harmonic function, $u(x)$. Here, if our neighborhood's average value is always the maximum value throughout the region, then our harmonic function's value is uniform throughout the region. In other words, there is a constant value throughout the region.

While the average of our neighborhood is considered the max in a constant function as shown, a function like this will not always occur. Here,