1. Let $f(x)$ be a 2π-period function on the interval $[-\pi, \pi]$ where $f(x) = \begin{cases} -1 & -\pi < x \le 0 \\ 1 & 0 < x \le \pi \end{cases}$

   (a) Plot the function on the interval $[-3\pi, 3\pi]$
   (b) Plot its (infinite) Fourier series on $[-3\pi, 3\pi]$
   (c) Find the Fourier series of $f(x)$

2. Let $f(x) = x^2$ be a 2π-periodic function on the interval $[-\pi, \pi]$.

   (a) Derive its Fourier series
   (b) Use Maple of Matlab to plot its finite Fourier series on $[-\pi, \pi]$ for $N = 10, 20, 50$ together with $f(x)$
   (c) Use your Fourier series from part (a) to show that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$

   Let us begin

3. In the solution of the heat equation, we end up solving $X'' = -\lambda X$. Show that if $\lambda < 0$ or $\lambda = 0$ there is only the trivial solution ($X(x) = 0$).

   Here, we have the equation:

$$X'' = -\lambda X \tag{1}$$

   We want to use this equation and set our boundary conditions as $X(0) = X(L) = 0$. Now, we must find an equation where after two derivatives on the right, we obtain a similar function on the left. On the left, we have a sign, coefficient, and function of x. Let us write a general solution for our equation:

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \tag{2}$$

   Here, we can make three assumptions ***via trichotomy***: $\lambda < 0, \lambda = 0$, or $\lambda > 0$. Let us look at the first two examples:

   (a) $\lambda < 0$

   Here, let us consider the case when λ is negative. Let us consider rewriting λ:

$$\lambda < 0 \tag{3}$$
$$\lambda \cdot -1 > 0 \cdot -1 \tag{4}$$
$$-1 \cdot \lambda > 0 \tag{5}$$

   Now, let us plug in our found value into our general equation:

$$X(x) = A\cos(\sqrt{-1 \cdot \lambda}x) + B\sin(\sqrt{-1 \cdot \lambda}x) \tag{6}$$

   Let us separate the terms under the radical:

$$X(x) = A\cos(\sqrt{-1 \cdot \lambda}x) + B\sin(\sqrt{-1 \cdot \lambda}x) \tag{7}$$
$$= A\cos(\sqrt{-1}\sqrt{\lambda}x) + B\sin(\sqrt{-1}\sqrt{\lambda}x) \tag{8}$$
$$= A\cos(i\sqrt{\lambda}x) + B\sin(i\sqrt{\lambda}x) \tag{9}$$

   Here, in our expression, we see we are taking the square root of a negative number, which would give us an imaginary number. Here, we are evaluating our general solution with real numbers, therefore, the following form:

$$X(x) = A\cos(i\sqrt{\lambda}x) + B\sin(i\sqrt{\lambda}x) \tag{10}$$

   Where $X(x)$ is a real number would only have the trivial solution $X(x) = 0$.

   (b) $\lambda = 0$

   Here, let us consider the case when λ is zero. Now, let us write our general equation:

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \tag{11}$$

   Here, since $\lambda = 0$, we can evaluate our equation:

$$X(x) = A\cos(0) + B\sin(0) \tag{12}$$
$$= A \tag{13}$$

   Now, let us evaluate our boundary condition for $X(x) = A$. First, we let $X(0) = 0$:

$$X(0) = 0 = A \tag{14}$$

   Here, we know A is 0. For the second condition, let us write:

$$X(L) = 0 = A \tag{15}$$

   Here, we will always have the trivial solution, $X(x) = 0$.

4. Show that $u(x,t) = e^{-\lambda^2 a^2 t} [A\cos(\lambda x) + B\sin(\lambda x)]$
5. Solve $u_t = u_{xx}$ given $u(0,t) = u(1,t) = 0$ for $t \geq 0$ and $u(x,0) = 1$ for $0 \leq x \leq 1$
6. Find the solution to the previous problem if $u(x,0) = x - x^2$ for $0 \leq x \leq 1$
7. Solve $u_t = u_{xx}$ given $u(0,t) = u(1,t) = 0$ for $t \geq 0$ and $u(x,0) = 10^{-5}\sin(10^6 \pi x)$ for $0 \leq x \leq 1$. Determine $u(x,2)$ and $u(x,-2)$ and look at their magnitudes. Note that when $t = -2$, we are looking at the backward heat equation and given the magnitude of $u(x,-2)$, what can you say about the solution to the backward heat equation?