1. Solve $\Delta u = 0$ on $x \in [0, 4]$, $y \in [0, 3]$, with $u(x, 0) = u(x, 3) = 0$ where

    (a) $u_x(0, y) = 0$ and $u_x(4, y) = \cos(\pi y)$

    (b) $u(0, y) = 1$ and $u(4, y) = 0$ (Hint: Translate the x coordinate so that $u(0, y) = 0$. This means that x goes between -4 and 0. Now go back to 0 to 4 with a y-axis flip.)

Here, let us first solve part a) of the problem.

    (a) First, let us assume our equation is separable:

$$u(x, y) = X(x)Y(y) \tag{1}$$

Let us take note of our u:

$$u_{xx} + u_{yy} = 0 \tag{2}$$
$$X''Y + XY'' = 0 \tag{3}$$
$$XY'' = -X''Y \tag{4}$$
$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda \tag{5}$$

Here, let us consider our boundary conditions:

$$u(x, 0) = 0 = X(0)Y(y) \tag{6}$$

Here, we either have $X(0) = 0$ or $Y(y) = 0$. We do not want to assume $Y(y) = 0$ because that would make $u(x, y) = 0 \; \forall y \in [0, 3]$. Therefore, we consider $X(0) = 0$. We will follow this assumption for the upcoming boundary conditions.

$$u(x, y) = X(x)Y(y) \tag{7}$$
$$u(x, 0) = X(x)Y(0) = 0 \Rightarrow Y(0) = 0 \tag{8}$$
$$u(x, 3) = X(x)Y(3) = 0 \Rightarrow Y(3) = 0 \tag{9}$$

Now, let us consider $u_x(x, y)$ as $X'(x)Y(y)$:

$$u_x(x, y) = X'(0)Y(y) \tag{10}$$
$$u_x(0, y) = X'(0)Y(y) = 0 \Rightarrow X'(0) = 0 \tag{11}$$

Now, let us write:

$$-\frac{X''}{X} = \frac{Y''}{Y} = -\lambda \tag{12}$$

    (b) Next we solve for $Y(y)$ since we have more information on $Y(y)$.

$$\frac{Y''}{Y} = -\lambda \tag{13}$$
$$Y'' = -\lambda Y \tag{14}$$

Here, let us write out our general equation:

$$Y_n(y) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \tag{15}$$

Here, we know $Y(0) = Y(3) = 0$. Let us input our y:

$$Y_n(0) = 0 = B \tag{16}$$
$$Y_n(3) = 0 = A\sin(\sqrt{\lambda}3) \tag{17}$$

Here, let us not consider A as 0, instead, the inside of our sine function to be 0.

$$\sqrt{\lambda}3 = n\pi \tag{18}$$
$$\sqrt{\lambda} = \frac{n\pi}{3} \tag{19}$$
$$\lambda_n = \left(\frac{n\pi}{3}\right)^2 \tag{20}$$

Now we have the following:

$$Y_n(y) = \sin\left(\frac{n\pi y}{3}\right) \tag{21}$$

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(c) Now, let us solve for X:

$$\frac{X''}{X} = \lambda \tag{22}$$

$$X'' = \lambda X \tag{23}$$

Here, we solved for λ in the previous step:

$$X_n'' = \left(\frac{n\pi}{3}\right)^2 X_n \tag{24}$$

$$X_n(x) = C \sinh\left(\frac{n\pi x}{3}\right) + D \cosh\left(\frac{n\pi x}{3}\right) \tag{25}$$

Here, let us find the derivative for X and find $X'(0)$:

$$X'(0) = \frac{n\pi x}{3} C = 0 \tag{26}$$

$$= C = 0 \tag{27}$$

Therefore, since $C = 0$, we get:

$$X_n(x) = D \cosh\left(\frac{n\pi x}{3}\right) \tag{28}$$

(d) Now, let us combine u_n and u:

$$u_n(x, y) = D \cosh\left(\frac{n\pi x}{3}\right) \sin\left(\frac{n\pi y}{3}\right) \tag{29}$$

By linearity, let us write:

$$u(x, y) = \sum_{n=1}^{\infty} D_n \cosh\left(\frac{n\pi x}{3}\right) \sin\left(\frac{n\pi y}{3}\right) \tag{30}$$

(e) Here, let us use our boundary conditions to find our coefficients.

$$u_x(4, y) = \cos(\pi y) \tag{31}$$

Here, let us derive our $u(x, y)$ to get $u_x(x, y)$

$$u_x(x, y) = \sum_{n=1}^{\infty} D_n \left(\frac{n\pi}{3}\right) \sinh\left(\frac{n\pi x}{3}\right) \sin\left(\frac{n\pi y}{3}\right) \tag{32}$$

$$u_x(4, y) = \sum_{n=1}^{\infty} D_n \left(\frac{n\pi}{3}\right) \sinh\left(\frac{n\pi 4}{3}\right) \sin\left(\frac{n\pi y}{3}\right) \tag{33}$$

Here, let us consider our coefficient in the equation:

$$D_n \left(\frac{n\pi}{3}\right) \sinh\left(\frac{4\pi n}{3}\right) = \frac{2}{3} \int_0^3 \cos(\pi y) \sin\left(\frac{n\pi y}{3}\right) dy \tag{34}$$

$$D_n = \frac{2}{3} \frac{3}{n\pi} \frac{1}{\sinh\left(\frac{4\pi n}{3}\right)} \int_0^3 \cos(\pi y) \sin\left(\frac{n\pi y}{3}\right) \tag{35}$$

$$= \frac{2}{n\pi \sinh\left(\frac{4\pi n}{3}\right)} \int_0^3 \cos(\pi y) \sin\left(\frac{n\pi y}{3}\right) \tag{36}$$

Here, let us isolate our integral and solve for that. Afterwards, let us combine our function.

$$\int_0^3 \cos(\pi y) \sin\left(\frac{n\pi y}{3}\right) \tag{37}$$

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2. Solve $u_{tt} = 4u_{xx}$ on $x \in [0,3]$, $t \in [0, \infty)$, with $u(0,t) = u(3,t) = 0$ where

  (a) $u(x,0) = 4\sin(2\pi x) + 7\sin(6\pi x) - 2\sin(\pi x)$, $u_t(x,0) = 0$

  (b) $u(x,0) = x(3 - x)$, $u_t(x,0) = \sin(\pi x)$

Let us begin with our boundary conditions.

  (a) Let us assume our equation is separable.

$$u_{tt} = 2^2 u_{xx} \tag{38}$$

$$XT'' = 2^2 X''T \tag{39}$$

$$\frac{T''}{2^2 T} = \frac{X''}{X} = -\lambda \tag{40}$$

  (b) Here, since we have more information regarding X, let us solve for X:

$$\frac{X''}{X} = -\lambda \tag{41}$$

$$X'' = -\lambda X \tag{42}$$

In this equation, we want to consider cos and sin for our general equation.

$$X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \tag{43}$$

Here, we know $X(0) = X(3) = 0$, so let us write:

$$X(0) = B = 0 \tag{44}$$

$$X(3) = A\sin(\sqrt{\lambda}3) = 0 \tag{45}$$

Here, let us find when our sine function is 0.

$$\sqrt{\lambda}3 = n\pi \tag{46}$$

$$\sqrt{\lambda} = \frac{n\pi}{3} \tag{47}$$

$$\lambda = \left(\frac{n\pi}{3}\right)^2 \tag{48}$$

Now, let us write our general equation for X:

$$X_n(x) = \sin\left(\frac{n\pi x}{3}\right) \tag{49}$$

  (c) Next, let us solve for T:

$$\frac{T''}{2^2 T} = -\lambda \tag{50}$$

$$T'' = -2^2 T\lambda \tag{51}$$

Here, let us write the general equation:

$$T_n(t) = C_n \sin\left(\frac{2n\pi t}{L}\right) + D_n \cos\left(\frac{2n\pi t}{L}\right) \tag{52}$$

  (d) Here, let us combine to find u_n and u:

$$u_n(x,t) = \sin\left(\frac{n\pi x}{3}\right)\left[C_n \sin\left(\frac{2n\pi t}{L}\right) + D_n \cos\left(\frac{2n\pi t}{L}\right)\right] \tag{53}$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{3}\right)\sin\left(\frac{2n\pi t}{L}\right) + D_n \sin\left(\frac{n\pi x}{3}\right)\cos\left(\frac{2n\pi t}{L}\right) \tag{54}$$

  (e) Here, let us find the coefficients using the Initial Conditions:
- $u(x,0) = 4\sin(2\pi x) + 7\sin(6\pi x) - 2\sin(\pi x)$
- $u_t(x,0) = 0$

Using this information, find our t partial from our summation series:

$$u_t(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{3}\right)\frac{2n\pi}{L}\cos\left(\frac{2n\pi t}{L}\right) - D_n \sin\left(\frac{n\pi x}{3}\right)\frac{2n\pi}{L}\sin\left(\frac{2n\pi t}{L}\right) \tag{55}$$

From here, let us plug in our initial condition, $u_t(x,0) = 0$:

$$u_t(x,0) = \sum_{n=1}^{\infty} C_n \frac{2n\pi}{L}\sin\left(\frac{n\pi x}{3}\right) = 0 \tag{56}$$

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