PDE Homework 6

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1. Determine the solution to the heat equation $u_t = \alpha^2 u_{xx}$ with $\alpha \neq 1$ (We did the $\alpha = 1$ case in class).

Now, let us consider the following steps:

(a) Solve for $u_t = u_{xx}$

$$u_t = \alpha^2 u_{xx} \tag{1}$$
$$F[u_t] = F[\alpha^2 u_{xx}] \tag{2}$$
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t e^{-ix\xi} \text{ dx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \alpha^2 u_{xx} e^{-ix\xi} \text{ dx} \tag{3}$$
$$\frac{\partial}{\partial t} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-ix\xi} \text{ dx} \right] = (i\xi)^2 \frac{\alpha^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-ix\xi} \text{ dx} \tag{4}$$
$$\frac{\partial}{\partial t} \hat{u}(\xi, t) = -\xi^2 \alpha^2 \hat{u}(\xi, t) \tag{5}$$

Our initial condition becomes:

$$F[u(x,0)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{-ix\xi} \text{ dx} \tag{6}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} \text{ dx} \tag{7}$$
$$= \hat{f}(\xi) \tag{8}$$

(b) Solve $\hat{u}_t = -\xi^2 \alpha^2 \hat{u}$,

$\hat{u}(\xi, 0) = \hat{f}(\xi)$. Here, let us write the general form of $\hat{u}$:

$$\hat{u}(\xi, t) = A(\xi) e^{-\xi^2 \alpha^2 t} \tag{9}$$

Here, let us use our initial condition to find $A(\xi)$

$$\hat{u}(\xi, 0) = \hat{f}(\xi) = A(\xi) \tag{10}$$
$$\hat{u}(\xi, t) = \hat{f}(\xi) e^{-\xi^2 \alpha^2 t} \tag{11}$$

(c) Retransform

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi,t) e^{ix\xi} \mathrm{d}\xi \tag{12}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\hat{f}(\xi) e^{-\xi^2\alpha^2 t}\right) e^{ix\xi} \mathrm{d}\xi \tag{13}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iy\xi} \mathrm{d}y\right] e^{-\xi^2\alpha^2 t}\right) e^{ix\xi} \mathrm{d}\xi \tag{14}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{-iy\xi} e^{-\xi^2\alpha^2 t} e^{ix\xi} \ \mathrm{d}\xi \ \mathrm{d}y \tag{15}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{-iy\xi - \xi^2\alpha^2 t + ix\xi} \ \mathrm{d}\xi \ \mathrm{d}y \tag{16}$$

Here, let us rename our inner integral:

$$Q = \int_{-\infty}^{\infty} e^{-iy\xi - \xi^2\alpha^2 t + ix\xi} \ d\xi \tag{17}$$

Here, let us take a look at the exponent in our equation. Let us pull out a $-\alpha^2 t$ and complete the square:

$$-iy\xi - \xi^2\alpha^2 t + ix\xi = -\alpha^2 t\left(\left(\xi - \frac{i(x-y)}{2\alpha^2 t}\right)^2 - \frac{i^2(x-y)^2}{4\alpha^4 t^2}\right) \tag{18}$$

Now, let us substitute back into Q

$$Q = \int_{-\infty}^{\infty} e^{-\alpha^2 t\left(\left(\xi - \frac{i(x-y)}{2\alpha^2 t}\right)^2 - \frac{i^2(x-y)^2}{4\alpha^4 t^2}\right)} \ d\xi \tag{19}$$

$$= \int_{-\infty}^{\infty} e^{-\alpha^2 t\left(\xi - \frac{i(x-y)}{2\alpha^2 t}\right)^2 - \frac{(x-y)^2}{4\alpha^2 t}} \ d\xi \tag{20}$$

Notice that part of our exponent has terms that does not belong to the intergral, therefore let us separate and evaluate the integral:

$$Q = \int_{-\infty}^{\infty} e^{-\alpha^2 t\left(\xi - \frac{i(x-y)}{2\alpha^2 t}\right)^2} e^{-\frac{(x-y)^2}{4\alpha^2 t}} \mathrm{d}\xi \tag{21}$$

$$= e^{-\frac{(x-y)^2}{4\alpha^2 t}} \int_{-\infty}^{\infty} e^{-\alpha^2 t\left(\xi - \frac{i(x-y)}{2\alpha^2 t}\right)^2} \mathrm{d}\xi \tag{22}$$

$$= e^{-\frac{(x-y)^2}{4\alpha^2 t}} \tag{23}$$

Now, let us return Q to our equation:

$$u(x,t) = \frac{1}{2} \int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{4\alpha^2 t}} \tag{24}$$

2. Use Fourier Transforms to determine the solution to $u_t = cu_x$ is $u(x,t) = f(x+ct)$ where $u(x,0) = f(x)$.

Here, let us consider the given equation:

$$u_t = cu_x \tag{1}$$

Here, let us use Fourier Transform to transform the equation:

$$F[u_t] = F[cu_x] \tag{2}$$

Here, we are deriving the right side once. Since we're doing one derivative, we pick up one $ci\xi$.

$$\hat{u}_t = ci\xi\hat{u} \tag{3}$$

Here, let us consider the following equation:

$$u(x,0) = f(x) \tag{4}$$
$$\hat{u}(\xi,0) = \hat{f}(\xi) \tag{5}$$

Now, let us consider our equation $u_t = cu_x$. Here, recall we are picking up $ci\xi$ on the right. Let us attempt to write a general form for our equation:

$$\hat{u}_t = ci\xi\hat{u} \tag{6}$$
$$\hat{u}(\xi,t) = A(\xi)e^{ci\xi} \tag{7}$$

Here, we want to find a way to solve for our general solution. If we consider what we wrote for $hatu(\xi,0)$, we want to find:

$$\hat{(}\xi,0) = \hat{f}(\xi) = A(\xi) \tag{8}$$

So, our exponential must disappear at $t = 0$, which means we have the following:

$$\hat{u}(\xi,t) = A(\xi)e^{ic\xi t} \tag{9}$$

Here, let us rewrite $A(\xi)$ as $f(\xi)$

$$\hat{u}(\xi,t) = f(\xi)e^{ic\xi t} \tag{10}$$

Now, let us take our equation and retransform back.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(\xi,t)e^{ix\xi}\ \mathrm{d}\xi \tag{11}$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ic\xi t}e^{ix\xi}\ \mathrm{d}\xi \tag{12}$$

Here, let us combine our exponents.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)e^{i\xi(x+ct)}\ \mathrm{d}\xi \tag{13}$$

Here, we found that $u(x+t)$ is $f(x+ct) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)e^{i\xi(x+ct)}\ \mathrm{d}\xi$

3. Determine d'Alembert's solution for $u_{tt} = c^2 u_{xx}$

Here, let us consider the fourier transform on both sides of the equation:

$$F[u_{tt}] = F[c^2 u_{xx}] \tag{1}$$

Here, we are deriving u_{xx} twice, therefore:

$$\begin{aligned} \hat{u}_{tt} &= c^2 (i\xi)^2 \hat{u} \qquad (2) \\ &= -c^2 \xi^2 \hat{u} \qquad (3) \end{aligned}$$

Here, let us consider $\hat{u}(\xi, 0)$ and $\hat{u}_t(\xi, 0)$. Here, let us write:

$$\hat{u}(\xi, 0) = \hat{f}(\xi) \tag{4}$$

Next, let us consider $F[u_(x, 0)]$:

$$\begin{aligned} \hat{u}_t(\xi, 0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x, 0) e^{-ix\xi} \mathrm{d}x \qquad (5) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ix\xi} \mathrm{d}x \qquad (6) \\ &= \hat{g}(\xi) \qquad (7) \end{aligned}$$

Next, let us solve for $\hat{u}_{tt} = -c^2 \xi^2 \hat{u}$. Recall we have $\hat{f}(\xi)$ and $\hat{g}(\xi)$. Here, we want to find the general form for our equations. Here, let us consider finding the general form for $\hat{u}$:

$$\hat{u}(\xi, t) = A(\xi) e^{ci\xi t} + B(\xi) e^{-ci\xi t} \tag{8}$$

Now, let us find $\hat{u}(\xi, 0)$

$$\hat{u}(\xi, 0) = A(\xi) + B(\xi) = \hat{f}(\xi) \tag{9}$$

Here, we found an equation for $f(\xi)$. Next, let us find the t partial and find $\hat{g}$:

$$\begin{aligned} \hat{u}_t(\xi, t) &= ci\xi A(\xi) e^{ci\xi t} - ci\xi B(\xi) e^{-ci\xi t} \qquad (10) \\ \hat{u}_t(\xi, 0) &= ci\xi A(\xi) - ci\xi B(\xi) = \hat{g}(\xi) \qquad (11) \end{aligned}$$

Here, let us rewrite $\hat{f}$ and $\hat{g}$:

$$\begin{aligned} ci\xi \hat{f}(\xi) &= ci\xi A(\xi) + ci\xi B(\xi) \qquad (12) \\ \hat{g}(\xi) &= ci\xi A(\xi) - ci\xi B(\xi) \qquad (13) \end{aligned}$$

Here, let us add these terms together,

$$\begin{aligned} ci\xi \hat{f}(\xi) + \hat{g}(\xi) &= 2ci\xi A(\xi) \qquad (14) \\ \frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2ci\xi} &= A(\xi) \qquad (15) \end{aligned}$$

Similarly for B:

$$ci\xi\hat{f}(\xi) - \hat{g}(\xi) = 2ci\xi B(\xi) \tag{16}$$

$$\frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2ci\xi} = B(\xi) \tag{17}$$

Next, let us substitute in for both $A(\xi)$ and $B(\xi)$ using our found equation, $\hat{u}(\xi, t)$.

$$\hat{u}(\xi, t) = \left(\frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2ci\xi}\right) e^{ci\xi t} + \left(\frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2ci\xi}\right) e^{-ci\xi t} \tag{18}$$

Now, let us retransform our equation back.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi, t) e^{ix\xi} \mathrm{d}\xi \tag{19}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\left(\frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2ci\xi}\right) e^{ci\xi t} + \left(\frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2ci\xi}\right) e^{-ci\xi t}\right) e^{ix\xi} \tag{20}$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\hat{f}(\xi)\left(e^{ci\xi t} + e^{-ci\xi t}\right) e^{ix\xi} + \frac{\hat{g}(\xi)}{ci\xi}\left(e^{ci\xi t} - e^{-ci\xi t}\right) e^{ix\xi}\right) \mathrm{d}\xi \tag{21}$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\hat{f}(\xi)\left(e^{i\xi(ct+x)} + e^{i\xi(x-ct)}\right) + \frac{\hat{g}(\xi)}{ci\xi}\left(e^{i\xi(ct+x)} - e^{i\xi(x-ct)}\right)\right) \mathrm{d}\xi \tag{22}$$

Here, note that we have the transforms for $f(x+ct)$ and $f(x-ct)$ present in our equation. Let us write:

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{g}(\xi)}{ci\xi}\left(e^{i\xi(ct+x)} - e^{i\xi(x-ct)}\right) \tag{23}$$

Here, let us consider our integral. Let us write:

$$\frac{\hat{g}(\xi)}{ci\xi} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{g(x)e^{-ix\xi}}{i\xi} \mathrm{d}x \tag{24}$$

Here, let us create a table for our integration.

$\frac{e^{-x\xi}}{-i\xi}$	$\int_{-\infty}^{x} g(y)\mathrm{d}y$
$e^{-x\xi}$	$g(x)\mathrm{d}x$

Here, let us continue:

$$= \frac{1}{c\sqrt{2\pi}}\left(\int_{-\infty}^{x} g(y)\mathrm{d}y \frac{e^{-ix\xi}}{-i\xi}\Big|_{-\infty}^{\infty}\right) - \int_{-\infty}^{\infty}\int_{-\infty}^{x} g(y)\mathrm{d}y\ e^{-ix\xi}\mathrm{d}x \tag{25}$$

$$\frac{\hat{g}(\xi)}{ci\xi} = \hat{h}(\xi) \tag{26}$$

Here, let us replace $\hat{g}$ and substitute into line 22:

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}\left(e^{i\xi(ct+x)} - e^{i\xi(x-ct)}\right) \mathrm{d}\xi \tag{27}$$

$$= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2}(h(x+ct) - h(x-ct)) \tag{28}$$

Here, let us modify h and write h in terms of g:

$$= \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\left(\int_{-\infty}^{x+ct} g(y)\mathrm{d}y \;-\int_{-\infty}^{x-ct} g(y)\mathrm{d}y\right) \tag{29}$$

Here, let us swap the integral and combine both integrals for g:

$$u(x,t) = \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\mathrm{d}y \tag{30}$$

4. Derive d'Alembert's formula for $u_{tt} = u_{xx}$ by assuming that $u(x,t) = v(x+t, x-t) = v(y,z)$. Next show that the wave equation yields $v_{yz} = 0$ and hence $v = A(y) + B(z)$ and solve for A and B using the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

Here, let us take a look at our assumption:

$$u(x,t) = v(x+t, x-t) = v(y,z) \tag{1}$$

Here, we have the following relationship:

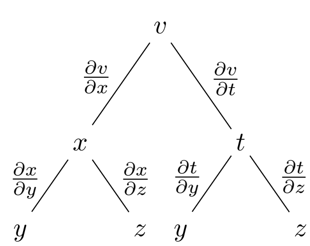
$$\begin{cases} y & = x+t \\ z & = x-t \end{cases} \tag{2}$$

Next, we want to show that the wave equation yields $v_{yz} = 0$. In orderfor the equation to be zero, that means v_y does not contain any $z's$, so when you differentiate v_y once more, then the non-z terms zero out. The same can be argued for v_z, where v_z does not contain any $y's$ and will zero out.

Therefore, v is a function of $y+z$ and we can write v as $v = A(y) + B(z)$.

Here, let us find the partial of v with respect to y and z.

Let us consider the following tree:



Here, let us consider our tree and find v_y:

$$v_y = \frac{\partial v}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y} \tag{3}$$

Earlier, we found the following system of equations:

$$\begin{cases} y & = x+t \\ z & = x-t \end{cases} \tag{4}$$

Here, notice we can isolate x or t by adding or subtracting the two equations together. First, let us add the equations to obtain the following:

$$y + z = 2x \tag{5}$$
$$x = \frac{1}{2}(y + z) \tag{6}$$

Using this information, we can find $\frac{\partial x}{\partial y}$:

$$\frac{\partial x}{\partial y} = \frac{1}{2} \tag{7}$$

Now, to find t, we subtract the equations:

$$y - z = 2t \tag{8}$$
$$t = \frac{1}{2}(y - z) \tag{9}$$

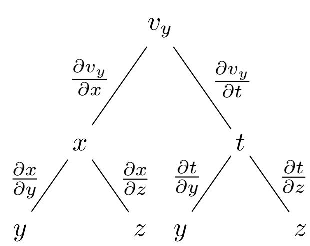
Similarly, let us find $\frac{\partial t}{\partial y}$

$$\frac{\partial t}{\partial y} = \frac{1}{2} \tag{10}$$

Here, now that we know $\frac{\partial x}{\partial y}$ and $\frac{\partial t}{\partial y}$, let us substitute this into line 3:

$$v_y = \frac{1}{2}v_x + \frac{1}{2}v_t \tag{11}$$

Now, let us find v_{yz}. Let us rewrite our tree:



Here, let us find v_{yz} using our tree.

$$v_{yz} = \frac{\partial v_y}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial v_y}{\partial t}\frac{\partial t}{\partial z} \tag{12}$$
$$= \frac{1}{2}v_{xx} + \frac{1}{2}v_{tx} \tag{13}$$

Now, we still have the following terms for x and t:

$$x = \frac{1}{2}(y + z) \tag{14}$$
$$t = \frac{1}{2}(y - z) \tag{15}$$

So let us substitute these back into the equation:

$$v_{yz} = \frac{1}{2}\left(v_{xx} + v_{tx}\right) - \frac{1}{2}\left(v_{xt} + v_{tt}\right) \tag{16}$$
$$= \frac{1}{2}v_{xx} - \frac{1}{2}v_{tt} \tag{17}$$

Recall from our initial conditions, we have $v_{tt} = u_{xx}$. Now, we can say $v = A(y) + B(z)$ since $v_{yz} = v_{xx} - v_{tt} = 0$.

Let us reconsider what we know:

- $u(x, 0) = f(x)$
- $u_t(x, 0) = g(x)$
- $V = A(y) + B(z)$
- $u(x, t) = v(x + t, x - t)$

Here, let us plug in for $u(x, 0)$ and substitute for y and z:

$$v = A(y) + B(z) \tag{18}$$
$$= A(x + t) + B(x - t) \tag{19}$$
$$u(x, 0) = A(x) + B(x) = f(x) \tag{20}$$

Now, let us find u_t, which requests for the t-partial, $u_t = v_t$. Let us use $A(x + t) + B(x - t)$ and find the derivative in terms of A and B:

$$u_t = A'(x + t) \cdot 1 - B'(x - t) \cdot 1 \tag{21}$$
$$u_t(x, 0) = A'(x) - B'(x) = g(x) \tag{22}$$

Here, we found both $g(x)$ and $f(x)$:

$$f(x) = A(x) + B(x) \tag{23}$$
$$g(x) = A'(x) - B'(x) \tag{24}$$

Here, let us integrate the second line:

$$h(x) = \int g(x)\mathrm{d}x \tag{25}$$

Here, we want to integrate over an area while keeping g spanning over the entire number line.

$$h(x) = \int_{-\infty}^{x} g(y)\mathrm{d}y \tag{26}$$

Here, let us consider $g(y)$'s behavior as $-\infty$ tends to infinity. As g tends to $-\infty$, then g will head towards 0. Now, let us consider integrating $A'(x) - B'(x)$

$$A(x) - B(x) = \int_{-\infty}^{x} g(y)\mathrm{d}y \tag{27}$$

Here, recall $A(x) + B(x) = f(x)$. Here, let this function and the previous function together:

$$A(x) + B(x) = f(x) \tag{28}$$

$$A(x) - B(x) = \int_{-\infty}^{x} g(y)\mathrm{d}y \tag{29}$$

$$2A(x) = f(x) + \int_{-\infty}^{x} g(y)\mathrm{d}y \tag{30}$$

$$2B(x) = f(x) - \int_{-\infty}^{x} g(y)\mathrm{d}y \tag{31}$$

Here, we want to plug our previous two functions into v. See how $A(x+t) + B(x-t)$ has $x+t$ and $x-t$. Let us swap these into our new functions:

$$v = \frac{1}{2}\left(f(x+t) + \int_{-\infty}^{x+t} g(y)\right)\mathrm{d}y + \frac{1}{2}\left(f(x-t) - \int_{-\infty}^{x-t} g(y)\mathrm{d}y\right) \tag{32}$$

$$v = \frac{1}{2}\left(f(x+t) + \int_{-\infty}^{x+t} g(y)\right)\mathrm{d}y + \frac{1}{2}\left(f(x-t) + \int_{x-t}^{-\infty} g(y)\mathrm{d}y\right) \tag{33}$$

Here, notice how our integrations are similar. The bottom and top intervals are the same respectively, we can combine them:

$$v = \frac{1}{2}\left(f(x-t) + f(x+t) + \int_{x-t}^{x+t} g(y)\mathrm{d}y\right) \tag{34}$$

Here, we have derive D'Alembert's Formula