Partial Differential Equations - Class Notes

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Wave Equation

Here, let us consider the following conditions:

- $u_{tt} = u_{xx}$, $t \in [0, \infty), x \in (-\infty, \infty)$
- $\lim\limits_{x \to \pm\infty} u(x,t) = 0$
- $u(x,0) = f(x)$
- $u_t(x,0) = g(x)$

Note: Two initial conditions for wave: Heat's condition ($u(x,0) = f(x)$) and $u_t(x,0) = g(x)$

Now, let us begin:

1. Let us solve F:

$$F[u_{tt}] = F[u_{xx}] \tag{1}$$
$$\Rightarrow \hat{u}_{tt} = (i\xi)^2 \hat{u} \tag{2}$$
$$\Rightarrow \hat{u}_{tt} = -\xi^2 \hat{u} \tag{3}$$
$$\hat{u}(\xi, 0) = \hat{f}(\xi) \tag{4}$$

Now, let us consider $\hat{u}_t(\xi, 0)$:

$$F[u_t(x,0)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x,0) e^{-ix\xi} \text{ dx} \tag{5}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ix\xi} \text{ dx} \tag{6}$$
$$= \hat{g}(\xi) \tag{7}$$

2. Solve $\hat{u}_{tt} = -\xi^2 \hat{u}$ with the following conditions:
   - $\hat{u}(\xi, 0) = \hat{f}(\xi)$
   - $\hat{u}_t(\xi, 0) = \hat{g}(\xi)$

Now let us write the general form:

$$\hat{u}(\xi, t) = A(\xi) \sin(\xi t) + B(\xi) \cos(\xi t) \tag{8}$$

Here, we do not want to use sine and cosine because we will multiply by expontentials later on.

$$\hat{u}(\xi, t) = A(\xi) e^{i\xi t} + B(\xi) e^{-i\xi t} \tag{9}$$
$$\hat{u}(\xi, 0) = A(\xi) + B(\xi) = \hat{f}(\xi) \tag{10}$$

Here, let us find the t partial,

$$\hat{u}_t(\xi, t) = i\xi A(\xi) e^{i\xi t} - i\xi B(\xi) e^{-i\xi t} \tag{11}$$
$$\hat{u}(\xi, 0) = i\xi A(\xi) - i\xi B(\xi) = \hat{g}(\xi) \tag{12}$$

Here, let us take the equation with $\hat{f}(\xi)$ and multiply it by $i\xi$:

$$2i\xi A(\xi) = i\xi \hat{f}(\xi) + \hat{g}(\xi) \tag{13}$$
$$a(\xi) = \frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2i\xi} \tag{14}$$

Now, let us subtract to find B:

$$2i\xi B(\xi) = i\xi \hat{f}(\xi) - \hat{g}(\xi) \tag{15}$$
$$B(\xi) = \frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2i\xi} \tag{16}$$

Here, substitute in our terms:

$$\hat{u}(\xi, t) = \left[\frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2i\xi}\right] e^{i\xi t} + \left[\frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2i\xi}\right] e^{-i\xi t} \tag{17}$$

3. Retransform

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi,t) e^{ix\xi} \; \mathrm{d}\xi && (18)\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2i\xi}\right] e^{i\xi t} + \left[\frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2i\xi}\right] e^{-i\xi t} \; \mathrm{d}\xi && (19)\\
&= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) \left(e^{i\xi t} + e^{-i\xi t}\right) e^{ix\xi} + \frac{\hat{g}(\xi)}{i\xi} \left(e^{i\xi t} - e^{-i\xi t}\right) e^{ix\xi} \; \mathrm{d}\xi && (20)\\
&= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) \left[e^{i\xi(x+t)} + e^{i\xi(x-t)}\right] \; \mathrm{d}\xi + \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{g}(\xi)}{i\xi} \left[e^{i\xi(x+t)} - e^{i\xi(x-t)}\right] \; \mathrm{d}\xi && (21)
\end{aligned}
$$

We know that

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} \; \mathrm{d}\xi && (22)\\
f(x+t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi(x+t)} \; \mathrm{d}\xi && (23)\\
f(x-t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi(x-t)} \; \mathrm{d}\xi && (24)
\end{aligned}
$$

From the previous two equations, let us write:

$$
\begin{aligned}
&\frac{1}{2}\left[f(x+t) + f(x-t)\right] && (25)\\
&\qquad = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{g}(\xi)}{i\xi} \left(e^{i\xi(x+t)} - e^{i\xi(x-t)}\right) \; \mathrm{d}\xi && (26)
\end{aligned}
$$

Now, let us write:

$$
\begin{aligned}
\frac{\hat{g}(\xi)}{i\xi} &= \frac{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ix\xi} \; \mathrm{dx}}{i\xi} && (27)\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{g(x) e^{-ix\xi}}{i\xi} \; \mathrm{dx} && (28)
\end{aligned}
$$

Here, we consider integral by parts: $u = \frac{e^{-x\xi}}{-i\xi} \Rightarrow du = e^{-i\xi} \; \mathrm{dx}$ and $dv = g(x) \; \mathrm{dx} \Rightarrow v = \int_{-\infty}^{x} g(y) \; \mathrm{dy}$.

$$
\begin{aligned}
&= -\frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{x} g(y) \; \mathrm{dy} \; \frac{e^{-i\xi x}}{-i\xi}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \int_{-\infty}^{x} g(y) \; \mathrm{dy} \; e^{-ix\xi} \; \mathrm{dx}\right] && (29)\\
&= \hat{h}(\xi) && (30)
\end{aligned}
$$

Now, from the two equations, let us consider $f(x-t)$:

$$
\begin{aligned}
&= \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}(\xi) \left(e^{i\xi(x+t)} - e^{i\xi(x-t)}\right) \; textd\xi && (31)\\
&= \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\left[h(x+t) - h(x-t)\right] && (32)\\
&= \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\left[\int_{-\infty}^{x+t} g(y) \; \mathrm{dy} \; - \int_{-\infty}^{x-t} g(y) \; \mathrm{dy}\right] && (33)\\
&= \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(y) \; \mathrm{dy} && (34)
\end{aligned}
$$

This is the solution to the wave equation on an infinite domain called D'Alembert's Formula.

Wave Equation Solutions

- $u_{tt} = c^2 u_{xx}$
- $x \in (-infty, \infty), t \in [0, \infty)$
- $u(x,0) = f(x)$
- $u_t(x,0) = g(x)$
- $u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y) \; \mathrm{dy}$

1. Conservation of Energy

$$\frac{dE}{dt} = 0 \tag{35}$$

Here, let us consider the energy as:

$$E = \frac{1}{2}\int_{-\infty}^{\infty} (u_t^2 + c^2 u_x^2) \text{ dx} \tag{36}$$

Here, the first term is kinetic energy and the second term is the potential energy. Let us derive our E:

$$\frac{dE}{dt} = \frac{d}{dt}\frac{1}{2}\int_{-\infty}^{\infty} (u_t^2 + c^2 u_x^2) \text{ dx} \tag{37}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} 2u_t u_{tt} + 2c^2 u_x u_{xt} \text{ dx} \tag{38}$$

$$= \int_{-\infty}^{\infty} u_t u_{tt} + c^2 u_x u_{xt} \text{ dx} \tag{39}$$

$$= \int_{-\infty}^{\infty} u_t c^2 u_{xx} + c^2 u_x u_{xt} \text{ dx} \tag{40}$$

$$= c^2 \int_{-\infty}^{\infty} u_t u_{xx} + u_x u_{xt} \text{ dx} \tag{41}$$

Here, let us integrate u_{xx} and differentiate u_t:

$$c^2 \left[u_t u_x \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} -u_{tx} u_x + u_x u_{xt} \text{ dx} \right] = 0 \tag{42}$$

Here, this shows our conservation of energy.

2. **Domain of Dependence / Range of Influence**

   How does the solution at a point depend on the initial condition?

   The domain of dependence is the interval between these two points:

$$[x_0 - ct_0, x_0 + ct_0] \tag{43}$$