1. Solve $\Delta u = 0$ on $x \in [0,4]$, $y \in [0,3]$, with $u(x,0) = u(x,3) = 0$ where

   (a) $u_x(0,y) = 0$ and $u_x(4,y) = \cos(\pi y)$

   (b) $u(0,y) = 1$ and $u(4,y) = 0$ (Hint: Translate the x coordinate so that $u(0,y) = 0$. This means that x goes between -4 and 0. Now go back to 0 to 4 with a y-axis flip.)

Here, let us first solve part a) of the problem.

(a) First, let us assume our equation is separable:

$$u(x,y) = X(x)Y(y) \tag{1}$$

Let us take note of our u:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && (2)\\ X''Y + XY'' &= 0 && (3)\\ XY'' &= -X''Y && (4)\\ \frac{Y''}{Y} &= -\frac{X''}{X} = -\lambda && (5) \end{aligned}$$

Here, let us consider our boundary conditions:

$$u(x,0) = 0 = X(0)Y(y) \tag{6}$$

Here, we either have $X(0) = 0$ or $Y(y) = 0$. We do not want to assume $Y(y) = 0$ because that would make $u(x,y) = 0\ \forall y \in [0,3]$. Therefore, we consider $X(0) = 0$. We will follow this assumption for the upcoming boundary conditions.

$$\begin{aligned} u(x,y) &= X(x)Y(y) && (7)\\ u(x,0) &= X(x)Y(0) = 0 \Rightarrow Y(0) = 0 && (8)\\ u(x,3) &= X(x)Y(3) = 0 \Rightarrow Y(3) = 0 && (9) \end{aligned}$$

Now, let us consider $u_x(x,y)$ as $X'(x)Y(y)$:

$$\begin{aligned} u_x(x,y) &= X'(0)Y(y) && (10)\\ u_x(0,y) &= X'(0)Y(y) = 0 \Rightarrow X'(0) = 0 && (11) \end{aligned}$$

Now, let us write:

$$-\frac{X''}{X} = \frac{Y''}{Y} = -\lambda \tag{12}$$

(b) Next we solve for $Y(y)$ since we have more information on $Y(y)$.

$$\begin{aligned} \frac{Y''}{Y} &= -\lambda && (13)\\ Y'' &= -\lambda Y && (14) \end{aligned}$$

Here, let us write out our general equation:

$$Y_n(y) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \tag{15}$$

Here, we know $Y(0) = Y(3) = 0$. Let us input our y:

$$\begin{aligned} Y_n(0) &= 0 = B && (16)\\ Y_n(3) &= 0 = A\sin(\sqrt{\lambda}3) && (17) \end{aligned}$$

Here, let us not consider A as 0, instead, the inside of our sine function to be 0.

$$\begin{aligned} \sqrt{\lambda}3 &= n\pi && (18)\\ \sqrt{\lambda} &= \frac{n\pi}{3} && (19)\\ \lambda_n &= \left(\frac{n\pi}{3}\right)^2 && (20) \end{aligned}$$

Now we have the following:

$$Y_n(y) = \sin\left(\frac{n\pi y}{3}\right) \tag{21}$$

(c) Now, let us solve for X:

$$\frac{X''}{X} = \lambda \tag{22}$$
$$X'' = \lambda X \tag{23}$$

Here, we solved for λ in the previous step:

$$X_n'' = \left(\frac{n\pi}{3}\right)^2 X_n \tag{24}$$
$$X_n(x) = C\sinh\left(\frac{n\pi x}{3}\right) + D\cosh\left(\frac{n\pi x}{3}\right) \tag{25}$$

Here, let us find the derivative for X and find $X'(0)$:

$$X'(0) = \frac{n\pi x}{3}C = 0 \tag{26}$$
$$= C = 0 \tag{27}$$

Therefore, since $C = 0$, we get:

$$X_n(x) = D\cosh\left(\frac{n\pi x}{3}\right) \tag{28}$$

(d) Now, let us combine u_n and u:

$$u_n(x,y) = D\cosh\left(\frac{n\pi x}{3}\right)\sin\left(\frac{n\pi y}{3}\right) \tag{29}$$

By linearity, let us write:

$$u(x,y) = \sum_{n=1}^{\infty} D_n \cosh\left(\frac{n\pi x}{3}\right)\sin\left(\frac{n\pi y}{3}\right) \tag{30}$$

(e) Here, let us use our boundary conditions to find our coefficients.

$$u_x(4,y) = \cos(\pi y) \tag{31}$$

Here, let us derive our $u(x,y)$ to get $u_x(x,y)$

$$u_x(x,y) = \sum_{n=1}^{\infty} D_n\left(\frac{n\pi}{3}\right)\sinh\left(\frac{n\pi x}{3}\right)\sin\left(\frac{n\pi y}{3}\right) \tag{32}$$
$$u_x(4,y) = \sum_{n=1}^{\infty} D_n\left(\frac{n\pi}{3}\right)\sinh\left(\frac{n\pi 4}{3}\right)\sin\left(\frac{n\pi y}{3}\right) \tag{33}$$

Here, let us consider our coefficient in the equation:

$$D_n\left(\frac{n\pi}{3}\right)\sinh\left(\frac{4\pi n}{3}\right) = \frac{2}{3}\int_0^3 \cos(\pi y)\sin\left(\frac{n\pi y}{3}\right)dy \tag{34}$$
$$D_n = \frac{2}{3}\frac{3}{n\pi}\frac{1}{\sinh\left(\frac{4\pi n}{3}\right)}\int_0^3 \cos(\pi y)\sin\left(\frac{n\pi y}{3}\right) \tag{35}$$
$$= \frac{2}{n\pi\sinh\left(\frac{4\pi n}{3}\right)}\int_0^3 \cos(\pi y)\sin\left(\frac{n\pi y}{3}\right) \tag{36}$$

Here, let us isolate our integral and solve for that. Afterwards, let us combine our function.

$$\int_0^3 \cos(\pi y)\sin\left(\frac{n\pi y}{3}\right) \tag{37}$$

Here, let us use trig identities to separate our product:

$$\int_0^3 \cos(\pi y)\sin\left(\frac{n\pi y}{3}\right) = \int_0^3 \frac{\sin\left(\frac{\pi n y - \pi 3y}{3}\right) + \sin\left(\frac{\pi n y + \pi 3 y}{3}\right)}{2} \tag{38}$$
$$= \frac{1}{2}\int_0^3 \sin\left(\frac{(n-3)\pi y}{3}\right)dy + \frac{1}{2}\int_0^3 \sin\left(\frac{(n+3)\pi y}{3}\right)dy \tag{39}$$

From here, let us consider u substitution on both sides. On the left, let us consider $u = \frac{(n-3)\pi y}{3}$ and $du = \frac{(n-3)\pi y}{3} dy$. From here, let us replace our interval from 0 to $(n-3)\pi$. For the right integral, let us consider the same procedure and replace $v = \frac{(n+3)\pi y}{3}$ and $vu = \frac{(n+3)\pi}{3} dy$. From here, let us change the interval to 0 to $(n+3)\pi$:

$$= \frac{1}{2}\frac{3}{(n-3)\pi}\int_0^{\pi(n-3)} \sin(u)du + \frac{1}{2}\frac{3}{(n+3)\pi}\int_0^{\pi(n+3)} \sin(v)dv \tag{40}$$

$$= -\frac{3}{2(n-3)}\Bigg[\cos u\Bigg]_0^{\pi(n-3)} - \frac{3}{2(n+3)}\Bigg[\cos u\Bigg]_0^{\pi(n-3)} \tag{41}$$

$$= -\frac{3}{2(n-3)\pi}\left[\cos(\pi((n-3)) - 1\right] - \frac{3}{2(n+3)\pi}\left[\cos(\pi(n+3)) - 1\right] \tag{42}$$

$$= -\frac{3}{2(n-3)\pi}\left[\cos(\pi n - \pi 3) - 1\right] - \frac{3}{2(n+3)\pi}\left[\cos(\pi n + \pi 3) - 1\right] \tag{43}$$

$$= -\frac{3}{2(n-3)\pi}\left[-\cos(\pi n) - 1\right] - \frac{3}{2(n+3)\pi}\left[-\cos(\pi n) - 1\right] \tag{44}$$

$$= \frac{3(\cos(\pi n) + 1)}{2(n-3)\pi} + \frac{3(\cos(\pi n) + 1)}{2(n+3)\pi} \tag{45}$$

$$= \frac{3n(\cos(\pi n) + 1)}{\pi(n^2 - 9)} \tag{46}$$

Here, let us plug our integral back into our equation at line (36):

$$= \frac{2}{n\pi \sinh(\frac{4\pi n}{3})}\frac{3n(\cos(\pi n) + 1)}{\pi(n^2 - 9)} \tag{47}$$

$$= \frac{6(\cos(\pi n) + 1)}{\pi^2(n^2 - 9)\sinh(\frac{4\pi n}{3})} \tag{48}$$

From here, let us write:

$$u(x, y) = \sum_{n=1}^{\infty} \frac{6(-1^n + 1)}{\pi^2(n^2 - 9)\sinh(\frac{4\pi n}{3})}\cosh(\frac{n\pi x}{3})\sin(\frac{n\pi y}{3}) \tag{49}$$

From here, allow us to backtrack and consider new conditions: $u(0, y) = 1$ and $u(4, y) = 0$. From here, we should consider shifting one of our functions. Let us start from part c, knowing $Y_n(y) = \sin\left(\frac{n\pi y}{3}\right)$ and $\lambda_n = \left(\frac{n\pi}{3}\right)^2$:

(d) Let us solve for $X(x)$

$$X(x) = C\sinh(\sqrt{\lambda}x) + D\cosh(\sqrt{\lambda}x) \tag{50}$$

$$X(0) = 0 = D \tag{51}$$

Here, let us shift our x to get $X(4) = 0$:

$$X(x) = C\sinh(\frac{n\pi}{3}(4 - x)) \tag{52}$$

$$X(4) = C\sinh(\frac{n\pi}{3}(4 - 4)) = 0 \tag{53}$$

(e) Here, let us combine our u and u_n:

$$u(x, y) = C_n \sinh\left(\frac{n\pi(4 - x)}{3}\right)\sin\left(\frac{4\pi y}{3}\right) \tag{54}$$

By linearity, let us write:

$$u_n(x, y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi(4 - x)}{3}\right)\sin\left(\frac{n\pi y}{3}\right) \tag{55}$$

From our initial conditions, we know $u(0, y) = 1$, so let us write:

$$u_n(0, y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi 4}{3}\right)\sin\left(\frac{n\pi y}{3}\right) \tag{56}$$

From here, let us find our coefficient for sin:

$$C_n \sinh\left(\frac{4n\pi}{3}\right) = \frac{2}{3}\int_0^3 \sin\left(\frac{n\pi y}{3}\right) dy \tag{57}$$

$$= -\frac{2}{3}\frac{3}{n\pi}\left[\cos\left(\frac{n\pi y}{3}\right)\right]_0^3 \tag{58}$$

$$C_n \sinh\left(\frac{4n\pi}{3}\right) = -\frac{2}{n\pi}\left[\cos(n\pi) - 1\right] \tag{59}$$

$$C_n = \frac{2(1-\cos(n\pi))}{n\pi \sinh(\frac{4n\pi}{3})} \tag{60}$$

From here, let us write:

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi \sinh(\frac{4n\pi}{3})} \sinh\left(\frac{4n\pi(4-x)}{3}\right) \sin\left(\frac{n\pi y}{3}\right) \tag{61}$$

2. Solve $u_{tt} = 4u_{xx}$ on $x \in [0, 3]$, $t \in [0, \infty)$, with $u(0, t) = u(3, t) = 0$ where

   (a) $u(x, 0) = 4\sin(2\pi x) + 7\sin(6\pi x) - 2\sin(\pi x)$, $u_t(x, 0) = 0$

   (b) $u(x, 0) = x(3 - x)$, $u_t(x, 0) = \sin(\pi x)$

Let us begin with our boundary conditions.

(a) Let us assume our equation is separable.

$$u_{tt} = 2^2 u_{xx} \tag{62}$$

$$XT'' = 2^2 X''T \tag{63}$$

$$\frac{T''}{2^2T} = \frac{X''}{X} = -\lambda \tag{64}$$

(b) Here, since we have more information regarding X, let us solve for X:

$$\frac{X''}{X} = -\lambda \tag{65}$$

$$X'' = -\lambda X \tag{66}$$

In this equation, we want to consider cos and sin for our general equation.

$$X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \tag{67}$$

Here, we know $X(0) = X(3) = 0$, so let us write:

$$X(0) = B = 0 \tag{68}$$

$$X(3) = A\sin(\sqrt{\lambda}3) = 0 \tag{69}$$

Here, let us find when our sine function is 0.

$$\sqrt{\lambda}3 = n\pi \tag{70}$$

$$\sqrt{\lambda} = \frac{n\pi}{3} \tag{71}$$

$$\lambda = \left(\frac{n\pi}{3}\right)^2 \tag{72}$$

Now, let us write our general equation for X:

$$X_n(x) = \sin\left(\frac{n\pi x}{3}\right) \tag{73}$$

(c) Next, let us solve for T:

$$\frac{T''}{2^2T} = -\lambda \tag{74}$$

$$T'' = -2^2T\lambda \tag{75}$$

Here, let us write the general equation:

$$T_n(t) = C_n \sin\left(\frac{2n\pi t}{3}\right) + D_n \cos\left(\frac{2n\pi t}{3}\right) \tag{76}$$

(d) Here, let us combine to find u_n and u:

$$u_n(x, t) = \sin\left(\frac{n\pi x}{3}\right)\left[C_n \sin\left(\frac{2n\pi t}{3}\right) + D_n \cos\left(\frac{2n\pi t}{3}\right)\right] \tag{77}$$

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{3}\right)\sin\left(\frac{2n\pi t}{3}\right) + D_n \sin\left(\frac{n\pi x}{3}\right)\cos\left(\frac{2n\pi t}{3}\right) \tag{78}$$

(e) Here, let us find the coefficients using the Initial Conditions:

- $u(x, 0) = 4\sin(2\pi x) + 7\sin(6\pi x) - 2\sin(\pi x)$
- $u_t(x, 0) = 0$

Using this information, find our t partial from our summation series:

$$u_t(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{3}\right)\frac{2n\pi}{3}\cos\left(\frac{2n\pi t}{3}\right) - D_n \sin\left(\frac{n\pi x}{3}\right)\frac{2n\pi}{3}\sin\left(\frac{2n\pi t}{3}\right) \tag{79}$$

From here, let us plug in our initial condition, $u_t(x,0) = 0$:

$$u_t(x,0) = \sum_{n=1}^{\infty} C_n \frac{2n\pi}{3}\sin\left(\frac{n\pi x}{3}\right) \tag{80}$$

From here, let us recall our terms:

- $4\sin(2\pi x) + 7\sin(6\pi x) - 2\sin(\pi x)$

Using these terms, we know the following:

- $D_3 = -2$
- $D_6 = 4$
- $D_{18} = 7$
- $D_n = 0, n \in \mathbb{N}, n \neq 3, 6, 18$

Let us plug in our coefficients:

$$u(x,t) = -2\sin(\pi x)\cos(2\pi t) + 4\sin(2\pi x)\cos(4\pi t) + 7\sin(6\pi x)\cos(12\pi t) \tag{81}$$

Now, let us reconsider our initial conditions once more:

(e) $u(x,0) = x(3-x), u_t(x,0) = \sin(\pi x)$

Here, let us consider finding our coefficient where $u(x,0) = x(3-x)$:

$$\begin{aligned} D_n &= \frac{2}{3}\int_0^3 x(3-x)\sin\left(\frac{n\pi x}{3}\right) &(82)\\ &= \frac{2}{3}\int_0^3 3x\sin\left(\frac{n\pi x}{3}\right) - x^2\sin\left(\frac{n\pi x}{3}\right) &(83)\\ &= 2\int_0^3 x\sin\left(\frac{n\pi x}{3}\right) - \frac{2}{3}\int_0^3 x^2\sin\left(\frac{n\pi x}{3}\right) &(84)\end{aligned}$$

Here, let us create two integration tables:

x	$\sin\left(\frac{n\pi x}{3}\right)$
1	$-\frac{3}{n\pi}\cos\left(\frac{n\pi x}{3}\right)$
0	$-\left(\frac{3}{n\pi}\right)^2\cos\left(\frac{n\pi x}{3}\right)$

x^2	$\sin\left(\frac{n\pi x}{3}\right)$
$2x$	$-\frac{3}{n\pi}\cos\left(\frac{n\pi x}{3}\right)$
2	$-\left(\frac{3}{n\pi}\right)^2\sin\left(\frac{n\pi x}{3}\right)$
0	$\left(\frac{3}{n\pi}\right)^3\cos\left(\frac{n\pi x}{3}\right)$

Using our table, let us write out our integral:

$$D_n = 2\left[-\frac{3x}{n\pi}\cos\left(\frac{n\pi x}{3}\right) + \left(\frac{3}{n\pi}\right)^2\cos\left(\frac{n\pi x}{3}\right)\right]_0^3 \tag{85}$$

$$-\frac{2}{3}\left[-\frac{x^2 3}{n\pi}\cos\left(\frac{n\pi x}{3}\right) + 2x\left(\frac{3}{n\pi}\right)^2\sin\left(\frac{n\pi x}{3}\right) - 2\left(\frac{3}{n\pi}\right)^3\cos\left(\frac{n\pi x}{3}\right)\right]_0^3 \tag{86}$$

$$D_n = 2\left[-\frac{9}{n\pi}\cos(n\pi) + \left(\frac{3}{n\pi}\right)^2\cos(n\pi) - \left(\frac{3}{n\pi}\right)^2\right] + 2\left[\frac{9}{n\pi}\cos(n\pi) + \frac{18}{n^3\pi^3}\cos(n\pi) - \frac{27}{n^3\pi^3}\right] \tag{87}$$

Here, let us simplify:

$$\begin{aligned} D_n &= 2\left[\frac{9}{n^2\pi^2}\cos(n\pi) - \frac{9}{n^2\pi^2} + \frac{18}{n^3\pi^3}\cos(n\pi) - \frac{27}{n^3\pi^3}\right] &(88)\\ &= \frac{9}{n^3\pi^3}\left[n\pi\cos(n\pi) - n\pi + 2\cos(n\pi) - 3\right] &(89)\\ &= \frac{9}{n^3\pi^3}\left[2\cos(n\pi) + n\pi\cos(n\pi) - n\pi - 3\right] &(90)\end{aligned}$$

If we consider $n \in \mathbb{Z}$, then let us rewrite:

$$D_n = \frac{9}{n^3\pi^3}\left[2(-1)^n + n\pi(-1)^n - n\pi - 3\right] \tag{91}$$

From here, let us write our second condition:

$$u_t(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{3}\right)\frac{2n\pi}{3}\cos\left(\frac{2n\pi t}{3}\right) - D_n \sin\left(\frac{n\pi x}{3}\right)\frac{2n\pi}{3}\sin\left(\frac{2n\pi t}{3}\right) \tag{92}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{3}\right)\frac{2n\pi}{3} \tag{93}$$

Here, $C_3 = \frac{1}{2\pi}$. Let us write:

$$u(x,t) = \sin(\pi x)\cos(2\pi t) - \frac{6}{n^2\pi^2}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{3}\right)\sin\left(\frac{2n\pi t}{3}\right) \tag{94}$$