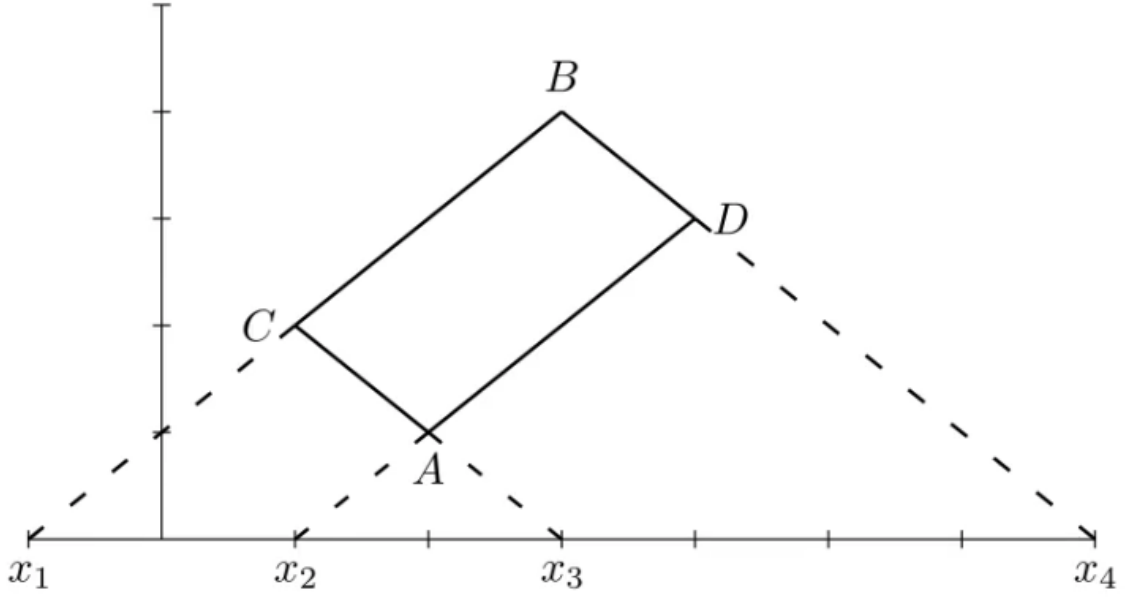


1. Use D'Alembert's formula to show the parallelogram property of the wave equation mentioned in class.



$$u(A) + u(B) = u(C) + u(D) \quad (1)$$

Note that our slope depends on  $c$ .

Here, let us consider D'Alembert's Formula:

$$\frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} g(y) dy \quad (2)$$

Now, let us consider using D'Alembert's Formula to generate the following equations:

$$u(A) = \frac{1}{2}[f(x_2) + f(x_3)] + \frac{1}{2c} \int_{x_2}^{x_3} g(y) dy \quad (3)$$

$$u(B) = \frac{1}{2}[f(x_1) + f(x_4)] + \frac{1}{2c} \int_{x_1}^{x_4} g(y) dy \quad (4)$$

$$u(C) = \frac{1}{2}[f(x_1) + f(x_3)] + \frac{1}{2c} \int_{x_1}^{x_3} g(y) dy \quad (5)$$

$$u(D) = \frac{1}{2}[f(x_2) + f(x_4)] + \frac{1}{2c} \int_{x_2}^{x_4} g(y) dy \quad (6)$$

From here, let us evaluate  $u(A) + u(B)$  and  $u(C) + u(D)$

$$u(A) + u(B) = \frac{1}{2}[f(x_2) + f(x_3)] + \frac{1}{2c} \int_{x_2}^{x_3} g(y) dy + \frac{1}{2}[f(x_1) + f(x_4)] + \frac{1}{2c} \int_{x_1}^{x_4} g(y) dy \quad (7)$$

$$= \frac{1}{2} \left( f(x_1) + f(x_4) + f(x_2) + f(x_3) + \frac{1}{c} \left[ \int_{x_1}^{x_4} g(y) dy + \int_{x_2}^{x_3} g(y) dy \right] \right) \quad (8)$$

Next, evaluate  $u(C) + u(D)$ :

$$u(C) + u(D) = \frac{1}{2}[f(x_1) + f(x_3)] + \frac{1}{2c} \int_{x_1}^{x_3} g(y) dy + \frac{1}{2}[f(x_2) + f(x_4)] + \frac{1}{2c} \int_{x_2}^{x_4} g(y) dy \quad (9)$$

$$= \frac{1}{2} \left( f(x_1) + f(x_3) + f(x_2) + f(x_4) + \frac{1}{c} \left[ \int_{x_1}^{x_3} g(y) dy + \int_{x_2}^{x_4} g(y) dy \right] \right) \quad (10)$$

2. If  $f(x)$  and  $g(x)$  are changed on the region  $x \in [0, 4]$ , on which region in the  $(x, t)$ -plane will the solutions of  $u_{tt} = 9u_{xx}$  be altered?

Here, we are given a wave equation on the  $x$  boundary  $[0, 4]$  and a constant  $3^2$ .  $f(x)$  and  $g(x)$  are given to determine the initial condition for our system. When changing our initial conditions, we change the Fourier Series solutions for the given problem. Altering  $f(x)$  and  $g(x)$

Here, let us consider our wave equation,  $u_{tt} = 9u_{xx}$ , where  $\sqrt{c} = 3$ .

Here, let us consider our characteristic line as we pass through  $(0, 0)$ . Let us write  $x \pm 3t = 0$ , which would give us  $t = \pm \frac{1}{3}x$ .

Now, looking for the characteristic line passing through  $(3, 0)$ , we can write  $x \pm 2t = 3$ , yielding

3. The solution to the non-homogeneous Laplace equation  $\Delta u = f(x, y)$  on  $x \in (-\infty, \infty), y \in (-\infty, \infty)$  is:

$$u(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x - \xi, y - \eta) f(\xi, \eta) d\xi d\eta \quad (1)$$

where

$$k(x, y) = -\frac{1}{2\pi} \ln \left( \sqrt{x^2 + y^2} \right) \quad (2)$$

Show that if  $f(\xi, \eta) = \delta(\xi)\delta(\eta)$ , then  $\Delta u = 0$  for  $(x, y) \neq (0, 0)$ .

Let us consider the given equation 1). Here, let us use given assumption,  $f(\xi, \eta) = \delta(\xi)\delta(\eta)$  and substitute it into 1)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x - \xi, y - \eta) f(\xi, \eta) d\xi d\eta = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x - \xi, y - \eta) \delta(\xi)\delta(\eta) d\xi d\eta \quad (3)$$

Let us consider our function,  $k$ . Equation 2) defines the function of  $k$ . Let us evaluate our function with the given parameters,  $x - \xi$  and  $y - \eta$ :

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x - \xi, y - \eta) \delta(\xi)\delta(\eta) d\xi d\eta = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{1}{2\pi} \left( \ln \sqrt{(x - \xi)^2 + (y - \eta)^2} \right) \delta(\xi)\delta(\eta) d\xi d\eta \quad (4)$$

Here, let us consider our  $\delta$  function and ways to manipulate the function. Here, we have the property:

$$\int_{-\infty}^{\infty} \delta(x - y) f(y) dy = f(x) \quad (5)$$

If we apply it to equation 4, we get:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{1}{2\pi} \left( \ln \sqrt{(x - \xi)^2 + (y - \eta)^2} \right) \delta(\xi)\delta(\eta) d\xi d\eta = -\frac{1}{2\pi} \ln \left( \sqrt{x^2 + y^2} \right) \quad (6)$$

Now, let us take the  $x$  and  $y$  partial of line 6)

$$u(x, y) = -\frac{1}{2\pi} \ln \left( \sqrt{x^2 + y^2} \right) \quad (7)$$

$$u_{xx}(x, y) + u_{yy}(x, y) = \left( -\frac{1}{2\pi} \frac{x}{x^2 + y^2} \right)_x + \left( -\frac{1}{2\pi} \frac{y}{x^2 + y^2} \right)_y \quad (8)$$

$$= \left( -\frac{1}{2\pi} \frac{y^2 - x^2}{(x^2 + y^2)^2} \right) \left( -\frac{1}{2\pi} \frac{x^2 - y^2}{(x^2 + y^2)^2} \right) \quad (9)$$

$$= \left( \frac{1}{2\pi} \frac{x^2 - y^2}{(x^2 + y^2)^2} \right) \left( \frac{1}{2\pi} \frac{y^2 - x^2}{(x^2 + y^2)^2} \right) \quad (10)$$

$$= \frac{1}{2\pi} \left( \frac{x^2 - y^2 + y^2 - x^2}{(x^2 + y^2)^2} \right) \quad (11)$$

$$= 0 \quad (12)$$

4. Show the following:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} dx = 1$$

We want to find the integral of the following:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} dx \quad (1)$$

First, let us move the constant out of our integral:

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-x^2/4t} dx \quad (2)$$

From here, let us rename our constant on the outside of our integral as  $\zeta$ :

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-x^2/4t} dx = \zeta \int_{-\infty}^{\infty} e^{-x^2/4t} dx \quad (3)$$

Here, let us focus on our integral. First, let us square our integral and change our variables in the second integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2/4t} dx \quad (4)$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2/4t} dx \int_{-\infty}^{\infty} e^{-y^2/4t} dy \quad (5)$$

From here, let us find the product of our integrals then combine our powers:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/4t} e^{-y^2/4t} dx dy \quad (6)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/4t} dx dy \quad (7)$$

$$(8)$$

Here, let us write our integral and variables in terms of polar coordinates:

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2/4t} r \, dr d\theta \quad (9)$$

$$= \int_0^{\infty} \int_0^{2\pi} e^{-r^2/4t} r \, d\theta dr \quad (10)$$

$$= 2\pi \int_0^{\infty} r e^{-r^2/4t} \, dr \quad (11)$$

$$(12)$$

Here, let us perform u-substitution, where we write  $u = \frac{r^2}{4t}$  and  $du = \frac{r}{2t} dr$

$$I^2 = 4\pi t \int_0^{\infty} e^{-u} \, du \quad (13)$$

$$I^2 = 4\pi t \quad (14)$$

$$I = \sqrt{4\pi t} \quad (15)$$

Here, let us plug our evaluation back to line 3 to find the solution:

$$\frac{1}{\sqrt{4\pi t}} \sqrt{4\pi t} = 1 \quad (16)$$

5. We know that the solution to the 2 -  $D$  heat equation  $u_t = u_{xx} + u_{yy}$ , with  $u(x, y, 0) = f(x, y)$  is

$$u(x, y, t) = \frac{1}{4\pi t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) e^{-\frac{(x-\xi)^2 + (y-\eta)^2}{4t}} d\xi d\eta \quad (1)$$

If

$$f(x, y) = \begin{cases} 1 & 2 \leq r \leq 4, r = \sqrt{x^2 + y^2} \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Sketch  $u(x, y, t)$  for different  $t$  values, say  $t = 0, 5, 100, \infty$