<u>April 1, 2022</u>

$$\xi'(t) = \frac{f(u_L) - f(u_R)}{u_L - u_R} \tag{1}$$

For some example $f(u) = \frac{u^2}{2}$, $u_L = 1, u_R = 0$

$$\xi'(t) = \frac{\frac{1}{2} - 0}{1 - 0} \tag{2}$$
$$= \frac{1}{2} \tag{3}$$

There are other initial conditions that still lead to two solutions.

$$u_t + uu_x = 0 \tag{4}$$
$$u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & e \geq 0 \end{cases} \tag{5}$$

Riemann Problem. The slope of our characteristic line is $\frac{1}{u}$.

In our solution, we have the verticals on the left side and slope $= 1$ on the right side, so the solutions do not collide. To remediate this, we add a shock in between and extend both solutions to the shock line.

R-H Jump Condition

$$\xi'(t) = \frac{0 - \frac{1}{2}}{0 - 1} \tag{6}$$
$$= \frac{1}{2} \tag{7}$$

Another solution is to make a fan (paper fan)

There would be no shock and the solution is continuous. The R-H jump condition is not used.

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0\frac{x}{t} < 1 \\ 1 & \frac{x}{t} \geq 1 \end{cases} \tag{8}$$

Conservation Law: $u_t + [f(u)]_x = 0$.

If f is smooth: $u_t + f'(u)u_x = 0$.

$f'(u)$ is the speed of the characteristic.

Slope of characteristic $= \frac{1}{f'(u)}$

<u>Note:</u> If the solution is continuous, the $R - H$ condition gives the slope of a characteristic, not the slope of shocks.

What is the actual solution to the last problem?

$$u_t + uu_x = \epsilon u_{xx} \tag{9}$$

Therefore, if $\epsilon > 0$ is a smoothing term (as in heat) $\rightarrow C^\infty$.

Which solution is the solution that you get if you solve the last equation and let $\epsilon \rightarrow 0$?

This ends up giving us the lax entropy condition:

The characteristic curves can enter a shock as time increases, but they cannot exit (or be created) from a shock.

Solution one violates the lax entropy condition, therefore solution two is the correct solution.

Theorem: There exists a unique solution to any conservation law $u_t + [f(u)]_x = 0$, $u(x, 0) = g(x)$, $x \in (-\infty, \infty), t \in [0, \infty)$ whose shocks satisfy the $R - H$ jump condition and the lax entropy condition.

Note: The fan is called a carefaction wave.

For any general first order equation $F(x, u(x), \nabla(x)) = 0$