1. Show the following:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \mathrm{d}x = 1$$

We want to find the integral of the following:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \mathrm{d}x \tag{1}$$

First, let us move the constant out of our integral:

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-x^2/4t} \mathrm{d}x \tag{2}$$

From here, let us rename our constant on the outside of our integral as ζ:

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-x^2/4t} \mathrm{d}x = \zeta \int_{-\infty}^{\infty} e^{-x^2/4t} \mathrm{d}x \tag{3}$$

Here, let us focus on our integral. First, let us square our integral and change our variables in the second integral:

$$\begin{align} I &= \int_{-\infty}^{\infty} e^{-x^2/4t} \mathrm{d}x \tag{4} \\ I^2 &= \int_{-\infty}^{\infty} e^{-x^2/4t} \mathrm{d}x \int_{-\infty}^{\infty} e^{-y^2/4t} \mathrm{d}y \tag{5} \end{align}$$

From here, let us find the product of our integrals then combine our powers:

$$\begin{align} I^2 &= \int_{-\infty}^{\infty} \int_{\infty}^{\infty} e^{-x^2/4t} e^{-y^2/4t} \mathrm{d}x\mathrm{d}y \tag{6} \\ &= \int_{-\infty}^{\infty} \int_{\infty}^{\infty} e^{-(x^2+y^2)/4t} \mathrm{d}x\mathrm{d}y \tag{7} \\ & \tag{8} \end{align}$$

Here, let us write our integral and variables in terms of polar coordinates:

$$\begin{align} I^2 &= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2/4t} r \ \mathrm{d}r\mathrm{d}\theta \tag{9} \\ &= \int_{0}^{\infty} \int_{0}^{2\pi} e^{-r^2/4t} r \ \mathrm{d}\theta\mathrm{d}r \tag{10} \\ &= 2\pi \int_{0}^{\infty} e^{-r^2/4t} r \ \theta\mathrm{d}r \tag{11} \\ & \tag{12} \end{align}$$

Here, let us perform u-substitution, where we write $u = r^2/4t$ and $\mathrm{d}u = \frac{1}{2t}$

2. We know that the solution to the $2-D$ heat equation $u_t = u_{xx} + u_{yy}$, with $u(x, y, 0) = f(x, y)$ is

$$u(x,y,t) = \frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4t}}\,\mathrm{d}\xi\mathrm{d}\eta \tag{1}$$

If

$$f(x,y) = \begin{cases} 1 & 2 \le r \le 4, r = \sqrt{x^2+y^2} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Sketch $u(x, y, t)$ for different t values, say $t = 0, 5, 100, \infty$