1 Conservation Laws

Recall how we mentioned heat is conserved, accumulated heat is heat in - heat our.

1-D Conservation:

- $u(x,t)$: Quantity that is conserved: energy, mass, momentum, ...

$$g(x,t) = \text{ flux} = f(u(x,t)) \tag{1}$$

Here, flux is dependent on the gradient. Before, we have:

$$g(x,t) = g(u_x(x,t)) \tag{2}$$

This was our gradient.

Conservation Law

$$\text{Accumulation} = \text{in - out}$$

$$\int_{x_0}^{x_1} u(x,t_1)\mathrm{d}x - \int_{x_0}^{x_1} u(x,t_0)\mathrm{d}x = u(x_0,t) \tag{3}$$

$$\int_{x_0}^{x} [u(x,t_1) - u(x,t_0)]\mathrm{d}x = \int_{t_0}^{t_1} [q(x_0,t) - q(x_1,t)]\mathrm{d}t \tag{4}$$

$$\int_{x_0}^{x_1}\int_{t_0}^{t_1} q_t(x,t)\mathrm{d}t\ \mathrm{d}x = -\int_{t_0}^{t_1}\int_{x_0}^{x_1} q_x(x,t)\mathrm{d}x\ \mathrm{d}t \tag{5}$$

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} [u_t(x,t) + u_x(x,t)]\mathrm{d}x\ \mathrm{d}t = -\int_{t_0}^{t_1}\int_{x_0}^{x_1} q_x(x,t)\mathrm{d}x\ \mathrm{d}t \tag{6}$$

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} [u_t(x,t) + q_x(x,t)]\mathrm{d}x\ \mathrm{d}t = 0 \tag{7}$$

Here, $u_t + q_x = 0$ if u_t and q_x are continuous.

Since $q = f(u)$, we get

$$u_t + [f(u)]_x = 0 \tag{8}$$

This is considered the conservation law. This is non-linear, first order.

<u>Ex:</u> Burger's Equation: For gas flow down a pipe.

$$u_t + \left(\frac{u^2}{2}\right)_x = \epsilon u_{xx} \tag{9}$$

Here, ϵ is the viscosity and u is the momentum. The viscosity of gases tend towards zero, therefore let us consider $\epsilon = 0$.

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{10}$$

Here, let $f(u) = \frac{u^2}{2}$:

$$u_t + uu_x = 0 \tag{11}$$

Domain: $x \in (-\infty, \infty), t \in [0, \infty)$

Initial condition: $u(x, 0) = g(x)$

Recall: Transport equation:

$u_t = cu_x \Rightarrow u_t - cu_x = 0$

$u(x, 0) = g(x)$.

The slope of the characteristic $= -\frac{1}{c}$

The values of the solution is constant along characteristic.

March 30, 2022

1.1 Burger's Equation

$$u_t + uu_x = 0 \tag{12}$$

The slope is $\frac{1}{u}$.

Recall, the slope of the characteristic of the equation $u_t - cu_x = 0$ is $-\frac{1}{c}$.

Ex:

$$u(x, 0) = \begin{cases} 1 & x < 0 \\ 1 - x & 0 \leq x < 1 \\ 0 & x \geq 1 \end{cases} \tag{13}$$

At $t = 1$, the solution is no longer continuous, therefore u_x is not defined at $x = 1, t = 1$.

For $t \geq 1$, there is no differentiable solution (everywhere).

If we allow solutions that aren't defined only for a single curve $\xi(t)$ of discontinuity, then we have an infinite number of solutions. From physics, we know that gas velocity (momentum) looks like this:

The shock in velocity causes a sonic boom

$$\text{Slope } = 2 \tag{14}$$

$$\xi'(t) = \frac{1}{2} \tag{15}$$

$u_t + [f(u)]_x = 0$ has no classical solution (existence problem) and ∞ number of solutions if we don't care about the equation being satisfied on shocks (uniqueness problem)

We would like:

1. a unique solution to exist

2. We want this solution to be what we observe

Define a weak solution to be $u(x,t)$ where

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1}[u_t+[f(u)]_x]\mathrm{d}x\mathrm{d}t=0 \tag{16}$$

where u_t and $[f(u)]_x$ are measures (they can contain δ functions)

This idea will lead to a formula for shock speed, $\xi'(t)$.

The integral in $x-$direction is enough.

$$\int_{x_0}^{x_1}[u_t+[f(u)]_x]\ \mathrm{d}x|_{t=t_0}=0 \tag{17}$$

Here,

$$\int_{x_0}^{x_1}u_t\ \mathrm{d}x=-\int_{x_0}^{x_1}[f(u)]_x\ \mathrm{d}x|_{t=t_0} \tag{18}$$

$$\frac{d}{dt}\int_{x_0}^{x_1}u\ \mathrm{d}x|_{t=t_0}=f(u(x_0,t_0)-f(u(x_1,t_0))) \tag{19}$$

$$\frac{d}{dt}\left[\int_{x_0}^{\xi(t)}u(x,t)\ \mathrm{d}x+\int_{\xi(t)}^{x_1}u(x,t)\ \mathrm{d}x\right]|_{t=t_0}=f(u_L)-f(u_R) \tag{20}$$

Using the second fundamental theorem of calculus,

$$\int_{x_0}^{\xi(t)}u_t(x,t)\mathrm{d}x+u(\xi(t),t)\xi'(t)+\int_{\xi(t)}^{x_1}u_t(x,t)\mathrm{d}x-u(\xi(t),t)\xi'(t)|_{t=t_0}=f(u_L)-f(u_R) \tag{21}$$

You cannot cancel the second and fourth term, as the second term is on the left of the shock and the fourth term is on the right of the shock.

Take lim as $x_0\to\xi(t)^-$,

$$\int_{x_0}^{\xi(t)}u_t(x,t)\mathrm{d}x=0 \tag{22}$$

Take lim as $x_1\to\xi(t)^+$,

$$\int^{x_1}u_t(x,t)\mathrm{d}x=0 \tag{23}$$

We now have:

$$u_L\xi'(t_0)-u_R\xi'(t_0)=f(u_L)-f(u_R) \tag{24}$$

t_0 is arbitrary.

$$\xi'(t)=\frac{f(u_L)-f(u_R)}{u_L-u_R} \tag{25}$$

Here, we have Rankine-Hugniot jump condition.

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$$\xi'(t) = \frac{f(u_L) - f(u_R)}{u_L - u_R} \tag{26}$$

For some example $f(u) = \frac{u^2}{2}$, $u_L = 1, u_R = 0$

$$\begin{aligned} \xi'(t) &= \frac{\frac{1}{2} - 0}{1 - 0} &(27)\\ &= \frac{1}{2} &(28)\end{aligned}$$

There are other initial conditions that still lead to two solutions.

$$\begin{aligned} u_t + uu_x &= 0 &(29)\\ u(x,0) &= \begin{cases} 0 & x < 0 \\ 1 & e \geq 0 \end{cases} &(30)\end{aligned}$$

Riemann Problem. The slope of our characteristic line is $\frac{1}{u}$.

In our solution, we have the verticals on the left side and slope $= 1$ on the right side, so the solutions do not collide. To remediate this, we add a shock in between and extend both solutions to the shock line.

R-H Jump Condition

$$\begin{aligned} \xi'(t) &= \frac{0 - \frac{1}{2}}{0 - 1} &(31)\\ &= \frac{1}{2} &(32)\end{aligned}$$

Another solution is to make a fan (paper fan)

There would be no shock and the solution is continuous. The R-H jump condition is not used.

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0\frac{x}{t} < 1 \\ 1 & \frac{x}{t} \geq 1 \end{cases} \tag{33}$$

Conservation Law: $u_t + [f(u)]_x = 0$.

If f is smooth: $u_t + f'(u)u_x = 0$.

$f'(u)$ is the speed of the characteristic.

Slope of characteristic $= \frac{1}{f'(u)}$

Note: If the solution is continuous, the $R - H$ condition gives the slope of a characteristic, not the slope of shocks.

What is the actual solution to the last problem?

$$u_t + uu_x = \epsilon u_{xx} \tag{34}$$

Therefore, if $\epsilon > 0$ is a smoothing term (as in heat) $\to C^\infty$.

Which solution is the solution that you get if you solve the last equation and let $\epsilon \to 0$?

This ends up giving us the lax entropy condition:

The characteristic curves can enter a shock as time increases, but they cannot exit (or be created) from a shock.

Solution one violates the lax entropy condition, therefore solution two is the correct solution.

<u>Theorem:</u> There exists a unique solution to any conservation law $u_t + [f(u)]_x = 0$, $u(x,0) = g(x)$, $x \in (-\infty, \infty), t \in [0, \infty)$ whose shocks satisfy the $R - H$ jump condition and the lax entropy condition.

<u>Note:</u> The fan is called a carefaction wave.

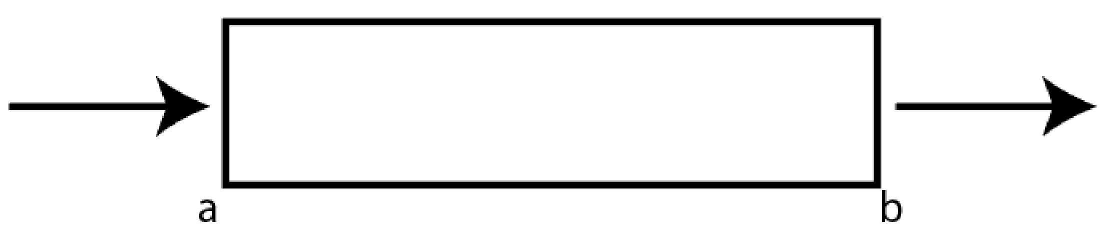
For any general first order equation $F(\vec{x}, u(\vec{x}), \nabla(\vec{x})) = 0$

$\vec{x} = (x,t)$ is our conservation law, you always have characteristic curves. The difficulty lies with how to resolve what happens when characteristic collide.

1.2 Traffic Flow

<u>**April 11, 2022**</u> Conservation of Cars

Traffic in - Traffic out = Accumulated traffic



Let $\rho(x,t)$ be the density and $q(x,t)$ is the flux.

$$\frac{d}{dt}\int_a^b \rho(x,t)\ \mathrm{d}x = q(a,t) - q(b,t) \tag{35}$$

$$= -\int_a^b q_x(x,t)\ \mathrm{d}x \tag{36}$$

$$\int_a^b (\rho_t + q_x)\ \mathrm{d}x = 0 \tag{37}$$

$$\rho_t + q_x = 0 \tag{38}$$

Here, u is car velocity:

$$q = pu \tag{39}$$

Now,

$$\rho_t + [\rho u]_x = 0 \tag{40}$$

Both ρ and u are a function of x, therefore there is a product rule that comes into play.

$$\rho_t + [c(\rho)]\rho_x = 0 \tag{41}$$

We have yet to determine c. So far, this equation is similar to the Transport Equation, where we use c to find the speed of the equation.

$c(\rho)$ will give the speed of the characteristic.

In general, the car velocity should be a decreasing function of ρ.

When $\rho = 0$, the car moves the fastest $= u_{\max}$

When $\rho = \rho_{\max} \Rightarrow u = 0$

The simplest relationship which satisfies these is:

$$u(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right) \tag{42}$$

This tells me,

$$\begin{align} q(\rho) &= u_{\max}\rho\left(1 - \frac{\rho}{\rho_{\max}}\right) \tag{43} \\ &= u_{\max}\rho - \frac{u_{\max}}{\rho_{\max}}\rho^2 \tag{44} \end{align}$$

Hit maximum velocity at $\frac{\rho_{textmax}}{2}$

$$\begin{align} q_x &= u_{\max}\rho_x - 2\frac{u_{\max}}{\rho_{\max}}\rho\rho_x \tag{45} \\ &= u_{\max}\left[1 - \frac{2}{\rho_{\max}}\rho\right]\rho_x \tag{46} \end{align}$$

Here, this shows our critical point is at $\frac{\rho_{\max}}{2}$. Let us redefine this equation as $c(\rho)\rho_x$.

$$\rho_t + u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)\rho_x = 0 \tag{47}$$

<u>Ex:</u> When a red light turns green.

$$\rho(x,t) = \begin{cases} \rho_{\max} & x < 0 \\ 0 & x > 0 \end{cases} \tag{48}$$

Recall, the speed of our characteristic:

$$c(\rho) = u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right) \tag{49}$$

The slope of our characteristic is:

$$= \frac{1}{u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)} \tag{50}$$

When we have $\rho = \rho_{\max}$, our slope is $-\frac{1}{u_{\max}}$.

When we have $\rho = 0$, our slope is $\frac{1}{u_{\max}}$.

$$\rho(x,t) = \begin{cases} \rho_{\max} & x < -u_{\max}t \\ \frac{\rho_{\max}}{2}\left(1 - \frac{x}{u_{\max}t}\right) & -u_{\max}t < x < u_{\max}t \\ 0 & x > u_{\max}t \end{cases} \tag{51}$$

<u>Ex:</u> When the light turns red, hit bumper-to-bumper traffic.

For $x < 0$, we have $\rho(x,0) = \rho_0$.

For $x > 0$, we have $\rho(x,0) = \rho_{\max}$.

Here, let us write:

$$u_t + [f(u)]_x = 0 \tag{52}$$

$$\xi'(\aleph) = \frac{f(u_L) - f(u_R)}{u_L - u_R} \tag{53}$$

Our q is:

$$q(\rho) = u_{\max}\rho\left(1 - \frac{\rho}{\rho_{\max}}\right) \tag{54}$$

If we consider our graph, a shock will form and we will have:

$$\xi'(t) = \frac{q(\rho_L) - q(\rho_R)}{\rho_L - \rho_R} \tag{55}$$

$$= \frac{u_{\max}\rho_0\left(1 - \frac{\rho_0}{\rho_{\max} - 0}\right)}{\rho_0 - \rho_{\max}} < 0 \tag{56}$$

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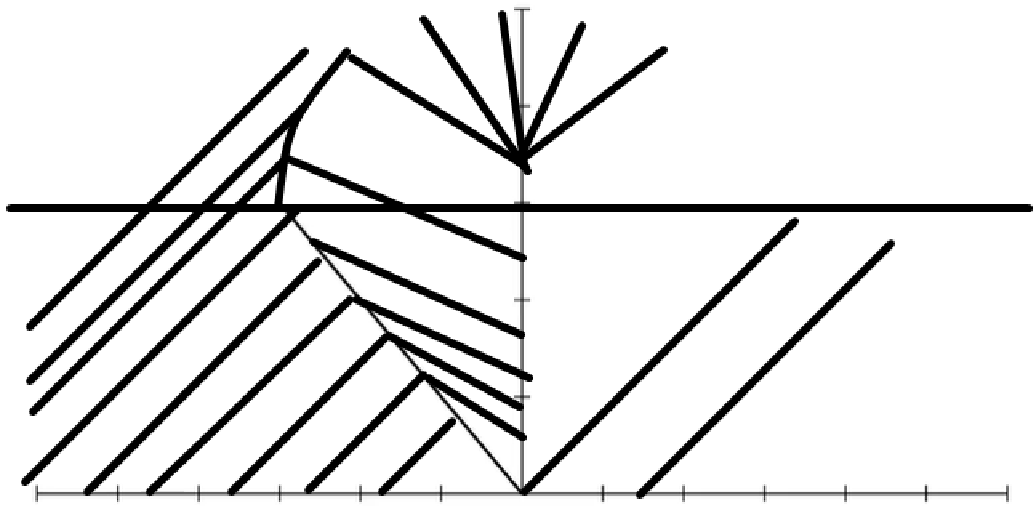
<u>Ex:</u> Green light turns red, then green



1.3 Method of Characteristics

We will look at agns? of the form:

$$a(x,t)u_x + b(x,t)u_t + c(x,t)u = 0 \tag{57}$$
$$u(x,0) = f(x) \tag{58}$$

Let:

- $x(t)$: Moving observer ¡- Location of observer
- $u(x,t)$: function of x and t

How does u change from the observer's perspective?

How does a point in front of one move?

Find $\frac{du}{dt}$:

$$\frac{du(x(t),t)}{dt} = \frac{du}{dt} + \frac{\partial u}{\partial x}\frac{dx}{dt} \tag{59}$$
$$= u_t + u_x \frac{dx}{dt} \tag{60}$$

<u>Ex:</u> Transport Equation

$$u_t + cu_x = 0 \tag{61}$$
$$\frac{dx}{dt} = c \tag{62}$$
$$\frac{du}{dt} = 0 \tag{63}$$

The observer is moving at speed c and u is not changing.

Let us consider the following:

$$u(x,0) = f(x) \tag{64}$$

$$x(0) = x_0 \tag{65}$$

$$u(0) = u_0 \tag{66}$$

Now, let us consider:

$$\frac{du}{dt} = u_t + \frac{dx}{dt}u_x \tag{67}$$

$$\frac{dx}{dt} = c \tag{68}$$

$$x = ct + k \tag{69}$$

$$x = ct + x_0 \tag{70}$$

$$\tag{71}$$

$$\frac{du}{dt} = 0 \tag{72}$$

$$u = u_0 \tag{73}$$

$$= u(x_0, 0) \tag{74}$$

$$= f(x_0) \tag{75}$$

$$= f(x - ct) \tag{76}$$

<u>Note:</u>

$$u_t + cu_x = 0 \tag{77}$$

$$\langle 1, c\rangle \cdot \langle u_t, u_x\rangle = 0 \tag{78}$$

$$\langle 1, c\rangle \cdot \langle \nabla\rangle = 0 \tag{79}$$

Here, we found the directional derivative.

<u>Ex:</u> $u_t + cu_x = 1$ and $u(x,0) = \sin x$

$$\frac{dx}{dt} = c \Rightarrow x = ct + x_0 \tag{80}$$

$$\frac{du}{dt} = 1 \Rightarrow u = t + B \tag{81}$$

$$u(x_0, t) = u_0 \tag{82}$$

$$= t + u_0 = t + u(x_0, 0) \tag{83}$$

$$= t + \sin x_0 \tag{84}$$

$$= t + \sin(x - ct) \tag{85}$$

Ex: $u_t + cu_x + au = 0$, $u(x,0) = f(x)$

$$u_t + cu_x = -au \tag{86}$$
$$\langle 1, c\rangle \cdot \langle u_t, u_x\rangle = -au \tag{87}$$

If $a > 0$, directional derivative $= -au \Rightarrow$ decay

$$\frac{dx}{dt} = c \Rightarrow x = ct + x_0 \tag{88}$$

The following is an exponential decay (Refer to Differential Equation):

$$\frac{du}{dt} = -au \tag{89}$$
$$u = De^{-at} = u_0 e^{-at} \tag{90}$$

When we plug in $t = 0$, we should get $f(x)$:

$$u = f(x_0)e^{-at} \tag{91}$$
$$= f(x - ct)e^{-at} \tag{92}$$

Ex: $u_t + xu_x = 0$, $u(x,0) = f(x)$

$$\frac{dx}{dt} = x \Rightarrow x = x_0 e^t \tag{93}$$
$$\frac{du}{dt} = 0 \Rightarrow u = u_0 = f(x_0) = \left(\frac{x}{e^t}\right) = f(xe^{-t}) \tag{94}$$

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$$\frac{du}{dt} = \frac{dx}{dt}u_x + u_t \tag{95}$$

We can also introduce a new variable,

$$a(x,t)u_x + b(x,t)u_t + c(x,t)u = 0 \tag{96}$$
$$\frac{dx}{ds} = a(x,t) \tag{97}$$

$$\frac{dx}{ds} = a(x,t) \tag{98}$$
$$\frac{dt}{ds} = b(x,t) \tag{99}$$
$$\frac{dy}{ds} = -c(x,t)u \tag{100}$$
$$\frac{du}{ds} = \frac{dx}{ds}u_x + \frac{dt}{ds}u_t \tag{101}$$

$$\begin{tikzcd} & u \arrow[dl, dash, "\frac{\partial u}{\partial x}"'] \arrow[dr, dash, "\frac{\partial u}{\partial t}"] & \\ x \arrow[d, dash, "\frac{\partial x}{\partial s}"] & & t \arrow[d, dash, "\frac{\partial t}{\partial s}"] \\ s & & s \end{tikzcd}$$

<u>Ex:</u> $xu_x + u_t + tu = 0 \Rightarrow xu_x + u_t = -tu$

$$\frac{dx}{ds} = x \Rightarrow x = ce^s = x_0 e^s = x_0 e^t \tag{102}$$
$$\frac{dt}{ds} = 1 \Rightarrow t = s + t_0 = s \tag{103}$$
$$\frac{du}{ds} = -tu = -su \Rightarrow u = f\left(xe^{-t}\right) e^{-\frac{1}{2}t^2} \tag{104}$$

To find the third differential, we did separation of variables:

$$\int \frac{1}{u} du = \int -s ds \tag{105}$$
$$\ln|u| = -\frac{1}{2}s^2 + c \tag{106}$$
$$u = e^{-\frac{1}{2}s^2 + c} \tag{107}$$
$$u = ce^{-\frac{1}{2}s^2} \tag{108}$$
$$u = u_0 e^{-\frac{1}{2}t^2} \tag{109}$$
$$u = f(x_0) e^{-\frac{1}{2}t^2} \tag{110}$$
$$u = f\left(xe^{-t}\right) e^{-\frac{1}{2}t^2} \tag{111}$$

Because $u(x0) = f(x)$ and $x_0 = \frac{x}{e^t} = xe^{-t}$

<u>Ex:</u> $2xtu_x + u_t = u$, $u(x, 0) = x$

Because $\frac{dt}{ds} = 1$, we do not consider s.

$$\frac{dx}{dt} = 2xt \Rightarrow x = x_0 e^{t^2} \tag{112}$$
$$\frac{du}{dt} = u \Rightarrow u = u_0 e^t = f(x_0) e^t = f(xe^{-t^2}) e^t = xe^{-t^2} e^t \tag{113}$$

Recall $u_0 = f(x)$.

Here, we found $\frac{dx}{dt}$ through separation of variables:

$$\frac{dx}{dt} = 2xt \tag{114}$$
$$\int \frac{1}{x} = \int 2t dt \tag{115}$$
$$\ln|x| = t^2 + c \tag{116}$$
$$x = ce^{t^2} \tag{117}$$
$$x = x_0 e^{t^2} \tag{118}$$

Ex: $u^2\frac{du}{dx}+\frac{du}{dt}=0$, $u(x,0)=\sqrt{x}$.

$$\frac{dx}{dt}=u^2 \Rightarrow x=u^2t+c=u^2t+x_0 \Rightarrow x_0=x-u^2t \tag{119}$$
$$\frac{du}{dt}=0 \Rightarrow u=u_0=f(x_0)=f(x-u^2t)=\sqrt{x-u^2t}=\sqrt{\frac{x}{1+t}} \tag{120}$$

Here, we solve for u as:

$$u=\sqrt{x-u^2t} \tag{121}$$
$$u^2=x-u^2t \tag{122}$$
$$u^2(1+t)=x \tag{123}$$
$$u^2=\frac{x}{1+t} \tag{124}$$
$$u=\sqrt{\frac{x}{1+t}} \tag{125}$$

Ex: $e^{t^2}u_t+tu_x=0$, $u(x,0)=f(x)$.

Here, let us divide out our term in front of u_t to get $u_t+te^{-t^2}u_x=0$,

$$\frac{dx}{dt}=te^{-t^2} \Rightarrow x= \tag{126}$$
$$\frac{du}{dt}=0 \Rightarrow u=u_0=f(x_0)=f(x+\frac{1}{2}e^{-t^2}-\frac{1}{2}) \tag{127}$$

To solve for x,

$$x=\int te^{-t^2} \tag{128}$$
$$=-\frac{1}{2}\int e^w \mathrm{d}w \tag{129}$$
$$=-\frac{1}{2}e^w+c \tag{130}$$
$$=-\frac{1}{2}e^{-t^2}+C \tag{131}$$

Here, we want our term to zero out:

$$x_0=-\frac{1}{2}+c \Rightarrow c=x_0+\frac{1}{2} \tag{132}$$
$$x=-\frac{1}{2}e^{t^2}+x_0+\frac{1}{2} \tag{133}$$
$$x_0=x+\frac{1}{2}e^{-t^2}-\frac{1}{2} \tag{134}$$

Ex: $u_t+tu_x=u^2$, $u(x,0)=f(x)$

$$\frac{dx}{dt}=t \Rightarrow x=\frac{1}{2}t^2+c=\frac{1}{2}t^2+x_0 \Rightarrow x_0=x-\frac{1}{2}t^2 \tag{135}$$
$$\frac{du}{dt}=u^2 \Rightarrow u= \tag{136}$$

Here, to solve the second term,

$$\frac{du}{dt} = u^2 \tag{137}$$

$$\int \frac{1}{u^2}\mathrm{d}u = \int \mathrm{d}t \tag{138}$$

$$-\frac{1}{u} = t + c \tag{139}$$

$$\frac{1}{u} = -t + c \tag{140}$$

$$u = \frac{1}{c - t} \tag{141}$$

$$= \frac{1}{\frac{1}{u_0} - t} \tag{142}$$

$$= \frac{u_0}{1 - u_0 t} \tag{143}$$

$$= \frac{f(x_0)}{1 - f(x_0)t} \tag{144}$$

$$= \frac{f\left(x - \frac{1}{2}t^2\right)}{1 - f\left(x - \frac{1}{2}t^2\right)t} \tag{145}$$

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<u>Ex:</u> $tu_x - xu_t = u \quad u(x,0) = f(x)$

$$\frac{dx}{ds} = t \tag{146}$$

$$\frac{dt}{ds} = -x \tag{147}$$

$$\frac{du}{ds} = u \tag{148}$$

Here, the issue with this format is that the first two terms have three variables. We must find x_0 and to get x_0, we must get to x. To solve this, we will look at parametric equations. Let us combine the first two lines to get:

$$\frac{dx}{dt} = -\frac{t}{x} \tag{149}$$

From here, we can solve this via separation of variables

$$\int x\mathrm{d}x = \int -t\mathrm{d}t \tag{150}$$

$$\frac{1}{2}x^2 = -\frac{1}{2}t^2 + C \tag{151}$$

$$x^2 = -t^2 + C \tag{152}$$

$$x^2 + t^2 = C \tag{153}$$

$$x^2 + t^2 = x_0^2 \tag{154}$$

Now, to solve for our third term, we would get an exponential via separation of variables:

$$\frac{du}{ds} = u \Rightarrow u = u_0 e^s \tag{155}$$

Here, if we want to find t for s, we write:

$$\frac{dx}{ds} = t \Rightarrow \frac{d^2x}{ds^2} = \frac{dt}{ds} = -x \Rightarrow x(s) = a\cos s + b\sin s \tag{156}$$
$$\Rightarrow x(0) = x_0 \to a = x_0 \tag{157}$$
$$\frac{dt}{ds} = -x \Rightarrow \frac{d^2t}{ds^2} = -\frac{dx}{ds} = -t \Rightarrow t(s) = b\cos s - a\sin s \tag{158}$$
$$\Rightarrow t(0) = 0 \to b = 0 \tag{159}$$

Since want to start t at 0, sin would belong to t.

Now, we know the following:

$$x = x_0\cos s \tag{160}$$
$$t = -x_0\sin s \tag{161}$$
$$\frac{t}{x} = -\tan s \tag{162}$$

So, to go back to $\frac{du}{ds}$,

$$\frac{du}{ds} = u \Rightarrow u = u_0e^s = f(x_0)e^{\arctan\frac{-t}{x}} \tag{163}$$
$$= f\left(\sqrt{x^2+t^2}\right)e^{\arctan\frac{-t}{x}} \tag{164}$$

<u>Ex:</u> $xu_x + tu_t = 2u \quad u(x_0, 1) = f(x)$

$$\frac{dx}{ds} = x \Rightarrow_1 x = x_0e^s \Rightarrow_4 x_0 = \frac{x}{e^s} = \frac{x}{t} \tag{165}$$
$$\frac{dt}{ds} = t \Rightarrow_2 t = t_0e^s = e^s \tag{166}$$
$$\frac{du}{ds} = 2u \Rightarrow_3 u = u_0e^{2s} \Rightarrow_5 u = f(x_0)e^{2s} = f\left(\frac{x}{t}\right)(e^s)^2 = f\left(\frac{x}{t}\right)t^2 \tag{167}$$

<u>Ex:</u> $u_t + tu_x = 0$, $u(x,0) = f(x)$

$$\frac{dx}{dt} = t \Rightarrow_1 x = \frac{1}{2}t^2 + x_0 \tag{168}$$
$$\frac{du}{dt} = 0 \Rightarrow u = u_0 = f(x_0) = f\left(x - \frac{1}{2}t^2\right) \tag{169}$$

<u>Ex:</u> $u_t + tu_x = xt$, $u(x,0) = f(x)$

$$\frac{dx}{dt} = t \Rightarrow_1 x = \frac{1}{2}t^2 + x_0 \tag{170}$$
$$\frac{du}{dt} = xt = t\left(\frac{1}{2}t^2 + x_0\right) = \frac{1}{2}t^3 + x_0t \tag{171}$$
$$\Rightarrow_2 u = \frac{1}{8}t^4 + \frac{1}{2}x_0t^2 + u_0 = \frac{1}{8}t^4 + \frac{1}{2}\left(x - \frac{1}{2}t^2\right)t^2 + f\left(x - \frac{1}{2}t^2\right) \tag{172}$$

Ex: $u_t + xu_x = x$, $u(x,0) = f(x)$

$$\frac{dx}{dt} = x \Rightarrow_1 x_0e^t \tag{173}$$
$$\frac{du}{dt} = x =_2 x_0e^t \Rightarrow u = x_0e^t + C = x_0e^t + u_0 - x_0 = x + f(xe^{-t}) - xe^{-t} \tag{174}$$

Ex: $xu_x + u_t = t$, $u(x,0) = x^2$

$$\frac{dx}{dt} = x \Rightarrow_1 x = x_0e^t \tag{175}$$
$$\frac{du}{dt} = t \Rightarrow_2 u = \frac{1}{2}t^2 + u_0 = \frac{1}{2}t^2 + f\left(xe^{-t}\right) = \frac{1}{2}t^2 + \left(xe^{-t}\right)^2 \tag{176}$$

Ex: $xu_t - 2xtu_x = 2tu$, $u(x,0) = f(x)$

Let us rewrite our equation as $u_t - 2tu_x = \frac{2tu}{x}$

$$\frac{dx}{dt} = -2t \Rightarrow_1 x = -t^2 + x_0 \tag{177}$$
$$\frac{du}{dt} = \frac{2tu}{x} =_2 \frac{2tu}{x_0 - t^2} \tag{178}$$

For the second term, we would have to separate:

$$\int \frac{1}{u}\mathrm{d}u = \int \frac{2t}{x_0 - t^2}\mathrm{d}t \tag{179}$$
$$\ln|u| = \int \frac{2t}{x_0 - t^2}\mathrm{d}t \tag{180}$$

Here, our $w = x_0t^2$ and $dw = -2tdt$

$$\ln|u| = -\ln|x_0 - t^2| + C \tag{181}$$
$$u = ce^{-\ln|x_0 - t^2|} \tag{182}$$
$$u = ce^{\ln|x_0 - t^2|^{-1}} \tag{183}$$
$$u = \frac{c}{x_0 - t^2} \tag{184}$$

Here, we know x_0 and we can find c since plugging in 0 should give us u_0:

$$= \frac{x_0u_0}{x_0 - t^2} \tag{185}$$
$$u = \frac{(x + t^2)f(x_0 + t^2)}{x} \tag{186}$$

2 Wave Equation on Semi-Infinite Domain

- $x \in [0, \infty), t \in [0, \infty)$
- $u_{tt} = c^2u_{xx}$
- $u(x,0) = f(x)$
- $u_t(x,0) = g(x)$

- $u(0,t)=0$

Recall: If $x \in (-\infty, \infty)$, we use d'Alembert's Formula:

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\ \mathrm{d}y \tag{187}$$

We would like to use the solution to the wave equation for $x \in (-\infty, \infty)$ to help solve the wave equation when $x \in [0, \infty)$.

To do this, we use the odd extension of the initial conditions:

$$\widetilde{f}(x) = \begin{cases} f(x) & x > 0 \\ 0 & x = 0 \\ -f(-x) & x < 0 \end{cases} \tag{188}$$

$$\widetilde{g}(x) = \begin{cases} f(x) & x > 0 \\ 0 & x = 0 \\ -f(-x) & x < 0 \end{cases} \tag{189}$$

This system can be solved using d'Alembert's Formula:

$$u(x,t) = \frac{1}{2}\left[\widetilde{f}(x+ct) + \widetilde{f}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \widetilde{g}(y)\ \mathrm{d}y \tag{190}$$

<u>Note:</u> This solves some PDE on $[0, \infty)$, since it solves it on $(-\infty, \infty)$.

<u>Note:</u> $u(0,t) = \frac{1}{2}\left[\widetilde{f}(ct) + \widetilde{f}(-ct)\right] + \frac{1}{2}\int_{-ct}^{ct} \widetilde{g}(y)\ \mathrm{d}y$, but our integral will zero out since it is odd. In addition, since our functions are odd, the $\widetilde{f}$ will cancel out as well. **<u>Case 1</u>**: $x - ct > 0$

$$u(x,t) = \frac{1}{2}\left[\widetilde{f}(x+ct) + \widetilde{f}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \widetilde{g}(y)\ \mathrm{d}y \tag{191}$$

$$= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\ \mathrm{d}y \tag{192}$$

Staying on the right, we do not hit a wall and nothing changes.

<u>Case 2</u>: $x - ct < 0$

$$u(x,t) = \frac{1}{2}\left[\widetilde{f}(x+ct) + \widetilde{f}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \widetilde{g}(y)\ \mathrm{d}y \tag{193}$$

$$= \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\left[\int_{x-ct}^{0} \widetilde{g}(y)\ \mathrm{d}y + \int_{0}^{x+ct} \widetilde{g}(y)\ \mathrm{d}y\right] \tag{194}$$

$$= \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\left[-\int_{x-ct}^{0} g(-y)\ \mathrm{d}y + \int_{0}^{x+ct} g(y)\ \mathrm{d}y\right] \tag{195}$$

Here, let us perform substitution with $w = -y$,

$$= \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\left[\int_{ct-x}^{0} g(w)\ \mathrm{d}w + \int_{0}^{x+ct} g(y)\ \mathrm{d}y\right] \tag{196}$$

$$= \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\left[\int_{ct-x}^{x+ct} g(y)\ \mathrm{d}y\right] \tag{197}$$

If we look at the domain of dependence, the left line reflect back to our domain and the line is represented as $ct - x$.

<u>Ex:</u> $u_{tt} = u_{xx}, x \in [0, \infty)$

$$u(x,0) = \begin{cases} 1 & 4 < x < 5 \\ 0 & \text{otherwise} \end{cases} \tag{198}$$

$$u_t(x,0) = 0 \tag{199}$$

3 D'Alembert's Formula on a Bounded Domain

$$u_{tt} = c^2 u_{xx} \quad u(x,0) = f(x) \quad u_t(x,0) = g(x) \quad 0 \le x \le L, t \in [0,\infty)$$

How do we find the solution to the wave equation on $(-\infty, \infty)$ to find a solution on $[0, L]$?

We extend the initial conditions to be odd and periodic with period $2L$.

$$\widetilde{f}(x) = \begin{cases} f(x) & 0 < x < L \\ 0 & x = 0 \\ -f(-x) & -L < x < 0 \end{cases} \tag{200}$$

Recall, we considered boundary conditions. Here, let us define boundary conditions as:

$$u(0,t) = u(L,t) = 0$$

Here, let us enforce $\widetilde{f}(x+2L) = \widetilde{f}(x)$ to force periodicity. For $\widetilde{g}(x)$, let us write:

$$\widetilde{g}(x) = \begin{cases} g(x) & 0 < x \le L \\ 0 & x = 0 \\ -g(-x) & -L < x < 0 \end{cases} \tag{201}$$

The solution will be:

$$\begin{align} u(x,t) &= \frac{1}{2}\left[\widetilde{f}(x+ct) + \widetilde{f}(x-ct)\right] + \frac{1}{2}\int_{x-ct}^{x+ct} \widetilde{g}(y)\ \mathrm{d}y \tag{202} \\ u(0,t) &= \frac{1}{2}\left[\widetilde{f}(ct) + \widetilde{f}(-ct)\right] \tag{203} \\ u(L,t) &= \frac{1}{2}\left[\widetilde{f}(L+ct) + \widetilde{f}(L-ct)\right] + \frac{1}{2c}\int_{L-ct}^{L+ct} \widetilde{g}(y)\ \mathrm{d}y \tag{204} \\ &= \frac{1}{2}\left[\widetilde{f}(ct-L) + \widetilde{f}(L-ct)\right] + \frac{1}{2c}\left(\int_{L-ct}^{0} \widetilde{g}(y)\ \mathrm{d}y + \int_{0}^{L+ct} \widetilde{g}(y)\ \mathrm{d}y\right) \tag{205} \\ &= \frac{1}{2}\left[\widetilde{-f}(L-ct) + \widetilde{f}(L-ct)\right] + \frac{1}{2c}\left(\int_{ct-L}^{0} \widetilde{g}(y)\ \mathrm{d}y + \int_{0}^{L+ct} \widetilde{g}(y)\ \mathrm{d}y\right) \tag{206} \\ &= \frac{1}{2c}\left(\int_{ct+L}^{0} \widetilde{g}(y)\ \mathrm{d}y + \int_{0}^{L+ct} \widetilde{g}(y)\ \mathrm{d}y\right) \tag{207} \end{align}$$

We added an integral of length $2L$, which is 0 since $\widetilde{g}$ is odd.