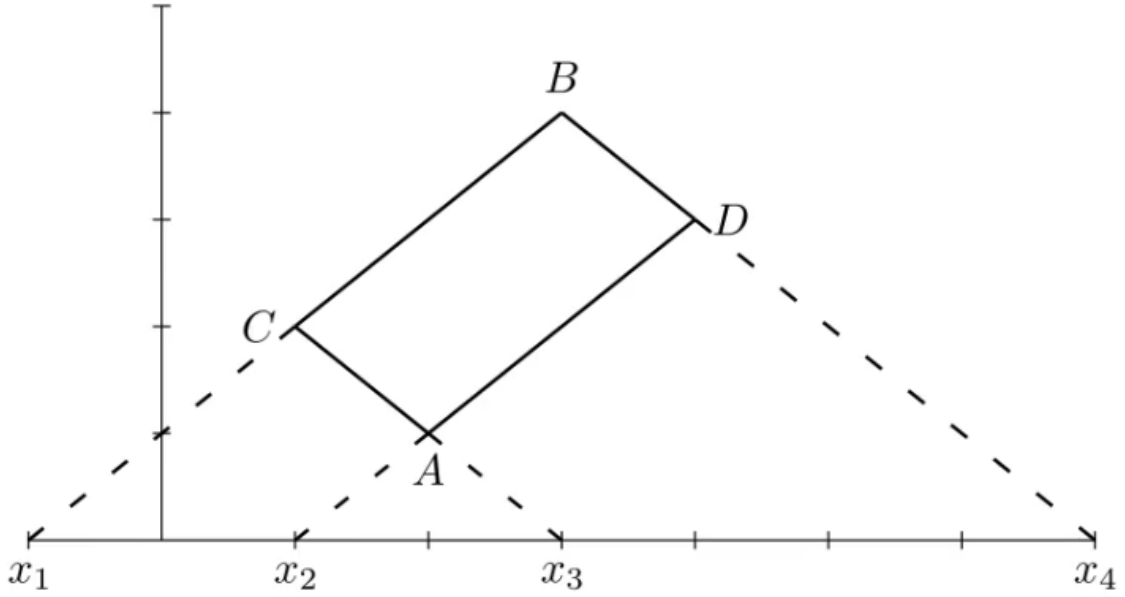


1. Use D'Alembert's formula to show the parallelogram property of the wave equation mentioned in class.



$$u(A) + u(B) = u(C) + u(D) \quad (1)$$

Note that our slope depends on c . Now, let us consider D'Alembert's Formula:

$$\frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} g(y) dy \quad (2)$$

Now, let us consider using D'Alembert's Formula to generate the following equations:

$$u(A) = \frac{1}{2}[f(x_2) + f(x_3)] + \frac{1}{2c} \int_{x_2}^{x_3} g(y) dy \quad (3)$$

$$u(B) = \frac{1}{2}[f(x_1) + f(x_4)] + \frac{1}{2c} \int_{x_1}^{x_4} g(y) dy \quad (4)$$

$$u(C) = \frac{1}{2}[f(x_1) + f(x_3)] + \frac{1}{2c} \int_{x_1}^{x_3} g(y) dy \quad (5)$$

$$u(D) = \frac{1}{2}[f(x_2) + f(x_4)] + \frac{1}{2c} \int_{x_2}^{x_4} g(y) dy \quad (6)$$

From here, let us evaluate $u(A) + u(B)$ and $u(C) + u(D)$

$$u(A) + u(B) = \frac{1}{2}[f(x_2) + f(x_3)] + \frac{1}{2c} \int_{x_2}^{x_3} g(y) dy + \frac{1}{2}[f(x_1) + f(x_4)] + \frac{1}{2c} \int_{x_1}^{x_4} g(y) dy \quad (7)$$

$$= \frac{1}{2} \left(f(x_1) + f(x_4) + f(x_2) + f(x_3) + \frac{1}{c} \left[\int_{x_1}^{x_4} g(y) dy + \int_{x_2}^{x_3} g(y) dy \right] \right) \quad (8)$$

Next, evaluate $u(C) + u(D)$:

$$u(C) + u(D) = \frac{1}{2}[f(x_1) + f(x_3)] + \frac{1}{2c} \int_{x_1}^{x_3} g(y) dy + \frac{1}{2}[f(x_2) + f(x_4)] + \frac{1}{2c} \int_{x_2}^{x_4} g(y) dy \quad (9)$$

$$= \frac{1}{2} \left(f(x_1) + f(x_3) + f(x_2) + f(x_4) + \frac{1}{c} \left[\int_{x_1}^{x_3} g(y) dy + \int_{x_2}^{x_4} g(y) dy \right] \right) \quad (10)$$

If we analyze the regions of our integral, we can observe the interval length of the integral for $u(A) + u(B)$ spans over 10 units. In addition, $u(C) + u(D)$ also spans over 10 intervals once again. Here, both intervals are equal. Therefore,

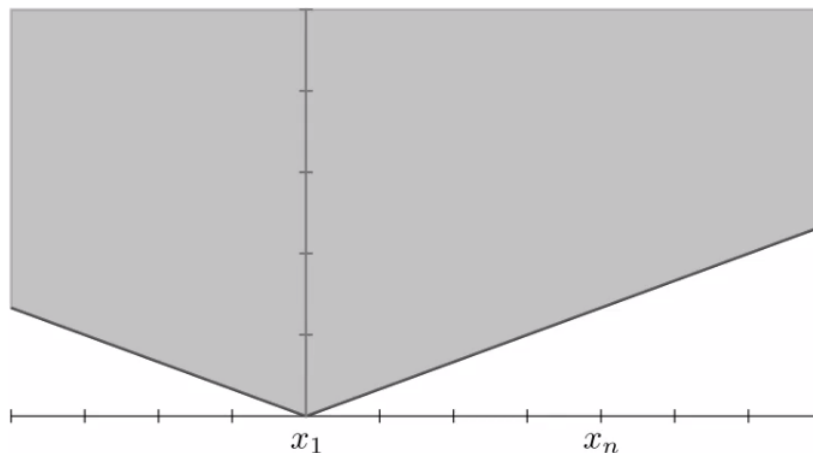
$$u(A) + u(B) = u(C) + u(D) \quad (11)$$

2. If $f(x)$ and $g(x)$ are changed on the region $x \in [0, 4]$, on which region in the (x, t) -plane will the solutions of $u_{tt} = 9u_{xx}$ be altered?

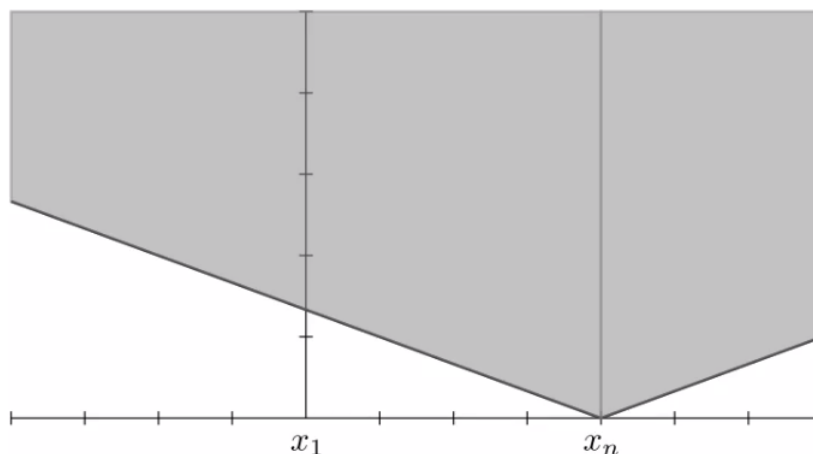
Here, we are given a wave equation on the x boundary $[0, 4]$ and a constant 3^2 .

Here, let us consider our wave equation, $u_{tt} = 9u_{xx}$, where $\sqrt{c} = 3$.

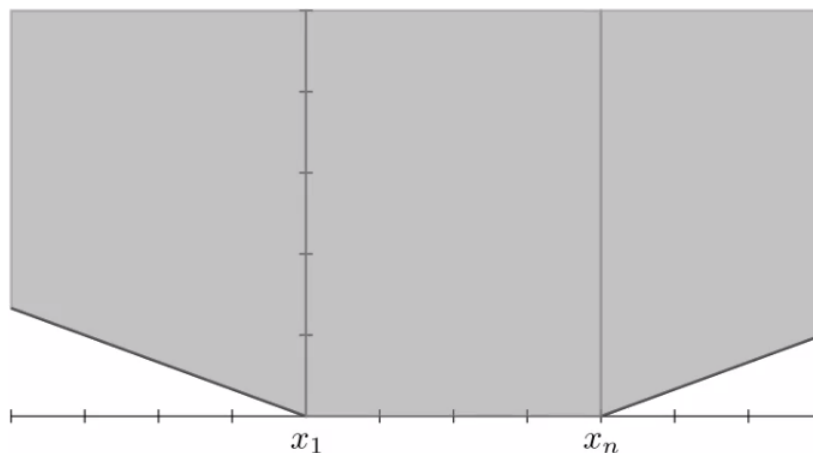
Here, the slope of our characteristic line is $\frac{1}{3}$. If we alter $f(x)$ and $g(x)$, we are changing the range of influence of our equation. For instance, let us pick $x_1 = 1$ for $f(x)$, we would obtain a range of influence such as the following:



Next, let us consider another point within our boundary, such as x_n , let us consider how our graph appears at $f(x_n)$:



In essence, our range of influence as we tweak $f(x)$ and $g(x)$ lies between the points x_1 and x_n :



3. The solution to the non-homogeneous Laplace equation $\Delta u = f(x, y)$ on $x \in (-\infty, \infty), y \in (-\infty, \infty)$ is:

$$u(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x - \xi, y - \eta) f(\xi, \eta) d\xi d\eta \quad (1)$$

where

$$k(x, y) = -\frac{1}{2\pi} \ln \left(\sqrt{x^2 + y^2} \right) \quad (2)$$

Show that if $f(\xi, \eta) = \delta(\xi)\delta(\eta)$, then $\Delta u = 0$ for $(x, y) \neq (0, 0)$.

Let us consider the given equation 1). Here, let us use given assumption, $f(\xi, \eta) = \delta(\xi)\delta(\eta)$ and substitute it into 1)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x - \xi, y - \eta) f(\xi, \eta) d\xi d\eta = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x - \xi, y - \eta) \delta(\xi)\delta(\eta) d\xi d\eta \quad (3)$$

Let us consider our function, k . Equation 2) defines the function of k . Let us evaluate our function with the given parameters, $x - \xi$ and $y - \eta$:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x - \xi, y - \eta) \delta(\xi)\delta(\eta) d\xi d\eta = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{1}{2\pi} \left(\ln \sqrt{(x - \xi)^2 + (y - \eta)^2} \right) \delta(\xi)\delta(\eta) d\xi d\eta \quad (4)$$

Here, let us consider our δ function and ways to manipulate the function. Here, we have the property:

$$\int_{-\infty}^{\infty} \delta(x - y) f(y) dy = f(x) \quad (5)$$

If we apply it to equation 4, we get:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{1}{2\pi} \left(\ln \sqrt{(x - \xi)^2 + (y - \eta)^2} \right) \delta(\xi)\delta(\eta) d\xi d\eta = -\frac{1}{2\pi} \ln \left(\sqrt{x^2 + y^2} \right) \quad (6)$$

Now, let us take the x and y partial of line 6)

$$u(x, y) = -\frac{1}{2\pi} \ln \left(\sqrt{x^2 + y^2} \right) \quad (7)$$

$$u_{xx}(x, y) + u_{yy}(x, y) = \left(-\frac{1}{2\pi} \frac{x}{x^2 + y^2} \right)_x + \left(-\frac{1}{2\pi} \frac{y}{x^2 + y^2} \right)_y \quad (8)$$

$$= \left(-\frac{1}{2\pi} \frac{y^2 - x^2}{(x^2 + y^2)^2} \right) \left(-\frac{1}{2\pi} \frac{x^2 - y^2}{(x^2 + y^2)^2} \right) \quad (9)$$

$$= \left(\frac{1}{2\pi} \frac{x^2 - y^2}{(x^2 + y^2)^2} \right) \left(\frac{1}{2\pi} \frac{y^2 - x^2}{(x^2 + y^2)^2} \right) \quad (10)$$

$$= \frac{1}{2\pi} \left(\frac{x^2 - y^2 + y^2 - x^2}{(x^2 + y^2)^2} \right) \quad (11)$$

$$= 0 \quad (12)$$

4. Show the following:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} dx = 1$$

We want to find the integral of the following:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} dx \quad (1)$$

First, let us move the constant out of our integral:

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-x^2/4t} dx \quad (2)$$

From here, let us rename our constant on the outside of our integral as ζ :

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-x^2/4t} dx = \zeta \int_{-\infty}^{\infty} e^{-x^2/4t} dx \quad (3)$$

Here, let us focus on our integral. First, let us square our integral and change our variables in the second integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2/4t} dx \quad (4)$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2/4t} dx \int_{-\infty}^{\infty} e^{-y^2/4t} dy \quad (5)$$

From here, let us find the product of our integrals then combine our powers:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/4t} e^{-y^2/4t} dx dy \quad (6)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/4t} dx dy \quad (7)$$

$$(8)$$

Here, let us write our integral and variables in terms of polar coordinates:

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2/4t} r \, dr d\theta \quad (9)$$

$$= \int_0^{\infty} \int_0^{2\pi} e^{-r^2/4t} r \, d\theta dr \quad (10)$$

$$= 2\pi \int_0^{\infty} r e^{-r^2/4t} \, dr \quad (11)$$

$$(12)$$

Here, let us perform u-substitution, where we write $u = \frac{r^2}{4t}$ and $du = \frac{r}{2t} dr$

$$I^2 = 4\pi t \int_0^{\infty} e^{-u} \, du \quad (13)$$

$$I^2 = 4\pi t \quad (14)$$

$$I = \sqrt{4\pi t} \quad (15)$$

Here, let us plug our evaluation back to line 3 to find the solution:

$$\frac{1}{\sqrt{4\pi t}} \sqrt{4\pi t} = 1 \quad (16)$$

5. We know that the solution to the 2-D heat equation $u_t = u_{xx} + u_{yy}$, with $u(x, y, 0) = f(x, y)$ is

$$u(x, y, t) = \frac{1}{4\pi t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) e^{-\frac{(x-\xi)^2 + (y-\eta)^2}{4t}} d\xi d\eta \quad (1)$$

If

$$f(x, y) = \begin{cases} 1 & 2 \leq r \leq 4, r = \sqrt{x^2 + y^2} \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Sketch $u(x, y, t)$ for different t values, say $t = 0, 5, 100, \infty$

Note, as t increases, the sharp edges near the top of the cylinder tend to smooth out. The drawings do not accurately represent this detail.

