1. Let $f(x)$ be a 2π-period function on the interval $[-\pi, \pi]$ where $f(x) = \begin{cases} -1 & -\pi < x \leq 0 \\ 1 & 0 < x \leq \pi \end{cases}$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \text{ dx} \tag{1}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \text{ dx} \tag{2}$$

$$b_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \text{ dx} \tag{3}$$

(a) Plot the function on the interval $[-3\pi, 3\pi]$

(b) Plot its (infinite) Fourier series on $[-3\pi, 3\pi]$

(c) Find the Fourier series of $f(x)$

Here, let us consider the symmetry of our function.

When we look at the graph of $f(x)$, we can see there is a reflection about the origin, making the function odd. sin is also an odd function, therefore a_n is an even function.

Looking at b_n, cos is an even function, therefore b_n becomes an odd function.

Finally, b_0 is always an odd function. When we integrate these three coefficients, we lose b_n and b_0, but keep a_n. Since a_n is even, we can write:

$$a_n = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \text{ dx} \tag{4}$$

Here, we are looking at the interval from 0 to L. Our given function, $f(x)$ runs from $-\pi$ to π, therefore our integral is:

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} 1 \cdot \sin\left(\frac{n\pi x}{\pi}\right) \text{ dx} \tag{5}$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \sin(nx) \text{ dx} \tag{6}$$

From here, we can compute our integral:

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} \sin(nx) \text{ dx} \tag{7}$$

$$= -\frac{2}{\pi n} \cos(nx) \Big|_{0}^{\pi} \tag{8}$$

$$= \frac{2}{\pi n}(1 - \cos(n\pi)) \tag{9}$$

Here, we found our coefficient, a_n. Now, since $f(x)$ is odd, we are only interested in the following:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \tag{10}$$

$$= \sum_{n=1}^{\infty} \frac{2}{\pi n}(1 - \cos(n\pi)) \sin\left(\frac{n\pi x}{L}\right) \tag{11}$$

Here, since our interval is $-\pi$ to π, so let us write:

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n}(1 - \cos(n\pi)) \sin\left(\frac{n\pi x}{\pi}\right) \tag{12}$$

$$= \sum_{n=1}^{\infty} \frac{2}{\pi n}(1 - \cos(n\pi)) \sin(nx) \tag{13}$$

Here, we have our Fourier series.

2. Let $f(x) = x^2$ be a 2π-periodic function on the interval $[-\pi, \pi]$.

   (a) Derive its Fourier series

   Let us consider the symmetry of our function. Our function, $f(x)$, is an even function. Therefore, we have the following coefficients:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ \mathrm{dx} \tag{14}$$

$$b_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ \mathrm{dx} \tag{15}$$

   Since $f(x)$ is even, we can write:

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ \mathrm{dx} \tag{16}$$

$$b_0 = \frac{1}{L}\int_{0}^{L} f(x)\ \mathrm{dx} \tag{17}$$

   In addition, since we also know our interval and our function, we can write:

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} x^2\cos(nx)\ \mathrm{dx} \tag{18}$$

$$b_0 = \frac{1}{\pi}\int_{0}^{\pi} x^2\ \mathrm{dx} \tag{19}$$

   First, let us find the integral of b_n. Let us rewrite b_n first:

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} x^2\cos(nx)\ \mathrm{dx} \tag{20}$$

   Here, we want to do integration by parts. We want x^2 as our derived function since we can derive that function to 0.

x^2	$\cos(nx)$
$2x$	$\frac{1}{n}\sin(nx)$
2	$-\frac{1}{n^2}\cos(nx)$
0	$-\frac{1}{n^3}\sin(nx)$

   Here, we can write our integral as the following:

$$b_n = \frac{2}{\pi}\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]_0^{\pi} \tag{21}$$

$$= \frac{2}{\pi n}\left[x^2\sin(nx) + \frac{2x}{n}\cos(nx) - \frac{2}{n^2}\sin(nx)\right]_0^{\pi} \tag{22}$$

$$= \frac{2}{\pi n}\left[\pi^2\sin(\pi x) + \frac{2\pi}{n}\cos(n\pi) - \frac{2}{n^2}\sin(n\pi)\right] - \frac{2}{n\pi}\left[0^2\sin(0) + \frac{0}{n}\cos(0) - \frac{2}{n^2}\sin(0)\right] \tag{23}$$

   Here, the entire right term zeroes out. On the left, $\sin(n\pi)$ zeroes out, leaving us with:

$$b_n = \frac{4}{n^2}\cos(n\pi) \tag{24}$$

   Now, let us find b_0:

$$b_0 = \frac{1}{\pi}\int_0^{\pi} x^2\ \mathrm{dx} \tag{25}$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \tag{26}$$

$$= \frac{1}{3\pi}\left[x^3\right]_0^{\pi} \tag{27}$$

$$= \frac{1}{3\pi}\left[\pi^3 - 0\right] \tag{28}$$

$$= \frac{\pi^2}{3} \tag{29}$$

   Now that we have our coefficients, we can write:

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(nx)\sin(nx) \tag{30}$$

(b) Use Maple of Matlab to plot its finite Fourier series on $[-\pi, \pi]$ for $N = 10, 20, 50$ together with $f(x)$

(c) Use your Fourier series from part (a) to show that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$.

3. In the solution of the heat equation, we end up solving $X'' = -\lambda X$. Show that if $\lambda < 0$ or $\lambda = 0$ there is only the trivial solution ($X(x) = 0$).

   Here, we have the equation:

$$X'' = -\lambda X \tag{31}$$

   We want to use this equation and set our boundary conditions as $X(0) = X(L) = 0$. Now, we must find an equation where after two derivatives on the right, we obtain a similar function on the left. On the left, we have a sign, coefficient, and function of x. Let us write a general solution for our equation:

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \tag{32}$$

   Here, we can make three assumptions ***via trichotomy***: $\lambda < 0, \lambda = 0$, or $\lambda > 0$. Let us look at the first two examples:

   (a) $\lambda < 0$

   Here, let us consider the case when λ is negative. Let us consider rewriting λ:

$$\lambda < 0 \tag{33}$$
$$\lambda \cdot -1 > 0 \cdot -1 \tag{34}$$
$$-1 \cdot \lambda > 0 \tag{35}$$

   Now, let us plug in our found value into our general equation:

$$X(x) = A\cos(\sqrt{-1 \cdot \lambda}x) + B\sin(\sqrt{-1 \cdot \lambda}x) \tag{36}$$

   Let us separate the terms under the radical:

$$X(x) = A\cos(\sqrt{-1 \cdot \lambda}x) + B\sin(\sqrt{-1 \cdot \lambda}x) \tag{37}$$
$$= A\cos(\sqrt{-1}\sqrt{\lambda}x) + B\sin(\sqrt{-1}\sqrt{\lambda}x) \tag{38}$$
$$= A\cos(i\sqrt{\lambda}x) + B\sin(i\sqrt{\lambda}x) \tag{39}$$

   Here, in our expression, we see we are taking the square root of a negative number, which would give us an imaginary number. Here, we are evaluating our general solution with real numbers, therefore, the following form:

$$X(x) = A\cos(i\sqrt{\lambda}x) + B\sin(i\sqrt{\lambda}x) \tag{40}$$

   Where $X(x)$ is a real number would only have the trivial solution $X(x) = 0$.

   (b) $\lambda = 0$

   Here, let us consider the case when λ is zero. Now, let us write our general equation:

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \tag{41}$$

   Here, since $\lambda = 0$, we can evaluate our equation:

$$X(x) = A\cos(0) + B\sin(0) \tag{42}$$
$$= A \tag{43}$$

   Now, let us evaluate our boundary condition for $X(x) = A$. First, we let $X(0) = 0$:

$$X(0) = 0 = A \tag{44}$$

   Here, we know A is 0. For the second condition, let us write:

$$X(L) = 0 = A \tag{45}$$

   Here, we will always have the trivial solution, $X(x) = 0$.

4. Show that $u(x,t) = e^{-\lambda^2 a^2 t}\left[A\cos(\lambda x) + B\sin(\lambda x)\right]$

5. Solve $u_t = u_{xx}$ given $u(0,t) = u(1,t) = 0$ for $t \geq 0$ and $u(x,0) = 1$ for $0 \leq x \leq 1$

   Let us consider the following conditions:

   - $u_t = u_{xx}$
   - $u(0,t) = 0, t \geq 0$
   - $u(1,t) = 0, t \geq 0$
   - $u(x,0) = 1, 0 \leq x \leq 1$

   Let us begin finding our solution.

   (a) Let us assume our solution is seperable

6. Find the solution to the previous problem if $u(x,0) = x - x^2$ for $0 \leq x \leq 1$

   - $u_t = u_{xx}$
   - $u(0,t) = 0, t \geq 0$
   - $u(1,t) = 0, t \geq 0$
   - $u(x,0) = x - x^2, 0 \leq x \leq 1$

7. Solve $u_t = u_{xx}$ given $u(0,t) = u(1,t) = 0$ for $t \geq 0$ and $u(x,0) = 10^{-5}\sin(10^6 \pi x)$ for $0 \leq x \leq 1$. Determine $u(x,2)$ and $u(x,-2)$ and look at their magnitudes. Note that when $t = -2$, we are looking at the backward heat equation and given the magnitude of $u(x,-2)$, what can you say about the solution to the backward heat equation?

   Let us consider the following conditions:

   - $u_t = u_{xx}$
   - $u(0,t) = 0, t \geq 0$
   - $u(1,t) = 0, t \geq 0$
   - $u(x,0) = 10^{-5}\sin(10^6 \pi x), 0 \leq x \leq 1$
   - Determine the following and look at their magnitudes
     - $u(x,2)$
     - $u(x,-2)$