1. Given the conservation law $u_t + [\cos u]_x = 0$, sketch the characteristic curves, where

   (a) $u(x,0) = \begin{cases} \frac{\pi}{2} & x < 0 \\ 0 & x \geq 0 \end{cases}$

   (b) $u(x,0) = \begin{cases} \frac{\pi}{6} & x < 0 \\ \frac{\pi}{2} & x \geq 0 \end{cases}$

   Here, we are given the conservation law. If we differentiate on x, we find:

$$u_t - u_x \sin u = 0 \tag{1}$$

   From here, let us use a third variable to solve our equation:

$$\frac{dt}{ds} = 1 \tag{2}$$
$$\frac{dx}{ds} = -\sin u \tag{3}$$
$$\frac{du}{ds} = 0 \tag{4}$$

   From here, we would let $t = s$.

   (a) Here, we would want to consider $x(0) = x_0$ and $u(0) = f(x)$, which is given as $u(x,0)$. If we make this assumption for $\frac{dx}{ds}$, we would get:

$$\frac{dx}{ds} = -\sin u \tag{5}$$
$$\frac{dx}{ds} = \begin{cases} \sin \frac{\pi}{2} & x_0 < 0 \\ -\sin 0 & x_0 \geq 0 \end{cases} \tag{6}$$

   From here, if we evaluate our terms, we get:

$$\frac{dx}{ds} = \begin{cases} -1 & x_0 < 0 \\ 0 & p \geq 0 \end{cases} \tag{7}$$

   Now, if we multiply both sides by ds and integrate, we would get:

$$x(s) = \begin{cases} -s + x_0 & x_0 < 0 \\ x_0 & x_0 \geq 0 \end{cases} \tag{8}$$

   Here, recall $t = s$,

$$x(t) = \begin{cases} -t + x_0 & x_0 < 0 \\ x_0 & x_0 \geq 0 \end{cases} \tag{9}$$

   (b) Now, let us consider $u(x,0)$ for part b) and write for $\frac{dx}{ds}$:

$$\frac{dx}{ds} = \begin{cases} -\sin \frac{\pi}{6} & x_0 < 0 \\ -\sin \frac{\pi}{2} & x_0 \geq 0 \end{cases} \tag{10}$$

When we integrate both sides with respect to their variables, we get:

$$x(s) = \begin{cases} -0.5s + x_0 & x_0 < 0 \\ -s + x_0 & x_0 \geq 0 \end{cases} \tag{11}$$

$$x(t) = \begin{cases} -0.5t + x_0 & x_0 < 0 \\ -t + x_0 & x_0 \geq 0 \end{cases} \tag{12}$$

2. Solve the following equations using the method of characteristics

   (a) $u_t + 7u_x = t \quad u(x, 0) = \sin x$

   (b) $u_t + xu_x + 2u = 0 \quad u(x, 0) = x^3$

   (c) $u_t + 2xtu_x = u \quad u(x, 0) = 1 - x$

   (d) $tu_t + 2xu_x = t\sin(\pi t) \quad u(x, 1) = \cos x$

   (a) Part a

   (b) Here, let us consider rewriting our equation:

$$u_t + xu_x = -2u \tag{1}$$

From here, we can find:

$$\frac{dx}{ds} = x \tag{2}$$

$$\frac{dt}{ds} = 1 \tag{3}$$

$$\frac{du}{ds} = -2u \tag{4}$$

Here, finding $\frac{dx}{ds}$ is a matter of solving a differential equation to get:

$$\frac{dx}{ds} = x \tag{5}$$

$$x = ce^s \tag{6}$$

Next, we can also find our second term for $\frac{dt}{ds}$ and find:

$$\frac{dt}{ds} = 1 \tag{7}$$

$$dt = ds \tag{8}$$

$$t = s + t_0 = s \tag{9}$$

Note since $s = t$, we can replace this for what we found with x. Now, to find $\frac{dt}{ds}$, let us take a similar approach as with $\frac{dx}{ds}$ and write:

$$\frac{du}{ds} = -2u \tag{10}$$

$$u = de^{-2s} \tag{11}$$

Here, recall $s = t$. We also are given $u(0)$ as x^3:

$$u = de^{-2s} \tag{12}$$
$$u = u_0 e^{-2t} \tag{13}$$
$$u = f\left(x_0\right) e^{-2t} \tag{14}$$

From here, let us take a look back at x and find x_0:

$$x = ce^{t} \tag{15}$$
$$x = x_0 e^{t} \tag{16}$$
$$xe^{-t} = x_0 \tag{17}$$

Now, if we plug x_0 into u, we can find:

$$u = f\left(x_0\right) e^{-2t} \tag{18}$$
$$u = f\left(xe^{-t}\right) e^{-2t} \tag{19}$$
$$u = \left(xe^{-t}\right)^3 e^{-2t} \tag{20}$$
$$u = x^3 e^{-3t} e^{-2t} \tag{21}$$
$$u = x^3 e^{-5t} \tag{22}$$

(c) Here, let us rewrite our equation:

$$u_t + 2xtu_x = u \tag{23}$$
$$u(x, 0) = 1 - x \tag{24}$$

Here, note that u_x has t as a coefficient. Let us skip over the change of variable $(t = s)$ and write:

$$\frac{dx}{dt} = 2xt \tag{25}$$
$$\frac{du}{dt} = u \tag{26}$$

First, let us consider solving for $\frac{dx}{dt}$. Here, let us solve the differential equation:

$$\frac{dx}{dt} = 2xt \tag{27}$$
$$\int \frac{1}{x}\ dx = \int 2t\ dt \tag{28}$$
$$\ln|x| = t^2 + c \tag{29}$$
$$x = ce^{t^2} \tag{30}$$