Partial Differential Equations - Class Notes

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March 22, 2022

Sidenotes

January 19, 2022

What is a PDE?

A PDE is an equation which contains partial derivatives of an unknown function and we want to find that unknown function.
Example: $F(t, x, y, z, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x \partial y}, \ldots) = 0.$
Note, the first partial derivatives are considered 1^{st} ordered partials
whereas the second ordered partials are considered 2^{nd} ordered partials.
The variables that are not u are considered independent variables and u is considered a dependent variable.
What PDEs do we study?
Generally, we restrict our attention to equations that model some phenomenom from physics, engineering, economics, geology, ... etc. We can use physical intuition to help guide the math.

Classification of PDEs

1. Order of PDE: Highest derivative.
   Example: $\frac{\partial^3 u}{\partial x^3} - \sin(y)u^7 = 3$ is a third order PDE.
   Example: $(\frac{\partial y}{\partial t})^5 - \frac{\partial^2 y}{\partial x \partial t} = e^x$ is a second order PDE.
2. Number of independent variables.
   Example: $\frac{du}{dt} = \frac{\partial^2 u}{\partial x^2}$ has two independent variables: t, x.
   This is the $1-D$ heat equation.
   Example: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \Delta u$ has 4 independent variables.
   This is the $3-D$ heat equation. Δu is Laplacian of u.
   Notation
   $\Delta u = \nabla^2 u = \nabla \cdot \nabla u = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$
   $\Delta u = 0$ is considered Laplace's equation.
3. Linear vs non-linear
   A linear PDE is any equation of the form $L[u(x)] = f(x)$ where $f(x)$ is a known function is a linear partial differential operator.

Definition: A differential operator is any rule that takes a function as its input and returns an expression that involves the derivatives of that function.
Example:

$$u(x,t) \qquad v(x,t) \tag{1}$$

$$O[u] = \frac{\partial^2 u}{\partial x^2} + \sin x + \pi - 7e^{tu} \tag{2}$$

$$O[u+3v] = \frac{\partial^2}{\partial x^2}(u+3v) + \sin x + \pi - 7e^{tu+3tv} \tag{3}$$

$$= \frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 v}{\partial x^2} + \sin x + \pi - 7e^{tu+3tv} \tag{4}$$

Definition: A linear operator, L, is an operator that has the property:

$$L[au+bv] = aL[u] + bL[v] \tag{5}$$

Where a and b are constants.
Theorem: If u and v are vectors and L is linear, then L can be represented by a matrix.
Theorem: If L is linear ordinary operator, it must take the form:

$$L[u] = f_0(x)u + f_1(x)u' + f_2(x)u'' + \ldots + f_n(x)u^{(n)} \tag{6}$$

Where the f_i's are known functions.
Definition: A linear ODE is any ODE of the form where $f(x)$ is known is the following:

$$L[u] = f(x) \tag{7}$$

If $f(x) = 0$, then the equation is homogeneous. Otherwise, the equation is non-homogeneous.
Ex: $(u')^2 = 0 \Rightarrow u' = 0 \rightarrow$ linear, homogeneous.
Theorem: If L is a linear partial differential operator, it must take the form (x is a vector with n unknowns)

$$L[u(x)] = f_0(x)u + \sum_{i=1}^{n} f_i(x)\frac{\partial u}{\partial x_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \ldots \tag{8}$$

Definition: A linear PDE is any PDE of the form

$$L[u(x)] = f(x) \tag{9}$$

If $f(x) = 0$, the equation is homogeneous, else it is non-homogeneous.
Ex: $u_t = 4u_x$ - Linear, homogeneous.

January 21, 2022

Example:

$$u_{tt} = u_{xx} + uyy \quad \text{Linear, homogeneous} \tag{10}$$
$$\cos(xt) = u + u_t + u_{xyz} \quad \text{Linear, non-homogeneous} \tag{11}$$
$$u_t u_{xt} = 0 \quad \text{non-linear} \tag{12}$$
$$u_{xt} + e^x \cos t \ u_t = 0 \quad \text{linear, homogeneous} \tag{13}$$
$$u_t + u_{xx} + ue^u = 0 \quad \text{non-linear} \tag{14}$$

Note: You can add linear combinations of solutions to linear homogeneous equations and still get a solution. Example: $u_x = u_t$.
Some solutions to this are:

1. $u_1(x,t) = 3$
2. $u_2(x,t) = x + t$
3. $u_3(x,t) = e^{x+t}\cos(x+t)$
4. $\vdots$
   $Au_1 + Bu_2 + Cu_3$ is also a solution.

How do we solve an ODE?

1. Use some technique to find an explicit solution.
2. Use power series and determine the coefficients
$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{15}$$
3. Laplace Transforms

How do we solve PDEs?

1. Try to locate an explicit solution
2. We don't use power series, instead, we use a trigonometric series $\Rightarrow$ Fourier Series.
$$y(x) = \sum_{n=0}^{\infty} a_n \sin(nx) + b_n \cos(nx) \tag{16}$$
3. Laplace Transforms are good if the domain is $[0, \infty)$.
   Fourier Transforms are good if the domain is $(-\infty, \infty)$.
4. Reduce the PDE to a system of ODEs.

Initial Condiction

1. For ODEs, to solve a 1^{st} order equation, you need $y(0)$.
   2^{nd} order $\rightarrow y(0), y'(0)$
   3^{rd} order $\rightarrow y(0), y'(0), y''(0)$
   $\vdots$
   n^{th} order $\rightarrow y(0), y'(0), y''(0), \ldots, y^{(n-1)}(0)$
2. For PDEs, it's more complicated $\Rightarrow$ it depends on the PDE.
   Example: $u(x,t), x \in [a,b], t \in [0, \infty)$
   If $u_t = u_{xx}$
3. Boundary conditions:
$$u(a,t) = g_1(t) \tag{17}$$
$$u(b,t) = g_2(t) \tag{18}$$
   If $u_{tt} = u_{xx}$, we must specify:
   (a) Initial Conditions
$$u(x,0) = f_1(x) \tag{19}$$
$$u_t(x,0) = f_2(x) \tag{20}$$

(b) Boundary Conditions

$$u(a,t) = g_1(t) \tag{21}$$
$$u(b,t) = g_2(t) \tag{22}$$

1-D Heat Equation
Assume cross sections are uniform Imagine a cross section:

O o==========o L

Assume cross sections are uniform and the lateral sides are well insulated $\Rightarrow$ heat only flows in the x-direction.
We need the following:

- $u(x,t)$: Temperature of rod at position x and time t.
- $u(x,0)$: Initial temperature
- $u(0,t)$ and $u(L,t)$: Boundary Conditions

Definition:

- $g(x,t)$: heat flux (energy / area time)
- $Q(x,t)$: heat energy density (energy / volume)
- A : Cross sectional area
- C_P : Heat capacity or specific heat
- ρ : Density
- K : Thermal conductivity

We want to find an equation for the temperature evolution. We will use conservation of energy : Look at a little Δx section of the rod starting at x_0.

Δx
o=====|o|=====o
x_0 $x_0 \Delta x$

Conservation of energy : heat in - heat out = heat accumulated
Heat in $=' qA\Delta t' = A\int_{t_0}^{t_0+\Delta t} q(x_0,t)$ dt
Heat out $= A\int_{t_0}^{t_0+\Delta t} q(x_0+\Delta x,t)$ dt
Heat Accumulated $= QA\Delta x|_{t_0+\Delta t} - QA\Delta x|_{t_0}$

$$= A\int_{x_0}^{x_0+\Delta x} Q(x,t_0+\Delta t)\ \mathrm{dx} - A\int_{x_0}^{x_0+\Delta x} Q(x,t_0)\ \mathrm{dx} \tag{23}$$

January 24, 2022

Heat Equation
Conservation of energy
Heat in - heat out = heat accumulated

$$A\int_{t_0}^{t_0\to\Delta t} g(x_0,t)\ \mathrm{dt}\ -A\int_{t_0}^{t_0\to\Delta t} q(x_0+\Delta x,t)\ \mathrm{dt}\ = A\int_{t_0}^{t_0\to\Delta t} Q(x,t_0+\Delta t)\ \mathrm{dx}\ -A\int_{t_0}^{t_0\to\Delta t} Q(x,t_0)\ \mathrm{dx} \tag{24}$$

Let us simplify and divide by A. Then, let us combine the integrals:

$$\int_{t_0}^{t_0\to\Delta t}[q(x_0,t)-q(x_0+\Delta x,t)]\ \mathrm{dt}\ = \int_{t_0}^{t_0\to\Delta t}[Q(x,t_0+\Delta t)-Q(x,t_0)]\ \mathrm{dx} \tag{25}$$

Divide by $\Delta x\Delta t$ and take limit as $\Delta x,\Delta t\to 0$

$$\lim_{\Delta t,\Delta x\to 0}\frac{1}{\Delta x\Delta t}\int_{t_0}^{t_0\to\Delta t}[q(x_0,t)-q(x_0+\Delta x,t)]\ \mathrm{dt}\ = \lim_{\Delta t,\Delta x\to 0}\frac{1}{\Delta x\Delta t}\int_{t_0}^{t_0\to\Delta t}[Q(x,t_0+\Delta t)-Q(x,t_0)]\ \mathrm{dx} \tag{26}$$

$$\lim_{\Delta t}\frac{1}{\Delta t}\int_{t_0}^{t_0\to\Delta t}[\lim_{\Delta x\to 0}\frac{q(x_0,t)-q(x_0+\Delta x,t)}{\Delta x}]\ \mathrm{dt}\ = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_{t_0}^{t_0\to\Delta t}\lim_{\Delta t\to 0}\frac{Q(x,t_0+\Delta t)-Q(x,t_0)}{\Delta t}\ \mathrm{dx} \tag{27}$$

On the left side, we see the order is a bit difference. We want the delta to come first, such as in the difference quotient. The eft is now $-q_x(x_0,t)$ and the right is $Q_t(x,t_0)$.

$$\lim_{\Delta t\to}\frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t}-q_x(x_0 t)\ \mathrm{dt}\ = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x}Q_t(x,t_0)\ \mathrm{dx} \tag{28}$$

$$\lim_{\Delta t\to 0}-q_x(x_0,t_0+\Delta t) = \lim_{\Delta x\to 0}Q_t(x_0+\Delta x,t_0) \tag{29}$$

At step 28, we used the fundamental theorem of calculus and derived both sides.

$$-q_x(x_0,t_0) = Q_t(x_0,t_0) \tag{30}$$

Since x_0 and t_0 are arbitrary, $-q_x(x,t) = Q_t(x,t)$
q and Q are related to u:

$$Q=\rho c_p u \qquad\qquad q=-Ku_x \tag{31}$$

$$-q_x = Q_t \Rightarrow Ku_{xx} = \rho c_p u_t \tag{32}$$

$$\Rightarrow u_t = \frac{k}{\rho c_p}u_{xx} \tag{33}$$

$$\Rightarrow u_t = \alpha^2 u_{xx} \tag{34}$$

$$\alpha = \sqrt{\frac{K}{\rho c_p}} \tag{35}$$

α is thermal diffusivity
$u_t = \alpha^2 u_{xx} \leftarrow$ 1-D heat equation (diffusivity equation)
We have a steady-state: $(t\to\infty)$, $u_t = 0 \Rightarrow u_{xx} = 0 \Rightarrow$ straight line
1-D: $-q_x = Q_t \Rightarrow -\nabla\cdot\vec{q} = Q_t$, $\vec{q}$ is a vector.

$$q=-K\nabla u \Rightarrow -\nabla\cdot(-K\nabla u) = \rho c_p u_t \tag{36}$$

$$\Rightarrow K\Delta u = \rho c_p u_t \tag{37}$$

$$\Rightarrow u_t = \alpha^2\Delta u \tag{38}$$

What about a steady-state? $u_t = 0$

$$\Delta u = 0 \tag{39}$$

Here, we have Laplace's equation.
Note: It is not dependent on time.
The Wave Equation $u(x,t)$ is the height of the rope. We use Newton's 2^{nd} law on small segments of rope.

- $\rho =$ density of rope.
- dm $= \rho$ dx

$$F = ma \tag{40}$$

$$T\sin(\theta(x+\Delta x)) - T\sin(\theta(x)) = \int_x^{x+\Delta x}u_{tt}\ \mathrm{dm} \tag{41}$$

$$T[\sin(\theta(x+\Delta x)) - \sin(\theta(x))] = \rho\int_x^{x+\Delta x}u_{tt}\ \mathrm{dx} \tag{42}$$

Let us assume θ is small, $\sin\theta \approx \tan\theta$

$$T[\tan(\theta(x+\Delta x)) - \tan(\theta(x))] = \rho \int_x^{x+\Delta x} u_{tt} \text{ dx} \tag{43}$$

Also, $\tan(\theta(x)) = u_x(x,t)$.

$$T[u_x(x+\Delta x,t) - u_x(x,t)] = \rho \int_x^{x+\Delta x} u_{tt} \text{ dx} \tag{44}$$

Now, let us divide both sides by Δx and take the limit as $\Delta x \to 0$

$$\lim_{\Delta x\to 0} T\Big[\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}\Big] = \rho \lim_{\Delta x\to 0} \frac{\int_x^{x+\Delta x} u_{tt} \text{ dx}}{\Delta x} \tag{45}$$

On the left side, we have $u_x x$ and the right side we have $u_{tt}(x+\Delta x,t)$.

$$Tu_{xx}(x,t) = \rho u_{tt}(x,t) \tag{46}$$

$$u_{tt} = \frac{T}{\rho}u_{xx} = c^2 u_{xx} \tag{47}$$

$$c = \sqrt{\frac{T}{\rho}} = \text{ wave speed} \tag{48}$$

On the left, we have the $1-D$ wave equation which is used for light, sound, rope, etc.
In 2-D, it corresponds to a vibrating membrane (drum)

$$u_{tt} = c^2 \Delta u \tag{49}$$

Remark:

$$u_t = u_{xx} \quad \text{Heat Equation} \tag{50}$$

$$u_{xx} + u_{yy} = 0 \quad \text{Laplace Equation} \tag{51}$$

$$u_{tt} = u_{xx} \quad \text{wave} \tag{52}$$

Here, we can replace:
u_t with t
u_x with x
u_{xx} with x^2

1. $t = x^2$ parabola
2. $x^2 + y^2 = 0$ ellipse
3. $t^2 = x^2$ hyperbolas

So, the equations behave like the following:

1. The Heat Equation is called parabolic
2. The Laplace Equation is called elliptic
3. The Wave Equation is called hyperbolic

January 26, 2022

Approximating Functions with other Functions

1. Prove Series

$$f(x) = \sum_{n=0}^{M} a_n x^n \quad \text{Finite Power Series} \tag{53}$$

This is not the best way to approximate a function.

We choose the a_n's so that the power series is "close" to $f(x)$ which means we want to minimize the error.

We increase M to get a better approximation.

The problem begins when you change M, the values of a_n's change as well. Therefore, recalculating is a lot of work.

If we let $M \to \infty$ and if $f \in C^\infty$, so then $a_n = \frac{f^{(n)}(0)}{n!}$ and we get the Taylor series.

Note: C^∞: C means Continuous and the ∞ indicates the number of derivatives that are continuous.

Problem: This is only good inside the radius of convergence.

A Fourier Series is a trigonometric polynomial

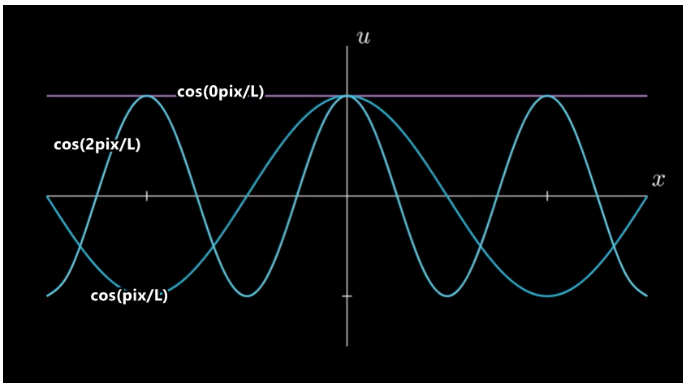
$$\sum_{n=0}^{M} a_n \sin\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right) \longleftarrow \text{period} = 2L \tag{54}$$

We use Fourier Series for a function on a bounded interval and we will use $x \in [-L, L]$

Advantages of Fourier Series

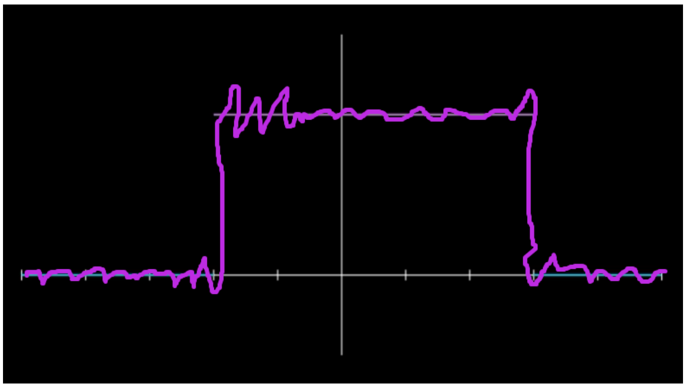
1. If M increases, we only need to calculate the new a_n's and b_n's. This property is due to the fact that the basis functions are orthogonal.
2. If $M = \infty$ and f is continuous, then the Fourier Series $= f(x) \forall x \in (-L, L)$. Our interval must be open for the case that $f(-L) \neq f(L)$.

Basis Functions : $\sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)$

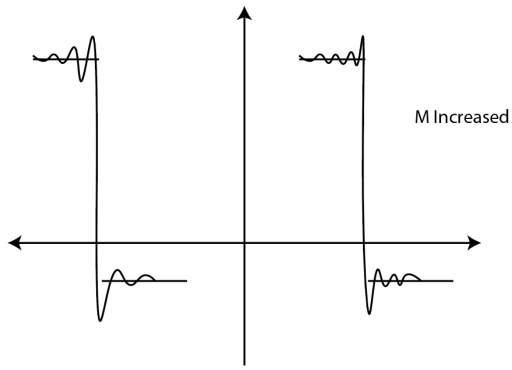


What happens if you use a Fourier Series on a discontinuous function?

$$f(x) = \begin{cases} 1 & x \in (-4, 6) \\ 0 & x \in [-10, -4] \bigcup [6, 10] \end{cases} \tag{55}$$

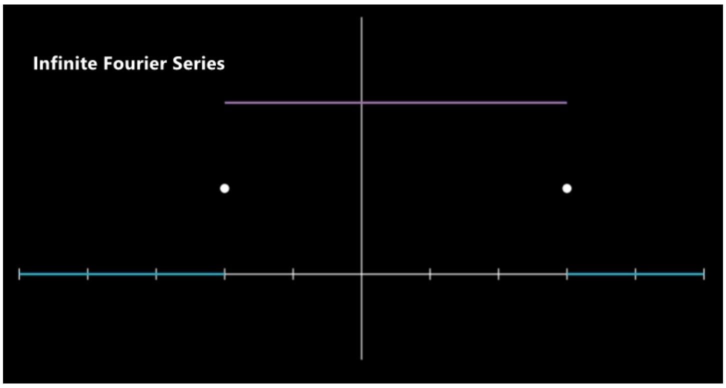


The Oscillations around the discontinuities are called Gibbs phenomenon. As M increases, the oscillation's amplitude does not change. However, the oscillations do get progressively closer to the discontinuities.



If $M = \infty$, then we have:

$$\text{Fourier Series } = \begin{cases} f(x) & = \text{ if } x \text{ is a point of continuity} \\ \lim_{c \to 0^+} \frac{f(x+c)+f(x-c)}{2} & \text{ if x is a point of discontinuity} \end{cases} \tag{56}$$



Orthogonality
Recall: The vectors

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \text{and} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{57}$$

are orthogonal if the dot product is zero.

$$u \circ v = \sum_{i=1}^{n} u_i v_i = 0 \tag{58}$$

We want to generalize this to function $x \in [-L, L]$.

Definition: Two functions $f(x)$ and $g(x)$ are orthogonal on $[a, b]$ if

$$\int_a^b f(x)g(x) \text{ dx } = 0 \tag{59}$$

Theorem: All basis functions in the Fourier Series are mutually orthogonal

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \text{ dx } = 0 \quad n \neq m \tag{60}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \text{ dx } = 0 \quad n \neq m \tag{61}$$

What happens if $m = n$?

$$\int_{-L}^{L} \sin^2\left(\frac{m\pi x}{L}\right)\,\mathrm{dx} \tag{62}$$

Here, we want to use the double angle formula: $\cos(2\theta) = 1 - 2\sin^2\theta$.

$$\begin{aligned}
\int_{-L}^{L} \sin^2\left(\frac{m\pi x}{L}\right)\,\mathrm{dx} &= \frac{1}{2}\int_{-L}^{L} 1 - \cos\left(\frac{2m\pi x}{L}\right)\,\mathrm{dx} && (63)\\
&= \frac{1}{2}\Big[x - \frac{L}{2m\pi}\sin\left(\frac{2m\pi x}{L}\right)\Big]_{-L}^{L} && (64)\\
&= \frac{1}{2}\Big[L - \frac{L}{2m\pi}\sin(2m\pi) - \Big(-L - \frac{2}{2m\pi}\sin(-2m\pi)\Big)\Big] && (65)\\
&= L && (66)
\end{aligned}$$

January 28, 2022

Similarly,

$$\int_{-L}^{L} \cos^2(\frac{n\pi x}{L})\,\mathrm{dx} = L \tag{67}$$

If $n = 0$,

$$\int_{-L}^{L} 1\,\mathrm{dx} = 2L \tag{68}$$

<u>Note:</u> You cannot differentiate the Fourier Series term-by-term $f'(x)$ like you can with Taylor series.

Let's show $\cos(\frac{n\pi x}{L})$ and $\sin(\frac{m\pi x}{L})$ are orthogonal on $[-L, L]$.

$$\begin{aligned}
\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{dx} &= \frac{1}{2}\int_{-L}^{L} \sin\left(\frac{(m+n)\pi x}{L}\right) + \sin\left(\frac{(m-n)\pi x}{L}\right)\,\mathrm{dx} && (69)\\
&= -\frac{1}{2}\Big[\frac{L}{(m+n)\pi}\cos\left(\frac{(m+n)\pi x}{L}\right) + \frac{L}{(m-n)\pi}\cos\left(\frac{(m-n)\pi x}{L}\right)\Big]_{-L}^{L} && (70)
\end{aligned}$$

Here, we expand our difference and notice we have even and odd functions.

In general, the coefficients are:

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{dx} && (71)\\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{dx} && (72)\\
b_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{dx} && (73)
\end{aligned}$$

<u>Example:</u> $f(x) = x, \quad x \in [-3, 3]$.

Find the Fourier Series for f.

$$a_n = \frac{1}{3}\int_{-3}^{3} x\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{dx} \tag{74}$$

Here, we want to integrate by parts:

x	$\sin\left(\frac{n\pi x}{L}\right)$
1	$-\frac{3}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$
0	$-\frac{9}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)$

<u>Note:</u> $L = 3$.

$$\begin{aligned}
&= \frac{1}{3}\Big[-\frac{3x}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\Big]_{-3}^{3} + \Big[\left(\frac{3}{n\pi}\right)^2\sin\left(\frac{n\pi x}{3}\right)\Big]_{-3}^{3} && (75)\\
&= \frac{1}{3}\left[-\frac{9}{n\pi}\cos(n\pi) + \frac{9}{n^2\pi^2}\sin(n\pi) - \left(+\frac{9}{n\pi}\cos(-n\pi) + \frac{9}{n^2\pi^2}\sin(-n\pi)\right)\right] && (76)\\
&= -\frac{6}{n\pi}\cos(n\pi) && (77)
\end{aligned}$$

January 31, 2022

The Fourier Series is not valid at $x = \pm 3$ since it is not continuous at ± 3.

Let's say $f(x)$ is odd, then $f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$

Let's say $f(x)$ is even, then $f(x) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right)$

If we are only interested in the behavior of $f(x)$ on $[0, L$, then we can either use a Fourier Sine Series $f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$ or a Fourier series $f(x) = \sum_{n=0}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right)$.

Solving the Heat Equation

$$u_t = \alpha^2 u_{xx} \tag{78}$$

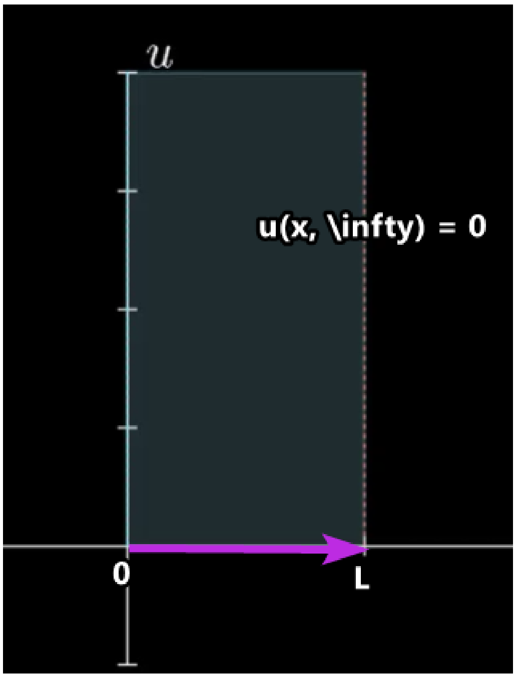
0 o============o L $u(x,t)$: temp

Initial condition:

$$u(x,0) = f(x) \tag{79}$$

Boundary conditions

$$u(0,t) = u(L,t) = 0 \tag{80}$$



So, whatever we get, we better have $\lim_{t\to\infty} u(x,t) = 0$. The method of reflection? relies on two things:

1. Fourier Series
2. Linearity

Method

1. Try a solution of the form

$$u(x,t) = X(x)T(t) \leftarrow \text{ Assume the solution is separable} \tag{81}$$

Boundary Conditions: Here, we conclude $X(0)$ is 0 because we want $T(t)$ to change as t changes.

$$u(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0 \tag{82}$$
$$u(L,t) = 0 \Rightarrow X(L)T(t) = 0 \Rightarrow X(L) = 0 \tag{83}$$
$$U_t = \alpha^2 u_{xx} \Rightarrow X(x)T'(t) = \alpha^2 X''(x)T(t) \tag{84}$$

Here, we divide by X, T, α^2.

$$\Rightarrow \frac{T'(t)}{\alpha^2 T(t)} = \frac{X''}{X(x)} = -\lambda \tag{85}$$

Here, λ is a constant.

2.

$$\frac{X''}{X(x)} = -\lambda \tag{86}$$
$$X''(x) = -\lambda X(x) \tag{87}$$

Here, we know $(x) = X(L) = 0$. We call every $(\lambda, X(x))$ pair that satisfies this equation an eigenvalue/eigenfunction pair for the differential equation.

Assume $\lambda > 0$

$$X'' = -\lambda x \tag{88}$$
$$x(0) = x(L) = 0 \tag{89}$$
$$\Rightarrow X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \tag{90}$$
$$\Rightarrow A\cos 0 + B\sin 0 = 0 \tag{91}$$
$$\Rightarrow A = 0 \tag{92}$$
$$\Rightarrow X(x) = B\sin(\sqrt{\lambda}x) \tag{93}$$
$$X(L) = 0 \Rightarrow B\sin(\sqrt{\lambda}L) = 0 \tag{94}$$
$$\Rightarrow \sin(\sqrt{\lambda}L) = 0 \tag{95}$$
$$\Rightarrow \sqrt{\lambda}L = n\pi, n \in \mathbb{Z}^+ \tag{96}$$
$$\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{97}$$
$$\Rightarrow_n (x) = \sin\left(\frac{n\pi x}{L}\right) \tag{98}$$

February 2, 2022

$$u(x,t) = X(x)T(t) \tag{99}$$
$$\frac{X''}{x} = \frac{T'}{\alpha^2 T} = -\lambda \tag{100}$$
$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{101}$$
$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \tag{102}$$

Here, we have a bsis function for Fourier Sine Series.

3. Solve for T

$$\frac{T'}{\alpha^2 T} = -\lambda \tag{103}$$
$$T' = -\alpha^2 \lambda T \tag{104}$$
$$T_n' = -\alpha^2 \lambda_n T_n \tag{105}$$
$$= -\alpha^2 \left(\frac{n\pi}{L}\right)^2 T \tag{106}$$

If we have something like $y' = ky$, we know that this derives from $y = e^{kx}$.

$$T_n(t) = e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 T} \tag{107}$$

4. Combine for u_n

$$u_n(x,t) = X_n(x)T_n(t) \tag{108}$$
$$= \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 T} \tag{109}$$

Each one of the $n's$ will yield a different u. We also know that $n \in \mathbb{N}$. We can take as many u's and add them all together. We find our u_n's and use it to find u.

By linearity,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \tag{110}$$
$$= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 T} \tag{111}$$

5. Satisfy the initial condition

$$u(x,0) = f(x) \tag{112}$$
$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \tag{113}$$

Line 113) is considered the Fourier Sine Series.

The A_n's are the coefficients of the Fourier Sine Series of $f(x)$.

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{114}$$
$$= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{115}$$

<u>Ex:</u> Solve with the following conditions:

1. $u_t = 4u_{xx}$
2. $u(0,t) = u(3,t) = 0$
3. $u(x,0) = 5\sin\left(\frac{2\pi x}{3}\right) - 7\sin(4\pi x)$

Now, let us perform the five steps to solve our equation:

1. Assume $u(x,t) = X(x)T(t)$
   Boundary Conditions :
   - $u(0,t) = 0$, then $X(0)T(t) = 0$. Here either $X(0)$ or $T(t)$ is 0, and we want $X(0) = 0$ here.
   - $(3,t) = 0$, then $X(3)T(t) = 0$, following the same logic, we have $X(3) = 0$.

$$u_t = 4u_{xx} \tag{116}$$
$$XT' = 4X''T \tag{117}$$
$$\frac{T'}{4T} = \frac{X''}{X} = -\lambda \tag{118}$$

2. Now, since we know more information regarding X, let us solve for X.

$$\frac{X''}{X} = -\lambda \tag{119}$$
$$X'' = -\lambda X, \quad X(0) = X(3) = 0 \tag{120}$$

Let us assume $\lambda > 0$. Here, we want an X'' where deriving twice gives us $-X$. Assume $\lambda > 0$

$$X = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \tag{121}$$

Set $X(0) = 0$

$$X = A \tag{122}$$

Now, let us find $X(3) = 0$:

$$0 = A\sin(\sqrt{\lambda}3) \tag{123}$$
$$\sqrt{\lambda}3 = n\pi \tag{124}$$
$$\lambda_n = \left(\frac{n\pi}{3}\right)^2 \tag{125}$$
$$X_n(x) = \sin\left(\frac{n\pi x}{3}\right) \tag{126}$$

3. Now, let us find T.

$$\frac{T'}{4T} = -\lambda \tag{127}$$

$$T_n' = -4\left(\frac{n\pi}{3}\right)^2 T_n \tag{128}$$

$$T_n(t) = e^{-4\left(\frac{n^2\pi^2}{9}\right)t} \tag{129}$$

4. Combine to find u_n and u

$$u_n(x,t) = X_n(x)T_n(t) \tag{130}$$

$$= \sin\left(\frac{n\pi x}{3}\right) e^{-4\left(\frac{n^2\pi^2}{9}\right)t} \tag{131}$$

By linearity,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{3}\right) e^{-4\left(\frac{n^2\pi^2}{9}\right)t} \tag{132}$$

5. Use the initial conditions to find A_n's

$$u(x,0) = 5\sin(\frac{2\pi x}{3}) - \sin(4\pi x) \tag{133}$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{3}\right) \tag{134}$$

$$A_n = \frac{2}{3}\int_0^3 5\left[\sin\left(\frac{2\pi x}{3}\right) - 7\sin(4\pi x)\right]\sin(\frac{n\pi x}{3})\ \mathrm{dx} \tag{135}$$

Lets look at our initial condition on line 133). The first one is $n = 2$, so $A_2 = 5$. In addition, the second term is at $A_12 = -7$. Therefore, we have $A_n = 0 \forall n$ except $n = 2, 12$.

Now, let us look at our linearity equation.

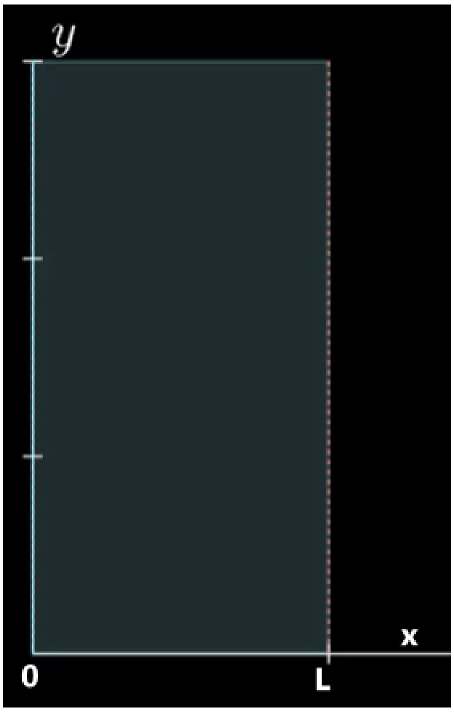
$$u(x,t) = 5\sin\left(\frac{2\pi x}{3}\right) e^{-\frac{16\pi^2}{9}t} - 7\sin(4\pi x)\, e^{-64\pi^2 t} \tag{136}$$

Here, this is our final solution.

Laplace's Equation 1-D : $u_{xx} = 0 \Rightarrow u = ax + b$

If $u(0) = u(L) = 0$, then that would force our function to be $u = 0$. This is the steady state solution. If our function is in the form of $ax + b$, then $u = 0$ is the only solution for the function to hit 0 twice in this fashion.

2-D: $\Delta u = 0 \Rightarrow u_{xx} + u_{yy} = 0$



We have two types of boundary conditions:

(a) Specify u on the perimeter

- $u(0, y) = 0$
- $u(L, y) = 0$
- $u(x, 0) = 0$
- $u(x, M) = f(x)$

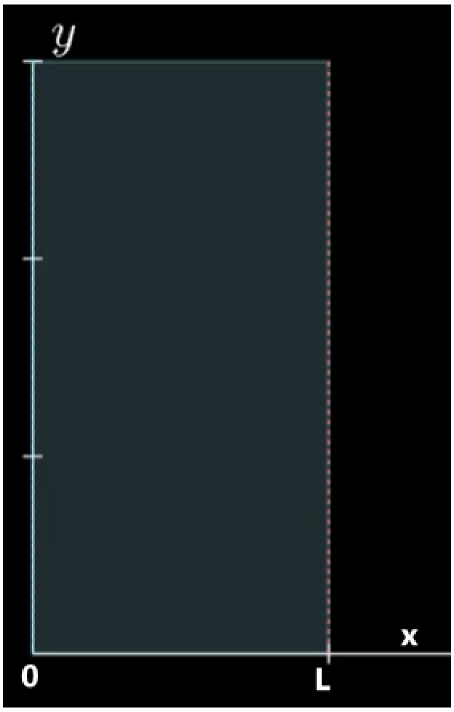
(b) Nuemann conditions : Specify the direction derivative in the normal direction on the boundary.

- $u_x(0, y) = 0$
- $u_x(L, y) = 0$
- $u_y(x, 0) = 0$
- $u_y(x, M) = \widetilde{f}(x)$
- $u(0, 0) \ = T$

This is if we know the heat flux $\vec{q} \cdot \vec{n}$ on the boundary.

February 4, 2022

Solving Laplace's Equation



$$u_{xx} + u_{yy} = 0$$

- $u_x(0, y) = 0$
- $u_x(L, y) = 0$
- $u(x, 0) = 0$
- $u(x, M) = f(x)$

1. Assume $u(x, y) = X(x)Y(y)$

   Boundary Conditions

$$u(x, y) = X(x)Y(y) \tag{137}$$
$$\Rightarrow X'(x)Y(y) \tag{138}$$

Now, let us write our boundary condition:

$$U_x(0, y) = 0 \tag{139}$$
$$\Rightarrow X'(0)Y(y) = 0 \tag{140}$$
$$\Rightarrow X'(0) = 0 \tag{141}$$

Now, let us find the next item,

$$u_x(L, y) = 0 \tag{142}$$
$$\Rightarrow X'(L)Y(y) = 0 \tag{143}$$
$$\Rightarrow X'(L) = 0 \tag{144}$$

Now, the next two items do not have a derivative:

$$u(x, 0) = 0 \tag{145}$$
$$\Rightarrow X(x)Y(0) = 0 \tag{146}$$
$$\Rightarrow Y(0) = 0 \tag{147}$$

Now, let us write:

$$u_{xx} + u_{yy} = 0 \tag{148}$$
$$\Rightarrow X''Y + XY'' = 0 \tag{149}$$
$$\Rightarrow X''Y = -XY'' \tag{150}$$
$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \tag{151}$$

2. Solve for X (Note: We solve for X first here, since we have more information about X).

$$\frac{X''}{X} = -\lambda \tag{152}$$
$$\Rightarrow X'' = -\lambda X, \quad X'(0) = X'(L) = 0 \tag{153}$$
$$\lambda > 0 \Rightarrow x(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \tag{154}$$
$$\Rightarrow X'(x) = A\sqrt{\lambda}\cos(\sqrt{\lambda}x) - B\sqrt{\lambda}\sin(\sqrt{\lambda}x) \tag{155}$$
$$X'(0) = 0 \Rightarrow A\sqrt{\lambda} = 0 \tag{156}$$

Now, if we rewrite out equation, we have:

$$X(x) = B\cos(\sqrt{\lambda}x) \tag{157}$$

Next, we want to find $X'(L) = 0$:

$$0 = -B\sqrt{\lambda}\sin(\sqrt{\lambda}L) \tag{158}$$
$$\sqrt{\lambda}L = n\pi \tag{159}$$
$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{160}$$
$$\Rightarrow X_n(x) = \cos\left(\frac{n\pi x}{L}\right) \tag{161}$$

If $\lambda = 0$

$$\frac{X_0''}{X_0} \Rightarrow X_0'' = 0 \tag{162}$$
$$\Rightarrow X_0(x) = Ax + B \tag{163}$$
$$\Rightarrow X_0'(x) = A \tag{164}$$
$$\Rightarrow X_0'(0) = 0 \tag{165}$$
$$\Rightarrow A = 0 \tag{166}$$
$$\Rightarrow X_0'(L) = 0 \tag{167}$$
$$\Rightarrow A = 0 \tag{168}$$

Neither conditions tell us more information about B,

$$\Rightarrow X_0(x) = B_0 \tag{169}$$

3. Now, we want to solve for Y: $-\frac{Y''}{Y} = -\lambda$

$$Y'' = \lambda y \tag{170}$$
$$Y'' = \left(\frac{n\pi}{L}\right)^2 Y_n, \quad Y_n(0) = 0 \tag{171}$$
$$Y_n(y) = Ce^{\frac{n\pi}{L}y} + De^{-\frac{n\pi}{L}}y \tag{172}$$
$$Y_n(0) = 0 \Rightarrow C + D = 0 \tag{173}$$

Here, we do not have an additional condition that could help use solve this equality. Let us consider the hyperbolic sin and cos:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{174}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{175}$$

Instead of writing Y in the same fashion we solved for X, we use the hyperbolic sinh and *cosh*

$$Y_n(y) = C\sinh\left(\frac{n\pi y}{L}\right) + D\cosh\left(\frac{n\pi y}{L}\right) \tag{176}$$
$$Y_n(0) = 0 \Rightarrow D = 0 \tag{177}$$
$$Y_n(y) = \sinh\left(\frac{n\pi y}{L}\right) \tag{178}$$

Now, let us write:

$$\frac{Y_0''}{Y_0} = \lambda_0 \tag{179}$$
$$\Rightarrow Y_0'' = 0 \tag{180}$$
$$\Rightarrow Y_0 = Cy + D \tag{181}$$
$$\Rightarrow Y_0(0) = 0 \tag{182}$$
$$\Rightarrow D = 0 \tag{183}$$
$$\Rightarrow Y_0(y) = C_0 y \tag{184}$$

4. Combine to find u_n and u:

$$u_n(x,y) = X_n(x)Y_n(y) = \begin{cases} \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right) & n \geq 1 \\ B_0 C_0 y & n = 0 \end{cases} \tag{185}$$

By linearity,

$$u(x,y) = \widetilde{B}_0 y + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right) \tag{186}$$

5. Here, use the final boundary condition to find the coefficients.

$$u(x,M) = f(x) \tag{187}$$

$$u(x,M) = \widetilde{B}_0 M + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi M}{L}\right) \tag{188}$$

This is our Fourier Cosine Series for $f(x)$. Here, we can say a few things about this equation,

- $b_0 = \widetilde{B}_0 M$
- $b_n = B_n \sinh\left(\frac{n\pi M}{L}\right)$

$$\widetilde{B}_0 M = \frac{2}{2L}\int_0^L f(x)\ \mathrm{dx} \tag{189}$$

$$\widetilde{B}_0 = \frac{1}{ML}\int_0^L f(x)\ \mathrm{dx} \tag{190}$$

Next, let us find:

$$B_n \sinh\left(\frac{n\pi M}{L}\right) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ \mathrm{dx} \tag{191}$$

$$= \frac{2}{L\sinh\left(\frac{n\pi M}{L}\right)}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ \mathrm{dx} \tag{192}$$

<u>Ex:</u> Solve $\Delta u = 0$

- $(x,y) = 0$
- $(2,y) = 0$
- $(x,0) = 0$
- $(x,3) = 4\sin(5x)$

1. Assume $u(x,y) = X(x)Y(y)$

Here, let us look at our boundary conditions:

$$u(0,y) = 0 \tag{193}$$

$$X(0)Y(y) = 0 \tag{194}$$

$$X(0) = 0 \tag{195}$$

Here, let us look at our next boundary conditions:

$$u(2,y) = 0 \tag{196}$$

$$X(2)Y(y) = 0 \tag{197}$$

$$X(2) = 0 \tag{198}$$

Here, let us look at our next boundary conditions:

$$u(x,0) = 0 \tag{199}$$

$$X(x)Y(0) = 0 \tag{200}$$

$$Y(x) = 0 \tag{201}$$

Now, we can write:

$$u_{xx} + u_{yy} = 0 \tag{202}$$

$$X''Y + XY'' = 0 \tag{203}$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \tag{204}$$

2. Now, let us solve for x:

$$\frac{X''}{X} = -\lambda \tag{205}$$
$$X'' = -\lambda X, \quad X(0) = X(2) = 0 \tag{206}$$
$$\lambda > 0 \Rightarrow X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \tag{207}$$
$$X(0) = B = 0 \tag{208}$$
$$X(2) = A\sin(\sqrt{\lambda}2) = 0 \tag{209}$$
$$= \lambda 2 = n\pi \tag{210}$$
$$= \lambda_n = \left(\frac{n\pi}{2}\right)^2 \tag{211}$$
$$= X_n(x) = \sin(\frac{n\pi x}{2}) \tag{212}$$

3. Let us solve for y:

$$\frac{Y_n''}{Y_n} = \lambda_n \tag{213}$$
$$Y_n'' = \left(\frac{n\pi}{2}\right)^2 Y_n, \quad Y_n(0) = 0 \tag{214}$$
$$Y_n(y) = C\sinh\left(\frac{n\pi y}{2}\right) + D\cosh\left(\frac{n\pi y}{2}\right) \tag{215}$$
$$= Y_n(0) = 0 \Rightarrow D = 0 \tag{216}$$
$$Y_n(y) = \sinh\left(\frac{n\pi y}{2}\right) \tag{217}$$

We are picking a constant for this last term later, so we can drop C.

Here, let us write out our equation for the following function,

February 7, 2022

- $\Delta u = 0$
- $u(0, y) = 0$
- $u(2, y) = 0$
- $u(x, 0) = 0$
- $u(x, 3) = 4$
- $\lambda_n = \left(\frac{n\pi}{2}\right)^2$
- $X_n(x) = \sin\left(\frac{n\pi x}{2}\right)$
- $Y_n(x) = \sinh\left(\frac{n\pi y}{2}\right)$

4. Combine to find u_n and u

$$u_n(x, y) = \sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi y}{2}\right) \tag{218}$$

By linearity,

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi y}{2}\right) \tag{219}$$

5. Find coefficients using last boundary conditions

$$u(x, y) = 4\sin(5\pi x) \tag{220}$$
$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi 3}{2}\right) \tag{221}$$
$$= 4\sin(5\pi x) \tag{222}$$

Recall, the coefficient of the Fourier Sine Series is anything but sin. Notice our last line with discrete number, our coeficient must equal 4 and our n is 10, therefore, we write:

$$A_{10}\sinh\left(\frac{10\pi 3}{2}\right) = 4 \tag{223}$$
$$\Rightarrow A_1 0 \frac{4}{\sinh(15\pi)} \tag{224}$$

The Wave Equation

$$u_{tt} = c^2 u_{xx} \tag{225}$$

Boundary conditions:

$$u(0,t) = u(L,t) = 0 \tag{226}$$

Initial Conditions:

$$u(x,0) = f(x) \tag{227}$$
$$u_t(x,0) = g(x) \tag{228}$$

Here, $g \in C^2$, $g(0) = g(L)$.

1. Assume separable:

$$u(x,t) \tag{229}$$

   Boundary conditions:

$$u(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0 \tag{230}$$
$$u(L,t) = 0 \Rightarrow X(L)T(t) = 0 \Rightarrow X(L) = 0 \tag{231}$$

   Now, let us rewrite our variables:

$$u_{tt} = c^2 u_{xx} \tag{232}$$
$$XT'' = c^2 X'' T \tag{233}$$
$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda \tag{234}$$

2. Solve for X:

$$\frac{X''}{X} = -\lambda \tag{235}$$
$$X'' = -\lambda x \tag{236}$$
$$X(0) = X(L) = 0 \tag{237}$$

   Here, let us write our general equation:

$$X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \tag{238}$$
$$X(0) = 0 \Rightarrow B = 0 \tag{239}$$
$$X(L) = A\sin(\sqrt{\lambda}L) = 0 \tag{240}$$
$$= \sqrt{\lambda}L = n\pi \tag{241}$$

3. Solve for T:

$$\frac{T''}{c^2 T_n} = -\lambda_n \tag{242}$$
$$T_n'' = -c^2 \left(\frac{n\pi}{L}\right)^2 T_n \tag{243}$$
$$T_n(t) = C_n \sin\left(\frac{cn\pi t}{L}\right) + D_n \cos\left(\frac{cn\pi t}{L}\right) \tag{244}$$

4. Combine to find u_n and u

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[C_n \sin\left(\frac{cn\pi z}{L}\right) + D_n \cos\left(\frac{cn\pi z}{L}\right)\right] \tag{245}$$
$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{cn\pi z}{L}\right) + D_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{cn\pi z}{L}\right) \tag{246}$$

5. Find coefficients using Inidial Conditions

$$u(x,0) = f(x) \tag{247}$$

$$= \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \tag{248}$$

$$D_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{dx} \tag{249}$$

$$u_t(x,0) = g(x) \tag{250}$$

$$u_t(x,t) \tag{251}$$

Here, we took the t partial from line 246.

$$u_t(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\frac{cn\pi}{L}\cos\left(\frac{cn\pi t}{L}\right) - D_n \sin\left(\frac{n\pi x}{L}\right)\frac{cn\pi}{L}\sin\left(\frac{cn\pi t}{L}\right) \tag{252}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\frac{cn\pi}{L} = g(x) \tag{253}$$

Here, the non-sin terms are the coefficients of the Fourier Sine Series.

$$C_n\frac{cn\pi}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{dx} \tag{254}$$

$$C_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{dx} \tag{255}$$

<u>Theorem:</u> You can differentiate a Fourier Series if it represents a C^2 function that is the same as the endpoints.

<u>Ex:</u> Solve $u_{tt} = u_{xx}$

- $u(0,t) = 0$
- $u(4,t) = 0$
- $u(x,0) = 2\sin(3\pi x) - \frac{1}{5}\sin\left(\frac{7\pi x}{2}\right)$
- $u_t(x,0) = 0$

1. Assume separable

$$u(x,t) = X(x)T(t) \tag{256}$$

Here, we have our boundary conditions,

$$u(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0 \tag{257}$$

$$u(4,t) = 0 \Rightarrow X(4)T(t) = 0 \Rightarrow X(4) = 0 \tag{258}$$

Now, let us separate:

$$u_{tt} = u_{xx} \tag{259}$$

$$XT'' = X''T \tag{260}$$

$$\frac{T''}{T} = \frac{X''}{X} = -\lambda \tag{261}$$

2. Solve for x:

$$\frac{X''}{X} = -\lambda \tag{262}$$

$$X'' = -\lambda x \tag{263}$$

$$X(0) = X(4) = 0 \tag{264}$$

$$X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \tag{265}$$

$$X(0) = B = 0 \tag{266}$$

$$X(4) = A\sin(\sqrt{\lambda}4) = 0 \tag{267}$$

$$= \sqrt{\lambda}4 = n\pi \tag{268}$$

$$= \lambda_n = \left(\frac{n\pi}{4}\right)^2 \tag{269}$$

$$= X_n(x) = \sin\left(\frac{n\pi x}{4}\right) \tag{270}$$

3. Solve for T:

$$\frac{T_n''}{T_n} = -\lambda_n \tag{271}$$

$$T_n'' = -\left(\frac{n\pi}{4}\right)^2 T_n \tag{272}$$

Here, we have the negative sign, therefore we use sine and cosine:

$$T_n(t) = C_n \sin\left(\frac{n\pi t}{4}\right) \tag{273}$$

4. Combine to get u_n and u

$$u_n(x,t) = \sin\left(\frac{n\pi x}{4}\right) + D_n \cos\left(\frac{n\pi t}{4}\right) \tag{274}$$

By linearity,

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{4}\right) + D_n \sin\left(\frac{n\pi t}{4}\right) \cos\left(\frac{n\pi t}{4}\right) \tag{275}$$

5. Use the initial conditions to find the coefficients

$$u(x,0) = 2\sin(3\pi x) - \frac{1}{5}\sin\left(\frac{7\pi x}{2}\right) \tag{276}$$

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{4}\right) \tag{277}$$

$$D_{12} = 2, D_{14} = -\frac{1}{5}, D_n = 0\ \forall n, n \neq 12, 14 \tag{278}$$

$$u_t(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{4}\right) \frac{n\pi}{4} \cos\left(\frac{n\pi t}{4}\right) - D_n \sin\left(\frac{n\pi x}{4}\right) \frac{n\pi}{4} \sin\left(\frac{n\pi t}{4}\right) \tag{279}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{4}\right) \frac{n\pi}{4} = g(x) = 0 \tag{280}$$

$$C_n = 0\ \forall n \tag{281}$$

$$u(x,t) = 2\sin(3\pi x)\cos(3\pi t) - \frac{1}{5}\sin\left(\frac{7\pi x}{2}\right)\cos\left(\frac{7\pi t}{2}\right) \tag{282}$$

February 9, 2022

More general Boundary Conditions

Before, we used to deal with boundary conditions where u starts and end at 0. Now, let us consider boundary conditions where u can start at any number.

The steady state solution is the following:

$$\frac{T_2 - T_1}{L}x + T_1 \tag{283}$$

Ideas, we want to try the following: $u(x,t) = w(x,t) + u(x,\infty)$. Note that ∞ is the steady state. Here, as time goes to infinity, $w(x,t)$ cancels out. We want to solve for w:

We must specify w.

$$u_t = \alpha^2 u_{xx} \tag{284}$$
$$w_t = \alpha^2 w_{xx} \tag{285}$$

We want to find out more about the boundary conditions. We also need the initial conditions to solve this.

Boundary Conditions

- $u(0,t) = T_1$
- $u(L,t) = T_2$

Let us consider the first boundary condition.

$$w(0,t) + u(0,\infty) = T_1 \tag{286}$$

Here, we know that for the steady state, x is T_1 at $x = 0$. Therefore,

$$w(0,t) = 0 \tag{287}$$

We repeat with our second bullet.

$$u(L,t) = T_2 \Rightarrow \tag{288}$$
$$w(L,t) + u(L,\infty) = T_2 \tag{289}$$
$$w(L,t) = 0 \tag{290}$$

Initial Conditions

$$u(x,0) = f(x) \Rightarrow \tag{291}$$
$$w(x,0) + u(x,\infty) = f(x) \tag{292}$$
$$w(x,0) = f(x) - u(x,\infty) \tag{293}$$
$$w(x,0) = f(x) - \left[\frac{T_2 - T_1}{L}x + T_1\right] \tag{294}$$

Ex:

Solve $u_t = u_{xx}$, $u(0,t) = 2$, $u(4,t) = 3$, $u(x,0) = -6\sin(\pi x) + \frac{x}{4} + 2$.

First, find the steady-state solution:

$$u(x,\infty) = \frac{3-2}{4}x + 2 \tag{295}$$
$$= \frac{x}{4} + 2 \tag{296}$$

Now, we assume $u(x,t) = w(x,t) + u(x,\infty)$. We can make the following assumption:

$$u_t = u_{xx} \Rightarrow w_t = w_{xx} \tag{297}$$

Boundary Conditions

$$u(0,t) = 2 \Rightarrow w(0,t) = u(0,t) - u(0,\infty) = 2 - 2 = 0 \tag{298}$$
$$u(4,t) = 3 \Rightarrow w(4,t) = u(4,t) - u(4,\infty) = 3 - 3 = 0 \tag{299}$$

Here, we plug in our x into our steady-state solution and get 2, 3.

Initial Conditions

$$\begin{aligned}
w(x,0) &= u(x,0) - u(x,\infty) && (300)\\
&= -\sin(\pi x) + \frac{x}{4} + 2 - \left(\frac{x}{4}+2\right) && (301)\\
&= -\sin(\pi x) && (302)
\end{aligned}$$

Now, solve for w:

1. Assume $w(x,t) = X(x)T(t)$

   Boundary Conditions

$$\begin{aligned}
&w(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0 && (303)\\
&w(4,t) = 0 \Rightarrow X(4)T(t) = 0 \Rightarrow X(4) = 0 && (304)\\
&w_t = w_{xx} \Rightarrow XT' \qquad = X''T \Rightarrow \frac{T'}{T} = \frac{X''}{X} = -\lambda && (305)
\end{aligned}$$

2. Solve for X :

$$\begin{aligned}
\frac{X''}{X} &= -\lambda \Rightarrow && (306)\\
X'' &= -\lambda X && (307)\\
X(0) &= X(4) = 0 && (308)
\end{aligned}$$

   Here, let us write our general equation:

$$\begin{aligned}
X(x) &= A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) && (309)\\
X(0) = 0 &\Rightarrow B = 0 && (310)\\
X(4) = 0 &\Rightarrow A\sin(\sqrt{\lambda}4) = 0 && (311)\\
&\Rightarrow \sqrt{\lambda}4 = n\pi && (312)\\
&\Rightarrow \lambda_n = \left(\frac{n\pi}{4}\right)^2 && (313)\\
&\Rightarrow X_n(x) = \sin\left(\frac{n\pi x}{4}\right) && (314)
\end{aligned}$$

3. Solve for T:

$$\begin{aligned}
\frac{T_n'}{T_n} &= -\lambda_n && (315)\\
T_n' &= -\left(\frac{n\pi}{4}\right)^2 T_n && (316)\\
T_n(t) &= e^{-\left(\frac{n\pi}{4}\right)^2 t} && (317)
\end{aligned}$$

4. Combine to find w_n and w:

$$w_n(x,t) = \sin\left(\frac{n\pi x}{4}\right) e^{-\left(\frac{n\pi}{4}\right)^2 t} \qquad (318)$$

   By linearity,

$$w(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{4}\right) e^{-\left(\frac{n\pi}{4}\right)^2 t} \qquad (319)$$

5. Find coefficients using Initial Condition

$$\begin{aligned}
w(x,0) &= -\sin(\pi x) && (320)\\
w(x,0) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{4}\right) && (321)\\
&= -6\sin(\pi x) && (322)\\
w(x,t) &= -6\sin(\pi x)e^{-\pi^2 t} && (323)
\end{aligned}$$

Here, $a_4 = -6$.

$$u(x,t) = -6\sin(\pi x)e^{-\pi^2 t} + \frac{x}{4} + 2 \tag{324}$$

Laplace's Equation General Dirichlet Boundary Conditions

- $u(x,0) = f_1(x)$
- $u(x,M) = f_2(x)$
- $u(0,y) = f_3(y)$
- $u(L,y) = f_4(y)$

Write our solution as the following:

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y) \tag{325}$$

- $\Delta u_1 = 0$
  - $u_1(x,0) = f_1(x)$
  - $u_1(x,M) = 0$
  - $u_1(0,y) = 0$
  - $u_1(L,y) = 0$
- $\Delta u_1 = 0$
  - $u_2(x,0) = 0$
  - $u_2(x,M) = f_2(x)$
  - $u_2(0,y) = 0$
  - $u_2(L,y) = 0$
- $\Delta u_1 = 0$
  - $u_3(x,0) = 0$
  - $u_3(x,M) = 0$
  - $u_3(0,y) = f_3(y)$
  - $u_3(L,y) = 0$
- $\Delta u_1 = 0$
  - $u_4(x,0) = 0$
  - $u_4(x,M) = 0$
  - $u_4(0,y) = 0$
  - $u_4(L,y) = f_4(y)$

This method works because Laplace's equation is linear.

We have already seen that for u_2 :

$$u_2(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right) \tag{326}$$

$$u_2(x,M) = f_2(x) \tag{327}$$

$$\Rightarrow \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi M}{L}\right) \tag{328}$$

$$= f(x) \tag{329}$$

Here, $B_n \sinh\left(\frac{n\pi M}{L}\right)$ is the coefficient for Laplace.

$$B_n \sinh\left(\frac{n\pi M}{L}\right) = \frac{2}{L}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)\ \mathrm{dx} \tag{330}$$

$$B_n = \frac{2}{L\sinh\left(\frac{n\pi M}{L}\right)}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)\ \mathrm{dx} \tag{331}$$

Similarly, for u_4,

$$u_4(x,y) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi y}{M}\right)\sinh\left(\frac{n\pi x}{L}\right) \tag{332}$$

$$u_4(L,y) = f_4(y) \tag{333}$$

$$\Rightarrow \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi y}{M}\right)\sinh\left(\frac{n\pi L}{M}\right) \tag{334}$$

$$= f_4(y) \tag{335}$$

February 11, 2022

Recall we are consider $u = u_1 + u + 2 + u_3 + u_4$. Let us write:

$$u_4(x, y) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi y}{M}\right) \sinh\left(\frac{n\pi x}{M}\right) \tag{336}$$
$$u_4(L, y) = f_4(y) \tag{337}$$
$$= \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi y}{M}\right) \sinh\left(\frac{n\pi L}{M}\right) = f_4(y) \tag{338}$$

Recall, our coefficient is D_n and the sinh function.

$$D_n \sinh\left(\frac{n\pi L}{M}\right) = \frac{2}{M}\int_0^M f_4(y) \sin\left(\frac{n\pi y}{M}\right) dy \tag{339}$$
$$D_n = \frac{2}{M \sinh\left(\frac{n\pi L}{M}\right)} \int_0^M f_4(y) \sin\left(\frac{n\pi y}{M}\right) dy \tag{340}$$

Let's look at u_1:

- $\Delta u_1 = 0$
- $u_1(x, 0) = f_1(x)$
- $u_1(x, M) = 0$
- $u_1(0, y) = 0$
- $u_1(L, y) = 0$

1. Here, let us consider $\Delta u_1 = 0$:

$$\frac{X''}{X} = -\frac{Y''}{y} = -\lambda \tag{341}$$
$$u_1(x, y) = X(x)Y(y) \tag{342}$$

Boundary Conditions

$$u_(x, M) = 0 \Rightarrow X(x)Y(M) = 0 \Rightarrow Y(M) = 0 \tag{343}$$
$$u_(0, M) = 0 \Rightarrow X(0)Y(M) = 0 \Rightarrow X(0) = 0 \tag{344}$$
$$u_(L, M) = 0 \Rightarrow X(L)Y(M) = 0 \Rightarrow X(L) = 0 \tag{345}$$
$$\tag{346}$$

2. $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$
3. Solve for y:

$$\frac{Y''}{Y_n} = \lambda_n \tag{347}$$
$$Y_n'' = \left(\frac{n\pi}{L}\right)^2 Y_n \tag{348}$$
$$Y_n(y) = C \sinh\left(\frac{n\pi y}{L}\right) + D \cosh\left(\frac{n\pi y}{L}\right) \tag{349}$$

Let us see what we have with $Y(m) = 0$:

$$C \sinh\left(\frac{n\pi M}{L}\right) + D \cosh\left(\frac{n\pi M}{L}\right) = 0 \tag{350}$$

Here, this does not work for us. Let us go back and change our $Y_n(y)$:

$$Y_n(y) = C \sinh\left(\frac{n\pi(M-y)}{L}\right) + D \cosh\left(\frac{n\pi(M-y)}{L}\right) \tag{351}$$

Now, let us use our Y:

$$Y_n(M) = C \sinh\left(\frac{n\pi(M-M)}{L}\right) + D \cosh\left(\frac{n\pi(M-M)}{L}\right) \tag{352}$$
$$= D = 0 \tag{353}$$
$$Y_n(y) = \sinh\left(\frac{n\pi(M-y)}{L}\right) \tag{354}$$

4. Let us combine:

$$u_m(x,y) = \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi(M-y)}{L}\right) \tag{355}$$

By linearity

$$u_1(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi(M-y)}{L}\right) \tag{356}$$

5. Find coefficients:

$$u_1(x,0) = f_1(x) \tag{357}$$

$$= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi M}{L}\right) = f_1(x) \tag{358}$$

Once more, we have our coefficient with A_n and sinh.

$$A_n \sinh\left(\frac{n\pi M}{L}\right) = \frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{dx} \tag{359}$$

$$A_n = \frac{2}{L\sinh\left(\frac{n\pi M}{L}\right)}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{dx} \tag{360}$$

Wave Equation

t	#	
	#	
$u = H_1$	#	$u = H_2$
	#	
	0 L	

$u(x,0) = f(x)$
$u_t(x,0) = g(x)$

Steady-state:

$$u_t = 0 \Rightarrow u_{tt} = 0 \Rightarrow u_{xx} = 0 \Rightarrow u = \frac{H_2 - H_1}{L}x + H_1 \tag{361}$$

Try a solution of the form : $u(x,t) = w(x,t) + u(x,\infty)$. Therefore, $w(x,t) = u(x,t) - u(x,\infty)$.

$$u_{tt} = c^2 u_{xx} \Rightarrow w_{tt} = c^2 w_{xx} \tag{362}$$

Boundary conditions:

$$w(0,t) = u(0,t) - u(0,\infty) = H_1 - H_1 = 0 \tag{363}$$

$$w(L,t) = u(L,t) - u(L,\infty) = H_2 - H_2 = 0 \tag{364}$$

Initial Conditions

$$w(x,0) = u(x,0) - u(x,\infty) = f(X) - \frac{H_2 - H_1}{L}x + H_1 \tag{365}$$

$$w_t(x,t) = u_t(x,t) \Rightarrow w_t(x,0) = u_t(x,0) = g(x) \tag{366}$$

Laplace's Equation in Polar Coordinates

Let's say we want to solve $\Delta u = 0$ with Dirichlet Boundary Conditions on a disk or annulus.

Problem: $\Delta u = u_{xx} + u_{yy} \leftarrow$ in terms of x and y.

We must find it in terms of r and θ.

$$u(x,y) \to u(r,\theta) \tag{367}$$

$$x = r\cos\theta \tag{368}$$

$$y = r\sin\theta \tag{369}$$

$$r = \sqrt{x^2 + y^2} \tag{370}$$

$$\theta = \arctan\frac{y}{x} \tag{371}$$

$$\tan\theta = \frac{y}{x} \tag{372}$$

We are going to find : u_x, u_{xx}, u_{yy}

1. u_x :

$$u(x,y) = u(x(r,\theta), y(r,\theta)) = u(r,\theta) = u(r(x,y), \theta(x,y)) \tag{373}$$

Here, we break our chain rule as the following:

$$\begin{array}{cccc} & & u & \\ & | & & | \\ & r & & \theta \\ | & | & | & | \\ x & y & x & y \end{array}$$

According to our tree, we have two routes.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} && (374) \\ &= u_r \frac{x}{\sqrt{x^2+y^2}} + u_\theta \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} && (375) \end{aligned}$$

Note that we know r from line 370. We can rewrite r as $(x^2+y^2)^{\frac{1}{2}}$.

Now, let us multiply our equation by $\frac{x^2}{x^2}$.

$$\begin{aligned} &= u_r \frac{x}{\sqrt{x^2+y^2}} + u_\theta \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{x^2}{x^2} && (376) \\ &= u_r \frac{r\cos\theta}{r} - u_\theta \frac{r\sin\theta}{r^2} && (377) \\ &= u_r \cos\theta - u_\theta \frac{\sin\theta}{r} && (378) \end{aligned}$$

February 14, 2021

Idea: What if we are on a disk?

$$\begin{aligned} \Delta u &= 0 \Rightarrow u_{xx} + u_{yy} = 0 && (379) \\ u_x &= u_r \cos\theta u_\theta \frac{\sin\theta}{r} && (380) \\ u_{xx} &= && (381) \\ &= \frac{\partial}{\partial x}[u_x] && (382) \\ &= \frac{\partial u_x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u_x}{\partial \theta}\frac{\partial \theta}{\partial x} && (383) \\ &= \frac{\partial}{\partial r}\left[u_r\cos\theta - u_\theta\frac{\sin\theta}{r}\right] + \frac{\partial}{\partial\theta}\left[u_r\cos\theta - u_\theta\frac{\sin\theta}{r}\right]\left[-\frac{\sin\theta}{r}\right] && (384) \\ &= \left[u_{rr}\cos\theta - u_{\theta r}\frac{\sin\theta}{r} + u_\theta\frac{\sin\theta}{r^2}\right]\cos\theta + \left[u_{r\theta}\cos\theta - u_r\sin\theta - u_{\theta\theta}\frac{\sin\theta}{r} - u_\theta\frac{\cos\theta}{r}\right]\frac{\sin\theta}{r} && (385) \\ &= u_{rr}\cos^2\theta - 2u_{\theta r}\frac{\sin\theta\cos\theta}{r} + 2u_\theta\frac{\sin\theta\cos\theta}{r^2} + u_r\frac{\sin^2\theta}{r} + u_{\theta\theta}\frac{\sin^2\theta}{r^2} && (386) \\ u_{yy} &= u_{rr}\sin^2\theta + 2u_{\theta r}\frac{\sin\theta\cos\theta}{r} - 2u_\theta\frac{\sin\theta\cos\theta}{r^2} + u_r\frac{\cos^2\theta}{r} + u_{\theta\theta}\frac{\cos^2\theta}{r^2} && (387) \\ \Delta u &= u_{xx} + u_{yy} && (388) \\ &= u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0 && (389) \end{aligned}$$

<u>Solving Laplace's Equation in Polar Coordinates</u>

If we have an open disk, akin to a washer, we have two boundaries: The innter boundary and outer boundary.

1. Assume $u(r,\theta) = R(r)\Theta(\theta)$

$$\begin{aligned} u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} &= 0 && (390) \\ R''\Theta + \frac{R'\Theta}{r} + \frac{R\Theta''}{r^2} &= 0 && (391) \\ R''\Theta + \frac{R'\Theta}{r} &= -\frac{R\Theta''}{r^2} && (392) \\ \frac{r^2}{R''}R + \frac{rR'}{R} &= -\frac{\Theta''}{\Theta} = \lambda && (393) \end{aligned}$$

2. Solve for Θ : $-\frac{\Theta''}{\Theta} = \lambda \Rightarrow \Theta'' = -\lambda\Theta$ If $\lambda > 0$, then we have:

$$\Theta(\theta) = A\sin(\sqrt{\lambda}\theta) + B\cos(\sqrt{\lambda}\theta) \tag{394}$$

Since the solution is periodic in terms of θ : $\begin{cases} \Theta(0) &= \Theta(2\pi) \\ \Theta'(0) &= \Theta'(2\pi) \end{cases}$

$$\begin{aligned} \Theta(0) = \Theta(2\pi) \Rightarrow B = A\sin(\sqrt{\lambda}2\pi) + B\cos(\sqrt{\lambda}2\pi) && (395) \\ \Theta'(\theta) = A\sqrt{\lambda}\cos(\sqrt{\lambda}\theta) - B\sqrt{\lambda}\sin(\sqrt{\lambda}\theta) && (396) \\ \Theta'(0) = \Theta'(2\pi) \Rightarrow A\sqrt{\lambda} = A\sqrt{\lambda}\cos(\sqrt{\lambda}2\pi) - B\sqrt{\lambda}\sin(\sqrt{\lambda}2\pi) && (397) \end{aligned}$$

These equations are satisfied when $\sqrt{\lambda}2\pi = 2\pi n$, $n \in \mathbb{Z}^+$. Then, $\lambda_n = n^2$. If we consider n, then we can also write $\Theta_n(\theta) = A_n\sin(n\theta) + B_n\cos(n\theta)$.

If $\lambda = 0 : \Theta_0'' = A\theta + B$.

$$\begin{aligned} \Theta_0(0) = \Theta_0(2\pi) &\Rightarrow B = A2\pi + \theta && (398) \\ &\Rightarrow A = 0 && (399) \\ \Theta_0'(\theta) = 0 &\Rightarrow \Theta'(0) = \Theta'(2\pi) && (400) \\ &\Rightarrow \Theta_0(\theta) = B_0 && (401) \end{aligned}$$

Note: $\lambda < 0$ yields the trivial solution.

3. Solve for R

$$\begin{aligned} r^2\frac{R_n''}{R_n} + r\frac{R_n'}{R_n} = \lambda_n &\Rightarrow r^2R + rR_n' = n^2R_n && (402) \\ &\Rightarrow r^2R_n'' + rR_n' - n^2R_n = 0 && (403) \end{aligned}$$

Here, let our guess be $R_n(r) = r^m$. Let us plug our guess in:

$$\begin{aligned} r^2m(m-1)r^{m-2} + rmr^{m-1} - n^2r^m &= 0 && (404) \\ r^2\left(m(m-1) + m - n^2\right) &= 0 && (405) \\ r^2\left(m^2 - n^2\right) &= 0 \Rightarrow m = \pm n && (406) \end{aligned}$$

If $n > 0$,

$$R_n(r) = C_nr^n + D_nr^{-n} \tag{407}$$

If $n = 0$,

$$\begin{aligned} r^2R_0'' + rR_0' &= 0 && (408) \\ R_0(r) &= C_0r^0 + D_0\ln r && (409) \\ &= C_0 + D_0\ln r && (410) \end{aligned}$$

4. Combine to find u_n and u:

$$u_n(r,\theta) = \begin{cases} B_0(C_0 + D_0\ln r) & n = 0 \\ C_nr^n + D_nr^{-n}\left(A_n\cos(n\theta) + B_n\cos(n\theta)\right) & n \in Z^+ \end{cases} \tag{411}$$

By linearity,

$$\begin{aligned} u(n\theta) &= B_0(C_0 + D_0\ln r) + \sum_{n=1}^{\infty}\left(C_nr^n + D_nr^{-n}\right)\left(A_n\sin(n\theta) + B_n\cos(n\theta)\right) && (412) \\ &= c_0 + d_0\ln r + \sum_{n=1}^{\infty}(a_nr^n + b_nr^{-n})\sin(n\theta) + (c_nr^n + d_nr^{-n})\cos(n\theta) && (413) \end{aligned}$$

February 16, 2022

5. Let us find the oefficients using the boundary conditions.

$$u(R_1,\theta) = g_1(\theta) \tag{414}$$

$$\Rightarrow g_1(\theta) = \underline{C_0 + d_0 \ln R_1} + \sum_{n=1}^{\infty} \left[(\underline{a_n R_1^n + b_n R_1^{-n}}) \sin(n\theta) + (\underline{C_n R_1^n + d_n R_1^{-n}}) \cos(n\theta) \right] \tag{415}$$

$$u(R_2,\theta) = G_2(\theta) \tag{416}$$

$$\Rightarrow g_2(\theta) = \underline{C_0 + d_0 \ln R_2} + \sum_{n=1}^{\infty} \left[(\underline{a_n R_2^n + b_n R_2^{-n}}) \sin(n\theta) + (\underline{C_n R_2^n + d_n R_2^{-n}}) \cos(n\theta) \right] \tag{417}$$

Underlines book-scan: $B_0, A_n, B_n, \widetilde{B}_0, \widetilde{A}_n, \widetilde{B}_n$:

$$\begin{cases} B_0 & = c_0 + d_0 \ln R_1 \\ \widetilde{B}_0 & = c_0 + d_0 \ln R_1 2 \end{cases} \tag{418}$$

$$\begin{cases} A_n & = a_n R_1^n + b_n R_1^{-n} \\ \widetilde{A}_n & = a_n R_2^n + b_n R_2^{-n} \end{cases} \tag{419}$$

$$\begin{cases} B_n & = c_n R_1^n + d_n R_1^{-n} \\ \widetilde{B}_n & = c_n R_2^n + d_n R_2^{-n} \end{cases} \tag{420}$$

<u>Ex:</u> Solve $\Delta u = 0$, where

- $u(1,\theta) = 3\sin(2\theta)$
- $u(2,\theta) = 7\cos(5\theta)$

1. Assume $u(r,\theta) = R(r)\Theta(\theta)$

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = R''\Theta + \frac{R'\Theta}{r} + \frac{R\Theta''}{r^2} = 0 \tag{421}$$

$$\Rightarrow R''\Theta + \frac{R'\Theta}{r} = -\frac{R\Theta''}{r^2} \tag{422}$$

$$\Rightarrow r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda \tag{423}$$

2. Solve for Θ : $-\frac{\Theta''}{\Theta} = \lambda \Rightarrow \Theta'' = -\lambda\Theta$.

   If $\lambda > 0$, then

$$\Theta(\theta) = A\sin(\sqrt{\lambda}\theta) + B\cos(\sqrt{\lambda}\theta) \tag{424}$$

$$\Theta'(\theta) = A\sqrt{\lambda}\cos(\sqrt{\lambda}\theta) - B\sqrt{\lambda}\sin(\sqrt{\lambda}\theta) \tag{425}$$

$$\sqrt{\lambda}2\pi = 2n\pi \Rightarrow \lambda_n = n^2, n \in \mathbb{Z}^+ \begin{cases} \Theta(0) = \Theta(2\pi) & \Rightarrow B = A\sin(\sqrt{\lambda}2\pi) + B\cos(\sqrt{\lambda}2\pi) \\ \Theta' = \Theta'(2\pi) & \Rightarrow A\sqrt{\lambda} = A\sqrt{\lambda}\cos(\sqrt{\lambda}2\pi) - B\sqrt{\lambda}\sin(\sqrt{\lambda}2\pi) \end{cases} \tag{426}$$

$$= n^2 \Rightarrow \Theta(n)(\theta) = A_n \sin(n\theta) + B_n\cos(n\theta) \tag{427}$$

   If $\lambda = 0$, then the second derivative is 0.

$$\Theta_0'' \Rightarrow \Theta_0(\theta) = A_0\Theta + B_0 \tag{428}$$

$$\Rightarrow \Theta_0'(\theta) = A_0 \tag{429}$$

$$\Rightarrow \Theta_0(0) = \Theta_0(2\pi) \Rightarrow B_0 = 2\pi A_0 + B_0 \Rightarrow A_0 = 0 \tag{430}$$

$$\Rightarrow \Theta_0'(0) = \Theta_0'(2\pi) = 0 \tag{431}$$

3. Solve for R : $r^2\frac{R_n''}{R_n} + \frac{rR_n'}{R_n} = \lambda_n$

$$r^2R_n'' + rR_n' - n^2R_n = 0 \tag{432}$$

$$\tag{433}$$

   Try $R_n(r) = R^m$, then

$$r^2m(m-1)r^{m-2} + rmr^{m-1} - n^2r^m = 0 \tag{434}$$

$$r^m\left[m(m-1) + m - n^2\right] = 0 \tag{435}$$

$$m^n - n^2 = 0 \tag{436}$$

$$m = \pm n \tag{437}$$

Next, let us write:

$$\Rightarrow \begin{cases} R_n(r) & = C_n r^n + D_n r^{-n}, n \in \mathbb{Z}^+ \\ R_n(r) & = C_0 + D_0 \ln r \end{cases} \tag{438}$$

4. Combine to obtain u_n and u,

$$u_n(r,\theta) = \begin{cases} B_0(C_0 + D_0 \ln r) & n = 0 \\ (C_n r^n + D_n r^{-n})(A_n \sin(n\theta) + B_n \cos(n\theta)) & n \in Z^+ \end{cases} \tag{439}$$

By linearity,

$$u(r,\theta) = c_0 + d_0 \ln r + \sum_{n=1}^{\infty} \big((C_n r^n + D_n r^{-n})(A_n \sin(n\theta) + B_n \cos(n\theta))\big) \tag{440}$$

$$= c_0 + d_0 \ln r + \sum_{n=1}^{\infty} \big((a_n r^n + b_n r^{-n}) \sin(n\theta) + (c_n r^n + d_n r^{-n}) \cos(n\theta)\big) \tag{441}$$

5. Find coefficients using BCs :

$$u(1,\theta) = 3\sin(2\theta) \tag{442}$$

$$u(2,\theta) = 7\cos(5\theta) \tag{443}$$

$$u(1,\theta) = c_0 + d_0 \ln(1) + \sum_{n=1}^{\infty} [(a_n + b_n)\sin(n\theta) + (c_n + d_n)\cos(n\theta)] \tag{444}$$

$$\begin{cases} c_0 & = 0 \\ c_n + d_n & = 0 \;\; \forall n \\ a_2 + b_2 & = 3 \\ a_n + b_n & = 0 \;\; \forall n, n \neq 2 \end{cases} \tag{445}$$

Now, let us write:

$$u(2,\theta) = 7\cos(5\theta) \tag{446}$$

$$u(2,\theta) = c_0 + d_0 \ln 2 + \sum_{n=1}^{\infty} \big[(a_n 2^n + b_n 2^{-n})\sin(n\theta) + (c_n 2^n + d_n 2^{-n})\cos(n\theta)\big] = 7\cos(5\theta) \tag{447}$$

$$\begin{cases} c_0 + d_0 \ln 2 & = 0 \Rightarrow d_0 = 0 \\ a_n 2^n + b_n 2^{-n} & = 0 \;\forall n \\ c_5 2^5 + d_5 2^{-5} & = 7 \\ c_n 2^n + d_n 2^{-n} & = 0 \;\forall n, n \neq 5 \end{cases} \tag{448}$$

If $n \neq 5$:

$$\begin{cases} c_n + d_n & = 0 \\ c_n 2^n + d_n 2^{-n} & = 0 \end{cases} \Rightarrow c_n = d_n = 0 \tag{449}$$

If $n \neq 2$:

$$\begin{cases} a_n + b_n & = 0 \\ a_n 2^n + b_n 2^{-n} & = 0 \end{cases} \Rightarrow a_n = b_n = 0 \tag{450}$$

If $n = 5$,

$$\begin{cases} c_5 2^5 + d_5 2^{-5} & = 7 \\ c_5 + d_5 & = 0 \Rightarrow d_5 = -c_5 \end{cases} \tag{451}$$

$$c_5 2^5 - c_5 2^{-5} = 7 \tag{452}$$

$$c_5(32 - \frac{1}{32}) = 7 \tag{453}$$

$$c_5 = \frac{7}{32 - \frac{1}{32}} \tag{454}$$

If $n = 2$,

$$4(a_2 + b_2 = 3) \tag{455}$$

$$-a_2 2^2 + b_2 2^{-3} = 0 \tag{456}$$

$$\frac{15}{4} b_2 = 12 \Rightarrow b_2 = \frac{48}{15} = \frac{16}{5} \tag{457}$$

$$\Rightarrow a_2 = 3 - b_2 = 3 - \frac{16}{5} = -\frac{1}{5} \tag{458}$$

6. $u_y(x,0)=0 \Rightarrow X(x)Y'(0)=0 \Rightarrow Y'(0)=0$

February 23, 2022

Heat and Wave Equations in Polar Coordinates

Heat Equation:

$$u_t = \alpha^2 \Delta u \tag{459}$$

Let $\alpha = 1$, then let us write:

$$u_t = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} \tag{460}$$

Wave Equation:

$$u_{tt} = c^2 \Delta u \tag{461}$$

Here, let $c = 1$:

$$u_{tt} = u_r r + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} \tag{462}$$

In the last two equations, we worked with three variables: t, r, θ.

$$u_t = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} \tag{463}$$

Here, assume $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$

$$R\Theta T' = R''\Theta T + \frac{R'\Theta T}{r} + \frac{R\Theta'' T}{r^2} \tag{464}$$

$$\frac{T'}{T} = \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = -\lambda \tag{465}$$

$$\frac{T'}{T} = -\lambda \tag{466}$$

Here, let us put a pin on T and solve for the second part of the equation:

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = -\lambda \tag{467}$$

$$\frac{r^2R''}{R} + \frac{rR'}{R} + \frac{\Theta''}{\Theta} = -\lambda r^2 \tag{468}$$

$$\frac{r^2R''}{R} + \frac{rR'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta} = \mu \tag{469}$$

We now have separate ODEs for each of the functions T, R, Θ. The solution for Θ looks like Laplace in polar. Recall, we set λ as n^2:

$$\frac{r^2R''}{R} + \frac{rR'}{R} + \lambda r^2 n^2 \tag{470}$$

$$r^2R'' + rR' + (\lambda r^2 - n^2)R = 0R'' + \frac{R'}{r} + (\lambda - \frac{n^2}{r^2})R \qquad = 0 \tag{471}$$

Here, this is Bessel's Equation.

We use the power series to solve Bessel's Equation and obtain:

$$R_n(r) = \sum_{i=1}^{\infty} \frac{(-\lambda)^i r^{2i+n}}{2^{n+2}(i+n)!i!} \tag{472}$$

If $\lambda = 1$, we get the Bessel function:

$$J_n(x) = \sum_{i=1}^{\infty} \frac{(-1)^i x^{2i+n}}{2^{n+2}(i+n)!i!} \tag{473}$$

Laplace in Spherical Coordinates

$$\Delta u = u_{xx} + u_{yy} + u_{zz} \tag{474}$$

Using the chain rule, we obtain:

$$\Delta u = u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{p^2}u_{\phi\phi} + \frac{\cot\theta}{\rho^2}u_\phi + \frac{1}{\rho^2\sin^2\phi}u_{\theta\theta} = 0 \tag{475}$$

Here, assume $u(\rho, \theta, \Phi) = P(\rho)\Theta(\theta)\Phi(\phi)$:

Equation for Θ

$$(1-x^2)\Theta''(x) - 2\Theta'(x) + n(n+1)\Theta(x) = 0 \tag{476}$$

Where $x = \cos(\theta)$. This equation is Legendre's Equation.

The solutions are Legendre Polynomials $P_n(x)$:

n	$P_n(x)$
0	1
1	x
2	$\frac{3x^2-1}{2}$
3	$\frac{5x^3-3x}{2}$
4	$\frac{35x^4-30x^2+3}{8}$

Legendre Polynomials ? make an orthogonal set on $[-1, 1]$

$$\int_{-1}^{1} P_m(x)P_n(x)\ \mathrm{dx} = 0, m \neq n \tag{477}$$

Laplace's Equation: $\Delta u = 0$, $u \in \Omega \subset \mathbb{R}$, $x \in \widetilde{\Omega} \subseteq \mathbb{R}^n$

Definition: A function that satisfies Laplace's Equation is called a harmonic function.

Definition: Let the ball of radius r centered on point x_0 be:

$$B_r(x_0) = \{x : ||x - x_0||_2 \leq r\} \tag{478}$$

Here, let us revisit what the norm notation indicates:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \tag{479}$$

$$||x||_{\ell_2} = \sqrt[2]{x_1^2 + x_2^2 + \ldots + x_n^2} \text{ Standard/Euclidean Norm} \tag{480}$$

$$||x||_{\ell_1} = |x_1| + |x_2| + \ldots + |x_n| \text{ Take } |x_n| \text{ for odd powers} \tag{481}$$

$$||x||_{\ell_4} = \sqrt[4]{x_1^4 + x_2^4 + \ldots + x_n^4} \tag{482}$$

$$||x||_{\ell_\infty} = \max\{|x_1|, |x_2|, \ldots, |x_n|\} \tag{483}$$

Here, let us write:

$$||f||_{L_1} = \int_\Omega |f(x)|\ \mathrm{dx} \tag{484}$$

$$||f||_{L_2} = \sqrt{\int_\Omega f(x)^2\ \mathrm{dx}} \tag{485}$$

$$||f||_{L_n} = \sqrt[n]{\int_\Omega f(x)^n\ \mathrm{dx}} \tag{486}$$

$$||f||_{L_\infty} = \text{essup } |f(x)| \tag{487}$$

Essential Supremum, $x \in \Omega$.

ℓ_2 and L_2 are the only two that correspond to a Hilbert space.

Theorem: If u is harmonic and $B_r(x_0) \subset \Omega$, then then average value of u in the ball equals $u(x_0)$.

$$u(x_0) = \frac{\int_{B_r(x_0)} u(x)\ \mathrm{dx}}{\int_{B_r(x_0)} 1\ \mathrm{dr}} \tag{488}$$

February 25, 2022

Note: These theorems are t? for any ball of any radians in dimensions n. The theorems don't care about the shape of Ω or Boundary Conditions.

Ex: $n = 1$, $\Delta u = u_{xx} = 0$

Here, the function is a linear function

$$u = Ax + B \tag{489}$$

When $n = 1$, we are working with an interval $[x_0 - r, x_0 + r]$.

Theorem 1:

$$u(x_0) = \frac{\int_{x_0-r}^{x_0+r} u(x)\ \mathrm{dx}}{\int_{x_0-r}^{x_0+r} 1\ \mathrm{dx}} \tag{490}$$

$$= \frac{\int_{x_0-r}^{x_0+r} u(x)\ \mathrm{dx}}{2r} \tag{491}$$

Here, we have the definition for average.

Theorem:

$$u(x_0) = \frac{u(x_0 - r) + u(x_0 + r)}{2} \tag{492}$$

Gauss' MVT in Complex Analysis

If $f(z)$ is analytic, then

$$f(z_0) = \frac{\oint_c f(z)\ \mathrm{dz}}{2\pi r} \tag{493}$$

Fourier Series are for $f(x)$ where x is defined over a finite interval. Fourier Transforms are for $f(x)$ where x is defined on $(-\infty, \infty)$.

There is a different form of the Fourier Series.

Here, let us consider Euler's Formula

$$e^{ix} = \cos x + i \sin x \tag{494}$$

$$e^{ix} = \cos x - i \sin x \tag{495}$$

When we combine both formulas, we get

$$\begin{cases} \cos x &= \frac{e^{ix} + e^{-ix}}{2} \\ \sin x &= \frac{e^{ix} - e^{-ix}}{2i} = \frac{\sin ix}{2i} \end{cases} \tag{496}$$

Now, let us rewrite Fourier Series:

$$f(X) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) + B_n \cos\left(\frac{n\pi x}{L}\right) \tag{497}$$

Here, let us replace our Fourier Series with terms we found,

$$f(x) = \sum_{n=0}^{\infty} A_n \frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2i} + B_n \frac{e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}}{2} \tag{498}$$

$$= \sum_{n=0}^{\infty} \left[\frac{A_n}{2i} + \frac{B_n}{2}\right] e^{\frac{in\pi x}{L}} + \left[-\frac{A_n}{2i} + \frac{B_n}{2}\right] e^{-\frac{in\pi x}{L}} \tag{499}$$

$$= \sum_{n=-\infty}^{\infty} \alpha_n e^{\frac{in\pi x}{L}} \tag{500}$$

Here, we found an alternative Fourier Series where

$$\alpha_n = \frac{A_n}{2i} + \frac{B_n}{2} n = 0, 1, 2, \ldots \tag{501}$$

$$\alpha_n = -\frac{A_n}{2i} + \frac{B_n}{2} n = 0, -1, -2, \ldots \tag{502}$$

In the alternative Fourier Series, there is a basis function aside α_n

The basis functions are almost orthogonal

$$\int_{-L}^{L} e^{\frac{im\pi x}{L}} e^{\frac{in\pi x}{L}} \text{ dx} = \int_{-L}^{L} e^{\frac{i\pi x(m+n)}{L}} \text{ dx} \tag{503}$$

$$= \frac{L}{i\pi(m+n)} e^{\frac{i\pi x(m+n)}{L}} \Big|_{-L}^{L} \tag{504}$$

$$= \frac{L}{i\pi(n+m)} \left[e^{i\pi(m+n)} - e^{-i\pi(m+n)} \right] \tag{505}$$

$$= \frac{L}{i\pi(m+n)} 2i\sin(\pi(m+n)) = 0 \tag{506}$$

If $m = -n$, then we get:

$$\int_{-L}^{L} e^{\frac{im\pi x}{L}} e^{\frac{-im\pi x}{L}} \text{ dx} = \int_{-L}^{L} 1 \text{ dx} = 2L \tag{507}$$

To find α_n:

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{\frac{in\pi x}{L}} \tag{508}$$

Here, let us multiply by $e^{-\frac{in\pi x}{L}}$ and integrate.

$$\int_{-L}^{L} f(x) e^{-\frac{ik\pi x}{L}} \text{ dx} = \sum_{n=-\infty}^{\infty} \alpha_n \int_{-L}^{L} e^{\frac{in\pi x}{L}} e^{\frac{-ik\pi x}{L}} \text{ dx} \tag{509}$$

Here, the integral is 0 except when $k = n$.

$$\int_{-L}^{L} f(x) e^{-\frac{n\pi x}{L}} \text{ dx} = \alpha_n 2L \tag{510}$$

$$\alpha_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} \text{ dx} \tag{511}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{\frac{in\pi x}{L}} \tag{512}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} \text{ dx} e^{\frac{in\pi x}{L}} \tag{513}$$

<u>Fourier Transform</u>

Define $\xi_n = \frac{n\pi}{L}$, $\Delta\xi = \frac{\pi}{L}$.

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta\xi}{2\pi} \int_{-L}^{L} f(x) \text{ dx} e^{i\xi nx} \tag{514}$$

This is a Riemann Sum.

Now let $L \to \infty, \Delta\xi \to d\xi$, replace ξ_n with ξ and $\sum \to \int$.

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \text{ dx } e^{i\xi x} d\xi \tag{515}$$

Define the Fourier Transform

$$F[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \text{ dx} = \hat{f}(\xi) \tag{516}$$

$$f^{-1}[\hat{f}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} \text{ d}\xi = f(x) \tag{517}$$

The first line is the Fourier Transform, the second line is the Inverse Fourier Transform.

Note: Laplace Transform

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t) e^{-st} \text{ dt} \tag{518}$$

$$f(t) = \frac{1}{\sqrt{2\pi i}} \int_{c-i\infty}^{c+i\infty} F(s) e^{st} \text{ ds} \tag{519}$$

The first line is the Laplace Transform, whereas the second line is the Inverse Laplace Transform.

We use Laplace Transforms on $[0, \infty)$.

Note: We use Fourier Transforms for functions $f(x)$ where

$$\int_{-\infty}^{\infty} |f(x)| \text{ dx } < \infty \tag{520}$$

Note: $c < \infty$ indicates finite.

February 28, 2022

Fourier Transform

$$F[f(x)] = \hat{f}(\xi) \tag{521}$$

Here, $F[f]$ represents the frequencies in f.

Panseval's Equality

1. if $x \in [-L, L]$

$$\frac{1}{2L}\int_{-L}^{L}[f(x)]^2 \text{ dx} = \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \tag{522}$$

   On the left integral, we have the inner product of f with itself. On the right side, we have the coefficients of Fourier Series

2. If $x \in (-\infty, \infty)$

$$\int_{-\infty}^{\infty}[f(x)]^2 \text{ dx} = \int_{-\infty}^{\infty}[\hat{f}(\xi)]^2 \text{ d}\xi \tag{523}$$

Key Property of the Fourier Transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ix\xi} \text{ dx} \tag{524}$$

$$F\left[\frac{df}{dx}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{df}{dx}e^{-ix\xi} \text{ dx} \tag{525}$$

Here, let us integrate our derivative to get $f(x)$.

$$\begin{array}{c|c} u = e^{-ix\xi} & f(x) \\ \hline du = -i\xi e^{-ix\xi} & \frac{df}{dx} \text{ dx} \end{array}$$

$$F\left[\frac{df}{dx}\right] = \frac{1}{\sqrt{2\pi}}\left[f(x)e^{-ix\xi}\Big|_{-\infty}^{\infty} + i\xi\int_{-\infty}^{\infty} f(x)e^{-ix\xi} \text{ dx}\right] \tag{526}$$

Recall, the L_1 norm of f is finite, allowing us to remove the term on the left in our brackets. Here, we are left with the integral:

$$F\left[\frac{df}{dx}\right] = \frac{i\xi}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ix\xi} \text{ dx} \tag{527}$$

$$= i\xi\hat{f}(\xi) \tag{528}$$

So a derivative in real space corresponds to multiplication in Fourier Space.

$$F\left[\frac{d^n f}{dx^n}\right] = (i\xi)^n\hat{f}(\xi) \tag{529}$$

We can use the Fourier Transform to help solve any linear PDE where the domain of a spatial variable is $(-\infty, \infty)$.

Linear Equations with Infinite Domains

1. The Transport Equation

$$u_t = cu_x \tag{530}$$

   (a) First order equation
   (b) $x \in (-\infty, \infty)$
   (c) $t \in [0, \infty)$
   (d) In essence, $u(x, 0) = f(x)$

Here, let us guess $u(x,t) = v(x+ct)$. Solutions of this form are called travelling wave equations.

Here, let us establish $\eta = x + ct$

When finding our partials, we run through the following tree:

$$\begin{array}{c} u \\ | \\ v \\ \eta \\ \begin{array}{cc} | & | \\ x & t \end{array} \end{array}$$

Let's show that this satisfies $u_t = cu_x$.

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{dv}{d\eta}\frac{\partial \eta}{\partial t} = \frac{dv}{d\eta} \cdot c \tag{531}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial t} = \frac{dv}{d\eta}\frac{\partial \eta}{\partial x} = \frac{dv}{d\eta} \cdot 1 \tag{532}$$

$$u_t = cu_x \tag{533}$$

Any function of the form $u = v(x+ct)$ is a solution to $u_t = cu_x$.

Let's look at the initial condition:

$$u(x,0) = f(x) \tag{534}$$

$$u(x,0) = v(x) \tag{535}$$

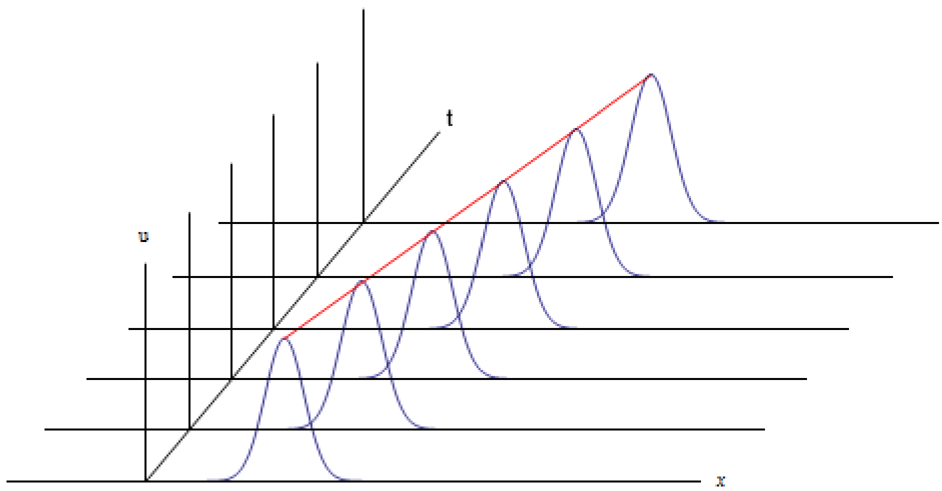
Here, $v(x) = f(X)$. In addition, $u = v(x+ct) = f(x+ct)$.

Ex:

$$u_t = -3u_X \tag{536}$$

$$u(x,0) = e^{-x^2} \tag{537}$$

$$u(x,t) = e^{-(x-3t)^2} \tag{538}$$



The solution translates as time increases, which is why it's called a travelling wave solution.

In this particular case, if $x - 3t =$ constant, then u is fixed.

Remarks

1. The parallel lines are called characteristic curves
2. The slope of the characteristic line is $-\frac{1}{c}$ for $u_t = cu_x$. c is the speed of the solution. It tells us how fast the waves are translated in the $x-$direction.

Lemma

$$\int_{-\infty}^{\infty} e^{-x^2}\, \mathrm{dx} = \sqrt{\pi} \tag{539}$$

Proof. Let us look at our equation squared:

$$
\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-x^2}\ \mathrm{dx} \cdot \int_{-\infty}^{\infty} e^{-y^2}\ \mathrm{dy} && (540)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}\cdot e^{-y^2}\ \mathrm{dx\ dy} && (541)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\ \mathrm{dx\ dy} && (542)\\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\ \mathrm{dr\ d}\theta && (543)
\end{aligned}
$$

Here, use polar to find our integral. □

March 2, 2022

$$
I = \int_{-\infty}^{\infty} e^{-x^2}\ \mathrm{dx} = \sqrt{\pi} \qquad (544)
$$

Let $u = r^2$

$$
\begin{aligned}
I^2 &= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\ \mathrm{dr\ d}\theta && (545)\\
&= \int_{0}^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_0^{\infty}\ \mathrm{d}\theta && (546)\\
&= \int^{2\pi} 0\frac{1}{2}\ \mathrm{d}\theta && (547)\\
&= \frac{1}{2}\cdot 2\pi && (548)\\
&= \pi && (549)
\end{aligned}
$$

Solving Linear Constant Coefficient PDEs with Fourier Transform

Assume $u(x,t)$ with $x \in (-\infty, \infty)$

1. Fourier Transform with respect to x
2. Solve the resulting ODE in T
3. Retransform to go back into real space.

We need $\lim_{x\to\pm\infty} u(x,t) = 0$. This will replace the boundary conditions we have used before.

Laplace Equation

- $u_{xx} + u_{yy} = 0$
- $x \in (-\infty, \infty)$
- $y \in (-\infty, \infty)$
- $\lim_{\substack{x\to\pm\infty \\ y\to\pm\infty}} u(x,y) = 0$

Here, the solution is $u(x,y) = 0$. Using MVT (use ball with large radius).

Heat Equation

Let us consider the following conditions:

- $u_t = u_{xx}$
- $u(x,0) = f(x)$
- $\lim_{x\to\pm\infty} u(x,t) = 0$
- $t \in [0, \infty)$
- $x \in (-\infty, \infty)$

Now, let us consider the following steps:

1. Solve for $u_t = u_{xx}$

$$u_t = u_{xx} \tag{550}$$
$$F[u_t] = F[u_{xx}] \tag{551}$$
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t e^{-ix\xi} \text{ dx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{xx} e^{-ix\xi} \text{ dx} \tag{552}$$
$$\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-ix\xi} \text{ dx}\right] = (i\xi)^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-ix\xi} \text{ dx} \tag{553}$$
$$\frac{\partial}{\partial t}\hat{u}(\xi, t) = -\xi^2 \hat{u}(\xi, t) \tag{554}$$

Our initial condition becomes:

$$F[u(x, 0)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, 0) e^{-ix\xi} \text{ dx} \tag{555}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} \text{ dx} \tag{556}$$
$$= \hat{f}(\xi) \tag{557}$$

2. Solve $\hat{u}_t = -\xi^2 \hat{u}$, $\hat{u}(\xi, 0) = \hat{f}(\xi)$. Here, let us write the general form of $\hat{u}$:

$$\hat{u}(\xi, t) = A(\xi) e^{-\xi^2 t} \tag{558}$$

Here, let us use our initial condition to find $A(\xi)$

$$\hat{u}(\xi, 0)\hat{f}(\xi) = A(\xi) \tag{559}$$
$$\hat{u}(\xi, t) = \hat{f}(\xi) e^{-\xi^2 t} \tag{560}$$

3. Retransform

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi, t) e^{-ix\xi} \text{ d}\xi \tag{561}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-\xi^2 t} e^{ix\xi} \text{ d}\xi \tag{562}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iy\xi} \text{ dy } e^{-\xi^2 t} e^{ix\xi} \text{ d}\xi \tag{563}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) e^{-iy\xi} e^{-\xi^2 t} e^{ix\xi} \text{ dy d}\xi \tag{564}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{-iy\xi} e^{-\xi^2 t} e^{ix\xi} \text{ d}\xi \text{ dy} \tag{565}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{-\xi^2 t + i\xi(x-y)} \text{ d}\xi \text{ dy} \tag{566}$$

Here, let us focus our attention on the inner integral, where we rewrite it as Q.

$$Q = \int_{-\infty}^{\infty} e^{-\xi^2 t + i\xi(x-y)} \text{ d}\xi \tag{567}$$

Let's look at the following:

$$-\xi^2 t + i\xi(x-y) = -t\left[\xi^2 - \frac{i\xi(x-y)}{t}\right] \tag{568}$$
$$= -t\left[\left(\xi - \frac{i(x-y)}{2t}\right)^2 - \frac{i^2(x-y)^2}{4t^2}\right] \tag{569}$$
$$= -t\left[\left(\xi - \frac{i(x-y)}{2t}\right)^2 + \frac{(x-y)^2}{4t^2}\right] \tag{570}$$

Now, we have the following for Q:

$$
\begin{aligned}
Q &= \int_{-\infty}^{\infty} e^{-t\left[\left(\xi - \frac{i(x-y)}{2t}\right)^2 + \frac{(x-y)^2}{4t^2}\right]} \, \mathrm{d}\xi && (571)\\
&= \int_{-\infty}^{\infty} e^{-t\left(\xi - \frac{i(x-y)}{2t}\right)^2 - \frac{(x-y)^2}{4t}} \, \mathrm{d}\xi && (572)\\
&= \int_{-\infty}^{\infty} e^{-t\left(\xi - \frac{i(x-y)}{2t}\right)^2} e^{-\frac{(x-y)^2}{4t}} \, \mathrm{d}\xi && (573)\\
&= e^{-\frac{(x-y)^2}{4t}} \int_{-\infty}^{\infty} e^{-t\left(\xi - \frac{i(x-y)}{2t}\right)^2} \, \mathrm{d}\xi && (574)
\end{aligned}
$$

Here, let us consider the following substitution:

$$
\begin{aligned}
w &= \sqrt{t}\left(\xi - \frac{i(x-y)}{2t}\right) && (575)\\
dw &= \sqrt{t} \, \mathrm{d}\xi && (576)
\end{aligned}
$$

Now, let us write:

$$
\begin{aligned}
e^{-\frac{(x-y)^2}{4t}} \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-w^2} \, \mathrm{dw} &= \sqrt{\frac{\pi}{t}} e^{-\frac{(x-y)^2}{4t}} && (577)\\
u(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \sqrt{\frac{\pi}{t}} e^{-\frac{(x-y)^2}{4t}} \, \mathrm{dy} && (578)\\
&= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{ty}} \, \mathrm{dy} && (579)
\end{aligned}
$$

March 4, 2022

Wave Equation

Here, let us consider the following conditions:

- $u_{tt} = u_{xx}$, $t \in [0, \infty), x \in (-\infty, \infty)$
- $\lim_{x \to \pm\infty} u(x,t) = 0$
- $u(x,0) = f(x)$
- $u_t(x,0) = g(x)$

Note: Two initial conditions for wave: Heat's condition ($u(x,0) = f(x)$) and $u_t(x,0) = g(x)$

Now, let us begin:

1. Let us solve F:

$$
\begin{aligned}
&F[u_{tt}] = F[u_{xx}] && (580)\\
&\Rightarrow \hat{u}_{tt} = (i\xi)^2 \hat{u} && (581)\\
&\Rightarrow \hat{u}_{tt} = -\xi^2 \hat{u} && (582)\\
&\hat{u}(\xi, 0) = \hat{f}(\xi) && (583)
\end{aligned}
$$

Now, let us consider $\hat{u}_t(\xi, 0)$:

$$
\begin{aligned}
F[u_t(x,0)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x,0) e^{-ix\xi} \, \mathrm{dx} && (584)\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ix\xi} \, \mathrm{dx} && (585)\\
&= \hat{g}(\xi) && (586)
\end{aligned}
$$

2. Solve $\hat{u}_{tt} = -\xi^2 \hat{u}$ with the following conditions:
   - $\hat{u}(\xi, 0) = \hat{f}(\xi)$
   - $\hat{u}_t(\xi, 0) = \hat{g}(\xi)$

Now let us write the general form:

$$\hat{u}(\xi,t) = A(\xi)\sin(\xi t) + B(\xi)\cos(\xi t) \tag{587}$$

Here, we do not want to use sine and cosine because we will multiply by expontentials later on.

$$\hat{u}(\xi,t) = A(\xi)e^{i\xi t} + B(\xi)e^{-i\xi t} \tag{588}$$
$$\hat{u}(\xi,0) = A(\xi) + B(\xi) = \hat{f}(\xi) \tag{589}$$

Here, let us find the t partial,

$$\hat{u}_t(\xi,t) = i\xi A(\xi)e^{i\xi t} - i\xi B(\xi)e^{-i\xi t} \tag{590}$$
$$\hat{u}(\xi,0) = i\xi A(\xi) - i\xi B(\xi) = \hat{g}(\xi) \tag{591}$$

Here, let us take the equation with $\hat{f}(\xi)$ and multiply it by $i\xi$:

$$2i\xi A(\xi) = i\xi \hat{f}(\xi) + \hat{g}(\xi) \tag{592}$$
$$a(\xi) = \frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2i\xi} \tag{593}$$

Now, let us subtract to find B:

$$2i\xi B(\xi) = i\xi \hat{f}(\xi) - \hat{g}(\xi) \tag{594}$$
$$B(\xi) = \frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2i\xi} \tag{595}$$

Here, substitute in our terms:

$$\hat{u}(\xi,t) = \left[\frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2i\xi}\right]e^{i\xi t} + \left[\frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2i\xi}\right]e^{-i\xi t} \tag{596}$$

3. Retransform

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(\xi,t)e^{ix\xi}\ \mathrm{d}\xi && (597)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left[\frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2i\xi}\right]e^{i\xi t} + \left[\frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2i\xi}\right]e^{-i\xi t}\ \mathrm{d}\xi && (598)\\
&= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)\left(e^{i\xi t} + e^{-i\xi t}\right)e^{ix\xi} + \frac{\hat{g}(\xi)}{i\xi}\left(e^{i\xi t} - e^{-i\xi t}\right)e^{ix\xi}\ \mathrm{d}\xi && (599)\\
&= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)\left[e^{i\xi(x+t)} + e^{i\xi(x-t)}\right]\ \mathrm{d}\xi + \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\hat{g}(\xi)}{i\xi}\left[e^{i\xi(x+t)} - e^{i\xi(x-t)}\right]\ \mathrm{d}\xi && (600)
\end{aligned}$$

We know that

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}\ \mathrm{d}\xi \tag{601}$$
$$f(x+t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi(x+t)}\ \mathrm{d}\xi \tag{602}$$
$$f(x-t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi(x-t)}\ \mathrm{d}\xi \tag{603}$$

From the previous two equations, let us write:

$$\frac{1}{2}\left[f(x+t) + f(x-t)\right] \tag{604}$$
$$= \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\hat{g}(\xi)}{i\xi}\left(e^{i\xi(x+t)} - e^{i\xi(x-t)}\right)\ \mathrm{d}\xi \tag{605}$$

Now, let us write:

$$\frac{\hat{g}(\xi)}{i\xi} = \frac{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-ix\xi}\ \mathrm{dx}}{i\xi} \tag{606}$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{g(x)e^{-ix\xi}}{i\xi}\ \mathrm{dx} \tag{607}$$

Here, we consider integral by parts: $u = \frac{e^{-x\xi}}{-i\xi} \Rightarrow du = e^{-i\xi}$ dx and $dv = g(x)$ dx $\Rightarrow v = \int_{-\infty}^{x} g(y)$ dy.

$$= -\frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{x} g(y) \text{ dy } \frac{e^{-i\xi x}}{-i\xi} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \int_{-\infty}^{x} g(y) \text{ dy } e^{-ix\xi} \text{ dx} \right] \tag{608}$$

$$= \hat{h}(\xi) \tag{609}$$

Now, from the two equations, let us consider $f(x-t)$:

$$= \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}(\xi) \left(e^{i\xi(x+t)} - e^{i\xi(x-t)} \right) \ textd\xi \tag{610}$$

$$= \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}[h(x+t) - h(x-t)] \tag{611}$$

$$= \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2} \left[\int_{-\infty}^{x+t} g(y) \text{ dy } - \int_{-\infty}^{x-t} g(y) \text{ dy} \right] \tag{612}$$

$$= \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} g(y) \text{ dy} \tag{613}$$

This is the solution to the wave equation on an infinite domain called D'Alembert's Formula.

Wave Equation Solutions

- $u_{tt} = c^2 u_{xx}$
- $x \in (-\infty, \infty), t \in [0, \infty)$
- $u(x,0) = f(x)$
- $u_t(x,0) = g(x)$
- $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)$ dy

1. Conservation of Energy

$$\frac{dE}{dt} = 0 \tag{614}$$

Here, let us consider the energy as:

$$E = \frac{1}{2} \int_{-\infty}^{\infty} (u_t^2 + c^2 u_x^2) \text{ dx} \tag{615}$$

Here, the first term is kinetic energy and the second term is the potential energy. Let us derive our E:

$$\frac{dE}{dt} = \frac{d}{dt}\frac{1}{2} \int_{-\infty}^{\infty} (u_t^2 + c^2 u_x^2) \text{ dx} \tag{616}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} 2u_t u_{tt} + 2c^2 u_x u_{xt} \text{ dx} \tag{617}$$

$$= \int_{-\infty}^{\infty} u_t u_{tt} + c^2 u_x u_{xt} \text{ dx} \tag{618}$$

$$= \int_{-\infty}^{\infty} u_t c^2 u_{xx} + c^2 u_x u_{xt} \text{ dx} \tag{619}$$

$$= c^2 \int_{-\infty}^{\infty} u_t u_{xx} + u_x u_{xt} \text{ dx} \tag{620}$$

Here, let us integrate u_{xx} and differentiate u_t:

$$c^2 \left[u_t u_x \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} -u_{tx} u_x + u_x u_{xt} \text{ dx} \right] = 0 \tag{621}$$

Here, this shows our conservation of energy.

2. **Domain of Dependence / Range of Influence**

   How does the solution at a point depend on the initial condition?

   The domain of dependence is the interval between these two points:

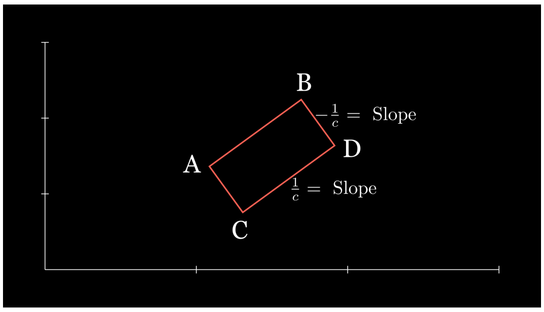
$$[x_0 - ct_0, x_0 + ct_0] \tag{622}$$

If we change the initial condition of a point, where will u be altered?

The range of influence of the initial condition at point x_0 is

$$\left\{(x,t) : \frac{|x-x_0|}{t} \leq c\right\} \tag{623}$$

3. Parallelogram Property (Also valid on $x \in [a,b]$).



$$u(A) + u(D) = u(B) + u(C) \tag{624}$$

4. Reversal of Time

   You can solve the wave equation in backward time

   (a) $x \in (-\infty, \infty)$

   (b) $x \in [a,b]$

   (c) $x \in \Omega \subseteq \mathbb{R}^n$

5. Expanding to Multi-Dimensions

   We cannot expand D'Alembert's Formula in any way to $x \in \mathbb{R}^n, n \leq 2$.

   There are formulas for solving the wave equation for $n \leq 2$.

   The wave equation and the transport equation are both called hyperbolic equations because characteristics are involved in the solution of both.

   Here, let us write:

$$\begin{array}{l|l} \text{Transport} & x+ct \\ \text{Wave} & x+ct, x-ct \end{array}$$

$$\begin{array}{l|ll} \text{Transport} & u_t = cu_x & u(x,0) = f(x) \\ \text{Wave} & U_t = CU_x & U(x,0) = F(x) \end{array}$$
$$\text{Where } U = \begin{bmatrix} u_t \\ u_x \end{bmatrix}$$

   Now, let us consider:

$$U_t = CU_x \Rightarrow \begin{bmatrix} u_{tt} \\ u_{xt} \end{bmatrix} = \begin{bmatrix} 0 & c^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_{tx} \\ u_{xx} \end{bmatrix}, \tag{625}$$
$$U(x,0) = \begin{bmatrix} u_t(x,0) \\ u_x(x,0) \end{bmatrix} = \begin{bmatrix} g(x) \\ f'(x) \end{bmatrix} \tag{626}$$

Wave Equation Solutions

- $u_t = u_x x$
- $x \in (-\infty, \infty)$, $t \in [0, \infty)$
- $u(x,0) = f(x)$
- $u(x,t) = \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{4t}}$ dy

Let the initial condition be a "delta function," $\delta(x)$.

What is a delta function, $\delta(x)$?

It has two main properties:

1. $\delta(x) = 0$, $x \neq 0$.

2. $\int_{-\infty}^{\infty} \delta(x) \text{ dx } = 1$

The "mass" is centered at $x = 0$. The delta function is not a function because $\delta(0) =?$. Actually, the delta function is a measure.

Calculations with Delta Functions

$$\int_{-\infty}^{\infty} \delta(y)g(x-y) \text{ dy} = \int_{-\infty}^{\infty} \delta(x-y)g(y) \text{ dy} = g(x) \tag{627}$$

Here, $\delta(y)$ is zero except when $y = 0$ and $\delta(x-y) = 0$ except when $x = y$.

Here, we have a convolution $\delta * g$, where our variables can switch.

What do we expect when $f(x) = \delta(x)$?

When $t = 0$, our area is the t axis: $|$, however, as $t \to \infty$, then the area slowly flattens, akin to a candle.

Mathematically, what do we expect?

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \delta(y) e^{-\frac{(x-y)^2}{4t}} \text{ dy} && (628) \\ &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} && (629) \end{aligned}$$

The t's impact in the fraction reduces the amplitude and the t in the exponent flattens out the curve.

This is the Gaussian Normal Distributions

What if $f(x) = 7\delta(x) + 5\delta(x-3)$?

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} [7\delta(y) + 5\delta(y-3)] e^{-\frac{(x-y)^2}{4t}} \text{ dy} && (630) \\ &= \frac{1}{\sqrt{4\pi t}} \left[\int_{-\infty}^{\infty} 7\delta(y) e^{-\frac{(x-y)^2}{4t}} \text{ dy} + \int_{-\infty}^{\infty} 5\delta(y-3) e^{-\frac{(x-y)^2}{4t}} \text{ dy} \right] && (631) \\ &= \frac{1}{\sqrt{4\pi t}} \left[7e^{-\frac{x^2}{4t}} + 5e^{-\frac{(x-3)^2}{4t}} \right] && (632) \end{aligned}$$

So for a general $f(x)$, think of $f(x)$ as a bunch of delta functions.

Conservation of Energy

The amount of heat stamp? constant

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{\infty} u(x,t) \text{ dx} &= \int_{-\infty}^{\infty} u(x,t) \text{ dx} && (633) \\ &= \alpha^2 \int_{-\infty}^{\infty} u_{xx}(x,t) \text{ dx} && (634) \\ &= \alpha^2 u_x(x,t) \Big|_{-\infty}^{\infty} && (635) \end{aligned}$$

Recall We know $\lim_{x \pm \infty} u(x,t) = 0$, therefore the rate of change at both infinities is zero.

$$\alpha^2 u_x(x,t) \Big|_{-\infty}^{\infty} = 0 \tag{636}$$

Dependence on Initial Condition

The entire initial condition affects the solution at any point.

Range of Influence is the entire (x,t) plane.

The solution to the heat equation for any fixed $t > 0$ is C^∞ even if the initial condition is discontinuous.

Reversal of Time : Disaster

Solving the heat equation in backwards time does not work because even slight changes in the initial condition lead to drastically different solutions. Equations that exhibit this behavior are called unstable or ill-posed

Expansion to Multi-Dimensions

The solution to the heat equation can easily extend to $x \in \mathbb{R}^n (u_t = \Delta u)$.

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(y) e^{-\frac{(|x-y|^2)}{4t}} \, \mathrm{d}y_1 \mathrm{d}y_2 \dots \mathrm{d}y_n \tag{637}$$

Here, $|x-y|$ can be onsidered the norm.

$$|x-y|^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2 \tag{638}$$

This is how heat looks like in multiple dimensions.

We know the solution to $u_t = u_{xx}, u(x,0) = f(x)$ is

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{4t}} \, \mathrm{dy} \tag{639}$$

$$= \int_{-\infty}^{\infty} K_t(x-y) f(y) \, \mathrm{dy} \tag{640}$$

Where we have Green's Function:

$$K_t(x-y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \tag{641}$$

K_t is called the heat kernel.

Project Information

If you want to mess with an image, you have to think about how computers view images. The image is stored. When we see it, we see an image, but what does the computer see?

Computers stores images as a point of pixels akin to a grid. Pixels contain RGB value information and location. Pictures are generally large.

The first thing is: An image is a grid of numbers (Array, matrix). This is what the computer sees.

Color: Saturation, hue, not the focus.

Focus: Grayscale, same functions, can be extended into color. Add extra dimensions.

Idea: Have a grid of values with pixels and a value at each point (Tells us the gray scale, point of scale between black and white).

Image Processing

Let's say we have a black and white digital image.

Here, we have:

- $u(x,y)$: u (int) is the darkness (grayscale) of the image at point (x,y).
- $u = 0$ black, $u = 255$ = white

Steps to our process:

1. Remove Noise. In the case of our project, we will remove Salt and Pepper noise.
2. Remove Blur.

We know Heat, Laplace, Wave, and Transport.

We want to use Heat.

Here, we want to remove the 'spikes' in our image. The idea is if we apply heat to our point, the peak of the spike reduces in magnitude and spreads out.

When we smoothen our picture, we also blur our ideal picture.

1. First step: Take an image and blur it.

   The heat equation blurs things. This will cause the salt and pepper noise fade.

Here, let us consider an initial image with noise. Our initial image has $t = 0$. Then, we apply

$$u_t = u_{xx} + u_{yy} \tag{642}$$
$$u(x, y, 0) = f(x, y) \tag{643}$$

Here, $f(x, y)$ is our image, the initial condition.

However, when we apply our heat function, we have a pro and a con:

- Pro: Salt and pepper noise are pretty much gone.
- Con: Whole image is blurred.

If we just have one point of noise, Let's say we have an $m \times n$ white grid and black dot in the center, then let us consider the following:

$$f(x, y) = \delta\left(x - \frac{L}{2}\right)\delta\left(y - \frac{M}{2}\right) \tag{644}$$

Here, we have the following conditions:

(a) $u(x, y, 0) = f(x, y)$

and

(a) $u(0, y, t) = f(0, y)$

(b) $u(L, y, t) = f(L, y)$

(c) $u(x, 0, t) = f(x, 0)$

(d) $u(x, M, t) = f(x, M)$

March 11, 2022

How do you get rid of Salt and Pepper noise without blurring the image at the same time?

The heat equation removes the noise, but blurs as well.

Selective blur: Blur in some direction. It all depends on the boundary.

If we are parallel to the boundary, we could care less.

We want to blur in the direction perpendicular to the gradient.

We are going to try to modify the heat equation so that it does not blur the edge.

Here, we define:

- η : Direction of gradient
- ξ : Direction normal to gradient

To blur only in the direction perpendicular to ∇u, use

$$u_t = u_{xx} + u_{yy} - u_{yy} \tag{645}$$
$$u_t = u_{xx} \tag{646}$$

Here, Δu is $u_{xx} + u_{yy}$ and the component in the direction of ∇u is u_{yy}.

The equation that blurs only in the direction normal to the gradient is

$$u_t = \Delta u - u_{\eta\eta} \tag{647}$$
$$= u_{\xi\xi} + u_{\eta\eta} - u_{\eta\eta} \tag{648}$$
$$= u_{\xi\xi} \tag{649}$$

<u>Note:</u> $\Delta u = u_{xx} + u_{yy} = u_{\xi\xi} + u_{\eta\eta}$ since $\xi \perp \eta$ and they are unit vectors.

How do we express $u_{\eta\eta}$ in terms of $u_x, u_y, u_{xx}, u_{yy}, u_{xy}$?

Let $\vec{n}$ be a unit vector.

$$u_n = \nabla u \cdot \vec{n} \tag{650}$$

This is called the directional derivative (Calc III)

$$\begin{aligned}
u_{nn} &= (u_n)_n && (651)\\
&= \nabla u_n \cdot \vec{n} && (652)\\
&= \nabla(\nabla u \cdot \vec{n}) \cdot \vec{n} && (653)\\
&= \nabla\nabla u \vec{n} \cdot \vec{n} && (654)
\end{aligned}$$

Here, $\nabla\nabla u$ is the tensor. The tensor is also called the Hessian Matrix.

$\vec{u} \cdot \vec{v} = \vec{v}^T \vec{u}$

$\vec{n} = n_1\hat{i} + n_2\hat{j}$

$$\begin{aligned}
&= \begin{bmatrix} n_1 & n_2 \end{bmatrix} \begin{bmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} && (655)\\
&= \begin{bmatrix} n_1 & n_2 \end{bmatrix} \begin{bmatrix} u_{xx}n_1 + u_{xy}n_2 \\ u_{xy}n_1 + u_{yy}n_2 \end{bmatrix} && (656)\\
&= n_1(u_{xx}n_1 + u_{xy}n_2) + n_2(u_{xy}n_1 + u_{yy}n_2) && (657)\\
&= n_1^2 u_{xx} + 2n_1n_2u_{xy} + n_2^2 u_{yy} && (658)
\end{aligned}$$

We know the following:

$$\begin{aligned}
\vec{\nabla} &= \frac{\nabla}{||\nabla u||} && (659)\\
&= \frac{\langle u_x, u_y \rangle}{\sqrt{u_x^2 + u_y^2}} && (660)\\
&= \left\langle \frac{u_x}{\sqrt{u_x^2 + u_y^2}}, \frac{u_y}{\sqrt{u_x^2 + u_y^2}} \right\rangle && (661)\\
\vec{\xi} &= \left\langle -\frac{u_y}{\sqrt{u_x^2 + u_y^2}}, \frac{u_x}{\sqrt{u_x^2 + u_y^2}} \right\rangle && (662)
\end{aligned}$$

Here, let us call the first vector n_1 and the second n_2. Now:

$$\begin{aligned}
u_{\xi\xi} &= \frac{u_y^2}{u_x^2 + u_y^2}u_{xx} - \frac{2u_xu_y}{u_x^2 + u_y^2}u_{xy} + \frac{u_x^2}{u_x^2 + u_y^2}u_{yy} && (663)\\
&= \frac{u_y^2u_{xx} - 2u_xu_yu_{xy} + u_x^2u_{yy}}{u_x^2 + u_y^2} && (664)
\end{aligned}$$

Here, let us write:

$$\begin{aligned}
u_t &= u_{\xi\xi} && (665)\\
&\Downarrow && (666)\\
u_t &= \frac{u_y^2u_{xx} - 2u_xu_yu_{xy} + u_x^2u_{yy}}{u_x^2 + u_y^2} && (667)
\end{aligned}$$

This was given credit to James Sethian (Berkeley) in 1988. This is called the Level Set Equation (Mean Curvature Equation)

Idea: We have a picture with three different circles with different sizes. The circle with the smallest radius has the largest curvature. The Mean Curvature Equation attacks the circles with the smallest curvatures first, so our salt and pepper noise, which can be seen as a minute circle, is attacked first.

When the level set equation is applied to an image, the boundaries of each level set move with speed proportional to their curvature.

One thing to note about two circles: Smaller circles become smaller at a faster rate than bigger circles since the radius is smaller and the curvature is bigger.

Grayson's Theorem (For 2-D level sets)

For any closed simple curve, as it evolves under the influence of $u_t = u_{\xi\xi}$, the curve the curve becomes more and more circle-like and disappears as a single point.

March 21, 2022

Effect on an image with Salt and Pepper Noise

When applying our our algorith, the circle around the smile is thicker since inside boundary of circle has large curvature and goes in faster.

The dots are like circles with really large curvatures so they go away quickly.

At a small T: Eliminates noise. At a large T: The image shrinks and mutates.

Where we have a curved line running through the boundaries of the picture, when do you stop? When the image looks the "Best."

Can we modify level set to work better?

Idea: use $u_t = u_{\xi\xi} - \alpha(u - f(x, y))$, where $\alpha > 0$.

u is the current image, f is the initial image. The first term pushes away, modifies, and distorts, whereas the second term wants to bring us back to the original image.

Let's consider $u_t = -\alpha(u - f(x))$, where our straight line is $f(x)$ is the modified image and u is the original. Where the gaps are bigger in our image, the change is greater.

If you run this method, the $u_{\xi\xi}$ will tend to blur in the orthogonal direction (to gradient) and $-\alpha(u - f)$ will tend toward the initial picture.

There two terms compete with each other and eventually the .e.net stabilizes as $t \to \infty$ (steady-state)

The idea is that you get a method that removes the noise but doesn't deviate too much from the original image. α is arbitrary, so you find the α that works best.

<u>Deblurring an Image</u>

We know that the heat equation will blur an image. $u_t = u_{xx} + u_{yy}$.

To deblur, perhaps we could run the backward heat equation $u_t = -(u_{xx} + u_{yy})$, but we know this is unstable and ill-posed.

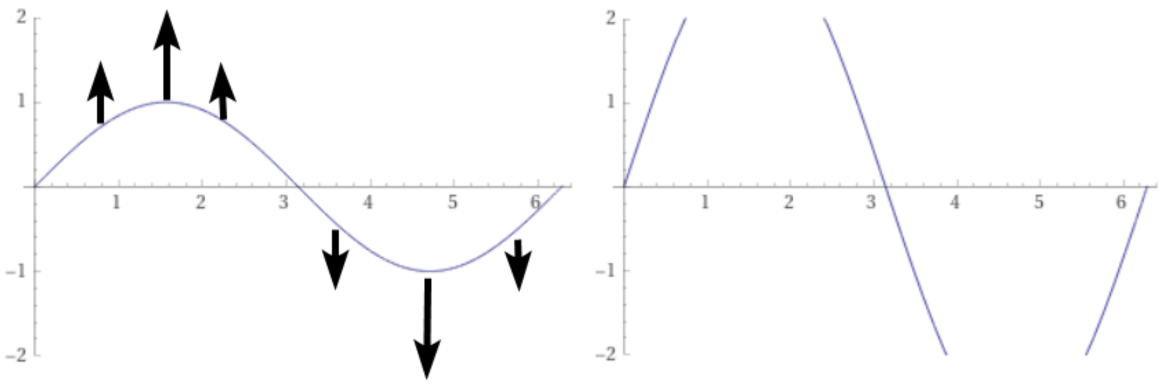
Perhaps we could tweak the backward heat equation?

<u>Idea</u> $u_t = -\alpha u_{xx}, \alpha > 0$ is unstable.

However, $u_t = -|u_x|u_{xx}$ is not ill-posed

Why is our new equation not ill-posed? Let us consider the following function:

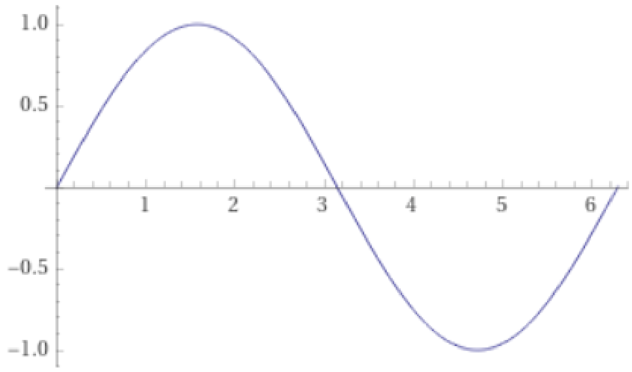
$$\begin{aligned} u_t &= -u_{xx} \\ u(x, 0) &= \sin x \\ u_{xx} &= -\sin x \\ u_t &= \sin x \end{aligned}$$



$u(x, t) = e^t \sin x$, the amplitude tends to infinity.

If we apply the shock filter, then:

$$\begin{aligned} u_t &= -|u_x|u_{xx} \\ u(x, 0) &= \sin x \\ u_t &= |cos x| \sin x \end{aligned}$$



Here, where $u_x = 0$, there is no change. Opposed to our first image, our graph will no longer increase its amplitude drastically. The image is stable at the peaks now and the change is greatest at the midpoints. As time goes on, the curve would beome boxy.

As time goes on, the new image will start taking the save of square sine waves, but will eventually exhibit a graph akin to discontinuous lines at the peaks with a point at the mid point, similar to the infinite fourier.

Note:

1. The solution does not blow up
2. Discontinuities are encouraged