Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{dx}$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ \mathrm{dx}$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ \mathrm{dx}$$
$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

Laplace's Equation

Recall the Fourier Sine Series:

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ \mathrm{dx}$$
$$A_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ \mathrm{dx}$$

Recall general solutions: sine, exp, sinh

Fill in the blanks at last step.

Steady State Solution

Here, T is our u at the start or end, creates a line.

$$v(x) = \frac{T_2 - T_1}{L}x + T_1$$
$$u(x,t) = w(x,t) + v(x)$$

Continue with $w(x,t)$ using the interval.

$$w(x,t) = u(x,t) - v(x)$$

Use the appropriate condition $(e.g. u(x,0))$

Continue to solve as usual with w. Once finished,

$$u(x,t) = w(x,t) + v(x)$$

Trig Derivatives

- $f(x) = \sin x \Rightarrow f'(x) = \cos x$
- $f(x) = \cos x \Rightarrow f'(x) = -\sin x$
- $\int \sin x\ \mathrm{dx} = -\cos x + C$
- $\int \cos x\ \mathrm{dx} = \sin x + C$

Misc

$$u_t = u_{xx} \tag{1}$$
$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} y^2 e^{-\frac{(x-y)^2}{4t}}\ \mathrm{d}y \tag{2}$$

Fourier Transformation

For a given $f(x)$, Fourier Transform is defined as:

$$F[f] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\ \mathrm{dx} = \hat{f}(\xi) \tag{3}$$
$$f^{-1}[\hat{f}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}\ \mathrm{d}\xi = f(x) \tag{4}$$

Euler's Formula is defined as:

$$e^{-ix} = \cos x + i\sin x \tag{5}$$

D'Alembert's Formula

Used in Wave Equation Solutions, $u_{tt} = c^2 u_{xx}$, $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $u(x,t)$:

$$\frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(y)\ \mathrm{d}y \tag{6}$$

Transport Equation

The Transport Equation

$$u_t = cu_x \tag{7}$$

1. First order equation
2. $x \in (-\infty, \infty)$
3. $t \in [0, \infty)$
4. In essence, $u(x,0) = f(x)$

Here, let us guess $u(x,t) = v(x+ct)$. Solutions of this form are called travelling wave equations.

Here, let us establish $\eta = x + ct$

Delta Functions

$$\int_{-\infty}^{\infty} \delta(y)g(x-y)\ \mathrm{d}y = \int_{-\infty}^{\infty} \delta(x-y)g(y)\ \mathrm{d}y = g(x) \tag{8}$$

Gaussian Normal Distributions

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \delta(y) e^{-\frac{(x-y)^2}{4t}} \, \mathrm{d}y \tag{9}$$
$$= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \tag{10}$$

For $\delta(x-0)$.

Integral

$$\int_{-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x = \sqrt{\pi} \tag{11}$$

Range/Domain

Slope: $\frac{1}{c}$

Range - upsidedown n. Lines: $x + ct = x_0$, $x - ct = x_0$

$$\left\{(x,t) : \frac{|x - x_0|}{t} \leq c\right\} \tag{12}$$

Domain - upsidedown u. Lines: $x - ct = u_1(x,t)$, $x + ct = u_2(x,t)$

Harmonic Functions

$\delta u = 0$

Find u_x, u_{xx}, u_y, u_{yy} and find the min/max on the boundaries

For non-obvious values, find critical number with respect to variable.

Shock Filter

$$\xi'(t) = \frac{f(u_L) - f(u_R)}{u_L - u_R} = c \tag{13}$$

Here, the slope of shock is $\frac{1}{c}$