

1. Let $f(x)$ be a 2π -period function on the interval $[-\pi, \pi]$ where $f(x) = \begin{cases} -1 & -\pi < x \leq 0 \\ 1 & 0 < x \leq \pi \end{cases}$

$$f(x) = a_n \sin\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right) \quad (1)$$

- (a) Plot the function on the interval $[-3\pi, 3\pi]$
(b) Plot its (infinite) Fourier series on $[-3\pi, 3\pi]$
(c) Find the Fourier series of $f(x)$ Here, let us consider a few points:
- $L = -\pi$

2. Let $f(x) = x^2$ be a 2π -periodic function on the interval $[-\pi, \pi]$.

- (a) Derive its Fourier series
- (b) Use Maple or Matlab to plot its finite Fourier series on $[-\pi, \pi]$ for $N = 10, 20, 50$ together with $f(x)$
- (c) Use your Fourier series from part (a) to show that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$

Let us begin

3. In the solution of the heat equation, we end up solving $X'' = -\lambda X$. Show that if $\lambda < 0$ or $\lambda = 0$ there is only the trivial solution ($X(x) = 0$).

Here, we have the equation:

$$X'' = -\lambda X \quad (2)$$

We want to use this equation and set our boundary conditions as $X(0) = X(L) = 0$. Now, we must find an equation where after two derivatives on the right, we obtain a similar function on the left. On the left, we have a sign, coefficient, and function of x . Let us write a general solution for our equation:

$$X(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x) \quad (3)$$

Here, we can make three assumptions *via trichotomy*: $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$. Let us look at the first two examples:

(a) $\lambda < 0$

Here, let us consider the case when λ is negative. Let us consider rewriting λ :

$$\lambda < 0 \quad (4)$$

$$\lambda \cdot -1 > 0 \cdot -1 \quad (5)$$

$$-1 \cdot \lambda > 0 \quad (6)$$

Now, let us plug in our found value into our general equation:

$$X(x) = A \cos(\sqrt{-1 \cdot \lambda}x) + B \sin(\sqrt{-1 \cdot \lambda}x) \quad (7)$$

Let us separate the terms under the radical:

$$X(x) = A \cos(\sqrt{-1 \cdot \lambda}x) + B \sin(\sqrt{-1 \cdot \lambda}x) \quad (8)$$

$$= A \cos(\sqrt{-1}\sqrt{\lambda}x) + B \sin(\sqrt{-1}\sqrt{\lambda}x) \quad (9)$$

$$= A \cos(i\sqrt{\lambda}x) + B \sin(i\sqrt{\lambda}x) \quad (10)$$

Here, in our expression, we see we are taking the square root of a negative number, which would give us an imaginary number. Here, we are evaluating our general solution with real numbers, therefore, the following form:

$$X(x) = A \cos(i\sqrt{\lambda}x) + B \sin(i\sqrt{\lambda}x) \quad (11)$$

Where $X(x)$ is a real number would only have the trivial solution $X(x) = 0$.

(b) $\lambda = 0$

Here, let us consider the case when λ is zero. Now, let us write our general equation:

$$X(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x) \quad (12)$$

Here, since $\lambda = 0$, we can evaluate our equation:

$$X(x) = A \cos(0) + B \sin(0) \quad (13)$$

$$= A \quad (14)$$

Now, let us evaluate our boundary condition for $X(x) = A$. First, we let $X(0) = 0$:

$$X(0) = 0 = A \quad (15)$$

Here, we know A is 0. For the second condition, let us write:

$$X(L) = 0 = A \quad (16)$$

Here, we will always have the trivial solution, $X(x) = 0$.

4. Show that $u(x, t) = e^{-\lambda^2 a^2 t} [A \cos(\lambda x) + B \sin(\lambda x)]$

5. Solve $u_t = u_{xx}$ given $u(0, t) = u(1, t) = 0$ for $t \geq 0$ and $u(x, 0) = 1$ for $0 \leq x \leq 1$

6. Find the solution to the previous problem if $u(x, 0) = x - x^2$ for $0 \leq x \leq 1$

7. Solve $u_t = u_{xx}$ given $u(0, t) = u(1, t) = 0$ for $t \geq 0$ and $u(x, 0) = 10^{-5} \sin(10^6 \pi x)$ for $0 \leq x \leq 1$. Determine $u(x, 2)$ and $u(x, -2)$ and look at their magnitudes. Note that when $t = -2$, we are looking at the backward heat equation and given the magnitude of $u(x, -2)$, what can you say about the solution to the backward heat equation?