Partial Differential Equations - Homework 8

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1. Given the conservation law $u_t + [\cos u]_x = 0$, sketch the characteristic curves, where

   (a) $u(x,0) = \begin{cases} \frac{\pi}{2} & x < 0 \\ 0 & x \geq 0 \end{cases}$

   (b) $u(x,0) = \begin{cases} \frac{\pi}{6} & x < 0 \\ \frac{\pi}{2} & x \geq 0 \end{cases}$

   Here, we are given the conservation law. If we differentiate on x, we find:

$$u_t - u_x \sin u = 0 \tag{1}$$

   From here, let us use a third variable to solve our equation:

$$\frac{dt}{ds} = 1 \tag{2}$$

$$\frac{dx}{ds} = -\sin u \tag{3}$$

$$\frac{du}{ds} = 0 \tag{4}$$

   From here, we would let $t = s$.

   (a) Here, we would want to consider $x(0) = x_0$ and $u(0) = f(x)$, which is given as $u(x,0)$. If we make this assumption for $\frac{dx}{ds}$, we would get:

$$\frac{dx}{ds} = -\sin u \tag{5}$$

$$\frac{dx}{ds} = \begin{cases} \sin \frac{\pi}{2} & x_0 < 0 \\ -\sin 0 & x_0 \geq 0 \end{cases} \tag{6}$$

   From here, if we evaluate our terms, we get:

$$\frac{dx}{ds} = \begin{cases} -1 & x_0 < 0 \\ 0 & p \geq 0 \end{cases} \tag{7}$$

   Now, if we multiply both sides by ds and integrate, we would get:

$$x(s) = \begin{cases} -s + x_0 & x_0 < 0 \\ x_0 & x_0 \geq 0 \end{cases} \tag{8}$$

Here, recall $t = s$,

$$x(t) = \begin{cases} -t + x_0 & x_0 < 0 \\ x_0 & x_0 \geq 0 \end{cases} \tag{9}$$

(b) Now, let us consider $u(x, 0)$ for part b) and write for $\frac{dx}{ds}$:

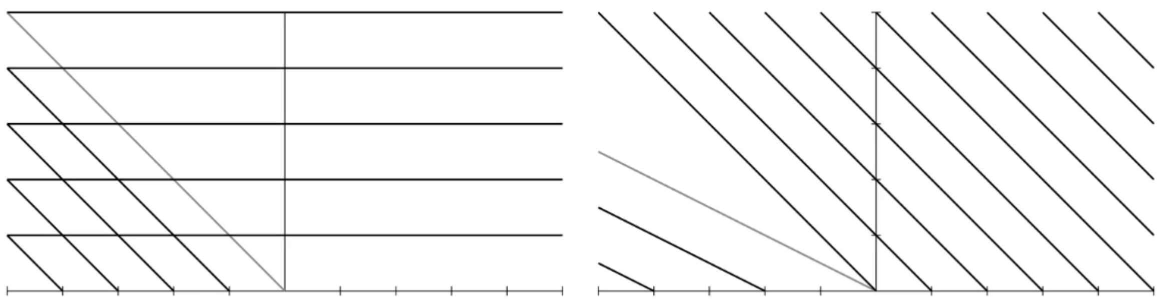
$$\frac{dx}{ds} = \begin{cases} -\sin\frac{\pi}{6} & x_0 < 0 \\ -\sin\frac{\pi}{2} & x_0 \geq 0 \end{cases} \tag{10}$$

When we integrate both sides with respect to their variables, we get:

$$x(s) = \begin{cases} -0.5s + x_0 & x_0 < 0 \\ -s + x_0 & x_0 \geq 0 \end{cases} \tag{11}$$

$$x(t) = \begin{cases} -0.5t + x_0 & x_0 < 0 \\ -t + x_0 & x_0 \geq 0 \end{cases} \tag{12}$$

Here are both characteristic lines plotted. Note that an extra line at $x_0 = 0$ is added to show both equations at x_0:



2. Solve the following equations using the method of characteristics

(a) $u_t + 7u_x = t \quad u(x,0) = \sin x$

(b) $u_t + xu_x + 2u = 0 \quad u(x,0) = x^3$

(c) $u_t + 2xtu_x = u \quad u(x,0) = 1 - x$

(d) $tu_t + 2xu_x = t\sin(\pi t) \quad u(x,1) = \cos x$

(a) Let us rewrite our given problem:

$$u_t + 7u_x = t \tag{1}$$
$$u(x,0) = \sin x \tag{2}$$

Here, let us derive information from our given statement:

$$\frac{dx}{dt} = 7 \tag{3}$$
$$\frac{du}{dt} = t \tag{4}$$

Here, let us solve for $\frac{dx}{dt}$:

$$\frac{dx}{dt} = 7 \tag{5}$$
$$\int dx = \int 7 \tag{6}$$
$$x = x_0 e^{7t} \tag{7}$$
$$x_0 = xe^{-7t} \tag{8}$$

Now that we have solved for x. Now, let us consider $\frac{du}{dt}$

$$\frac{du}{dt} = t \tag{9}$$
$$\int du = \int tdt \tag{10}$$
$$u = \frac{1}{2}t^2 + u_0 \tag{11}$$
$$u = \frac{1}{2}t^2 + f(x_0) \tag{12}$$
$$u = \frac{1}{2}t^2 + f\left(xe^{-7t}\right) \tag{13}$$
$$u = \frac{1}{2}t^2 + \sin\left(xe^{-7t}\right) \tag{14}$$
$$\tag{15}$$

(b) Here, let us consider rewriting our equation:

$$u_t + xu_x = -2u \tag{16}$$

From here, we can find:

$$\frac{dx}{ds} = x \tag{17}$$
$$\frac{dt}{ds} = 1 \tag{18}$$
$$\frac{du}{ds} = -2u \tag{19}$$

Here, finding $\frac{dx}{ds}$ is a matter of solving a differential equation to get:

$$\frac{dx}{ds} = x \tag{20}$$
$$x = ce^{s} \tag{21}$$

Next, we can also find our second term for $\frac{dt}{ds}$ and find:

$$\frac{dt}{ds} = 1 \tag{22}$$
$$dt = ds \tag{23}$$
$$t = s + t_0 = s \tag{24}$$

Note since $s = t$, we can replace this for what we found with x. Now, to find $\frac{dt}{ds}$, let us take a similar approach as with $\frac{dx}{ds}$ and write:

$$\frac{du}{ds} = -2u \tag{25}$$
$$u = de^{-2s} \tag{26}$$

Here, recall $s = t$. We also are given $u(0)$ as x^3:

$$u = de^{-2s} \tag{27}$$
$$u = u_0 e^{-2t} \tag{28}$$
$$u = f\left(x_0\right) e^{-2t} \tag{29}$$

From here, let us take a look back at x and find x_0:

$$x = ce^{t} \tag{30}$$
$$x = x_0 e^{t} \tag{31}$$
$$xe^{-t} = x_0 \tag{32}$$

Now, if we plug x_0 into u, we can find:

$$u = f\left(x_0\right) e^{-2t} \tag{33}$$
$$u = f\left(xe^{-t}\right) e^{-2t} \tag{34}$$
$$u = \left(xe^{-t}\right)^3 e^{-2t} \tag{35}$$
$$u = x^3 e^{-3t} e^{-2t} \tag{36}$$
$$u = x^3 e^{-5t} \tag{37}$$

(c) Here, let us rewrite our equation:

$$u_t + 2xtu_x = u \tag{38}$$
$$u(x,0) = 1 - x \tag{39}$$

Here, note that u_x has t as a coefficient. Let us skip over the change of variable $(t = s)$ and write:

$$\frac{dx}{dt} = 2xt \tag{40}$$
$$\frac{du}{dt} = u \tag{41}$$

First, let us consider solving for $\frac{dx}{dt}$. Here, let us solve the differential equation:

$$\frac{dx}{dt} = 2xt \tag{42}$$
$$\int \frac{1}{x}\ dx = \int 2t\ dt \tag{43}$$
$$\ln|x| = t^2 + c \tag{44}$$
$$x = ce^{t^2} \tag{45}$$
$$x = x_0 e^{t^2} \tag{46}$$
$$x_0 = xe^{-t^2} \tag{47}$$

Now, let us solve for $\frac{du}{dt}$:

$$\frac{du}{dt} = u \tag{48}$$
$$u = u_0 e^t \tag{49}$$
$$u = f(x_0)e^t \tag{50}$$

Here, $u(x,0)$ is defined as $1 - x$. Let us substitute this in to solve for u:

$$u = \left(1 - xe^{-t^2}\right)e^t \tag{51}$$
$$u = e^t - xe^{-t^2+t} \tag{52}$$

(d) Here, let us rewrite our given problem:

$$tu_t + 2xu_x = t\sin(\pi t) \tag{53}$$
$$u(x,1) = \cos x \tag{54}$$

First, let us rewrite our given equation:

$$u_t + \frac{1}{t}2xu_x = \sin(\pi t) \tag{55}$$

Here, we eliminated to coefficient in front of u_t, so let us only consider $\frac{dx}{dt}$ and $\frac{du}{dt}$:

$$\frac{dx}{dt} = \frac{1}{t}2x \tag{56}$$
$$\frac{du}{dt} = \sin(\pi t) \tag{57}$$

From here, let us solve for x:

$$\frac{dx}{dt} = \frac{1}{t}2x \tag{58}$$

$$\int \frac{1}{x}dx = \int \frac{2}{t}dt \tag{59}$$

$$\ln|x| = 2\ln|t| + x_0 \tag{60}$$

$$\ln|x| = \ln|t^2| + x_0 \tag{61}$$

$$e^{\ln|x|} = x_0 e^{\ln t^2} \tag{62}$$

$$x = x_0 t^2 \tag{63}$$

$$xt^{-2} = x_0 \tag{64}$$

Now, if we consider solving for $\frac{du}{dt}$, we get:

$$\frac{du}{dt} = \sin(\pi t) \tag{65}$$

$$\int du = \int \sin(\pi t)dt \tag{66}$$

$$u = -\frac{1}{\pi}\cos(\pi t) + u_0 \tag{67}$$

$$u = -\frac{1}{\pi}\cos(\pi t) + f(x_0) \tag{68}$$

$$u = -\frac{1}{\pi}\cos(\pi t) + \cos\left(xt^{-2}\right) \tag{69}$$