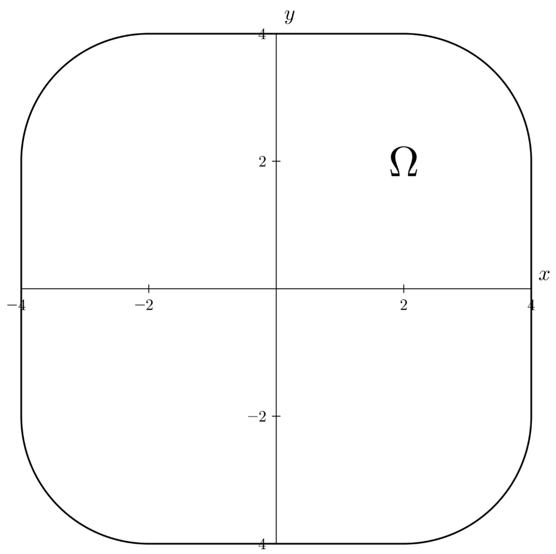
1. Prove the maximum principle using the Mean Value Theorem(s). If $\Delta u = 0$ on a bounded domain Ω, show that

$$\max_{x \in \Omega} u(x) = \max_{x \in \partial\Omega} u(x) \tag{1}$$

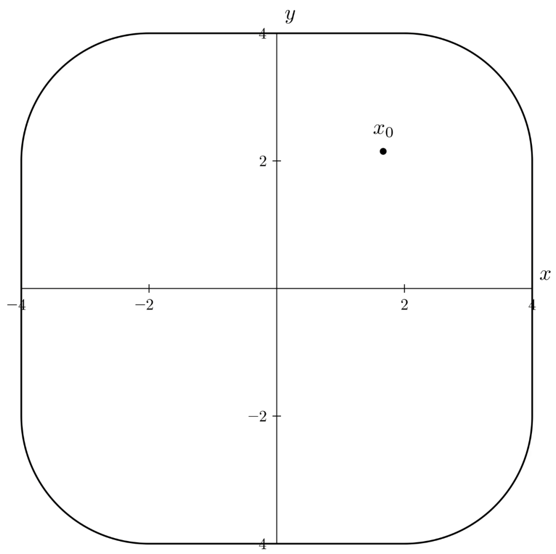
In other words, the max of a harmonic function is attained on its boundary. Hint: Use proof by contradiction.

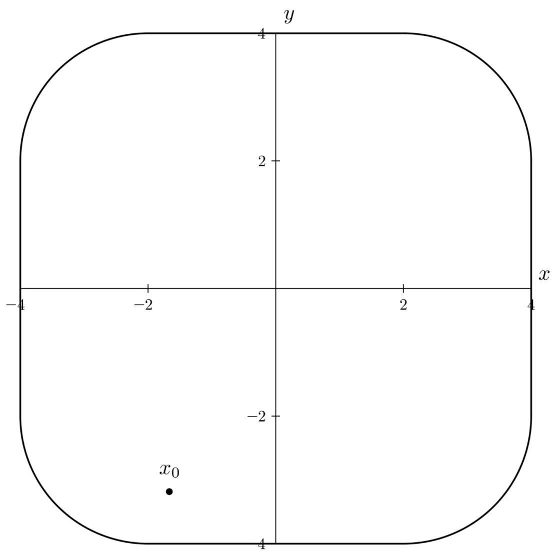
Here, let us consider our given statement from line 1. We are given the assumption that for a point in a given area, Ω, there is a point that gives us the max u(point).

Here, let us consider the area Ω, where Ω is any closed area. For the sake of coding these graphs, let us assume our boundary assumes the shape of a rounded rectangle

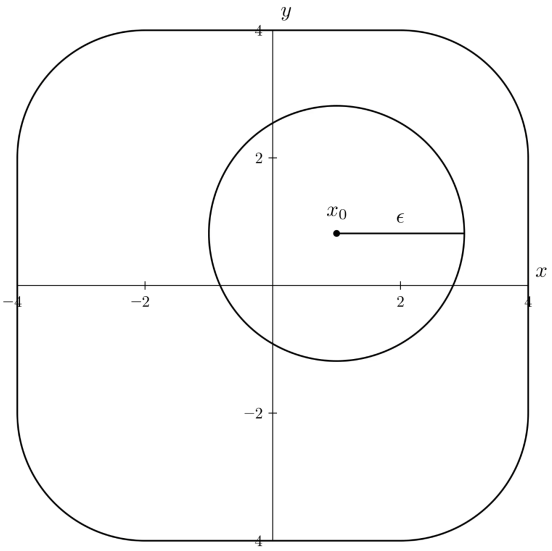


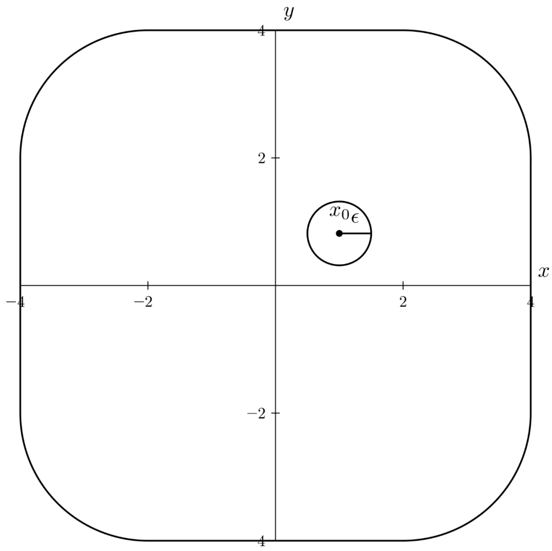
Now, let us consider a point within this interval, x_0. Here, let us consider point x_0 to be an arbitrary point within our area, Ω:



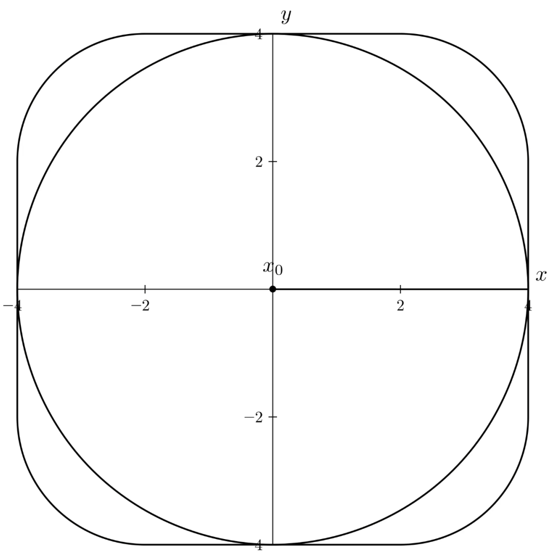


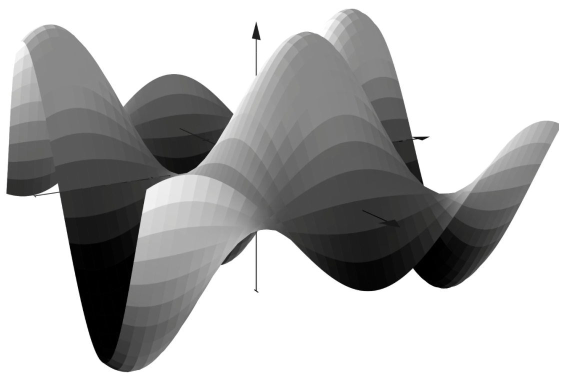
To begin, let us consider a neighborhood, or ball, around our point, x_0. Here, our ball around x_0, has a radius ϵ, where $\epsilon > 0$, and the area of our ball is a subset of Ω.



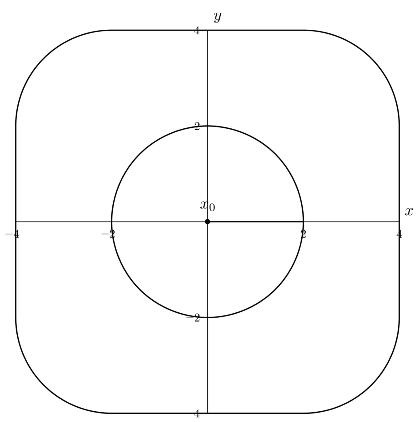


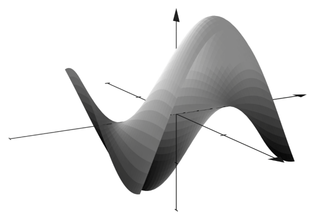
Now, let us consider what is $u(x)$. Here, $u(x)$ is a harmonic function, allowing us to make the assumption that the average value of the function within its neighborhood is the center point of our neighborhood. In this case, our neighborhood is centered about x_0.

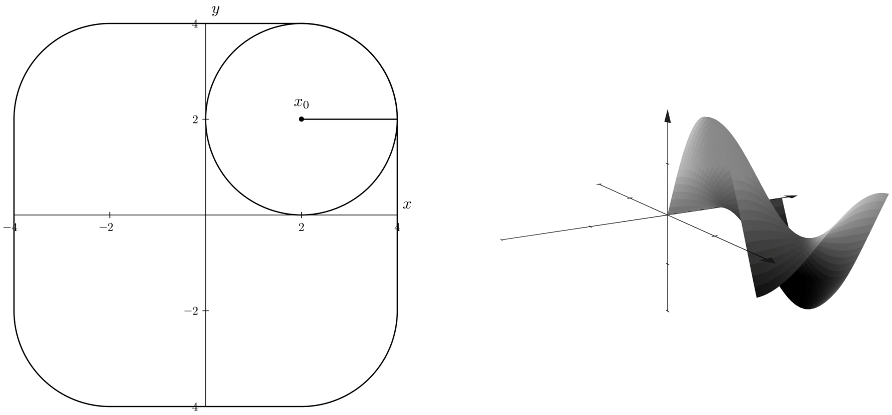




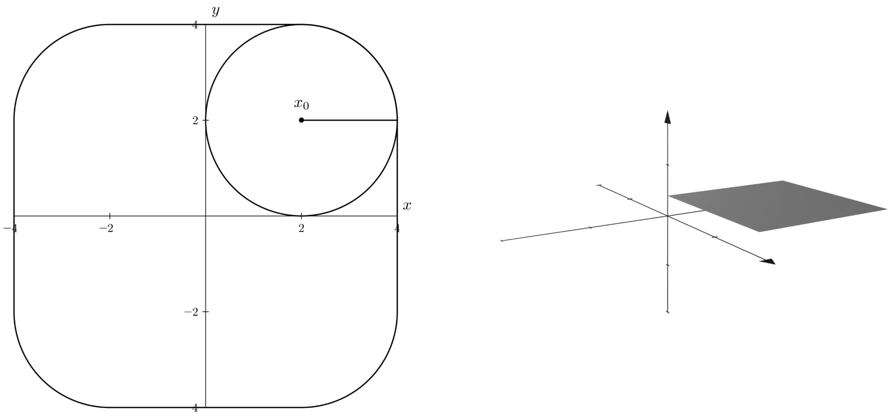
Here, let us consider the contents of u within nour neighborhood. We can change the size of our neighborhood, as long as the size does not exceed Ω's boundaries, and move the center of our neighborhood throughout the region.

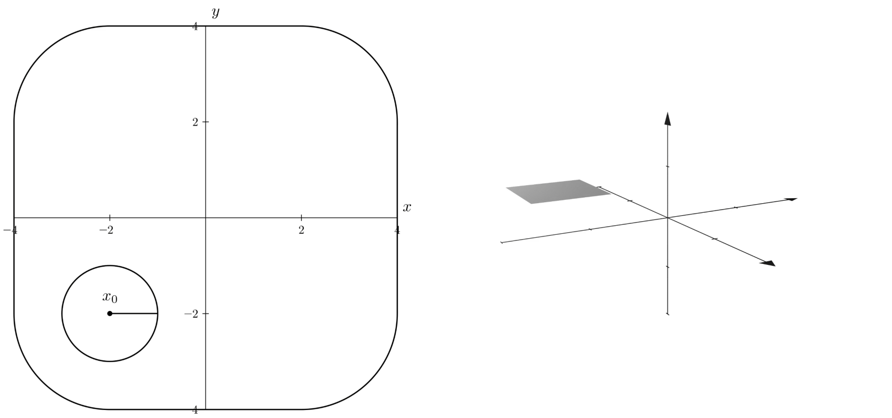






Here, let us consider a function where the average value of its neighborhood is also the maximum value of our harmonic function, $u(x)$. Here, if our neighborhood's average value is always the maximum value throughout the region, then our harmonic function's value is uniform throughout the region. In other words, there is a constant value throughout the region.





While the average of our neighborhood is considered the max in a constant function as shown, a function like this will not always occur.

Here, we have a contradiction, as a function will only have an average max within its interior when the function is constant. However, this cannot always happen, therefore we will not see the max in the interior, rather on the boundary.

2. Plot the given functions and find their Fourier Transforms

(a) $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

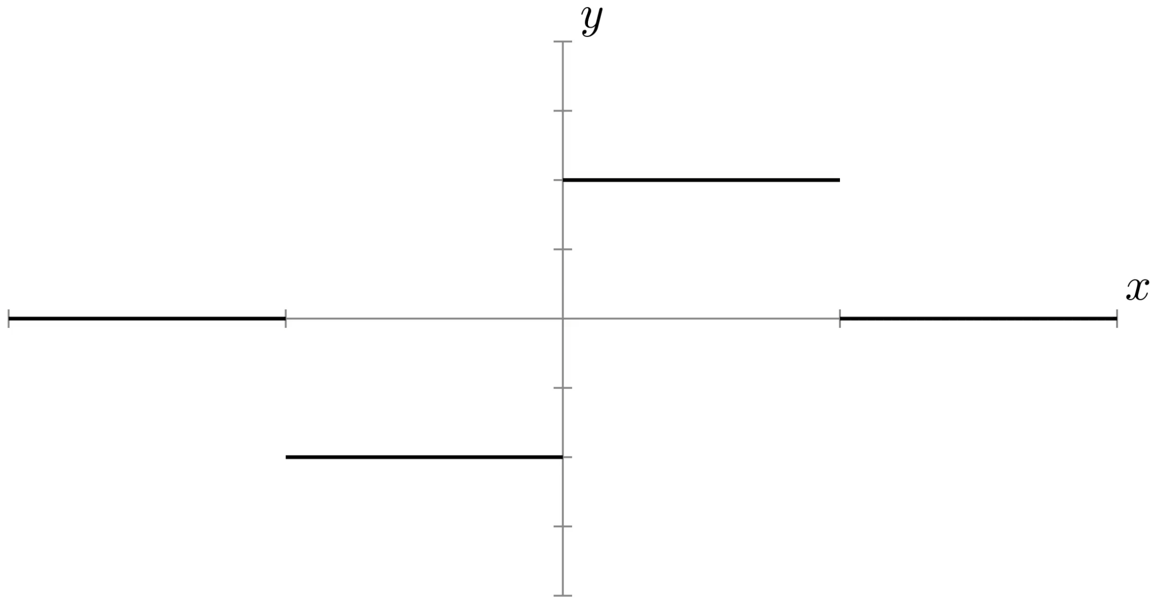
(b) $f(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Here, let us consider both problems individually.

(a) First, let us consider our first given equation:

$$f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, let us plot our given function:



Now, let us find the Fourier Transform of this problem.

Here, let us consider our domain of integration, where f_i is the i^{th} position. Let us informally write:

$$f_1(x) = \int_{-1}^{0} -1 \text{ dx} \tag{1}$$

$$f_2(x) = \int_{0}^{1} 1 \text{ dx} \tag{2}$$

$$f_3(x) = \int_{-\infty}^{-1} 0 \text{ dx} + \int_{1}^{\infty} 0 \text{ dx} \tag{3}$$

Here, let us write our definition of the Fourier Transform:

$$F[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \text{ dx} \tag{4}$$

From here, let us split our integral:

$$F[f] = \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{0} -e^{-i\xi x} \text{ dx} + \int_{0}^{1} e^{-i\xi x} \text{ dx} + \int_{-\infty}^{-1} 0e^{-i\xi x} \text{ dx} + \int_{1}^{\infty} 0e^{-i\xi x} \text{ dx} \right] \tag{5}$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\int_{-1}^{0} e^{-i\xi x} \text{ dx} + \int_{0}^{1} e^{-i\xi x} \text{ dx} \right] \tag{6}$$

Here, let us integrate our integrals:

$$F[f] = \frac{1}{\sqrt{2\pi}}\left[-\int_{-1}^{0} e^{-i\xi x}\ \mathrm{dx} + \int_{0}^{1} e^{-i\xi x}\ \mathrm{dx}\right] \tag{7}$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{1}{-i\xi}e^{-i\xi x}\Big|_{-1}^{0} + \frac{1}{-i\xi}e^{-i\xi x}\Big|_{0}^{1}\right] \tag{8}$$

$$= \frac{1}{i\xi\sqrt{2\pi}}\left[e^{-i\xi x}\Big|_{-1}^{0} - e^{-i\xi x}\Big|_{0}^{1}\right] \tag{9}$$

$$= \frac{1}{i\xi\sqrt{2\pi}}\left[\left(e^{-i\xi 0} - e^{-i\xi(-1)}\right) - \left(e^{-i\xi 1} - e^{-i\xi 0}\right)\right] \tag{10}$$

$$\tag{11}$$

Here, let us evaluate our expressions and simplify:

$$= \frac{1}{i\xi\sqrt{2\pi}}\left[\left(1 - e^{i\xi}\right) - \left(e^{-i\xi} - 1\right)\right] \tag{12}$$

$$= \frac{1}{i\xi\sqrt{2\pi}}\left[1 - e^{i\xi} - e^{-i\xi} + 1\right] \tag{13}$$

$$= \frac{2}{i\xi\sqrt{2}\sqrt{\pi}}\left[-e^{i\xi} - e^{-i\xi}\right] \tag{14}$$

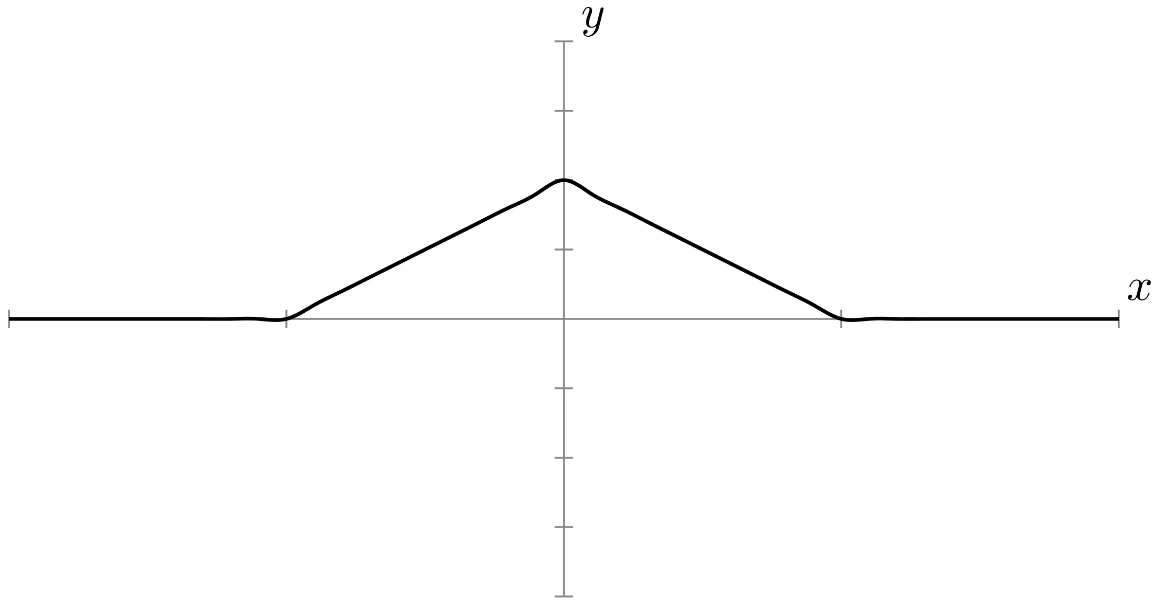
$$= \frac{\sqrt{2}}{\xi\sqrt{\pi}}\frac{e^{i\xi} + e^{-i\xi}}{i} \tag{15}$$

$$= \frac{\sqrt{2}}{\xi\sqrt{\pi}}\frac{2\cos(\xi)}{i} \tag{16}$$

(b) Now, let us consider our second given equation:

$$f(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, let us plot our given function:



Now, let us find the Fourier Transform of this problem.

Here, let us write our equation and further divide our function:

$$f_1(x) = \int_0^1 1 - x \text{ dx} \tag{1}$$

$$f_2(x) = \int_{-1}^0 1 + x \text{ dx} \tag{2}$$

$$f_3(x) = \int_1^\infty 0 \text{ dx } + \int_{-\infty}^{-1} 0 \text{ dx} \tag{3}$$

Here, let us use the definition of the Fourier Transform from 4) is the previous part. First, let us split our integral:

$$F[f] = \frac{1}{\sqrt{2\pi}} \left[\int_0^1 (1-x)e^{-i\xi x} \text{ dx} + \int_{-1}^0 (1+x)e^{-i\xi x} \text{ dx} + \int_1^\infty 0e^{-i\xi x} \text{ dx } + \int_{-\infty}^{-1} 0e^{-i\xi x} \text{ dx} \right] \tag{4}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^1 (1-x)e^{-i\xi x} \text{ dx} + \int_{-1}^0 (1+x)e^{-i\xi x} \text{ dx} \right] \tag{5}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^1 e^{-i\xi x} - xe^{-i\xi x} \text{ dx} + \int_{-1}^0 e^{-i\xi x} + xe^{-i\xi x} \text{ dx} \right] \tag{6}$$

Before proceeding, let us create a table of integration:

x	$e^{-i\xi x}$
1	$\frac{1}{-i\xi}e^{-i\xi x}$
0	$\frac{1}{i^2\xi^2}e^{-i\xi x}$

Here, we have our integration by parts. Now, let us proceed with our integrals:

$$F[f] = \frac{1}{\sqrt{2\pi}} \left[\left[\left(\frac{1}{-i\xi}e^{-i\xi x} \right) - \left(\frac{x}{-i\xi}e^{-i\xi x} + \frac{1}{\xi^2}e^{-i\xi x} \right) \right]_0^1 + \left[\left(\frac{1}{-i\xi}e^{-i\xi x} \right) + \left(\frac{x}{-i\xi}e^{-i\xi x} + \frac{1}{\xi^2}e^{-i\xi x} \right) \right]_{-1}^0 \right] \tag{7}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left[-\frac{1}{i\xi}e^{-i\xi x} + \frac{x}{i\xi}e^{-i\xi x} - \frac{1}{\xi^2}e^{-i\xi x} \right]_0^1 + \left[-\frac{1}{i\xi}e^{-i\xi x} - \frac{x}{i\xi}e^{-i\xi x} + \frac{1}{\xi^2}e^{-i\xi x} \right]_{-1}^0 \right] \tag{8}$$

Here, let us evaluate both integrals side-by-side:

Let us consider the integral on the left:

$$\left[-\frac{1}{i\xi}e^{-i\xi x}+\frac{x}{i\xi}e^{-i\xi x}-\frac{1}{\xi^2}e^{-i\xi x}\right]_0^1 \tag{9}$$

$$\left(-\frac{1}{i\xi}e^{-i\xi}+\frac{1}{i\xi}e^{-i\xi}-\frac{1}{\xi^2}e^{-i\xi}\right)+\left(\frac{1}{i\xi}+\frac{1}{\xi^2}\right) \tag{10}$$

Now, let us consider the integral on the right:

$$\left[-\frac{1}{i\xi}e^{-i\xi x}-\frac{x}{i\xi}e^{-i\xi x}+\frac{1}{\xi^2}e^{-i\xi x}\right]_{-1}^0 \tag{11}$$

$$\left(-\frac{1}{i\xi}+\frac{1}{\xi^2}\right)+\left(\frac{1}{i\xi}e^{i\xi}-\frac{1}{i\xi}e^{i\xi}-\frac{1}{\xi^2}e^{i\xi}\right) \tag{12}$$

Now, let us plug in our parts back into our integral:

$$\begin{aligned}
F[f] &= \frac{1}{\sqrt{2\pi}}\left[\left(-\frac{1}{i\xi}e^{-i\xi}+\frac{1}{i\xi}e^{-i\xi}-\frac{1}{\xi^2}e^{-i\xi}\right)+\left(\frac{1}{i\xi}+\frac{1}{\xi^2}\right)+\left(-\frac{1}{i\xi}+\frac{1}{\xi^2}\right)+\left(\frac{1}{i\xi}e^{i\xi}-\frac{1}{i\xi}e^{i\xi}-\frac{1}{\xi^2}e^{i\xi}\right)\right] && (13)\\
&= \frac{1}{\sqrt{2\pi}}\left[\left(-\frac{1}{i\xi}e^{-i\xi}+\frac{1}{i\xi}e^{-i\xi}-\frac{1}{\xi^2}e^{-i\xi}\right)+\left(\frac{1}{i\xi}e^{i\xi}-\frac{1}{i\xi}e^{i\xi}-\frac{1}{\xi^2}e^{i\xi}\right)+\left(\frac{1}{i\xi}+\frac{1}{\xi^2}\right)+\left(-\frac{1}{i\xi}+\frac{1}{\xi^2}\right)\right] && (14)\\
&= \frac{1}{\sqrt{2\pi}}\left[\left(-\frac{1}{\xi^2}e^{-i\xi}\right)+\left(-\frac{1}{\xi^2}e^{i\xi}\right)+\left(\frac{1}{i\xi}+\frac{1}{\xi^2}\right)+\left(-\frac{1}{i\xi}+\frac{1}{\xi^2}\right)\right] && (15)
\end{aligned}$$

From here, let us shift our terms around then group them.

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\left[\left(-\frac{1}{\xi^2}e^{-i\xi}\right)+\left(-\frac{1}{\xi^2}e^{i\xi}\right)+\left(\frac{1}{i\xi}+\frac{1}{\xi^2}\right)+\left(-\frac{1}{i\xi}+\frac{1}{\xi^2}\right)\right] && (16)\\
&= \frac{1}{\sqrt{2\pi}}\left[\left(-\frac{1}{\xi^2}e^{-i\xi}-\frac{1}{\xi^2}e^{i\xi}\right)+\left(\frac{2}{\xi^2}\right)\right] && (17)
\end{aligned}$$

From here, let us rewrite our exponential and factor our terms.

$$\begin{aligned}
&= -\frac{1}{\sqrt{2\pi}\xi^2}\left(e^{-i\xi}+e^{i\xi}\right)+\frac{2}{\sqrt{2\pi}\xi^2} && (18)\\
&= -\frac{\sqrt{2}}{\sqrt{\pi}\xi^2}\cos(\xi)+\frac{\sqrt{2}}{\sqrt{\pi}\xi^2} && (19)\\
&= -\frac{\sqrt{2}}{\sqrt{\pi}\xi^2}(\cos(\xi)+1) && (20)
\end{aligned}$$

3. Show that the convolution of two functions can be written as either:

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\ \mathrm{d}\xi$$

or

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\ \mathrm{d}\xi$$

In other words, show they are equivalent.

Proof. Here, let us prove both equations are equivalent,

Here, let us work with the first equation to get to the second equation.

Here, let us consider our function:

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\ \mathrm{d}\xi \tag{1}$$

Here, let us consider substituting the inside of our equation with the following:

$$\begin{align} u &= x - \xi \tag{2} \\ \mathrm{d}u &= -\mathrm{d}\xi \tag{3} \\ -\mathrm{d}u &= \mathrm{d}\xi \tag{4} \end{align}$$

Here, instead of using u, let us use $x - u$.

Furthermore, let us consider our domain of integration and input them into u:

$$\begin{align} u(x) &= x - \xi \tag{5} \\ u(\infty) &= x - \infty \tag{6} \\ &= -\infty \tag{7} \\ u(-\infty) &= x - -\infty \tag{8} \\ &= x + \infty \tag{9} \\ &= \infty \tag{10} \end{align}$$

Now, let us execute the substitution in our first equation:

$$\begin{align} (f * g)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\mathrm{d}\xi \tag{11} \\ &= -\frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(u)g(x-u)\mathrm{d}u \tag{12} \end{align}$$

Here, let us flip our integral,

$$\begin{align} &= -\frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(u)g(x-u)\mathrm{d}u \tag{13} \\ (f * g)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)g(x-u)\mathrm{d}u \tag{14} \end{align}$$

Here, using u substitution, we have reached the second given equation by using the first equation. The convolution of the two functions are equivalent. $\square$