

1. Plot the given functions and find their Fourier Transforms

$$(a) f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$(b) f(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, let us consider both problems individually.

(a) First, let us consider our first given equation:

$$f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, let us plot our given function:

First image here

Now, let us find the Fourier Transform of this problem.

Here, let us consider our domain of integration, where f_i is the i^{th} position. Let us informally write:

$$f_1(x) = \int_{-1}^0 -1 \, dx \quad (1)$$

$$f_2(x) = \int_0^1 1 \, dx \quad (2)$$

$$f_3(x) = \int_{-\infty}^{-1} 0 \, dx + \int_1^{\infty} 0 \, dx \quad (3)$$

Here, let us write our definition of the Fourier Transform:

$$F[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, dx \quad (4)$$

From here, let us split our integral:

$$F[f] = \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^0 -e^{-i\xi x} \, dx + \int_0^1 e^{-i\xi x} \, dx + \int_{-\infty}^{-1} 0e^{-i\xi x} \, dx + \int_1^{\infty} 0e^{-i\xi x} \, dx \right] \quad (5)$$

$$= \frac{1}{\sqrt{2\pi}} \left[- \int_{-1}^0 e^{-i\xi x} \, dx + \int_0^1 e^{-i\xi x} \, dx \right] \quad (6)$$

Here, let us integrate our integrals:

$$F[f] = \frac{1}{\sqrt{2\pi}} \left[- \int_{-1}^0 e^{-i\xi x} \, dx + \int_0^1 e^{-i\xi x} \, dx \right] \quad (7)$$

$$= \frac{1}{\sqrt{2\pi}} \left[- \frac{1}{-i\xi} e^{-i\xi x} \Big|_{-1}^0 + \frac{1}{-i\xi} e^{-i\xi x} \Big|_0^1 \right] \quad (8)$$

$$= \frac{1}{i\xi\sqrt{2\pi}} \left[e^{-i\xi x} \Big|_{-1}^0 - e^{-i\xi x} \Big|_0^1 \right] \quad (9)$$

$$= \frac{1}{i\xi\sqrt{2\pi}} \left[\left(e^{-i\xi 0} - e^{-i\xi(-1)} \right) - \left(e^{-i\xi 1} - e^{-i\xi 0} \right) \right] \quad (10)$$

$$(11)$$

Here, let us evaluate our expressions and simplify:

$$= \frac{1}{i\xi\sqrt{2\pi}} [(1 - e^{i\xi}) - (e^{-i\xi} - 1)] \quad (12)$$

$$= \frac{1}{i\xi\sqrt{2\pi}} [1 - e^{i\xi} - e^{-i\xi} + 1] \quad (13)$$

$$= \frac{2}{i\xi\sqrt{2}\sqrt{\pi}} [-e^{i\xi} - e^{-i\xi}] \quad (14)$$

$$= \frac{\sqrt{2}}{\xi\sqrt{\pi}} \frac{e^{i\xi} + e^{-i\xi}}{i} \quad (15)$$

(b) Now, let us consider our second given equation:

$$f(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, let us plot our given function:

Now, let us find the Fourier Transform of this problem.

Here, let us write our equation and further divide our function:

$$f_1(x) = \int_0^1 1 - x \, dx \quad (1)$$

$$f_2(x) = \int_{-1}^0 1 + x \, dx \quad (2)$$

$$f_3(x) = \int_1^\infty 0 \, dx + \int_{-\infty}^{-1} 0 \, dx \quad (3)$$

Here, let us use the definition of the Fourier Transform from 4) is the previous part. First, let us split our integral:

$$F[f] = \frac{1}{\sqrt{2\pi}} \left[\int_0^1 (1 - x)e^{-i\xi x} \, dx + \int_{-1}^0 (1 + x)e^{-i\xi x} \, dx + \int_1^\infty 0e^{-i\xi x} \, dx + \int_{-\infty}^{-1} 0e^{-i\xi x} \, dx \right] \quad (4)$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^1 (1 - x)e^{-i\xi x} \, dx + \int_{-1}^0 (1 + x)e^{-i\xi x} \, dx \right] \quad (5)$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^1 e^{-i\xi x} - xe^{-i\xi x} \, dx + \int_{-1}^0 e^{-i\xi x} + xe^{-i\xi x} \, dx \right] \quad (6)$$

Before proceeding, let us create a table of integration:

x	$e^{-i\xi x}$
1	$\frac{1}{-i\xi} e^{-i\xi x}$
0	$\frac{1}{i^2 \xi^2} e^{-i\xi x}$

Here, we have our integration by parts. Now, let us proceed with our integrals:

$$F[f] = \frac{1}{\sqrt{2\pi}} \left[\left[\left(\frac{1}{-i\xi} e^{-i\xi x} \right) - \left(\frac{x}{-i\xi} e^{-i\xi x} + \frac{1}{\xi^2} e^{-i\xi x} \right) \right]_0^1 + \left[\left(\frac{1}{-i\xi} e^{-i\xi x} \right) + \left(\frac{x}{-i\xi} e^{-i\xi x} + \frac{1}{\xi^2} e^{-i\xi x} \right) \right]_{-1}^0 \right] \quad (7)$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left[-\frac{1}{i\xi} e^{-i\xi x} + \frac{x}{i\xi} e^{-i\xi x} - \frac{1}{\xi^2} e^{-i\xi x} \right]_0^1 + \left[-\frac{1}{i\xi} e^{-i\xi x} - \frac{x}{i\xi} e^{-i\xi x} + \frac{1}{\xi^2} e^{-i\xi x} \right]_{-1}^0 \right] \quad (8)$$

Here, let us evaluate both integrals side-by-side:

Let us consider the integral on the left:

$$\left[-\frac{1}{i\xi}e^{-i\xi x} + \frac{x}{i\xi}e^{-i\xi x} - \frac{1}{\xi^2}e^{-i\xi x} \right]_0^1 \quad (9)$$

$$\left(-\frac{1}{i\xi}e^{-i\xi} + \frac{1}{i\xi}e^{-i\xi} - \frac{1}{\xi^2}e^{-i\xi} \right) + \left(\frac{1}{i\xi} + \frac{1}{\xi^2} \right) \quad (10)$$

Now, let us consider the integral on the right:

$$\left[-\frac{1}{i\xi}e^{-i\xi x} - \frac{x}{i\xi}e^{-i\xi x} + \frac{1}{\xi^2}e^{-i\xi x} \right]_{-1}^0 \quad (11)$$

$$\left(-\frac{1}{i\xi} + \frac{1}{\xi^2} \right) + \left(\frac{1}{i\xi}e^{i\xi} - \frac{1}{i\xi}e^{i\xi} - \frac{1}{\xi^2}e^{i\xi} \right) \quad (12)$$

Now, let us plug in our parts back into our integral:

$$F[f] = \frac{1}{\sqrt{2\pi}} \left[\left(-\frac{1}{i\xi}e^{-i\xi} + \frac{1}{i\xi}e^{-i\xi} - \frac{1}{\xi^2}e^{-i\xi} \right) + \left(\frac{1}{i\xi} + \frac{1}{\xi^2} \right) + \left(-\frac{1}{i\xi} + \frac{1}{\xi^2} \right) + \left(\frac{1}{i\xi}e^{i\xi} - \frac{1}{i\xi}e^{i\xi} - \frac{1}{\xi^2}e^{i\xi} \right) \right] \quad (13)$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(-\frac{1}{i\xi}e^{-i\xi} + \frac{1}{i\xi}e^{-i\xi} - \frac{1}{\xi^2}e^{-i\xi} \right) + \left(\frac{1}{i\xi}e^{i\xi} - \frac{1}{i\xi}e^{i\xi} - \frac{1}{\xi^2}e^{i\xi} \right) + \left(\frac{1}{i\xi} + \frac{1}{\xi^2} \right) + \left(-\frac{1}{i\xi} + \frac{1}{\xi^2} \right) \right] \quad (14)$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(-\frac{1}{\xi^2}e^{-i\xi} \right) + \left(-\frac{1}{\xi^2}e^{i\xi} \right) + \left(\frac{1}{i\xi} + \frac{1}{\xi^2} \right) + \left(-\frac{1}{i\xi} + \frac{1}{\xi^2} \right) \right] \quad (15)$$

From here, let us shift our terms around then group them.

$$= \frac{1}{\sqrt{2\pi}} \left[\left(-\frac{1}{\xi^2}e^{-i\xi} \right) + \left(-\frac{1}{\xi^2}e^{i\xi} \right) + \left(\frac{1}{i\xi} + \frac{1}{\xi^2} \right) + \left(-\frac{1}{i\xi} + \frac{1}{\xi^2} \right) \right] \quad (16)$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(-\frac{1}{\xi^2}e^{-i\xi} - \frac{1}{\xi^2}e^{i\xi} \right) + \left(\frac{2}{\xi^2} \right) \right] \quad (17)$$

From here, let us distribute our fraction and factor our terms.

$$= -\frac{1}{\sqrt{2\pi}\xi^2} (e^{-i\xi} + e^{i\xi}) + \frac{2}{\sqrt{2\pi}\xi^2} \quad (18)$$

$$= -\frac{\sqrt{2}}{\sqrt{\pi}\xi^2} \cos(\xi) + \frac{\sqrt{2}}{\sqrt{\pi}\xi^2} \quad (19)$$