1. Solve $\Delta u = 0$ on $x \in [0, 2]$, $y \in [0, 5]$, with $u_x(0, y) = \cos(3\pi y)$, $u_x(2, y) = 0$, $u_y(x, 0) = \sin(\pi x)$, $u_y(x, 5) = 0$ and $u(0, 0) = 3$.

   Here, let us write out our given conditions (Neumann):

   (a) $x \in [0, 2], y \in [0, 5]$

   (b) $\Delta u = 0 \Rightarrow u_{xx} + u_{yy} = 0$

   (c) $u_x(0, y) = \cos(3\pi y)$

   (d) $u_x(2, y) = 0$

   (e) $u_y(x, 0) = \sin(\pi x)$

   (f) $u_y(x, 5) = 0$

   (g) $u\ (0, 0) = 3$

   Now, let us begin solving for our equation.

   (a) First, let us assume our equation is separable.

$$u_{xx} + u_{yy} = 0 \tag{1}$$
$$u_{xx} = -u_{yy} \tag{2}$$
$$X''Y = -XY'' \tag{3}$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \tag{4}$$

   Let us consider u_1 and u_2, where $u_{1x}(0, y) = 0$ and $u_{2y}(x, 0) = 0$.

   (b) Here, let us solve for X_1:

$$\frac{X''}{X} = -\lambda \tag{5}$$
$$X'' = -\lambda X \tag{6}$$

   Here, the general equation for this form is sine and cosine:

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \tag{7}$$

   Now, since we have information on u_{1x}, let us find the first derivative:

$$X'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x) \tag{8}$$

   Now, let us solve for $u_{1x}(0, y)$

$$X'(0) = -A\sqrt{\lambda}\sin(0) + B\sqrt{\lambda}\cos(0) \tag{9}$$
$$X'(x) = A\sqrt{\lambda} = 0 \tag{10}$$
$$= B = 0 \tag{11}$$

Now, we have:

$$X'(x) = A\sqrt{\lambda}\sin(\sqrt{\lambda}x) \tag{12}$$
$$X'(2) = A\sqrt{\lambda}\sin(\sqrt{\lambda}2) = 0 \tag{13}$$
$$= \sqrt{\lambda}2 = n\pi \tag{14}$$
$$= \sqrt{\lambda} = \frac{n\pi}{2} \tag{15}$$
$$= \lambda_{1n} = \left(\frac{n\pi}{2}\right)^2 \tag{16}$$

Here, we have:

$$X_1(x) = A\cos\left(\frac{n\pi x}{2}\right) \tag{17}$$

(c) Now, let us solve for Y_1:

$$\frac{Y''}{Y} = \lambda \tag{18}$$

Here, we must use sinh and cosh and shift our variable:

$$Y_1(y) = C\cosh(\sqrt{\lambda}(5-y) + D\sinh(\sqrt{\lambda}(5-y)) \tag{19}$$

Now, let us take the first derivative:

$$Y\prime_{1y}(y) = C\sqrt{\lambda}\sinh(\sqrt{\lambda}(5-y)) - D\sqrt{\lambda}\cosh(\sqrt{\lambda}(5-y)) \tag{20}$$

Here, let $y = 5$,

$$Y\prime_{1y}(5) = -D\sqrt{\lambda} = 0 \tag{21}$$
$$= D = 0 \tag{22}$$

Now, let us write again:

$$Y_1(y) = C\sinh(\sqrt{\lambda}(5-y)) \tag{23}$$

Then let us input our λ:

$$Y_{1n}(y) = C\cosh\left(\frac{n\pi(5-y)}{2}\right) \tag{24}$$

(d) Now, if we combine our functions, we can write:

$$u_{1n}(x,y) = \cos\left(\frac{n\pi x}{2}\right)\cosh\left(\frac{n\pi(5-y)}{2}\right) \tag{25}$$

By linearity, let us write:

$$u_1(x,y) = \sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{2}\right)\cosh\left(\frac{n\pi(5-y)}{2}\right) \tag{26}$$

(b) Let us go back and solve for $u_2(x, y)$, starting with Y. Consider our separable equation:

$$
\begin{aligned}
\frac{X''}{X} &= -\frac{Y''}{Y} && (27)\\
\frac{Y''}{Y} &= -\frac{X''}{X} = -\lambda && (28)\\
& && (29)
\end{aligned}
$$

Here, we can perform the same series of steps to solve for Y_2 as we solved for X_1, swapping our L from 2 to 5 in our new case.

$$
\begin{aligned}
\lambda_{2n} &= \left(\frac{n\pi}{5}\right)^2 && (30)\\
Y_2(x) &= A\cos\left(\frac{n\pi x}{5}\right) && (31)
\end{aligned}
$$

(c) Similarly, let us write the solution for X_2 as we solved for Y_1:

$$X_{2n}(x) = C\cosh\left(\frac{n\pi(2-y)}{5}\right) \tag{32}$$

(d) Again, let us combine u_2 and u_{2n}:

$$u_{2n}(x, y) = \cos\left(\frac{n\pi x}{5}\right)\cosh\left(\frac{n\pi(2-y)}{5}\right) \tag{33}$$

By linearity,

$$u_2(x, y) = \sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{5}\right)\cosh\left(\frac{n\pi(2-y)}{5}\right) \tag{34}$$

(e) Here, let us combine our u_1 and u_2:

$$u(x, y) = \sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{2}\right)\cosh\left(\frac{n\pi(5-y)}{2}\right) + \cos\left(\frac{n\pi x}{5}\right)\cosh\left(\frac{n\pi(2-y)}{5}\right) \tag{35}$$

2. Solve $u_t = 9u_{xx}$ on $x \in [0,2]$ if $u(0,t) = 4$, $u(2,t) = 8$ and $u(x,0) = 3\sin(5\pi x) - 11\sin(9\pi x) + 2x + 4$

Here, let us write out our given conditions:

(a) $x \in [0,2]$

(b) $u_t = 9u_x x$

(c) $u(0,t) = 4$

(d) $u(2,t) = 8$

(e) $u(x,0) = 3\sin(5\pi x) - 11\sin(9\pi x) + 2x + 4$

Let us consider general boundaries, as u does not start and end at 0. We have $T_1 = 4$ and $T_2 = 8$.

Now, our line can be described as the following:

$$\frac{8-4}{2}x + 4 \tag{1}$$
$$2x + 4 \tag{2}$$

Here, let us solve for $u(x,t) = w(x,t) + u(x,\infty)$. To begin, let us consider our steady state condition as well:

$$u(0,t) = 4 \Rightarrow w(0,t) = u(0,t) - u(0,\infty) = 4 - 4 = 0 \tag{3}$$
$$u(2,t) = 8 \Rightarrow w(2,t) = u(2,t) - u(2,\infty) = 8 - 8 = 0 \tag{4}$$

Now, let us plug in our x into our steady-state solution and get the next two solutions:

(a) Assume $w(x,t) = X(x)T(t)$

$$XT' = 9X''T \tag{5}$$
$$\frac{T'}{9T} = \frac{X''}{X} = -\lambda \tag{6}$$

(b) Here, let us solve for X:

$$\frac{X''}{X} = -\lambda \tag{7}$$
$$X'' = -\lambda X \tag{8}$$

Here, we want to use the general cosine and sine form:

$$X(x) = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right) \tag{9}$$

Here, let us write our conditions:

$$X(0) = B = 0 \tag{10}$$
$$X(x) = A\sin\left(\sqrt{\lambda}x\right) \tag{11}$$
$$X(2) = A\sin\left(\sqrt{\lambda}2\right) = 0 \tag{12}$$
$$\Rightarrow \sqrt{\lambda}2 = n\pi \tag{13}$$
$$\Rightarrow \sqrt{\lambda} = \frac{n\pi}{2} \tag{14}$$
$$\Rightarrow \lambda = \left(\frac{n\pi}{2}\right) \tag{15}$$
$$X(x) = A\sin\left(\frac{n\pi x}{2}\right) \tag{16}$$

(c) Let us solve for T:

$$\frac{T_n'}{3^2 T_n} = -\lambda_n \tag{17}$$
$$T_n' = -\lambda_n T_n 3^2 \tag{18}$$

Here, we want to consider the general form from an expontential. Let us write:

$$T_n(t) = e^{-\left(\frac{n\pi 3}{2}\right)^2 t} \tag{19}$$

(d) Combine and find w_n and w:

$$w_n(x,t) = \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n\pi 3}{2}\right)^2 t} \tag{20}$$

By linearity,

$$w(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n\pi 3}{2}\right)^2 t} \tag{21}$$

Recall the characteristic of our line, $2x+4$. When we solve for $u(x,0)$ in terms of w, we get $w(x,0) = 3\sin(5\pi x) - 11\sin(9\pi x)$:

$$w(x,0) = 3\sin(5\pi x) - 11\sin(9\pi x) \tag{22}$$
$$w(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) \tag{23}$$
$$= 3\sin(5\pi x) - 11\sin(9\pi x) \tag{24}$$

Here, we found $A_{10} = 3$ and $A_{18} = 11$. We can write:

$$u(x,t) = 3\sin(5\pi x) e^{-\left(\frac{n\pi 3}{2}\right)^2 t} - 11\sin(9\pi x) e^{-\left(\frac{n\pi 3}{2}\right)^2 t} + 2x + 4 \tag{25}$$

3. Solve $u_{tt} = u_{xx}$ on $x \in [0,1]$ if $u(0,t) = 5$, $u(1,t) = 2$, $u(x,0) = x(1-x) - 3x + 5$ and $u_t(x,0) = 4$.

Here, let us write out our given conditions:

(a) $x \in [0,1]$

(b) $u_{tt} = u_{xx}$

(c) $u\ (0,t) = 5$

(d) $u\ (1,t) = 2$

(e) $u\ (x,0) = x(1-x) - 3x + 5$

(f) $u_t(x,0) = 4$

Here, similar to the last problem, let us consider w through T_1 and T_1:

$$\frac{T_1 - T_2}{L} = (2-5)x + 2 \tag{1}$$
$$= 3x - 5 \tag{2}$$

Now, let us consider our steady state:

$$u(0,t) = 5 \Rightarrow w(0,t) = u(0,t) - u(0,\infty) = 5 - 5 = 0 \tag{3}$$
$$u(0,t) = 5 \Rightarrow w(0,t) = u(0,t) - u(0,\infty) = 2 - 2 = 0 \tag{4}$$

Now, let us begin:

(a) Let us assume $w(x,t) = X(x)T(t)$

$$XT'' = X''T \tag{5}$$
$$\frac{T''}{T} = \frac{X''}{X} = -\lambda \tag{6}$$

(b) Let us solve for X

$$X'' = -\lambda X \tag{7}$$

Here, let us use the general cosine, sine form:

$$X(x) = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right) \tag{8}$$
$$X(0) = 0 = B \tag{9}$$
$$X(x) = A\sin\left(\sqrt{\lambda}x\right) \tag{10}$$
$$X(1) = 0 = A\sin\left(\sqrt{\lambda}1\right) \tag{11}$$
$$n\pi = \sqrt{\lambda}1 \tag{12}$$
$$\sqrt{\lambda} = n\pi \tag{13}$$
$$\lambda_n = (n\pi)^2 \tag{14}$$
$$X_n(x) = \sin(n\pi x) \tag{15}$$

(c) Let us solve for T_n

$$T_n''(t) = -(n\pi)^2 T \tag{16}$$
$$T_n(t) = C_n\cos(n\pi t) + D_n\sin(n\pi t) \tag{17}$$

(d) Combine and find w_n and w

Here, let us combined our values:

$$w_n(x,t) = \sin(n\pi x)\left[C_n\cos(n\pi t) + D_n\sin(n\pi t)\right] \tag{18}$$

By linearity,

$$\begin{aligned} w(x,t) &= \sum_{n=1}^{\infty} \sin(n\pi x)\left[C_n\cos(n\pi t) + D_n\sin(n\pi t)\right] &(19)\\ &= \sum_{n=1}^{\infty} C_n\sin(n\pi x)\cos(n\pi t) + D_n\sin(n\pi x)\sin(n\pi t) &(20)\end{aligned}$$

(e) Let us find the coefficients using the initial condition:

$$\begin{aligned} w(x,t) &= \sum_{n=1}^{\infty} C_n\sin(n\pi x)\cos(n\pi t) + D_n\sin(n\pi x)\sin(n\pi t) &(21)\\ w_t(x,t) &= \sum_{n=1}^{\infty} -C_n n\pi\sin(n\pi x)\sin(n\pi t) + D_n n\pi\sin(n\pi x)\cos(n\pi t) &(22)\\ w_t(x,0) &= \sum_{n=1}^{\infty} D_n n\pi\sin(n\pi x) = 4 &(23)\end{aligned}$$

Here, let us integrate:

$$\begin{aligned} D_n n\pi &= 2\int_0^1 4\sin(n\pi x)\ \mathrm{dx} &(24)\\ D_n &= \frac{8}{n\pi}\int_0^1 \sin(n\pi x)\ \mathrm{dx} &(25)\\ &= -\frac{8}{n\pi}\frac{1}{n\pi}\cos(n\pi x)\Big|_0^1 &(26)\\ &= -\frac{8}{n^2\pi^2}\cos(n\pi x)\Big|_0^1 &(27)\\ &= -\frac{8}{n^2\pi^2}(\cos(n\pi) - 1) &(28)\\ &= \frac{8}{n^2\pi^2}(1 - \cos(n\pi)) &(29)\\ &\Rightarrow \frac{8}{n^2\pi^2}(1 - (-1)^n) &(30)\end{aligned}$$

Now, let us find $w(x,0)$:

$$\begin{aligned} w(x,t) &= \sum_{n=1}^{\infty} C_n\sin(n\pi x)\cos(n\pi t) + \frac{8}{n^2\pi^2}(1-(-1)^n)\sin(n\pi x)\sin(n\pi t) &(31)\\ w(x,0) &= \sum_{n=1}^{\infty} C_n\sin(n\pi x) = x - x^2 &(32)\end{aligned}$$

Here, let us integrate:

$$C_n = 2\int_0^1 x\sin(n\pi x) - x^2\sin(n\pi x) \tag{33}$$

Let us create our integration tables:

x	$\sin(n\pi x)$
1	$-\frac{1}{n\pi}\cos(n\pi x)$
0	$-\frac{1}{n^2\pi^2}\sin(n\pi x)$

x^2	$\sin(n\pi x)$
$2x$	$-\frac{1}{n\pi}\cos(n\pi x)$
2	$-\frac{1}{n^2\pi^2}\sin(n\pi x)$
0	$\frac{1}{n^3\pi^3}\cos(n\pi x)$

Here, we have:

$$C_n = 2\left(\frac{x}{n\pi}\cos(n\pi x) - \frac{1}{n^2\pi^2}\sin(n\pi x) - \frac{x^2}{n\pi}\cos(n\pi x) + \frac{2x}{n^2\pi^2}\sin(n\pi x) + \frac{2}{n^3\pi^3}\cos(n\pi x)\right)\Big|_0^1 \tag{34}$$

$$= 2\left(\frac{1}{n\pi}\cos(n\pi) - \frac{1}{n\pi}\cos(n\pi) - \frac{2}{n^3\pi^3}\cos(n\pi) + \frac{2}{n^3\pi^3}\right) \tag{35}$$

$$= 2\left(-\frac{2}{n^3\pi^3}\cos(n\pi) + \frac{2}{n^3\pi^3}\right) \tag{36}$$

$$= 4\left(\frac{1-\cos(n\pi)}{n^3\pi^3}\right) \tag{37}$$

$$= \left(\frac{4-4\cos(n\pi)}{n^3\pi^3}\right) \tag{38}$$

Now, our heat equation is:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4-4\cos(n\pi)}{n^3\pi^3}\sin(n\pi x)\cos(n\pi t) + \frac{8}{n^2\pi^2}(1-(-1)^n)\sin(n\pi x)\sin(n\pi t) - 3x + 5 \tag{39}$$

4. Solve $\Delta u = 0$ on $x^2 + y^2 \leq 25$, where $u(5, \theta) = 7\sin(3\theta) - 6\sin(8\theta)$ and u is bounded when $r = 0$.

Here, let us consider $x^2 + y^2 \leq 25$. From our assumptions, we know r is bounded between $[0, 5]$.

Here, we have $u_{xx} + u_{yy} = 0$. First, let us write our u_x:

$$u_x = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \tag{1}$$
$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r} \tag{2}$$

Here, let us write our u_{xx} and u_{yy} in terms of polar coordinates:

$$u_{xx} = u_{rr}\cos^2\theta - 2u_{\theta r}\frac{\sin\theta\cos\theta}{r} + 2u_\theta\frac{\sin\theta\cos\theta}{r^2} + u_r\frac{\sin^2\theta}{r} + u_{\theta\theta}\frac{\sin^2\theta}{r^2} \tag{3}$$
$$u_{yy} = u_{rr}\sin^2\theta + 2u_{\theta r}\frac{\sin\theta\cos\theta}{r} - 2u_\theta\frac{\sin\theta\cos\theta}{r^2} + u_r\frac{\cos^2\theta}{r} + u_{\theta\theta}\frac{\cos^2\theta}{r^2} \tag{4}$$
$$\Delta u = u_{xx} + u_{yy} \tag{5}$$
$$\Delta u = u_{rr} + \frac{u_r}{r} + u_{\theta\theta}r^2 = 0 \tag{6}$$

Here, let us consider our inner and outer boundary:

(a) Assume $u(r, \theta) = R(r)\Theta(\theta)$

$$R''\Theta + \frac{R'\Theta}{r} + \frac{R\Theta''}{r^2} = 0 \tag{7}$$
$$r^2\frac{R}{R''} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda \tag{8}$$

(b) Here, let us solve for Θ:

$$\Theta'' = -\lambda\Theta \tag{9}$$

If $\lambda > 0$, then

$$\Theta(\theta) = A\sin(\sqrt{\lambda}\theta) + B\cos(\sqrt{\lambda}\theta) \tag{10}$$
$$\Theta'(\theta) = A\sqrt{\lambda}\cos(\sqrt{\lambda}\theta) - B\sqrt{\lambda}\sin(\sqrt{\lambda}\theta) \tag{11}$$
$$\sqrt{\lambda}2\pi = 2n\pi \Rightarrow \lambda_n = n^2, n \in \mathbb{Z}^+ \begin{cases} \Theta(0) = \Theta(2\pi) & \Rightarrow B = A\sin(\sqrt{\lambda}2\pi) + B\cos(\sqrt{\lambda}2\pi) \\ \Theta' = \Theta'(2\pi) & \Rightarrow A\sqrt{\lambda} = A\sqrt{\lambda}\cos(\sqrt{\lambda}2\pi) - B\sqrt{\lambda}\sin(\sqrt{\lambda}2\pi) \end{cases} \tag{12}$$
$$= n^2 \Rightarrow \Theta(n)(\theta) = A_n\sin(n\theta) + B_n\cos(n\theta) \tag{13}$$

If $\lambda = 0$, then the second derivative is 0.

$$\Theta_0'' \Rightarrow \Theta_0(\theta) = A_0\Theta + B_0 \tag{14}$$
$$\Rightarrow \Theta_0'(\theta) = A_0 \tag{15}$$
$$\Rightarrow \Theta_0(0) = \Theta_0(2\pi) \Rightarrow B_0 = 2\pi A_0 + B_0 \Rightarrow A_0 = 0 \tag{16}$$
$$\Rightarrow \Theta_0'(0) = \Theta_0'(2\pi) = 0 \tag{17}$$

(c) Next, we solve for R:

$$r^2\frac{R_n''}{R_n} + r\frac{R_n'}{R_n} = \lambda_n \tag{18}$$

Here, let us consider the following homogeneous equation of our equation:

$$r^2 R_n'' + rR_n' - n^2 R_n = 0 \tag{19}$$

$$\tag{20}$$

Try $R_n(r) = R^m$, then

$$r^2 m(m-1)r^{m-2} + rmr^{m-1} - n^2 r^m = 0 \tag{21}$$

$$r^m \left[m(m-1) + m - n^2\right] = 0 \tag{22}$$

$$m^n - n^2 = 0 \tag{23}$$

$$m = \pm n \tag{24}$$

Next, let us write:

$$\Rightarrow \begin{cases} R_n(r) & = C_n r^n + D_n r^{-n}, n \in \mathbb{Z}^+ \\ R_0(r) & = C_0 + D_0 \ln r \end{cases} \tag{25}$$

Recall our interval for r is $[0, 5]$.

(d) Combine to find u_n and u:

$$u_n(r,\theta) = \begin{cases} B_0(C_0 + D_0 \ln r) & n = 0 \\ C_n r^n + D_n r^{-n} \left(A_n \cos(n\theta) + B_n \cos(n\theta)\right) & n \in Z^+ \end{cases} \tag{26}$$

By linearity,

$$u(r,\theta) = c_0 + d_0 \ln r + \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n}) \sin(n\theta) + (c_n r^n + d_n r^{-n}) \cos(n\theta) \tag{27}$$

(e) Next, let us find the coefficients using our boundary condition:

$$u(5,\theta) = 7\sin(3\theta) - 6\sin(8\theta) \tag{28}$$

Now, let us write:

$$u(5,\theta) = c_0 + d_0 \ln 5 + \sum_{n=1}^{\infty} (a_n 5^n + b_n r^{-n}) \sin(n\theta) + (c_n 5^n + d_n 5^{-n}) \cos(n\theta) \tag{29}$$

Here, for our coefficients, let us write:

$$\begin{cases} c_0 + d_0 \ln 5 & = 0 \\ c_n 5^n + d_n 5^{-n} & = 0 \ \forall n \\ a_3 5^3 + b_3 5^{-3} & = 7, n = 3 \\ a_8 5^8 + b_8 5^{-8} & = -6, n = 8 \\ a_n 5^n + b_n 5^{-n} & = 0 \ \forall n, n \neq 3, 8 \end{cases} \tag{30}$$

If $n \neq 5$:

$$c_0 + d_0 \ln 5 = 0 \Rightarrow c_0 = d_0 = 0 \tag{31}$$

$$c_n + d_n = 0 \Rightarrow c_0 = d_0 = 0 \tag{32}$$

If $n \neq 3, 8$:

$$a_n 5^n + b_n 5^{-n} = 0 \Rightarrow a_n = b_n = 0 \tag{33}$$

If $n = 3$

$$a_3 5^3 + b_3 5^{-3} = 7 \tag{34}$$

If $n = 8$

$$a_8 5^8 + b_8 5^{-8} = -6 \tag{35}$$

From here, let us write u:

$$u(r, \theta) = \sum_{n=1}^{\infty} (a_n 5^n + b_n 5^{-n}) \sin(n\theta) + 7 \sin(3\theta) - 6 \sin(8\theta) \tag{36}$$