

Partial Differential Equations - Class Notes

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1 Chapter 1

Sidenotes

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What is a PDE?

A PDE is an equation which contains partial derivatives of an unknown function and we want to find that unknown function.

Example: $F(t, x, y, z, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x \partial y}, \dots) = 0$.

Note, the first partial derivatives are considered 1st ordered partials whereas the second ordered partials are considered 2nd ordered partials.

The variables that are not u are considered independent variables and u is considered a dependent variable.

What PDEs do we study?

Generally, we restrict our attention to equations that model some phenomenon from physics, engineering, economics, geology, ... etc. We can use physical intuition to help guide the math.

Classification of PDEs

1. Order of PDE: Highest derivative.

Example: $\frac{\partial^3 u}{\partial x^3} - \sin(y)u^7 = 3$ is a third order PDE.

Example: $(\frac{\partial y}{\partial t})^5 - \frac{\partial^2 y}{\partial x \partial t} = e^x$ is a second order PDE.

2. Number of independent variables.

Example: $\frac{du}{dt} = \frac{\partial^2 u}{\partial x^2}$ has two independent variables: t, x .

This is the 1 - D heat equation.

Example: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \Delta u$ has 4 independent variables.

This is the 3 - D heat equation. Δu is Laplacian of u .

Notation

$\Delta u = \nabla^2 u = \nabla \cdot \nabla u = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

$\Delta u = 0$ is considered Laplace's equation.

3. Linear vs non-linear

A linear PDE is any equation of the form $L[u(x)] = f(x)$ where $f(x)$ is a known function is a linear partial differential operator.

Definition: A differential operator is any rule that takes a function as its input and returns an expression that involves the derivatives of that function.

Example:

$$u(x, t) \quad v(x, t) \tag{1}$$

$$O[u] = \frac{\partial^2 u}{\partial x^2} + \sin x + \pi - 7e^{tu} \tag{2}$$

$$O[u + 3v] = \frac{\partial^2}{\partial x^2}(u + 3v) + \sin x + \pi - 7e^{tu+3tv} \tag{3}$$

$$= \frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 v}{\partial x^2} + \sin x + \pi - 7e^{tu+3tv} \tag{4}$$

Definition: A linear operator, L , is an operator that has the property:

$$L[au + bv] = aL[u] + bL[v] \tag{5}$$

Where a and b are constants.

Theorem: If u and v are vectors and L is linear, then L can be represented by a matrix.

Theorem: If L is linear ordinary operator, it must take the form:

$$L[u] = f_0(x)u + f_1(x)u' + f_2(x)u'' + \dots + f_n(x)u^{(n)} \tag{6}$$

Where the f_i 's are known functions.

Definition: A linear ODE is any ODE of the form where $f(x)$ is known is the following:

$$L[u] = f(x) \tag{7}$$

If $f(x) = 0$, then the equation is homogeneous. Otherwise, the equation is non-homogeneous.

Ex: $(u')^2 = 0 \Rightarrow u' = 0 \rightarrow$ linear, homogeneous.

Theorem: If L is a linear partial differential operator, it must take the form (x is a vector with n unknowns)

$$L[u(x)] = f_0(x)u + \sum_{i=1}^n f_i(x)\frac{\partial u}{\partial x_i} + \sum_{i=1}^n \sum_{j=1}^n f_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \dots \tag{8}$$

Definition: A linear PDE is any PDE of the form

$$L[u(x)] = f(x) \tag{9}$$

If $f(x) = 0$, the equation is homogeneous, else it is non-homogeneous.

Ex: $u_t = 4u_x$ - Linear, homogeneous.

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Example:

$$u_{tt} = u_{xx} + uyy \quad \text{Linear, homogeneous} \quad (10)$$

$$\cos(xt) = u + u_t + u_{xyz} \quad \text{Linear, non-homogeneous} \quad (11)$$

$$u_t u_{xt} = 0 \quad \text{non-linear} \quad (12)$$

$$u_{xt} + e^x \cos t \, u_t = 0 \quad \text{linear, homogeneous} \quad (13)$$

$$u_t + u_{xx} + ue^u = 0 \quad \text{non-linear} \quad (14)$$

Note: You can add linear combinations of solutions to linear homogeneous equations and still get a solution.

Example: $u_x = u_t$.

Some solutions to this are:

1. $u_1(x, t) = 3$
 2. $u_2(x, t) = x + t$
 3. $u_3(x, t) = e^{x+t} \cos(x + t)$
 4. \vdots
- $Au_1 + Bu_2 + Cu_3$ is also a solution.

How do we solve an ODE?

1. Use some technique to find an explicit solution.
2. Use power series and determine the coefficients

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad (15)$$

3. Laplace Transforms

How do we solve PDEs?

1. Try to locate an explicit solution
2. We don't use power series, instead, we use a trigonometric series \Rightarrow Fourier Series.

$$y(x) = \sum_{n=0}^{\infty} a_n \sin(nx) + b_n \cos(nx) \quad (16)$$

3. Laplace Transforms are good if the domain is $[0, \infty)$.
Fourier Transforms are good if the domain is $(-\infty, \infty)$.
4. Reduce the PDE to a system of ODEs.

Initial Condition

1. For ODEs, to solve a 1st order equation, you need $y(0)$.
 2^{nd} order $\rightarrow y(0), y'(0)$
 3^{rd} order $\rightarrow y(0), y'(0), y''(0)$
 \vdots
 n^{th} order $\rightarrow y(0), y'(0), y''(0), \dots, y^{(n-1)}(0)$
2. For PDEs, it's more complicated \Rightarrow it depends on the PDE.
Example: $u(x, t), x \in [a, b], t \in [0, \infty)$
If $u_t = u_{xx}$
3. Boundary conditions:

$$u(a, t) = g_1(t) \quad (17)$$

$$u(b, t) = g_2(t) \quad (18)$$

If $u_{tt} = u_{xx}$, we must specify:

(a) Initial Conditions

$$u(x, 0) = f_1(x) \quad (19)$$

$$u_t(x, 0) = f_2(x) \quad (20)$$

(b) Boundary Conditions

$$u(a, t) = g_1(t) \quad (21)$$

$$u(b, t) = g_2(t) \quad (22)$$

1-D Heat Equation

Assume cross sections are uniform Imagine a cross section:

We have a steady-state: $(t \rightarrow \infty)$, $u_t = 0 \Rightarrow u_{xx} = 0 \Rightarrow$ straight line
1-D: $-q_x = Q_t \Rightarrow -\nabla \cdot \vec{q} = Q_t$, \vec{q} is a vector.

$$q = -K \nabla u \Rightarrow -\nabla \cdot (-K \nabla u) = \rho c_p u_t \quad (36)$$

$$\Rightarrow K \Delta u = \rho c_p u_t \quad (37)$$

$$\Rightarrow u_t = \alpha^2 \Delta u \quad (38)$$

What about a steady-state? $u_t = 0$

$$\Delta u = 0 \quad (39)$$

Here, we have Laplace's equation.

Note: It is not dependent on time.

The Wave Equation $u(x, t)$ is the height of the rope. We use Newton's 2^{nd} law on small segments of rope.

- ρ = density of rope.
- $dm = \rho \, dx$

$$F = ma \quad (40)$$

$$T \sin(\theta(x + \Delta x)) - T \sin(\theta(x)) = \int_x^{x+\Delta x} u_{tt} \, dm \quad (41)$$

$$T[\sin(\theta(x + \Delta x)) - \sin(\theta(x))] = \rho \int_x^{x+\Delta x} u_{tt} \, dx \quad (42)$$

Let us assume θ is small, $\sin \theta \approx \tan \theta$

$$T[\tan(\theta(x + \Delta x)) - \tan(\theta(x))] = \rho \int_x^{x+\Delta x} u_{tt} \, dx \quad (43)$$

Also, $\tan(\theta(x)) = u_x(x, t)$.

$$T[u_x(x + \Delta x, t) - u_x(x, t)] = \rho \int_x^{x+\Delta x} u_{tt} \, dx \quad (44)$$

Now, let us divide both sides by Δx and take the limit as $\Delta x \rightarrow 0$

$$\lim_{\Delta x \rightarrow 0} T \left[\frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x} \right] = \rho \lim_{\Delta x \rightarrow 0} \frac{\int_x^{x+\Delta x} u_{tt} \, dx}{\Delta x} \quad (45)$$

On the left side, we have u_{xx} and the right side we have $u_{tt}(x + \Delta x, t)$.

$$T u_{xx}(x, t) = \rho u_{tt}(x, t) \quad (46)$$

$$u_{tt} = \frac{T}{\rho} u_{xx} = c^2 u_{xx} \quad (47)$$

$$c = \sqrt{\frac{T}{\rho}} = \text{wave speed} \quad (48)$$

On the left, we have the 1 - D wave equation which is used for light, sound, rope, etc.
In 2-D, it corresponds to a vibrating membrane (drum)

$$u_{tt} = c^2 \Delta u \quad (49)$$

Remark:

$$u_t = u_{xx} \quad \text{Heat Equation} \quad (50)$$

$$u_{xx} + u_{yy} = 0 \quad \text{Laplace Equation} \quad (51)$$

$$u_{tt} = u_{xx} \quad \text{wave} \quad (52)$$

Here, we can replace:

u_t with t

u_x with x

u_{xx} with x^2

1. $t = x^2$ parabola
2. $x^2 + y^2 = 0$ ellipse
3. $t^2 = x^2$ hyperbolas

So, the equations behave like the following:

1. The Heat Equation is called parabolic
2. The Laplace Equation is called elliptic
3. The Wave Equation is called hyperbolic