

1. Solve  $\Delta u = 0$  on  $x \in [0, 4]$ ,  $y \in [0, 3]$ , with  $u(x, 0) = u(x, 3) = 0$  where

- (a)  $u_x(0, y) = 0$  and  $u_x(4, y) = \cos(\pi y)$
- (b)  $u(0, y) = 1$  and  $u(4, y) = 0$  (Hint: Translate the  $x$  coordinate so that  $u(0, y) = 0$ . This means that  $x$  goes between  $-4$  and  $0$ . Now go back to  $0$  to  $4$  with a  $y$ -axis flip.)

Here, let us first solve part a) of the problem.

(a) First, let us assume our equation is separable:

$$u(x, y) = X(x)Y(y) \quad (1)$$

Let us take note of our  $u$ :

$$u_{xx} + u_{yy} = 0 \quad (2)$$

$$X''Y + XY'' = 0 \quad (3)$$

$$XY'' = -X''Y \quad (4)$$

$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda \quad (5)$$

Here, let us consider our boundary conditions:

$$u(x, 0) = 0 = X(0)Y(y) \quad (6)$$

Here, we either have  $X(0) = 0$  or  $Y(y) = 0$ . We do not want to assume  $Y(y) = 0$  because that would make  $u(x, y) = 0 \forall y \in [0, 3]$ . Therefore, we consider  $X(0) = 0$ . We will follow this assumption for the upcoming boundary conditions.

$$u(x, y) = X(x)Y(y) \quad (7)$$

$$u(x, 0) = X(x)Y(0) = 0 \Rightarrow Y(0) = 0 \quad (8)$$

$$u(x, 3) = X(x)Y(3) = 0 \Rightarrow Y(3) = 0 \quad (9)$$

Now, let us consider  $u_x(x, y)$  as  $X'(x)Y(y)$ :

$$u_x(x, y) = X'(x)Y(y) \quad (10)$$

$$u_x(0, y) = X'(0)Y(y) = 0 \Rightarrow X'(0) = 0 \quad (11)$$

Now, let us write:

$$-\frac{X''}{X} = \frac{Y''}{Y} = -\lambda \quad (12)$$

(b) Next we solve for  $Y(y)$  since we have more information on  $Y(y)$ .

$$\frac{Y''}{Y} = -\lambda \quad (13)$$

$$Y'' = -\lambda Y \quad (14)$$

Here, let us write out our general equation:

$$Y_n(y) = A \sin(\sqrt{\lambda}x) + B \cos(\sqrt{\lambda}x) \quad (15)$$

Here, we know  $Y(0) = Y(3) = 0$ . Let us input our  $y$ :

$$Y_n(0) = 0 = B \quad (16)$$

$$Y_n(3) = 0 = A \sin(\sqrt{\lambda}3) \quad (17)$$

Here, let us not consider  $A$  as 0, instead, the inside of our sine function to be 0.

$$\sqrt{\lambda}3 = n\pi \quad (18)$$

$$\sqrt{\lambda} = \frac{n\pi}{3} \quad (19)$$

$$\lambda_n = \left(\frac{n\pi}{3}\right)^2 \quad (20)$$

Now we have the following:

$$Y_n(y) = \sin\left(\frac{n\pi y}{3}\right) \quad (21)$$

(c) Now, let us solve for  $X$ :

$$\frac{X''}{X} = \lambda \quad (22)$$

$$X'' = \lambda X \quad (23)$$

Here, we solved for  $\lambda$  in the previous step:

$$X_n'' = \left(\frac{n\pi}{3}\right)^2 X_n \quad (24)$$

$$X_n(x) = C \sinh\left(\frac{n\pi x}{3}\right) + D \cosh\left(\frac{n\pi x}{3}\right) \quad (25)$$

Here, let us find the derivative for  $X$  and find  $X'(0)$ :

$$X'(0) = \frac{n\pi x}{3} C = 0 \quad (26)$$

$$= C = 0 \quad (27)$$

Therefore, since  $C = 0$ , we get:

$$X_n(x) = D \cosh\left(\frac{n\pi x}{3}\right) \quad (28)$$

(d) Now, let us combine  $u_n$  and  $u$ :

$$u_n(x, y) = D \cosh\left(\frac{n\pi x}{3}\right) \sin\left(\frac{n\pi y}{3}\right) \quad (29)$$

By linearity, let us write:

$$u(x, y) = \sum_{n=1}^{\infty} D_n \cosh\left(\frac{n\pi x}{3}\right) \sin\left(\frac{n\pi y}{3}\right) \quad (30)$$

(e) Here, let us use our boundary conditions to find our coefficients.

$$u_x(4, y) = \cos(\pi y) \quad (31)$$

Here, let us derive our  $u(x, y)$  to get  $u_x(x, y)$

$$u_x(x, y) = \sum_{n=1}^{\infty} D_n \left(\frac{n\pi}{3}\right) \sinh\left(\frac{n\pi x}{3}\right) \sin\left(\frac{n\pi y}{3}\right) \quad (32)$$

$$u_x(4, y) = \sum_{n=1}^{\infty} D_n \left(\frac{n\pi}{3}\right) \sinh\left(\frac{n\pi 4}{3}\right) \sin\left(\frac{n\pi y}{3}\right) \quad (33)$$

Here, let us consider our coefficient in the equation:

$$D_n \left(\frac{n\pi}{3}\right) \sinh\left(\frac{4\pi n}{3}\right) = \frac{2}{3} \int_0^3 \cos(\pi y) \sin\left(\frac{n\pi y}{3}\right) dy \quad (34)$$

$$D_n = \frac{2}{3} \frac{3}{n\pi} \frac{1}{\sinh\left(\frac{4\pi n}{3}\right)} \int_0^3 \cos(\pi y) \sin\left(\frac{n\pi y}{3}\right) dy \quad (35)$$

$$= \frac{2}{n\pi \sinh\left(\frac{4\pi n}{3}\right)} \int_0^3 \cos(\pi y) \sin\left(\frac{n\pi y}{3}\right) dy \quad (36)$$

Here, let us isolate our integral and solve for that. Afterwards, let us combine our function.

$$\int_0^3 \cos(\pi y) \sin\left(\frac{n\pi y}{3}\right) dy \quad (37)$$

Here, let us use trig identities to separate our product:

$$\int_0^3 \cos(\pi y) \sin\left(\frac{n\pi y}{3}\right) dy = \int_0^3 \frac{\sin\left(\frac{\pi n y - \pi 3 y}{3}\right) + \sin\left(\frac{\pi n y + \pi 3 y}{3}\right)}{2} dy \quad (38)$$

$$= \frac{1}{2} \int_0^3 \sin\left(\frac{(n-3)\pi y}{3}\right) dy + \frac{1}{2} \int_0^3 \sin\left(\frac{(n+3)\pi y}{3}\right) dy \quad (39)$$

From here, let us consider  $u$  substitution on both sides. On the left, let us consider  $u = \frac{(n-3)\pi y}{3}$  and  $du = \frac{(n-3)\pi}{3} dy$ . From here, let us replace our interval from 0 to  $(n-3)\pi$ . For the right integral, let us consider the same procedure and replace  $v = \frac{(n+3)\pi y}{3}$  and  $dv = \frac{(n+3)\pi}{3} dy$ . From here, let us change the interval to 0 to  $(n+3)\pi$ :

$$= \frac{1}{2} \frac{3}{(n-3)\pi} \int_0^{\pi(n-3)} \sin(u) du + \frac{1}{2} \frac{3}{(n+3)\pi} \int_0^{\pi(n+3)} \sin(v) dv \quad (40)$$

$$= -\frac{3}{2(n-3)\pi} \left[ \cos u \right]_0^{\pi(n-3)} - \frac{3}{2(n+3)\pi} \left[ \cos v \right]_0^{\pi(n+3)} \quad (41)$$

$$= -\frac{3}{2(n-3)\pi} [\cos(\pi(n-3)) - 1] - \frac{3}{2(n+3)\pi} [\cos(\pi(n+3)) - 1] \quad (42)$$

$$= -\frac{3}{2(n-3)\pi} [\cos(\pi n - \pi 3) - 1] - \frac{3}{2(n+3)\pi} [\cos(\pi n + \pi 3) - 1] \quad (43)$$

$$= -\frac{3}{2(n-3)\pi} [-\cos(\pi n) - 1] - \frac{3}{2(n+3)\pi} [-\cos(\pi n) - 1] \quad (44)$$

$$= \frac{3(\cos(\pi n) + 1)}{2(n-3)\pi} + \frac{3(\cos(\pi n) + 1)}{2(n+3)\pi} \quad (45)$$

$$= \frac{3n(\cos(\pi n) + 1)}{\pi(n^2 - 9)} \quad (46)$$

Here, let us plug our integral back into our equation at line (36):

$$= \frac{2}{n\pi \sinh(\frac{4\pi n}{3})} \frac{3n(\cos(\pi n) + 1)}{\pi(n^2 - 9)} \quad (47)$$

$$= \frac{6(\cos(\pi n) + 1)}{\pi^2(n^2 - 9) \sinh(\frac{4\pi n}{3})} \quad (48)$$

From here, let us write:

$$u(x, y) = \sum_{n=1}^{\infty} \frac{6(-1^n + 1)}{\pi^2(n^2 - 9) \sinh(\frac{4\pi n}{3})} \cosh(\frac{n\pi x}{3}) \sin(\frac{n\pi y}{3}) \quad (49)$$

From here, allow us to backtrack and consider new conditions:  $u(0, y) = 1$  and  $u(4, y) = 0$ . From here, we should consider shifting one of our functions. Let us start from part c, knowing  $Y_n(y) = \sin(\frac{n\pi y}{3})$  and  $\lambda_n = (\frac{n\pi}{3})^2$ :

(d) Let us solve for  $X(x)$

$$X(x) = C \sinh(\sqrt{\lambda}x) + D \cosh(\sqrt{\lambda}x) \quad (50)$$

$$X(0) = 0 = D \quad (51)$$

Here, let us shift our  $x$  to get  $X(4) = 0$ :

$$X(x) = C \sinh(\frac{n\pi}{3}(4 - x)) \quad (52)$$

$$X(4) = C \sinh(\frac{n\pi}{3}(4 - 4)) = 0 \quad (53)$$

(e) Here, let us combine our  $u$  and  $u_n$ :

$$u(x, y) = C_n \sinh\left(\frac{n\pi(4 - x)}{3}\right) \sin\left(\frac{4\pi y}{3}\right) \quad (54)$$

By linearity, let us write:

$$u_n(x, y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi(4 - x)}{3}\right) \sin\left(\frac{n\pi y}{3}\right) \quad (55)$$

From our initial conditions, we know  $u(0, y) = 1$ , so let us write:

$$u_n(0, y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi 4}{3}\right) \sin\left(\frac{n\pi y}{3}\right) \quad (56)$$

From here, let us find our coefficient for sin:

$$C_n \sinh\left(\frac{4n\pi}{3}\right) = \frac{2}{3} \int_0^3 \sin\left(\frac{n\pi y}{3}\right) dy \quad (57)$$

$$= -\frac{2}{3} \frac{3}{n\pi} \left[ \cos\left(\frac{n\pi y}{3}\right) \right]_0^3 \quad (58)$$

$$C_n \sinh\left(\frac{4n\pi}{3}\right) = -\frac{2}{n\pi} [\cos(n\pi) - 1] \quad (59)$$

$$C_n = \frac{2(1 - \cos(n\pi))}{n\pi \sinh(\frac{4n\pi}{3})} \quad (60)$$

From here, let us write:

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2(1 - (-1)^n)}{n\pi \sinh(\frac{4n\pi}{3})} \sinh\left(\frac{4n\pi(4-x)}{3}\right) \sin\left(\frac{n\pi y}{3}\right) \quad (61)$$

2. Solve  $u_{tt} = 4u_{xx}$  on  $x \in [0, 3]$ ,  $t \in [0, \infty)$ , with  $u(0, t) = u(3, t) = 0$  where

(a)  $u(x, 0) = 4 \sin(2\pi x) + 7 \sin(6\pi x) - 2 \sin(\pi x)$ ,  $u_t(x, 0) = 0$

(b)  $u(x, 0) = x(3 - x)$ ,  $u_t(x, 0) = \sin(\pi x)$

Let us begin with our boundary conditions.

(a) Let us assume our equation is separable.

$$u_{tt} = 2^2 u_{xx} \quad (62)$$

$$XT'' = 2^2 X''T \quad (63)$$

$$\frac{T''}{2^2 T} = \frac{X''}{X} = -\lambda \quad (64)$$

(b) Here, since we have more information regarding  $X$ , let us solve for  $X$ :

$$\frac{X''}{X} = -\lambda \quad (65)$$

$$X'' = -\lambda X \quad (66)$$

In this equation, we want to consider  $\cos$  and  $\sin$  for our general equation.

$$X(x) = A \sin(\sqrt{\lambda}x) + B \cos(\sqrt{\lambda}x) \quad (67)$$

Here, we know  $X(0) = X(3) = 0$ , so let us write:

$$X(0) = B = 0 \quad (68)$$

$$X(3) = A \sin(\sqrt{\lambda}3) = 0 \quad (69)$$

Here, let us find when our sine function is 0.

$$\sqrt{\lambda}3 = n\pi \quad (70)$$

$$\sqrt{\lambda} = \frac{n\pi}{3} \quad (71)$$

$$\lambda = \left(\frac{n\pi}{3}\right)^2 \quad (72)$$

Now, let us write our general equation for  $X$ :

$$X_n(x) = \sin\left(\frac{n\pi x}{3}\right) \quad (73)$$

(c) Next, let us solve for  $T$ :

$$\frac{T''}{2^2 T} = -\lambda \quad (74)$$

$$T'' = -2^2 T \lambda \quad (75)$$

Here, let us write the general equation:

$$T_n(t) = C_n \sin\left(\frac{2n\pi t}{3}\right) + D_n \cos\left(\frac{2n\pi t}{3}\right) \quad (76)$$

(d) Here, let us combine to find  $u_n$  and  $u$ :

$$u_n(x, t) = \sin\left(\frac{n\pi x}{3}\right) \left[ C_n \sin\left(\frac{2n\pi t}{3}\right) + D_n \cos\left(\frac{2n\pi t}{3}\right) \right] \quad (77)$$

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{3}\right) \sin\left(\frac{2n\pi t}{3}\right) + D_n \sin\left(\frac{n\pi x}{3}\right) \cos\left(\frac{2n\pi t}{3}\right) \quad (78)$$

(e) Here, let us find the coefficients using the Initial Conditions:

- $u(x, 0) = 4 \sin(2\pi x) + 7 \sin(6\pi x) - 2 \sin(\pi x)$
- $u_t(x, 0) = 0$

Using this information, find our  $t$  partial from our summation series:

$$u_t(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{3}\right) \frac{2n\pi}{3} \cos\left(\frac{2n\pi t}{3}\right) - D_n \sin\left(\frac{n\pi x}{3}\right) \frac{2n\pi}{3} \sin\left(\frac{2n\pi t}{3}\right) \quad (79)$$

From here, let us plug in our initial condition,  $u_t(x, 0) = 0$ :

$$u_t(x, 0) = \sum_{n=1}^{\infty} C_n \frac{2n\pi}{3} \sin\left(\frac{n\pi x}{3}\right) \quad (80)$$

From here, let us recall our terms:

- $4 \sin(2\pi x) + 7 \sin(6\pi x) - 2 \sin(\pi x)$

Using these terms, we know the following:

- $D_3 = -2$
- $D_6 = 4$
- $D_{18} = 7$
- $D_n = 0, n \in \mathbb{N}, n \neq 3, 6, 18$

Let us plug in our coefficients:

$$u(x, t) = -2 \sin(\pi x) \cos(2\pi t) + 4 \sin(2\pi x) \cos(4\pi t) + 7 \sin(6\pi x) \cos(12\pi t) \quad (81)$$

Now, let us reconsider our initial conditions once more:

(e)  $u(x, 0) = x(3 - x), u_t(x, 0) = \sin(\pi x)$

Here, let us consider finding our coefficient where  $u(x, 0) = x(3 - x)$ :

$$D_n = \frac{2}{3} \int_0^3 x(3 - x) \sin\left(\frac{n\pi x}{3}\right) \quad (82)$$

$$= \frac{2}{3} \int_0^3 3x \sin\left(\frac{n\pi x}{3}\right) - x^2 \sin\left(\frac{n\pi x}{3}\right) \quad (83)$$

$$= 2 \int_0^3 x \sin\left(\frac{n\pi x}{3}\right) - \frac{2}{3} \int_0^3 x^2 \sin\left(\frac{n\pi x}{3}\right) \quad (84)$$

Here, let us create two integration tables:

$$x \mid \sin\left(\frac{n\pi x}{3}\right)$$