1. Categorize the following equations by:
    - Order
    - Number of independent variables
    - Linear vs Non-linear. If linear, is it homogeneous or non-homogeneous?

    (a) $u_{xx} + u_{yy} + u_{zz} = f(y,t)$
    - Second Order
    - 4: x, y, z, t
    - Linear - Non-homogeneous

    (b) $u_{tt} = u_{tx} + t^2 u_x$
    - Second Order
    - 2: x, t
    - Linear, Homogeneous

    (c) $(u_y)^4 + (u_x)^5 = 7$
    - First Order
    - 2: x, y
    - Non-linear

    (d) $u_t - \sqrt{1 + (u_y)^2} = 0$
    - First Order
    - 2: y, t
    - Non-linear

    (e) $u_t + (u^2)_x = 0$
    - First order
    - 2: x, t
    - Non-linear

    (f) $u_t + \frac{\partial^2}{\partial x^2} u^3 - \frac{\partial}{\partial y} u^{\frac{5}{2}} = 0$
    - Second Order
    - 3: x, y, t
    - Non-linear

    (g) $u_t - u u_y + 6u_{xx} = 4\cos t$
    - Second Order
    - 3: x, y, t
    - Non-linear

    (h) $0 = \nabla \cdot \nabla u$ (Where u is dependent on n variables $x_1, x_2, \ldots, x_n$).

    $$0 = \frac{\partial^2 u}{\partial x_1^2} + \frac{p^2 u}{\partial x_2^2} + \ldots + \frac{p^2 u}{\partial x_n^2}$$

    - Second Order
    - n variables.
    - Non-linear

    (i) $\left(\frac{\partial^4 u}{\partial t \partial x^2 \partial y}\right)^2 = g(x,t)$
    - Fourth order
    - 3: x, y, t
    - Non-linear

    (j) $u_t = \frac{u_{xx}(u_y)^2 - 2u_x u_y u_{xy} + u_{yy}(u_x)^2}{(u_y)^2 + (u_x)^2}$
    - Second Order
    - 2: x, y
    - Non-linear

    (k) $\sqrt{u_x + u_y} = e^{xt}$
    - First Order
    - 3: x, y, t
    - Non-linear

2. Derive the heat equation for a 2-D region in the following ways:

(a) Do this over a differential square $\Delta x \Delta y$, generalizing the argument from the notes.
Let us derive the heat equation for a 2-D region. We want to consider the conservation of energy where we can consider heat accumulated with heat in - heat out. Let us consider the transfer of heat about the x axis:

$$q_x(x, y, t)\Delta y \Delta z \Delta t \tag{1}$$

Here, we want to consider finding the heat accumulated through the heat in and heat out, therefore we want to consider x_0 and $x_0 + \Delta$

$$q_x(x_0, y, t)\Delta y \Delta z \Delta t - q_x(x_0 + \Delta x, y, t)\Delta y \Delta z \Delta t \tag{2}$$

Here, we want to consider the deltas in 2). We have our heat function in the x direction, q_x. We have a function with variables y, t. Instead of keeping them in their form, let us find the integral and integrate in terms of y and t:

$$\Delta z \int\int q_x(x_0, y, t)\text{dydt} - \Delta z \int\int q_x(x_0 + \Delta x, y, t)\text{dydt} \tag{3}$$

Let us take note of the integral. We are finding the area over the span of y and t, these are our intervals. In addition, let us combine the integrals:

$$\Delta z \int_{t_0}^{t_0+\Delta t} \int_{y_0}^{y_0+\Delta y} q_x(x_0, y, t) - q_x(x_0 + \Delta x, y, t) \text{ dy dt} \tag{4}$$

Now for the y direction, we can repeat the previous steps to obtain the following equation:

$$\Delta z \int_{t_0}^{t_0+\Delta t} \int_{x_0}^{x_0+\Delta x} q_y(x, y_0, t) - q_y(x, y_0 + \Delta y, t) \text{ dx dt} \tag{5}$$

Now, let us combine both equations:

$$\Delta z \int_{t_0}^{t_0+\Delta t} \int_{y_0}^{y_0+\Delta y} q_x(x_0, y, t) - q_x(x_0 + \Delta x, y, t) \text{ dy dt} + \Delta z \int_{t_0}^{t_0+\Delta t} \int_{x_0}^{x_0+\Delta x} q_y(x, y_0, t) - q_y(x, y_0 + \Delta y, t) \text{ dx dt} \tag{6}$$

Here, let us divide line 4) by $\frac{1}{\Delta x \Delta y \Delta z \Delta t}$ and take the limit of each of these variables at they approach 0 and simplify Δz immediately:

$$\lim_{\Delta x, \Delta y, \Delta t \to 0} \frac{1}{\Delta x \Delta y \Delta t}\Big(\int_{t_0}^{t_0+\Delta t} \int_{y_0}^{y_0+\Delta y} q_x(x_0, y, t) - q_x(x_0 + \Delta x, y, t) \text{ dy dt} \tag{7}$$

$$+ \int_{t_0}^{t_0+\Delta t} \int_{x_0}^{x_0+\Delta x} q_y(x, y_0, t) - q_y(x, y_0 + \Delta y, t) \text{ dx dt}\Big) \tag{8}$$

$$= \lim_{\Delta x, \Delta y, \Delta t \to 0} \Big(\int_{t_0}^{t_0+\Delta t} \int_{y_0}^{y_0+\Delta y} \frac{q_x(x_0, y, t) - q_x(x_0 + \Delta x, y, t)}{\Delta x \Delta y \Delta t} \text{ dy dt} \tag{9}$$

$$+ \int_{t_0}^{t_0+\Delta t} \int_{x_0}^{x_0+\Delta x} q_y(x, y_0, t) - q_y(x, y_0 + \Delta y, t) \text{ dx dt}\Big) \tag{10}$$

$$\tag{11}$$

(b) Do this over any small area by using the divergence theorem.

The Divergence Theorem

- In 3-D: Let $\vec{F}$ be any vector field, then

$$\int\int\int_\Omega \nabla \cdot \vec{F} dV = \int\int_R \vec{F} \cdot \vec{n} \text{ dA}\vec{f}$$

where Ω is any bounded, simple 3-D region, R is the surface of the 3-D region, and $\vec{n}$ is the unit outward normal.

- In 2-D:

$$\int\int_R \nabla \cdot \vec{F} \text{ da } = \oint \vec{F} \cdot \vec{n} \text{ dS} \tag{12}$$

Where R is a simple 2-D region, C is the boundary of the region, and $\vec{n}$ is the unit normal.

- In 1-D: The Funtamental Theorem of Calculus

$$\int_L \frac{\partial f}{\partial x} \text{ dx } = f(b) - f(a) \tag{13}$$

note that here we are integrating along a line segment L which is $[a, b]$

1. cube: $q_1 =$ _

$$q_1(x, y, t)\Delta y \Delta z \Delta t \tag{14}$$

Here, we have t and y in our function, we cannot multiply Delta y and Delta t so we multiply by dy and dz, they become the double integrals

$$\Delta z \int_{t_0}^{t_0+\Delta t} \int_{y_0}^{y_0+\Delta y} q_1(x_0, y, t) dydt \tag{15}$$

This is the heat in on the left side of the cube, and heat out is the right side. x_0 is the heat in and $x_0 + \Delta x$ is the heat out in this cube.

Now, we want to change our q to q_2 whih is going upwards.

$$q_2(x, y, t)\Delta x \Delta z \Delta t \tag{16}$$

Now, we have x and t in the function, so we repeat the same process.

Now, we want to consider the other side as heat accumulated,

$$Q\Delta x \Delta y \Delta z \tag{17}$$

Q is a function of x, y. Here, we want to change x and y. We also want to compute our ending heat - starting heat (t_0).

$$\Delta z \int \int Q(x, y, t_0) dxdy \tag{18}$$

Refer to January 24 notes Divide by three variables this time around. Get to $-nabla \cdot q = Q_t$.

$$- < \partial/\partial x, \partial/\partial y > \cdot < q_1, q_2 > \tag{19}$$

2.

$$\int \int Q_t dxdy - \oint_c \vec{q} \cdot \vec{n} ds \tag{20}$$

$$-\nabla \cdot \vec{q} = Q_t \tag{21}$$

$$-\int_R \int \nabla \cdot \vec{q} da = \int \int_R Q_t DA \tag{22}$$

$$\int \int_R (Q_t + \nabla \cdot \vec{q}) dA = 0 \tag{23}$$

$$\int_a^b f(x) dx = 0 \tag{24}$$