1. Let $f(x)$ be a 2π-period function on the interval $[-\pi, \pi]$ where $f(x) = \begin{cases} -1 & -\pi < x \leq 0 \\ 1 & 0 < x \leq \pi \end{cases}$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ \mathrm{dx} \tag{1}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ \mathrm{dx} \tag{2}$$

$$b_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ \mathrm{dx} \tag{3}$$

(a) Plot the function on the interval $[-3\pi, 3\pi]$

(b) Plot its (infinite) Fourier series on $[-3\pi, 3\pi]$

(c) Find the Fourier series of $f(x)$

Here, let us consider the symmetry of our function.

When we look at the graph of $f(x)$, we can see there is a reflection about the origin, making the function odd. sin is also an odd function, therefore a_n is an even function.

Looking at b_n, cos is an even function, therefore b_n becomes an odd function.

Finally, b_0 is always an odd function. When we integrate these three coefficients, we lose b_n and b_0, but keep a_n. Since a_n is even, we can write:

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ \mathrm{dx} \tag{4}$$

Here, we are looking at the interval from 0 to L. Our given function, $f(x)$ runs from $-\pi$ to π, therefore our integral is:

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_{0}^{\pi} 1 \cdot \sin\left(\frac{n\pi x}{\pi}\right)\ \mathrm{dx} && (5) \\ &= \frac{2}{\pi}\int_{0}^{\pi} \sin(nx)\ \mathrm{dx} && (6) \end{aligned}$$

From here, we can compute our integral:

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_{0}^{\pi} \sin(nx)\ \mathrm{dx} && (7) \\ &= -\frac{2}{\pi n}\cos(nx)\Big|_{0}^{\pi} && (8) \\ &= \frac{2}{\pi n}(1 - \cos(n\pi)) && (9) \end{aligned}$$

Here, we found our coefficient, a_n. Now, since $f(x)$ is odd, we are only interested in the following:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) && (10) \\ &= \sum_{n=1}^{\infty} \frac{2}{\pi n}(1 - \cos(n\pi))\sin\left(\frac{n\pi x}{L}\right) && (11) \end{aligned}$$

Here, since our interval is $-\pi$ to π, so let us write:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{2}{\pi n}(1 - \cos(n\pi))\sin\left(\frac{n\pi x}{\pi}\right) && (12) \\ &= \sum_{n=1}^{\infty} \frac{2}{\pi n}(1 - \cos(n\pi))\sin(nx) && (13) \end{aligned}$$

Here, we have our Fourier series.

2. Let $f(x) = x^2$ be a 2π-periodic function on the interval $[-\pi, \pi]$.

(a) Derive its Fourier series

Let us consider the symmetry of our function. Our function, $f(x)$, is an even function. Therefore, we have the following coefficients:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ \mathrm{dx} \tag{14}$$

$$b_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ \mathrm{dx} \tag{15}$$

Since $f(x)$ is even, we can write:

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ \mathrm{dx} \tag{16}$$

$$b_0 = \frac{1}{L}\int_{0}^{L} f(x)\ \mathrm{dx} \tag{17}$$

In addition, since we also know our interval and our function, we can write:

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} x^2\cos(nx)\ \mathrm{dx} \tag{18}$$

$$b_0 = \frac{1}{\pi}\int_{0}^{\pi} x^2\ \mathrm{dx} \tag{19}$$

First, let us find the integral of b_n. Let us rewrite b_n first:

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} x^2\cos(nx)\ \mathrm{dx} \tag{20}$$

Here, we want to do integration by parts. We want x^2 as our derived function since we can derive that function to 0.

x^2	$\cos(nx)$
$2x$	$\frac{1}{n}\sin(nx)$
2	$-\frac{1}{n^2}\cos(nx)$
0	$-\frac{1}{n^3}\sin(nx)$

Here, we can write our integral as the following:

$$b_n = \frac{2}{\pi}\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]_0^{\pi} \tag{21}$$

$$= \frac{2}{\pi n}\left[x^2\sin(nx) + \frac{2x}{n}\cos(nx) - \frac{2}{n^2}\sin(nx)\right]_0^{\pi} \tag{22}$$

$$= \frac{2}{\pi n}\left[\pi^2\sin(\pi x) + \frac{2\pi}{n}\cos(n\pi) - \frac{2}{n^2}\sin(n\pi)\right] - \frac{2}{n\pi}\left[0^2\sin(0) + \frac{0}{n}\cos(0) - \frac{2}{n^2}\sin(0)\right] \tag{23}$$

Here, the entire right term zeroes out. On the left, $\sin(n\pi)$ zeroes out, leaving us with:

$$b_n = \frac{4}{n^2}\cos(n\pi) \tag{24}$$

Now, let us find b_0:

$$b_0 = \frac{1}{\pi}\int_0^{\pi} x^2\ \mathrm{dx} \tag{25}$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \tag{26}$$

$$= \frac{1}{3\pi}\left[x^3\right]_0^{\pi} \tag{27}$$

$$= \frac{1}{3\pi}\left[\pi^3 - 0\right] \tag{28}$$

$$= \frac{\pi^2}{3} \tag{29}$$

Now that we have our coefficients, we can write:

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(nx)\sin(nx) \tag{30}$$

(b) Use Maple of Matlab to plot its finite Fourier series on $[-\pi, \pi]$ for $N = 10, 20, 50$ together with $f(x)$

(c) Use your Fourier series from part (a) to show that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$.

3. In the solution of the heat equation, we end up solving $X'' = -\lambda X$. Show that if $\lambda < 0$ or $\lambda = 0$ there is only the trivial solution ($X(x) = 0$).

Here, we have the equation:

$$X'' = -\lambda X \tag{31}$$

We want to use this equation and set our boundary conditions as $X(0) = X(L) = 0$. Now, we must find an equation where after two derivatives on the right, we obtain a similar function on the left. On the left, we have a sign, coefficient, and function of x. Let us write a general solution for our equation:

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \tag{32}$$

Here, we can make three assumptions ***via trichotomy***: $\lambda < 0, \lambda = 0$, or $\lambda > 0$. Let us look at the first two examples:

(a) $\lambda < 0$

Here, let us consider the case when λ is negative. Let us consider rewriting λ:

$$\lambda < 0 \tag{33}$$
$$\lambda \cdot -1 > 0 \cdot -1 \tag{34}$$
$$-1 \cdot \lambda > 0 \tag{35}$$

Now, let us plug in our found value into our general equation:

$$X(x) = A\cos(\sqrt{-1 \cdot \lambda}x) + B\sin(\sqrt{-1 \cdot \lambda}x) \tag{36}$$

Let us separate the terms under the radical:

$$X(x) = A\cos(\sqrt{-1 \cdot \lambda}x) + B\sin(\sqrt{-1 \cdot \lambda}x) \tag{37}$$
$$= A\cos(\sqrt{-1}\sqrt{\lambda}x) + B\sin(\sqrt{-1}\sqrt{\lambda}x) \tag{38}$$
$$= A\cos(i\sqrt{\lambda}x) + B\sin(i\sqrt{\lambda}x) \tag{39}$$

Here, in our expression, we see we are taking the square root of a negative number, which would give us an imaginary number. Here, we are evaluating our general solution with real numbers, therefore, the following form:

$$X(x) = A\cos(i\sqrt{\lambda}x) + B\sin(i\sqrt{\lambda}x) \tag{40}$$

Where $X(x)$ is a real number would only have the trivial solution $X(x) = 0$.

(b) $\lambda = 0$

Here, let us consider the case when λ is zero. Now, let us write our general equation:

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \tag{41}$$

Here, since $\lambda = 0$, we can evaluate our equation:

$$X(x) = A\cos(0) + B\sin(0) \tag{42}$$
$$= A \tag{43}$$

Now, let us evaluate our boundary condition for $X(x) = A$. First, we let $X(0) = 0$:

$$X(0) = 0 = A \tag{44}$$

Here, we know A is 0. For the second condition, let us write:

$$X(L) = 0 = A \tag{45}$$

Here, we will always have the trivial solution, $X(x) = 0$.

4. Show that $u(x,t) = e^{-\lambda^2 a^2 t}\,[A\cos(\lambda x) + B\sin(\lambda x)]$

5. Solve $u_t = u_{xx}$ given $u(0,t) = u(1,t) = 0$ for $t \geq 0$ and $u(x,0) = 1$ for $0 \leq x \leq 1$

Let us consider the following conditions:

- $u_t = u_{xx}$
- $u(0,t) = 0, t \geq 0$
- $u(1,t) = 0, t \geq 0$
- $u(x,0) = 1, 0 \leq x \leq 1$

Let us begin finding our solution.

(a) Let us assume our solution is seperable. Therefore, we can write $u(x,t) = X(x)T(x)$. Now, using our initial conditions, let us write:

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0 \tag{46}$$
$$u(1,t) = X(1)T(t) = 0 \Rightarrow X(1) = 0 \tag{47}$$
$$u(x,0) = X(x)T(0) = 0 \Rightarrow T(0) = 0 \tag{48}$$

Now that we have used our initial conditions, let us write:

$$u_t = u_{xx} \tag{49}$$
$$XT' = X''T \tag{50}$$
$$\frac{T'}{T} = \frac{X''}{X} = -\lambda \tag{51}$$

(b) Here, we have more information regarding X, so let us write:

$$\frac{X''}{X} = -\lambda \tag{52}$$
$$X'' = -\lambda X \tag{53}$$

Here, we know $X(0) = X(1) = 0$. We want to write the general form of our equation as the following:

$$X(x) = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right) \tag{54}$$

Here, we can input a condition for our general statement. Let us find $X(0)$ first:

$$X(0) = 0 = A\sin(0) + B\cos(0) \tag{55}$$
$$0 = B \tag{56}$$
$$X(x) = A\sin(\sqrt{\lambda}x) \tag{57}$$

We also know $X(1) = 0$:

$$X(1) = 0 = A\sin(\sqrt{\lambda}) \tag{58}$$

Now, if A is 0, then our answer is trivial, therefore we want the inside of sin to be $n\pi$:

$$n\pi = \sqrt{\lambda} \tag{59}$$
$$n^2\pi^2 = \lambda_n \tag{60}$$

Therefore, we can write:

$$X_n(x) = \sin(n\pi x) \tag{61}$$

(c) Now, let us find T:

$$\frac{T'}{T} = -\lambda \tag{62}$$
$$\frac{T'}{T} = -n\pi \tag{63}$$
$$T_n' = -n\pi T \tag{64}$$
$$T_n = e^{-n\pi t} \tag{65}$$

(d) Now, let us combine to find u_n

$$u_n(x,t) = X_n(x)T_n(t) \tag{66}$$
$$= \sin(n\pi x)e^{-n\pi t} \tag{67}$$

By linearity,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx\pi)e^{-n\pi t} \tag{68}$$

(e) Here, we would use an initial condition to find A_n. We know $u(x,0) = 1$, so let us write:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx\pi) = 1 \tag{69}$$
$$A_n = 2 \in_0^1 \sin(n\pi x) \text{ dx} \tag{70}$$

6. Find the solution to the previous problem if $u(x,0) = x - x^2$ for $0 \leq x \leq 1$

- $u_t = u_{xx}$
- $u(0,t) = 0, t \geq 0$
- $u(1,t) = 0, t \geq 0$
- $u(x,0) = x - x^2, 0 \leq x \leq 1$

Here, by following the same steps as the previous problem, we would reach the same conclusion up to step e. At step e, we want to replace our condition with the fourth bullet:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx\pi)e^{-n\pi t} \tag{71}$$

Simarly as the end of the last question,

$$A_n = 2\int_0^1 (x - x^2)\sin(n\pi x) \text{ dx} \tag{72}$$

7. Solve $u_t = u_{xx}$ given $u(0,t) = u(1,t) = 0$ for $t \geq 0$ and $u(x,0) = 10^{-5}\sin(10^6\pi x)$ for $0 \leq x \leq 1$. Determine $u(x,2)$ and $u(x,-2)$ and look at their magnitudes. Note that when $t = -2$, we are looking at the backward heat equation and given the magnitude of $u(x,-2)$, what can you say about the solution to the backward heat equation?

   Let us consider the following conditions:

   - $u_t = u_{xx}$
   - $u(0,t) = 0, t \geq 0$
   - $u(1,t) = 0, t \geq 0$
   - $u(x,0) = 10^{-5}\sin(10^6\pi x), 0 \leq x \leq 1$
   - Determine the following and look at their magnitudes
     - $u(x,2)$
     - $u(x,-2)$

   Now, let us begin:

   (a) First, let us assume our equation is separable:

$$X(x)T'(t) = X''T(t) \tag{73}$$

   Using our boundary conditions, we can find the following:

$$u(0,t) = 0 = X(0)T(t) \Rightarrow X(0) = 0 \tag{74}$$
$$u(1,t) = 0 = X(1)T(t) \Rightarrow X(1) = 0 \tag{75}$$

   Now, let us separate:

$$u_t = u_{xx} \tag{76}$$
$$XT' = X''T \tag{77}$$
$$\frac{T'}{T} = \frac{X''}{X} = -\lambda \tag{78}$$

   (b) Here, let us solve for X:

$$X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \tag{79}$$
$$X(0) = 0 = B \tag{80}$$
$$X(1) = 0 = A\sin(\sqrt{\lambda}) \tag{81}$$
$$n\pi = \sqrt{\lambda} \tag{82}$$
$$n^2\pi^2 = \lambda \tag{83}$$
$$X_n(x) = \sin(n\pi x) \tag{84}$$

   (c) Now, let us solve for T:

$$\frac{T'}{T} = -n^2\pi^2 \tag{85}$$
$$T' = -n^2\pi^2 T_n \tag{86}$$
$$T_n = e^{-n^2\pi^2 t} \tag{87}$$

   (d) Now, let us combine both T_n and X_n:

$$u_n(x,t) = X_n(x)T_n(x) \tag{88}$$
$$= \sin(n\pi x)e^{-n^2\pi^2 t} \tag{89}$$

   By linearity,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)e^{-n^2\pi^2 t} \tag{90}$$

8. Here, let us use our initial condition to find A_n:

$$2 \cdot 10^{-5}\int_0^1 \sin(10^6\pi x)\sin(n\pi x)\ \mathrm{dx} \tag{91}$$

   Here, $A_{10^6} = 10^{-5}$, so let us write:

$$u(x,t) = 10^{-5}\sin(10^6\pi x)e^{-n^2\pi^2 t} \tag{92}$$
$$u(x,2) = 10^{-5}\sin(10^6\pi x)e^{-2n^2\pi^2} \tag{93}$$

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