1. Solve $\Delta u = 0$ on $x \in [0,4]$, $y \in [0,3]$, with $u(x,0) = u(x,3) = 0$ where

   (a) $u_x(0,y) = 0$ and $u_x(4,y) = \cos(\pi y)$

   (b) $u(0,y) = 1$ and $u(4,y) = 0$ (Hint: Translate the x coordinate so that $u(0,y) = 0$. This means that x goes between -4 and 0. Now go back to 0 to 4 with a y-axis flip.)

Here, let us first solve part a) of the problem.

(a) First, let us assume our equation is separable:

$$u(x,y) = X(x)Y(y) \tag{1}$$

Here, let us consider our boundary conditions:

$$u(x,0) = 0 = X(0)Y(y) \tag{2}$$

Here, we either have $X(0) = 0$ or $Y(y) = 0$. We do not want to assume $Y(y) = 0$ because that would make $u(x,y) = 0\ \forall y \in [0,3]$. Therefore, we consider $X(0) = 0$. We will follow this assumption for the upcoming boundary conditions.

$$u(x,y) = X(x)Y(y) \tag{3}$$
$$u(x,0) = X(x)Y(0) = 0 \Rightarrow Y(0) = 0 \tag{4}$$
$$u(x,3) = X(x)Y(3) = 0 \Rightarrow Y(3) = 0 \tag{5}$$

Now, let us consider $u_x(x,y)$ as $X'(x)Y(y)$:

$$u_x(x,y) = X'(0)Y(y) \tag{6}$$
$$u_x(0,y) = X'(0)Y(y) = 0 \Rightarrow X'(0) = 0 \tag{7}$$

Now, let us write:

$$-\frac{X''}{X} = \frac{Y''}{Y} = -\lambda \tag{8}$$

(b) Next, we solve for X, first solving for $Y(y)$ since we have more information on $Y(y)$.

$$\frac{Y''}{Y} = -\lambda \tag{9}$$
$$Y'' = -\lambda Y \tag{10}$$

Here, let us write out our general equation:

$$Y_n(y) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \tag{11}$$

Here, we know $Y(0) = Y(3) = 0$. Let us input our y:

$$Y_n(0) = 0 = B \tag{12}$$
$$Y_n(3) = 0 = A\sin(\sqrt{\lambda}3) \tag{13}$$

Here, let us not consider A as 0, instead, the inside of our sine function to be 0.

$$\sqrt{\lambda}3 = n\pi \tag{14}$$
$$\sqrt{\lambda} = \frac{n\pi}{3} \tag{15}$$
$$\lambda_n = \left(\frac{n\pi}{3}\right)^2 \tag{16}$$

2. Solve $u_{tt} = 4u_{xx}$ on $x \in [0, 3]$, $t \in [0, \infty)$, with $u(0, t) = u(3, t) = 0$ where

   (a) $u(x, 0) = 4\sin(2\pi x) + 7\sin(6\pi x) - 2\sin(\pi x)$, $u_t(x, 0) = 0$

   (b) $u(x, 0) = x(3 - x)$, $u_t(x, 0) = \sin(\pi x)$