1. Solve $\Delta u = 0$ on $x \in [0,4]$, $y \in [0,3]$, with $u(x,0) = u(x,3) = 0$ where

   (a) $u_x(0,y) = 0$ and $u_x(4,y) = \cos(\pi y)$

   (b) $u(0,y) = 1$ and $u(4,y) = 0$ (Hint: Translate the x coordinate so that $u(0,y) = 0$. This means that x goes between -4 and 0. Now go back to 0 to 4 with a y-axis flip.)

Here, let us first solve part a) of the problem.

(a) First, let us assume our equation is separable:

$$u(x,y) = X(x)Y(y) \tag{1}$$

Let us take note of our u:

$$\begin{align}
u_{xx} + u_{yy} &= 0 \tag{2}\\
X''Y + XY'' &= 0 \tag{3}\\
XY'' &= -X''Y \tag{4}\\
\frac{Y''}{Y} &= -\frac{X''}{X} = -\lambda \tag{5}
\end{align}$$

Here, let us consider our boundary conditions:

$$u(x,0) = 0 = X(0)Y(y) \tag{6}$$

Here, we either have $X(0) = 0$ or $Y(y) = 0$. We do not want to assume $Y(y) = 0$ because that would make $u(x,y) = 0\ \forall y \in [0,3]$. Therefore, we consider $X(0) = 0$. We will follow this assumption for the upcoming boundary conditions.

$$\begin{align}
u(x,y) &= X(x)Y(y) \tag{7}\\
u(x,0) &= X(x)Y(0) = 0 \Rightarrow Y(0) = 0 \tag{8}\\
u(x,3) &= X(x)Y(3) = 0 \Rightarrow Y(3) = 0 \tag{9}
\end{align}$$

Now, let us consider $u_x(x,y)$ as $X'(x)Y(y)$:

$$\begin{align}
u_x(x,y) &= X'(0)Y(y) \tag{10}\\
u_x(0,y) &= X'(0)Y(y) = 0 \Rightarrow X'(0) = 0 \tag{11}
\end{align}$$

Now, let us write:

$$-\frac{X''}{X} = \frac{Y''}{Y} = -\lambda \tag{12}$$

(b) Next we solve for $Y(y)$ since we have more information on $Y(y)$.

$$\begin{align}
\frac{Y''}{Y} &= -\lambda \tag{13}\\
Y'' &= -\lambda Y \tag{14}
\end{align}$$

Here, let us write out our general equation:

$$Y_n(y) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \tag{15}$$

Here, we know $Y(0) = Y(3) = 0$. Let us input our y:

$$\begin{align}
Y_n(0) &= 0 = B \tag{16}\\
Y_n(3) &= 0 = A\sin(\sqrt{\lambda}3) \tag{17}
\end{align}$$

Here, let us not consider A as 0, instead, the inside of our sine function to be 0.

$$\begin{align}
\sqrt{\lambda}3 &= n\pi \tag{18}\\
\sqrt{\lambda} &= \frac{n\pi}{3} \tag{19}\\
\lambda_n &= \left(\frac{n\pi}{3}\right)^2 \tag{20}
\end{align}$$

Now we have the following:

$$Y_n(y) = \sin\left(\frac{n\pi y}{3}\right) \tag{21}$$

(c) Now, let us solve for X:

$$\frac{X''}{X} = \lambda \tag{22}$$

$$X'' = \lambda X \tag{23}$$

Here, we solved for λ in the previous step:

$$X_n'' = \left(\frac{n\pi}{3}\right)^2 X_n \tag{24}$$

$$X_n(x) = C \sinh\left(\frac{n\pi x}{3}\right) + D \cosh\left(\frac{n\pi x}{3}\right) \tag{25}$$

2. Solve $u_{tt} = 4u_{xx}$ on $x \in [0,3]$, $t \in [0,\infty)$, with $u(0,t) = u(3,t) = 0$ where

   (a) $u(x,0) = 4\sin(2\pi x) + 7\sin(6\pi x) - 2\sin(\pi x)$, $u_t(x,0) = 0$

   (b) $u(x,0) = x(3-x)$, $u_t(x,0) = \sin(\pi x)$

   Let us begin with our boundary conditions.

   (a) Let us assume our equation is separable.

$$u_{tt} = 2^2 u_{xx} \tag{26}$$

$$XT'' = 2^2 X''T \tag{27}$$

$$\frac{T''}{2^2 T} = \frac{X''}{X} = -\lambda \tag{28}$$

   (b) Here, since we have more information regarding X, let us solve for X:

$$\frac{X''}{X} = -\lambda \tag{29}$$

$$X'' = -\lambda X \tag{30}$$

   In this equation, we want to consider cos and sin for our general equation.

$$X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \tag{31}$$

   Here, we know $X(0) = X(3) = 0$, so let us write:

$$X(0) = B = 0 \tag{32}$$

$$X(3) = A\sin(\sqrt{\lambda}3) = 0 \tag{33}$$

   Here, let us find when our sine function is 0.

$$\sqrt{\lambda}3 = n\pi \tag{34}$$

$$\sqrt{\lambda} = \frac{n\pi}{3} \tag{35}$$

$$\lambda = \left(\frac{n\pi}{3}\right)^2 \tag{36}$$

   Now, let us write our general equation for X:

$$X_n(x) = \sin\left(\frac{n\pi x}{3}\right) \tag{37}$$

   (c) Next, let us solve for T:

$$\frac{T''}{2^2 T} = -\lambda \tag{38}$$

$$T'' = -2^2 T\lambda \tag{39}$$

   Here, let us write the general equation:

$$T_n(t) = C_n \sin\left(\frac{2n\pi t}{L}\right) + D_n \cos\left(\frac{2n\pi t}{L}\right) \tag{40}$$

   (d) Here, let us combine to find u_n and u:

$$u_n(x,t) = \sin\left(\frac{n\pi x}{3}\right)\left[C_n \sin\left(\frac{2n\pi t}{L}\right) + D_n \cos\left(\frac{2n\pi t}{L}\right)\right] \tag{41}$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{3}\right)\sin\left(\frac{2n\pi t}{L}\right) + D_n \sin\left(\frac{n\pi x}{3}\right)\cos\left(\frac{2n\pi t}{L}\right) \tag{42}$$

   (e) Here, let us find the coefficients using the Initial Conditions:

   - $u(x,0) = 4\sin(2\pi x) + 7\sin(6\pi x) - 2\sin(\pi x)$
   - $u_t(x,0) = 0$

   Using this information, find our t partial from our summation series:

$$u_t(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{3}\right)\frac{2n\pi}{L}\cos\left(\frac{2n\pi t}{L}\right) - D_n \sin\left(\frac{n\pi x}{3}\right)\frac{2n\pi}{L}\sin\left(\frac{2n\pi t}{L}\right) \tag{43}$$

   From here, let us plug in our initial condition, $u_t(x,0) = 0$:

$$u_t(x,0) = \sum_{n=1}^{\infty} C_n \frac{2n\pi}{L}\sin\left(\frac{n\pi x}{3}\right) = 0 \tag{44}$$