

1. Determine the solution to the heat equation $u_t = \alpha^2 u_{xx}$ with $\alpha \neq 1$ (We did the $\alpha = 1$ case in class).

Now, let us consider the following steps:

- (a) Solve for $u_t = u_{xx}$

$$u_t = \alpha^2 u_{xx} \quad (1)$$

$$F[u_t] = F[\alpha^2 u_{xx}] \quad (2)$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t e^{-ix\xi} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \alpha^2 u_{xx} e^{-ix\xi} dx \quad (3)$$

$$\frac{\partial}{\partial t} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-ix\xi} dx \right] = (i\xi)^2 \frac{\alpha^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-ix\xi} dx \quad (4)$$

$$\frac{\partial}{\partial t} \hat{u}(\xi, t) = -\xi^2 \alpha^2 \hat{u}(\xi, t) \quad (5)$$

Our initial condition becomes:

$$F[u(x, 0)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, 0) e^{-ix\xi} dx \quad (6)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx \quad (7)$$

$$= \hat{f}(\xi) \quad (8)$$

- (b) Solve $\hat{u}_t = -\xi^2 \alpha^2 \hat{u}$,

$\hat{u}(\xi, 0) = \hat{f}(\xi)$. Here, let us write the general form of \hat{u} :

$$\hat{u}(\xi, t) = A(\xi) e^{-\xi^2 t} \quad (9)$$

Here, let us use our initial condition to find $A(\xi)$

$$\hat{u}(\xi, 0) \hat{f}(\xi) = A(\xi) \quad (10)$$

$$\hat{u}(\xi, t) = \hat{f}(\xi) e^{-\xi^2 t} \quad (11)$$

- (c) Retransform

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi, t) e^{-ix\xi} d\xi \quad (12)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-\xi^2 t} e^{ix\xi} d\xi \quad (13)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iy\xi} dy e^{-\xi^2 t} e^{ix\xi} d\xi \quad (14)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) e^{-iy\xi} e^{-\xi^2 t} e^{ix\xi} dy d\xi \quad (15)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{-iy\xi} e^{-\xi^2 t} e^{ix\xi} d\xi dy \quad (16)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{-\xi^2 t + i\xi(x-y)} d\xi dy \quad (17)$$

Here, let us focus our attention on the inner integral, where we rewrite it as Q .

$$Q = \int_{-\infty}^{\infty} e^{-\xi^2 t + i\xi(x-y)} d\xi \quad (18)$$

Let's look at the following:

$$-\xi^2 t + i\xi(x-y) = -t \left[\xi^2 - \frac{i\xi(x-y)}{t} \right] \quad (19)$$

$$= -t \left[\left(\xi - \frac{i(x-y)}{2t} \right)^2 - \frac{i^2(x-y)^2}{4t^2} \right] \quad (20)$$

$$= -t \left[\left(\xi - \frac{i(x-y)}{2t} \right)^2 + \frac{(x-y)^2}{4t^2} \right] \quad (21)$$

Now, we have the following for Q:

$$Q = \int_{-\infty}^{\infty} e^{-t \left[\left(\xi - \frac{i(x-y)}{2t} \right)^2 + \frac{(x-y)^2}{4t^2} \right]} d\xi \quad (22)$$

$$= \int_{-\infty}^{\infty} e^{-t \left(\xi - \frac{i(x-y)}{2t} \right)^2 - \frac{(x-y)^2}{4t}} d\xi \quad (23)$$

$$= \int_{-\infty}^{\infty} e^{-t \left(\xi - \frac{i(x-y)}{2t} \right)^2} e^{-\frac{(x-y)^2}{4t}} d\xi \quad (24)$$

$$= e^{-\frac{(x-y)^2}{4t}} \int_{-\infty}^{\infty} e^{-t \left(\xi - \frac{i(x-y)}{2t} \right)^2} d\xi \quad (25)$$

Here, let us consider the following substitution:

$$w = \sqrt{t} \left(\xi - \frac{i(x-y)}{2t} \right) \quad (26)$$

$$dw = \sqrt{t} d\xi \quad (27)$$

Now, let us write:

$$e^{-\frac{(x-y)^2}{4t}} \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-w^2} dw = \sqrt{\frac{\pi}{t}} e^{-\frac{(x-y)^2}{4t}} \quad (28)$$

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \sqrt{\frac{\pi}{t}} e^{-\frac{(x-y)^2}{4t}} dy \quad (29)$$

$$= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{4t}} dy \quad (30)$$

2. Use Fourier Transforms to determine the solution to $u_t = cu_x$ is $u(x, t) = f(x + ct)$ where $u(x, 0) = f(x)$.

Here, let us consider the given equation:

$$u_t = cu_x \quad (1)$$

Here, let us use Fourier Transform to transform the equation:

$$F[u_t] = F[cu_x] \quad (2)$$

Here, we are deriving the right side once. Since we're doing one derivative, we pick up one $ci\xi$.

$$\hat{u}_t = ci\xi\hat{u} \quad (3)$$

Here, let us consider the following equation:

$$u(x, 0) = f(x) \quad (4)$$

$$\hat{u}(\xi, 0) = \hat{f}(\xi) \quad (5)$$

Now, let us consider our equation $u_t = cu_x$. Here, recall we are picking up $ci\xi$ on the right. Let us attempt to write a general form for our equation:

$$\hat{u}_t = ci\xi\hat{u} \quad (6)$$

$$\hat{u}(\xi, t) = A(\xi)e^{ci\xi t} \quad (7)$$

Here, we want to find a way to solve for our general solution. If we consider what we wrote for $\hat{u}(\xi, 0)$, we want to find:

$$\hat{u}(\xi, 0) = \hat{f}(\xi) = A(\xi) \quad (8)$$

So, our exponential must disappear at $t = 0$, which means we have the following:

$$\hat{u}(\xi, t) = A(\xi)e^{ic\xi t} \quad (9)$$

Here, let us rewrite $A(\xi)$ as $f(\xi)$

$$\hat{u}(\xi, t) = f(\xi)e^{ic\xi t} \quad (10)$$

Now, let us take our equation and retransform back.

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi, t) e^{ix\xi} d\xi \quad (11)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ic\xi t} e^{ix\xi} d\xi \quad (12)$$

Here, let us combine our exponents.

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{i\xi(x+ct)} d\xi \quad (13)$$

3. Determine d'Alembert's solution for $u_{tt} = c^2 u_{xx}$

4. Derive d'Alembert's formula for $u_{tt} = u_{xx}$ by assuming that $u(x, t) = v(x+t, x-t) = v(y, z)$. Next show that the wave equation yields $v_{yz} = 0$ and hence $v = A(y) + B(z)$ and solve for A and B using the initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$.

Here, let us take a look at our assumption:

$$u(x, t) = v(x+t, x-t) = v(y, z) \quad (1)$$

Here, we have the following relationship:

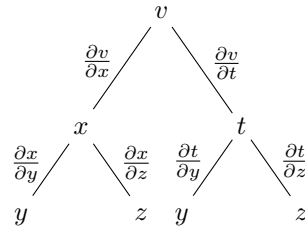
$$\begin{cases} y &= x+t \\ z &= x-t \end{cases} \quad (2)$$

Next, we want to show that the wave equation yields $v_{yz} = 0$. In order for the equation to be zero, that means v_y does not contain any z 's, so when you differentiate v_y once more, then the non- z terms zero out. The same can be argued for v_z , where v_z does not contain any y 's and will zero out.

Therefore, v is a function of $y + z$ and we can write v as $v = A(y) + B(z)$.

Here, let us find the partial of v with respect to y and z .

Let us consider the following tree:



Here, let us consider our tree and find v_y :

$$v_y = \frac{\partial v}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial v}{\partial t} \frac{\partial t}{\partial y} \quad (3)$$

Earlier, we found the following system of equations:

$$\begin{cases} y &= x+t \\ z &= x-t \end{cases} \quad (4)$$

Here, notice we can isolate x or t by adding or subtracting the two equations together. First, let us add the equations to obtain the following:

$$y + z = 2x \quad (5)$$

$$x = \frac{1}{2}(y + z) \quad (6)$$

Using this information, we can find $\frac{\partial x}{\partial y}$:

$$\frac{\partial x}{\partial y} = \frac{1}{2} \quad (7)$$

Now, to find t , we subtract the equations:

$$y - z = 2t \quad (8)$$

$$t = \frac{1}{2}(y - z) \quad (9)$$

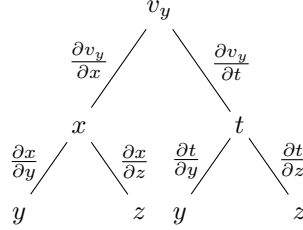
Similarly, let us find $\frac{\partial t}{\partial y}$

$$\frac{\partial t}{\partial y} = \frac{1}{2} \quad (10)$$

Here, now that we know $\frac{\partial x}{\partial y}$ and $\frac{\partial t}{\partial y}$, let us substitute this into line 3:

$$v_y = \frac{1}{2}v_x + \frac{1}{2}v_t \quad (11)$$

Now, let us find v_{yz} . Let us rewrite our tree:



Here, let us find v_{yz} using our tree.

$$v_{yz} = \frac{\partial v_y}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial v_y}{\partial t} \frac{\partial t}{\partial z} \quad (12)$$

$$= \frac{1}{2}v_{xx} + \frac{1}{2}v_{tx} \quad (13)$$

Now, we still have the following terms for x and t :

$$x = \frac{1}{2}(y + z) \quad (14)$$

$$t = \frac{1}{2}(y - z) \quad (15)$$

So let us substitute these back into the equation:

$$v_{yz} = \frac{1}{2}(v_{xx} + v_{tx}) - \frac{1}{2}(v_{xt} + v_{tt}) \quad (16)$$

$$= \frac{1}{2}v_{xx} - \frac{1}{2}v_{tt} \quad (17)$$

Recall from our initial conditions, we have $v_{tt} = u_{xx}$. Now, we can say $v = A(y) + B(z)$ since $v_{yz} = v_{xx} - v_{tt} = 0$.

Let us reconsider what we know:

- $u(x, 0) = f(x)$
- $u_t(x, 0) = g(x)$
- $V = A(y) + B(z)$
- $u(x, t) = v(x + t, x - t)$

Here, let us plug in for $u(x, 0)$ and substitute for y and z :

$$v = A(y) + B(z) \quad (18)$$

$$= A(x + t) + B(x - t) \quad (19)$$

$$u(x, 0) = A(x) + B(x) = f(x) \quad (20)$$

Now, let us find u_t , which requests for the t -partial, $u_t = v_t$. Let us use $A(x + t) + B(x - t)$ and find the derivative in

terms of A and B :

$$u_t = A'(x+t) \cdot 1 - B'(x-t) \cdot 1 \quad (21)$$

$$u_t(x, 0) = A'(x) - B'(x) = g(x) \quad (22)$$

Here, we found both $g(x)$ and $f(x)$:

$$f(x) = A(x) + B(x) \quad (23)$$

$$g(x) = A'(x) - B'(x) \quad (24)$$

Here, let us integrate the second line:

$$h(x) = \int g(x) dx \quad (25)$$

Here, we want to integrate over an area while keeping g spanning over the entire number line.

$$h(x) = \int_{-\infty}^x g(y) dy \quad (26)$$

Here, let us consider $g(y)$'s behavior as $-\infty$ tends to infinity. As g tends to $-\infty$, then g will head towards 0. Now, let us consider integrating $A'(x) - B'(x)$

$$A(x) - B(x) = \int_{-\infty}^x g(y) dy \quad (27)$$

Here, recall $A(x) + B(x) = f(x)$. Here, let this function and the previous function together:

$$A(x) + B(x) = f(x) \quad (28)$$

$$A(x) - B(x) = \int_{-\infty}^x g(y) dy \quad (29)$$

$$2A(x) = f(x) + \int_{-\infty}^x g(y) dy \quad (30)$$

$$2B(x) = f(x) - \int_{-\infty}^x g(y) dy \quad (31)$$

Here, we want to plug our previous two functions into v . See how $A(x+t) + B(x-t)$ has $x+t$ and $x-t$. Let us swap these into our new functions:

$$v = \frac{1}{2} \left(f(x+t) + \int_{-\infty}^{x+t} g(y) dy \right) + \frac{1}{2} \left(f(x-t) - \int_{-\infty}^{x-t} g(y) dy \right) \quad (32)$$

$$v = \frac{1}{2} \left(f(x+t) + \int_{-\infty}^{x+t} g(y) dy \right) + \frac{1}{2} \left(f(x-t) + \int_{x-t}^{-\infty} g(y) dy \right) \quad (33)$$

Here, notice how our integrations are similar. The bottom and top intervals are the same respectively, we can combine them:

$$v = \frac{1}{2} \left(f(x-t) + f(x+t) + \int_{x-t}^{x+t} g(y) dy \right) \quad (34)$$

Here, we have derive D'Alembert's Formula.