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Contents

1 Convention

Light Cone Coordinate

The signature we use are $(-,+)$. We define

$$U = t - x \quad , \quad V = t + x \tag{1.1}$$

The transformation of the derivative, the metric and the vector are :

$$\partial_U = \frac{1}{2}\left(\partial_t + \partial_x\right) \quad , \quad \partial_V = \frac{1}{2}\left(\partial_t - \partial_x\right) \tag{1.2}$$

$$dxdt = \frac{1}{2}dUdV \tag{1.3}$$

$$\Box = -\partial_t^2 + \partial_x^2 = -4\partial_U\partial_V \tag{1.4}$$

$$g_{UU} = g_{VV} = 0 \quad , \quad g_{UV} = g_{VU} = -\frac{1}{2} \tag{1.5}$$

$$A_U = \frac{1}{2}(A_t - A_x) \quad , \quad A_V = \frac{1}{2}(A_t + A_x) \tag{1.6}$$

2 Review

2.1 Local 2D Klein-Gordon field

The action is

$$S = \int d^2x \, \frac{1}{2}\phi\Box\phi \tag{2.1}$$

The corresponding path integral :

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi \; e^{\,i\int \frac{1}{2}\phi\Box\phi + \int J\phi} \\ &= \int \mathcal{D}\phi \; e^{\,i\int \frac{1}{2}\left(\phi - i\Box^{-1}J\right)\Box\left(\phi - i\Box^{-1}J\right) + i\int \frac{1}{2}J\Box^{-1}J} \\ &= N e^{\,i\int \frac{1}{2}J\Box^{-1}J} \end{aligned} \tag{2.2}$$

Hence, we obtain the propagator is :

$$\int \frac{d^2k}{4\pi^2}\left(-\frac{i}{k^2}\right)e^{-ik\cdot x} = \int \frac{dk_U dk_V}{4\pi^2}\,\frac{i}{2k_U k_V}e^{-ik_U U - ik_V V} \tag{2.3}$$

where we note there is Jacobian : $|\frac{\partial(k_0,k_1)}{\partial(k_U,k_V)}| = 2$. For simplifying the notation, we write $k_U = \Omega$ and $k_V = \bar{\Omega}$. Thus, the propagator is rewritten as :

$$G(U,V) = \int \frac{d\Omega d\bar{\Omega}}{4\pi^2}\frac{i}{2\Omega\bar{\Omega} + i\epsilon}e^{-i\Omega U - i\bar{\Omega}V} \tag{2.4}$$

2.2 Nonlocal 2D Klein-Gordon Field

Consider massless two dimensional action :

$$S = \int d^2x \; \frac{1}{2}\phi \Box e^{-\ell^2 \Box}\phi \tag{2.5}$$

From path integral formalism, the propagator is :

$$\begin{aligned} G &= \int \frac{d^2k}{4\pi^2} \left(-i\frac{e^{-\ell^2 k^2}}{k^2 - i\epsilon} \right) e^{ik\cdot x} \\ &= \int \frac{d\Omega d\bar{\Omega}}{4\pi^2} \left(i\frac{e^{4\ell^2 \Omega\bar{\Omega}}}{2\Omega\bar{\Omega} + i\epsilon} \right) e^{-i\Omega U - i\bar{\Omega}V} \end{aligned} \tag{2.6}$$

We attempt to represent :

$$i\frac{e^{4\ell^2\Omega\bar{\Omega}}}{2\Omega\bar{\Omega} + i\epsilon} = 2\int_{-i\ell^2}^{\infty} d\tau \; e^{\,i\tau(4\Omega\bar{\Omega}+i\epsilon)} = 2\int_{\ell_E^2}^{\infty} d\tau \; e^{\,i\tau(4\Omega\bar{\Omega}+i\epsilon)} \tag{2.7}$$

in which we define the analytic continuation:

$$\boxed{\ell_E^2 = -i\ell^2} \tag{2.8}$$

With this analytic continuation, we obtain the out going propagator:

$$G_{\text{out}}(U,V) = \int_0^{|V|/4\ell_E^2} \frac{d\Omega}{2\pi}\,\frac{1}{2\Omega}\left[\Theta(V)e^{-i\Omega U} + \Theta(-V)e^{i\Omega U}\right] \tag{2.9}$$

Introduce the Fourier expansion of the field (with only out going mode)

$$\hat{\phi}_{out}(U,V) = \int_0^{\infty} \frac{d\Omega}{\sqrt{4\pi\Omega}} \; \hat{a}_\Omega(V)e^{-i\Omega U} + \hat{a}_\Omega^\dagger(V)e^{i\Omega U} \tag{2.10}$$

From the out going propagator, we can derive the commutator relation:

$$[\hat{a}_\Omega(V), \hat{a}_{\Omega'}^\dagger(V')] = \Theta(|\Delta V| - 4\ell_E^2\Omega)\delta(\Omega - \Omega') \tag{2.11}$$

3 From Path Integral to Operator Formalism

In this section, we use the path integral of the stringy model to derive the commutator relation. Then, we investigate whether the commutator relation eq(2.11) can be reproduced from the path integral.

First, we consider the commutator relation derived from the symmetry : $\phi \to \phi + \epsilon$. We will see that commutator relation eq(2.11) agrees with it.

Second, we consider the commutator relation stems from the space time translation symmetry: $\phi \to \phi + \epsilon^\mu \partial_\mu \phi$. Unfortunately, we are unable to show whether eq(2.11) is consistent with it or not. Note that translation symmetry of space time derives the stress tensor, which also define the Hamiltonian. As we will shown the Hamiltonian can be written as infinite sum. It is not clear if we can write down a close form of the Hamiltonian.

Third, we may consider a postulated Hamiltonian and use it to reproduce the Heisenberg equation.

3.1 Bracket from Field Translation Symmetry

Consider the field translation symmetry : $\phi \to \phi + \epsilon$ where ϵ is an infinitesimal constant. Promote ϵ to be local :

$$\int \mathcal{D}\phi\, e^{\,iS}\mathcal{O} = \int \mathcal{D}\phi\, e^{\,iS}\left(1 + i\int d^2x\, \epsilon(x)\Box e^{-\ell^2\Box}\phi(x)\right)\left(\mathcal{O} + \int d^2y \frac{\delta\mathcal{O}}{\delta\phi(y)}\epsilon(y)\right) \tag{3.1}$$

Hence, we obtain :

$$\Box e^{-\ell^2\Box}\langle\phi(x)\mathcal{O}\rangle = i\left\langle\frac{\delta\mathcal{O}}{\delta\phi(x)}\right\rangle \tag{3.2}$$

Now, consider the case that $\mathcal{O} = \phi(U', V')$. Then,

$$\begin{aligned}
&4\partial_U\partial_V e^{4\ell^2\partial_U\partial_V}\langle\phi(U,V)\phi(U',V')\rangle = -i2\delta(U-U')\delta(V-V') \\
\longrightarrow &\int_{V'-\epsilon}^{V'+\epsilon} dV\,\partial_U\partial_V e^{4\ell^2\partial_U\partial_V}\langle\phi(U,V)\phi(U',V')\rangle = -\frac{i}{2}\delta(U-U')\int_{V'-\epsilon}^{V'+\epsilon} dV\,\delta(V-V') \\
\longrightarrow &\,\partial_U e^{4\ell^2\partial_U\partial_V}\langle\phi(U,V)\phi(U',V')\rangle\Big|_{V'-\epsilon}^{V'+\epsilon} = -\frac{i}{2}\delta(U-U') \\
\longrightarrow &\left\langle\left[\partial_U e^{4\ell^2\partial_U\partial_V}\phi(U,V), \phi(U',V)\right]\right\rangle = -\frac{i}{2}\delta(U-U')
\end{aligned} \tag{3.3}$$

Thus, we may expect to reproduce the bracket :

$$\left[\partial_U e^{4\ell^2\partial_U\partial_V}\hat{\phi}(U,V), \hat{\phi}(U',V)\right] = -\frac{i}{2}\delta(U-U') \tag{3.4}$$

in the operator formalism.

In the rest of this subsection, we write $\hat{\phi}$ as ϕ for convenience. The following calculation shows that the operator formalism can reproduce the commutator relation eq(3.4) by using eq(2.11):

$$
\begin{aligned}
&\left[\partial_U e^{4\ell^2\partial_U\partial_V}\phi(U,V),\phi(U',V')\right] \\
&=\left[\int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}}\left[(-i\Omega)a_\Omega(V+4\ell_E^2\Omega)e^{-i\Omega U}+(i\Omega)a_\Omega^\dagger(V-4\ell_E^2\Omega)e^{-i\Omega U}\right],\phi(U',V')\right] \\
&=\int_0^\infty\int_0^\infty \frac{d\Omega d\Omega'}{4\pi\sqrt{\Omega\Omega'}}-i\Omega\left\{\left[a_\Omega(V+4\ell_E^2\Omega),a_{\Omega'}^\dagger(V')\right]e^{-i\Omega U+i\Omega' U'}+\left[a_{\Omega'}(V'),a_\Omega^\dagger(V-4\ell_E^2\Omega)\right]e^{i\Omega U-i\Omega' U'}\right\} \\
&=\int_0^\infty \frac{d\Omega}{4\pi i}\left[\Theta(V'-V-8\ell_E^2\Omega)+\Theta(V-V')\right]e^{-i\Omega(U-U')} \\
&\qquad\qquad+\left[\Theta(V-V'-8\ell_E^2\Omega)+\Theta(V'-V)\right]e^{i\Omega(U-U')}
\end{aligned}
\tag{3.5}
$$

Hence, as $V=V'$, we have

$$
\boxed{\left[\partial_U e^{4\ell^2\partial_U\partial_V}\phi(U,V),\phi(U',V')\right]=-\frac{i}{2}\Theta(0)\delta(U-U')}
\tag{3.6}
$$

If we define $\Theta(0)=1$, we can reproduce the desired commutation relation.

3.2 Stress Tensor

Let a theory possess translation symmetry: $x^\mu\to x^\mu+\epsilon^\mu$. We define the stress tensor through:

$$
\delta\phi=\epsilon^\mu(x)\partial_\mu\phi\longrightarrow\delta S=-\int dx\ \epsilon^\nu\partial_\mu T^\mu{}_\nu
\tag{3.7}
$$

If we take the variation $\delta\phi=\epsilon^\mu(x)\partial_\mu\phi$ of the path integral:

$$
\int\mathcal{D}\phi\ e^{iS}\mathcal{O}=\int\mathcal{D}\phi\ e^{iS}\left(1-i\int dx\ \epsilon^\nu(x)\partial_\mu T^\mu{}_\nu\right)\left(\mathcal{O}+\int dy\ \frac{\delta\mathcal{O}}{\delta\phi(y)}\epsilon^\sigma(y)\partial_\sigma\phi\right)
\tag{3.8}
$$

where $\mathcal{O}$ is an arbitrary operator. Hence, we have:

$$
\langle\partial_\mu T^\mu{}_\nu(x)\mathcal{O}\rangle=-i\left\langle\partial_\nu\phi(x)\frac{\delta\mathcal{O}}{\delta\phi(x)}\right\rangle
\tag{3.9}
$$

In the following discussion we may switch to the light cone coordinate, $x=(U,V)$. We can take $\mathcal{O}=\phi(y_1)\mathcal{O}_1$. For concreteness, we further take $\mathcal{O}_1=\phi(y_2)$ with $V_2<<V_1$. Then, we have:

$$
\int_I dU\int_{V_1-\epsilon}^{V_1+\epsilon}dV\ \partial_\mu\langle T^\mu{}_\nu(x)\phi(y_1)\phi(y_2)\rangle=\int_I dU\int_{V_1-\epsilon}^{V_1+\epsilon}dV\ \ -i\delta(x-y_1)\partial_\nu\langle\phi(x)\phi(y_2)\rangle
\tag{3.10}
$$

where we let $I=(U_a,U_b)$. Since we assume $V_2<<V_1$, V_2 is outside the integral and $\delta(V-V_2)$ vanishes under the integral. For the L.H.S.,

$$
\int_I dU\ \left\langle T^V{}_\nu(x)\phi(y_1)\phi(y_2)\right\rangle\Big|_{(U,V_1-\epsilon)}^{(U,V_1+\epsilon)}+\int_{V_1-\epsilon}^{V_1+\epsilon}dV\ \left\langle T^U{}_\nu(x)\phi(y_1)\phi(y_2)\right\rangle\Big|_{(U_a,V)}^{(U_b,V)}
\tag{3.11}
$$

As $\epsilon \to 0$, the second term vanishes if we assume the integrand is continuous. For the R.H.S.,

$$\int_I dU - 2i\delta(U-U_1)\left\langle \partial_\nu \phi(V_1,U)\phi(y_2)\right\rangle \tag{3.12}$$

where the factor 2 comes from the Jacobian of the delta function. Since the interval I is arbitrary, we can take it out:

$$\begin{aligned}\left\langle {T^V}_\mu(x)\phi(y_1)\phi(y_2)\right\rangle\Big|_{V_1-\epsilon}^{V_1+\epsilon} &= -2i\delta(U-U_1)\left\langle \partial_\mu \phi(V_1,U)\phi(y_2)\right\rangle \\ \longrightarrow \left\langle \left[{T^V}_\mu(V_1,U),\phi(V_1,U_1)\right]\phi(y_2)\right\rangle &= -2i\delta(U-U_1)\left\langle \partial_\mu \phi(V_1,U)\phi(y_2)\right\rangle\end{aligned} \tag{3.13}$$

Since $\mathcal{O}_1$ can be taken arbitrarily, we may expect in the operator formalism:

$$\left[\hat{T}^V_\mu(U,V),\hat{\phi}(V,U')\right] = -2i\delta(U-U')\partial_\mu\hat{\phi}(U,V) \tag{3.14}$$

In appendix A, we derive:

$$\begin{aligned}{T^U}_\mu &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}(-1)^{n+l}\frac{(4\ell^2)^n}{n!}\partial_\mu\partial_U^{n-l}\phi\partial_V^{n+1}\partial_U^l\phi - \sum_{n=0}^{\infty}(-1)^n\frac{(4\ell^2)^n}{n!}\delta^U_\mu\partial_U^{n+1}\phi\partial_V^{n+1}\phi \\ {T^V}_\mu &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}(-1)^{n+l}\frac{(4\ell^2)^n}{n!}\partial_\mu\partial_V^{n-l}\phi\partial_U^{n+1}\partial_V^l\phi - \sum_{n=0}^{\infty}(-1)^n\frac{(4\ell^2)^n}{n!}\delta^V_\mu\partial_U^{n+1}\phi\partial_V^{n+1}\phi\end{aligned} \tag{3.15}$$

Hence, we obtain:

$${T^V}_V = \sum_{n=1}^{\infty}\sum_{l=1}^{n}(-1)^{n+l}\frac{(4\ell^2)^n}{n!}\left(\partial_V^{n-l+1}\phi\right)\left(\partial_U^{n+1}\partial_V^l\phi\right) \tag{3.16}$$

The first three terms of ${T^V}_V$ are:

$$\ell^0 : 0 \tag{3.17}$$

$$\ell^2 : 4\ell^2\partial_V\phi\partial_U^2\partial_V\phi \tag{3.18}$$

$$\ell^4 : -8\ell^4\left(\partial_V^2\phi\partial_U^3\partial_V\phi - \partial_V\phi\partial_U^3\partial_V^2\phi\right) \tag{3.19}$$

The vanishing leading term is consistent with the local QFT. In order to check whether eq(3.14) is correct or not, we may want to calculate these first three terms. To make the calculation easier, we could rewrite the Pauli-Jordan function (**we write ϕ instead of $\hat{\phi}$ for convenience in the following calculations**):

$$\begin{aligned}\Delta(\Delta V,\Delta U) &= [\phi_{out}(U,V),\phi_{out}(V',U')] \\ &= \int_0^{|\Delta V|/4\ell_E^2}\frac{d\Omega}{4\pi\Omega}\left[e^{-i\Omega(U-U')} - e^{+i\Omega(U-U')}\right] \\ &= \int_0^{\frac{|\Delta V|\Delta U}{4\ell_E^2}}\frac{dx}{2\pi i}\frac{\sin x}{x}\end{aligned} \tag{3.20}$$

In the last equality, we introduce $x = \Omega(U - U')$ and $\Delta U = U - U'$. Note that under the limit $\ell_E^2 \to 0$, $\Delta(\Delta V, \Delta V) \to -\frac{i}{4} sgn(\Delta U)$ which is consistent with the local QFT. With this form, we can calculate commutator relation eq(3.14) at the second order eq(3.18)(we omit the subscript *out* of ϕ):

$$\begin{aligned}
&\left[\partial_V\phi\partial_U^2\partial_V\phi(U,V),\phi(V',U')\right]\Big|_{V=V'} \\
=&\partial_V\phi(U,V)\left[\partial_U^2\partial_V\phi(U,V),\phi(V',U')\right] + \left[\partial_V\phi(U,V),\phi(V',U')\right]\partial_U^2\partial_V\phi(U,V)\Bigg|_{V=V'} \\
=&\partial_V\phi(U,V)\partial_U^2\partial_V\left(\int_0^{\frac{|\Delta V|\Delta U}{4\ell_E^2}}\frac{dx}{2\pi i}\frac{\sin x}{x}\right) + \partial_V\left(\int_0^{\frac{|\Delta V|\Delta U}{4\ell_E^2}}\frac{dx}{2\pi i}\frac{\sin x}{x}\right)\partial_U^2\partial_V\phi(U,V)\Bigg|_{V=V'} \\
=&\partial_V\phi(U,V)\left[-\frac{sgn(\Delta V)}{128\pi i\ell_E^4}\Delta V^2\Delta U + \mathcal{O}(\Delta V^4\Delta U^3)\right] + \left[\frac{sgn(\Delta V)}{8\pi i\ell_E^2}\Delta U + \mathcal{O}(\Delta V^2\Delta U^3)\right]\partial_U^2\partial_V\phi\Bigg|_{V=V'}
\end{aligned} \tag{3.21}$$

As $V \to V'$, the $\mathcal{O}$ terms vanishes directly; however, since there is $sgn(\Delta V)$, the two limits $V' \to V^{\pm}$ are different. If we just simply take $sgn(0) = 0$, then we get the trivial answer:

$$\left[\partial_V\phi\partial_U^2\partial_V\phi(U,V),\phi(V,U')\right] = 0 \quad , \quad \text{if we take } sgn(0) = 0 \tag{3.22}$$

By the similar way, we can calculate eq(3.14) perturbatively. However, we are fail to calculate all the terms and sum them up in an exact form.

It's not clear whether ${T^V}_V$ constructed from the path integral is the Hamiltonian density under the commutator relation eq(2.11).

3.3 Construction of the Hamiltonian

Although it's not clear the Hamiltonian constructed from the path integral can give the eq(3.14), we can attempt to build a Hamiltonian s.t. $[\hat{H},\hat{\phi}] = -i\partial_V\hat{\phi}$. In the following discussion, for the sake of convenience, we write ϕ instead of $\hat{\phi}$.

We postulate the Hamiltonian :

$$\mathcal{H}(U,V) = \partial_V\phi\partial_U e^{i4\ell_E^2\partial_U\partial_V}\phi + \partial_U e^{-i4\ell_E^2\partial_U\partial_V}\phi\partial_V\phi \quad , \quad H(V) = \int_{-\infty}^{\infty} dU\, \mathcal{H} \tag{3.23}$$

where the field operators in $\mathcal{H}$ are all evaluated at (U,V). We first calculate the mode expansion of the Hamiltonian. The first term is:

$$
\begin{aligned}
&\int dU\ \partial_V\phi\partial_U e^{i4\ell_E^2\partial_U\partial_V}\phi \\
=&\int_{-\infty}^{\infty} dU \int_0^{\infty} \frac{d\Omega d\Omega'}{4\pi\sqrt{\Omega\Omega'}}(-i\Omega)\left(\dot{a}_{\Omega'}(V)e^{-i\Omega' U}+\dot{a}^\dagger_{\Omega'}(V)e^{i\Omega' U}\right)\left(a_\Omega(V+4\ell_E^2\Omega)e^{-i\Omega U}-a^\dagger_\Omega(V-4\ell_E^2\Omega)e^{i\Omega U}\right) \\
=&i\int_0^\infty \frac{d\Omega}{2}\ \dot{a}_\Omega(V)a^\dagger_\Omega(V-4\ell_E^2\Omega)-\dot{a}^\dagger_\Omega(V)a_\Omega(V+4\ell_E^2\Omega) \\
=&i\int_0^\infty \frac{d\Omega}{2}\ a^\dagger_\Omega(V-4\ell_E^2\Omega)\dot{a}_\Omega(V)-\dot{a}^\dagger_\Omega(V)a_\Omega(V+4\ell_E^2\Omega)+\delta(0)\Theta(0)
\end{aligned}
\tag{3.24}
$$

We've used the commutation relation in the last equality. Similarly,

$$
\begin{aligned}
&\int dU\ \partial_U e^{-i4\ell_E^2\partial_U\partial_V}\phi\partial_V\phi \\
=&i\int_0^\infty \frac{d\Omega}{2}\ a^\dagger_\Omega(V+4\ell_E^2\Omega)\dot{a}_\Omega(V)-a_\Omega(V-4\ell_E^2\Omega)\dot{a}^\dagger_\Omega(V) \\
=&i\int_0^\infty \frac{d\Omega}{2}\ a^\dagger_\Omega(V+4\ell_E^2\Omega)\dot{a}_\Omega(V)-\dot{a}^\dagger_\Omega(V)a_\Omega(V-4\ell_E^2\Omega)-\delta(0)\Theta(0)
\end{aligned}
\tag{3.25}
$$

As a result, we have :

$$
\begin{aligned}
H(V)=i\int_0^\infty \frac{d\Omega}{2}\ & a^\dagger_\Omega(V-4\ell_E^2\Omega)\dot{a}_\Omega(V)+a^\dagger_\Omega(V+4\ell_E^2\Omega)\dot{a}_\Omega(V) \\
&-\dot{a}^\dagger_\Omega(V)a_\Omega(V+4\ell_E^2\Omega)-\dot{a}^\dagger_\Omega(V)a_\Omega(V-4\ell_E^2\Omega)
\end{aligned}
\tag{3.26}
$$

We may abbreviate it as :

$$
\boxed{H(V)=i\int_0^\infty \frac{d\Omega}{2}\ a^\dagger_\Omega(V\pm4\ell_E^2\Omega)\dot{a}_\Omega(V)-\dot{a}^\dagger_\Omega(V)a_\Omega(V\pm4\ell_E^2\Omega)}
\tag{3.27}
$$

We list some property of this Hamiltonian :

1. $H(V)$ is hermitian. It can be seen from the definition of $\mathcal{H}$, i.e. $\mathcal{H}=\mathcal{H}^\dagger$. The hermiticity is also preserved in the expression of the mode expansion eq(3.26).

2. We can reproduce the E.O.M. in the sense: $[H,\phi]=-i\partial_V\phi$. We can compute :

$$
\begin{aligned}
&[H(V),\phi(U',V')] \\
=&\frac{i}{2}\int_0^\infty d\Omega\ \left[a^\dagger_\Omega(V\pm4\ell_E^2\Omega)\dot{a}_\Omega(V)-\dot{a}^\dagger_\Omega(V)a_\Omega(V\pm4\ell_E^2\Omega),\phi(V',U')\right] \\
=&\frac{i}{2}\int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}}\ -\Theta(|V-V'\pm4\ell_E^2\Omega|-4\ell_E^2\Omega)\left(\dot{a}_\Omega(V)e^{-i\Omega U'}+\dot{a}^\dagger_\Omega(V)e^{+i\Omega U'}\right) \\
&+\partial_V\Theta(|V-V'|-4\ell_E^2\Omega)\left(a_\Omega(V\pm4\ell_E^2\Omega)e^{-i\Omega U'}+a^\dagger_\Omega(V\pm4\ell_E^2\Omega)e^{+i\Omega U'}\right)
\end{aligned}
\tag{3.28}
$$

We note that $\partial_V\Theta(|V-V'|-4\ell_E^2\Omega)=\delta(|V-V'|-4\ell_E^2\Omega)sgn(V-V')$. If we again set $sgn(0)=0$, we have:

$$
[H(V),\phi(V,U')]=-i\partial_V\phi(V,U')
\tag{3.29}
$$

4 Physical Hilbert Space

In this section, we will propose the definition of the physical state. Before solving such state, we explore the coherent states and their properties. We will define the physical state from the equation of motion after being analytic continuation on $\ell^2 \to -i\ell^2 = \ell_E^2$. After the physical space is constructed, we investigate some simple solutions and study their physical meanings.

4.1 Coherent State

In this section, we aim to find the eigenstates of $a_\Omega(V)$.

For a complex scalar field, $\psi_\Omega(V)$, we note that :

$$\begin{aligned}&\left[a_\Omega(V), \left(\int_0^\infty d\Omega' \int dV'\ \psi_{\Omega'}(V') a^\dagger_{\Omega'}(V')\right)^n\right]\\ =&n\left(\int dV'\ \psi_\Omega(V')\Theta(|V-V'|-4\ell_E^2\Omega)\right)\left(\int_0^\infty d\Omega' \int dV'\ \psi_{\Omega'}(V') a^\dagger_{\Omega'}(V')\right)^{n-1}\end{aligned} \tag{4.1}$$

where the integral range of V' is taken to be $(-\infty, \infty)$. We define the coherent state:

$$|\psi\rangle := \exp\left(\int_0^\infty d\Omega \int dV \psi_\Omega(V) a^\dagger_\Omega(V)\right)|0\rangle \tag{4.2}$$

In this definition, $|\psi\rangle$ is a vector valued functional. To be explicit, we should write $|\Psi[\psi_\Omega(V)]\rangle$ meaning a state is dependent on the complex function $\psi_\Omega(V)$. For the sake of simplicity, we just write $|\psi\rangle$. We can calculate:

$$\begin{aligned} a_\Omega(V)|\psi\rangle &= \left[a_\Omega(V), \exp\left(\int_0^\infty d\Omega' \int dV' \psi_{\Omega'}(V') a^\dagger_{\Omega'}(V')\right)\right]|0\rangle \\ &= \sum_{n=0}^\infty \frac{1}{n!}\left[a_\Omega(V), \left(\int_0^\infty d\Omega' \int dV' \psi_{\Omega'}(V') a^\dagger_{\Omega'}(V')\right)^n\right]|0\rangle \\ &= \int dV' \psi_\Omega(V')\Theta(|\Delta V| - 4\ell_E^2\Omega)\sum_{n=0}^\infty \frac{1}{n!}\left(\int_0^\infty d\Omega' \int dV' \psi_{\Omega'}(V') a^\dagger_{\Omega'}(V')\right)^n|0\rangle \\ &= \int dV' \psi_\Omega(V')\Theta(|\Delta V| - 4\ell_E^2\Omega)|\psi\rangle \end{aligned} \tag{4.3}$$

Hence, this vector indeed acts like a coherent state whose eigenvalue is a function depending on V

and Ω. Moreover, we can calculate the inner product:

$$
\begin{aligned}
\langle\psi|\phi\rangle &= \langle 0|\exp\left(\int_0^\infty d\Omega\int dV\ \psi^*_\Omega(V)a_\Omega(V)\right)\exp\left(\int_0^\infty d\Omega\int dV\ \phi_\Omega(V)a^\dagger_\Omega(V)\right)|0\rangle \\
&= \sum_{n=0}^\infty\sum_{m=0}^\infty\frac{1}{n!m!}\langle 0|\left(\int_0^\infty d\Omega\int dV\ \psi^*_\Omega(V)a_\Omega(V)\right)^n\left(\int_0^\infty d\Omega'\int dV'\ \phi_{\Omega'}(V')a^\dagger_{\Omega'}(V')\right)^m|0\rangle \\
&= \sum_{n=0}^\infty\frac{1}{n!}\langle 0|\left(\int_0^\infty d\Omega\int dVdV'\ \psi^*_\Omega(V)\phi_\Omega(V')\Theta(|\Delta V|-4\ell_E^2\Omega)\right)^n|0\rangle \\
&= \exp\left(\int_0^\infty d\Omega\int dVdV'\psi^*_\Omega(V)\phi_\Omega(V')\Theta(|\Delta V|-4\ell_E^2\Omega)\right)
\end{aligned}
\tag{4.4}
$$

we see that the inner product is always non-negative.

Now, we denote that:

$$
\mathcal{H}_a = span\left\{\ a^\dagger_{\Omega_1}(V_1)\cdots a^\dagger_{\Omega_n}(V_n)|0\rangle\ |\ n\in\mathbb{N}\right\}\quad,\quad \mathcal{H}_{coh} = span\left\{\ |\psi\rangle\ |\ \psi_\Omega(V):\mathbb{R}^2\to\mathbb{C}\right\} \tag{4.5}
$$

Next, we are going to prove that these coherent states form a complete basis:

$$
\boxed{\mathcal{H}_a \simeq \mathcal{H}_{coh}} \tag{4.6}
$$

By definition, $\mathcal{H}_{coh}\subset\mathcal{H}_a$. Hence, in order to prove this, we should show that each vector in $\mathcal{H}_a$ can be written as the superposition in $\mathcal{H}_{coh}$.

We consider the path integral:

$$
\int\mathcal{D}\psi\mathcal{D}\psi^*\ e^{-\int d\Omega dV\ \psi_\Omega(V)\psi^*_\Omega(V)}\left[\psi^*_{\Omega_1}(V_1)\cdots\psi^*_{\Omega_n}(V_n)\right]|\psi\rangle \tag{4.7}
$$

where we define the measurement in the path integral to be $\mathcal{D}\psi = \Pi_{\Omega,V}\frac{d[\psi_\Omega(V)]}{\sqrt{2\pi i}}$, in which Ω varies from 0 to ∞ and V from $-\infty$ to ∞. Hence, the path integral becomes:

$$
\int\prod_{\Omega,V}\frac{d[\psi_\Omega(V)]}{\sqrt{2\pi i}}\frac{d[\psi^*_\Omega(V)]}{\sqrt{2\pi i}}\ e^{-\psi_\Omega(V)\psi^*_\Omega(V)}\left[\psi^*_{\Omega_1}(V_1)\cdots\psi^*_{\Omega_n}(V_n)\right]e^{\psi_\Omega(V)a^\dagger_\Omega(V)}|0\rangle \tag{4.8}
$$

Expend the exponential $e^{\psi a^\dagger}$ and use the identity:

$$
\int\frac{dzdz^*}{2\pi i}e^{-zz^*}z^{*n}z^m = \delta_{m,n}\Gamma(n+1) \tag{4.9}
$$

Then, we find that the path integral is equal to:

$$
a^\dagger_{\Omega_1}(V_1)\cdots a^\dagger_{\Omega_n}(V_n)|0\rangle \tag{4.10}
$$

Therefore, it indicates that any vector in $\mathcal{H}_a$ can be written as the superposition of the vectors in $\mathcal{H}_{coh}$. As a result, we've shown that $\mathcal{H}_a\simeq\mathcal{H}_{coh}$.

4.2 Equation of Motion as a Constraint

We proposed the definition of the physical states as the state satisfies the following equation:

$$\partial_U \partial_V e^{i\ell_E^2 \partial_U \partial_V} \hat{\phi}^{(+)}(U,V)|\text{phy}\rangle = 0 \tag{4.11}$$

where the + sign on the field operator represents the annihilate part only. We denote the vector space as $\mathcal{H}_{phy}$. Clearly, following this definition, we have:

$$\langle\text{phy}|\partial_U \partial_V e^{i\ell_E^2 \partial_U \partial_V} \hat{\phi}(U,V)|\text{phy}\rangle = 0 \tag{4.12}$$

reproducing the equation motion. In order to find such state, we attempt to act a coherent state on it:

$$\begin{aligned} &\partial_U \partial_V e^{i\ell_E^2 \partial_U \partial_V} \hat{\phi}^{(+)}(U,V)|\psi\rangle \\ =&\int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}}(-i\Omega)\dot{a}_\Omega^\dagger(V+4\ell_E^2\Omega)e^{-i\Omega U}|\psi\rangle \\ =&-i\int_0^\infty d\Omega\,\sqrt{\frac{\Omega}{4\pi}}\int dV'\;\psi_\Omega(V')\partial_V\Theta\left(|V'-V-4\ell_E^2\Omega|-4\ell_E^2\Omega\right)e^{-i\Omega U}|\psi\rangle \end{aligned} \tag{4.13}$$

Using the relation,

$$\partial_V\Theta\left(|V'-V-4\ell_E^2\Omega|-4\ell_E^2\Omega\right) = \delta(|V'-V-4\ell_E^2\Omega|-4\ell_E^2\Omega)sgn(V-V'+4\ell_E^2\Omega) \tag{4.14}$$

we finally obtain:

$$\partial_U \partial_V e^{i\ell_E^2 \partial_U \partial_V} \hat{\phi}^{(+)}(U,V)|\psi\rangle = -i\int_0^\infty d\Omega\; e^{-i\Omega U}\sqrt{\frac{\Omega}{4\pi}}\left[\psi_\Omega(V)-\psi_\Omega(V+8\ell_E^2\Omega)\right]|\psi\rangle \tag{4.15}$$

For a physical state, we require that the above equation vanishes. Besides, since $\mathcal{H}_a \simeq \mathcal{H}_{coh}$, $\mathcal{H}_{phy}$ is all the coherent states such that:

$$-i\int_0^\infty d\Omega\; e^{-i\Omega U}\sqrt{\frac{\Omega}{4\pi}}\left[\psi_\Omega(V)-\psi_\Omega(V+8\ell_E^2\Omega)\right] = 0 \tag{4.16}$$

A kind of simple solutions of the physical space is ψ_Ω being periodic in V direction with period $8\ell_E^2\Omega$:

$$\psi_\Omega(V) = \psi_\Omega(V+8\ell_E^2\Omega) \tag{4.17}$$

We could consider the example:

$$\psi_\Omega(V) = f_\Omega \sum_{m=-\infty}^{\infty} \delta(V - m\tau_\Omega - \zeta_\Omega) \tag{4.18}$$

where we define:

$$\tau_\Omega = 8\ell_E^2\Omega \tag{4.19}$$

and ζ_Ω is a constant depend on Ω. Therefore, the coherent state reads:

$$|\psi\rangle = \exp\left(\int_0^\infty d\Omega\ f_\Omega \sum_{m=\infty}^{\infty} a^\dagger_\Omega(m\tau_\Omega + \zeta_\Omega)\right)|0\rangle := |f\rangle \tag{4.20}$$

Now, we can use the path integral to change into the basis of $\mathcal{H}_a$:

$$\begin{aligned}
&\int \mathcal{D}f\mathcal{D}f^*\ e^{-\int_0^\infty d\Omega\ f_\Omega f^*_\Omega} \prod_k f^*_{\Omega_k}|f\rangle \\
=&\int \prod_{\Omega\in(0,\infty)} \frac{df_\Omega df^*_\Omega}{2\pi i}\ e^{-f_\Omega f^*_\Omega}\left(\prod_k f^*_{\Omega_k}\right) e^{f_\Omega \sum_m a^\dagger_\Omega(m\tau_\Omega+\zeta_\Omega)}|0\rangle \\
=&\prod_k \sum_{m=-\infty}^{\infty} a^\dagger_{\Omega_k}(m\tau_{\Omega_k} + \zeta_{\Omega_k})|0\rangle
\end{aligned} \tag{4.21}$$

If we just insert one f^* into the path integral, we obtain a "one particle state":

$$|1_{\zeta_\Omega}\rangle := \sum_{m=-\infty}^{\infty} a^\dagger_\Omega(m\tau_\Omega + \zeta_\Omega)|0\rangle \tag{4.22}$$

One can directly verify that it satisfies the physical states condition. We can further calculate the norm of this state:

$$\langle 1_{\zeta_\Omega}|1_{\zeta_\Omega}\rangle = \delta(0)\sum_{m,n}\Theta(|m-n|-\frac{1}{2}) = \delta(0)\sum_{m\neq n} 1 \tag{4.23}$$

where the $\delta(0)$ comes from $\delta(\Omega-\Omega)$. The norm is positive but divergent. However, we could attempt to carry the regularization by Riemann Zeta function: $\sum_{m\neq n} = \sum_n\sum_m - \sum_{n=m} = \zeta^2(0) - \zeta(0) = \frac{3}{4}$, which is still positive.

A general one particle state with certain frequency can be defined by a function $f_\Omega(z)$:

$$|1_{f_\Omega}\rangle := \int_0^{\tau_\Omega} dz\ f_\Omega(z)\sum_n a^\dagger_\Omega(z + n\tau_\Omega)|0\rangle \tag{4.24}$$

Recall in the usual 2D Klein Gordon field, we have $\langle 0|\hat{\phi}_{out}(U,V)a^\dagger_\Omega|0\rangle \sim e^{-i\Omega U}$ giving the wave function of one particle state. Hence, we may want to calculate:

$$\langle 0|\hat{\phi}_{out}(U,V)|1_{f_\Omega}\rangle = \frac{e^{-i\Omega U}}{\sqrt{4\pi\Omega}}\int_0^{\tau_\Omega} dz\ f_\Omega(z)\underbrace{\sum_n \Theta(|V - z - n\tau_\Omega| - \tau_\Omega/2)}_{\Psi_\Omega(V;z)} \tag{4.25}$$

In the local QFT, Ψ_Ω should be a constant which is V independent. In the stringy model, $\Psi_\Omega(V;z)$ diverges since we don't normalize it well. However, we could analyze its Fourier mode and see whether it agrees with the usual field theory. In appendix B, we derive:

$$\mathcal{F}\left\{\Psi_\Omega(V;z)\right\}(\bar{\Omega}) = -3\pi\delta(\bar{\Omega}) \tag{4.26}$$

which indicates the spectrum along V direction is trivial and the same with local QFT.

5 Conclusion and Discussion

In section 3, we've derived the bracket from the symmetry: $\phi \to \phi + \epsilon$ by path integral. However, the same method didn't succeed for the symmetry: $\phi \to \phi + \epsilon \cdot \partial\phi$. Consequently, we are fail to derive the non-perturbative form of the Hamiltonian from the path integral formalism. However, we could postulate a exact Hamiltonian, which is hermitian and can reproduce E.O.M.

In section 4, we construct the coherent state for the stringy model. Further, we've proved that the coherent state is complete. We propose to treat the E.O.M. as a constraint eq(4.11) to construct the physical space. Using coherent state can easily solve the physical space. Moreover, we propose "one particle state" which is defined by infinite many $a^\dagger_\Omega(V)$ with interval $8\ell_E^2\Omega$. Unfortunately, we found its Fourier spectrum is not interesting. The physical meaning of this state is not clear enough.

Appendices

A Nonlocal Stress Tensor

In this appendix, we derive eq(3.15). Recall that the action we consider is:

$$S=\int d^2x\,\frac{1}{2}\phi\Box e^{-\ell^2\Box}\phi=\int dUdV\sum_{n=0}^{\infty}(-1)^n\frac{(4\ell^2)^n}{n!}\partial_U^{n+1}\phi\partial_V^{n+1}\phi \tag{A.1}$$

Consider the translation symmetry: $\phi\to\phi+\epsilon^\mu\partial_\mu\phi$. The variation of action is:

$$\delta S=\int dUdV\sum_{n=0}^{\infty}(-1)^n\frac{(4\ell^2)^n}{n!}\left(\underbrace{\partial_U^{n+1}\left(\epsilon^\mu\partial_\mu\phi\right)\partial_V^{n+1}\phi}_{a}+\underbrace{\partial_U^{n+1}\phi\partial_V^{n+1}\left(\epsilon^\mu\partial_\mu\phi\right)}_{b}\right) \tag{A.2}$$

We first calculate the a term in the integrand.

$$\begin{aligned}
&\partial_U^{n+1}\left(\epsilon^\mu\partial_\mu\phi\right)\partial_V^{n+1}\phi\\
=&\sum_{k=0}^{n+1}\binom{n+1}{k}\left(\partial_U^k\epsilon^\mu\right)\left(\partial_\mu\partial_U^{n+1-k}\phi\right)\partial_V^{n+1}\phi\\
=&\sum_{k=1}^{n+1}\binom{n+1}{k}\left(\partial_U^k\epsilon^\mu\right)\left(\partial_\mu\partial_U^{n+1-k}\phi\right)\partial_V^{n+1}\phi+\epsilon_\mu\left(\partial^\mu\partial_U^{n+1}\phi\right)\partial_V^{n+1}\phi\\
=&\underbrace{\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}\epsilon^\mu\partial_U^k\left(\partial_\mu\partial_U^{n+1-k}\phi\partial_V^{n+1}\phi\right)}_{a_1}+\underbrace{\epsilon_\mu\left(\partial^\mu\partial_U^{n+1}\phi\right)\partial_V^{n+1}\phi}_{a_2}+(\text{Boundary Terms})
\end{aligned} \tag{A.3}$$

We prefer to write a_1 term into the form $\epsilon^\mu\partial_U(\cdots)$. To achieve it, we let ∂_U^k act on the rest of terms and sum them together:

$$a_1=\epsilon^\mu\partial_U\left(\sum_{l=0}^{n}F_l^n\partial^\mu\partial_U^{n-l}\phi\partial_V^{n+1}\partial_U^l\phi\right) \tag{A.4}$$

where the coefficient

$$F_l^n=(-1)^{l+1}\sum_{m=0}^{n-l}(-1)^m\binom{n+1}{l+m+1}\binom{m+l}{m}=(-1)^{l+1} \tag{A.5}$$

which is independent of n. The above identity can be shown by Mathmatica.

The result of b term can be obtained by shifting $U\leftrightarrow V$. As a result, we get the variation of the action:

$$\begin{aligned}
\delta S=-\int dUdV\,\epsilon^\mu&\left[\partial_U\left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}(-1)^{n+l}\frac{(4\ell^2)^n}{n!}\partial_\mu\partial_U^{n-l}\phi\partial_V^{n+1}\partial_U^l\phi\right)+(U\leftrightarrow V)\right]\\
&+\epsilon^\mu\partial_\nu\left(\sum_{n=0}^{\infty}(-1)^n\frac{(4\ell^2)^n}{n!}\delta_\mu^\nu\partial_U^{n+1}\phi\partial_V^{n+1}\phi\right)
\end{aligned} \tag{A.6}$$

From this, according to eq(3.7), we can see the stress tensor is eq(3.15).

B Derivation of eq(4.26)

We first calculate

$$
\begin{aligned}
&\int dV\ \Theta\left(|V-z-m\tau_\Omega|-\frac{\tau_\Omega}{2}\right)e^{-i\bar{\Omega}V}\\
=&e^{-i\bar{\Omega}z}\int dV\ \Theta\left(|V-m\tau_\Omega|-\frac{\tau_\Omega}{2}\right)e^{-i\bar{\Omega}V}\\
=&e^{-i\bar{\Omega}z}\left(\int dV\ \Theta\left(V-(m+\frac{1}{2})\tau_\Omega\right)e^{-i(\bar{\Omega}-i\epsilon)V}+\int dV\ \Theta\left((m-\frac{1}{2})\tau_\Omega-V\right)e^{-i(\bar{\Omega}+i\epsilon)V}\right)\\
=&e^{-i\bar{\Omega}z}\left(-2e^{-im\tau_\Omega\bar{\Omega}}\frac{\sin\left(\tau_\Omega\bar{\Omega}/2\right)}{\bar{\Omega}}+2\pi\delta(\bar{\Omega})\right)
\end{aligned}
\tag{B.1}
$$

where we've introduced the infinitesimal positive number ϵ to make the integral converge and used the identity $\frac{1}{x\pm i\epsilon}=\mathcal{P}\frac{1}{x}\mp i\pi\delta(x)$. Recall the identity of the Dirac comb:

$$
\sum_m e^{im\tau_\Omega\bar{\Omega}}=\frac{2\pi}{\tau_\Omega}\sum_m\delta\left(\bar{\Omega}-\frac{2\pi m}{\tau_\Omega}\right)
\tag{B.2}
$$

Now, as we sum over m:

$$
\begin{aligned}
&\int dV\ \sum_m\Theta\left(|V-z-m\tau_\Omega|-\frac{\tau_\Omega}{2}\right)e^{-i\bar{\Omega}V}\\
=&e^{-i\bar{\Omega}z}\sum_m\left(-2e^{-im\tau_\Omega\bar{\Omega}}\frac{\sin\left(\tau_\Omega\bar{\Omega}/2\right)}{\bar{\Omega}}+2\pi\delta(\bar{\Omega})\right)\\
=&e^{-i\bar{\Omega}z}\left(-\frac{4\pi}{\tau_\Omega}\sum_m\delta\left(\bar{\Omega}-\frac{2\pi m}{\tau_\Omega}\right)\frac{\sin\left(\tau_\Omega\bar{\Omega}/2\right)}{\bar{\Omega}}+2\pi\delta(\bar{\Omega})\sum_m 1\right)\\
=&2\pi e^{-i\bar{\Omega}z}\delta(\bar{\Omega})\left(-1+\sum_m 1\right)\\
=&2\pi\delta(\bar{\Omega})\left(-1+\sum_m 1\right)
\end{aligned}
\tag{B.3}
$$

If we use Riemann zeta function to regularize $\sum_m\to\zeta(0)=-\frac{1}{2}$, we obtain $-3\pi\delta(\bar{\Omega})$.