Limited Information Strategies for Topological Games

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Abstract

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- In a topological game, a property of the topological space being played upon is characterized by whether one player or another has a "winning strategy", a strategy which cannot be countered by any possible play by the opponent.
- The presenter's research involves investigating several topological games from the literature for properties characterized by the existence of winning "limited information" strategies, such as Markov strategies which only require knowledge of the round number and only the most recent move of the opponent.

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What is Topology?

- Topology is, put simply, the study of mathematical "spaces".
- Most of us have learned about the (usual) topology of the real line and the *xy*-plane in calculus.
- Topology is chiefly concerned with the "structure" of mathematical spaces. An example of a topological observation is that removing a point from the real line splits it into two pieces, while removing a point from the real plane does not.
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Why study Topology?

- One of the primary uses of topology is as a toolkit for other mathematicians. Topological facts are often cited within proofs in other mathematical fields.
- TODO: Consider a proof of the Fundamental Theorem of Algebra?
- However, topology is also emerging as powerful tool in data analysis.
  - A data analysist is given a finite number of data points: ordered lists of numbers (n-dimensional vectors).
  - These points may be embedded in the Euclidean topological space $\mathbb{R}^n$: by “fattening” them up, we gain insight as to the structure of the source of the data.
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- For this talk, I'll stick with one familiar topological space and one (most likely) unfamiliar one.
- When I'm talking about the **xy-plane**, I'm referring to the usual space of ordered pairs of real numbers from calculus. TODO: Add picture.
- We'll also use another example of a topological space, known by set-theoretic topologists as the "sequential fan". However, let's just call it the **Milky Way space**. TODO: Add picture

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What's Game Theory?

- Game theory is a powerful tool of use to anyone interested in the study of strategic decision-making: economists, biologists, logicians, political scientists...
- Within game theory, there are two main types of two-player games: **simultaneous** and **sequential**.
- Simultaneous games include the famous **Prisoner's Dilemma**: should a prisoner testify against his partner in exchange for a light sentence, knowing that his partner is simultaneously given the same option? TODO: Add picture

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- However, my research is concerned with **sequential games.** Tic-tac-toe and Chess are handy examples we're all familiar with. TODO: Add pictures
- Mathematically, we can model sequential games by tracking the decisions made by each player during each round. TODO: Add gameplay records for Tic-tac-toe and Chess.

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And what's this about Infinite-Length Games?

- A good game designer would avoid this, but mathematically we can consider games which aren't required to terminate in a victory for either player.
- In that case, the game never ends, but as long as the players involved have a gameplan, we can consider the result of them sticking to their gameplan: the sequence of choices made by each player. TODO: Add choices for tic-tac-toe on the integer lattice.
- If the game doesn't end, we'll have a rule to judge how each player did throughout the game, and declare a winner that way.

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- Example game: Player I and Player II take turns picking positive integers 2 - 9. A player wins as soon as if the product of all chosen numbers equals a multiple of 18. If the game never ends, Player I wins as long as she chose 9 at least once during the game; otherwise Player II wins. TODO: Have picture of two players picking numbers.

- While it's easy to imagine this game never ending (both players always picking 5 would do it), we can say that Player II has a **winning strategy**:
    - Player I can't play any number besides 5 or 7 unless it results in a multiple of 18 - otherwise Player II can make the multiple of 18 on the next turn.
    - If Player II always plays 7 in response to 5 or 7 being played by Player I, then Player I can never make a multiple of 18 on her own.
- Thus one winning strategy for Player II is to always respond with 7 if Player I chooses 5 or 7, and to pick an appopriate number to make a multiple of 18 otherwise.
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So it's sort of like boxing...

- You can think of these games like a **boxing match** with infinite rounds. TODO: Add Punchout screenshot
- Each boxer takes turns swinging at each other. If the swing knocks the other guy out, that's a Win by KO.
- If neither boxer manages to KO the other, then we turn to the judges to evaluate based on how they played throughout the entire game. One of the players must get a Win by Decision.

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Topological Darts in the xy-plane

- Consider a game of "Topological Darts" played in the xy-plane. During each round:
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  - Player D responds by throwing a **D**art at the dartboard (aka picking a point on the dartboard). TODO add picture

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Player B has a winning strategy for Topological Darts when played in the xy-plane. TODO: Add picture.

Topological Darts in the Milky Way Space

- The interior of the circular dartboards in the xy-plane represent topological objects known as **open sets.** What these open sets look like depend on the topological space.
- In our so-called Milky Way Space, the dartboards / open sets placed around the point ∞ look like this: TODO Add picture

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Player D has a winning strategy for Topological Darts when played in the Milky Way Space. TODO: Add picture

So what?

- We care about these games because they provide very slick ways of describing possible structures of a topological space.
  - A topological space X is an "α_2 Fréchet-Urysohn" space if for each subset A of X, and each point $x \in \overline{A}$, there exists a sequence of points in A converging to x, and for each countable collection of sequences coverging to x, there is yet another sequence converging to x which intersects each of these infinitely many times.

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- It's much simpler to say this:
  - A topological space X is an "α_2 Fréchet-Urysohn" space if Player D has no winning strategy in a game of Topological Darts played in X.
- So we know the *xy*-plane is "α_2 Fréchet-Urysohn" (we found a winning strategy for Player B, so Player D doesn't have one), but the Milky Way Space isn't (we found a winning strategy for Player D).

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Consquences of Limited Information

- My reseach is concerned with the consquences of one player having limited information in a topological game.
- So far we've assumed both players have perfect memories. But what happens if a player can only remember the most recent move of her opponent?

In the *xy*-plane, this is of no consequence to player B. TODO add picture.

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But in the Milky Way Space, this lack of information hurts Player D. TODO add picture.

- So even though Player D has a winning perfect information strategy, Player D lacks a winning "tactical" strategy relying on the most recent move of the opponent.

- The precense or absence of perfect information strategies in a topological game characterize some structure of the space played upon - same goes for limited information strategies.
  - A ("countably-tight", "locally-compact") topological space X is **"meta-Lindelöf"** when Player B has a winning strategy for Topological Darts played in the "one-point compactification" of X.
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So what have you done?

asdf

Thank you!

Any questions?