Limited Information Strategies for Topological Games

GSC Scholars' Forum / AU Research Week 2013

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http://www.stevenclontz.com/aurw-2013/

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Abstract

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- In a topological game, a property of the topological space being played upon is characterized by whether one player or another has a “winning strategy”, a strategy which cannot be countered by any possible play by the opponent.
- The presenter’s research involves investigating several topological games from the literature for properties characterized by the existence of winning “limited information” strategies.

Might want some background first...

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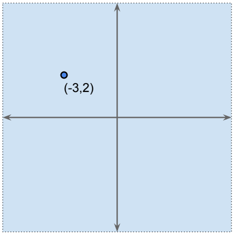
What is Topology?

- Topology is, put simply, the study of mathematical "spaces".
- Most of us have learned about the (usual) topology of the real line and the xy-plane in calculus.



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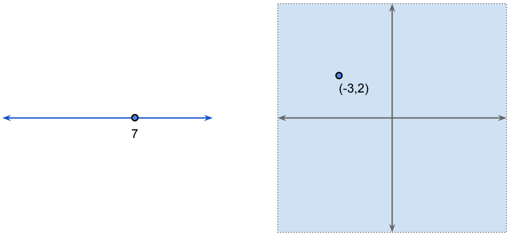
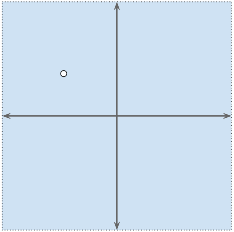


- Topology is chiefly concerned with the "structure" of mathematical spaces.
- A simple example of a topological observation is that removing a point from the real line splits it into two separated pieces, while removing a point from the real plane does not.



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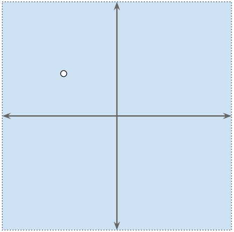


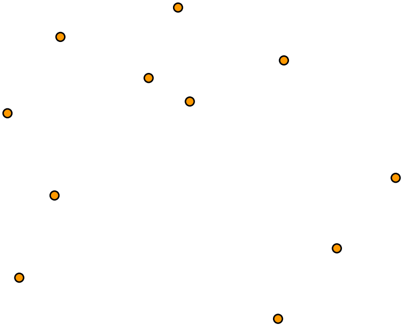
Why study Topology?

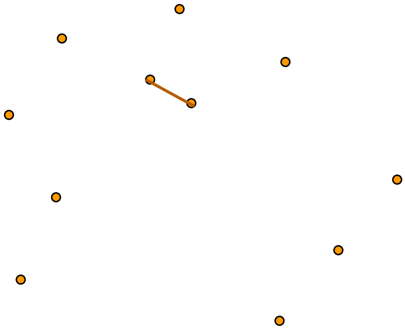
- One of the primary uses of topology is as a toolkit for other mathematicians. Topological facts are often cited within proofs in other mathematical fields.

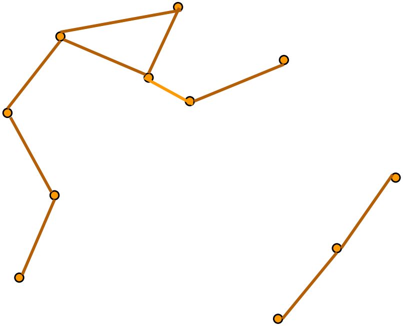
Fundamental Theorem of Algebra: Every polynomial $p(z)$ of degree n has at least one complex root.
Trick for Proof: Compare the topological winding numbers of the curves $q_0(t) = p(0) \neq 0$ and $q_n(t) = p(Re^{2\pi it}) \neq 0$ around the origin $0 + 0i$.

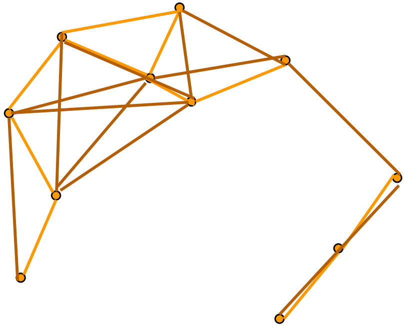
- However, topology is also emerging as powerful tool in data analysis.
  - A data analysist is given a finite number of data points: ordered lists of numbers (n-dimensional vectors).
  - These points may be embedded in the Euclidean topological space $\mathbb{R}^n$: by defining a tolerance, you can "connect the dots" to get a collection of simplices approximating the data source.

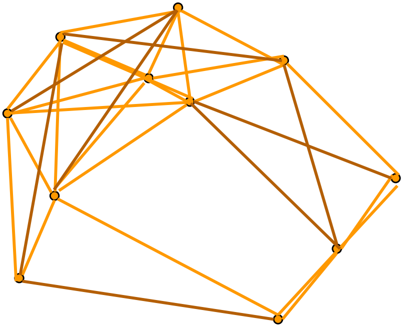
See: <http://bit.ly/wVejrq>

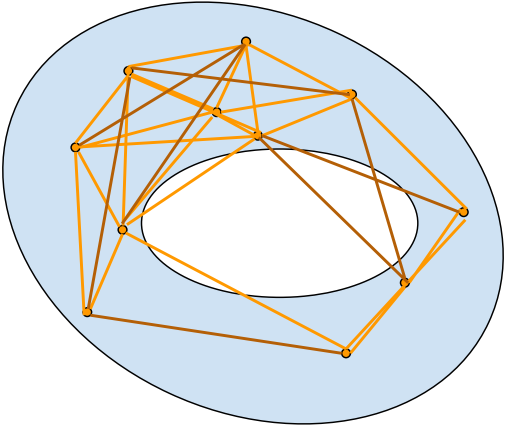












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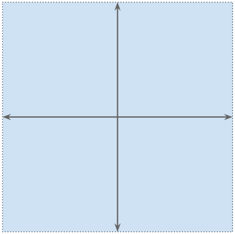
- When I'm talking about the **xy-plane**, I'm referring to the usual space of ordered pairs of real numbers from calculus.



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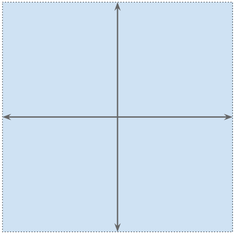
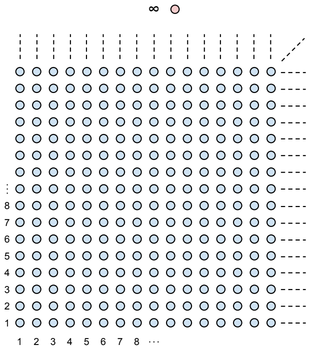


- We'll also use another example of a topological space, which I'll call the **Milky Way space.**



See *Katsuya Eda et al, Sequential fans in topology* <http://bit.ly/X6WpQm>

What's Game Theory?

- Game theory is a powerful tool of use to anyone interested in the study of strategic decision-making: economists, biologists, logicians, political scientists...
- Within game theory, there are two main types of two-player games: **simultaneous** and **sequential.**

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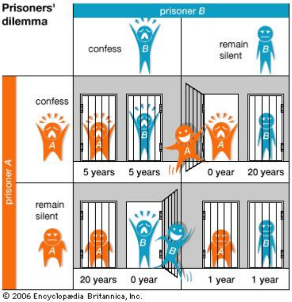
- Within game theory, there are two main types of two-player games: **simultaneous** and **sequential**.

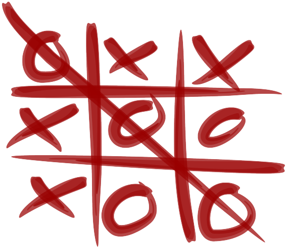
- Simultaneous games include the famous **Prisoner's Dilemma**: should a prisoner testify against his partner in exchange for a light sentence, knowing that his partner is simultaneously given the same option?



See: <http://bit.ly/XaPSCR>

What's a Sequential Game?

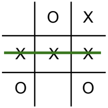
- However, my research is concerned with **sequential games**. Tic-tac-toe and Chess are handy examples we're all familiar with.



- Mathematically, we can model sequential games by tracking the decisions made by each player during each round.

Round	Player X	Player O
1	center	top
2	top-right	bottom-left
3	right	bottom-right
4	left | WIN	

Round	White	Black
1	e4	e5
2	Qh5	Nc6
3	Bc4	Nf6
4	Qxf7#	



And what's this about Infinite-Length Games?

- A good game designer would avoid this, but mathematically we can consider games which aren't required to terminate in a victory for either player after finitely-many moves.
- In that case, the game never ends, but as long as the players involved have a gameplan, we can consider the result of them sticking to their gameplan: the sequence of choices made by each player.
- If the game doesn't end, we'll have a rule to judge how each player did throughout the game, and declare a winner that way.

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Topological Games

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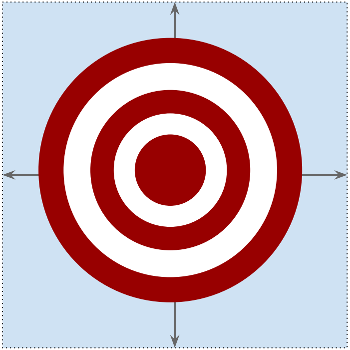
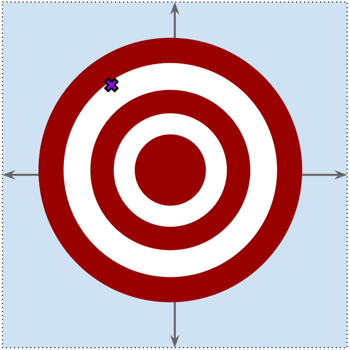
Topological Darts in the xy-plane

- Consider a game of "Topological Darts" played in the xy-plane.

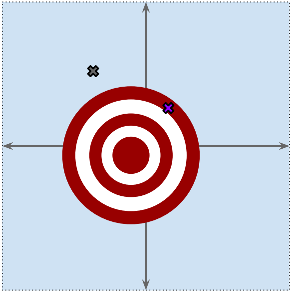
Player O places a circular dartboard ("**O**") of any size on the plane so that it covers the point $(0, 0)$.

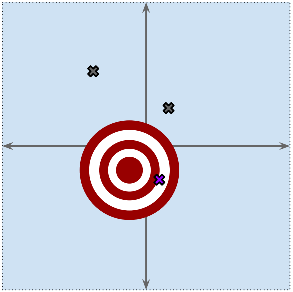


Player P responds by throwing a **P**ointy dart at the dartboard (picks a point on the plane within the dartboard).



The game continues like this for every round.





- Player O automatically wins if Player P ever misses the dartboard.
- Of course, if Player P never misses the dartboard, the game never ends!

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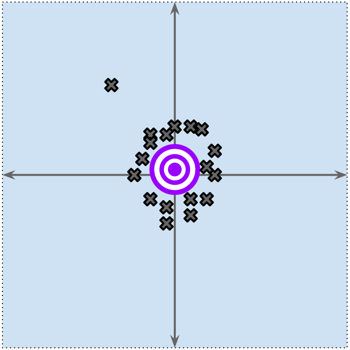
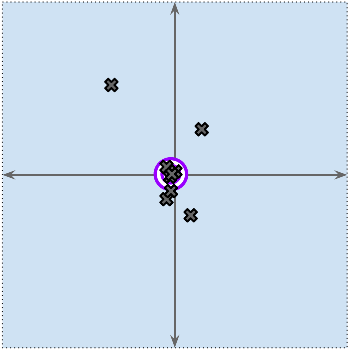
In that case, if Player P can show a new dartboard covering $(0, 0)$ that misses infintely many of the thrown darts, she wins.



Otherwise, Player O is the victor. (Since the points **converge** towards the origin, a dartboard of any size contains all but the first few:)



- Player O has a **winning (unbeatable) strategy** for Topological Darts when played in the *xy*-plane:

Round	Board Picker (O)	Dart Thrower (P)	Dist. from $(0,0)$
1	$x^2+y^2<1$	(x_1,y_1)	<1
2	$x^2+y^2<\frac{1}{4}$	(x_2,y_2)	$<\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$x^2+y^2<\frac{1}{n^2}$	(x_n,y_n)	$<\frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Topological Darts in the Milky Way Space

- Of course, these darts and dartboards are just some glitter and macaroni covering what's really going on mathematically.
- The interior of the circular dartboards in the *xy*-plane represent topological objects known as **open sets**. What these open sets look like depend on the topological space.

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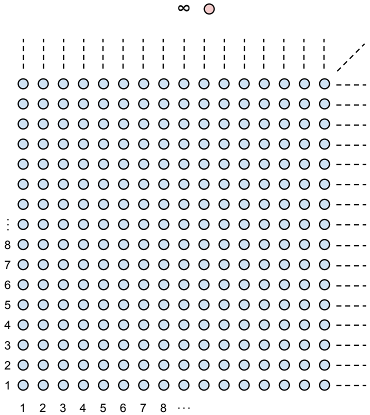
In our so-called Milky Way Space, the dartboards / open sets placed around the point ∞ look like



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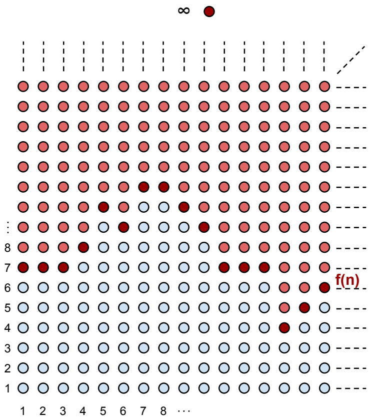
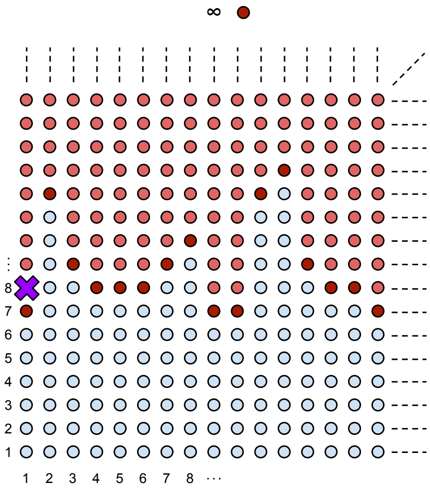


Player P has a **winning strategy** for Topological Darts when played in the Milky Way Space.

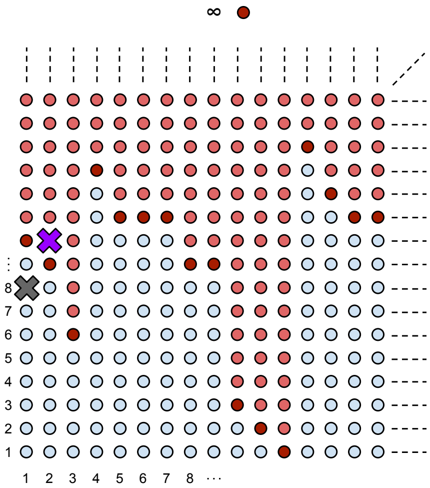
Round	Board Picker (O)	Dart Thrower (P)
1	f_1	$(1, f_1(1) + 1)$
2	f_2	$(2, f_2(2) + 1)$
$\vdots$	$\vdots$	$\vdots$
n	f_n	$(n, f_n(n) + 1)$
$\vdots$	$\vdots$	$\vdots$

The dartboard given by $g(n) = f_n(n) + 3$ misses all points played by P.

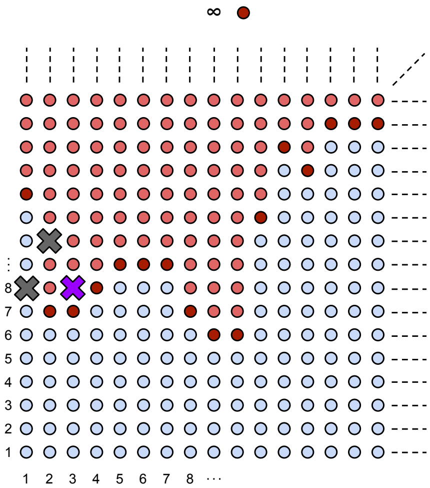
Round 1



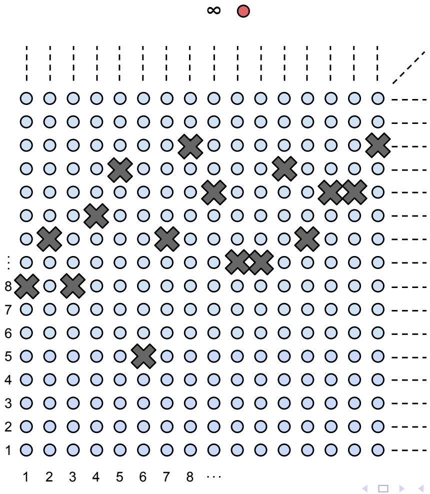
Round 2



Round 3



After the game...



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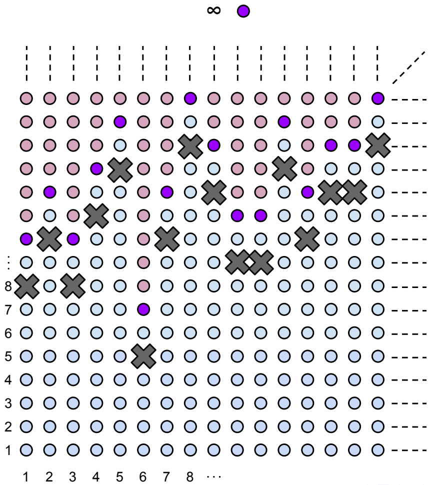

So what?

- We care about these games because they provide very slick ways of describing possible structures of a topological space.
  - A topological space X is an "α_2 Fréchet-Urysohn" space if for each subset A of X, and each point $x \in \overline{A}$, there exists a sequence of points in A converging to x, and for each countable collection of sequences coverging to x, there is yet another sequence converging to x which intersects each of these infinitely many times.

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- It's much simpler to say this:
  - A topological space X is an "α_2 Fréchet-Urysohn" space if and only if Player P cannot find a winning strategy in a game of Topological Darts played in X.
- So we know the *xy*-plane is "α_2 Fréchet-Urysohn" (we found a winning strategy for Player O, so Player P doesn't have one), but the Milky Way Space isn't (we found a winning strategy for Player P).

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Consquences of Limited Information

- So far we've assumed both players have perfect memories. But what happens if a player can only remember (for example) the most recent move of her opponent?
- My reseach is concerned with the consquences of one player having this sort of "limited information" in a topological game.

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In the *xy*-plane, Player O has a **tactical** strategy for Topological Darts which relies on only the most recent move of her opponent:

- During round (?), Player O sees that Player P has just thrown a dart at $(x_?, y_?)$.
- Although Player O doesn't know anything else about what's happened during the game, she places a dartboard with center $(0, 0)$ and radius $\frac{\sqrt{x_?^2+y_?^2}}{2}$.

Here's how it plays out:

	Player O	Player P	Dist. from $(0,0)$
1	$x^2+y^2<1$	(x_1,y_1)	<1
2	$x^2+y^2<\frac{x_1^2+y_1^2}{4}$	(x_2,y_2)	$<\frac{\sqrt{x_1^2+y_1^2}}{2}<\frac{1}{2}$
3	$x^2+y^2<\frac{x_2^2+y_2^2}{4}$	(x_3,y_3)	$<\frac{\sqrt{x_2^2+y_2^2}}{2}<\frac{1}{2}\frac{1}{2}=\frac{1}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$x^2+y^2<\frac{x_{n-1}^2+y_{n-1}^2}{4}$	(x_n,y_n)	$<\frac{\sqrt{x_{n-1}^2+y_{n-1}^2}}{2}<\frac{1}{2}\frac{1}{2^{n-1}}=\frac{1}{2^n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

But in the Milky Way Space, Player P does *not* have a winning tactical strategy, *even though she has a unbeatable perfect information strategy*.

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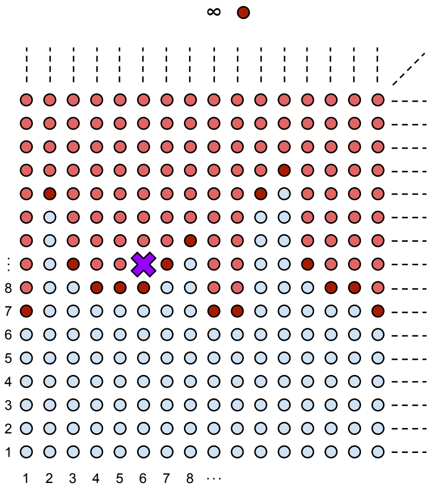
Round ?



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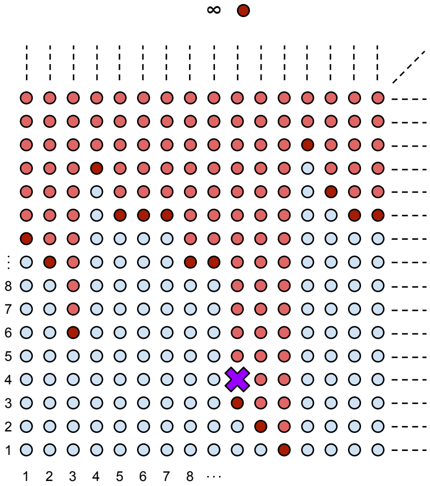
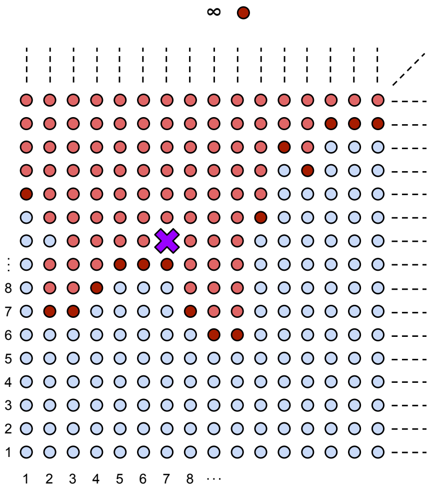


Round ?#!?@



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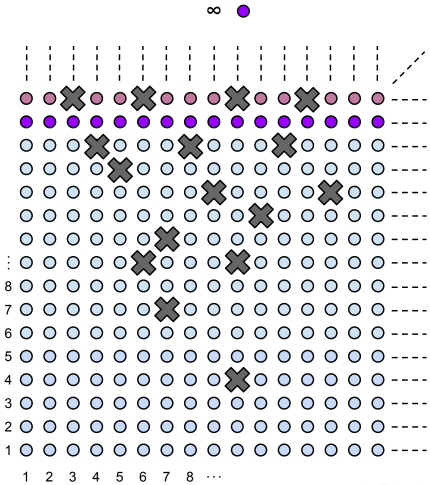

- The presence or absence of perfect information strategies in a topological game characterize some structure of the space played upon - same goes for limited information strategies.

Assume all spaces are countably-tight, locally-compact. X^* is the one-point compactification of X.

- A topological space X is **metaLindelöf** if and only if Player O has a winning strategy for Topological Darts played in X^*.
- A topological space X is **metacompact** if and only if Player O has a winning **tactical** strategy for Topological Darts played in X^*.

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So what have you done?

My main questions:

- "How can we strengthen existing results from the literature concerning topological games?"
- "What common topological properties are characterized by the existance or absence of limited information strategies?"

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Gary Gruenhage (Auburn) has shown that, for locally compact spaces,

$$X \text{ is metacompact} \Leftrightarrow O \uparrow_{\text{tactic}} Con_{O,P}(X^*, \infty)$$

$Con_{O,P}(X, x)$ is Topological Darts.

- I have shown further that

$$X \text{ is metacompact} \Leftrightarrow O \uparrow_{\text{tactic}} Clus_{O,P}(X^*, \infty)$$

- I've also proven there exists a space which suggests the following conjecture:

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Thank you!

- Thanks goes to my advisor Gary Gruenhage, and regular attendees of the set-theoretic topology seminar for their feedback and support.
- A BIG thank-you goes to the Graduate Student Council for organizing the Graduate Scholars' Forum and assisting with AU Research Week.
- And most of all, thanks to the volunteer judges for taking time out of their busy schedules to support graduate student research at Auburn!

Any questions?

Full presentation available on: <http://www.stevenclontz.com/AURW2013/>

Example of a nontopological infinite game

Postscript: Here's a simple example of a nontopological infinite-length game.

- Example game: Player I and Player II take turns picking positive integers 2 - 9. A player wins as soon as if the product of all chosen numbers equals a multiple of 18. If the game never ends, Player I wins as long as she chose 9 at least once during the game; otherwise Player II wins.

- While it's easy to imagine this game never ending (both players always picking 5 would do it), we can say that Player II has a **winning strategy**:
  - Player I can't play any number besides 5 or 7 unless it results in a multiple of 18 - otherwise Player II can make the multiple of 18 on the next turn.
  - If Player II always plays 7 in response to 5 or 7 being played by Player I, then Player I can never make a multiple of 18 on her own.
- Thus one winning strategy for Player II is to always respond with 7 if Player I chooses 5 or 7, and to pick an appopriate number to make a multiple of 18 otherwise.
  - The result of any game where Player II sticks to this strategy either involves Player II making a multiple of 18, or Player I never choosing 9!

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