Chapter 1

Rigid Body Equilibrium

1.1 Introduction

In this chapter we will investigate the equilibrium of simple rigid bodies like your book, phone, or pencil. The important difference between rigid bodies and the particles of `[cross-reference to target(s) "Chapter_03" missing or not unique]` is that rigid bodies have the potential to rotate around a point or axis, while particles do not.

For rigid body equilibrium, we need to maintain translational equilibrium with

$$\sum \mathbf{F} = 0 \tag{1.1.1}$$

and also maintain a balance of rotational forces and couple-moments with a new equilibrium equation

$$\sum \mathbf{M} = 0. \tag{1.1.2}$$

1.2 Degree of Freedom

Degrees of freedom refers to the number of independent parameters or values required to specify the *state* of an object.

The *state* of a *particle* is completely specified by its location in space, while the state of a rigid body includes its location in space and also its orientation.

Two-dimensional rigid bodies in the xy plane have three degrees of freedom. Position can be characterized by the x and y coordinates of a point on the object, and orientation by angle θ_z about an axis perpendicular to the plane. The complete movement of the body can be defined by two linear displacements Δx and Δy, and one angular displacement $\Delta\theta_z$.

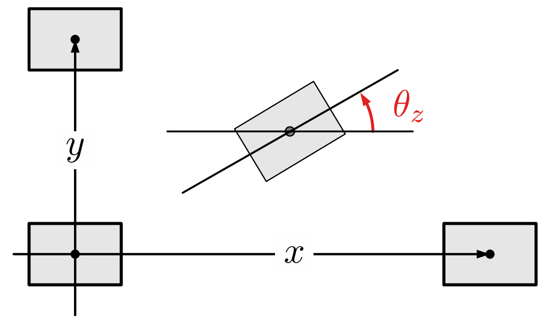


Figure 1.2.1 Two-dimensional rigid bodies have three degrees of freedom.

Three-dimensional rigid bodies have six degrees of freedom, which can be specified with three orthogonal coordinates x, y and z, and three angles of rotation, θ_x, θ_y and θ_z. Movement of the body is defined by three translations Δx, Δy and Δz, and three rotations $\Delta\theta_x$, $\Delta\theta_y$ and θ_z.

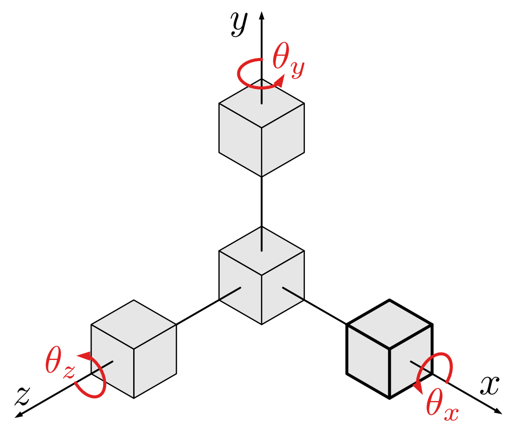


Figure 1.2.2 Three-dimensional rigid body have six degrees of freedom - three translations and three rotations.

For a body to be in static equilibrium, all possible movements of the body need to be adequately restrained. If a degree of freedom is not restrained, the body is in an unstable state, free to move in one or more ways. Stability is highly desirable for reasons of human safety, and bodies are often restrained by *redundant* restraints so that if one were to fail, the body would still remain stable. If the restraints correctly interpreted, then equal constraints and degrees of freedom create a stable system, and the values of the reaction forces and moments can be determined using equilibrium equations. If the number of restraints exceeds the number of degrees of freedom, the body is in equilibrium but you will need techniques we won't cover in statics to determine the reactions.

1.3 Free Body Diagrams

Key Questions

- What are the five steps to create a free-body diagram?
- What are degrees of freedom, and how do they relate to stability?
- Which reaction forces and couple-moments come from each support type?
- What are the typical support force components and couple-moment components which can be modeled from the various types of supports?

Free body diagrams are the tool that engineers use to identify the forces and moments that influence an object. They will be used extensively in statics, and you will use them again in other engineering courses so your effort to master them now is worthwhile. Although the concept is simple, students often have great difficulty with them.

Drawing a correct free-body diagram is the first and most important step in the process of solving an equilibrium problem. It is the basis for all the equilibrium equations you will write; if your free-body diagram is incorrect then your equations, analysis, and solutions will be wrong as well.

A good free-body diagram is neat and clearly drawn and contains all the information necessary to solve the equilibrium. You should take your time and think carefully about the free body diagram before you begin to write and solve equations. A straightedge, protractor and colored pencils all can help. You will inevitably make mistakes that will lead to confusion or incorrect answers; you are also encouraged to think about these errors and identify any misunderstandings to avoid them in the future.

Every equilibrium problem begins by drawing and labeling a free-body diagram!

Creating Free Body Diagrams. The basic process for drawing a free body diagrams is

1. *Select and isolate an object.*

   The "free-body" in free-body diagram means that the body to be analyzed must be free from the supports that are physically holding it in place

   Simply sketch a quick outline of the object as if it is floating in space disconnected from everything. Do *not* draw free-body diagram forces on top of your problem drawing — the body needs to be drawn free of its supports.

2. *Select a reference frame.*

   Select a right-handed coordinate system to use as a reference for your equilibrium equations. If you are using something other than a horizontal x axis and vertical y axis, indicate it on your diagram.

Look ahead and select a coordinate system which minimizes the number of unknown force components in your equations. The choice is technically arbitrary, but a good choice will simplify your calculations and reduce your effort. If you and another student pick different reference systems, you should both get the same answer, while expressing your work with different components.

3. *Identify all loads.*

   Add vectors arrows representing the applied forces and couple-moments acting on the body. These are often obvious. Include the body's weight if it is non-negligible. If a vector has a known line of action, draw the arrow in that direction; if its sense is unknown, assume one. Every vector should have a descriptive variable name and a clear arrowhead indicating its direction.

4. *Identify all reactions.*

   Traverse the perimeter of the object and wherever a support was removed when isolating the body, replace it with the forces and/or couple-moments which it provides. Label each reaction with a descriptive variable name and a clear arrowhead. Again, if a vector's direction is unknown just assume one.

   The reaction forces and moments provided by common two-dimensional supports are shown in Figure **??** and three dimensional support in Figure **??**. Identifying the correct reaction forces and couple-moments coming from supports is perhaps the most challenging step in the entire equilibrium process.

5. *Label the diagram.*

   Verify that every dimension, angle, force, and moment is labeled with either a value or a symbolic name if the value is unknown. Supply the information needed for your calculations, but don't clutter the diagram up with unneeded information; This diagram should be a "stand-alone" presentation.

Drawing good free-body diagrams is surprisingly tricky and requires practice. Study the examples, think hard about them, do lots of problems, and learn from your errors.

Two-dimensional Reactions. Supports supply reaction forces and moment which prevent bodies from moving when loaded. In the most basic terms, forces prevent translation, and moments prevent rotation.

The reactions supplied by a support depend on the nature of the particular support. For example in a top view, a door hinge allows the door to rotate freely but prevents it from translating. We model this as a frictionless pin that supplies a perpendicular pair of reaction forces, but no reaction moment. We can evaluate all the other physical supports in a similar way to come up with the table below. You will notice that some two-dimensional supports only restrain one degree of freedom and others restrain up to three degrees of freedom. The number of degrees of freedom directly correlates to the number of unknowns created by the support.

The table below shows typical two-dimensional support methods and the corresponding reaction forces and moments supplied each.

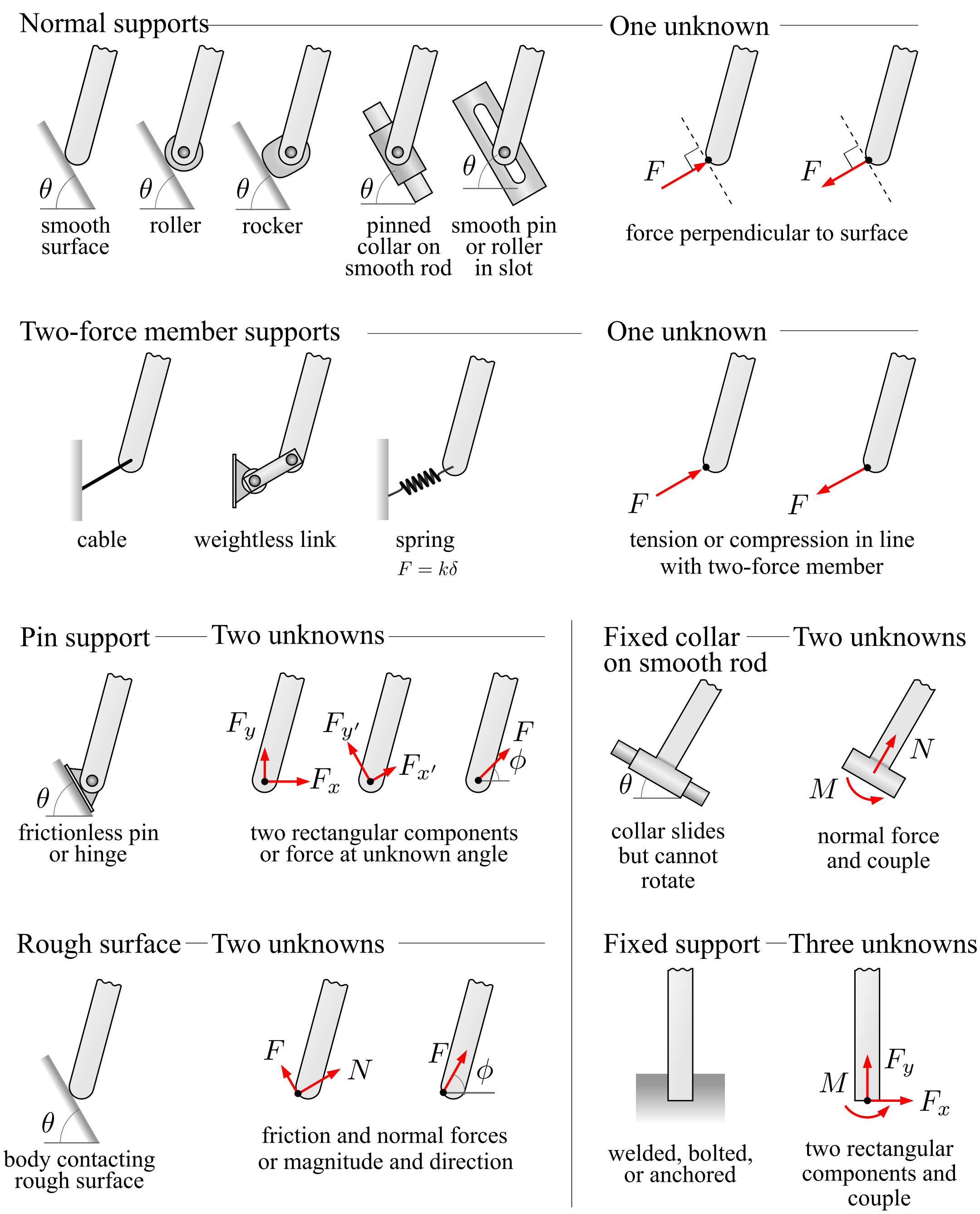


Figure 1.3.1 Table of common two-dimensional supports and their representation on free body diagrams.

Three-dimensional Reactions. The main added complexity with three-dimensional objects is that there are more possible ways the the object can move, and also more possible ways to restrain it. The table below show the types of supports which are available and the corresponding reaction forces and moment. As before, your free-body digrams should show the reactions supplied by the constraints, not the constraints themselves.

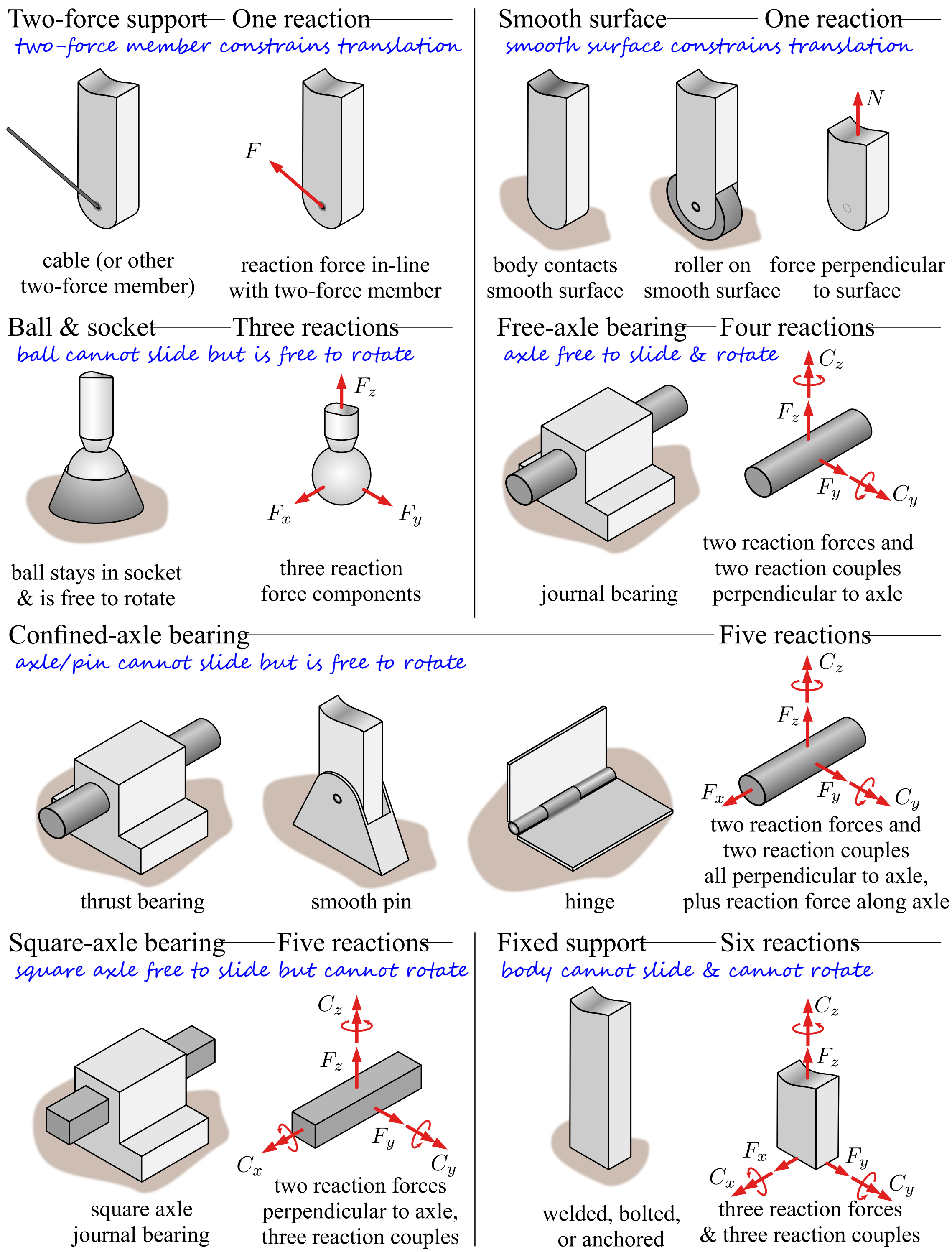


Figure 1.3.2 Table of common three-dimensional supports and their associated reactions.

One new issue we face in three-dimensional problems is that reaction couples may be **available** but not **engaged**.

A support which provides a non-zero reaction is said to be engaged. Picture a crate sitting at rest on a horizontal surface with a cable attached to the top of the crate. If the cable is slack, the reaction of the cable would be available but not *engaged*. Instead, the floor would be supporting the full weight of the crate. If we were to remove the floor, the cable would be engaged and support the weight of the crate.

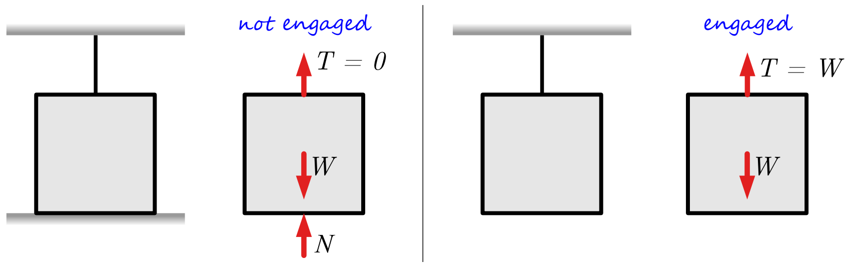


Figure 1.3.3 Available and Engaged reactions.

To get a feel for how reaction couples engage, pick up your laptop or a heavy book and hold it horizontally with your left hand. Can you feel your hand supplying an upward force to support the weight *and* a counter-clockwise reaction couple to keep it horizontal? Now add a similar support by gripping with your right hand. How do the forces and couple-moments change? You should have felt the force of your left hand decrease as your right hand picked up half the weight, and also noticed that the reaction couple from your left hand was no longer needed.

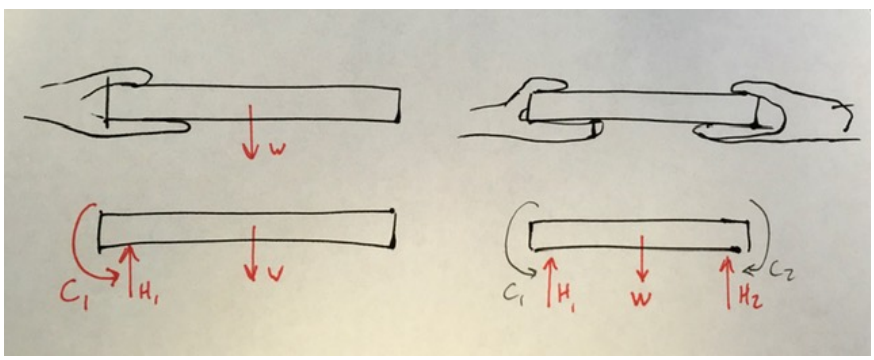


Figure 1.3.4 One hand holding an object versus two hands holding the same object.

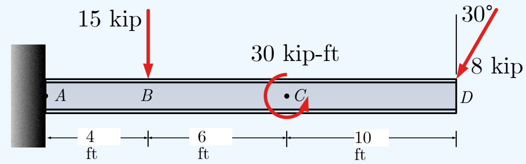
The vertical force in your right hand engaged instead of the couple-moment of your left hand. The reaction couples from both hands are available, but the vertical forces engage first and are sufficient for equilibrium. This phenomena is described by the saying "reaction forces engage before reaction couple-moments".

Free Body Diagram Examples. Given that there several options for representing reaction forces and couple-moments from a support, there are different, equally valid options for drawing free-body diagrams. With experience you will learn which representation to choose to simplify the equilibrium calculations.

Possible free-body diagrams for two common situations are shown in the next two examples.

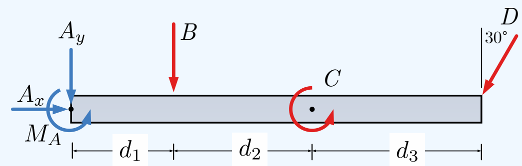
Example 1.3.5 Fixed support.

The cantilevered beam is embedded into a fixed vertical wall at A. Draw a neat, labeled, correct free-body diagram of the beam and identify the knowns and the unknowns.



Solution.

Begin by drawing a neat rectangle to represent the beam disconnected from its supports, then add all the known forces and couple-moments. Label the magnitudes of the loads and the known dimensions symbolically.



Choose the standard xy coordinate system, since it aligns well with the forces.

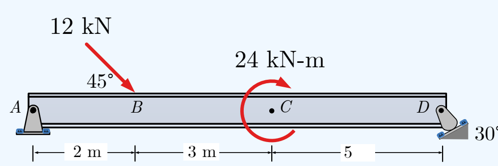
The wall at A is a fixed support which prevents the beam from translating up, down, left or right, or rotating in the plane of the page. These constraints are represented by two perpendicular forces and a concentrated moment, as shown in Figure **??**. Label these unknowns as well.

The knowns in this problem are the magnitudes and directions of moment **C**, forces **B**, and **D** and the dimensions of the beam. The unknowns are the two force components A_x and A_y and the scalar moment M_A caused by the fixed connection. If you prefer, you may represent force **A** as a force of unknown magnitude acting at an unknown direction. Whether you represent it as x and y components or as a magnitude and direction, there are two unknowns associated with force **A**.

The three unknown reactions can be found using the three independent equations of equilibrium we will discuss later in this chapter.

Example 1.3.6 Frictionless pin and roller.

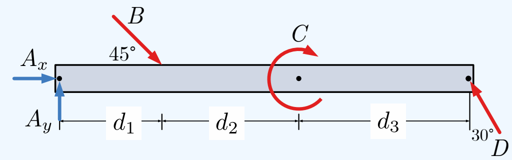
The beam is supported by a frictionless pin at A and a rocker at D. Draw a neat, labeled, correct free-body diagram of the beam and identify the knowns and the unknowns.



Solution. In this problem, the knowns are the magnitude and direction of force **B** and moment **C** and the dimensions of the beam.

The constraints are the frictionless pin at A and the rocker at D. The pin prevents translation but not rotation, which means two it has two unknowns, represented by either magnitude and direction, or by two orthogonal components. The rocker provides a force perpendicular to the surface it rests on, which is 30° from the horizontal. This means that the line of action of force **D** is 30° from the vertical, giving us its direction but not its sense or magnitude

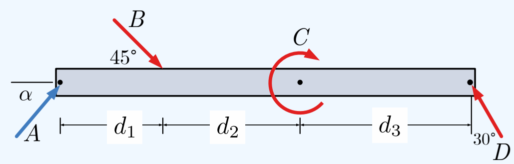
To draw the free body diagram, start with a neat rectangle to representing the beam disconnected from its supports, then draw and label known force B and moment C and the dimensions.



Add forces A_x and A_y representing vector **A** and force **D** at D, acting 30° from the vertical.

When a force has a known line of action as with force **D**, draw it acting along that line; don't break it into components. When it is not obvious which way a reaction force actually points along its lines of action, just make your best guess and place an arrowhead accordingly. Your calculations will confirm or refute your guess later.

As in the previous example, you could alternately represent force **A** as an unknown magnitude acting in an unknown direction, though there is no particular advantage to doing so in this case.



1.4 Equations of Equilibrium

Key Questions

- What is the definition of static equilibrium?
- How do I choose which are the most efficient equations to solve two-dimensional equilibrium problems?

In statics, our focus is on systems where both linear acceleration **a** and angular acceleration α are zero. These systems are frequently stationary, but could be moving with constant velocity.

Under these conditions `[cross-reference to target(s) "second-law" missing or not unique]` reduces to

$$\sum \mathbf{F} = 0, \tag{1.4.1}$$

and, `[cross-reference to target(s) "second-law-rotation" missing or not unique]` gives the similar equation

$$\sum \mathbf{M} = 0. \tag{1.4.2}$$

The first of these equations requires that all forces acting on an object balance and cancel each other out, and the second requires that all moments balance as well. Together, these two equations are the mathematical basis of this course and are sufficient to evaluate equilibrium for systems with up to six degrees of freedom.

These are vector equations; hidden within each are three independent scalar equations, one for each coordinate direction

$$\sum \mathbf{F} = 0 \implies \begin{cases} \Sigma F_x & = 0 \\ \Sigma F_y & = 0 \\ \Sigma F_z & = 0 \end{cases} \qquad \sum \mathbf{M} = 0 \implies \begin{cases} \Sigma M_x & = 0 \\ \Sigma M_y & = 0 \\ \Sigma M_z & = 0 \end{cases}. \tag{1.4.3}$$

Working with these scalar equations is often easier than using their vector equivalents, particularly in two-dimensional problems.

In many cases we do not need all six equations. We saw in `[cross-reference to target(s) "Chapter_03" missing or not unique]` that particle equilibrium problems can be solved using the force equilibrium equation alone, because particles have, at most, three degrees of freedom and are not subject to any rotation.

To analyze rigid bodies, which can rotate as well as translate, the moment equations are needed to address the additional degrees of freedom. Two-dimensional rigid bodies have only one degree of rotational freedom, so they can be solved using just one moment equilibrium equation, but to solve three-dimensional rigid bodies, which have six degrees of freedom, all three moment equations and all three force equations are required.

1.5 2D Rigid Body Equilibrium

Two-dimensional rigid bodies have three degrees of freedom, so they only require three independent equilibrium equations to solve. The six scalar equations of (**??**) can easily be reduced to three by eliminating the equations which refer to the unused z dimension. For objects in the xy plane there are no forces acting in the z direction to create moments about the x or y axes, so the reduced set of three equations is

$$\{1\} = \begin{cases} \sum F_x & = 0 \\ \sum F_y & = 0 \\ \sum M_A & = 0 \end{cases}.$$

where the subscript z has been replaced with a letter to indicate an arbitrary moment center in the xy plane instead of a perpendicular z axis.

This is not the only possible set of equilibrium equations. Either force equation can be replaced with a linearly independent moment equation about a point of your choosing [1], so the other possible sets are

$$\{2\} = \begin{cases} \sum F_x & = 0 \\ \sum M_B & = 0 \\ \sum M_A & = 0 \end{cases} \qquad \{3\} = \begin{cases} \sum M_C & = 0 \\ \sum F_y & = 0 \\ \sum M_A & = 0 \end{cases} \qquad \{4\} = \begin{cases} \sum M_C & = 0 \\ \sum M_B & = 0 \\ \sum M_A & = 0 \end{cases}.$$

For set four, moment centers A, B, and C must form a triangle to ensure the three equations are linearly independent.

You have a lot of flexibility when solving rigid-body equilibrium problems. In addition to choosing which set of equations to use, you are also free to rotate the coordinate system to any orientation you like, pick different points for moment centers, and solve the equations in any order or simultaneously.

This freedom raises several questions. Which equation set should you choose? Is one choice 'better' than another? Why bother rotating coordinate systems? How do you select moment centers? Students want to know "how to solve the problem," when in reality there are many ways to do it.

The actual task is to choose an efficient approach and carry it out. An efficient solution is one which avoids mathematical complications and makes the problem easy to solve. Complications include unpleasant geometries, unnecessary algebra, and particularly simultaneous equations, which are algebra intensive and error prone.

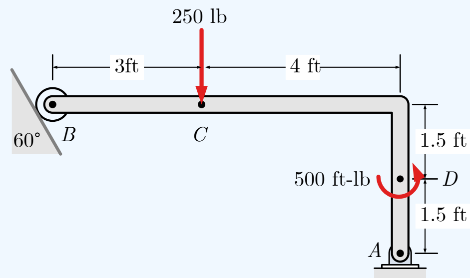
So how do you do set up an efficient approach? First, stop, think, and look for opportunities to make the solution more efficient. Here are some recommendations.

1. Equation set one is usually a good choice, and should be considered first.
2. Inspect your free-body diagram and identify the unknown values in the problem. These may be magnitudes, directions, angles or dimensions.
3. Align the coordinate system with at least one unknown force.
4. Take moments about the point where the lines of action of two unknown forces intersect, which eliminates them from the equation.
5. Solve equations with one unknown first.

[1] Labels A, B and C in these equations are representative. They don't have to correspond to points A, B and C on your problem.

Example 1.5.1 Pin and Roller.

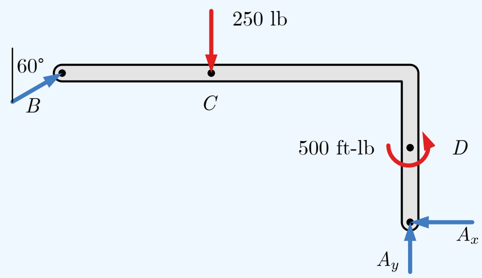
The L-shaped body is supported by a roller at B and a frictionless pin at A. The body supports a 100 lb vertical force at C and a 500 ft·lb couple-moment at D. Determine the reactions at A and B.



This problem will be solved three different ways, to demonstrate the advantages and disadvantages of different approaches.

Solution 1.

Solutions always start with a free-body diagram, showing all forces and moments acting on the object. Here, the known loads $C = 250$ lb (down) and $D = 500$ ft·lb (CCW) are red, and the unknown reactions A_x, A_y and B are blue.



The force at B is drawn along its known line-of-action perpendicular to the roller surface, and drawn pointing up and right because that will oppose the rotation of the frame about A caused by load C and moment D. The force at A is represented by unknown components A_x and A_y. The sense of these components is unknown, so we have arbitrarily assigned the arrowheads pointing left and up.

We have chosen the standard coordinate system with positive x to the right and positive y pointing up, and resolved force A into components in the x and y directions.

The magnitude of force B is unknown but its direction is known, so the x and y components of B can be expressed as

$$B_x = B\sin 60^\circ \qquad\qquad B_y = B\cos 60^\circ.$$

We choose to solve equation set $\{A\}$, and choose to take moments about point A, because unknowns A_x and A_y intersect there. Substituting the variables into the equation and solving for the unknowns gives

$$\begin{aligned} \sum F_x &= 0\\ B_x - A_x &= 0\\ A_x &= B\ \sin 60^\circ \end{aligned} \tag{1}$$

$$\sum F_y = 0$$

$$\begin{aligned} B_y - C + A_y &= 0 \\ A_y &= C - B\ \cos 60° \qquad (2) \end{aligned}$$

$$\begin{aligned} \sum M_A &= 0 \\ -B_x(3) - B_y(7) + C(4) + D &= 0 \\ 3B\cos 60° + 7B\sin 60° &= 4C + D \\ B(3\sin 60° + 7\cos 60°) &= 4C + D \\ B &= \frac{4C + D}{6.098} \qquad (3) \end{aligned}$$

Of these three equations only the third can be evaluated immediately, because we know C and D. In equations (1) and (2) unknowns A_x and A_y can't be found until B is known. Inserting the known values into (3) and solving for B gives

$$\begin{aligned} B &= \frac{4(250) + 500}{6.098} \\ &= \frac{1500 \text{ ft}\cdot\text{lb}}{6.098 \text{ ft}} \\ &= 246.0 \text{ lb} \end{aligned}$$

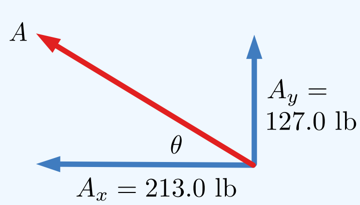
Now with the magnitude of B known, A_x and A_y can be found with (1) and (2).

$$\begin{aligned} A_x &= B\sin 60° \\ &= 246.0\sin 60° \\ &= 213.0 \text{ lb} \end{aligned}$$

$$\begin{aligned} A_y &= C - B\cos 60° \\ &= 250 - 246.0\cos 60° \\ &= 127.0 \text{ lb} \end{aligned}$$

The positive signs on these values indicate that the directions assumed on the free-body diagram were correct.

The magnitude and direction of force **A** can be found from the scalar components A_x and A_y using a rectangular to polar conversion.



$$A = \sqrt{A_x^2 + A_y^2} = 248.0 \text{ lb}$$

$$\theta = \tan^{-1}\left|\frac{A_y}{A_x}\right| = 30.8°$$

The final answer for **A** and **B**, with angles measured counter-clockwise from the positive x axis are

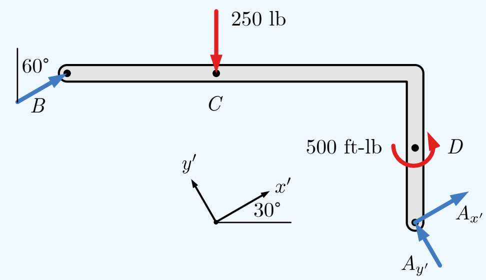
$$\mathbf{A} = 248.0 \text{ lb} \measuredangle 149.2°,$$

$$\mathbf{B} = 246.0 \text{ lb} \measuredangle 60°.$$

This solution demonstrates a fairly standard approach appropriate for many statics problems, and should be considered whenever the free body diagram contains a frictionless pin. Start by taking moments there.

Solution 2.

In this solution, we have rotated the coordinate system 30° to align it with force **B** and also chosen the components of force **A** to align with the new coordinate system.



There is no particular advantage to this approach over the first one, but with two unknown forces aligned with the x' direction, $A_{y'}$ can be found directly after breaking force C into components.

$$\begin{aligned} \sum F_{x'} &= 0 \\ B - C_{x'} + A_{x'} &= 0 \\ A_{x'} &= -B + C \sin 30° \end{aligned} \tag{1}$$

$$\begin{aligned} \sum F_{y'} &= 0 \\ C_{y'} + A_{y'} &= 0 \\ A_{y'} &= C \cos 30° \end{aligned} \tag{2}$$

$$\begin{aligned} \sum M_A &= 0 \\ -B_x(3) - B_y(7) + C(4) + D &= 0 \\ 3B \cos 60° + 7B \sin 60° &= 4C + D \\ B(3 \cos 60° + 7 \sin 60°) &= 4C + D \\ B &= \frac{4C + D}{7.56} \end{aligned} \tag{3}$$

Solving equation (2) yields

$$A_{y'} = 216.5 \text{ lb}.$$

Solving equation (3) yields the same result as previously

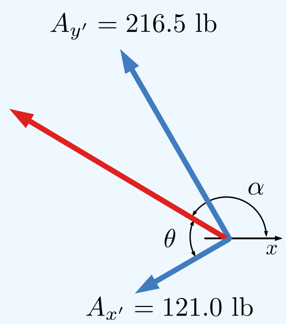
$$B = 246.0 \text{ lb}.$$

Substituting B and C into equation (1) yields

$$\begin{aligned} A_{x'} &= -B + C\sin 30° \\ &= -246.0 + 250\sin 30° \\ &= -121.0 \text{ lb} \end{aligned}$$

The negative sign on this result indicates that our assumed direction for $A_{x'}$ was incorrect, and that force actually points 180° to the assumed direction.

Resolving the $A_{x'}$ and $A_{y'}$ gives the magnitude and direction of force **A**.



$$A = \sqrt{A_{x'}^2 + A_{y'}^2} = 248.0 \text{ lb}$$

$$\theta = \tan^{-1}\left|\frac{A_y}{A_x}\right| = 60.8°$$

$$\alpha = 180° - (\theta - 30°) = 149.2°$$

Again, the final answer for **A** and **B**, with angles measured counter-clockwise from the positive x axis are

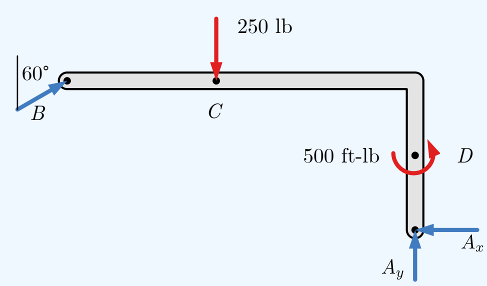
$$\mathbf{A} = 248.0 \text{ lb} \measuredangle\, 149.2°,$$

$$\mathbf{B} = 246.0 \text{ lb} \measuredangle\, 60°$$

This approach was slightly more difficult than solution 1 because of the additional trigonometry involved to find components in the rotated coordinate system.

Solution 3.

For this solution, we will use the same free-body diagram as solution one, but will use three moment equations, about points B, C and D.



$$\sum M_B = 0$$

$$\begin{aligned} -A_x(3) + A_y(7) - C(3) + D &= 0 \\ -3A_x + 7A_y &= 250 \qquad (1) \end{aligned}$$

$$\sum M_C = 0$$

$$
\begin{aligned}
-A_x(3) + A_y(4) - B_y(3) + D &= 0 \\
-3A_x + 4A_y - 3B\cos 60^\circ &= -D \\
3A_x - 4A_y + 1.5B &= 500 \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
\sum M_D &= 0 \\
-A_x(1.5) - B_x(1.5) - B_y(7) + C(4) + D &= 0 \\
1.5A_x + 1.5B\sin 60^\circ + 7B\cos 60^\circ &= 4C + D \\
1.5A_x + 4.799B &= 1500 \qquad (3)
\end{aligned}
$$

This set of three equations and three unknowns can be solved with some algebra.

Adding (1) and (2) gives

$$3A_y + 1.5B = 750 \qquad (4)$$

Dividing equation (2) by 2 and subtracting it from (3) gives

$$2A_y + 4.049B = 1250 \qquad (5)$$

Multiplying (4) by 2/3 and subtracting from (5) eliminates A_y and gives

$$3.049B = 750$$

$$B = 246.0 \text{ lb},$$

the same result as before.

Substituting B into (3) gives $A_x = 213.0$ lb, and substituting this into (1) gives $A_y = 127.0$ lb, again the same result as before.

An alternate approach is to set these three equations up for a matrix solution and use technology to do the algebra, as done here with Sage.

$$
\begin{bmatrix} -3 & 7 & 0 \\ 3 & -4 & 1.5 \\ 1.5 & 0 & 4.799 \end{bmatrix}
\begin{bmatrix} A_x \\ A_y \\ B \end{bmatrix}
=
\begin{bmatrix} 250 \\ 500 \\ 1500 \end{bmatrix}
$$

```
A = Matrix([[-3,7,0],[3,-4, 1.5],[1.5,0,4.799]])
B = vector([250, 500, 1500])
x = A.solve_right(B)
x
```

```
(213.020662512299, 127.008855362414, 245.982289275172)
```

This is a good example of an inefficient solution because of all the algebra involved. The issue here was the poor choice of B, C and D as moment centers. Whenever possible you should take moments about a point where the line of action of two unknowns intersect as was done in solution 1. This gives a moment equation which can be solved immediately for the third unknown.

1.6 3D Rigid Body Equilibrium

Key Questions

- What are the similarities and differences between solving two-dimensional and three-dimensional equilibrium problems?
- Why are some three-dimensional reaction couple-moments “available but not engaged”?
- What kinds of problems are solvable using linear algebra?

Three-dimensional systems are closer to reality than two-dimensional systems and the basic principles to solving both are the same, however they are generally harder solve because of the additional degrees of freedom involved and the difficulty visualizing and determining distances, forces and moments in three dimensions.

Three-dimensional problems are usually solved using vector algebra rather than the scalar approach used in the last section. The main differences are that directions are described with unit vectors rather than with angles, and moments are determined using the vector cross product rather scalar methods. Because they have more possible unknowns it is harder to find efficient equations to solve by hand. A problem might involve solving a system of up to six equations and six unknowns, in which case it is best solved using linear algebra and technology.

Resolving Forces and Moments into Components. To break two-dimensional forces into components, you likely used right-triangle trigonometry, sine and cosine. However, three-dimensional forces will likely need to be broken into components using `[cross-reference to target(s) "Chapter_02-unit-vectors" missing or not unique]`.

When summing moments, make sure to consider both the $\mathbf{r} \times \mathbf{F}$ moments and also the couple-moments with the following guidance:

1. First, choose any point in the system to sum moments around.
2. There are two general methods for summin gthe $\mathbf{r}\times\mathbf{F}$ moments. Both techniques will give you the same set of equations.
   (a) *Sum moments around each axis.*
   For relatively simple systems with few position and force vector components, you can find the cross product for each non-parallel position and force pair. Using this method requires you to resolve the direction of each cross product pair using the right-hand rule as covered in `[cross-reference to target(s) "Chapter_04" missing or not unique]`. Recall that there are up to six pairs of non-parallel components that you need to consider.
   (b) *Sum all moments around a point using vector determinants.*
   Choose a point in the system which is on the line of action of as many forces as possible, then set up each cross product as a determinant. After

computing the components coming from each determinant, combine the x, y, and z terms into each of the $\Sigma\mathbf{M}_x = 0$, $\Sigma\mathbf{M}_y = 0$, and $\Sigma\mathbf{M}_z = 0$ equations.

3. Finally, add the components of any couple-moments into the corresponding $\Sigma\mathbf{M}_x = 0$, $\Sigma\mathbf{M}_y = 0$, and $\Sigma\mathbf{M}_z = 0$ equations.

Solving for unknown values in equilibrium equations. Once you have formulated $\Sigma\mathbf{F} = 0$ and $\Sigma\mathbf{M} = 0$ equations in each of the x, y and z directions, you could be facing up to six equations and six unknown values.

Frequently these simultaneous equation sets can be solved with substitution, but it is often be easier to solve large equation sets with linear algebra. Note that the adjective "linear" specifies that the unknown values must be linear terms, which means that each unknown variable cannot be raised to a exponent, be an exponent, or buried inside of a sin or cos function. Luckily, most unknowns in equilibrium are linear terms, except for unknown angles. If you are not familiar with the use of linear algebra matrices to solve simultaneously equations, search the internet for *Solving Systems of Equations Using Linear Algebra* and you will find plenty of resources.

No matter how you choose to solve for the unknown values, any numeric values which come out to be negative indicate that your initial hypothesis of that vector's sense was incorrect.

Three-dimensional Equilibrium Examples. [TBD]

1.7 Stability and Determinacy

Key Questions

- What does *stable* mean for a rigid body?
- What does *determinate* mean for a rigid body?
- Does stability a depend on the external loads or only the reactions?
- How can I tell if a system is determinate?
- How can I decide if a problem is both stable and determinate, which makes it solvable statics?

Determinate vs. Indeterminate. A static system is **determinate** if it is possible to find the unknown reactions using the methods of statics, that is, by using equilibrium equations, otherwise it is **indeterminate**.

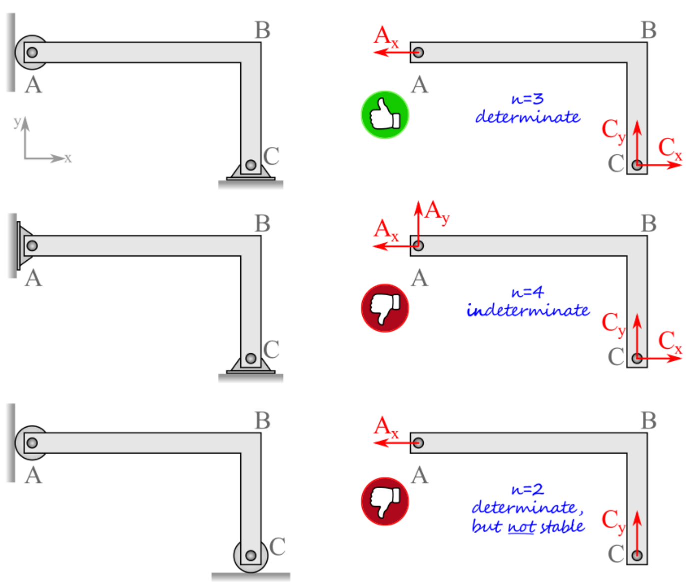
In order for a system to be determinate the number of unknown force and moment reaction components must be less than or equal to the number of independent equations of equilibrium available. Each equilibrium equation derives from a degree of freedom of the system, so there may be no more unknowns than degrees of freedom. This means that we can determine no more than three unknown reaction components in two-dimensional systems and no more than six in three-dimensional systems.

An indeterminate system with fewer reaction components than degrees of freedom is **under-constrained** and therefore unstable. On the other hand, if there are more reaction components than degrees of freedom, the system is both **over-constrained** and **indeterminate**. In terms of force and moment equations, there are more unknowns than than equilibrium equations so they can't all be determined. This is not to say that it is impossible to find all reaction force on an over-constrained system, just that you will not learn how to find them in this course.

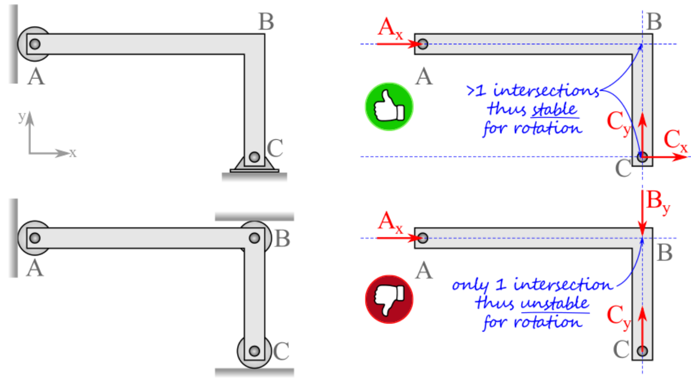
Stable vs. Unstable. A body in equilibrium is held in position by its supports, which restrict the body's motion and counteract the applied loads. When there are sufficient supports to restrain a body from moving, we say that the body is **stable**. A stable body is prevented from translating and rotating in all directions. A body which *can* move is **unstable** even if it is not currently moving, because the slightest change in load may take it out of equilibrium and initiate motion. Stability is loading independent i.e. a stable body is stable for *any* loading condition.

Rules to Validate a Stable and Determinate System. There are three rules to determine if a system is both stable and determinate. While, the rules below can technically be checked in any order, they have been sorted from the quickest to the most time consuming to speed up your analysis.

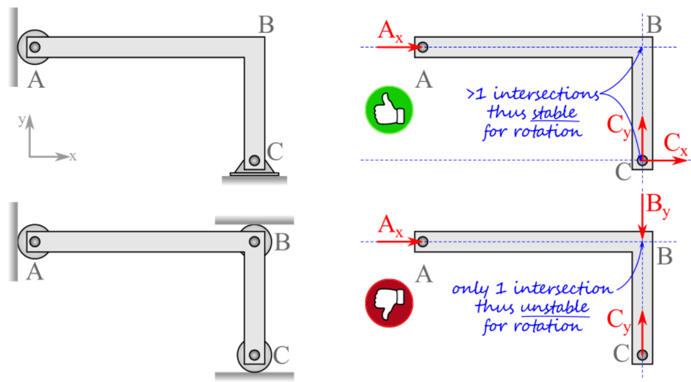
Rule 1: Are there exactly three reaction components on a two-dimensional body?
If YES, the system is determinate.
If NO, the system is indeterminate or not stable.



Rule 2: Are any reaction force components parallel to one another?
If YES, the system is unstable for translation.
If NO, the system is stable for translation.



Rule 3: Do the lines of action of the reaction forces intersect at a single point?
If YES, the system is unstable for rotation about the single intersection point.
If NO, the system is stable for rotation.



1.8 Equilibrium Examples

You can use these interactives to explore how the reactions supporting rigid bodies are affected by the loads applied. You can use the equations of equilibrium to solve for the unknown reactions, and check your work.

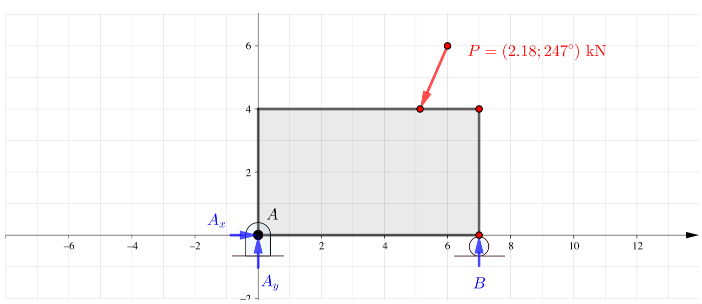


Figure 1.8.1 Rigid body Equilibrium

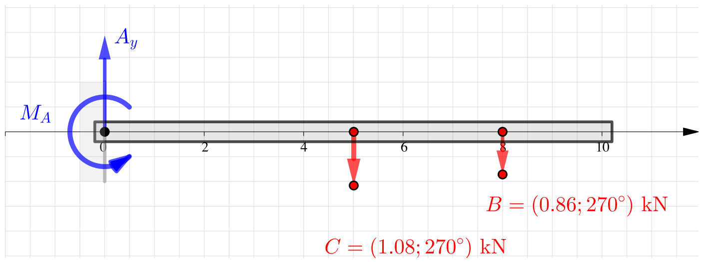


Figure 1.8.2 Cantilever beam

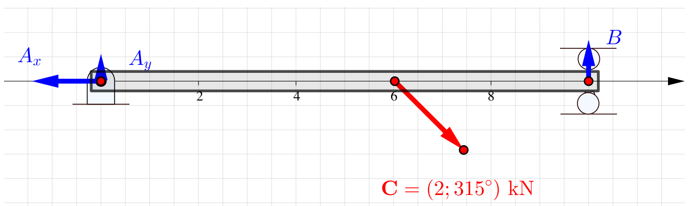


Figure 1.8.3 Beam with concentrated load

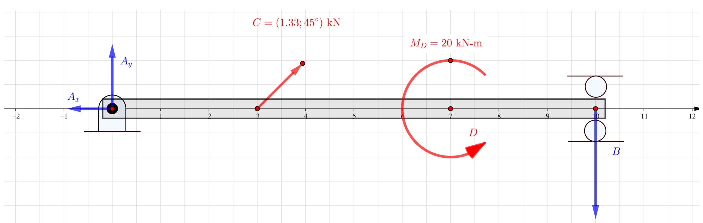


Figure 1.8.4 Beam with concentrated force and couple moment

1.9 Exercises