

## Sections 10.1 - 10.2 Overview

- Three-Dimensional Coordinates (10.1)

- Distance between points in 3D space

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Simple planes in 3D Space

$$x = a, y = b, z = c$$

- Spheres in 3D Space

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

- Vectors (10.2)

- Definition of a Vector

- \* A vector  $\mathbf{v} = \overrightarrow{v}$  is a mathematical object which stores length (magnitude) and direction, and can be thought of as a directed line segment.
- \* Two vectors with the same length and direction are considered equal, even if they aren't in the same position.
- \* We often (but not always) assume the initial point (the one without an arrow) lays at the origin.

- Component Form

$\langle v_x, v_y, v_z \rangle$  is equal to the vector with initial point at  $(0, 0, 0)$  and terminal point at  $(v_x, v_y, v_z)$ .

- 2D vs 3D Vectors

$$\langle a, b \rangle = \langle a, b, 0 \rangle$$

- Position Vector

If  $P = (a, b, c)$  is a point, then  $\mathbf{P} = \langle a, b, c \rangle$  is its **position vector**.

We assume  $(a, b, c) = \langle a, b, c \rangle$ .

- Vector Between Points

The vector from  $P_1 = (x_1, y_1, z_1)$  to  $P_2 = (x_2, y_2, z_2)$  is

$$\mathbf{P_1P_2} = \overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

- Length of a Vector

$$|\mathbf{v}| = |\langle v_1, v_2, v_3 \rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

- The Zero Vector

$$\mathbf{0} = \vec{0} = \langle 0, 0, 0 \rangle$$

- Vector Operations

- \* Addition

$$\langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$$

- \* Scalar Multiplication

$$k \langle v_1, v_2, v_3 \rangle = \langle kv_1, kv_2, kv_3 \rangle$$

- Vector Operation Properties

1.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2.  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3.  $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4.  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5.  $0\mathbf{u} = \mathbf{0}$
6.  $1\mathbf{u} = \mathbf{u}$
7.  $a(b\mathbf{u}) = (ab)\mathbf{u}$
8.  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9.  $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

- Unit Vectors

- \* A **unit vector** or **direction** is any vector whose length is 1.

- \* Standard unit vectors

- $\mathbf{i} = \langle 1, 0, 0 \rangle$
- $\mathbf{j} = \langle 0, 1, 0 \rangle$
- $\mathbf{k} = \langle 0, 0, 1 \rangle$

- \* Standard Unit Vector Form:

$$\langle v_x, v_y, v_z \rangle = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$$

- \* Length-Direction Form:

$$\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$$

### 10.3 The Dot Product

- Dot Product

$$\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1v_1 + u_2v_2 + u_3v_3$$

- Angle between vectors

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

- Alternate Dot Product formula

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$$

- Orthogonal Vectors

- $\mathbf{u}, \mathbf{v}$  are orthogonal if  $\mathbf{u} \cdot \mathbf{v} = 0$
- $\mathbf{u}, \mathbf{v}$  are orthogonal if the angle between them is  $\frac{\pi}{2} = 90^\circ$
- $\mathbf{0}$  is orthogonal to every vector

- Dot Product Properties

1.  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2.  $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3.  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4.  $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5.  $\mathbf{0} \cdot \mathbf{u} = 0$

- Projection Vector

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}$$

- Work

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos \theta$$

- Suggested Exercises for 10.3

- Finding and applying dot products: 1-8
- Work done by a constant vector force: 39-40

## 10.4 The Cross Product

- Right-hand rule

Any method for determining a special orthogonal direction used throughout mathematics and physics, with respect to an ordered pair of vectors  $\mathbf{u}, \mathbf{v}$

- Unit Normal Vector

The vector  $\mathbf{n}$  orthogonal to an ordered pair of vectors  $\mathbf{u}, \mathbf{v}$  following the right-hand rule

- Cross Product

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin \theta) \mathbf{n}$$

- Parallel Vectors

- $\mathbf{u}, \mathbf{v}$  are parallel if  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$
- $\mathbf{u}, \mathbf{v}$  are parallel if the angle between them is  $0 = 0^\circ$  or  $\pi = 180^\circ$
- $\mathbf{0}$  is parallel to every vector

- Cross Product Properties

1.  $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2.  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3.  $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
4.  $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
5.  $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
6.  $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

- Standard Unit Vector Cross Products

1.  $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
2.  $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
3.  $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

- Parallelogram Area The area of a parallelogram determined by  $\mathbf{u}, \mathbf{v}$  is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$$

- Determinants

- 2x2 Determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- 3x3 Determinant

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_3 b_2 c_1 + a_1 b_3 c_2 + a_2 b_1 c_3) \end{aligned}$$

- Computing Cross Products

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle \\ &= \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle \end{aligned}$$

Shortcut “long multiplication” method:

$$\frac{\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}}{\begin{vmatrix} u_2 v_3 - u_3 v_2 & u_3 v_1 - u_1 v_3 & u_1 v_2 - u_2 v_1 \end{vmatrix}} \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- Torque

$$\vec{\tau} = \mathbf{r} \times \mathbf{F} = (|\mathbf{r}||\mathbf{F}| \sin \theta) \mathbf{n}$$

- Triple Scalar (or “Box”) Product

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Its absolute value  $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$  gives the volume of a parallelepiped determined by the three vectors.

- **Suggested Exercises for 10.4**

- Finding cross products: 1-14
- Finding areas and unit normal vectors using cross products: 15-18
- Finding volumes using cross products: 19-22
- Computing torque: 25-26

### 10.5 Lines and Planes in Space

- Vector Equation for a Line

$$\mathbf{r}(t) = \mathbf{P}_0 + t\mathbf{v}$$

for  $-\infty < t < \infty$

- Parametric Equations for a Line

$$x = x_0 + tv_1, y = y_0 + tv_2, z = z_0 + tv_3$$

for  $-\infty < t < \infty$

- Line Passing through a pair of points

$$\mathbf{r}(t) = \mathbf{P}_0 + t(\mathbf{P}_0\mathbf{P}_1) = (1 - t)\mathbf{P}_0 + t\mathbf{P}_1$$

for  $-\infty < t < \infty$

- Line Segment joining a pair of points

$$\mathbf{r}(t) = \mathbf{P}_0 + t(\mathbf{P}_0\mathbf{P}_1) = (1 - t)\mathbf{P}_0 + t\mathbf{P}_1$$

for  $0 \leq t \leq 1$

- Distance from a Point to a Line

$$d = \frac{|\mathbf{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

- Equation for a Plane

$$\mathbf{n} \cdot (\mathbf{P}_0\mathbf{P}) = 0$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

- Line of Intersection of Two Planes

$$\mathbf{r}(t) = \mathbf{P} + t(\mathbf{n}_1 \times \mathbf{n}_2)$$

- Distance from a Point to a Plane

$$d = \frac{|\mathbf{PS} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

- **Suggested Exercises for 10.5**

- Finding parametric equations for lines: 1-12
- Finding parametrizations for line segments: 13-20
- Finding equations for planes: 21-26
- Distance from a point to a line: 33-38
- Distance from a point to a plane: 39-44



## 10.6 Cylinders and Quadratic Surfaces

- Sketching surfaces
  - To sketch a 3D surface, sketch planar cross-sections
    - \*  $z = c$  is parallel to  $xy$  plane
    - \*  $y = b$  is parallel to  $xz$  plane
    - \*  $x = a$  is parallel to  $yz$  plane
- Cylinders
  - A **cylinder** is any surface generated by moving a planar along a line normal to that plane.
  - A 3D surface defined by a function of only two variables results in a cylinder.
- Quadric Surfaces
  - A **quadric surface** is any surface defined by a second degree equation of  $x, y, z$ :
$$Ax^2 + Bx + Cy^2 + Dy + Ez^2 + Fz = G$$
  - Most helpful to consider the cross-sections in each of the coordinate planes.
- Ellipsoids
  - Cross-sections in the coordinate planes include
    - \* Three ellipses
- Elliptical Cone
  - Cross-sections in the coordinate planes include
    - \* Two double-lines
    - \* One point (with parallel ellipses)
  - Cross-sections parallel to the point cross-section are ellipses.

- Elliptical Paraboloid
  - Cross-sections in the coordinate planes include
    - \* Two parabolas
    - \* One point (with parallel ellipses)
  - Cross-sections parallel to the point cross-section are ellipses.
- Hyperbolic Paraboloid
  - Cross-sections in the coordinate planes include
    - \* Two parabolas (with parallel parabolas)
    - \* One double line (with parallel hyperbolas)
- Hyperboloid of One Sheet
  - Cross-sections in the coordinate planes include
    - \* Two hyperbolas
    - \* One ellipsis (with parallel hyperbolas)
- Hyperboloid of Two Sheets
  - Cross-sections in the coordinate planes include
    - \* Two hyperbola
    - \* One empty cross-section (with parallel hyperbolas)
  - Cross-sections parallel to the empty cross-section are ellipses.
- **Suggested Exercises for 10.6**
  - Identify surfaces from equations: 1-12
  - Sketching surfaces: 13-44

## 11.1 Vector Functions and their Derivatives

- Curves, Paths, and Vector Functions

- A **position function** maps a moment in time to a position on a path. It can be defined with **parametric equations**

$$x = x(t)$$

$$y = y(t)$$

$$z = z(t)$$

or with a **vector function**

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

- $x(t), y(t), z(t)$  are called **component functions**

- Vector Function Limits

- If the value of the vector function  $\mathbf{r}(t)$  becomes arbitrarily close to the vector  $\mathbf{L}$  as values of  $t$  close to  $t_0$  are plugged into the function, then the **limit of  $\mathbf{r}(t)$  as  $t$  approaches  $t_0$**  is  $\mathbf{L}$ , or

$$\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{L}$$

\* (Precise definition:) Suppose there exists a function  $\delta(\epsilon)$  so that for all positive numbers  $\epsilon > 0$  and all numbers  $t$  where  $|\mathbf{r}(t) - \mathbf{L}| < \epsilon$ , it follows that  $|t - t_0| < \delta(\epsilon)$ . Then we say that  $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{L}$ .

- Note that

$$\lim_{t \rightarrow t_0} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow t_0} f(t), \lim_{t \rightarrow t_0} g(t), \lim_{t \rightarrow t_0} h(t) \right\rangle$$

- Continuity of Vector Functions

- The function  $\mathbf{r}(t)$  is **continuous at a point  $t_0$**  if

$$\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$$

- The function  $\mathbf{r}(t)$  is **continuous** if

$$\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$$

for all  $t_0$  in its domain.

–  $\mathbf{r}(t)$  is continuous exactly when  $f(t), g(t), h(t)$  are all continuous.

- Derivatives of Vector Functions

$$- \frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

$$- \mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

–  $\mathbf{r}(t)$  is **differentiable** if  $\mathbf{r}'(t)$  is defined for every value of  $t$  is in its domain.

–  $\mathbf{r}(t)$  is **smooth** if  $\mathbf{r}(t)$  is differentiable,  $\mathbf{r}'(t)$  is continuous, and  $\mathbf{r}'(t) \neq 0$

–  $\mathbf{r}'(t_0)$  is a **tangent vector** to the curve where  $t = t_0$

– The **tangent line** to a curve given by the vector function:

$$\mathbf{l}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$$

- Vectors and Physics

– Position:  $\mathbf{r}(t)$

– Velocity:  $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}$

– Speed:  $|\mathbf{v}(t)|$

– Direction:  $\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$

\* (Remember that  $\mathbf{v} = |\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$ )

– Acceleration:  $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$

- Differentiation Rules for Vector Functions

1. Constant Function Rule

$$\frac{d}{dt}[\mathbf{C}] = \mathbf{0}$$

2. Constant Multiple Rules

$$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{C}] = f'(t)\mathbf{C}$$

3. Sum and Difference Rules

$$\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$$

## 4. Scalar Product Rule

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f(t)\mathbf{u}'(t) + f'(t)\mathbf{u}(t)$$

## 5. Dot Product Rule

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t)$$

## 6. Cross Product Rule

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$$

## 7. Chain Rule

$$\frac{d\mathbf{u}}{dt} = \frac{d}{dt}[\mathbf{u}(f(t))] = \mathbf{u}'(f(t))f'(t) = \frac{d\mathbf{u}}{df} \frac{df}{dt}$$

## • Derivative of a Constant Length Vector Function

- If  $|\mathbf{r}(t)| = c$  always, then

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

- Thus the derivative of a constant length vector function is perpendicular to the original.

## • Suggested Exercises for 11.1

- Position/Velocity/Acceleration Vectors: 1-14

## 11.2 Integrals of Vector Functions

- Antiderivatives of Vector Functions

- If  $\mathbf{R}'(t) = \mathbf{r}(t)$ , then  $\mathbf{R}(t)$  is an **antiderivative** of  $\mathbf{r}(t)$ .
- The **indefinite integral**  $\int \mathbf{r}(t) dt$  is the collection of all the antiderivatives of  $\mathbf{r}(t)$ .

$$\int \mathbf{r}(t) dt = \mathbf{R}(t) + \mathbf{C}$$

$$\int \mathbf{r}(t) dt = \left\langle \int x(t) dt, \int y(t) dt, \int z(t) dt \right\rangle$$

- Definite Integrals

$$\int_a^b \mathbf{r}(t) dt = \left\langle \int_a^b x(t) dt, \int_a^b y(t) dt, \int_a^b z(t) dt \right\rangle$$

$$\int_a^b \mathbf{r}(t) dt = [\mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

- Initial Value Problems

- If we know  $\mathbf{r}'(t)$  and  $\mathbf{r}(t_0)$ , then

$$\mathbf{r}(t) = \mathbf{R}'(t) + \mathbf{r}(t_0) - \mathbf{R}'(t_0)$$

- Ideal Projectile Motion

- Assume the following:
  - \* The acceleration acting on a projectile is  $\langle 0, -g \rangle$
  - \* The launch position is the origin  $\langle 0, 0 \rangle = \mathbf{0}$
  - \* The launch angle is  $\alpha$
  - \* The initial velocity is  $\mathbf{v}_0$ , and initial speed is  $v_0 = |\mathbf{v}_0|$
- This results in the initial value problem:

$$\mathbf{a}(t) = \langle 0, -g \rangle$$

$$\mathbf{v}(0) = \langle v_0 \cos \alpha, v_0 \sin \alpha \rangle$$

$$\mathbf{r}(0) = \langle 0, 0 \rangle$$

- The velocity function solves to

$$\mathbf{v}(t) = \langle v_0 \cos \alpha, -gt + v_0 \sin \alpha \rangle$$

- The position function solves to

$$\mathbf{r}(t) = \left\langle (v_0 \cos \alpha)t, -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t \right\rangle$$

with parametric equations

$$x = (v_0 \cos \alpha)t$$

$$y = -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t$$

- The parabolic position curve can be expressed as

$$y = -\left(\frac{g}{2v_0^2 \cos^2 \alpha}\right)x^2 + (\tan \alpha)x$$

- Properties of ideal projectile motion beginning at origin:

$$y_{max} = \frac{(v_0 \sin \alpha)^2}{2g}$$

$$t_{tot} = \frac{2v_0 \sin \alpha}{g}$$

$$R = \frac{v_0^2}{g} \sin 2\alpha$$

- If we assume the initial position is instead  $\mathbf{r}(0) = \langle x_0, y_0 \rangle$ , then the position function changes to

$$\mathbf{r}(t) = \left\langle (v_0 \cos \alpha)t + x_0, -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t + y_0 \right\rangle$$

- **Suggested Exercises for 11.2**

- Vector function integrals: 1-6
- Vector function initial value problems: 7-12
- Ideal projectile motion: 15-21

### 11.3 Arc Length in Space

- Arc Length along a Space Curve

- Approximation

$$L \approx \sum_{i=0}^n |\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)| = \sum_{i=1}^n \left| \frac{\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)}{\Delta t} \right| \Delta t$$

- Definition

$$L = \int_a^b \left| \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \right| dt = \int_a^b |\mathbf{v}(t)| dt$$

- Arclength Parameter

$$s(t) = \int_0^t |\mathbf{v}(\tau)| d\tau$$

$$\frac{ds}{dt} = |\mathbf{v}(t)| = \text{speed}$$

- Unit Tangent Vector

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

- Suggested Exercises for 11.3

- Unit tangent vectors and arc length: 1-8
  - Arc length parameter: 11-14



### 11.4 Curvature of a Curve

- Curvature

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

- Curvature of a Circle

- The curvature of a circle with radius  $a$  is constantly

$$\kappa = \frac{1}{a}$$

- Principal Unit Normal Vector

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds}$$

- Circles of Curvature

- The circle which:

1. is tangent to a curve at a point
2. has the same curvature as the curve at that point
3. lies on the concave side of the curve, in the direction of  $\mathbf{N}$

- Radius:  $a = \frac{1}{\kappa}$ .

- Center:  $\langle x_0, y_0 \rangle = \mathbf{r}(t_0) + a\mathbf{N}$ .

- Equations:

$$(x - x_0)^2 + (y - y_0)^2 = a^2$$

$$\mathbf{c}(t) = \langle a \sin t + x_0, a \cos t + y_0 \rangle, 0 \leq t \leq 2\pi$$

- Suggested Exercises for 11.4:

- Find  $\mathbf{T}, \mathbf{N}, \kappa$ : 1-4, 9-16
- Circles of Curvature: 21-22

## 11.5 Tangential and Normal Components of Acceleration

- Binormal Unit Vector

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

- Right-handed vector frames

- $\mathbf{i}, \mathbf{j}, \mathbf{k}$
- $\mathbf{T}, \mathbf{N}, \mathbf{B}$

- Tangential and Normal Components of Acceleration

$$\mathbf{a} = \left( \frac{d^2 s}{dt^2} \right) \mathbf{T} + \kappa \left( \frac{ds}{dt} \right)^2 \mathbf{N} + 0\mathbf{B}$$

- Tangential component

$$a_T = \frac{d^2 s}{dt^2} = \frac{d}{dt} |\mathbf{v}|$$

- Normal component

$$a_N = \kappa \left( \frac{ds}{dt} \right)^2 = \kappa |\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

- Torsion

- Magnitude of torsion

$$|\tau| = \left| \frac{d\mathbf{B}}{ds} \right|$$

- Signed torsion

$$\frac{d\mathbf{B}}{ds} = (-\tau)\mathbf{N}$$

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\frac{1}{|\mathbf{v}|} \left( \frac{d\mathbf{B}}{dt} \cdot \mathbf{N} \right)$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$$

- Suggested Exercises for 11.5:

- Finding tangential and normal components of acceleration: 1-6
- Finding  $\mathbf{B}$  and  $\tau$ : 9-16

## 11.6 Velocity and Acceleration in Polar Coordinates

- Polar Coordinates  $(r, \theta)$

- Cartesian to Polar

$$r = \sqrt{x^2 + y^2}, \theta = \text{Arctan}\left(\frac{y}{x}\right)$$

- Polar to Cartesian

$$x = r \cos \theta, y = r \sin \theta$$

- Cylindrical Coordinates  $(r, \theta, z)$

- Cartesian to Cylindrical

$$r = \sqrt{x^2 + y^2}, \theta = \text{Arctan}\left(\frac{y}{x}\right), z = z$$

- Cylindrical to Cartesian

$$x = r \cos \theta, y = r \sin \theta, z = z$$

- Polar/Cylindrical Unit Vectors

$$\mathbf{u}_r = \langle \cos \theta, \sin \theta \rangle, \mathbf{u}_\theta = \langle -\sin \theta, \cos \theta \rangle$$

- Cylindrical Right-handed frame

$$\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{k}$$

- Derivatives

$$\frac{d}{dt} [\mathbf{u}_r] = \dot{\mathbf{u}}_r = \dot{\theta} \mathbf{u}_\theta$$

$$\frac{d}{dt} [\mathbf{u}_\theta] = \dot{\mathbf{u}}_\theta = -\dot{\theta} \mathbf{u}_r$$

- Polar Position/Velocity/Acceleration

$$\mathbf{r} = r \mathbf{u}_r$$

$$\mathbf{v} = \dot{r} \mathbf{u}_r + r \dot{\theta} \mathbf{u}_\theta$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2) \mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \mathbf{u}_\theta$$

- Cylindrical Position/Velocity/Acceleration

$$\mathbf{r} = r\mathbf{u}_r + z\mathbf{k}$$

$$\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k}$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{k}$$

- **Suggested Exercises for 11.6:**

- Expressing  $\mathbf{v}$  and  $\mathbf{a}$  in terms of  $\mathbf{u}_r$  and  $\mathbf{u}_\theta$ : 1-5

## 12.1 Functions of Several Variables

- Real-Valued Functions

- A **real-valued function**  $f$  on with **domain**  $D \subset \mathbb{R}^n$  is a rule that assigns a real number

$$f(x_1, x_2, \dots, x_n) \in \mathbb{R}$$

to each  $(x_1, x_2, \dots, x_n) \in D$ .

- The domain of a function is assumed to be all of  $\mathbb{R}^n$  except where the function is not well-defined.
- The **range** of the function is

$$R = \{f(x_1, x_2, \dots, x_n) : (x_1, x_2, \dots, x_n) \in D\}$$

- Regions

- A subset of the  $xy$ -plane ( $\mathbb{R}^2$ ) or  $xyz$ -space ( $\mathbb{R}^3$ ) is known as a **region**.
- The **ball**  $B(p, \epsilon)$  is the set of points

$$B(p, \epsilon) = \{q \in \mathbb{R}^2 : \text{the distance between } p \text{ and } q \text{ is less than } \epsilon\}$$

Its **center** is the point  $p$  and its **radius** is  $\epsilon$ .

- A point  $p \in \mathbb{R}^2$  is known as an **interior point** of a region  $R$  if *there exists some ball* containing  $p$  that lies inside  $R$ .
- A point  $p \in \mathbb{R}^2$  is known as a **boundary point** of a region  $R$  if *every ball* containing  $p$  contains some points in  $R$  and some points not in  $R$ .
- A point  $p \in \mathbb{R}^2$  is known as an **exterior point** of a region  $R$  if *there exists some ball* containing  $p$  that lies outside  $R$ .
- The **interior** of  $R$  is the set

$$\text{int}(R) = \{p : p \text{ is an interior point of } R\}$$

- The **boundary** of  $R$  is the set

$$\text{bd}(R) = \{p : p \text{ is a boundary point of } R\}$$

- The **exterior** of  $R$  is the set

$$\text{ext}(R) = \{p : p \text{ is an exterior point of } R\}$$

- A region  $R$  is **open** if it doesn't contain any of its boundary.
- A region  $R$  is **closed** if it contains all of its boundary.
- A region  $R$  is **bounded** if it can be contained within a ball.
- A region  $R$  is **unbounded** if it cannot be contained within a ball.

- Sketching Functions

- Level curve

$$\{(x, y) : f(x, y) = c\}$$

- Surface  $z = f(x, y)$

$$\{(x, y, f(x, y)) : (x, y) \in \text{Dom}(f)\}$$

- Contour curve

$$\{(x, y, c) : f(x, y) = c\}$$

- Level surface

$$\{(x, y, z) : f(x, y, z) = c\}$$

- Suggested Exercises for 12.1:

- Identifying and describing domains, ranges, level curves, boundaries: 1-12
  - Relating level curves to graphs: 13-18
  - Sketching surfaces and level curves: 19-28
  - Finding level curves through a point: 29-32
  - Sketching level surfaces: 33-40
  - Finding level surfaces through a point: 41-44

## 12.2 Limits and Continuity in Higher Dimensions

- Limits

- If the value of the vector function  $f(P)$  becomes arbitrarily close to the number  $L$  as points  $P$  close to  $P_0$  are plugged into the function, then the **limit of  $f(P)$  as  $P$  approaches  $P_0$  is  $L$ :**

$$\lim_{P \rightarrow P_0} f(P) = L$$

- \* Precise definition:

Suppose there exists a function  $\delta(\epsilon)$  so that for all positive numbers  $\epsilon > 0$  and all numbers  $P$  where  $|f(P) - L| < \epsilon$ , it follows that  $|\mathbf{P} - \mathbf{P}_0| < \delta(\epsilon)$ . Then we say that  $\lim_{P \rightarrow P_0} f(P) = L$ .

- Note that values of  $f$  must approach  $L$  no matter which direction we approach  $p_0$ .

- Limit Laws

1. Sum/Difference Law

$$\lim_{p \rightarrow p_0} (f(p) \pm g(p)) = \lim_{p \rightarrow p_0} f(p) \pm \lim_{p \rightarrow p_0} g(p)$$

2. Product Law

$$\lim_{p \rightarrow p_0} (f(p) \cdot g(p)) = \lim_{p \rightarrow p_0} f(p) \cdot \lim_{p \rightarrow p_0} g(p)$$

3. Constant Multiple Law

$$\lim_{p \rightarrow p_0} (kf(p)) = k \lim_{p \rightarrow p_0} f(p)$$

4. Quotient Law

$$\lim_{p \rightarrow p_0} \frac{f(p)}{g(p)} = \frac{\lim_{p \rightarrow p_0} f(p)}{\lim_{p \rightarrow p_0} g(p)}$$

5. Power Law (for  $r, s \in \mathbb{Z}$ )

$$\lim_{p \rightarrow p_0} (f(p))^{r/s} = \left( \lim_{p \rightarrow p_0} f(p) \right)^{r/s}$$

- Computing Limits

- $\lim_{P \rightarrow P_0} f(x) = \lim_{x \rightarrow x_0} f(x)$
- Factoring and cancelling is a useful strategy.
- L'Hopital's Rule does not apply for multiple variable limits.

- Showing a Limit DNE

- If

$$\lim_{x \rightarrow x_0} h(x, f(x)) \neq \lim_{x \rightarrow x_0} h(x, g(x))$$

then  $\lim_{P \rightarrow P_0} h(x, y)$  DNE.

- Continuity

- A function  $f(p)$  is **continuous** if  $\lim_{p \rightarrow p_0} f(p) = f(p_0)$  for all points  $p_0$  in its domain.
- If a multi-variable function is composed of continuous single-variable functions, then it is also continuous.

- **Suggested Exercises for 12.2:**

- Computing limits: 1-26
- Showing limits don't exist: 35-42



## 12.3 Partial Derivatives

- Partial Derivatives

- The **partial derivative of  $f$  with respect to  $x_i$**  is the limit

$$\frac{\partial f}{\partial x_i} = f_{x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

- By the definition, we can see that to compute partial derivatives with respect to a variable, we can treat all other variables as constants and differentiate as normal.

- Higher Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left[ \frac{\partial f}{\partial y} \right] = (f_y)_x = f_{yx}$$

$$\frac{\partial^2 g}{\partial z \partial z} = \frac{\partial^2 g}{\partial z^2} = g_{zz}$$

- Mixed Derivative Theorem:

$$f_{xy} = f_{yx}$$

- **Suggested Exercises for 12.3:**

- Finding first-order partial derivatives: 1-38
- Finding second-order partial derivatives: 41-50
- Finding partial derivatives from the limit definition: 53-56

## 12.4 The Chain Rule

- Gradient Vector Function

$$\nabla f = \langle f_{x_1}, \dots, f_{x_n} \rangle = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$$

- Chain Rule

- For single variable functions:

$$\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt} + \dots$$

- For multi-variable functions:

$$\frac{\partial f}{\partial t_i} = \nabla f \cdot \frac{\partial \mathbf{r}}{\partial t_i} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots$$

- Differentiation by Substitution

- The Chain Rule can be avoided by “plugging in” functions and using single-variable calculus.

- Total Derivative

- If the variables  $x, y, z$  of a function  $f$  are dependent on each other, then

$$\frac{df}{dx} = \nabla f \cdot \frac{d\mathbf{r}}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial z} \frac{dz}{dx}$$

- Implicit Differentiation

- If  $f(x, y) = c$  defines  $y$  as a function of  $x$ , then

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

- Suggested Exercises for 12.4:

- Finding  $\frac{dw}{dt}$  for  $w = f(x(t), y(t), z(t))$ : 1-6
- Finding partial derivatives for compositions of multi-variable functions: 7-12, 33-38
- Using partial derivatives for implicit differentiation: 25-28

## 12.5 Directional Derivatives and Gradient Vectors

- Directional Derivative

- The **directional derivative of  $f$  in the direction of  $\mathbf{u}$**  is

$$\frac{df}{ds_{\mathbf{u}}} = \left( \frac{df}{ds} \right)_{\mathbf{u}} = D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$

- If  $\theta$  is the angle between  $\nabla f$  and  $\mathbf{u}$ , then

$$\frac{df}{ds_{\mathbf{u}}} = \nabla f \cdot \mathbf{u} = |\nabla f| |\mathbf{u}| \cos \theta = |\nabla f| \cos \theta$$

- The angle  $\theta$  between  $\nabla f$  and  $\mathbf{u}$  determines the value of the directional derivative at a fixed point  $p_0$ :

- \* Max:  $|\nabla f_{p_0}|$  at  $\theta = 0$
- \* Zero: 0 at  $\theta = \frac{\pi}{2}$
- \* Min:  $-|\nabla f_{p_0}|$  at  $\theta = \pi$

- Normal Vector to a Level Curve

- $\nabla f$  is normal to the level curve  $f(x, y) = c$  for every point  $(x, y)$  in the domain of  $f$ .

- Gradient Rules

1. Constant Multiple Rule

$$\nabla(kf) = k\nabla f$$

2. Sum Rule

$$\nabla(f + g) = \nabla f + \nabla g$$

3. Difference Rule

$$\nabla(f - g) = \nabla f - \nabla g$$

4. Product Rule

$$\nabla(fg) = g(\nabla f) + f(\nabla g)$$

5. Quotient Rule

$$\nabla \left( \frac{f}{g} \right) = \frac{g(\nabla f) - f(\nabla g)}{g^2}$$

- **Suggested Exercises for 12.5:**

- Finding  $\nabla f$  at a point: 1-8
- Finding directional derivatives: 9-16
- Finding the direction of maximal/minimal rate of change: 17-22
- Finding the direction of no instantaneous change: 27-28

## 12.6 Tangent Planes and Differentials

- Normal Vector to a Level Surface
  - $\nabla f$  is normal to the level surface  $f(x, y, z) = c$  for every point  $(x, y, z)$  in the domain of  $f$ .
- Normal Vector to the Surface  $z = f(x, y)$ 
  - If  $g(x, y, z) = f(x, y) - z$ , then

$$\nabla g = \langle f_x, f_y, -1 \rangle$$

is normal to the surface  $z = f(x, y)$  for every point  $(x, y)$  in the domain of  $f$ .

- Tangent Line to Curve of Intersection of Two Surfaces
  - If  $P_0$  is a point on two surfaces with normal vectors  $\mathbf{n}_1, \mathbf{n}_2$ , then the tangent line to the curve of intersection is given by

$$\mathbf{r}(t) = \mathbf{P}_0 + t(\mathbf{n}_1 \times \mathbf{n}_2)$$

- **Suggested Exercises for 12.6:**
  - Finding tangent planes & normal lines to surfaces of the form  $f(x, y, z) = c$ : 1-8
  - Finding tangent planes & normal lines to surfaces of the form  $z = f(x, y)$ : 9-12
  - Finding tangent lines to curves of intersection: 13-18

## 12.7 Extreme Values and Saddle Points

- Local Extreme Values

- Let  $f$  be a function of many variables defined on a region containing the point  $P_0$ .
  - \*  $f(P_0)$  is a **local maximum** if it is the largest nearby value (there exists an open region around  $P_0$  over which no greater value of  $f$  exists)
  - \*  $f(P_0)$  is a **local minimum** if it is the smallest nearby value (there exists an open region around  $P_0$  over which no lesser value of  $f$  exists)
- Local max/mins are also known as **local extrema**.

- Critical Points

- The **critical points** for a function  $f$  of many variables are the points in the domain where

$$\nabla f = 0 \text{ or } \nabla f \text{ DNE}$$

- Critical points occur when there is a horizontal tangent plane or no tangent plane.

- First Derivative Test for Local Extreme Values

- The local extreme values of a function always occur at critical points.

- Saddle Points

- Not every critical point gives a local extreme value.
- The **saddle points** of  $f$  are the critical points which don't yield local extreme values.

- Discriminant Function

- The **discriminant** (sometimes called “Hessian”) of  $f(x, y)$  is the function

$$f_D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

- Second Derivative Test for Local Extreme Values of  $f(x, y)$ 
  - If  $f_D(a, b) > 0$  and  $f_{xx}(a, b) < 0$ , then  $f(a, b)$  is a local maximum.
  - If  $f_D(a, b) > 0$  and  $f_{xx}(a, b) > 0$ , then  $f(a, b)$  is a local minimum.
  - If  $f_D(a, b) < 0$ , then  $f$  has a saddle point at  $(a, b)$ .
  - If  $f_D(a, b) = 0$ , then the test is inconclusive.
- Absolute Extrema on Closed and Bounded Regions
  - Let  $f$  be a function of many variables defined on a region containing the point  $P_0$ .
    - \*  $f(P_0)$  is the **absolute maximum** of  $f$  if it is the largest value in the range of  $f$
    - \*  $f(P_0)$  is the **absolute minimum** of  $f$  if it is the smallest value in the range of  $f$
  - Absolute max/mins are also known as **absolute extrema**.
  - Every continuous function of many variables with a closed and bounded domain has absolute extrema.
- Finding Absolute Extrema of  $f(x, y)$  on a Closed and Bounded Region  $D$ 
  - The following points are candidates for giving the absolute extrema:
    - \* Critical points within  $D$ .
    - \* Critical points on any of  $D$ 's boundary curves. (Find a relation of  $x$  and  $y$  and use that to make  $f$  a function of a single variable.)
    - \* Corners of  $D$ .
  - Plug each of these into  $f(x, y)$ . The largest of these is the absolute maximum, and the smallest of these is the absolute minimum.
- Suggested Exercises for 12.7:
  - Finding local max/min and saddle points: 1-30
  - Finding absolute max/min: 31-36

## 12.8 Lagrange Multipliers

- The Method of Lagrange Multipliers
  - The **Method of Lagrange Multipliers** says that if  $f(P)$  is a function of many variables which has an absolute extreme value on the restricted domain  $\{P : g(P) = c\}$ , and  $f, g$  are differentiable functions such that  $\nabla g \neq \mathbf{0}$ , then the absolute extreme value occurs satisfies

$$\nabla f = \lambda \nabla g \text{ and } g = c$$

for some real number  $\lambda$ .

- **Suggested Exercises for 12.8:**
  - Finding absolute extrema using the Method of Lagrange Multipliers: 1-30



### 13.1 Double and Iterated Integrals over Rectangles

- Volume as Integral of Area

- If  $A(x)$  is the area of a solid's cross-section, then its volume is

$$V = \int_a^b A(x) dx$$

- Double Integrals over Rectangles

- For a solid bounded above by  $z = f(x, y) \geq 0$  over the rectangle

$$R : a \leq x \leq b, c \leq y \leq d$$

its cross-sectional area at  $x$  is given by:

$$A(x) = \int_c^d f(x, y) dy$$

- Thus its volume is the **iterated integral**:

$$V = \int_a^b A(x) dx = \int_a^b \int_c^d f(x, y) dy dx$$

- Similarly, its cross-sectional area at  $y$  and volume may be given by:

$$A(y) = \int_a^b f(x, y) dx$$

$$V = \int_c^d A(y) dy = \int_c^d \int_a^b f(x, y) dx dy$$

- We also represent its volume as a **double integral**:

$$V = \iint_R f(x, y) dA$$

- If  $f(x, y) \not\geq 0$ , then the double integral represents **net volume**: volume above the  $xy$ -plane minus volume below the  $xy$ -plane.

- Suggested Exercises for 13.1:

- Evaluating iterated integrals with constant bounds: 1-12
- Evaluating double integrals over rectangles: 13-28

### 13.2 Double Integrals over General Regions

- Double Integrals over Nonrectangular Regions

- For a solid bounded above by  $z = f(x, y) \geq 0$  over the region

$$R : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$$

its cross-sectional area at  $x$  is given by:

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y) dy$$

- Thus its volume is the **iterated integral**:

$$V = \int_a^b A(x) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

- Similarly, for a solid bounded above by  $z = f(x, y) \geq 0$  over the region

$$R : h_1(y) \leq x \leq h_2(y), a \leq y \leq b$$

its cross-sectional area at  $x$  is given by:

$$A(y) = \int_{h_1(y)}^{h_2(y)} f(x, y) dx$$

$$V = \int_a^b A(y) dy = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

- We also represent its volume as a **double integral**:

$$V = \iint_R f(x, y) dA$$

- If  $f(x, y) \not\geq 0$ , then the double integral represents **net volume**: volume above the  $xy$ -plane minus volume below the  $xy$ -plane.

- Finding Limits of Integration

1. Sketch the region and label bounding curves
2. Determine if it is easier to describe bottom/top bounds

$$g_1(x) \leq y \leq g_2(x)$$

or left/right bounds

$$h_1(y) \leq x \leq h_2(y)$$

For  $g_1(x) \leq y \leq g_2(x)$ :

3. Find the  $x$ -limits of integration  $a, b$  by finding the leftmost, rightmost  $x$ -values in the region:

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

For  $h_1(y) \leq x \leq h_2(y)$ :

3. Find the  $y$ -limits of integration  $c, d$  by finding the bottommost, topmost  $y$ -values in the region:

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

- Swapping Variables of Integration

- You can only swap the order of integration of an iterated integral by first converting to a double-integral, and using the above steps.

- Properties of Double Integrals

1. Zero Integral

$$\iint_R 0 dA = 0$$

2. Constant Multiple

$$\iint_R cf(x, y) dA = c \iint_R f(x, y) dA$$

## 3. Sum/Difference

$$\iint_R f(x, y) \pm g(x, y) dA = \iint_R f(x, y) dA \pm \iint_R g(x, y) dA$$

## 4. Domination

If  $f(x, y) \leq g(x, y)$  for all  $(x, y) \in R$ , then

$$\iint_R f(x, y) dA \leq \iint_R g(x, y) dA$$

## 5. Additivity

If  $R$  can be split into two regions  $R_1, R_2$ , then

$$\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA$$

- **Suggested Exercises for 13.2:**

- Evaluating nonrectangular double integrals: 1-6, 11-14
- Finding limits of integration: 7-10, 33-44
- Swapping order of integration: 25-32

### 13.3 Area by Double Integration

- Areas of Regions in the Plane

- The area of a region  $R$  in the plane is

$$A = \iint_R dA = \iint_R 1 \, dA$$

- Average Value of a Function of Two Variables

- The average value of  $f(x, y)$  over the region  $R$  is defined to be

$$\text{Avg Val} = \frac{1}{\text{area of } R} \iint_R f(x, y) \, dA$$

- **Suggested Exercises for 13.3:**

- Finding areas of regions: 1-8
- Finding average values of functions: 15-18

### 13.5 Triple Integrals in Rectangular Coordinates

- Hypervolume as Integral of Volume

- A hypersolid is a region of  $\mathbb{R}^4$ , that is, a set of ordered 4-tuples  $(x, y, z, w)$ .
- If  $V(x)$  is the volume of a four-dimensional hypersolid's cross-section, then its hypervolume is

$$HV = \int_a^b V(x) dx$$

- Applications include modeling density within 3D space:  $(x, y, z, \delta)$ .
- Hypervolume in  $xyz\delta$ -space represents mass.

- Triple Integrals over Rectangular Boxes

- For a hypersolid bounded above by  $w = f(x, y, z) \geq 0$  over the rectangular box

$$D : a_1 \leq x \leq b_1, a_2 \leq y \leq b_2, a_3 \leq z \leq b_3$$

its cross-sectional volume at  $x$  is given by:

$$V(x) = \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x, y, z) dz dy$$

- Thus its hypervolume is the iterated integral:

$$HV = \int_{a_1}^{b_1} V(x) dx = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x, y, z) dz dy dx$$

- The *constant bounds* of this iterated integral and differentials may be swapped around.
- We also represent its hypervolume as the **triple integral**

$$HV = \iiint_D f(x, y, z) dV$$

- If  $w = f(x, y, z) \not\geq 0$ , then the triple integral represents net hypervolume.

- Triple Integrals over Other Solids

- For a general solid with bottom/top surface

$$h_1(x, y) \leq z \leq h_2(x, y)$$

and shadow in the  $xy$  plane bounded by

$$a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$$

the triple integral over the solid may be expressed by the iterated integral:

$$\iiint_D f(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x, y)}^{h_2(x, y)} f(x, y, z) dz dy dx$$

- Other orders of integration can be attained by using shadows in other coordinate planes and/or swapping order of integration for the shadow.

- Volumes of Regions in Space

- The volume of a solid  $D$  in space is

$$V = \iiint_D dV = \iiint_D 1 dV$$

- Average Value of a Function of Three Variables

- The average value of  $f(x, y, z)$  over the solid  $D$  is defined to be

$$\text{Avg Val} = \frac{1}{\text{volume of } D} \iiint_D f(x, y, z) dV$$

- Triple Integral Properties

- The properties for double integrals in Section 13.2 similarly hold for triple integrals.

- **Suggested Exercises for 13.5:**

- Evaluating triple integrals: 7-20
- Finding volumes of solids: 23-36
- Finding the average value of functions: 37-40

### 13.8 Substitution in Multiple Integrals

- Transformations

- Two similar regions in 2D space can be transformed by a “nice” pair of functions

$$\mathbf{r}(u, v) = \mathbf{r}(\mathbf{s}) = \langle x(\mathbf{s}), y(\mathbf{s}) \rangle = \langle x(u, v), y(u, v) \rangle$$

that map points in a  $uv$  plane to the  $xy$  plane.

- Two similar solids in 3D space can be transformed by a “nice” triple of functions

$$\mathbf{r}(u, v, w) = \mathbf{r}(\mathbf{s}) = \langle x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s}) \rangle = \langle x(u, v, w), y(u, v, w), z(u, v, w) \rangle$$

that map points in a  $uvw$  space to the  $xyz$  space.

- The Jacobian

- The Jacobian of a 2D transformation given by  $\mathbf{r}(u, v)$  is the determinant

$$\mathbf{r}_J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial \mathbf{r}}{\partial \mathbf{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

- The Jacobian of a 3D transformation given by  $\mathbf{r}(u, v, w)$  is the determinant

$$\mathbf{r}_J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{\partial \mathbf{r}}{\partial \mathbf{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- 2D Substitution

- Suppose that the region  $R$  in the  $xy$ -plane is the result of applying the transformation  $\mathbf{r}(u, v)$  to the region  $G$  in the  $uv$ -plane.
- Then it follows that

$$\iint_R f(x, y) \, dx \, dy = \iint_G f(x(u, v), y(u, v)) |\mathbf{r}_J(u, v)| \, du \, dv$$



- 3D Substitution

- Suppose that the solid  $D$  in  $xyz$  space is the result of applying the transformation  $\mathbf{r}(u, v, w)$  to the region  $H$  in  $uvw$  space.

- Then it follows that

$$\begin{aligned} & \iiint_D f(x, y, z) \, dx \, dy \, dz \\ &= \iiint_H f(x(u, v, w), y(u, v, w), z(u, v, w)) |\mathbf{r}_J(u, v, w)| \, du \, dv \, dw \end{aligned}$$

- **Suggested Exercises for 13.8:**

- 2D Jacobians, Transformations, and substitutions: 1-10

### 13.4 Double Integrals in Polar Form

- Integrating over Regions expressed using Polar Coordinates

- The polar coordinate transformation

$$\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta \rangle$$

from polar  $G$  into Cartesian  $R$  yields

$$\iint_R f(x, y) dA = \iint_G f(r \cos \theta, r \sin \theta) r dr d\theta$$

- **Suggested Exercises for 13.4:**

- Changing Cartesian integrals to polar integrals: 1-16
- Finding integrals over polar regions: 17-22

### 13.7 Triple Integrals in Cylindrical and Spherical Coordinates

- Cylindrical Coordinates

- The cylindrical coordinate transformation

$$\mathbf{r}(r, \theta, z) = \langle r \cos \theta, r \sin \theta, z \rangle$$

from cylindrical  $H$  into Cartesian  $D$  yields

$$\iiint_D f(x, y, z) dV = \iiint_H f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

- Spherical Coordinates

- The spherical coordinate transformation

$$\mathbf{r}(\rho, \phi, \theta) = \langle \rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi \rangle$$

from spherical  $H$  into Cartesian  $D$  yields

$$\iiint_D f(x, y, z) dV = \iiint_H f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\phi d\theta$$

- **Suggested Exercises for 13.7:**

- Cylindrical coordinate integrals: 1-20
- Finding integrals over polar regions: 21-38