## Sections 10.1 - 10.2 Overview

- Three-Dimensional Coordinates (10.1)
  - Distance between points in 3D space
  $$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
  - Simple planes in 3D Space
  $$x = a,\ y = b,\ z = c$$
  - Spheres in 3D Space
  $$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$
- Vectors (10.2)
  - Definition of a Vector
    * A vector $\mathbf{v} = \overrightarrow{v}$ is a mathematical object which stores length (magnitude) and direction, and can be thought of as a directed line segment.
    * Two vectors with the same length and direction are considered equal, even if they aren't in the same position.
    * We often (but not always) assume the initial point (the one without an arrow) lays at the origin.
  - Component Form
  $\langle v_x, v_y, v_z\rangle$ is equal to the vector with initial point at $(0, 0, 0)$ and terminal point at $(v_x, v_y, v_z)$.
  - 2D vs 3D Vectors
  $$\langle a, b\rangle = \langle a, b, 0\rangle$$
  - Position Vector
  If $P = (a, b, c)$ is a point, then $\mathbf{P} = \langle a, b, c\rangle$ is its **position vector**.
  We assume $(a, b, c) = \langle a, b, c\rangle$.
  - Vector Between Points
  The vector from $P_1 = (x_1, y_1, z_1)$ to $P_2 = (x_2, y_2, z_2)$ is
  $$\mathbf{P_1P_2} = \overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$$

- Length of a Vector

$$|\mathbf{v}| = |\langle v_1, v_2, v_3 \rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

- The Zero Vector

$$\mathbf{0} = \overrightarrow{0} = \langle 0, 0, 0 \rangle$$

- Vector Operations
  * Addition

$$\langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$$

  * Scalar Multiplication

$$k \langle v_1, v_2, v_3 \rangle = \langle kv_1, kv_2, kv_3 \rangle$$

- Vector Operation Properties
  1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
  4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
  5. $0\mathbf{u} = \mathbf{0}$
  6. $1\mathbf{u} = \mathbf{u}$
  7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
  8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
  9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$
- Unit Vectors
  * A **unit vector** or **direction** is any vector whose length is 1.
  * Standard unit vectors
    · $\mathbf{i} = \langle 1, 0, 0 \rangle$
    · $\mathbf{j} = \langle 0, 1, 0 \rangle$
    · $\mathbf{k} = \langle 0, 0, 1 \rangle$
  * Standard Unit Vector Form:

$$\langle v_x, v_y, v_z \rangle = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

  * Length-Direction Form:

$$\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$$

## 10.3 The Dot Product

- Dot Product

$$\mathbf{u}\cdot\mathbf{v} = \langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$$

- Angle between vectors

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

- Alternate Dot Product formula

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$$

- Orthogonal Vectors
  - $\mathbf{u}, \mathbf{v}$ are orthogonal if $\mathbf{u}\cdot\mathbf{v} = 0$
  - $\mathbf{u}, \mathbf{v}$ are orthogonal if the angle between them is $\frac{\pi}{2} = 90°$
  - $\mathbf{0}$ is orthogonal to every vector
- Dot Product Properties
  1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
  2. $(c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v}) = c(\mathbf{u}\cdot\mathbf{v})$
  3. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$
  4. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$
  5. $\mathbf{0}\cdot\mathbf{u} = 0$
- Projection Vector

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

- Work

$$W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos\theta$$

- **Suggested Exercises for 10.3**
  - Finding and applying dot products: 1-8
  - Work done by a constant vector force: 39-40

## 10.4 The Cross Product

- Right-hand rule

  Any method for determining a special orthogonal direction used throughout mathematics and physics, with respect to an ordered pair of vectors $\mathbf{u}, \mathbf{v}$

- Unit Normal Vector

  The vector $\mathbf{n}$ orthogonal to an ordered pair of vectors $\mathbf{u}, \mathbf{v}$ following the right-hand rule

- Cross Product

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin\theta)\mathbf{n}$$

- Parallel Vectors
  - $\mathbf{u}, \mathbf{v}$ are parallel if $\mathbf{u} \times \mathbf{v} = 0$
  - $\mathbf{u}, \mathbf{v}$ are parallel if the angle between them is $0 = 0°$ or $\pi = 180°$
  - $\mathbf{0}$ is parallel to every vector

- Cross Product Properties
  1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
  2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
  3. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
  4. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
  5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
  6. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

- Standard Unit Vector Cross Products
  1. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
  2. $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
  3. $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

- Parallelogram Area The area of a parallelogram determined by $\mathbf{u}, \mathbf{v}$ is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$$

- Determinants
  - 2x2 Determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

  - 3x3 Determinant

$$\begin{aligned}
\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_3b_2c_1 + a_1b_3c_2 + a_2b_1c_3)
\end{aligned}$$

- Computing Cross Products

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} &= \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle \\
&= \langle u_2v_3 - u_3v_2 \,,\, u_3v_1 - u_1v_3 \,,\, u_1v_2 - u_2v_1 \rangle
\end{aligned}$$

  Shortcut "long multiplication" method:

$$\begin{array}{rccccccl}
\langle & u_1 & , & u_2 & , & u_3 & \rangle \\
\times\langle & v_1 & , & v_2 & , & v_3 & \rangle \\
\hline
\langle & u_2v_3 - u_3v_2 & , & u_3v_1 - u_1v_3 & , & u_1v_2 - u_2v_1 & \rangle
\end{array}$$

- Torque

$$\overrightarrow{\tau} = \mathbf{r} \times \mathbf{F} = (|\mathbf{r}||\mathbf{F}|\sin\theta)\mathbf{n}$$

- Triple Scalar (or "Box") Product

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

  Its absolute value $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ gives the volume of a parallelpiped determined by the three vectors.

- **Suggested Exercises for 10.4**
  - Finding cross products: 1-14
  - Finding areas and unit normal vectors using cross products: 15-18
  - Finding volumes using cross products: 19-22
  - Computing torque: 25-26

## 10.5 Lines and Planes in Space

- Vector Equation for a Line
$$\mathbf{r}(t) = \mathbf{P_0} + t\mathbf{v}$$
for $-\infty < t < \infty$

- Parametric Equations for a Line
$$x = x_0 + tv_1, y = y_0 + tv_2, z = z_0 + tv_3$$
for $-\infty < t < \infty$

- Line Passing through a pair of points
$$\mathbf{r}(t) = \mathbf{P_0} + t(\mathbf{P_0P_1}) = (1-t)\mathbf{P_0} + t\mathbf{P_1}$$
for $-\infty < t < \infty$

- Line Segment joining a pair of points
$$\mathbf{r}(t) = \mathbf{P_0} + t(\mathbf{P_0P_1}) = (1-t)\mathbf{P_0} + t\mathbf{P_1}$$
for $0 \leq t \leq 1$

- Distance from a Point to a Line
$$d = \frac{|\mathbf{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

- Equation for a Plane
$$\mathbf{n} \cdot (\mathbf{P_0P}) = 0$$
$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

- Line of Intersection of Two Planes
$$\mathbf{r}(t) = \mathbf{P} + t(\mathbf{n_1} \times \mathbf{n_2})$$

- Distance from a Point to a Plane
$$d = \frac{|\mathbf{PS} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

- **Suggested Exercises for 10.5**
  - Finding parametric equations for lines: 1-12
  - Finding parametrizations for line segments: 13-20
  - Finding equations for planes: 21-26
  - Distance from a point to a line: 33-38
  - Distance from a point to a plane: 39-44

## 10.6 Cylinders and Quadratic Surfaces

- Sketching surfaces
  - To sketch a 3D surface, sketch planar cross-sections
    - $z = c$ is parallel to $xy$ plane
    - $y = b$ is parallel to $xz$ plane
    - $x = a$ is parallel to $yz$ plane
- Cylinders
  - A **cylinder** is any surface generated by moving a planar along a line normal to that plane.
  - A 3D surface defined by a function of only two variables results in a cylinder.
- Quadric Surfaces
  - A **quadric surface** is any surface defined by a second degree equation of $x, y, z$:
$$Ax^2 + Bx + Cy^2 + Dy + Ez^2 + Fz = G$$
  - Most helpful to consider the cross-sections in each of the coordinate planes.
- Ellipsoids
  - Cross-sections in the coordinate planes include
    - Three ellipses
- Elliptical Cone
  - Cross-sections in the coordinate planes include
    - Two double-lines
    - One point (with parallel ellipses)
  - Cross-sections parallel to the point cross-section are ellipses.

- Elliptical Paraboloid
  - Cross-sections in the coordinate planes include
    * Two parabolas
    * One point (with parallel ellipses)
  - Cross-sections parallel to the point cross-section are ellipses.
- Hyperbolic Paraboloid
  - Cross-sections in the coordinate planes include
    * Two parabolas (with parallel parabolas)
    * One double line (with parallel hyperbolas)
- Hyperboloid of One Sheet
  - Cross-sections in the coordinate planes include
    * Two hyperbolas
    * One ellipsis (with parallel hyperbolas)
- Hyperboloid of Two Sheets
  - Cross-sections in the coordinate planes include
    * Two hyperbola
    * One empty cross-section (with parallel hyperbolas)
  - Cross-sections parallel to the empty cross-section are ellipses.
- **Suggested Exercises for 10.6**
  - Identify surfaces from equations: 1-12
  - Sketching surfaces: 13-44

## 11.1 Vector Functions and their Derivatives

- Curves, Paths, and Vector Functions
  - A **position function** maps a moment in time to a position on a path. It can be defined with **parametric equations**

$$x = x(t)$$

$$y = y(t)$$

$$z = z(t)$$

    or with a **vector function**

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

  - $x(t), y(t), z(t)$ are called **component functions**
- Vector Function Limits
  - If the value of the vector function $\mathbf{r}(t)$ becomes arbitrarily close to the vector $\mathbf{L}$ as values of $t$ close to $t_0$ are plugged into the function, then the **limit of $\mathbf{r}(t)$ as $t$ approaches $t_0$** is $\mathbf{L}$, or

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

    * (Precise definition:) Suppose there exists a function $\delta(\epsilon)$ so that for all positive numbers $\epsilon > 0$ and all numbers $t$ where $|\mathbf{r}(t) - \mathbf{L}| < \epsilon$, it follows that $|t - t_0| < \delta(\epsilon)$. Then we say that $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$.
  - Note that

$$\lim_{t \to t_0} \mathbf{r}(t) = \left\langle \lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \right\rangle$$

- Continuity of Vector Functions
  - The function $\mathbf{r}(t)$ is **continuous at a point** $t_0$ if

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$$

  - The function $\mathbf{r}(t)$ is **continuous** if

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$$

    for all $t_0$ in its domain.

- $\mathbf{r}(t)$ is continuous exactly when $f(t), g(t), h(t)$ are all continuous.

- Derivatives of Vector Functions
  - $\dfrac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{\Delta t \to 0} \dfrac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$
  - $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$
  - $\mathbf{r}(t)$ is **differentiable** if $\mathbf{r}'(t)$ is defined for every value of $t$ is in its domain.
  - $\mathbf{r}(t)$ is **smooth** if $\mathbf{r}(t)$ is differentiable, $\mathbf{r}'(t)$ is continuous, and $\mathbf{r}'(t) \neq 0$
  - $\mathbf{r}'(t_0)$ is a **tangent vector** to the curve where $t = t_0$
  - The **tangent line** to a curve given by the vector function:

$$\mathbf{l}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$$

- Vectors and Physics
  - Position: $\mathbf{r}(t)$
  - Velocity: $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}$
  - Speed: $|\mathbf{v}(t)|$
  - Direction: $\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$
    * (Remember that $\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$)
  - Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$

- Differentiation Rules for Vector Functions
  1. Constant Function Rule
  $$\frac{d}{dt}[\mathbf{C}] = \mathbf{0}$$
  2. Constant Multiple Rules
  $$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$
  $$\frac{d}{dt}[f(t)\mathbf{C}] = f'(t)\mathbf{C}$$
  3. Sum and Difference Rules
  $$\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$$

4. Scalar Product Rule

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f(t)\mathbf{u}'(t) + f'(t)\mathbf{u}(t)$$

5. Dot Product Rule

$$\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}(t)\cdot\mathbf{v}'(t) + \mathbf{u}'(t)\cdot\mathbf{v}(t)$$

6. Cross Product Rule

$$\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}(t)\times\mathbf{v}'(t) + \mathbf{u}'(t)\times\mathbf{v}(t)$$

7. Chain Rule

$$\frac{d\mathbf{u}}{dt} = \frac{d}{dt}[\mathbf{u}(f(t))] = \mathbf{u}'(f(t))f'(t) = \frac{d\mathbf{u}}{df}\frac{df}{dt}$$

- Derivative of a Constant Length Vector Function
  - If $|\mathbf{r}(t)| = c$ always, then
  $$\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$$
  - Thus the derivative of a constant length vector function is perpindicular to the original.
- **Suggested Exercises for 11.1**
  - Position/Velocity/Acceleration Vectors: 1-14

## 11.2 Integrals of Vector Functions

- Antiderivatives of Vector Functions
  - If $\mathbf{R}'(t) = \mathbf{r}(t)$, then $\mathbf{R}(t)$ is an **antiderivative** of $\mathbf{r}(t)$.
  - The **indefinite integral** $\displaystyle\int \mathbf{r}(t)\,dt$ is the collection of all the antiderivatives of $\mathbf{r}(t)$.

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}$$

$$\int \mathbf{r}(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$$

- Definite Integrals

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

$$\int_a^b \mathbf{r}(t)dt = [\mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

- Initial Value Problems
  - If we know $\mathbf{r}'(t)$ and $\mathbf{r}(t_0)$, then

$$\mathbf{r}(t) = \mathbf{R}'(t) + \mathbf{r}(t_0) - \mathbf{R}'(t_0)$$

- Ideal Projectile Motion
  - Assume the following:
    * The acceleration acting on a projectile is $\langle 0, -g\rangle$
    * The launch position is the origin $\langle 0, 0\rangle = \mathbf{0}$
    * The launch angle is $\alpha$
    * The initial velocity is $\mathbf{v_0}$, and initial speed is $v_0 = |\mathbf{v_0}|$
  - This results in the initial value problem:

$$\mathbf{a}(t) = \langle 0, -g\rangle$$

$$\mathbf{v}(0) = \langle v_0 \cos\alpha, v_0 \sin\alpha\rangle$$

$$\mathbf{r}(0) = \langle 0, 0\rangle$$

- The velocity function solves to

$$\mathbf{v}(t) = \langle v_0 \cos\alpha, -gt + v_0 \sin\alpha\rangle$$

- The position function solves to

$$\mathbf{r}(t) = \left\langle (v_0 \cos\alpha)t, -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t \right\rangle$$

  with parametric equations

$$x = (v_0 \cos\alpha)t$$

$$y = -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t$$

- The parabolic position curve can be expressed as

$$y = -\left(\frac{g}{2v_0^2 \cos^2\alpha}\right)x^2 + (\tan\alpha)x$$

- Properties of ideal projectile motion beginning at origin:

$$y_{max} = \frac{(v_0 \sin\alpha)^2}{2g}$$

$$t_{tot} = \frac{2v_0 \sin\alpha}{g}$$

$$R = \frac{v_0^2}{g}\sin 2\alpha$$

- If we assume the initial position is instead $\mathbf{r}(0) = \langle x_0, y_0\rangle$, then the position function changes to

$$\mathbf{r}(t) = \left\langle (v_0 \cos\alpha)t + x_0, -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t + y_0 \right\rangle$$

- **Suggested Exercises for 11.2**
  - Vector function integrals: 1-6
  - Vector function initial value problems: 7-12
  - Ideal projectile motion: 15-21

## 11.3 Arc Length in Space

- Arc Length along a Space Curve
  - Approximation

$$L \approx \sum_{i=0}^{n} |\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)| = \sum_{i=1}^{n} \left| \frac{\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)}{\Delta t} \right| \Delta t$$

  - Definition

$$L = \int_a^b \left| \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \right| dt = \int_a^b |\mathbf{v}(t)|\, dt$$

  - Arclength Parameter

$$s(t) = \int_0^t |\mathbf{v}(\tau)| d\tau$$

$$\frac{ds}{dt} = |\mathbf{v}(t)| = \text{speed}$$

- Unit Tangent Vector

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

- **Suggested Exercises for 11.3**
  - Unit tangent vectors and arc length: 1-8
  - Arc length parameter: 11-14

## 11.4 Curvature of a Curve

- Curvature

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$$

- Curvature of a Circle
  - The curvature of a circle with radius $a$ is constantly

$$\kappa = \frac{1}{a}$$

- Principal Unit Normal Vector

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$$

- Circles of Curvature
  - The circle which:
    1. is tangent to a curve at a point
    2. has the same curvature as the curve at that point
    3. lies on the concave side of the curve, in the direction of $\mathbf{N}$
  - Radius: $a = \dfrac{1}{\kappa}$.
  - Center: $\langle x_0, y_0\rangle = \mathbf{r}(t_0) + a\mathbf{N}$.
  - Equations:

$$(x-x_0)^2 + (y-y_0)^2 = a^2$$

$$\mathbf{c}(t) = \langle a\sin t + x_0, a\cos t + y_0\rangle, 0 \le t \le 2\pi$$

- **Suggested Exercises for 11.4:**
  - Find $\mathbf{T}, \mathbf{N}, \kappa$: 1-4, 9-16
  - Circles of Curvature: 21-22

### 11.5 Tangental and Normal Components of Acceleration

- Binormal Unit Vector
$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$
- Right-handed vector frames
  - $\mathbf{i}, \mathbf{j}, \mathbf{k}$
  - $\mathbf{T}, \mathbf{N}, \mathbf{B}$
- Tangental and Normal Components of Acceleration
$$\mathbf{a} = \left(\frac{d^2s}{dt^2}\right)\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N} + 0\mathbf{B}$$
  - Tangental component
$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}|$$
  - Normal component
$$a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$
- Torsion
  - Magnitude of torsion
$$|\tau| = \left|\frac{d\mathbf{B}}{ds}\right|$$
  - Signed torsion
$$\frac{d\mathbf{B}}{ds} = (-\tau)\mathbf{N}$$
$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\frac{1}{|\mathbf{v}|}\left(\frac{d\mathbf{B}}{dt} \cdot \mathbf{N}\right)$$
$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$$
- **Suggested Exercises for 11.5:**
  - Finding tangental and normal components of acceleration: 1-6
  - Finding $\mathbf{B}$ and $\tau$: 9-16

## 11.6 Velocity and Acceleration in Polar Coordinates

- Polar Coordinates $(r, \theta)$
  - Cartesian to Polar
$$r = x^2 + y^2, \theta = \text{Arctan}\left(\frac{y}{x}\right)$$
  - Polar to Cartesian
$$x = r\cos\theta, y = r\sin\theta$$
- Cylindrical Coordinates $(r, \theta, z)$
  - Cartesian to Cylindrical
$$r = x^2 + y^2, \theta = \text{Arctan}\left(\frac{y}{x}\right), z = z$$
  - Cylindrical to Cartesian
$$x = r\cos\theta, y = r\sin\theta, z = z$$
- Polar/Cylindrical Unit Vectors
$$\mathbf{u}_r = \langle \cos\theta, \sin\theta \rangle, \mathbf{u}_\theta = \langle -\sin\theta, \cos\theta \rangle$$
  - Cylindrical Right-handed frame
$$\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{k}$$
  - Derivatives
$$\frac{d}{dt}[\mathbf{u}_r] = \dot{\mathbf{u}}_r = \dot{\theta}\mathbf{u}_\theta$$
$$\frac{d}{dt}[\mathbf{u}_\theta] = \dot{\mathbf{u}}_\theta = -\dot{\theta}\mathbf{u}_r$$
  - Polar Position/Velocity/Acceleration
$$\mathbf{r} = r\mathbf{u}_r$$
$$\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta$$
$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta$$

- Cylindrical Position/Velocity/Acceleration

$$\mathbf{r} = r\mathbf{u}_r + z\mathbf{k}$$

$$\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k}$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{k}$$

- **Suggested Exercises for 11.6:**
  - Expressing $\mathbf{v}$ and $\mathbf{a}$ in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$: 1-5

## 12.1 Functions of Several Variables

- Real-Valued Functions
  - A **real-valued function** $f$ on with **domain** $D \subset \mathbb{R}^n$ is a rule that assigns a real number
    $$f(x_1, x_2, \ldots, x_n) \in \mathbb{R}$$
    to each $(x_1, x_2, \ldots, x_n) \in D$.
  - The domain of a function is assumed to be all of $\mathbb{R}^n$ except where the function is not well-defined.
  - The **range** of the function is
    $$R = \{f(x_1, x_2, \ldots, x_n) : (x_1, x_2, \ldots, x_n) \in D\}$$
- Regions
  - A subset of the $xy$-plane ($\mathbb{R}^2$) or $xyz$-space ($\mathbb{R}^3$) is known as a **region**.
  - The **ball** $B(p, \epsilon)$ is the set of points
    $$B(p, \epsilon) = \{q \in \mathbb{R}^2 : \text{the distance between } p \text{ and } q \text{ is less than } \epsilon\}$$
    Its **center** is the point $p$ and its **radius** is $\epsilon$.
  - A point $p \in \mathbb{R}^2$ is known as an **interior point** of a region $R$ if *there exists some ball* containing $p$ that lies inside $R$.
  - A point $p \in \mathbb{R}^2$ is known as a **boundary point** of a region $R$ if *every ball* containing $p$ contains some points in $R$ and some points not in $R$.
  - A point $p \in \mathbb{R}^2$ is known as an **exterior point** of a region $R$ if *there exists some ball* containing $p$ that lies outside $R$.
  - The **interior** of $R$ is the set
    $$\text{int}(R) = \{p : p \text{ is an interior point of } R\}$$
  - The **boundary** of $R$ is the set
    $$\text{bd}(R) = \{p : p \text{ is a boundary point of } R\}$$
  - The **exterior** of $R$ is the set
    $$\text{ext}(R) = \{p : p \text{ is an exterior point of } R\}$$

  - A region $R$ is **open** if it doesn't contain any of its boundary.
  - A region $R$ is **closed** if it contains all of its boundary.
  - A region $R$ is **bounded** if it can be contained within a ball.
  - A region $R$ is **unbounded** if it cannot be contained within a ball.

- Sketching Functions

  - Level curve
$$\{(x, y) : f(x, y) = c\}$$
  - Surface $z = f(x, y)$
$$\{(x, y, f(x, y)) : (x, y) \in \mathrm{Dom}(f)\}$$
  - Contour curve
$$\{(x, y, c) : f(x, y) = c\}$$
  - Level surface
$$\{(x, y, z) : f(x, y, z) = c\}$$

- **Suggested Exercises for 12.1:**

  - Identifying and describing domains, ranges, level curves, boundaries: 1-12
  - Relating level curves to graphs: 13-18
  - Sketching surfaces and level curves: 19-28
  - Finding level curves through a point: 29-32
  - Sketching level surfaces: 33-40
  - Finding level surfaces through a point: 41-44

## 12.2 Limits and Continuity in Higher Dimensions

- Limits
  - If the value of the vector function $f(P)$ becomes arbitrarily close to the number $L$ as points $P$ close to $P_0$ are plugged into the function, then the **limit of** $f(P)$ **as** $P$ **approaches** $P_0$ is $L$:

$$\lim_{P\to P_0} f(P) = L$$

    * Precise definition:
      Suppose there exists a function $\delta(\epsilon)$ so that for all positive numbers $\epsilon > 0$ and all numbers $P$ where $|f(P)-L| < \epsilon$, it follows that $|\mathbf{P}-\mathbf{P_0}| < \delta(\epsilon)$. Then we say that $\lim_{P\to P_0} f(P) = L$.
  - Note that values of $f$ must approach $L$ no matter which direction we approach $p_0$.

- Limit Laws

  1. Sum/Difference Law

$$\lim_{p\to p_0}(f(p) \pm g(p)) = \lim_{p\to p_0} f(p) \pm \lim_{p\to p_0} g(p)$$

  2. Product Law

$$\lim_{p\to p_0}(f(p) \cdot g(p)) = \lim_{p\to p_0} f(p) \cdot \lim_{p\to p_0} g(p)$$

  3. Constant Multiple Law

$$\lim_{p\to p_0}(kf(p)) = k \lim_{p\to p_0} f(p)$$

  4. Quotient Law

$$\lim_{p\to p_0} \frac{f(p)}{g(p)} = \frac{\lim_{p\to p_0} f(p)}{\lim_{p\to p_0} g(p)}$$

  5. Power Law (for $r, s \in \mathbb{Z}$)

$$\lim_{p\to p_0}(f(p))^{r/s} = \left(\lim_{p\to p_0} f(p)\right)^{r/s}$$

- Computing Limits
  - $\lim_{P \to P_0} f(x) = \lim_{x \to x_0} f(x)$
  - Factoring and cancelling is a useful strategy.
  - L'Hopital's Rule does not apply for multiple variable limits.
- Showing a Limit DNE
  - If
    $$\lim_{x \to x_0} h(x, f(x)) \neq \lim_{x \to x_0} h(x, g(x))$$
    then $\lim_{P \to P_0} h(x, y)$ DNE.
- Continuity
  - A function $f(p)$ is **continuous** if $\lim_{p \to p_0} f(p) = f(p_0)$ for all points $p_0$ in its domain.
  - If a multi-variable function is composed of continuous single-variable functions, then it is also continuous.
- **Suggested Exercises for 12.2:**
  - Computing limits: 1-26
  - Showing limits don't exist: 35-42

## 12.3 Partial Derivatives

- Partial Derivatives
  - The **partial derivative of $f$ with respect to $x_i$** is the limit
$$\frac{\partial f}{\partial x_i} = f_{x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$
  - By the definition, we can see that to compute partial derivatives with respect to a variable, we can treat all other variables as constants and differentiate as normal.
- Higher Order Partial Derivatives
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = (f_y)_x = f_{yx}$$
$$\frac{\partial^2 g}{\partial z \partial z} = \frac{\partial^2 g}{\partial z^2} = g_{zz}$$
  - Mixed Derivative Theorem:
$$f_{xy} = f_{yx}$$
- **Suggested Exercises for 12.3:**
  - Finding first-order partial derivatives: 1-38
  - Finding second-order partial derivatives: 41-50
  - Finding partial derivatives from the limit definition: 53-56

## 12.4 The Chain Rule

- Gradient Vector Function

$$\nabla f = \langle f_{x_1}, \dots, f_{x_n} \rangle = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$$

- Chain Rule
  - For single variable functions:

$$\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \dots$$

  - For multi-variable functions:

$$\frac{\partial f}{\partial t_i} = \nabla f \cdot \frac{\partial \mathbf{r}}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \dots$$

- Differentiation by Substitution
  - The Chain Rule can be avoided by “plugging in” functions and using single-variable calculus.
- Total Derivative
  - If the variables $x, y, z$ of a function $f$ are dependent on each other, then

$$\frac{df}{dx} = \nabla f \cdot \frac{d\mathbf{r}}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx}$$

- Implicit Differentiation
  - If $f(x, y) = c$ defines $y$ as a function of $x$, then

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

- **Suggested Exercises for 12.4:**
  - Finding $\frac{dw}{dt}$ for $w = f(x(t), y(t), z(t))$: 1-6
  - Finding partial derivatives for compositions of multi-variable functions: 7-12, 33-38
  - Using partial derivatives for implicit differentiation: 25-28

## 12.5 Directional Derivatives and Gradient Vectors

- Directional Derivative
  - The **directional derivative of $f$ in the direction of $\mathbf{u}$** is

$$\frac{df}{ds_{\mathbf{u}}} = \left(\frac{df}{ds}\right)_{\mathbf{u}} = D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$

  - If $\theta$ is the angle between $\nabla f$ and $\mathbf{u}$, then

$$\frac{df}{ds_{\mathbf{u}}} = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$$

  - The angle $\theta$ between $\nabla f$ and $\mathbf{u}$ determines the value of the directional derivative at a fixed point $p_0$:
    * Max: $|\nabla f_{p_0}|$ at $\theta = 0$
    * Zero: $0$ at $\theta = \frac{\pi}{2}$
    * Min: $-|\nabla f_{p_0}|$ at $\theta = \pi$

- Normal Vector to a Level Curve
  - $\nabla f$ is normal to the level curve $f(x, y) = c$ for every point $(x, y)$ in the domain of $f$.

- Gradient Rules

1. Constant Multiple Rule
$$\nabla(kf) = k\nabla f$$
2. Sum Rule
$$\nabla(f + g) = \nabla f + \nabla g$$
3. Difference Rule
$$\nabla(f - g) = \nabla f - \nabla g$$
4. Product Rule
$$\nabla(fg) = g(\nabla f) + f(\nabla g)$$
5. Quotient Rule
$$\nabla\left(\frac{f}{g}\right) = \frac{g(\nabla f) - f(\nabla g)}{g^2}$$

- **Suggested Exercises for 12.5:**
  - Finding $\nabla f$ at a point: 1-8
  - Finding directional derivatives: 9-16
  - Finding the direction of maximal/minimal rate of change: 17-22
  - Finding the direction of no instantaneous change: 27-28

## 12.6 Tangent Planes and Differentials

- Normal Vector to a Level Surface
  - $\nabla f$ is normal to the level surface $f(x, y, z) = c$ for every point $(x, y, z)$ in the domain of $f$.
- Normal Vector to the Surface $z = f(x, y)$
  - If $g(x, y, z) = f(x, y) - z$, then

    $$\nabla g = \langle f_x, f_y, -1 \rangle$$

    is normal to the surface $z = f(x, y)$ for every point $(x, y)$ in the domain of $f$.
- Tangent Line to Curve of Intersection of Two Surfaces
  - If $P_0$ is a point on two surfaces with normal vectors $\mathbf{n_1}, \mathbf{n_2}$, then the tangent line to the curve of intersection is given by

    $$\mathbf{r}(t) = \mathbf{P_0} + t(\mathbf{n_1} \times \mathbf{n_2})$$
- **Suggested Exercises for 12.6:**
  - Finding tangent planes & normal lines to surfaces of the form $f(x, y, z) = c$: 1-8
  - Finding tangent planes & normal lines to surfaces of the form $z = f(x, y)$: 9-12
  - Finding tangent lines to curves of intersection: 13-18

## 12.7 Extreme Values and Saddle Points

- Local Extreme Values
  - Let $f$ be a function of many variables defined on a region containing the point $P_0$.
    * $f(P_0)$ is a **local maximum** if it is the largest nearby value (there exists an open region around $P_0$ over which no greater value of $f$ exists)
    * $f(P_0)$ is a **local minimum** if it is the smallest nearby value (there exists an open region around $P_0$ over which no lesser value of $f$ exists)
  - Local max/mins are also known as **local extrema**.
- Critical Points
  - The **critical points** for a function $f$ of many variables are the points in the domain where
$$\nabla f = 0 \text{ or } \nabla f \text{ DNE}$$
  - Critical points occur when there is a horizontal tangent plane or no tangent plane.
- First Derivative Test for Local Extreme Values
  - The local extreme values of a function always occur at critical points.
- Saddle Points
  - Not every critical point gives a local extreme value.
  - The **saddle points** of $f$ are the critical points which don't yield local extreme values.
- Discriminant Function
  - The **discriminant** (sometimes called "Hessian") of $f(x, y)$ is the function
$$f_D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

- Second Derivative Test for Local Extreme Values of $f(x, y)$
  - If $f_D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
  - If $f_D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
  - If $f_D(a, b) < 0$, then $f$ has a saddle point at $(a, b)$.
  - If $f_D(a, b) = 0$, then the test is inconclusive.
- Absolute Extrema on Closed and Bounded Regions
  - Let $f$ be a function of many variables defined on a region containing the point $P_0$.
    - $f(P_0)$ is the **absolute maximum** of $f$ if it is the largest value in the range of $f$
    - $f(P_0)$ is the **absolute minimum** of $f$ if it is the smallest value in the range of $f$
  - Absolute max/mins are also known as **absolute extrema**.
  - Every continuous function of many variables with a closed and bounded domain has absolute extrema.
- Finding Absolute Extrema of $f(x, y)$ on a Closed and Bounded Region $D$
  - The following points are candidates for giving the absolute extrema:
    - Critical points within $D$.
    - Critical points on any of $D$'s boundary curves. (Find a relation of $x$ and $y$ and use that to make $f$ a function of a single variable.)
    - Corners of $D$.
  - Plug each of these into $f(x, y)$. The largest of these is the absolute maximum, and the smallest of these is the absolute minimum.
- **Suggested Exercises for 12.7:**
  - Finding local max/min and saddle points: 1-30
  - Finding absolute max/min: 31-36

## 12.8 Lagrange Multipliers

- The Method of Lagrange Multipliers
  - The **Method of Lagrange Multipliers** says that if $f(P)$ is a function of many variables which has an absolute extreme value on the restricted domain $\{P : g(P) = c\}$, and $f, g$ are differentiable functions such that $\nabla g \neq \mathbf{0}$, then the absolute extreme value occurs satisfies

    $$\nabla f = \lambda \nabla g \text{ and } g = c$$

    for some real number $\lambda$.

- **Suggested Exercises for 12.8:**
  - Finding absolute extrema using the Method of Lagrange Multipliers: 1-30

## 13.1 Double and Iterated Integrals over Rectangles

- Volume as Integral of Area
  - If $A(x)$ is the area of a solid's cross-section, then its volume is
    $$V = \int_a^b A(x)\,dx$$
- Double Integrals over Rectangles
  - For a solid bounded above by $z = f(x,y) \geq 0$ over the rectangle
    $$R : a \leq x \leq b, c \leq y \leq d$$
    its cross-sectional area at $x$ is given by:
    $$A(x) = \int_c^d f(x,y)\,dy$$
  - Thus its volume is the **iterated integral**:
    $$V = \int_a^b A(x)\,dx = \int_a^b \int_c^d f(x,y)\,dy\,dx$$
  - Similarly, its cross-sectional area at $y$ and volume may be given by:
    $$A(y) = \int_a^b f(x,y)\,dx$$
    $$V = \int_c^d A(y)\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$$
  - We also represent its volume as a **double integral**:
    $$V = \iint_R f(x,y)\,dA$$
  - If $f(x,y) \ngeq 0$, then the double integral represents **net volume**: volume above the $xy$-plane minus volume below the $xy$-plane.
- **Suggested Exercises for 13.1:**
  - Evaluating iterated integrals with constant bounds: 1-12
  - Evaluating double integrals over rectangles: 13-28

## 13.2 Double Integrals over General Regions

- Double Integrals over Nonrectangular Regions
  - For a solid bounded above by $z = f(x, y) \geq 0$ over the region

    $$R : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$$

    its cross-sectional area at $x$ is given by:

    $$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

  - Thus its volume is the **iterated integral**:

    $$V = \int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

  - Similarly, for a solid bounded above by $z = f(x, y) \geq 0$ over the region

    $$R : h_1(y) \leq x \leq h_2(y), a \leq y \leq b$$

    its cross-sectional area at $x$ is given by:

    $$A(y) = \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx$$

    $$V = \int_a^b A(y)\, dy = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

  - We also represent its volume as a **double integral**:

    $$V = \iint_R f(x, y)\, dA$$

  - If $f(x, y) \not\geq 0$, then the double integral represents **net volume**: volume above the $xy$-plane minus volume below the $xy$-plane.

- Finding Limits of Integration

  1. Sketch the region and label bounding curves
  2. Determine if it is easier to describe bottom/top bounds

  $$g_1(x) \leq y \leq g_2(x)$$

  or left/right bounds

  $$h_1(y) \leq x \leq h_2(y)$$

  For $g_1(x) \leq y \leq g_2(x)$:

  3. Find the $x$-limits of integration $a, b$ by finding the leftmost, rightmost $x$-values in the region:

  $$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

  For $h_1(y) \leq x \leq h_2(y)$:

  3. Find the $y$-limits of integration $c, d$ by finding the bottommost, topmost $y$-values in the region:

  $$\iint\limits_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

- Swapping Variables of Integration

  – You can only swap the order of integration of an iterated integral by first converting to a double-integral, and using the above steps.

- Properties of Double Integrals

  1. Zero Integral

  $$\iint\limits_R 0\,dA = 0$$

  2. Constant Multiple

  $$\iint\limits_R cf(x,y)\,dA = c\iint\limits_R f(x,y)\,dA$$

3. Sum/Difference

$$\iint\limits_R f(x,y) \pm g(x,y)\,dA = \iint\limits_R f(x,y)\,dA \pm \iint\limits_R g(x,y)\,dA$$

4. Domination
If $f(x,y) \leq g(x,y)$ for all $(x,y) \in R$, then

$$\iint\limits_R f(x,y)\,dA \leq \iint\limits_R g(x,y)\,dA$$

5. Additivity
If $R$ can be split into two regions $R_1, R_2$, then

$$\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA$$

- **Suggested Exercises for 13.2:**
  - Evaluating nonrectangular double integrals: 1-6, 11-14
  - Finding limits of integration: 7-10, 33-44
  - Swapping order of integration: 25-32

## 13.3 Area by Double Integration

- Areas of Regions in the Plane
  - The area of a region $R$ in the plane is

$$A = \iint_R dA = \iint_R 1\, dA$$

- Average Value of a Function of Two Variables
  - The average value of $f(x, y)$ over the region $R$ is defined to be

$$\text{Avg Val} = \frac{1}{\text{area of } R} \iint_R f(x, y)\, dA$$

- **Suggested Exercises for 13.3:**
  - Finding areas of regions: 1-8
  - Finding average values of functions: 15-18

## 13.5 Triple Integrals in Rectangular Coordinates

- Hypervolume as Integral of Volume
  - A hypersolid is a region of $\mathbb{R}^4$, that is, a set of ordered 4-tuples $(x, y, z, w)$.
  - If $V(x)$ is the volume of a four-dimensional hypersolid's cross-section, then its hypervolume is
$$HV = \int_a^b V(x)\,dx$$
  - Applications include modeling density within 3D space: $(x, y, z, \delta)$.
  - Hypervolume in $xyz\delta$-space represents mass.

- Triple Integrals over Rectangular Boxes
  - For a hypersolid bounded above by $w = f(x, y, z) \geq 0$ over the rectangular box
$$D : a_1 \leq x \leq b_1, a_2 \leq y \leq b_2, a_3 \leq z \leq b_3$$
its cross-sectional volume at $x$ is given by:
$$V(x) = \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x, y, z)\,dz\,dy$$
  - Thus its hypervolume is the iterated integral:
$$HV = \int_{a_1}^{b_1} V(x)\,dx = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x, y, z)\,dz\,dy\,dx$$
  - The *constant bounds* of this iterated integral and differentials may be swapped around.
  - We also represent its hypervolume as the **triple integral**
$$HV = \iiint_D f(x, y, z)\,dV$$
  - If $w = f(x, y, z) \ngeq 0$, then the triple integral represents net hypervolume.

- Triple Integrals over Other Solids
  - For a general solid with bottom/top surface

    $$h_1(x,y) \leq z \leq h_2(x,y)$$

    and shadow in the $xy$ plane bounded by

    $$a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$$

    the triple integral over the solid may be expressed by the iterated integral:

    $$\iiint\limits_D f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$
  - Other orders of integration can be attained by using shadows in other coordinate planes and/or swapping order of integration for the shadow.
- Volumes of Regions in Space
  - The volume of a solid $D$ in space is

    $$V = \iiint\limits_D dV = \iiint\limits_D 1\,dV$$
- Average Value of a Function of Three Variables
  - The average value of $f(x,y,z)$ over the solid $D$ is defined to be

    $$\text{Avg Val} = \frac{1}{\text{volume of } D} \iiint\limits_D f(x,y,z)\,dV$$
- Triple Integral Properties
  - The properties for double integrals in Section 13.2 similarly hold for triple integrals.
- **Suggested Exercises for 13.5:**
  - Evaluating triple integrals: 7-20
  - Finding volumes of solids: 23-36
  - Finding the average value of functions: 37-40

## 13.8 Substitution in Multiple Integrals

- Transformations
  - Two similar regions in 2D space can be transformed by a "nice" pair of functions
    $$\mathbf{r}(u,v) = \mathbf{r}(\mathbf{s}) = \langle x(\mathbf{s}), y(\mathbf{s})\rangle = \langle x(u,v), y(u,v)\rangle$$
    that map points in a $uv$ plane to the $xy$ plane.
  - Two similar solids in 3D space can be transformed by a "nice" triple of functions
    $$\mathbf{r}(u,v,w) = \mathbf{r}(\mathbf{s}) = \langle x(\mathbf{s}), y(\mathbf{s}), z(\mathbf{s})\rangle = \langle x(u,v,w), y(u,v,w), z(u,v,w)\rangle$$
    that map points in a $uvw$ space to the $xyz$ space.
- The Jacobian
  - The Jacobian of a 2D transformation given by $\mathbf{r}(u,v)$ is the determinant
    $$\mathbf{r}_J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial \mathbf{r}}{\partial \mathbf{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$
  - The Jacobian of a 3D transformation given by $\mathbf{r}(u,v,w)$ is the determinant
    $$\mathbf{r}_J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{\partial \mathbf{r}}{\partial \mathbf{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$
- 2D Substitution
  - Suppose that the region $R$ in the $xy$-plane is the result of applying the transformation $\mathbf{r}(u,v)$ to the region $G$ in the $uv$-plane.
  - Then it follows that
    $$\iint\limits_R f(x,y)\,dx\,dy = \iint\limits_G f(x(u,v), y(u,v))|\mathbf{r}_J(u,v)|\,du\,dv$$

- 3D Substitution
  - Suppose that the solid $D$ in $xyz$ space is the result of applying the transformation $\mathbf{r}(u, v, w)$ to the region $H$ in $uvw$ space.
  - Then it follows that

$$\iiint_D f(x, y, z)\, dx\, dy\, dz$$

$$= \iiint_H f(x(u, v, w), y(u, v, w), z(u, v, w))|\mathbf{r}_J(u, v, w)|\, du\, dv\, dw$$

- **Suggested Exercises for 13.8:**
  - 2D Jacobians, Transformations, and substitutions: 1-10

## 13.4 Double Integrals in Polar Form

- Integrating over Regions expressed using Polar Coordinates
  - The polar coordinate transformation
    $$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle$$
    from polar $G$ into Cartesian $R$ yields
    $$\iint\limits_R f(x,y)\,dA = \iint\limits_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$
- **Suggested Exercises for 13.4:**
  - Changing Cartesian integrals to polar integrals: 1-16
  - Finding integrals over polar regions: 17-22

### 13.7 Triple Integrals in Cylindrical and Spherical Coordinates

- Cylindrical Coordinates
  - The cylindrical coordinate transformation

    $$\mathbf{r}(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$$

    from cylindrical $H$ into Cartesian $D$ yields

    $$\iiint\limits_D f(x,y,z)\,dV = \iiint\limits_H f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

- Spherical Coordinates
  - The spherical coordinate transformation

    $$\mathbf{r}(\rho,\phi,\theta) = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$$

    from spherical $H$ into Cartesian $D$ yields

    $$\iiint\limits_D f(x,y,z)\,dV = \iiint\limits_H f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

- **Suggested Exercises for 13.7:**
  - Cylindrical coordinate integrals: 1-20
  - Finding integrals over polar regions: 21-38

## 14.1 Line Integrals

- Line Integrals with Respect to Arclength
  - The area of the ribbon with base along the curve $C$ in $xyz$ space and height given by $f(x, y, z)$ is given by the **line integral of** $f(x, y, z)$ **over** $C$ **with respect to arclength** $s$:

$$\int_C f(x, y, z)\, ds$$

  - Arclength line integrals can be evaulated by finding a smooth parametrization $\mathbf{r}(s)$ of the curve $C$ with respect to arclength $s$ for $a \leq s \leq b$:

$$\int_C f(x, y, z)\, ds = \int_{s=a}^{s=b} f(x(s), y(s), z(s))\, ds$$

  - If $\mathbf{r}(t)$ is an arbitrary parametrization of $C$ for $a \leq t \leq b$, then

$$\int_C f(x, y, z)\, ds = \int_{t=a}^{t=b} f(x(t), y(t), z(t))|\mathbf{v}(t)|\, dt$$

- Additivity

$$\int_{C_1+C_2} f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds$$

- Reversing Arclength Line Integrals

$$\int_C f\, ds = \int_{-C} f\, ds$$

- **Suggested Exercises for 14.1:**
  - Identifying vector equations for graphs: 1-8
  - Evaluating line integrals: 9-22

## 14.2 Vector Fields, Work, Circulation, and Flux

- Line Integrals with Respect to Variables
  - The net projected area of the ribbon with base curve $C$ and height $f(x, y, z)$ with respect to the $x$-axis is given by the **line integral of** $f(x, y, z)$ **over** $C$ **with respect to** $x$:
  $$\int_C f(x, y, z)\, dx$$
  (similar for $y$, $z$)
  - Line integrals with respect to variables can be evaluated by finding a parametrization $\mathbf{r}(t)$ for the curve $C$:
  $$\int_C f(x, y, z)\, dx = \int_a^b f(x(t), y(t), z(t))\frac{dx}{dt}\, dt$$
  - Such integrals have the property
  $$\int_{-C} f\, dx = -\int_C f\, dx$$

- Vector Fields
  - A **vector field** is a function
  $$\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$$
  ($\mathbf{F} = \langle M, N, P\rangle$ for short) which assigns a vector to each point in its domain.
  - Gradient functions $\nabla f = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$ and transformations $\langle x(u, v, w), y(u, v, w), z(u, v, w)\rangle$ are examples of vector fields.

- Line Integrals of Vector Fields
  - The **line integral of** $\mathbf{F} = \langle M, N, P\rangle$ **over** $C$ is given by
  $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\, dx + N\, dx + P\, dz$$
  gives the sum of the line integrals of each component of $\mathbf{F}$ with respect to each variable $x, y, z$.

- These line integrals can be calculated by using parametrizations of $C$:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dx + P\,dz = \int_a^b \left( M\frac{dx}{dt} + N\frac{dx}{dt} + P\frac{dz}{dt} \right) dt$$

$$= \int_a^b \mathbf{F} \cdot \mathbf{v}\,dt = \int_a^b \mathbf{F} \cdot \mathbf{T}\,ds$$

- It follows that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

- Work over a Smooth Curve
  - Work is given by the product of force and displacement:

$$W = \mathbf{F} \cdot \mathbf{D}$$

  - So work over a smooth curve can be approximated by the Riemann sum:

$$W \approx \sum_{i=1}^{n} \mathbf{F}(x_i, y_i, z_i) \cdot \Delta\mathbf{r_i}$$

  - We limit this sum to infinity to define work over a smooth curve:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

- Flow
  - The **flow** of a fluid along a curve $C$ is defined to be the line integral

$$\text{Flow} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

  - If $C$ is closed (its starting point and ending point are the same), then the flow is also known as the **circulation**.

- Flux
  - The **flux** of $\mathbf{F}$ across $C$ is
$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds$$
where $\mathbf{n}(x, y)$ is the outward unit normal vector to $C$.
  - If $C$ is oriented counter-clockwise, then
$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C \mathbf{F} \cdot (\mathbf{k} \times \mathbf{T})\, ds$$
$$= \int_C \langle M, N \rangle \cdot \left\langle \frac{dy}{ds}, -\frac{dx}{ds} \right\rangle ds = \int_C M\, dy - N\, dx$$
- **Suggested Exercises for 14.2:**
  - Work over a curve: 7-22
  - Circulation, flow, and flux: 23-28, 37-40

### 14.3 Path Independence, Potential Functions, and Conservative Fields

- Several Equivalencies for Conservative Fields

  The following are all equivalent for piecewise smooth curves and vector fields with continuous first derivatives:

  - $\mathbf{F} = \langle M, N, P\rangle$ is a **conservative field**.
  - $\mathbf{F}\cdot d\mathbf{r} = M\,dx + N\,dy + P\,dz$ is **exact**.
  - $\int \mathbf{F}\cdot d\mathbf{r}$ is **path independent**: the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ only depends on the endpoints of the curve $C$.
  - There exists a **potential function** $f$ such that $\nabla f = \mathbf{F}$.
  - (Closed Loop Property of Conservative Fields) $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed loop $C$ in $D$.
  - (Fundamental Theorem of Line Integrals) $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = f(B) - f(A)$ for every path $C$ connecting $A$ to $B$.
  - (Component Test for Conservative Fields) $\dfrac{\partial P}{\partial y} = \dfrac{\partial N}{\partial z}$, $\dfrac{\partial M}{\partial z} = \dfrac{\partial P}{\partial x}$, and $\dfrac{\partial N}{\partial x} = \dfrac{\partial N}{\partial y}$.

- **Suggested Exercises for 14.3:**

  - Determining if a field is conservative: 1-6
  - Finding potential functions: 7-12
  - Evaluating integrals of differential forms: 13-22

## 14.4 Green's Theorem in the Plane

- Gradient Operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

- Divergence
  - The **divergence** of a planar vector field $\mathbf{F} = \langle M, N \rangle$ is given by
  $$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \nabla \cdot \mathbf{F}$$
  In physics, divergence is often called the **flux density**.

- Spin
  - The **spin** of a planar vector field $\mathbf{F} = \langle M, N \rangle$ is given by
  $$\operatorname{spin} \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$
  In physics, spin is often called the **circulation density**.
  - Spin is also the **k-component of curl**, defined in a later section.

- Simple Curves
  - A curve which does not cross itself is said to be **simple**.

- Green's Theorem in the Plane
  - Let $C$ be a piecewise smooth, simple closed curve enclosing the region $R$ and oriented counter-clockwise. Let $\mathbf{F} = \langle M, N \rangle$ be a vector field for which $M, N$ have continuous first partial derivatives in an open region containing $R$. Then:
  $$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \operatorname{div} \mathbf{F} \, dA$$
  $$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_R \operatorname{spin} \mathbf{F} \, dA$$

- **Suggested Exercises for 14.4:**
  - Using Green's Theorem to find circulation and flux: 5-14
  - Using Green's Theorem to evaluate line integrals: 17-20

## 14.5 Surfaces and Area

- Parametrization of Surfaces
  - Vector functions of two variables
    $$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$
    may be used to parametrize surfaces in $xyz$ space.
- Smooth Vector Functions
  - A surface parametrized by $\mathbf{r}(u,v)$ is called **smooth** if
    $$\mathbf{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right\rangle, \mathbf{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right\rangle$$
    are continuous and $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ on the interior of the surface.
- Surface Area of a Parametrized Surface
  - The area of a smooth surface with parametrizing vector function $\mathbf{r}(u,v)$ for a region $R$ in the $uv$ plane is given by
    $$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$
- Implicit Surface
  - Level surfaces $F(x,y,z) = c$ are sometimes called **implicit surfaces**.
  - If $\mathbf{p}$ is a unit vector normal a coordinate plane, then the surface area defined by $F(x,y,z)$ bounded by the cylinder given by a region $R$ in that coordinate plane is
    $$\iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA$$
- Surface Area Differential
  - The integral $\iint_S d\sigma$ is used to represent surface area, and $d\sigma$ is known as the surface area differential.
    $$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA$$

- **Suggested Exercises for 14.5:**
  - Finding parametrizations of surfaces: 1-16
  - Finding surface area: 17-26

## 14.6 Surface Integrals and Flux

- Surface Integrals
  - The **surface integral** of a function $G(x, y, z)$ over a surface $S$ is given by
  $$\iint_S G(x, y, z)\, d\sigma$$
  - This integral may be computed by parametrizing $S$ with
  $$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$$
  for $(u, v) \in R$ and evaluating
  $$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x(u, v), y(u, v), z(u, v))|\mathbf{r}_u \times \mathbf{r}_v|\, dA$$
  - Or, if $S$ is given by $F(x, y, z) = c$ with a shadow $R$ in a coordinate plane normal to the unit vector $\mathbf{p}$, the surface integral can be evaluated using
  $$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, z)\frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA$$
- Orientable Surfaces
  - A surface is said to be **orientable** if it is "two-sided". More technically, it is orientable if there exists a continuous normal unit vector field $\mathbf{n}$ to the surface.
  - An real-life example of a non-orientable surface is the Mobius strip formed by twisting a strip of paper together once and taping its ends together.
- Flux in Three Dimensions
  - The flux of a three dimensional vector field $\mathbf{F}$ across an oriented surface $S$ in the direction of $\mathbf{n}$ is given by the surface integral
  $$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$$
- **Suggested Exercises for 14.6:**
  - Evaluating surface integrals: 1-14
  - Three-dimensional flux: 15-24

## 14.7 Stokes' Theorem

- Curl
  - The **curl** of a vector field $\mathbf{F}$ is defined as
  $$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$$
  - Expanding this cross product, we see
  $$\operatorname{curl} \mathbf{F} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$
  - Recalling that, for a vector field in the $xy$ plane ($z = 0$),
  $$\operatorname{spin} \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$
  we see that the curl vector measures the vector field's spin about that point on the planes parallel to $x = 0$, $y = 0$, and $z = 0$ respectively.
- Stokes' Theorem
  - Recall that the counter-clockwise circulation about a curve $C$ in the plane bounding the region $R$ can be computed by
  $$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_R \operatorname{spin} \mathbf{F} \, dA$$
  - Noting that in $\mathbb{R}^2$
  $$\operatorname{spin} \mathbf{F} = \operatorname{curl} \mathbf{F} \cdot \mathbf{k} = \nabla \times \mathbf{F} \cdot \mathbf{k}$$
  in $\mathbb{R}^3$ we may define the counterclockwise spin with respect to the vector $\mathbf{v}$ to be
  $$\operatorname{spin}_{\mathbf{v}} \mathbf{F} = \operatorname{curl} \mathbf{F} \cdot \mathbf{v} = \nabla \times \mathbf{F} \cdot \mathbf{v}$$
  - If a curve $C$ in $\mathbb{R}^3$ is the boundary of a surface $S$, and we want to compute the counter-clockwise circulation with respect to unit normal vectors $\mathbf{n}$ on the surface, we may use
  $$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_S \operatorname{spin}_{\mathbf{n}} \mathbf{F} \, d\sigma = \iint_S (\operatorname{curl} \mathbf{F} \cdot \mathbf{n}) \, d\sigma = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

- Identities and Properties
  - Due to the Mixed Derivative Theorem,
  $$\operatorname{curl} \nabla f = \nabla \times \nabla f = \mathbf{0}$$
  - If $\nabla \times \mathbf{F} = \mathbf{0}$ for every point in a region $D$, then
  $$\int\limits_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0$$
  for every curve $C$ and surface $S$ within $D$.
- **Suggested Exercises for 14.7:**
  - Using Stokes' Theorem: 1-10

## 14.8 Divergence Theorem and a Unified Theory

- Divergence Theorem
  - Divergence in $\mathbb{R}^2$ was defined as
    $$\operatorname{div}\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \nabla \cdot \mathbf{F}$$
    and is defined in $\mathbb{R}^3$ as
    $$\operatorname{div}\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \nabla \cdot \mathbf{F}$$
  - In both cases it measures the tendency of the vector field to point outward from a point.
  - The Divergence Theorem lets us measure the flux on a closed surface $S$ by integrating over the divergence within its bounded region $D$:
    $$\text{Flux} = \iint\limits_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint\limits_D \operatorname{div}\mathbf{F}\,dV = \iiint\limits_D \nabla\cdot\mathbf{F}\,dV$$
- The Unified Theory
  - The unified theory notes that in order to compute circulation and flux over a closed curve or surface, we may consider the spin/curl and divergence over the region bounded by that curve or surface.
  - Let $C$ be a counter-clockwise closed curve in $\mathbb{R}^2$ bounding the region $R$.
    $$\text{Circulation of } \mathbf{F} \text{ around } C = \iint\limits_R \operatorname{spin}\mathbf{F}\,dA = \iint\limits_R \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\,dA$$
    $$\text{Flux of } \mathbf{F} \text{ across } C = \iint\limits_R \operatorname{div}\mathbf{F}\,dA$$
  - Let $C$ be a closed curve in $\mathbb{R}^3$ counter-clockwise to $\mathbf{n}$ bounding the surface $S$.
    $$\text{Circulation of } \mathbf{F} \text{ around } C = \iint\limits_S \operatorname{spin}_{\mathbf{n}}\mathbf{F}\,d\sigma = \iint\limits_R \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,d\sigma$$
  - Let $S$ be a closed surface in $\mathbb{R}^3$ bounding the solid $D$.
    $$\text{Flux of } \mathbf{F} \text{ across } S = \iiint\limits_D \operatorname{div}\mathbf{F}\,dV$$
- **Suggested Exercises for 14.8:**
  - Using the Divergence Theorem: 5-16