Game-theoretic strengthenings of Menger's property

AMS Sectional Meeting at UNCG

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November 9, 2014

The Menger property

Definition

A space X is Menger if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ of open covers of X there exists a sequence $\langle \mathcal{F}_0, \mathcal{F}_1, \ldots \rangle$ such that $\mathcal{F}_n \subseteq \mathcal{U}_n$, $|\mathcal{F}_n| < \omega$, and $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Proposition

X is σ-compact $\Rightarrow$ X is Menger $\Rightarrow$ X is Lindelöf.

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The Menger game

Game

Let $Men_{C,F}(X)$ denote the *Menger game* with players $\mathscr{C}$, $\mathscr{F}$. In round n, $\mathscr{C}$ chooses an open cover $\mathcal{C}_n$, followed by $\mathscr{F}$ choosing a finite subcollection $\mathcal{F}_n \subseteq \mathcal{C}_n$.
$\mathscr{F}$ wins the game, that is, $\mathscr{F} \uparrow Men_{C,F}(X)$ if $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover for the space X, and $\mathscr{C}$ wins otherwise.

Theorem (Hurewicz 1926, effectively)

X is Menger if and only if $\mathscr{C} \not\uparrow Men_{C,F}(X)$.

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Menger suspected that the subsets of the real line with his property were exactly the σ-compact spaces; however:

Theorem (Fremlin, Miller 1988)

There are ZFC examples of non-σ-compact subsets of the real line which are Menger.

But metrizable non-σ-compact Menger spaces will be *undetermined* for the Menger game.

Theorem (Telgarsky 1984, Scheepers 1995)

Let X be metrizable. $\mathscr{F} \uparrow Men_{C,F}(X)$ if and only if X is σ-compact.

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Note that for Lindelöf spaces, metrizability is characterized by regularity and secound countability.

Sketch of Scheeper's proof:

- Using second-countability and the winning strategy for $\mathscr{F}$, construct certain subsets K_s for $s \in \omega^{<\omega}$ such that $X = \bigcup_{s \in \omega^{<\omega}} K_s$.
- Using regularity, show that each K_s is compact.

By considering winning *limited-information strategies*, we'll be able to factor out this proof a bit.

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Limited information strategies

Definition

A *(perfect information) strategy* has knowledge of all the past moves of the opponent.

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A *k-tactical strategy* has knowledge of only the past k moves of the opponent.

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A *k-Markov strategy* has knowledge of only the past k moves of the opponent and the round number.

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Obviously,

$$\mathscr{A} \underset{k\text{-tact}}{\uparrow} G \Rightarrow \mathscr{A} \underset{k\text{-mark}}{\uparrow} G \Rightarrow \mathscr{A} \underset{\text{(perfect)}}{\uparrow} G$$

But tactical strategies aren't interesting for the Menger game.

Proposition

For any $k < \omega$, $\mathscr{F} \underset{k\text{-tact}}{\uparrow} Men_{C,F}(X)$ if and only if X is compact.

Effectively, $\mathscr{F}$ needs some sort of seed to prevent from being stuck in a loop: there's nothing stopping $\mathscr{C}$ from playing the same open cover during every round of the game.

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Comparitively, Markov strategies are very powerful.

Proposition

If X is σ-compact, then $\mathscr{F} \underset{1\text{-}mark}{\uparrow} Men_{C,F}(X)$.

Proof.

Let $X = \bigcup_{n<\omega} K_n$. During round n, $\mathscr{F}$ picks a finite subcollection of the last open cover played by $\mathscr{C}$ (the only one $\mathscr{F}$ remembers) which covers K_n. $\square$

Without assuming regularity, we can't quite reverse the implication, but we can get close.

Definition

A subset Y of X is *relatively compact* if for every open cover for X, there exists a finite subcollection which covers Y.

Proposition

If X is σ-relatively-compact, then $\mathscr{F} \underset{1\text{-mark}}{\uparrow} Men_{C,F}(X)$.

Proposition

For regular spaces, $Y \subseteq X$ is relatively compact if and only if $\overline{Y}$ is compact. So σ-relatively-compact regular spaces are exactly the σ-compact regular spaces.

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Theorem

$\mathscr{F} \underset{1\text{-}mark}{\uparrow} Men_{C,F}(X)$ *if and only if* X *is* σ*-relatively-compact.*

Proof.

Let $\sigma(\mathcal{U}, n)$ represent a 1-Markov strategy. For every open cover $\mathcal{U} \in \mathfrak{C}$, $\sigma(\mathcal{U}, n)$ witnesses relative compactness for the set

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

If X is not σ-relatively compact, fix $x \notin R_n$ for any $n < \omega$. Then $\mathscr{C}$ can beat σ by choosing $\mathcal{U}_n \in \mathfrak{C}$ during each round such that $x \notin \bigcup \sigma(\mathcal{U}_n, n)$. □

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Corollary

For regular spaces X, $\mathscr{F} \underset{1\text{-}mark}{\uparrow} Men_{C,F}(X)$ if and only if X is σ-compact.

We can complete Telgarsky's/Scheeper's result by showing the following:

Theorem

For second countable spaces X, $\mathscr{F} \uparrow Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{1\text{-}mark}{\uparrow} Men_{C,F}(X)$.

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Proof

It's sufficient to assume all covers contain only basic open sets, and since X is a second-countable space, there are only countably many finite collections of basic open sets.

Let σ be a perfect information strategy, and suppose we've defined open covers $\mathcal{U}_{s'}$ for $s' \leq s \in \omega^{<\omega}$. If $\mathcal{U}$ is an arbitrary open cover, then there are only countably many choices for the finite subcollection

$$\sigma(\mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{U}) \subseteq \mathcal{U}$$

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Proof (cont.)

Thus we may define open covers $\mathcal{U}_{s^\frown\langle n\rangle}$ for each $n < \omega$ such that for an arbitrary open cover $\mathcal{U}$,

$$\sigma(\mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{U}) = \sigma(\mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle n\rangle})$$

for some $n < \omega$.

Let $t : \omega \to \omega^{<\omega}$ be a bijection. During round n and seeing only the latest open cover $\mathcal{U}$, $\mathscr{F}$ may use the following 1-Markov strategy:

$$\tau(\mathcal{U}, n) = \sigma(\mathcal{U}_{t(n)\restriction 1}, \dots, \mathcal{U}_{t(n)}, \mathcal{U})$$

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Suppose there exists a counter-attack $\langle \mathcal{V}_0, \mathcal{V}_1, \dots \rangle$ which defeats the 1-Markov strategy τ. Then there exists $f : \omega \to \omega$ such that, if $\mathcal{V}^n = \mathcal{V}_{t^{-1}(f \restriction n)}$

$$\begin{aligned} x &\notin \bigcup \tau(\mathcal{V}^n, t^{-1}(f \restriction n)) \\ &= \bigcup \sigma(\mathcal{U}_{f\restriction 1}, \dots, \mathcal{U}_{f\restriction n}, \mathcal{V}^n) \\ &= \bigcup \sigma(\mathcal{U}_{f\restriction 1}, \dots, \mathcal{U}_{f\restriction n}, \mathcal{U}_{f\restriction (n+1)}) \end{aligned}$$

Thus $\langle \mathcal{U}_{f\restriction 1}, \mathcal{U}_{f\restriction 2}, \dots \rangle$ is a successful counter-attack by $\mathscr{C}$ against the perfect information strategy σ, showing $\mathscr{O} \uparrow Men_{C,F}(X) \Rightarrow \mathscr{O} \underset{\text{1-mark}}{\uparrow} Men_{C,F}(X)$. $\square$

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It's speculated that there are spaces X_k such that for the Banach-Mazur game, $\mathscr{N} \underset{k+1\text{-tact}}{\uparrow} BM_{E,N}(X_k)$ but $\mathscr{N} \underset{k\text{-tact}}{\not\uparrow} BM_{E,N}(X_k)$. (This is true for $k = 1$.)

Theorem

$\mathscr{F} \underset{k+2\text{-mark}}{\uparrow} Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men_{C,F}(X)$.

Proof.

$$\tau(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) = \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}, \dots, \mathcal{U}}_{k+1-m}, \underbrace{\mathcal{V}, \dots, \mathcal{V}}_{m+1} \rangle, (n+1)(k+2)+m)$$

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Having knowledge of *two* of an opponent's moves allows a player to react when the opponent changes her moves, something impossible to do using a 1-tactical or 1-Markov strategy.

Definition

Let $\kappa^\dagger = \kappa \cup \{\infty\}$ be the *one point Lindelöf-ication* of discrete κ: neighborhoods of ∞ are exactly the co-countable sets containing it.

$\kappa^\dagger$ is a simple space which is a regular and Lindelöf, but not second-countable or σ-compact. Thus $\mathscr{F} \underset{\text{1-mark}}{\not\uparrow} Men_{C,F}(\kappa^\dagger)$, but it's easy to see that $\mathscr{F} \uparrow Men_{C,F}(\kappa^\dagger)$. What about 2-Markov strategies?

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The game $Men_{C,F}(\kappa^\dagger)$ essentially involves choosing countable and finite subsets of κ. Conveniently, there already exists an infinite game also involving the countable and finite subsets of κ in the literature.

Game (Scheepers 1991)

Let $Fill_{C,F}^{\cup,\subset}(\kappa)$ denote the *strict union filling game* with two players $\mathscr{C}$, $\mathscr{F}$. In round 0, $\mathscr{C}$ chooses $C_0 \in [\kappa]^{\leq\omega}$, followed by $\mathscr{F}$ choosing $F_0 \in [\kappa]^{<\omega}$. In round $n+1$, $\mathscr{C}$ chooses $C_{n+1} \in [\kappa]^{\leq\omega}$ such that $C_{n+1} \supset C_n$, followed by $\mathscr{F}$ choosing $F_{n+1} \in [\kappa]^{<\omega}$. $\mathscr{F}$ wins the game if $\bigcup_{n<\omega} F_n \supseteq \bigcup_{n<\omega} C_n$; otherwise, $\mathscr{C}$ wins.

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Definition

For two functions f, g we say f is *almost compatible* with g ($f \wr g$) if $|\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(x) \neq g(x)\}| < \omega$.

Definition

$S(\kappa)$ states that there exist functions $f_A : A \to \omega$ for each $A \in [\kappa]^{\leq\omega}$ such that $|f_A^{-1}(n)| < \omega$ for all $n < \omega$ and $f_A \wr f_B$ for all $A, B \in [\kappa]^{\omega}$.

Theorem (Scheepers 1991)

$S(\omega_1)$; $\neg S(\kappa)$ for $\kappa > 2^\omega$; $Con(S(2^\omega) + \neg CH)$.

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Theorem (Scheepers 1991)

If $S(\kappa)$, then $\mathscr{F} \underset{2\text{-}tact}{\uparrow} Fill^{\cup,\subset}_{C,F}(\kappa)$.

We slightly adapt Scheeper's game to characterize $Men_{C,F}(\kappa^\dagger)$ purely combinatorially.

Definition

Let $Fill^{\cap}_{C,F}(\kappa)$ denote the *intersection filling game* analogous to $Fill^{\cup,\subset}_{C,F}(\kappa)$, except that $\mathscr{C}$ has no restriction on the countable sets she chooses, but $\mathscr{F}$ need only ensure that $\bigcup_{n<\omega} F_n \supseteq \bigcap_{n<\omega} C_n$ to win the game.

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Theorem

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Men_{C,F}(\kappa^\dagger)$ *if and only if* $\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa)$.

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Proof.

Let $f_A : A \to \omega$ witness $S(\kappa)$. Then we define the winning 2-Markov strategy σ as follows:

$$\sigma(\langle A \rangle, 0) = \{\alpha \in A : f_A(\alpha) = 0\}$$

$$\sigma(\langle A, B \rangle, n+1) = \{\alpha \in A \cap B : f_B(\alpha) \leq n+1 \text{ or } f_A(\alpha) \neq f_B(\alpha)\}$$

$\square$

Corollary

$\mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathit{Men}_{C,F}(\omega_1^{\dagger})$, but $\mathscr{F} \underset{\text{1-mark}}{\not\uparrow} \mathit{Men}_{C,F}(\omega_1^{\dagger})$.

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Question

Does $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa)$ imply $S(\kappa)$?

Question

Are $\mathscr{F} \uparrow Men_{C,F}(X)$ and $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$ distinct?

A “yes” for the first is a “yes” for the second, since $\neg S(\kappa)$ for $\kappa > 2^\omega$ but $\mathscr{F} \uparrow Men_{C,F}(\kappa^\dagger)$ for all κ.

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Are $\mathscr{F} \uparrow Men_{C,F}(X)$ and $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$ distinct?

A “yes” for the first is a “yes” for the second, since $\neg S(\kappa)$ for $\kappa > 2^\omega$ but $\mathscr{F} \uparrow Men_{C,F}(\kappa^\dagger)$ for all κ.

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Where does $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$ fit in with other properties between σ-(relatively-)compact and Menger?

Assuming T_3, properties which come to mind from the literature are implied by $\mathscr{F} \uparrow Men_{C,F}(X)$: e.g. Alster (Aurichi, Tall 2013), and thus productively Lindelöf (Alster 1988) and Hurewicz (Tall 2009).

Questions? Thanks for listening!