English Translation of "Stratégies gagnantes dans certains jeux topologiques"

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Abstract. We prove that on an α-favorable space for the Banach-Mazur game, there exists always an α-winning strategy depending only on α and β last moves. We give an example of a completely regular α-favorable space on which the player α has no winning strategy depending only on β last move.

Introduction. Rappelons que le jeu de *Banach-Mazur* sur un espace topologique $(X, \mathcal{F})$ est un jeu infini où deux joueurs α et β choisissent alternativement à chaque coup, un ouvert non vide contenu dans l'ouvert choisi par l'autre joueur au coup prècèdent; c'est le joueur β qui commence à jouer. Ainsi au cours d'une partie les joueurs α et β construisent deux suites d'ouverts non vides $(V_n)_{n\in N}$ et $(U_n)_{n\in N}$ respectivement, avec $V_n \supset U_n \supset V_{n+1}$; le joueur α gagne la partie si $\bigcap_{n\in N} U_n = \bigcap_{n\in N} V_n = \emptyset$.

Le jeu (ou l'espace X) est dit α-favorable si le joueur α possède une stratégie gagnante. L'interêt des espaces α-favorables tient au fait qu'ils forment une large classe d'espaces de Baire stable par produit et qui contient tous les cas classiques.

La notion de stratègie est utilisèe ici au sens des jeux à information parfaite, c'est-à-dire quá chaque coup les joueurs sont informés de tous les coups précédemment joués et un joueur peut tenir compte de ces informations dans la construction d'une stratégie. Le but de ce travail est d'étudier pour un jeu α-favorable donné l'existence de stratégies simples: plus préciseément on s'intéressera à trois types de stratégies:

Introduction. Recall that the Banach-Mazur game on a topological space $(X, \mathcal{F})$ is an infinite game where two players α and β alternately choose a non-empty open set which is a subset of the set previously chosen by the other player. β is the first player to move. The game results in two sequences, $(V_n)_{n\in N}$ chosen by α and $(U_n)_{n\in N}$ chosen by β, such that $V_n \supset U_n \supset V_{n+1}$; α wins the game in the case that $\bigcap_{n\in N} U_n = \bigcap_{n\in N} V_n = \emptyset$.

The game (or the space X) is called α-favorable if α has a winning strategy. What makes α-favorable spaces interesting is that they form a large class of Baire spaces with stable products which contains the classic cases.

The idea of strategy used here is as a perfect information game, that is to say, each round the players are informed of all previous plays, and a player may consider that information in the construction of a strategy. The aim of this work is to study if a game gives the existence of simple α-favorable strategies: to be precise, we focus on three specified types of strategies:

(I) Les stratégies σ dépendant seulement du dernier coup joué (par le joueur β), c'est-à-dire de la forme $\sigma(V_0, U_0, V_1, \ldots, U_{n-1}, V_n) = \tau(V_n)$.

(I) The strategies σ which only depend on the latest play made by β, that is to say, strategies of the form $\sigma(V_0, U_0, V_1, \ldots, U_{n-1}, V_n) = \tau(V_n)$.

(II) Les stratégies σ dépendant seulement des deux derniers coups joués (les derniers coups joués par les joueurs α et β respectivement) c'est-à-dire de la forme: $\sigma(V_0, U_0, V_1, \ldots, U_{n-1}, V_n) = \tau(U_{n-1}, V_n)$.

(II) The strategies σ which only depend on the latest plays by α and β, that is to say, of the form: $\sigma(V_0, U_0, V_1, \ldots, U_{n-1}, V_n) = \tau(U_{n-1}, V_n)$.

(III) Les stratégies σ dépendant seulement des deux dernier coups joués par le joueur β c'est-à-dire de la forme $\sigma(V_0, U_0, V_1, \ldots, U_{n-1}, V_n) = \tau(V_{n-1}, V_n)$

(III) The strategies σ which depend on only the last two plays made by β, that is, $\sigma(V_0, U_0, V_1, \ldots, U_{n-1}, V_n) = \tau(V_{n-1}, V_n)$

Nous attirons ici l'attention du lecteur à ce que certains auteurs appellent "faiblement α-favorable" ce que nous appelons "α-favorable", terme qu'ils rèservent alors pour les espaces possèdant une stratègie du type I.

We'd like to point out that some authors use "weakly α-favorable" to denote what we refer to as "α-favorable", which they reserve for when the space has a strategy of type I.

Le point de départ de ce travail était la question suivante: Est-ce que tout espace α-favorable admet une stratégie gagnante du type (I)? Cette question qui se pose naturellement a été reprise par W. G. Fleissner et K. Kunen puis D. H. Fremlin. Elle est justifiée par le fait que la réponse est positive dans tous les cas classiques. En particulier, un remarquable résultat de F. Galvin et R. Telgarsky affirme que si le joueur α posséde une stratégie gagnante qui ne épend que du dernier coup V_n et de son numéro n, alors il existe une stratégie gagnante du type (I). Enfin il découle d'un résultat de J. C. Oxtoby que le probléme analogue pour le joueur β a une réponse positive.

The starting point of this work was the question: Does every space admit a positive α-favorable strategy of type (I)? This natural question was posed by W. G. Fleissner, K. Kunen, and D. H. Fremlin. It is justified by the fact that the answer is positive in all classical cases. In particular, a remarkable result of F. Galvin and R. Telgarsky says that if the α player has a winning strategy that depends only on the last play and its V_n number n, then there exists a winning strategy of type (I). Finally it follows from a result of J. C. Oxtoby that the similar problem for player β has a positive response.

Dans ce travail on construit un espace topologique complètement régulier qui est α-favorable avec une stratégie gagnante de type (III) et sur lequel il n'existe aucune stratégie gagnante du type (I). Mais on démontre que sur tout espace α-favorable on peut construire une stratégie gagnante de type (II). (Nous avons appris par une correspondance récente que ce résultat a été obtenu indépendement par F. Galvin et R. Telgarsky [7]). En fait notre théorème sera démontré dans un cadre plus général que celui du jeu de Banach-Mazur, et qui est mieux adapté au ques a été réalisée par R. Telgarsky.

In this work we construct a completely regular topological space that has an α-favorable strategy of type (III) but no α-favorable strategy of type (I). However, it can be shown that on any α-favorable space we can find a strategy of type (II). (We learned from a recent correspondence that this result has been obtained independently by F. Galvin and R. Telgarsky [7].) In fact, our theorem will be proved in a framework more general than the Banach-Mazur game and is better adapted to the question asked by R. Telgarsky.

Je suis reconnaissant à D. H. Fremlin pour des discussions qui m'ont été très utiles pour ce travail.

I am grateful to D. H. Fremlin for our discussions which have been very helpful for this work.

1. Notations. On se donne deux ensembles E, F et on note $G = E \cup F$. Dans l'ensemble $G^{(N)}$ des suites finies d'éléments de G, on notes par $r^\frown s$ la concaténation des suites r et s. On désigne par $\mathcal{A}(G)$ le sous-ensemble de $G^{(N)}$ formé des *suites alternées* $r = (z_i)_{0 \leq i \leq n}$ (c'est-à-dire vérifiant: $z_{i+1} \in E \Leftrightarrow z_i \in F$) et par $\mathcal{A}(\mathcal{G})$ le sousensemble de G^N formé des suites alternées infinies. Si $(x_i)_{0 \leq i \leq n}$ et $(y_i)_{0 \leq i \leq n}$ sont deux suites finies de E et F respectivement on définit

1. Notations. We are given two sets E, F and denote $G = E \cup F$. We use $G^{(N)}$ to denote the set of finite sequences of elements of G, and we use $r^\frown s$ to denote the concatenation of the sequences r and s. We denote by $\mathcal{A}(G)$ the subset of $G^{(N)}$ of *alternating sequences* $r = (z_i)_{0 \leq i \leq n}$ satisfying $z_{i+1} \in E \Leftrightarrow z_i \in F$, and similarly let $\mathcal{A}(G)$ denote the subset of G^N (infinite sequences) which alternate. If $(x_i)_{0 \leq i \leq n}$ and $(y_i)_{0 \leq i \leq n}$ are two finite sequences of E and F respectively we define

$$\langle y_i; x_i \rangle_{0 \leq i \leq n} = (y_0, x_0, y_1, x_1, \ldots, y_n, x_n) \in \mathcal{A}(G)$$

$$\langle x_i; y_i \rangle_{0 \leq i \leq n} = (x_0, y_0, x_1, t_1, \ldots, x_n, y_n) \in \mathcal{A}(G)$$

On se donne deux relations $R_\alpha \subset F \times E$, et $R_\beta \subset E \times F$. On note $R = (R_\alpha \cup R_\beta) \subset G \times G$ et

We are given two relations $R_\alpha \subset F \times E$ and $R_\beta \subset E \times F$. We denote $R = (R_\alpha \cup R_\beta) \subset G \times G$ and

$$\mathcal{R} = \{(z_i)_{0 \leq i \leq n} \in \mathcal{A}(G) : z_1 R z_{i+1}, 0 \leq i \leq n\}$$

On désigne par $\mathcal{R}_{\beta,\alpha}$ (resp. par $\mathcal{R}_{\beta,\beta}$) le sous-ensemble de $\mathcal{R}$ formé des suites commençant par un élément de F et finissant par un élément de E (resp. de F); et par $\widetilde{\mathcal{R}}$ (resp. par $\widetilde{\mathcal{R}}_\beta$) l'ensemble des branches infinies de $\mathcal{R}$ (resp. de $\mathcal{R}_\beta = \mathcal{R}_{\beta,\alpha} \cup \mathcal{R}_{\beta,\beta}$). Si $\rho = \langle y_n ; x_n \rangle_{n \in N} \in \widetilde{\mathcal{R}}_\beta$ on appellera *sous-suite alternée* de ρ toute suite ρ' de la forme $\rho' = \langle y_{n_k} ; x_{m_k} \rangle_{k \in N}$ avec $n_k \leq m_k < n_{k+1}$ pour tout $k \in N$.

We denote by $\mathcal{R}_{\beta,\alpha}$ (resp. $\mathcal{R}_{\beta,\beta}$) the subset of $\mathcal{R}$ which form sequences starting with an element of F and ending with an element of E (resp. F). We denote the set $\widetilde{\mathcal{R}}$ (resp. $\widetilde{\mathcal{R}}_\beta$) the set of infinite branches of $\mathcal{R}$ (resp. $\mathcal{R}_\beta = \mathcal{R}_{\beta,\alpha} \cup \mathcal{R}_{\beta,\beta}$). If $\rho = \langle y_n ; x_n \rangle_{n \in N} \in \widetilde{\mathcal{R}}_\beta$ call its *alternating sequence* $\rho' = \langle y_{n_k} ; x_{m_k} \rangle_{k \in N}$ where $n_k \leq m_k < n_{k+1}$ for all $k \in N$.

2. Terminologie. Un *feu alternatif* (E, F, R, A) est la donnée d'un triplet (E, F, R) comme précédemment et d'un sous-ensemble A de $\widetilde{\mathcal{R}}_\beta$. On dira que β est le joueur I (celui qui commence les parties) et que α est le joueur II. La relation R sera dite la régle du jeu, $\widetilde{\mathcal{R}}_\beta$ l'ensemble des parties licites du jeu, et A lénsemble des parties licites gagnées par α.

2. Terminology. An *alternating light* (E, F, R, A) is given by a triple (E, F, R) as before and a subset $A \subseteq \widetilde{\mathcal{R}}_\beta$. We say that β is the first player (Player I) and that α is the second player (Player II). The relation R will be called the rules of the game, $\widetilde{\mathcal{R}}_\beta$ the set of legal plays of the game, and A all legal plays of the game won by α.

Une *stratégie* pour le joueur α est une application $\sigma : \mathcal{R}_{\beta,\beta} \to E$ vérifiant:

A *strategy* for the player α is a map $\sigma : \mathcal{R}_{\beta,\beta} \to E$ satisfying:

$$(r^\frown \sigma(r) \in \mathcal{R}_{\beta,\alpha}, \forall r \in \mathcal{R}_{\beta,\beta})$$

On dira qu'une stratégie σ pour le joueur α est une *tactique* si elle ne dépend que du dernier coup joué, c'est-á-dire s'il existe $\tau : F \to E$ tel que $\sigma(r^\frown y) = \tau(y)$ pour tout $r \in \mathcal{R}_{\beta,\alpha}$ et $y \in F$. De maniére analogue on dira que σ ne dépend que des deux derniers coups joués síl existe $\tau : R_\beta \to E$ tel que $\sigma(r^\frown s) = \tau(s)$ pour tous $r \in \mathcal{R}_{\beta,\beta}$ et $s \in R_\beta$.

L'ensemble $A(\sigma)$ des parties licites conformes à la stratégie σ est défini par:

We say a strategy σ for the player α is a *tactitic* if it only depends on the last move by β, that is to say, there exists $\tau : F \to E$ such that $\sigma(r^\frown y) = \tau(y)$ for all $r \in \mathcal{R}_{\beta,\alpha}$ and $y \in F$. Similarly, we say that σ depends only the last two moves if there exists $\tau : R_\beta \to E$ such that $\sigma(r^\frown s) = \tau(s)$ for all $r \in \mathcal{R}_{\beta,\beta}$ and $s \in R_\beta$.

The set $A(\sigma)$ of legal plays which follow the strategy σ is defined by:

$$A(\sigma) = \{\langle y_n; x_n\rangle_{n\in N} \in \widetilde{\mathcal{R}}_\beta : x_n = \sigma((\langle y_p, x_p\rangle_{o\leq p<n})^\frown y_n)\forall n \in N)\}$$

La *stratégie* σ est dite *gagnante* pour α si $A(\sigma) \subset A$. S'il existe une stratégie gagnante pour α le *jeu* est dit *α-favorable.*

The strategy σ is called a *winning strategy* for α if $A(\sigma) \subset A$. If there exists a winning strategy for α the game is called *α-favorable.*

3. Jeux asymptotiques pour α. (To be translated.)

3. Games asymptotic for α. (To be translated.)

4. Un countre-exemple.

`Notations 6.` Dans toute la suite on note par:

$\mathbb{R}^* = \mathbb{R} \setminus \{0\}$

$\mathcal{I}$: l'ensemble des intervalles de $\mathbb{R}^*$ qui sont bornés, ouverts, non vides et à extrémités rationnelles.

$\mathcal{D}$: la famille des parties au plus dénombrables et non vides de $\mathbb{R}^*$.

$\mathcal{S}$: l'ensemble des fonctions partielles $S : \mathcal{R}^* \to \{0,1\}$ qui sont surjectives (i.e. non constantes) et à domaine dénombrable. Cet ensemble sera muni de la relation d'inclusion $\subset$ induite par celle de $\mathbb{R}^* \times \{0,1\}$.

X : l'ensemble des applications $x : \mathcal{D} \to \mathbb{R}$ vérifiant: $\exists \Delta(x) \in \mathcal{D}$, $\exists \tau(x) \in \mathbb{R}^* \setminus \Delta(x)$ tels que:

4. A counter-example.

`Notations 6.` From now on we denote:

$\mathbb{R}^* = \mathbb{R} \setminus \{0\}$

$\mathcal{I}$: nonempty open intervals in $\mathbb{R}^*$ which are bounded by rational endpoints

$\mathcal{D}$: the family of nonempty countable subsets of $\mathbb{R}^*$.

$\mathcal{S}$: the family of nonconstant partial functions $S : \mathbb{R}^* \to \{0,1\}$ with countable domains. This set is given the partial order given by the inclusion relation $\subset$ induced by $\mathbb{R}^* \times \{0,1\}$.

X : the set of maps $x : \mathcal{D} \to \mathbb{R}$ which satisfy $\exists \Delta(x) \in \mathcal{D}$, $\exists \tau(x) \in \mathbb{R}^* \setminus \Delta(x)$ such that:

$$\begin{array}{lll} \text{(i)} & \forall D \subset \Delta(x), & x(D) = \tau(x); \\ \text{(ii)} & \forall D \not\subset \Delta(x), & x(D) = 0; \end{array}$$

On munit X de topologie de la convergence uniforme sur les parties dénombrables de $\mathcal{D}$.

Pour $I \in \mathcal{I}$, et $S \in \mathcal{S}$:

We equip X the topology of uniform convergence for the countable subsets in $\mathcal{D}$.

For $I \in \mathcal{I}$, and $S \in \mathcal{S}$:

(Translator's note: I have changed $\{f = y\}$ to $f^{-1}(y)$ throughout this paper.)

$$V[S, I] = \{x \in X : x(S^{-1}(1)) \in I \text{ and } x(\{t\}) = 0 \ \forall t \in S^{-1}(0)\}$$

`Proposition 7.` Sur l'espace complétement régulier X la famille

`Proposition 7.` On the completely regular space X, the family

$$\{V[S,I] : S \in \mathcal{S}, I \in \mathcal{I}\}$$

forme une base de topologie et on a:

forms a base for the topology on X such that:

$$(V[S_1,I_1] \supset V[S_2,I_2]) \Leftrightarrow (S_1 \subseteq S_2 \text{ and } I_1 \supset I_2)$$

Démonstration, Remarquons que si pour $\delta > 0$ on a $]-\delta,\delta[\cap I = \emptyset$ alors: $V[S,I] = \{x \in X : x(\{S = 1\}) \in I\} \cap \{x \in X : x(\{t\}) \in]-\delta,\delta[, \forall t \in \{S = 0\}\}$ puisqu'un élément de X ne prend qu'une seule valeur $\neq 0$ et que $0 \notin I$; done les $V[S,I]$ sont ouverts dans X.

Proof. There exists a $\delta > 0$ such that $(-\delta,\delta) \cap I = \emptyset$ and $V[S,I] = \{x \in X : x(S^{-1}(1)) \in I\} \cap \{x \in X : x(\{t\}) \in (-\delta,\delta)\ \forall t \in S^{-1}(0)\}$ since an element of X takes only a single value $\neq 0$ and $0 \notin I$; therefore the $V[S,I]$ are open in X.

Soit U un voisinage élémentaire d'un élément $x_0 \in X$:

Let U be a basic neighborhood about $x_0 \in X$:

$$U = \{x \in X : |x(D_i) - x_0(D_i)| < \epsilon, \forall i \in \mathbb{N}\}$$

où $\epsilon > 0$ et $\{D_i : i \in \mathbb{N}\}$ est une partie dénombrable de $\mathcal{D}$. Considérons l'élément S de $\mathcal{S}$ défini par $(\{S = 1\} = \Delta(x_0)$ et $\{S = 0\} = \bigcup_{i\in\mathbb{N}}(D_i \setminus \Delta(x_0)) \cup \{t_0\})$ oú t_0 est un élément quelconque de $\mathbb{R}^* \setminus (\bigcup_{i\in\mathbb{N}} D_i \cup \Delta(x_0))$ et soit $I \in \mathcal{I}$ tel que $\tau(x_0) \in I$ et $\mathrm{diam}(I) < \epsilon$, alors $V[S,I] \subset U$. En effet si $x \in V[S,I]$ alors $x(\Delta(x_0)) \neq 0$ done $\Delta(x_0) \subset \Delta(x)$; par suite:

for $\epsilon > 0$ and where $\{D_i : i \in \mathbb{N}\}$ is a countable subset of $\mathcal{D}$. Consider the element S of $\mathcal{S}$ defined by $(S^{-1}(1) = \Delta(x_0)$ and $S^{-1}(0) = \bigcup_{i\in\mathbb{N}}(D_i \setminus \Delta(x_0)) \cup \{t_0\})$ where t_0 is any element of $\mathbb{R}^* \setminus (\bigcup_{i\in\mathbb{N}} D_i \cup \Delta(x_0))$ and let $I \in \mathcal{I}$ such that $\tau(x_0) \in I$ and $\mathrm{diam}(I) < \epsilon$, then $V[S,I] \subset U$. Indeed, if $x \in V[S,I]$ then $x(\Delta(x_0)) \neq 0$ and therefore $\Delta(x_0) \subset \Delta(x)$; resulting in:

$$(D_i \subset \Delta(x_0) \subset \Delta(x)) \Rightarrow (x(D_i) \text{ and } x_0(D_i) \in I) \Rightarrow (|x(D_i) - x_0(D_i)| < \epsilon),$$

$$(D_i \not\subset \Delta(x_0) \Rightarrow (\exists t \in D_i \setminus \Delta(x_0) : x(\{t\}) = 0) \Rightarrow (x(D_i) = x_0(D_i) = 0)$$

et $x \in U$. Donc pour S, I ainsi définis on a: $x_0 \in V[S, I] \subset U$. Enfin si $x_0 \in V[S_1, I_1] \cap V[S_2, I_2]$ alors S_1 et S_2 sont nécessairement compatibles $S_1 \cup S_2 \in \mathcal{S}$) et $I_1 \cap I_2 \neq \emptyset$; done

and $x \in U$. So for S, I was well defined (?): $x_0 \in V[S, I] \subset U$. Finally, if $x_0 \in V[S_1, I_1] \cap V[S_2, I_2]$ then S_1 and S_2 are necessarily compatible ($S_1 \cup S_2 \in \mathcal{S}$) and $I_1 \cap I_2 \neq \emptyset$; therefore

$$x_0 \in V[S_1 \cup S_2, I_1 \cap I_2] \subset V[S_1, I_1] \cap V[S_2, I_2]$$

Supposons maintenant que $V_2 = V[S_2, I_2] \subset V[S_1, I_1] = V_1$:

Now suppose $V_2 = V[S_2, I_2] \subset V[S_1, I_1] = V_1$:

(a) Si $I_2 \not\subset I_1$ et $t \in I_2 \setminus (I_1 \cup \mathrm{Dom}S_1 \cup \mathrm{Dom}S_2) \neq \emptyset$ (puisque $I_2 \setminus I_1 = \emptyset$ ou $\notin \mathcal{D}$), alors pour l'élément x de X défini par: $(\tau(x) = t$ et $\Delta(x) = \{S_2 = 1\})$ on a: $x \in V_2 \setminus V_1$.

(a) If $I_2 \not\subset I_1$ and $t \in I_2 \setminus (I_1 \cup \mathrm{Dom}S_1 \cup \mathrm{Dom}S_2) \neq \emptyset$ (since $I_2 \setminus I_1 = \emptyset$ or $\notin \mathcal{D}$), then for the element x of X defined by: $(\tau(x) = t$ and $\Delta(x) = S_2^{-1}(1))$ and thus $x \in V_2 \setminus V_1$ (a contradiction).

(b) Si $S_1 \not\subset S_2$ alors on a:

(b) If $S_1 \not\subset S_2$ then where:

$$\exists t \in \mathrm{Dom}S_1 : (t \notin \mathrm{Dom}S_2) \text{ or } (t \in \mathrm{Dom}S_2 \text{ and } S_1(t) \neq S_2(t))$$

de sorte que, en posant $S(t) = 1 - S_1(t)$, on ait: $S = S_2 \cup \{t, S(t)\} \in \mathcal{S}$. Soient $r \in I_2 \setminus \mathrm{Dom}S$ et x l'élément de X défini par: $(\tau(x) = r$ et $\Delta(x) = \{S = 1\})$, alors $x \in V_2$ puisque $S_2 \subset S$ et $x \notin V_1$ puisque:

so that, by having $S(t) = 1 - S_1(t)$, we have: $S = S_2 \cup \{(t, S(t))\} \in \mathcal{S}$. Let $r \in I_2 \setminus \mathrm{Dom}S$ and x the element of X defined by: $(\tau(x) = r$ and $\Delta(x) = S^{-1}(1))$, then $x \in V_2$ since $S_2 \subset S$ and $x \notin V_1$ since:

$$x(\{t\}) = r \neq 0 \Leftrightarrow S_1(t) = 0$$

et

and

$$x(\{t\}) = 0 \notin I_1 \Leftrightarrow S_1(t) = 1$$

Donc $I_2 \subset I_1$ et $S_1 \subset S_2$ ce qui démontre l'équivalence annoncée puisque l'implication inverse est triviale. □

So $I_2 \subset I_1$ an $S_1 \subset S_2$ which proves the claimed equivalence since reverse implication is trivial. □

Théoréme 8. *Il existe sur X une stratégie gagnante pour le joueur α qui ne dépend que des deux derniers coups joués par le joueur β.*

Theorem 8. *There exists on X a winning strategy for the player α which depends on only the last two plays by β.*

Fixons pour tout $D \in \mathcal{D}$: une suite $(\Phi_n(D))_{n\in N}$ de parties dénombrables deux à deux disjointes de $\mathbb{R}^* \setminus D$, et une suite $(\phi_{n,D})_{n\in N}$ de bijections $\phi_{n,D} : D \to \Phi_n(D)$; et posons $\Phi(D) = \bigcup_{n\in N} \Phi_n(D) \subset \mathbb{R}^* \setminus D$, et $\mathcal{F}_{-1} = \emptyset$.

Fix for all $D \in \mathcal{D}$ a sequence $(\Phi_n(D))_{n\in N}$ of countable pairwise disjoint subsets of $\mathbb{R}^* \setminus D$ and a sequence $(\phi_{n,D})_{n\in N}$ of bijections $\phi_{n,D} : D \to \Phi_n(D)$; and let $\Phi(D) = \bigcup_{n\in N} \Phi_n(D) \subset \mathbb{R}^* \setminus D$, and $\mathcal{F}_{-1} = \emptyset$.

Pour tout $n \in N$, on définit:

For all $n \in N$, define:

$$\mathcal{T}_n = \{(T', T) \in \mathcal{S} \times \mathcal{S} : T' \subset T, \Phi(\mathrm{Dom}T') \subset \mathrm{Dom}T, T(\Phi_p(\mathrm{Dom}T')) = \{1\}, \forall p \geq n\}$$

$\mathcal{T} = \bigcup_{n\in N} \mathcal{T}_n$ et $\mathcal{F}$ l'ensemble des parties finies de $\mathcal{D}$. La démonstration du théoréme repose sur le lemme suivant:

$\mathcal{T} = \bigcup_{n\in N} \mathcal{T}_n$ and $\mathcal{F}$ all finite subsets of $\mathcal{D}$. The proof of the theorem is then based on the following lemma:

Lemme 9. *Il existe deux applications $f : \mathcal{S} \cup \mathcal{T} \to \mathcal{S}$ et $d : \mathcal{S} \cup \mathcal{T} \to \mathcal{F}$ vérifiant:*

(a) Si $T \in \mathcal{S}$ alors $(T, f(T)) \in \mathcal{T}_0$,

(b) Si $(T', T) \in \mathcal{T}_n$ alors $(T, f(T', T)) \in \mathcal{T}_{n+1}$.

(c) Si $f(\mathcal{T}_0) \subset \mathcal{T}_1$ et $f(\mathcal{T}_{n-1}, \mathcal{T}_n) \subset \mathcal{T}_{n+1}$ pour tout $n \geq 1$, alors

Lemma 9. *There exist two maps $f : \mathcal{S} \cup \mathcal{T} \to \mathcal{S}$ et $d : \mathcal{S} \cup \mathcal{T} \to \mathcal{F}$ such that:*

(a) If $T \in \mathcal{S}$ then $(T, f(T)) \in \mathcal{T}_0$.

(b) If $(T', T) \in \mathcal{T}_n$ then $(T, f(T', T)) \in \mathcal{T}_{n+1}$.

(c) If $f(\mathcal{T}_0) \subset \mathcal{T}_1$ and $f(\mathcal{T}_{n-1}, \mathcal{T}_n) \subset \mathcal{T}_{n+1}$ for all $n \geq 1$, then

$$d(T_{n-1}, T_n) = \{\mathrm{Dom}T_p : 0 \leq p \leq n\}$$

Démonstration. Pour $T \in \mathcal{S}$ et $D = \mathrm{Dom}T$, on définit $f(T) \in \mathcal{S}$ par:

Proof. For $T \in \mathcal{S}$ and $D = \mathrm{Dom}T$, we define $f(T) \in \mathcal{S}$ such that:

$$\mathrm{Dom}f(T) = D \cup \Phi(D);\ \ f(T)|_D = T;\ \ f(T)|_{\Phi(D)} = 1$$

Donc (a) est vérifié.

So (a) is verified.

Pour $(T', T) \in \mathcal{T}_n \setminus \mathcal{T}_{n-1}$ avec $D' = \mathrm{Dom}T' \subset \mathrm{Dom}T = D$ on définit $S = f(T', T)$ et $d(T', T)$ par:

For $(T', T) \in \mathcal{T}_n \setminus \mathcal{T}_{n-1}$ with $D' = \mathrm{Dom}T' \subset \mathrm{Dom}T = D$ we define $S = f(T', T)$ and $d(T', T)$ such that:

$$d(T', T) = \{l_p(T', T) : 0 \leq p \leq n+1\} \tag{1}$$

$$0 \leq p \leq n \Rightarrow l_p(T', T) = \phi^{-1}_{p,D'}(\phi_p(D') \cap T^{-1}(1)) \tag{2}$$

$$p \geq n+1 \Rightarrow l_p(T', T) = \mathrm{Dom}T = D \tag{3}$$

$$\mathrm{Dom}S = D \cup \Phi(D) \tag{4}$$

$$S|_D = T \tag{5}$$

$$\Phi_p(D) \cap S^{-1}(1) = \phi_{p,D}(l_p(T', T))\ \forall p \in \mathbb{N} \tag{6}$$

Il découle de (3) et (6) que $S|_{\Phi_p(D)} = 1$ pour tout $p \geq n+1$ donc $(T, f(T', T)) \in \mathcal{T}_{n+1}$ et (b) est vérifié.

It follows from (3) and (6) that $S|_{\Phi_p(D)} = 1$ for all $p \geq n+1$, so $(T, f(T', T)) \in \mathcal{T}_{n+1}$ and (b) is verified.

De plus $(T, f(T', T)) \notin \mathcal{T}_n$; en effet:

Furthermore $(T, f(T', T)) \notin \mathcal{T}_n$; in fact:

$$\begin{aligned}\Phi_n(D)\cap S^{-1}(1) &= \phi_{n,D}(l_n(T',T))\\ &= \phi_{n,D}(D')\\ &\subset \Phi_n(D)\end{aligned}$$

et la dernière inclusion est stricte puisque D' est contenu strictement dans D.

and the last inclusion is strict because D' is strictly contained in D.

Soit $(T_n)_{n\in\mathbb{N}}$ une suite satisfaisant les hypothéses de (c) et posons: $D_n = \mathrm{Dom}T_n$ et $S_n = f(T_n, T_{n+1})$. Comme $T_1|_{\Phi(D_0)} = S_0|_{\Phi(D_0)} = 1$ on a $(T_0, T_1) \in \mathcal{T}_0$ et d'après (b) on a alors $(T_n, T_{n+1}) \in \mathcal{T}_n \setminus \mathcal{T}_{n-1}$. Nous allons maintenant montrer par récurrence sur $n \geq 1$ que $l_p(T_{n-1}, T_n) = D_p$ pour $0 \leq p \leq n$. Pour $n = 1$, on a d'après (2) et (3):

Let $(T_n)_{n\in\mathbb{N}}$ be a sequence satisfying the hypotheses of (c) and let $D_n = \mathrm{Dom}T_n$ and $S_n = f(T_n, T_{n+1})$. As $T_1|_{\Phi(D_0)} = S_0|_{\Phi(D_0)} = 1$ with $(T_0, T_1) \in \mathcal{T}_0$ and by (b) was then $(T_n, T_{n+1}) \in \mathcal{T}_n \setminus \mathcal{T}_{n-1}$. We will now show by induction on $n \geq 1$ that $l_p(T_{n-1}, T_n) = D_p$ for $0 \leq p \leq n$. For $n = 1$, we have by (2) and (3):

$$l_0(T_0,T_1) = \phi_{0,D_0}^{-1}(\Phi_0(D_0)\cap T_1^{-1}(1)) = \phi_{0,D_0}^{-1}(\Phi_0(D_0)) = D_0$$

$$l_1(T_0,T_1) = D_1$$

Supposons la relation établie pour n alors d'après (3):

Suppose it holds for n, then by (3):

$$l_{n+1}(T_n,T_{n+1}) = D_{n+1}$$

et pour $p \leq n$ on a d'aprés (2) et (6):

and for $p \leq n$ by (2) and (6):

$$\begin{aligned}l_p(T_n,T_{n+1}) &= \phi_{p,D_n}^{-1}(\Phi_p(D_n)\cap S_n^{-1}(1)) = \phi_{p,D_n}^{-1}(\phi_{p,D_n}(l_p(T_{n-1},T_n)))\\ &= l_p(T_{n-1},T_n) = D_n\end{aligned}$$

Donc d'après (1) on a $d(T_{n-1}, T_n) = \{D_p : 0 \leq p \leq n\}$. □

Therefore, after (1) we have $d(T_{n-1}, T_n) = \{D_p : 0 \leq p \leq n\}$. □

`Démonstration du théorème 8.` Fixons pour tout $D \in \mathcal{D}$ une surjection $\vartheta_D : N \to D$. Si $(T', T) \in \mathcal{T}$ et $d(T', T) = \{D_p : 0 \leq p \leq n\}$ on choisit pour tout $J \in \mathcal{J}$ un élément $I = g(T', T, J)$ de $\mathcal{J}$ vérifiant

`Proof of Theorem 8.` Fix for all $D \in \mathcal{D}$ a surjection $\vartheta_D : N \to D$. If $(T', T) \in \mathcal{T}$ and $d(T', T) = \{D_p : 0 \leq p \leq n\}$ choose for all $J \in \mathcal{J}$ an element $I = g(T', T, J)$ of $\mathcal{J}$ such that

$$\operatorname{diam}(I) < \frac{1}{2}\operatorname{diam}(J) \tag{1}$$

$$\overline{I} \subset J \tag{2}$$

$$I \cap \{\vartheta_{D_p}(q) : 0 \leq p, q \leq n\} = \emptyset \tag{3}$$

On définit ainsi une application $g : \mathcal{T} \times \mathcal{J} \to \mathcal{J}$.

We also define an application $g : \mathcal{T} \times \mathcal{J} \to \mathcal{J}$.

Considérons maintenant la α-stratégie σ qui ne dépend que des deux derniers jeux de β et qui est définie par

Now consider the α-strategy σ which only depends on the last two moves by β and is defined by

$$\sigma(V[T, J]) = V[f(T), J]$$

$$\sigma(V[T_{n-1}, J_{n-1}], V[T_n, J_n]) = V[f(T_{n-1}, T_n), \underbrace{g(T_{n-1}, T_n, J_n)}_{I_n}] \quad V[T_{n+1}, J_{n+1}]$$

$$\sigma(V[T', J'], V[T, J]) = V[f(T', T), g(T', T, J)]$$

Si dans une partie compatible avec σ le joueur β a joué au n ème coup: $V[T_n, J_n] = V_n$ alors en posant: $D_n = \mathrm{Dom}T_n$ et $I_n = g(T_{n-1}, T_n, J_n)$ on a d'après (1) et (2) que $\bigcap_{n\in\mathbb{N}} I_n = \bigcap_{n\in\mathbb{N}} J_n = \{t\}$ et d'après (3) et lemme 9 (c) que $t \notin \bigcup_{p,q\in\mathbb{N}}\{\vartheta_{D_p}(q)\} = \bigcup_{n\in\mathbb{N}} D_n$. Donc l'élément a de X défini par ($\tau(a) = t$ et $\Delta(a) = \bigcup_{n\in\mathbb{N}}\{T_n = 1\}$) vérifie $a \in \bigcap_{n\in\mathbb{N}} V_n$ et par suite la stratégie σ est gagnante pour α. □

If a part is compatible with σ the player β plays on the n^{th} turn: $V[T_n, J_n] = V_n$ while having: $D_n = \mathrm{Dom}T_n$ and $I_n = g(T_{n-1}, T_n, J_n)$ after it has (1) and (2) that $\bigcap_{n\in\mathbb{N}} I_n = \bigcap_{n\in\mathbb{N}} J_n = \{t\}$ and by (3) and lemme 9 (c), $t \notin \bigcup_{p,q\in\mathbb{N}}\{\vartheta_{D_p}(q)\} = \bigcup_{n\in\mathbb{N}} D_n$. So the element a of X defined by ($\tau(a) = t$ et $\Delta(a) = \bigcup_{n\in\mathbb{N}} T_n^{-1}(1)$) satisfies $a \in \bigcap_{n\in\mathbb{N}} V_n$ and hence the strategy σ is α-favorable. □

Detailed Proofs of some Theorems

Proposition 7. The family

$$\{V[S,I] : S \in \mathcal{S}, I \in \mathcal{I}\}$$

forms a base for the topology on X such that:

$$(V[S_1,I_1] \supset V[S_2,I_2]) \Leftrightarrow (S_1 \supset S_2 \text{ and } I_1 \supset I_2)$$

Proof: Recall that basic sets in X are those of the form

$$B(x_0,\epsilon,\{D_n\}) = \{x \in X : |x(D) - x_0(D)| < \epsilon \text{ for all } D \in \{D_n\}\}$$

for $x_0 \in X$, $\epsilon > 0$, and a countable subset $\{D_n\} \subset \mathcal{D}$. Also recall that the proposed base is comprised of sets of the form

$$V[S,I] = \{x \in X : x(S^{-1}(1)) \in I \text{ and } x(\{t\}) = 0 \text{ for all } t \in S^{-1}(0)\}$$

where $S \in \mathcal{S}$ and $I \in \mathcal{I}$.

We will first find $x_0, \epsilon, \{D_n\}$ such that $B(x_0,\epsilon,\{D_n\}) = V[S,I]$ for any $V[S,I]$. Consider some $V[S,I]$ and choose $\epsilon > 0$ such that $(-\epsilon,\epsilon) \cap I = \emptyset$.

Detailed Proofs of some Theorems

Proposition 7. The family

$$\{V[S,I] : S \in \mathcal{S}, I \in \mathcal{I}\}$$

forms a base for the topology on X such that:

$$(V[S_1, I_1] \supseteq V[S_2, I_2]) \Leftrightarrow (S_1 \subseteq S_2 \text{ and } I_1 \supseteq I_2)$$

Proof: Recall that basic sets in X are those of the form

$$B(x_0, \epsilon, \{D_n\}) = \{x \in X : |x(D) - x_0(D)| < \epsilon \text{ for all } D \in \{D_n\}\}$$

for $x_0 \in X$, $\epsilon > 0$, and a countable subset $\{D_n\} \subset \mathcal{D}$. Also recall that the proposed base is comprised of sets of the form

$$V[S,I] = \{x \in X : x(S^{-1}(1)) \in I \text{ and } x(\{t\}) = 0 \text{ for all } t \in S^{-1}(0)\}$$

where $S \in \mathcal{S}$ and $I \in \mathcal{I}$.

Let $x_0 \in X$. We will first find $\epsilon, \{D_n\}$ such that $B(x_0, \epsilon, \{D_n\}) \subseteq V[S,I]$ for any $V[S,I]$ containing x_0. So consider such a $V[S,I]$ containing x_0 and choose $\delta > 0$ such that $(-\delta, \delta) \cap I = \emptyset$. Then

$$\begin{aligned} V[S,I] &= \{x \in X : x(S^{-1}(1)) \in I \text{ and } x(\{t\}) \in (-\delta,\delta) \text{ for all } t \in S^{-1}(0)\} \\ &= \{x \in X : x(S^{-1}(1)) \in I\} \cap \{x \in X : x(\{t\}) \in (-\delta,\delta) \text{ for all } t \in S^{-1}(0)\} \end{aligned}$$

Let J be an interval centered about $\tau(x_0) \in I$ such that $J \subseteq I$. Let $|J| = \operatorname{diam}(J)$ and let $\epsilon = \min(|J|/2, \delta)$. It follows that

$$\begin{aligned} V[S,I] &\supseteq \{x \in X : |x(S^{-1}(1)) - x_0(S^{-1}(1))| < |J|/2\} \cap \{x \in X : |x(S^{-1}(0)) - x_0(S^{-1}(0))| < \delta\} \\ &= B(x_0, |J|/2, \{S^{-1}(1)\}) \cap B(x_0, \delta, \{S^{-1}(0)\}) \\ &\supseteq B(x_0, \epsilon, \{S^{-1}(0), S^{-1}(1)\}) \end{aligned}$$

Now consider $x_0 \in B(x_0, \epsilon, \{D_n\})$. We need to find $V[S,I]$ such that $x_0 \in V[S,I] \subseteq B(x_0, \epsilon, \{D_n\})$.

Let $t_0 \in \mathbb{R}^* \setminus \left(\bigcup_{n \in \mathbb{N}} D_n\right) \setminus (\Delta(x_0))$. Then define

$$S(t) = \begin{cases} 0 : t \in \bigcup_{n\in\mathbb{N}}(D_n \setminus \Delta(x_0)) \cup \{t_0\} \\ 1 : t \in \Delta(x_0) \end{cases}$$

so that $S^{-1}(0) = \bigcup_{n\in\mathbb{N}}(D_n \setminus \Delta(x_0)) \cup \{t_0\}$ and $S^{-1}(1) = \Delta(x_0)$.

Now let $I \in \mathcal{I}$ be such that $\tau(x_0) \in I$ and $|I| < \epsilon$. Note

$$\begin{aligned} V[S,I] &= \{x \in X : x(S^{-1}(1)) \in I \text{ and } x(\{t\}) = 0 \text{ for all } t \in S^{-1}(0)\} \\ &= \{x \in X : x(\Delta(x_0)) \in I \text{ and } x(\{t\}) = 0 \text{ for all } t \in \textstyle\bigcup_{n\in\mathbb{N}}(D_n \setminus \Delta(x_0)) \cup \{t_0\}\} \end{aligned}$$

So $x_0 \in V[S,I]$ as $x_0(\Delta(x_0)) = \tau(x_0) \in I$, and $\left(\bigcup_{n\in\mathbb{N}}(D_n \setminus \Delta(x_0)) \cup \{t_0\}\right) \cap \Delta(x_0) = \emptyset$ which means $x_0(\{t\}) = 0$ for all $t \in \bigcup_{n\in\mathbb{N}}(D_n \setminus \Delta(x_0)) \cup \{t_0\}$.

To show that $V[S,I] \subseteq B(x_0, \epsilon, \{D_n\})$, let $x \in V[S,I]$. It follows that $x(\Delta(x_0)) \in I$ and thus $x(\Delta(x_0)) \neq 0$. So then $\Delta(x_0) \subseteq \Delta(x)$. Consider $D \in \{D_n\}$.

- In the case that $D \subseteq \Delta(x_0) \subseteq \Delta(x)$, note $x_0(D) = x_0(\Delta(x_0)) \in I$ and $x(D) = x(\Delta(x)) = \tau(x) = x(\Delta(x_0)) \in I$. Thus $|x(D) - x_0(D)| < |I| < \epsilon$.
- In the other case that $D \not\subseteq \Delta(x_0)$, choose $t \in D \setminus \Delta(x_0)$. It follows that $x(\{t\}) = 0$ as $t \in \bigcup_{n\in\mathbb{N}}(D_n \setminus \Delta(x_0)) \cup \{t_0\}$. Thus $D \not\subseteq \Delta(x)$, and $|x(D) - x_0(D)| = |0 - 0| < \epsilon$.

Thus $x \in \{x \in X : |x(D) - x_0(D)| < \epsilon \text{ for all } D \in \{D_n\}\} = B(x_0, \epsilon, \{D_n\})$ showing $V[S,I] \subseteq B(x_0, \epsilon, \{D_n\})$.

Let $V_n = V[S_n, I_n]$. To complete the proof that we have a base, we should show that for $x_0 \in V_1 \cap V_2$, we have some $V[S,I]$ where $x_0 \in V[S,I] \subseteq V_1 \cap V_2$.

Note that $x_0(S_1^{-1}(1)) \in I_1$ and $x_0(S_2^{-1}(1)) \in I_2$, and since $0 \notin I_1$ and $0 \notin I_2$, we have $x_0(S_1^{-1}(1)) = x_0(S_2^{-1}(1)) = \tau(x_0) \in I_1 \cap I_2$.

Then note that $x_0(\{t\}) = 0$ for all $t \in S_1^{-1}(0)$ and $t \in S_2^{-1}(0)$. For all $t \in \mathrm{Dom}S_1 \cap \mathrm{Dom}S_2$, if $x_0(\{t\}) = 0$, then $t \in S_1^{-1}(0)$ and $t \in S_2^{-1}(0)$ yielding $S_1(t) = S_2(t) = 0$. Otherwise, $x_0(\{t\}) = \tau(x_0) \neq 0$, which proves $t \notin S_1^{-1}(0)$ and $t \notin S_2^{-1}(0)$ yielding $S_1(t) = S_2(t) = 1$. This shows that $S_1 \cup S_2$ is a function, and thus

$$x_0 \in V[S_1 \cup S_2, I_1 \cap I_2] \subseteq V_1 \cup V_2$$

Finally, we must show the equivalence

$$(V_1 \supseteq V_2) \Leftrightarrow (S_1 \subseteq S_2 \text{ and } I_1 \supseteq I_2)$$

Assume $V_1 \supseteq V_2$.

(a) Suppose by way of contradiction that $I_2 \not\subseteq I_1$.

$$I_2 \setminus I_1 \setminus \mathrm{Dom} S_1 \setminus \mathrm{Dom} S_2 \neq \emptyset$$

as $I_2 \setminus I_1$ is uncountable and $\mathrm{Dom} S_1$, $\mathrm{Dom} S_2$ are countable. So let $t \in I_2 \setminus I_1 \setminus \mathrm{Dom} S_1 \setminus \mathrm{Dom} S_2$. Define $x_0 \in X$ by $\tau(x_0) = t$ and $\Delta(x_0) = S_2^{-1}(1)$. Note $x(D) = t$ or $0 \notin I_1$ for all $D \in \mathcal{D}$, so

$$x_0 \in \begin{array}{c} \{x \in X : x(S_2^{-1}(1)) \in I_2 \text{ and } x(\{t\}) = 0 \text{ for all } t \in S_2^{-1}(0)\} \\ \setminus \\ \{x \in X : x(S_1^{-1}(1)) \in I_1 \text{ and } x(\{t\}) = 0 \text{ for all } t \in S_1^{-1}(0)\} \end{array} = V_2 \setminus V_1 = \emptyset$$

Contradiction.

(b) Suppose by way of contradiction that $S_1 \not\subseteq S_2$. Then there exists $t \in \mathrm{Dom} S_1$ such that either $t \notin \mathrm{Dom} S_2$ or $S_2(t) = 1 - S_1(t)$. Either way, $S = S_2 \cup \{\langle t, 1 - S_1(t)\rangle\}$ is a function such that $S(t) = 1 - S_1(t)$.

Let $t \in I_2 \setminus \mathrm{Dom} S$ and define $x_0 \in X$ by $\tau(x_0) = t$ and $\Delta(x_0) = S^{-1}(1)$. Note

$$x_0(\{t\}) = t \neq 0 \Leftrightarrow S(t) = 1 \Leftrightarrow S_1(t) = 0$$

and

$$x_0(\{t\}) = 0 \notin I_1 \Leftrightarrow S(t) = 0 \Leftrightarrow S_1(t) = 1$$

So we have then that

$$x_0 \in \begin{array}{c} \{x \in X : x(S_2^{-1}(1)) \in I_2 \text{ and } x(\{t\}) = 0 \text{ for all } t \in S_2^{-1}(0)\} \\ \setminus \\ \{x \in X : x(S_1^{-1}(1)) \in I_1 \text{ and } x(\{t\}) = 0 \text{ for all } t \in S_1^{-1}(0)\} \end{array} = V_2 \setminus V_1 = \emptyset$$

Contradiction.

So by (a), (b) we have proven that forward implication. The backward implication is trivial: If $I_2 \subseteq I_1$ and $S_1 \subseteq S_2$, then

$$\begin{aligned} V_2 &= \{x \in X : x(S_2^{-1}(1)) \in I_2 \text{ and } x(\{t\}) = 0 \text{ for all } t \in S_2^{-1}(0)\} \\ &\subseteq \{x \in X : x(S_1^{-1}(1)) \in I_1 \text{ and } x(\{t\}) = 0 \text{ for all } t \in S_1^{-1}(0)\} \\ &= V_1 \end{aligned}$$

□

Lemma 9 There exist functions $f : \mathcal{S} \cup \mathcal{T} \to \mathcal{S}$ and $d : \mathcal{S} \cup \mathcal{T} \to \mathcal{F}$ such that

(a) $T \in \mathcal{S} \Rightarrow (T, f(T)) \in \mathcal{T}_0$

(b) $(T', T) \in \mathcal{T}_n \setminus \mathcal{T}_{n-1} \Rightarrow (T, f(T', T)) \in \mathcal{T}_{n+1} \setminus \mathcal{T}_n$

(c) If $f(\mathcal{T}_0) \subseteq \mathcal{T}_1$ and $f(\mathcal{T}_{n-1}, \mathcal{T}_n) \subseteq \mathcal{T}_{n+1}$ for all $n \geq 1$, then $d(\mathcal{T}_{n-1}, \mathcal{T}_n) = \{\mathrm{Dom}T_p : 0 \leq p \leq n\}$.

Proof: A few definitions:

- $\langle \Phi_{D,n} \rangle_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{D}$ disjoint from D.
- $\Phi_D = \bigcup_{n \in \mathbb{N}} \Phi_{D,n} \subset \mathbb{R}^* \setminus D$
- $\langle \phi_{D,n} \rangle_{n \in \mathbb{N}}$ is a sequence of bijections $D \to \Phi_{D,n}$.
- $\mathcal{T}_{-1} = \emptyset$
- $\mathcal{T}_n = \{(T', T) \in \mathcal{S} \times \mathcal{S} : (T' \subseteq T) \wedge (\Phi_{\mathrm{Dom}T'} \subseteq \mathrm{Dom}T) \wedge (T \restriction \Phi_{\mathrm{Dom}T',p} = 1, \forall p \geq n)\}$
- $\mathcal{F} \subset \mathcal{P}(\mathcal{D})$ contains exactly the finite subsets of $\mathcal{D}$.

We let $T', T \in \mathcal{S}$ and use the shorthand $D = \mathrm{Dom}T, D' = \mathrm{Dom}T'$.

We define $f(T)$ such that

- $\mathrm{Dom}f(T) = D \cup \Phi_D$
- $f(T) \restriction D = T$
- $f(T) \restriction \Phi_D$ is the constant function 1

We verify that this definition satisfies (a) by showing $(T, f(T)) \in \mathcal{T}_0$. $f(T)$ extends T, that is, $T \subseteq f(T)$. We included $\Phi_D = \Phi_{\mathrm{Dom}T}$ in the domain of $f(T)$, so $\Phi_{\mathrm{Dom}T} \subseteq \mathrm{Dom}f(T)$. Lastly, $f(T) \restriction \Phi_{\mathrm{Dom}T,p} = f(T) \restriction \Phi_D = 1$.

Now observe that $\mathcal{T}_{n-1} \subseteq \mathcal{T}_n$ for $n \in \mathcal{N}$, so for $(T', T) \in \mathcal{T}$, let $n(T', T)$ denote the least number such that $(T', T) \in \mathcal{T}_{n(T',T)}$.

We first define a helper function $l_p(T', T)$ such that (preserving the numbering of the original proof),

2. If $0 \leq p \leq n(T', T)$, then $l_p(T', T) = \phi_{D',p}^{-1}(\Phi_{D',p} \cap T^{-1}(1))$.
3. If $p > n(T', T)$, then $l_p(T', T) = D$.

We then define $f(T',T)$ such that

4. $\mathrm{Dom} f(T',T) = D \cup \Phi_D$
5. $f(T',T) \restriction D = T$
6. $f(T',T) \restriction \Phi_{D,p}(t) = \begin{cases} 1 \text{ if } t \in \phi_{D,p}(l_p(T',T)) \\ 0 \text{ otherwise} \end{cases}$

Note that (6) is equivalent to its statement in the original proof.

6. $\Phi_{D,p} \cap (f(T',T))^{-1}(1) = \phi_{D,p}(l_p(T',T))$

So to satisfy (b), we show that for $(T',T) \in \mathcal{T}_n \setminus \mathcal{T}_{n-1}$, $(T, f(T',T)) \in \mathcal{T}_{n+1}$. Note $f(T',T)$ extends T and $\Phi_{\mathrm{Dom}T} \subseteq f(T',T)$. So now assume $p \geq n+1$.

$$f(T',T) \restriction \Phi_{D,p}(t) = \begin{cases} 1 \text{ if } t \in \phi_{D,p}(l_p(T',T)) \\ 0 \text{ otherwise} \end{cases} = \begin{cases} 1 \text{ if } t \in \phi_{D,p}(D) \\ 0 \text{ otherwise} \end{cases} = \begin{cases} 1 \text{ if } t \in \Phi_{D,p} \\ 0 \text{ otherwise} \end{cases} = 1$$

Thus $f(T',T) \in \mathcal{T}_{n+1}$. Also we note

$$\begin{array}{rcll} (f(T',T))^{-1}(1) \cap \Phi_{D,n} & = & \phi_{D,n}(l_n(T',T)) & \text{by (6)} \\ & = & \phi_{D,n}(\phi^{-1}_{D',n}(\Phi_{D',n} \cap T^{-1}(1))) & \text{by (2)} \\ & = & \phi_{D,n}(\phi^{-1}_{D',n}(\Phi_{D',n})) & \text{by } (T',T) \in \mathcal{T}_n \\ & = & \phi_{D,n}(D') & \text{by definition of } \phi_{D',n} \\ & \subset & \Phi_{D,n} & \text{by } D' \subset D \end{array}$$

Thus there are elements of $\Phi_{D,n}$ which aren't sent to 1 by $f(T',T)$, and $(T, f(T',T)) \notin \mathcal{T}_n$, showing that $\mathcal{T}_{n+1} \setminus \mathcal{T}_n \neq \emptyset$.

Finally, we define $d(T',T)$ for $(T',T) \in \mathcal{T}_n \setminus \mathcal{T}_{n-1}$.

1. $d(T',T) = \{l_p(T',T) : 0 \leq p \leq n+1\}$

To satisfy (c), we should show that if $f(T_0) \subseteq T_1$ and $f(T_{n-1}, T_n) \subseteq T_{n+1}$ for all $n \geq 1$, then $d(T_{n-1}, T_n) = \{\mathrm{Dom} T_p : 0 \leq p \leq n\}$. We first show such a sequence satisfies $(T_n, T_{n+1}) \in \mathcal{T}_n \setminus \mathcal{T}_{n-1}$.

This follows simply from the observation that if $(T', T) \in \mathcal{T}_n$, then $(T', S) \in \mathcal{T}_n$ for any extension $S \supseteq T$. Therefore by (a), $(T_0, f(T_0)) \in \mathcal{T}_0 \Rightarrow (T_0, T_1) \in \mathcal{T}_0$. Then if we assume by way of induction that $(T_n, T_{n+1}) \in \mathcal{T}_n \setminus \mathcal{T}_{n-1}$, then by (b),

$$(T_{n+1}, f(T_n, T_{n+1})) \in \mathcal{T}_{n+1} \setminus \mathcal{T}_n \Rightarrow (T_{n+1}, T_{n+2}) \in \mathcal{T}_{n+1} \setminus \mathcal{T}_n$$

We will now show by induction that $l_p(T_{n-1}, T_n) = D_p$ for $0 \leq p \leq n$. If $n = 1$, then

$$\begin{aligned}
l_0(T_0, T_1) &= \phi^{-1}_{D_0,0}(\Phi_{D_0,0} \cap T_1^{-1}(1)) && \text{by (2) as } 0 = n(T_0, T_1) \\
&= \phi^{-1}_{D_0,0}(\Phi_{D_0,0} \cap (f(T_0))^{-1}(1)) && \text{as } T_1 \text{ extends } f(T_0) \\
&= \phi^{-1}_{D_0,0}(\Phi_{D_0,0}) && \text{by } f(T_0) \restriction \Phi_D = 1 \\
&= D_0
\end{aligned}$$

and by (3)

$$l_1(T_0, T_1) = D_1$$

So if it holds that $l_p(T_{n-1}, T_n) = D_p$ when $0 \leq p \leq n$, then for $0 \leq p < n+1$

$$\begin{aligned}
l_p(T_n, T_{n+1}) &= \phi^{-1}_{D_n,p}(\Phi_{D_n,p} \cap T^{-1}_{n+1}(1)) && \text{by (2) as } p \leq n = n(T_n, T_{n+1}) \\
&= \phi^{-1}_{D_n,p}(\Phi_{D_n,p} \cap (f(T_{n-1}, T_n))^{-1}(1)) && \text{as } T_{n+1} \text{ extends } f(T_{n-1}, T_n) \\
&= \phi^{-1}_{D_n,p}(\phi_{D_n,p}(l_p(T_{n-1}, T_n))) && \text{by (6)} \\
&= l_p(T_{n-1}, T_n) \\
&= D_p
\end{aligned}$$

and for $p = n + 1$, by (3),

$$l_{n+1}(T_n, T_{n+1}) = D_{n+1}$$

So finally, by (1), we have

$$\begin{aligned}
d(T_n, T_{n+1}) &= \{l_p(T_n, T_{n+1}) : 0 \leq p \leq n+1\} \\
&= \{D_p : 0 \leq p \leq n+1\}
\end{aligned}$$

□

Theorem 8. There exists on X a winning strategy for the player α which depends on only the last two plays by β.

Proof: For each $D \in \mathcal{D}$, enumerate its elements $D = \{\theta_i^D : i \in \mathbb{N}\}$. Let $e(T', T) = \{\theta_i^D : D \in d(T', T), 0 \leq i \leq |d(T', T)|\}$. We may then define $g : \mathcal{T} \times \mathcal{I} \to \mathcal{I}$ so that $g(T', T, J) = I$ where is some interval such that

1. $\mathrm{diam}(I) < \frac{1}{2}\mathrm{diam}(J)$
2. $\overline{I} \subseteq J$
3. $I \subseteq (e(T', T))^c$ (which we may do as $e(T', T)$ is finite)

We give α the strategy σ such that on the first turn, α plays

$$\sigma(V[T, J]) = V[f(T), J]$$

and on subsequent turns α plays

$$\sigma(V[T', J'], V[T, J]) = V[f(T', T), g(T', T, J)]$$

Let $I_n = g(T_{n-1}, T_n, J_n)$ and $D_n = \mathrm{Dom}T_n$. The game proceeds as follows:

$$\begin{array}{lccccccc}
\beta : & V[T_0, J_0] & & V[T_1, J_1] & & V[T_2, J_2] & & \dots \\
\alpha : & & V[f(T_0, J_0)] & & V[f(T_0, T_1), I_1] & & V[f(T_1, T_2), I_2] & \dots
\end{array}$$

Observe that $J_1 \supseteq \overline{I_1} \supseteq J_2 \supseteq \overline{I_2} \supseteq \dots$ with $\mathrm{diam}(\overline{I_n}) \to 0$, so $\bigcap_{n\in\mathbb{N}} J_n = \bigcap_{n\in\mathbb{N}} I_n = \{t\}$ for some t. Also, since $e(T_{n-1}, T_n)$ captures the first n elements of the first n domains of the $\{T_n\}$,

$$t \in \bigcap_{n\in\mathbb{N}} I_n \subseteq \bigcap_{n>0} (e(T_{n-1}, T_n))^c = \left(\bigcup_{n>0} e(T_{n-1}, T_n)\right)^c = \left(\bigcup_{n\in\mathbb{N}} D_n\right)^c$$

So $t \notin D_n$ for any n. Therefore let $a \in X$ be defined by $\tau(a) = t$ and $\Delta(a) = \bigcup_{n\in\mathbb{N}} T_n^{-1}(1)$. Consider any $V[T_n, J_n]$ and note that as $T_n^{-1}(1) \subseteq \Delta(a)$,

$$a(T_n^{-1}(1)) = \tau(a) = t \in I_n$$

And for $r \in T_n^{-1}(0)$, $\{r\} \not\subseteq \Delta(a)$, so $a(\{r\}) = 0$. So we have $a \in V[T_n, I_n]$ for all n, and α wins the game. $\square$

Lemma 11. If μ is an α-winning tactic then there exists $g : \mathcal{S} \times \mathcal{I} \to \mathcal{S} \times \mathcal{I}$ such that

(a) $(T, J) \prec g(T, J)$ for all $(T, J) \in \mathcal{S} \times \mathcal{I}$

(b) If $g(T_n, J_n) \prec (T_{n+1}, J_{n+1})$ for all $n \in \mathbb{N}$, then $\left(\bigcap_{n\in\mathbb{N}} I_n\right) \cap \left(\bigcup_{n\in\mathbb{N}} \mathrm{Dom}T_n\right) = \emptyset$.

Proof: Give pairs in $\mathcal{S} \times \mathcal{I}$ the transitive relation $\prec$ where

$$(T, J) \prec (S, I) \Leftrightarrow S \supseteq T \wedge \overline{I} \subseteq J \wedge \mathrm{diam} I < \frac{1}{2}\mathrm{diam} J$$

We then begin by noting that we may assume $\mu(V[T, J]) = V[h_1(T), h_2(J)]$ for some h_1, h_2.

By Proposition 7, $h_1(T,J)$ $h_2(T,J)$ throughout

$$V[T, J] \supseteq V[h_1(T), h_2(J))] \Leftrightarrow h_1(T) \supseteq T \wedge h_2(J) \subseteq J$$

We may additionally assume that

$$\overline{h_2(J)} \subseteq J \wedge \mathrm{diam} h_2(J) < \frac{1}{2}\mathrm{diam} J$$

since β could force this for every other turn anyway. Therefore, we assume $(T, J) \prec (h_1(T), h_2(J))$.

For each $S \in \mathcal{S}$, we let S^* be the function such that

$$\mathrm{Dom}S^* = \mathrm{Dom}S \text{ and } S^*(t) = 1 - S(t) \text{ for all } t \in \mathrm{Dom}S^*$$

Then, we define the functions g_1, g_2 by $g_1(T,J)$, $g_2(T,J)$ throughout

$$g_1(T) = [h_1([\underset{h_2(T,J)}{h_1(T)}]^*)]^* \text{ and } g_2(J) = h_2(\underset{h_1(T,J)}{h_2(J)})$$

We note for $t \in \mathrm{Dom}T$, we have

$$g_1(T)(t) = 1 - h_1([h_1(T)]^*)(t) = 1 - [h_1(T)]^*(t) = h_1(T)(t) = T(t)$$

and thus $g_1(T)$ extends T, and as the interval requirements are satisfied as well, we have

$$(T, J) \prec (g_1(T), g_2(J))$$

satisfying (a) for $g(T, J) = (g_1(T), g_2(J))$.

Now let (T_n, J_n) be a sequence such that $g(T_n, J_n) = (g_1(T_n), g_2(J_n)) \prec (T_{n+1}, J_{n+1})$ for all $n \in \mathbb{N}$ as in the hypothesis of (b), and denote $S_n = h_1(T_n)$ and $I_n = h_2(J_n)$. Let $V_n = V[T_n, J_n]$. Since

$$(T_n, J_n) \prec \underset{(h_1(T_n,J_n),\, h_2(T_n,J_n))}{\underset{\shortparallel}{h(T_n, J_n)}} \prec (T_{n+1}, J_{n+1}) \Rightarrow V_n \supseteq \mu(V_n) \supseteq V_{n+1}$$

it follows that V_n are the legal moves of β when α uses the winning strategy μ, and thus $\bigcap_{n\in\mathbb{N}} V_n \neq \emptyset$.

We claim that $h(S_n^*, I_n) \prec (S_{n+1}^*, I_{n+1})$. Let $t \in \mathrm{Dom} S_n^*$.

$$S_{n+1}^*(t) = 1 - S_{n+1}(t) = 1 - T_{n+1}(t) = 1 - g_1(T_n)(t) = h_1([h_1(T_n)]^*)(t) = h_1(S_n^*)(t)$$

and we have the result since the interval requirements are easily seen to be satisfied.

Now, similarly to before, since we have

$$(S_n^*, I_n) \prec h(S_n^*, I_n) \prec (S_{n+1}^*, I_{n+1}) \Rightarrow V[S_n^*, I_n] \supseteq \mu(V[S_n^*, I_n]) \supseteq V[S_{n+1}^*, I_{n+1}]$$

it follows that $V[S_n^*, I_n]$ are legal moves of β when α uses the winning strategy μ, and thus $\bigcap_{n\in\mathbb{N}} W_n \neq \emptyset$ for $W_n = V[S_n^*, I_n]$.

So choose $a \in \bigcap_{n\in\mathbb{N}} V_n$ and $b \in \bigcap_{n\in\mathbb{N}} W_n$. First note that since

$$J_0 \supseteq \overline{I_0} \supseteq J_1 \supseteq \overline{I_1} \supseteq \dots \text{ and } \mathrm{diam}\overline{I_n} \to 0$$

we have that

$$\bigcap_{n\in\mathbb{N}} J_n = \bigcap_{n\in\mathbb{N}} I_n = \{t\}$$

for some singleton $t \neq 0$. In fact, t must be $\tau(a) = \tau(b)$.

Then note that for all n,

$$a(T_n^{-1}(1)) \in J_n \Rightarrow a(T_n^{-1}(1)) = \tau(a) \Rightarrow T_n^{-1}(1) \subseteq \Delta(a)$$

and

$$b(S_n^{*-1}(1)) \in I_n \Rightarrow b(S_n^{*-1}(1)) = \tau(b) \Rightarrow S_n^{*-1}(1) \subseteq \Delta(b) \Rightarrow S_n^{-1}(0) \subseteq \Delta(b)$$

So since

$$\bigcup_{n\in\mathbb{N}} S_n^{-1}(0) = \bigcup_{n\in\mathbb{N}} T_n^{-1}(0)$$

and

$$\bigcup_{n\in\mathbb{N}} T_n^{-1}(0) \cup \bigcup_{n\in\mathbb{N}} T_n^{-1}(1) = \bigcup_{n\in\mathbb{N}} \mathrm{Dom} T_n$$

we have that

$$\bigcup_{n\in\mathbb{N}} \mathrm{Dom} T_n \subseteq \Delta(a) \cup \Delta(b) \not\ni t$$

and

$$\left(\bigcap_{n\in\mathbb{N}} I_n\right) \cap \left(\bigcup_{n\in\mathbb{N}} \mathrm{Dom} T_n\right) = (\{t\}) \cap \left(\bigcup_{n\in\mathbb{N}} \mathrm{Dom} T_n\right) = \emptyset$$

□

Lemma 12. If μ is an α-winning tactic then there exist $A \in \mathcal{S}$, $a \in \mathbb{R}^*$ and $\{J_n : n \in \mathbb{N}\} \subseteq \mathcal{I}$ such that:

(a) $\bigcap_{n\in\mathbb{N}} J_n = \{a\}$

(b) $\forall S \supseteq A\ \forall n \in \mathbb{N}\ \exists T \supseteq S(g_2(T, J_n) = J_{n+1})$

Proof. Suppose by way of contradiction that:

$$\exists(B, J)\ \forall(B', J') \succ (B, J)\ \exists S \supseteq B'\ \forall T \supseteq S(g_2(T, J) \neq J')$$

all $I \in \mathcal{J}$ s.t. $I \subset J$, diam $I < \frac{1}{2}$ diam J

Let $\mathcal{I} = \{I_n; n \in \mathbb{N}\}$ be an enumeration of $\mathcal{I}$. Let $B_0 = B$ and B_{n+1} be the S extending B'_n above so that

$$\forall T \supseteq B_{n+1}\,(g_2(T, J) \neq I_n) \tag{3}$$

Let $B_\infty = \bigcup_{n\in\mathbb{N}} B_n \in \mathcal{S}$ and note $g_2(B_\infty, J) \neq I_n$ for all $n \in \mathbb{N}$ since B_∞ extends each B_n. Contradiction as $g_2(B_\infty, J)$ must map to some element of $\mathcal{I}$. So we have

$$\forall(B, J)\ \exists(B', J') \succ (B, J)\ \forall S \succ B'\ \exists T \succ S(g_2(T, J) = J')$$

Let (A_0, J_0) be anything, with $(A_{n+1}, J_{n+1}) = (A'_n, J'_n)$. So $\{a\} = \bigcap_{n\in\mathbb{N}} J_n$ and $A = \bigcup_{n\in\mathbb{N}} A_n$, which trivially satisfies (a). It also satisfies (b) since for all $S \supseteq A$, $S \supseteq A_n$ for all n, so there exists a $T \supseteq S$ such that $g_2(T, J_n) = J'_n = J_{n+1}$. $\square$

`Theorem 10.` There does not exist an α-winning strategy on X which depends on only the most recent play by β.

Suppose by way of contradiction there exists an α-winning tactic. By Lemma 12, there are $A \in \mathcal{S}$, $a \in \mathbb{R}^*$, and $\{J_n : n \in \mathbb{N}\} \subseteq \mathcal{I}$ such that $J_n \supseteq J_{n+1}$, $\bigcap_{n\in\mathbb{N}} J_n = \{a\}$, and

$$\forall S \supseteq A \, \forall n \in \mathbb{N} \, \exists \gamma_n(S) \supseteq S \, (g_2(\gamma_n(S), J_n) = J_{n+1})$$

Then define

$$S_0 = \begin{cases} A & \text{if } a \in \mathrm{Dom} A \\ A \cup \{(a, 0)\} & \text{otherwise} \end{cases}$$

$$T_n = \gamma_n(S_n)$$

$$S_{n+1} = g_1(\gamma_n(S_n), J_n)$$

And thus we have

$$(T_n, J_n) \prec g(T_n, J_n) = g(\gamma_n(S_n), J_n) = (S_{n+1}, J_{n+1}) \prec (T_{n+1}, J_{n+1})$$

These (T_n, J_n) satisfy the hypothesis of Lemma 11b; however, note that

$$a \in \left(\bigcap_{n\in\mathbb{N}} J_n\right) \cap \mathrm{Dom} T_0 \subseteq \left(\bigcap_{n\in\mathbb{N}} J_n\right) \cap \left(\bigcup_{n\in\mathbb{N}} \mathrm{Dom} T_n\right) = \emptyset$$

a contradiction. □

Lemma 11. If μ is an α-winning tactic then there exists $g : \mathcal{S} \times \mathcal{I} \to \mathcal{S} \times \mathcal{I}$ such that

(a) $(T, J) \prec g(T, J)$ for all $(T, J) \in \mathcal{S} \times \mathcal{I}$

(b) If $g(T_n, J_n) \prec (T_{n+1}, J_{n+1})$ for all $n \in \mathbb{N}$, then $\left(\bigcap_{n\in\mathbb{N}} I_n\right) \cap \left(\bigcup_{n\in\mathbb{N}} \mathrm{Dom}T_n\right) = \emptyset$.

Lemma 12. If μ is an α-winning tactic then there exist $A \in \mathcal{S}$, $a \in \mathbb{R}^*$ and $\{J_n : n \in \mathbb{N}\} \subseteq \mathcal{I}$ such that:

(a) $\bigcap_{n\in\mathbb{N}} J_n = \{a\}$

(b) $\forall S \supseteq A\, \forall n \in \mathbb{N}\, \exists T \supseteq S(g_2(T, J_n) = J_{n+1})$