An Example of Gruenhage's Compact-Point Game for which K has a winning strategy, but no winning k-Markov strategy

We construct a ZFC example given by Gary Gruenhage, inpsired by a ZFC+$\neg$SH example due to Stephen Watson [3].

Theorem 1. *There exists a compact, zero-dimensional topological space X which has a point-countable cover $\mathcal{U} = \{U_c : c \in 2^\omega\}$ of clopen sets which is not the union of countably-many point-finite collections.*

Proof. Take a zero-dimensional Corson compact Y of weight 2^ω, which is not Eberlein compact. It follows by [1] that $Y^2 \setminus \Delta = \{(y_1, y_2) : y_1, y_2 \in Y, y_1 \neq y_2\}$ is metalindelöf, but not σ-metacompact.

Let $X = Y^2$, and let $\mathcal{V}$ be an open cover of X by 2^ω clopen sets. Since $Y^2 \setminus \Delta$ is metalindelöf and Δ is compact, let $\mathcal{V}'$ be a refinement of $\mathcal{U}$ such that it is a point-countable cover of $Y^2 \setminus \Delta$, and let $\mathcal{V}''$ be a refinement of $\mathcal{U}$ such that it is finite cover of Δ. Then $\mathcal{U} = \mathcal{V}' \cup \mathcal{V}''$ is point-countable on Y^2, and if it was the union of countably-many point-finite collections, so would $\mathcal{V}'$ (making $Y^2 \setminus \Delta$ σ-metacompact, contradiction). $\square$

Definition 2. Using the X from Theorem 1, let

$$\mathbb{X} = (X \times 2^{<\omega}) \cup 2^\omega$$

compose a topological sum of $2^{<\omega}$ copies of X along with a discrete copy of the Cantor Set 2^ω, and add open (in fact, compact) neighborhoods of the form:

$$B_c = c \cup (U_c \times \{c \upharpoonright n : n < \omega\})$$

as seen in Figure 1.

Definition 3. Let $S \in [2^\omega]^{<\omega}$ and $m < \omega$. Define

$$K_S = \bigcup_{c \in S} B_c$$

$$A = \{z^\frown\langle 1\rangle : z \in 1^{<\omega}\}$$

$$K_S^* = K_S \setminus (X \times A)$$

$$L_m = X \times 2^{<m}$$

and observe that every compact set is dominated by the compact set $K_S^* \cup L_m$ for some S, m.

Intuitively, K_S^* collects the branches of U_c converging up to each $c \in S$ while avoiding the copy of X for each s in an antichain A, and L_m collects the copies of X with $|s| < m$ at the base of the tree. (See Figure 2)

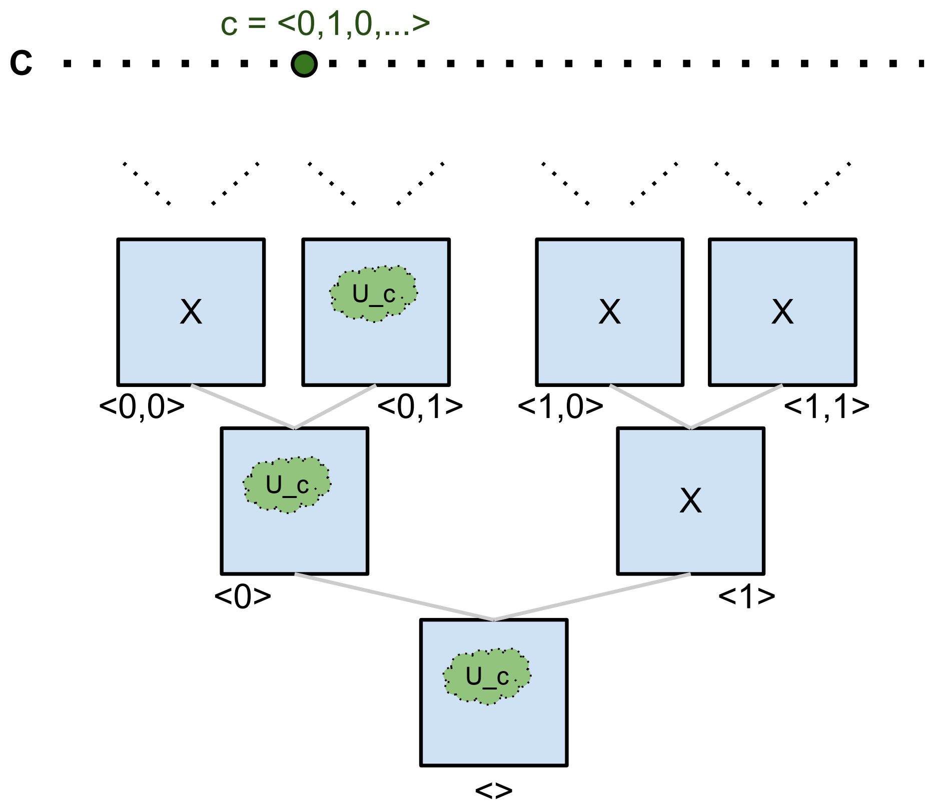

Figure 1: The Cantor Tree space $\mathbb{X}$

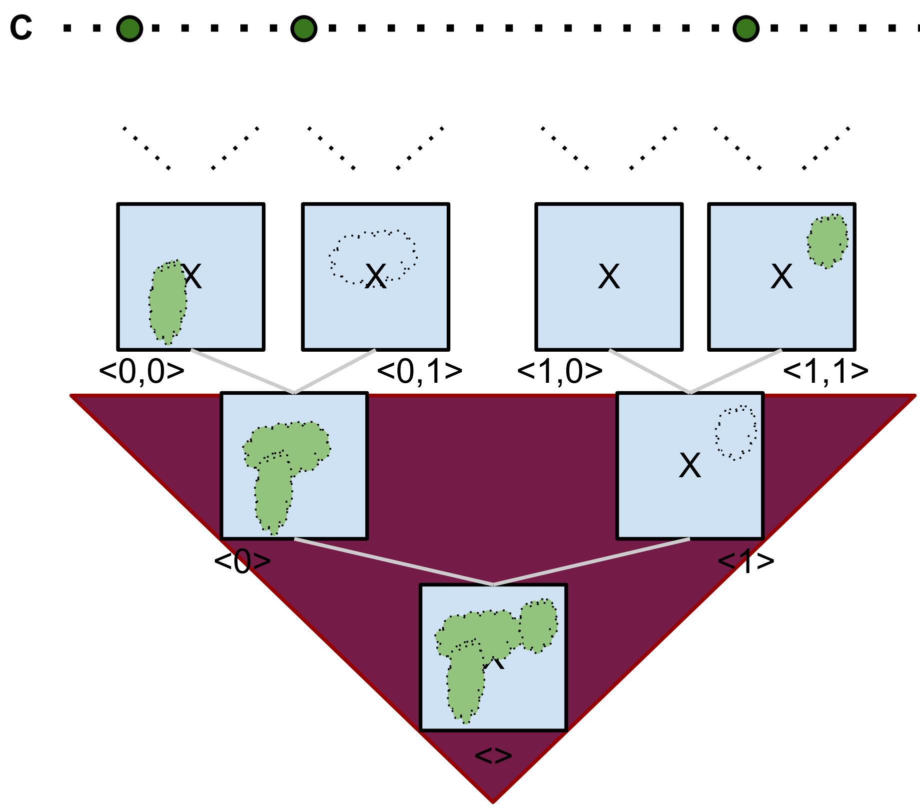

Figure 2: K_S^* and L_m

Definition 4. $LF_{K,P}(\mathbb{X})$ is a topological game consisting of players K and P. During each round, K chooses a compact subset of $\mathbb{X}$, and P chooses a point outside of any compact set previously played by K. K wins the game if the set of points chosen by P throughout all rounds of the game are locally finite in the space.

Definition 5. We say $I \uparrow G$ if Player I has a winning strategy in the game G.

We say $I \uparrow_{\text{tact}} G$ (resp. $I \uparrow_{k\text{-tact}} G$) if Player I has a winning tactical (resp. k-tactical) strategy in the game G, a strategy depending on only the (k) most recent move(s) of the opponent.

We say $I \uparrow_{\text{mark}} G$ (resp. $I \uparrow_{k\text{-mark}} G$) if Player I has a winning Markov (resp. k-Markov) strategy in the game G, a strategy depending on only the (k) most recent move(s) of the opponent and the current round number.

Proposition 6. *Without loss of generality, we may assume P always plays points in $X \times 2^{<\omega}$ throughout $LF_{K,P}(\mathbb{X})$.*

Proposition 7. $K \uparrow LF_{K,P}(\mathbb{X})$

Proof. Let $x \in X$. $C^x = \{c \in C : x \in U_c\}$ is a countable collection by the point-countability of $\mathcal{U}$, so label its elements as $\{c_n^x : n < \omega\}$.

K may use the strategy

$$\sigma(\langle x_0, s_0\rangle, \ldots, \langle x_{n-1}, s_{n-1}\rangle) = \bigcup_{i<n} K_{\{c_0^{x_i}, \ldots, c_{n-1}^{x_i}\}} \cup L_{|s_i|+1}$$

This is a winning strategy because each move $\langle x_i, s_i\rangle$ by P cannot be a part of a subsequence of the play converging to any $c_n^{x_i}$, since $K_{\{c_0^{x_i}, \ldots, c_n^{x_i}\}} \supseteq B_{c_n^{x_i}}$ was forbidden during round n. □

Theorem 8. $K \not\uparrow_{tact} LF_{K,P}(\mathbb{X})$.

Proof. This is actually a corollary of Gruenhage's theorem in [2]: $\mathbb{X}$ is locally compact (each point is either in some X or in some B_c) but not metacompact (any cover of 2^ω necessarily will have infinite overlap at some $X \times \{s\}$). However, we proceed with a direct game-theoretic proof.

Suppose that $\sigma(\langle x, s\rangle)$ was a winning tactical strategy for K and define the compact set

$$\sigma'(x, n) = \bigcup_{|s| \le n} \sigma(\langle x, s\rangle)$$

There exists some $f : 2^\omega \to \omega$ such that for all $x \in U_c$, $\sigma'(x, f(c))$ covers some $B_c \setminus L_m$. (If not, P counters by simply always playing in $B_c \setminus L_m$.)

Recall that $\mathcal{U}$ is not the union of the countably-many point-finite collections, and

$$\mathcal{U} = \bigcup_{n<\omega} \{U_c : f(c) = n\}$$

so we may choose n where $\mathcal{U}_n = \{U_c : f(c) = n\}$ is not point-finite. Fix x so that x belongs to each of $\{U_{c_0}, U_{c_1}, \dots\} \subseteq \mathcal{U}_n$.

For each c_i, $\sigma'(x, f(c_i)) = \sigma'(x, n)$ covers c_i. Thus $\sigma'(x, n) \supseteq \{c_0, c_1, \dots\}$ is a compact set covering a closed discrete subset, a contradiction. $\square$

Theorem 9. $K \not\uparrow_{2\text{-}tact} LF_{K,P}(\mathbb{X})$.

Proof. Suppose $\sigma(\langle x, s\rangle, \langle y, t\rangle)$ was a winning 2-tactical strategy. Without loss of generality we assume it ignores order. We may define $S(x, y, n) \in [2^\omega]^{<\omega}$ (increasing on n) and $n < m(x, y, n) < \omega$ such that for each (x, y),

$$\bigcup_{s,t\in 2^{\leq n}} \sigma(\langle x, s\rangle, \langle y, t\rangle) \subseteq K^*_{S(x,y,n)} \cup L_{m(x,y,n)}$$

and so we assume

$$\sigma(\langle x, s\rangle, \langle y, t\rangle) = K^*_{S(x,y,\max(|s|,|t|))} \cup L_{m(x,y,\max(|s|,|t|))}$$

Select an arbitrary point $x' \in X$. We define a tactical strategy

$$\tau(x, s) = K^*_{S(x,x',m(x,x',|s|)+1)} \cup L_{m(x,x',m(x,x',|s|)+1)}$$

We complete the proof by showing τ is a winning tactical strategy (a contradiction).

Suppose

$$\langle x_0, s_0\rangle, \langle x_1, s_1\rangle, \langle x_2, s_2\rangle, \dots$$

successfully countered τ by clustering at $c \in 2^\omega$ (the strategy trivially prevents clustering elsewhere). Let $z_n = \langle 0, \dots, 0\rangle$ with n zeros. We claim

$$\langle x_0, s_0\rangle, \langle x', z_{m(x_0,x',|s_0|)}{}^\frown\langle 1\rangle\rangle, \langle x_1, s_1\rangle, \langle x', z_{m(x_1,x',|s_1|)}{}^\frown\langle 1\rangle\rangle, \langle x_2, s_2\rangle, \langle x', z_{m(x_2,x',|s_2|)}{}^\frown\langle 1\rangle\rangle, \dots$$

is a successful counter to σ.

We will need the fact that, as $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ:

$$|s_i| < m(x_i, x', |s_i|) + 1 = |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|$$

$$< m(x_i, x', m(x_i, x', |s_i|) + 1) = m(x_i, x', |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|) \leq |s_{i+1}|$$

Note that $m(x, y, \max(|s|, |t|))$ is increasing throughout this play of the game versus σ:

$$m(x_i, x', \max(|s_i|, |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|))$$

$$
\begin{aligned}
&= m(x_i, x', |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|) \\
&\leq |s_{i+1}| \\
&< m(x_{i+1}, x', |s_{i+1}|) \\
&= m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|)) \\
&= |z_{m(x_{i+1},x',|s_{i+1}|)}| \\
&< |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle| \\
&< m(x_{i+1}, x', |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle|) \\
&= m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle|))
\end{aligned}
$$

We turn to showing that $\langle x', z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle\rangle$ is always a legal move. Since $z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle$ is on the antichain avoided by any K^*, the problem is reduced to showing that this move isn't forbidden by

$$L_{m(x_{i+1},x',\max(|s_{i+1}|,|z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|))}$$

which we can see here:

$$m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|)) = m(x_{i+1}, x', |s_{i+1}|) < |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle|$$

We can conclude by showing that $\langle x_{i+1}, s_{i+1}\rangle$ is always a legal move. We can see it avoids

$$L_{m(x_i,x',\max(|s_i|,|z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|))}$$

since

$$m(x_i, x', \max(|s_i|, |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|)) = m(x_i, x', |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|) \leq |s_{i+1}|$$

Since $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ, for $h \leq i$ it avoided

$$K^*_{S(x_h,x',m(x_h,x',|s_h|)+1)} = K^*_{S(x_h,x',\max(|s_h|,|z_{m(x_h,x',|s_h|)}{}^\frown\langle 1\rangle|))}$$

and when $h < i$, we see it avoids:

$$
\begin{aligned}
K^*_{S(x_{h+1},x',\max(|s_{h+1}|,|z_{m(x_h,x',|s_h|)}{}^\frown\langle 1\rangle|))} &= K^*_{S(x_{h+1},x',|s_{h+1}|)} \\
&\subseteq K^*_{S(x_{h+1},x',m(x_{h+1},x',|s_{h+1}|)+1)}
\end{aligned}
$$

This conclude the proof. $\square$

Theorem 10. $K \not\uparrow_{k\text{-}tact} LF_{K,P}(\mathbb{X})$.

Proof. The proof proceeds in parallel to the proof of $K \not\uparrow_{2\text{-tact}} LF_{K,P}(\mathbb{X})$, and in fact is just a generalization of said proof (at the cost of simplicity).

Suppose $\sigma(\langle x_0, s_0\rangle, \dots, \langle x_k, s_k\rangle)$ was a winning $(k+1)$-tactical strategy. Without loss of generality we assume it ignores order. We may define $S(x_0, \dots, x_k, n) \in [2^\omega]^{<\omega}$ (increasing on n) and $n < m(x_0, \dots, x_k, n) < \omega$ such that for each $(x_0, \dots, x_k)$,

$$\bigcup_{s_0,\dots,s_k \in 2^{\leq n}} \sigma(\langle x_0, s_0\rangle, \dots, \langle x_k, s_k\rangle) \subseteq K^*_{S(x_0,\dots,x_k,n)} \cup L_{m(x_0,\dots,x_k,n)}$$

and so we assume

$$\sigma(\langle x_0, s_0\rangle, \dots, \langle x_k, s_k\rangle) = K^*_{S(x_0,\dots,x_k,\max(|s_0|,\dots,|s_k|))} \cup L_{m(x_0,\dots,x_k,\max(|s_0|,\dots,|s_k|))}$$

Select an arbitrary point $x' \in X$. Let $M^0(x,n) = m(x, x', \dots, x', n)$ and $M^{i+1}(x,n) = M^0(x, M^i(x,n)+1)$. We define a tactical strategy

$$\tau(x,s) = K^*_{S(x,x',\dots,x',M^{k-1}(x,|s|)+1)} \cup L_{m(x,x',\dots,x',M^{k-1}(x,|s|)+1)}$$

We complete the proof by showing τ is a winning tactical strategy (a contradiction).

Suppose

$$\langle x_0, s_0\rangle, \langle x_1, s_1\rangle, \langle x_2, s_2\rangle, \dots$$

successfully countered τ by clustering at $c \in 2^\omega$ (the strategy trivially prevents clustering elsewhere). Let $z_n = \langle 0, \dots, 0\rangle$ with n zeros. We claim

$$\langle x_0, s_0\rangle, \langle x', z_{M^0(x_0,|s_0|)}{}^\frown\langle 1\rangle\rangle, \langle x', z_{M^1(x_0,|s_0|)}{}^\frown\langle 1\rangle\rangle, \dots, \langle x', z_{M^{k-1}(x_0,|s_0|)}{}^\frown\langle 1\rangle\rangle,$$

$$\langle x_1, s_1\rangle, \langle x', z_{M^0(x_1,|s_1|)}{}^\frown\langle 1\rangle\rangle, \langle x', z_{M^1(x_1,|s_1|)}{}^\frown\langle 1\rangle\rangle, \dots, \langle x', z_{M^{k-1}(x_1,|s_1|)}{}^\frown\langle 1\rangle\rangle, \dots$$

is a successful counter to σ.

We will need the fact that, as $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ:

$$\begin{aligned} |s_i| < M^0(x_i,|s_i|)+1 = |z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle| < M^0(x_i, M^0(x_i,|s_i|)+1)+1 \\ = M^1(x_i,|s_i|)+1 = |z_{M^1(x_i,|s_i|)}{}^\frown\langle 1\rangle| < \dots < |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle| \\ = M^{k-1}(x_i,|s_i|)+1 < m(x_i, x', \dots, x', M^{k-1}(x_i,|s_i|)+1) \leq |s_{i+1}| \end{aligned}$$

Note that $m(x_0, \dots, x_k, \max(|s_0|, \dots, |s_k|))$ is increasing throughout this play of the game versus σ:

$$m(x_i, x', \dots, x', \max(|s_i|, |z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle|, \dots, |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|))$$

$$
\begin{aligned}
&= m(x_i, x', \ldots, x', |z_{M^{k-1}(x_i,|s_i|)} {}^\frown \langle 1\rangle|) \\
&= m(x_i, x', \ldots, x', M^{k-1}(x_i, |s_i|) + 1) \\
&\leq |s_{i+1}| \\
&< M^0(x_{i+1}, |s_{i+1}|) \\
&= m(x_{i+1}, x', \ldots, x', |s_{i+1}|) \\
&= m(x_{i+1}, x', \ldots, x', \max(|s_{i+1}|, |z_{M^0(x_i,|s_i|)} {}^\frown \langle 1\rangle|, \ldots, |z_{M^{k-1}(x_i,|s_i|)} {}^\frown \langle 1\rangle|)) \\
&= |z_{m(x_{i+1},x',\ldots,x',|s_{i+1}|)}| \\
&= |z_{M^0(x_{i+1},|s_{i+1}|)}| \\
&< |z_{M^0(x_{i+1},|s_{i+1}|)} {}^\frown \langle 1\rangle| \\
&< m(x_{i+1}, x', \ldots, x', |z_{M^0(x_{i+1},|s_{i+1}|)} {}^\frown \langle 1\rangle|) \\
&= m(x_{i+1}, x', \ldots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1},|s_{i+1}|)} {}^\frown \langle 1\rangle|, |z_{M^1(x_i,|s_i|)} {}^\frown \langle 1\rangle|, \ldots, |z_{M^{k-1}(x_i,|s_i|)} {}^\frown \langle 1\rangle|)) \\
&\vdots \\
&< m(x_{i+1}, x', \ldots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1},|s_{i+1}|)} {}^\frown \langle 1\rangle|, \ldots, |z_{M^{k-1}(x_{i+1},|s_{i+1}|)} {}^\frown \langle 1\rangle|))
\end{aligned}
$$

We turn to showing that $\langle x', z_{M^j(x_{i+1},|s_{i+1}|)} {}^\frown \langle 1\rangle\rangle$ is always a legal move. Since $z_{M^j(x_{i+1},|s_{i+1}|)} {}^\frown \langle 1\rangle$ is on the antichain avoided by any K^*, the problem is reduced to showing that this move isn't forbidden by

$$
\begin{aligned}
&L_{m(x_{i+1},x',\ldots,x',\max(|s_{i+1}|,|z_{M^0(x_{i+1},|s_{i+1}|)} {}^\frown \langle 1\rangle|,\ldots,|z_{M^{j-1}(x_{i+1},|s_{i+1}|)} {}^\frown \langle 1\rangle|,|z_{M^j(x_i,|s_i|)} {}^\frown \langle 1\rangle|,\ldots,|z_{M^k(x_i,|s_i|)} {}^\frown \langle 1\rangle|))} \\
&= L_{m(x_{i+1},x',\ldots,x',|z_{M^{j-1}(x_{i+1},|s_{i+1}|)} {}^\frown \langle 1\rangle|)}
\end{aligned}
$$

which we can see here:

$$
\begin{aligned}
&m(x_{i+1}, x', \ldots, x', |z_{M^{j-1}(x_{i+1},|s_{i+1}|)} {}^\frown \langle 1\rangle|) \\
&= m(x_{i+1}, x', \ldots, x', M^{j-1}(x_{i+1}, |s_{i+1}|) + 1) \\
&= M^0(x_{i+1}, M^{j-1}(x_{i+1}, |s_{i+1}|) + 1) \\
&= M^j(x_{i+1}, s_{i+1}) \\
&< |z_{M^j(x_{i+1},|s_{i+1}|)} {}^\frown \langle 1\rangle|
\end{aligned}
$$

We can conclude by showing that $\langle x_{i+1}, s_{i+1}\rangle$ is always a legal move. We can see it avoids

$$
L_{m(x_i,x',\ldots,x',\max(|s_i|,|z_{M^0(x_i,|s_i|)} {}^\frown \langle 1\rangle|,\ldots,|z_{M^{k-1}(x_i,|s_i|)} {}^\frown \langle 1\rangle|))}
$$

since

$$m(x_i, x', \ldots, x', \max(|s_i|, |z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|))$$
$$= m(x_i, x', \ldots, x', |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|)$$
$$= m(x_i, x', \ldots, x', M^{k-1}(x_i, |s_i|) + 1)$$
$$\leq |s_{i+1}|$$

Since $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ, for $h \leq i$ it avoided

$$K^*_{S(x_h,x',\ldots,x',M^{k-1}(x_h,|s_h|)+1)}$$
$$= K^*_{S(x_h,x',\ldots,x',\max(|s_h|,|z_{M^0(x_h,|s_h|)}{}^\frown\langle 1\rangle|,\ldots,|z_{M^{k-1}(x_h,|s_h|)}{}^\frown\langle 1\rangle|))}$$

and when $h < i$, we see it avoids both:

$$K^*_{S(x_{h+1},x',\ldots,x',\max(|s_{h+1}|,|z_{M^0(x_{h+1},|s_{h+1}|)}{}^\frown\langle 1\rangle|,\ldots,|z_{M^{j-1}(x_{h+1},|s_{h+1}|)}{}^\frown\langle 1\rangle|,|z_{M^j(x_h,|s_h|)}{}^\frown\langle 1\rangle|,\ldots,|z_{M^k(x_h,|s_h|)}{}^\frown\langle 1\rangle|))}$$
$$= K^*_{S(x_{h+1},x',\ldots,x',|z_{M^{j-1}(x_{h+1},|s_{h+1}|)}{}^\frown\langle 1\rangle|)}$$
$$= K^*_{S(x_{h+1},x',\ldots,x',M^{j-1}(x_{h+1},|s_{h+1}|)+1)}$$
$$\subseteq K^*_{S(x_{h+1},x',\ldots,x',M^{k-1}(x_{h+1},|s_{h+1}|)+1)}$$

and:

$$K^*_{S(x_{h+1},x',\ldots,x',\max(|s_{h+1}|,|z_{M^0(x_h,|s_h|)}{}^\frown\langle 1\rangle|,\ldots,|z_{M^k(x_h,|s_h|)}{}^\frown\langle 1\rangle|))}$$
$$= K^*_{S(x_{h+1},x',\ldots,x',|s_{k+1}|)}$$
$$\subseteq K^*_{S(x_{h+1},x',\ldots,x',M^{k-1}(x_{h+1},|s_{h+1}|)+1)}$$

This concludes the proof. □

Corollary 11. *$K \not\uparrow_{k\text{-}mark} LF_{K,P}(\mathbb{X})$.*

Proof. Let $\sigma(\langle x_0, s_0\rangle, \ldots, \langle x_{k-1}, s_{k-1}\rangle, n)$ be a winning k-Markov strategy for K increasing on n. We define a k-tactical strategy

$$\tau(\langle x_0, s_0\rangle, \ldots, \langle x_{k-1}, s_{k-1}\rangle) = \sigma(\langle x_0, s_0\rangle, \ldots, \langle x_{k-1}, s_{k-1}\rangle, \max_{i<k}(|s_i|)) \cup L_{\max_{i<k}(|s_i|)+1}$$

and observe that since for any legal play of the game, the round number $n \leq \max_{i<k}(|s_i|)$, we know τ always yields supersets of σ, and is thus also a winning strategy, contradiction. □

References

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[3] S. Watson, *Locally compact normal meta-Lindelöf spaces may not be paracompact: an application of uniformization and Suslin lines*. Proc. Amer. Math. Soc. 98 (1986), no. 4, 676680.