Definition 1. A **uniform space** $\langle X, \mathcal{D}\rangle$ is a set X paired with a filter $\mathcal{D}$ (called its **uniformity**) of relations (called **entourages**) on X such that for each entourage $D \in \mathcal{D}$:

- D is reflexive, i.e., the diagonal $\Delta \subseteq D$.
- Its inverse $D^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in D\} \in \mathcal{D}$.
- There exists $\frac{1}{2}D \in \mathcal{D}$ such that

$$2(\frac{1}{2}D) = \frac{1}{2}D \circ \frac{1}{2}D = \{\langle x, z\rangle : \exists y(\langle x, y\rangle, \langle y, z\rangle \in \frac{1}{2}D)\} \subseteq D$$

Note that since $\mathcal{D}$ is a filter, for each $D \in \mathcal{D}$, the symmetric relation $D \cap D^{-1} \in \mathcal{D}$.

Proposition 2. *For each $D \in \mathcal{D}$ and $n < \omega$ there exists $\frac{1}{2^{n+1}}D \in \mathcal{D}$ such that*

$$2(\frac{1}{2^{n+1}}D) = \frac{1}{2^{n+1}}D \circ \frac{1}{2^{n+1}}D \subseteq \frac{1}{2^n}D$$

and if $2E \subseteq \frac{1}{2^n}D$, then $E \subseteq \frac{1}{2^{n+1}}D$.

Definition 3. For an entourage $D \in \mathcal{D}$, let $D[x] = \{y : (x, y) \in D\}$ be the **D-neighborhood** of x. The uniform topology for a uniform space $\langle X, \mathcal{D}\rangle$ is generated by the base $\{D[x] : x \in X, D \in \mathcal{D}\}$.

Theorem 4. *A space X is uniformizable (its topology is the uniform topology for some uniformity) if and only if X is completely regular ($T_{3\frac{1}{2}}$).*

Proposition 5. *If X is a uniform space, then for all $x \in X$ and symmetric entourages D:*

$$x \in \frac{1}{2}D[y] \text{ and } y \in \frac{1}{2}D[z] \Rightarrow x \in D[z]$$

and

$$\frac{1}{2}D[x] \subseteq \overline{\frac{1}{2}D[x]} \subseteq D[x]$$

Proof. The first is by definition of $\frac{1}{2}D$.

If $z \in \overline{\frac{1}{2}D[x]}$, it follows that there is $y \in \frac{1}{2}D[x] \cap \frac{1}{2}D[z]$ since $\frac{1}{2}D[z]$ is an open neighborhood of z. Thus $(x, z) \in D \Rightarrow z \in D[x] \Rightarrow \overline{\frac{1}{2}D[x]} \subseteq D[x]$. □

Definition 6. For a uniform space X, Bell's proximity game proceeds as follows.

In round 0, $\mathscr{D}$ chooses an entourage D_0, followed by $\mathscr{P}$ choosing a point $p_0 \in X$.

In round $n+1$, $\mathscr{D}$ chooses an entourage $D_{n+1} \subseteq D_n$, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in 4D_n[p_n]$.

Player $\mathscr{D}$ wins if either $\bigcap_{n<\omega} 4D_n[p_n] = \emptyset$ or $\langle p_0, p_1, \dots\rangle$ converges.

Definition 7. For a uniform space X, the simplified proximal game $Prox_{D,P}(X)$ can be defined as follows:

In round 0, $\mathscr{D}$ chooses a symmetric entourage D_0, followed by $\mathscr{P}$ choosing a point $p_0 \in X$.

In round $n+1$, $\mathscr{D}$ chooses a symmetric entourage D_{n+1}, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in \left(\bigcap_{m\leq n} D_m\right)[p_n]$.

Player $\mathscr{D}$ wins if either $\bigcap_{n<\omega}\left(\bigcap_{m\leq n} D_m\right)[p_n] = \emptyset$ or $\langle p_0, p_1, \ldots\rangle$ converges.

Theorem 8. *$\mathscr{D}$ has a winning perfect-information strategy in Bell's game if and only if $\mathscr{D} \uparrow Prox_{D,P}(X)$.*

Proof. Let σ be a winning perfect information strategy for $\mathscr{D}$ in Bell's game. We define a perfect information strategy τ in the simplified game to yield symmetric entourages $\tau(p \restriction n) = \sigma(p \restriction n) \cap (\sigma(p \restriction n))^{-1}$ for all partial attacks $p \restriction n$. Note that $\tau(p \restriction n) = \bigcap_{m\leq n} \tau(p \restriction m)$.

If p attacks τ in the simplified game, $p(n+1) \in \left(\bigcap_{m\leq n} \tau(p \restriction m)\right)[p(n)] = \tau(p \restriction n)[p(n)] \subseteq \sigma(p \restriction n)[p(n)] \subseteq 4\sigma(p \restriction n)[p(n)]$, so p attacks σ in Bell's game. Thus either p converges, or

$$\emptyset = \bigcap_{n<\omega} 4\sigma(p \restriction n)[p(n)] \supseteq \bigcap_{n<\omega} \tau(p \restriction n)[p(n)] = \bigcap_{n<\omega}\left(\bigcap_{m\leq n} \tau(p \restriction n)\right)[p(n)]$$

For the other direction, let σ be a winning perfect information strategy for $\mathscr{D}$ in the simplified game such that $\sigma(p \restriction n) = \bigcap_{m\leq n} \sigma(p \restriction m)$. Define the perfect information strategy τ in Bell's Game such that $4\tau(p \restriction n) \subseteq \sigma(p \restriction n)$ and $\tau(p \restriction n) = \bigcap_{m\leq n} \tau(p \restriction m)$ for all partial attacks $p \restriction n$.

If p attacks τ in Bell's game, $p(n) \in 4\tau(p \restriction n) \subseteq \sigma(p \restriction n) = \bigcap_{m\leq n} \sigma(p \restriction m)$, so p attacks σ in the simplified game. Thus either p converges, or

$$\emptyset = \bigcap_{n<\omega}\left(\bigcap_{m\leq n} \sigma(p \restriction n)\right)[p(n)] = \bigcap_{n<\omega} \sigma(p \restriction n)[p(n)] \supseteq \bigcap_{n<\omega} 4\tau(p \restriction n)[p(n)] \supseteq \bigcap_{n<\omega} \tau(p \restriction n)[p(n)]$$

□

Proposition 9. *$\mathscr{P}$ has a winning perfect-information strategy in Bell's game if and only if $\mathscr{P} \uparrow Prox_{D,P}(X)$.*

Proof. Similar to the previous. □

Definition 10. A uniform space is **proximal** if $\mathscr{D} \uparrow Prox_{D,P}(X)$.

Definition 11. For a space X and a point $x \in X$, the W**-convergence-game** $Con_{O,P}(X, x)$ proceeds as follows.

In round 0, $\mathscr{O}$ chooses a neighborhood U_n of x, followed by $\mathscr{P}$ choosing a point $p_n \in \bigcap_{m \leq n} U_m$.

Player $\mathscr{O}$ wins if $\langle p_0, p_1, \dots \rangle$ converges.

Definition 12. A space is W if $\mathscr{O} \uparrow Con_{O,P}(X, x)$ for all $x \in X$.

Definition 13. For each finite tuple $(m_0, \dots, m_{n-1})$, we define the k**-tactical fog-of-war**

$$T_k(\langle m_0, \dots, m_{n-1} \rangle) = \langle m_{n-k}, \dots, m_{n-1} \rangle$$

and the k**-Marköv fog-of-war**

$$M_k(\langle m_0, \dots, m_{n-1} \rangle) = \langle \langle m_{n-k}, \dots, m_{n-1} \rangle, n \rangle$$

So $P \underset{k\text{-tact}}{\uparrow} G$ if and only if there exists a winning strategy for P of the form $\sigma \circ T_k$, and $P \underset{k\text{-mark}}{\uparrow} G$ if and only if there exists a winning strategy of the form $\sigma \circ M_k$.

Theorem 14. *For all $x \in X$:*

- $\mathscr{D} \uparrow Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow Con_{O,P}(X, x)$
- $\mathscr{D} \underset{2k\text{-}tact}{\uparrow} Prox_{D,P}(X) \Rightarrow \mathscr{O} \underset{k\text{-}tact}{\uparrow} Con_{O,P}(X, x)$
- $\mathscr{D} \underset{2k\text{-}mark}{\uparrow} Prox_{D,P}(X) \Rightarrow \mathscr{O} \underset{k\text{-}mark}{\uparrow} Con_{O,P}(X, x)$

Proof. Let σ witness $\mathscr{D} \underset{2k\text{-tact}}{\uparrow} Prox_{D,P}(X)$ (resp. $\mathscr{D} \underset{2k\text{-mark}}{\uparrow} Prox_{D,P}(X)$, $\mathscr{D} \uparrow Prox_{D,P}(X)$). We define the k-tactical (resp. k-Marköv, perfect info) strategy τ such that

$$\tau \circ L_k(p) = \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(|p| - 1) \rangle)[x] \cap \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(|p| - 1), x \rangle)[x]$$

where L_{2k} is the $2k$-tactical fog-of-war (resp. $2k$-Marköv fog-of-war, identity) and L_k is the k-tactical fog-of-war (resp. k-Marköv fog-of-war, identity).

Let p attack τ. Consider the attack q against the winning strategy σ such that $q(2n) = x$ and $q(2n + 1) = p(n)$, and let $D_n = \sigma \circ L_{2k}(q)$ and $E_n = \bigcap_{m \leq n} D_n$.

Certainly, $x \in E_{2n}[x] = E_{2n}[q(2n)]$ for any $n < \omega$. Note also for any $n < \omega$ that

$$p(n) \in \bigcap_{m \leq n} \tau \circ L_k(p \restriction n)$$

$$= \bigcap_{m \leq n} (\sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(m-1) \rangle)[x] \cap \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(m-1), x \rangle)[x])$$

$$= \bigcap_{m \leq n} (D_{2m}[x] \cap D_{2m+1}[x]) = \bigcap_{m \leq 2n+1} D_m[x] = E_{2n+1}[x]$$

so by the symmetry of E_{2n+1}, $x \in E_{2n+1}[p(n)] = E_{2n+1}[q(2n+1)]$. Thus $x \in \bigcap_{n<\omega} E_n[q(n)] \neq \emptyset$, and since σ is a winning strategy, the attack q converges. Since $q(2n) = x$, q must converge to x. Thus its subsequence p converges to x, and τ is a winning strategy in $Con_{O,P}(X, x)$. □

Corollary 15. *For all $x \in X$:*

- $\mathscr{D} \underset{k\text{-tact}}{\uparrow} Prox_{D,P}(X) \Rightarrow \mathscr{O} \underset{k\text{-tact}}{\uparrow} Con_{O,P}(X, x)$
- $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Prox_{D,P}(X) \Rightarrow \mathscr{O} \underset{k\text{-mark}}{\uparrow} Con_{O,P}(X, x)$

Corollary 16. *All proximal spaces are W-spaces.*

Theorem 17. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete. Then*

- $\mathscr{O} \uparrow Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathscr{D} \uparrow Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \underset{k\text{-tact}}{\uparrow} Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathscr{D} \underset{k\text{-tact}}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \underset{k\text{-mark}}{\uparrow} Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathscr{D} \underset{k\text{-mark}}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$

Proof. Note that the topology on $X \cup \{\infty\}$ is induced by the uniformity with equivalence relation entourages $D(U) = \Delta \cup U^2$ for each open neighborhood U of ∞.

Let σ witness $\mathscr{D} \underset{k\text{-tact}}{\uparrow} Con_{O,P}(X \cap \{\infty\}, \infty)$ (resp. $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Con_{O,P}(X \cap \{\infty\}, \infty)$, $\mathscr{D} \uparrow Con_{O,P}(X \cap \{\infty\}, \infty)$). We define the k-tactical (resp. k-Marköv, perfect info) strategy τ such that

$$\tau \circ L(p) = D(\sigma \circ L(p))$$

where L is the k-tactical fog-of-war (resp. k-Marköv fog-of-war, identity).

Let $p \in (X \cup \{\infty\})^\omega$ attack τ such that $\bigcap_{n<\omega} \tau(p \upharpoonright n)[p(n)] \neq \emptyset$.

If $\infty \in \bigcap_{n<\omega} \tau(p \upharpoonright n)[p(n)]$, it follows that p is an attack on σ. Since σ is a winning strategy, it follows that q and its subsequence p must coverge to ∞.

Otherwise, $\infty \notin \tau(p \upharpoonright N)[p(N)]$ for some $N < \omega$, and then $\tau(p \upharpoonright N)[p(N)] = \{p(N)\}$ implies $p \to p(N)$.

Thus $\tau \circ L$ is a winning strategy. □

Corollary 18. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete. Then*

- $\mathscr{O} \uparrow Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \uparrow Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \underset{k\text{-}tact}{\uparrow} Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \underset{k\text{-}tact}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \underset{k\text{-}mark}{\uparrow} Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \underset{k\text{-}mark}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$

Proposition 19. *For any $x \in X$ and $k \geq 1$,*

- $\mathscr{O} \underset{k\text{-}tact}{\uparrow} Con_{O,P}(X, x) \Leftrightarrow \mathscr{O} \underset{tact}{\uparrow} Con_{O,P}(X, x)$
- $\mathscr{O} \underset{k\text{-}mark}{\uparrow} Con_{O,P}(X, x) \Leftrightarrow \mathscr{O} \underset{mark}{\uparrow} Con_{O,P}(X, x)$

Proof. If σ witnesses $\mathscr{O} \underset{k\text{-tact}}{\uparrow} Con_{O,P}(X, x)$, let $\tau(\emptyset) = \sigma(\emptyset)$ and

$$\tau(\langle q \rangle) = \bigcap_{i<k} \sigma(\langle \underbrace{x, \dots, x}_{k-i-1}, q, \underbrace{x, \dots, x}_{i} \rangle)$$

This is easily verified to be a winning strategy. The proof for $\mathscr{O} \underset{k\text{-mark}}{\uparrow} Con_{O,P}(X, x)$ is analogous. □

Corollary 20. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete, and $k \geq 1$. Then*

- $\mathscr{D} \underset{k\text{-}tact}{\uparrow} Prox_{D,P}(X \cup \{\infty\}) \Leftrightarrow O \underset{tact}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{D} \underset{k\text{-}mark}{\uparrow} Prox_{D,P}(X \cup \{\infty\}) \Leftrightarrow O \underset{mark}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$

Proposition 21. *For any uniform space X,*

- $\mathscr{O} \underset{k\text{-}tact}{\uparrow} Prox_{D,P}(X) \Leftrightarrow \mathscr{O} \underset{2\text{-}tact}{\uparrow} Prox_{D,P}(X)$
- $\mathscr{O} \underset{k\text{-}mark}{\uparrow} Prox_{D,P}(X) \Leftrightarrow \mathscr{O} \underset{2\text{-}mark}{\uparrow} Prox_{D,P}(X)$

Proof. If σ witnesses $\mathscr{O} \underset{k\text{-tact}}{\uparrow} Con_{O,P}(X, x)$, let $\tau(\emptyset) = \sigma(\emptyset)$ and

$$\tau(\langle q \rangle) = \bigcap_{i<k} \sigma(\langle \underbrace{q, \dots, q}_{i} \rangle)$$

$$\tau(\langle q, q' \rangle) = \bigcap_{i<k} \sigma(\langle \underbrace{q, \dots, q}_{k-i}, \underbrace{q', \dots, q'}_{i} \rangle)$$

This is easily verified to be a winning strategy. The proof for $\mathscr{O} \underset{k\text{-mark}}{\uparrow} Con_{O,P}(X, x)$ is analogous. $\square$

Definition 22. The absolute proximal game $aProx_{D,P}(X)$ is analogous to $Prox_{D,P}(X)$, except $\mathscr{D}$ may only win if p converges.

Definition 23. A **uniformly locally compact** space is a uniformizable space with a **uniformly compact entourage** M where $\overline{M[x]}$ is compact for all x.

Theorem 24. *For any uniformly locally compact space X, $\mathscr{D} \uparrow Prox_{D,P}(X) \Leftrightarrow \mathscr{D} \uparrow aProx_{D,P}(X)$*

Proof. Let M be a uniformly locally compact entourage. Let σ witness $\mathscr{D} \uparrow Prox_{D,P}(X)$ such that $\sigma(a) \subseteq M$ always (so $\overline{\sigma(a)[x]} \subseteq \overline{M[x]}$ is compact), and $a \supseteq b$ implies $\sigma(a) \subseteq \frac{1}{4}\sigma(b)$.

Let $\tau(p \upharpoonright n) = \frac{1}{2}\sigma(p \upharpoonright n)$. If p attacks τ in $aProx_{D,P}(X)$, then

$$p(n+1) \in \tau(p \upharpoonright n)[p(n)] = \frac{1}{2}\sigma(p \upharpoonright n)[p(n)]$$

and for

$$x \in \overline{\sigma(p \upharpoonright (n+1))[p(n+1)]} \subseteq \overline{\frac{1}{4}\sigma(p \upharpoonright n)[p(n+1)]} \subseteq \frac{1}{2}\sigma(p \upharpoonright n)[p(n+1)]$$

we can conclude $x \in \sigma(p \upharpoonright n)[p(n)]$. Thus

$$\sigma(p \upharpoonright (n+1))[p(n+1)] \subseteq \overline{\sigma(p \upharpoonright (n+1))[p(n+1)]} \subseteq \sigma(p \upharpoonright n)[p(n)]$$

Finally, note that p attacks the winning strategy σ in $Prox_{D,P}(X)$, but since the intersection of a chain of nonempty compact sets is nonempty:

$$\bigcap_{n<\omega} \sigma(p \upharpoonright n)[p(n)] = \bigcap_{n<\omega} \overline{\sigma(p \upharpoonright n)[p(n)]} \neq \emptyset$$

We conclude that p converges. $\square$

Corollary 25. *A uniformaly locally compact space X is proximal if and only if $\mathscr{D} \uparrow aProx_{D,P}(X)$.*

Theorem 26. *For any uniformly locally compact proximal space X, $\mathscr{O} \uparrow Clus_{O,P}(X,H)$ for all compact $H \subseteq X$.*

Proof. Let σ witness $\mathscr{D} \uparrow aProx_{D,P}(X)$ such that $p \supseteq q$ implies $\sigma(p) \subseteq \frac{1}{4}\sigma(q)$.

Let $o(t)$ be the subsequence of t consisting of its odd-indexed terms.

We define $T(\emptyset)$, etc. as follows:

- Let $\emptyset \in T(\emptyset)$.
- Choose $m_\emptyset < \omega$, $h_{\emptyset,i} \in H$ for $i < m_\emptyset$, and $h_{\emptyset,i,j} \in H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$ for $i, j < m_\emptyset$ such that
$$\{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}] : i < m_\emptyset\}$$
is a cover for H and such that for each $i < m_\emptyset$
$$\{\frac{1}{4}\sigma(\langle h_{\emptyset,i}\rangle)[h_{\emptyset,i,j}] : j < m_\emptyset\}$$
is a cover for $H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$.
- Let $\langle i\rangle \in T(\emptyset)$, $\langle i, h_{\emptyset,i}\rangle \in T(\emptyset)$, and $\langle i, h_{\emptyset,i}, j\rangle \in T(\emptyset)$ for $i, j < m_\emptyset$.

Suppose $T(a)$, etc. are defined. We then define $T(a^\frown\langle x\rangle)$, etc. for
$$x \in \bigcup_{s^\frown\langle i,h_{s,i},j\rangle \in \max(T(a))} \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$
as follows:

- Let $T(a) \subseteq T(a^\frown\langle x\rangle)$.
- Choose $t = s^\frown\langle i, h_{s,i}, j, x\rangle$ such that $s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(a))$ and $x \in \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$.
- Note that, assuming $o(s)^\frown\langle h_{s,i}\rangle$ is a legal partial attack against σ, then
$$x \in \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}] \subseteq \frac{1}{4}\sigma(o(s))[h_{s,i,j}]$$
and
$$h_{s,i,j} \in \overline{\frac{1}{4}\sigma(o(s))[h_{s,i}]} \subseteq \frac{1}{2}\sigma(o(s))[h_{s,i}]$$
implies
$$x \in \sigma(o(s))[h_{s,i}]$$
and thus $o(s)^\frown\langle h_{s,i}, x\rangle = o(t)$ is a legal partial attack against σ.
- Choose $m_t < \omega$, $h_{t,k} \in H \cap \overline{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]}$ for $k < m_t$, and $h_{t,k,l} \in H \cap \overline{\frac{1}{4}\sigma(t)[h_{t,k}]}$ for $k, l < m_t$ such that
$$\{\frac{1}{4}\sigma(o(t))[h_{t,k}] : k < m_t\}$$
is a cover for $H \cap \overline{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]}$ and such that for each $k < m_t$
$$\{\frac{1}{4}\sigma(o(t)^\frown\langle h_{t,k}\rangle)[h_{t,i,j}] : l < m_t\}$$
is a cover for $H \cap \overline{\frac{1}{4}\sigma(o(t))[h_{t,k}]}$.

- Note that, assuming $o(t)$ is a legal partial attack against σ, then
$$h_{t,k} \in \overline{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]} \subseteq \frac{1}{2}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$
and
$$x \in \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$
implies
$$h_{t,k} \in \sigma(o(s)^\frown\langle h_{s,i}\rangle)[x]$$
and thus $o(t)^\frown\langle h_{t,k}\rangle$ is a legal partial attack against σ.

- Let $t \in T(a^\frown\langle x\rangle)$, $t^\frown\langle k\rangle \in T(a^\frown\langle x\rangle)$, $t^\frown\langle k, h_{t,k}\rangle \in T(a^\frown\langle x\rangle)$, and $t^\frown\langle k, h_{t,k}, l\rangle \in T(a^\frown\langle x\rangle)$ for $k, l < m_t$.

- Note that assuming
$$\{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}] : s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(a))\}$$
covers H, then since
$$\{\frac{1}{4}\sigma(o(t)^\frown\langle h_{t,k}\rangle)[h_{t,k,l}] : s^\frown\langle i, h_{s,i}, j, x, k, h_{t,k}, l\rangle \in \max(T(a^\frown\langle x\rangle)) \setminus \max(T(a))\}$$
covers $H \cap \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$, we have that
$$\{\frac{1}{4}\sigma(o(t)^\frown\langle h_{t,k}\rangle)[h_{t,k,l}] : t^\frown\langle k, h_{t,k}, l\rangle \in \max(T(a^\frown\langle x\rangle))\}$$
covers H.

With this we may define the perfect information strategy τ for $\mathscr{O}$ in $Con_{O,P}(X, H)$ such that:
$$\tau(p \upharpoonright n) = \bigcup_{s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(p\upharpoonright n))} \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

If p attacks τ, then it follows that $T(p \upharpoonright n)$ is defined for all $n < \omega$, so let $T(p) = \bigcup_{n<\omega} T(p \upharpoonright n)$. We note $T(p)$ is an infinite tree with finite levels:

- $\emptyset$ has exactly $m_\emptyset$ successors $\langle i\rangle$.
- $s^\frown\langle i\rangle$ has exactly one successor $t^\frown\langle i, h_{s,i}\rangle$
- $s^\frown\langle i, h_{s,i}\rangle$ has exactly m_s successors $t^\frown\langle i, h_{s,i}, j\rangle$
- $s^\frown\langle i, h_{s,i}, j\rangle$ has either no successors or exactly one successor $t^\frown\langle i, h_{s,i}, j, x\rangle$

- $t = s^\frown\langle i, h_{s,i}, j, x\rangle$ has exactly m_t successors $t^\frown\langle k\rangle$

Let $q' = \langle i_0, h_0, j_0, x_0, i_1, h_1, j_1, x_1, \dots\rangle$ correspond to this infinite branch in $T(p)$, and let $q = o(q') = \langle h_0, x_0, h_1, x_1, \dots\rangle$. Note that by the construction of $T(p)$, q is an attack on the winning strategy σ in $aProx_{D,P}(X)$, so it must converge. Since every other term of q is in H, it must converge to H. Then since q is a subsequence of p, p must cluster at H. $\square$

Corollary 27. *For any uniformly locally compact proximal space, $\mathscr{O} \uparrow Con_{O,P}(X, H)$ for all compact $H \subseteq X$.*

Proof. $\mathscr{O} \uparrow Con_{O,P}(X, H)$ if and only if $\mathscr{O} \uparrow Clus_{O,P}(X, H)$. $\square$

Corollary 28. *A compact uniform space X is Corson compact if and only if it is proximal.*

Proof. A characterization of Corson compact is having a W-set diagonal. If X is proximal compact, then X^2 is proximal compact, and its compact diagonal is a W-set. $\square$

Theorem 29. *$\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X, H)$ if and only if there exists a countable base around H.*

Proof. Let $\{U_n : n < \omega\}$ be a countable base around H. We define the predetermined strategy $\sigma(n) = \bigcap_{m \leq n} U_m$. Let p attack $\sigma(n)$ - then if U is any neighborhood of H, we may choose $H \subseteq U_m \subseteq U$, and note that $\sigma(n) \subseteq U_m$ for $n \geq m$, and thus $p(n) \in U_m \subseteq U$ for all $n \geq m$. Thus σ is a winning strategy.

For the other direction, suppose there does not exist a countable base around H, and let $\sigma(n)$ be an arbitrary predetermined strategy. Since $\{\bigcap_{m \leq n} \sigma(m) : n < \omega\}$ is not a countable base around H, we may choose an open set U around H such that $\bigcap_{m \leq n} \sigma(m) \not\subseteq U$ for all $n < \omega$. We may easily verify that if $p(n) \in \bigcap_{m \leq n} \sigma(m) \setminus U$ for all $n < \omega$, then p is a successful counterattack to σ. □

Corollary 30. *X is first countable if and only if $\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X, x)$ for all $x \in X$*

Corollary 31. *$\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$ implies X is first countable.*

Definition 32. Scattered Eberlein compact spaces are known as **strong Eberlein compact** spaces.

Theorem 33 (folklore)**.** *Scattered compact first-countable spaces are metrizable.*

Corollary 34. *If X is scattered compact and $\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X, x)$ for all $x \in X$ (or $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$), then X is metrizable.*

Example 35. $\mathscr{D} \underset{\text{pre}}{\not\uparrow} Prox_{D,P}(\omega_1^*)$

Proof. There does not exist a countable base around ∞, so $\mathscr{O} \underset{\text{pre}}{\not\uparrow} Con_{O,P}(X, \omega_1)$. □

Example 36. $\mathscr{O} \underset{\text{tact}}{\uparrow} Con_{O,P}(\kappa^*, \infty)$ and $\mathscr{D} \underset{\text{tact}}{\uparrow} Prox_{D,P}(\kappa^*)$ for all cardinals κ

Proof. For $Con_{O,P}(\kappa^*, \infty)$, let $\sigma() = \sigma(\infty) = \kappa^*$ and $\sigma(x) = \kappa^* \setminus \{x\}$ otherwise. □

Theorem 37. *If H is a closed subset of X, then $\mathscr{D} \underset{limit}{\uparrow} Prox_{D,P}(X) \Rightarrow \mathscr{D} \underset{limit}{\uparrow} Prox_{D,P}(H)$ where $\underset{limit}{\uparrow}$ is any of $\uparrow$, $\underset{k\text{-}tact}{\uparrow}$, or $\underset{k\text{-}mark}{\uparrow}$.*

Proof. Let $\sigma \circ L$ witness $\mathscr{D} \underset{\text{limit}}{\uparrow} Prox_{D,P}(X)$. We define $\tau \circ L$ for $\mathscr{D}$ in $Prox_{D,P}(H)$ as follows:

$$\tau \circ L(p \restriction n) = \sigma \circ L(p \restriction n) \cap H^2$$

Let p attack $\tau \circ L$. p also attacks the winning strategy $\sigma \circ L$, so either

$$\bigcap_{n<\omega} \left(\bigcap_{m \leq n} \tau \circ L(p \restriction n) \right) [p(n)] \subseteq \bigcap_{n<\omega} \left(\bigcap_{m \leq n} \sigma \circ L(p \restriction n) \right) [p(n)] = \emptyset$$

or p converges in X, and thus converges in H. $\square$

Theorem 38. *If $\mathscr{D} \underset{limit}{\uparrow} Prox_{D,P}(X_i)$ for $i < \omega$, then $\mathscr{D} \underset{limit}{\uparrow} Prox_{D,P}(\prod_{i<\omega} X_i)$, where $\underset{limit}{\uparrow}$ is either $\uparrow$ or $\underset{k\text{-}mark}{\uparrow}$.*

Proof. A subbase for $\prod_{i<\omega} X_i$ is

$$\{\pi_i^{-1}(D) : i < \omega, D \in \mathcal{D}_i\}$$

where π_i is the natural projection from $\left(\prod_{i<\omega} X_i\right)^2$ onto X_i^2. (See Bell.)

For $p \in \left(\prod_{i<\omega} X_i\right)^\omega$, let $p_i \in X_i^\omega$ such that $p_i(n) = p(n)(i)$.

Let $\sigma_i \circ L$ witness $\mathscr{D} \underset{\text{limit}}{\uparrow} Prox_{D,P}(X_i)$ for $i < \omega$, and assume without loss of generality that $\sigma_i \circ L$ always yields X_i^2 before round i.

Then we define the strategy $\tau \circ L$ for $\mathscr{D}$ in $Prox_{D,P}(\prod_{i<\omega} X_i)$ as follows:

$$\tau \circ L(p \restriction n) = \bigcap_{i \leq n} \pi_i^{-1}(\sigma_i \circ L(p_i \restriction n))$$

Let p attack $\tau \circ L$. If $\bigcap_{n<\omega} \left(\bigcap_{m \leq n} \sigma_i(p_i \restriction n) \right) [p_i(n)] = \emptyset$ for any $i < \omega$, it easily follows that $\bigcap_{n<\omega} \left(\bigcap_{m \leq n} \tau(p \restriction n) \right) [p(n)] = \emptyset$.

Otherwise, we assume that for each $i < \omega$, p_i converges to some $x_i \in X_i$. Thus p converges to $x = \langle x_0, x_1, \ldots \rangle$. $\square$

Note: I expect I should be able to do some clever things assuming $S(\kappa, \omega, \omega)$ to get a similar result for sigma products of dimension κ.

Example 39. $\mathscr{D} \underset{\text{mark}}{\uparrow} Prox_{D,P}((\kappa^*)^\omega)$

Proof. $\mathscr{D} \underset{\text{tact}}{\uparrow} Prox_{D,P}(\kappa^*)$ + previous result □

Theorem 40. *For any predetermined absolutely proximal space X, $\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X,H)$ for all compact $H \subseteq X$.*

Proof. Let $\sigma(n)$ be a winning predetermined strategy for $\mathscr{D}$ in the absolutely proximal game such that $\sigma(n+1) \subseteq \frac{1}{4}\sigma(n)$. For a given tree T, let $\max(T)$ denote its maximal nodes.

First we define $T(0) \subseteq \omega^{\leq 2}$.

- Let $\emptyset \in T(0)$.
- Choose $m_\emptyset < \omega$, $h_{\langle i \rangle} \in H$ for $i < m_\emptyset$, and $h_{\langle i,j \rangle} \in H \cap \overline{\frac{1}{4}\sigma(0)[h_{\langle i \rangle}]}$ for $i, j < m_\emptyset$ such that
$$\left\{\frac{1}{4}\sigma(0)[h_{\langle i \rangle}] : i < m_\emptyset\right\}$$
is a cover for H and such that for each $i < m_\emptyset$
$$\left\{\frac{1}{4}\sigma(1)[h_{\langle i,j \rangle}] : j < m_\emptyset\right\}$$
is a cover for $H \cap \overline{\frac{1}{4}\sigma(0)[h_{\langle i \rangle}]}$.
- Let $\langle i \rangle$ and $\langle i, j \rangle$ be in $T(0)$ for $i, j < m_\emptyset$.

Now suppose $T(n) \subseteq \omega^{\leq 2n+2}$ is defined. We then define $T(n+1) \subseteq \omega^{\leq 2n+4}$ as follows:

- Let $T(n) \subseteq T(n+1)$.
- For each $t \in \max(T(n))$, choose $m_t < \omega$, $h_{t^\frown\langle i \rangle} \in H \cap \overline{\frac{1}{4}\sigma(2n+2)[h_t]}$ for $i < m_t$, and $h_{t^\frown\langle i,j \rangle} \in H \cap \overline{\frac{1}{4}\sigma(2n+3)[h_{t^\frown\langle i \rangle}]}$ for $i, j < m_t$ such that
$$\left\{\frac{1}{4}\sigma(2n+2)[h_{t^\frown\langle i \rangle}] : i < m_t\right\}$$
is a cover for $H \cap \overline{\frac{1}{4}\sigma(2n+1)[h_t]}$ and such that for each $i < m_t$
$$\left\{\frac{1}{4}\sigma(2n+3)[h_{t^\frown\langle i,j \rangle}] : i < m_t\right\}$$
is a cover for $H \cap \overline{\frac{1}{4}\sigma(2n+2)[h_{t^\frown\langle i \rangle}]}$.
- For each $t \in \max(T(n))$ and each $i, j < m_t$, put $t^\frown\langle i \rangle$ and $t^\frown\langle i, j \rangle$ in $T(n+1)$.

We now define the predetermined strategy τ for $\mathscr{O}$ in $Clus_{O,P}(X,H)$ such that:

$$\tau(n) = \bigcup_{t\in\max(T(n))} \frac{1}{4}\sigma(2n+1)[h_t]$$

In order for τ to be a legal strategy, we must show that $\tau(n)$ contains H. For $n=0$,

$$\tau(0) = \bigcup_{i,j<m_\emptyset} \frac{1}{4}\sigma(1)[h_{\langle i,j\rangle}]$$

Since $\left\{\frac{1}{4}\sigma(1)[h_{\langle i,j\rangle}] : j < m_\emptyset\right\}$ is a cover for $H\cap\overline{\frac{1}{4}\sigma(0)[h_{\langle i\rangle}]}$, and since $\left\{\frac{1}{4}\sigma(0)[h_{\langle i\rangle}] : i < m_\emptyset\right\}$ is a cover for H, $\tau(0)$ must contain H. A similar argument follows for $\tau(n)$.

Let p be an attack against τ such that $p(n) \in \bigcap_{m\leq n}\tau(m)$. If we can construct an attack q against σ which shares a subsequence with p, then p must cluster. To find such a q, we construct a new tree T'.

We begin by setting

$$T'(0) = \{s : s \leq \langle i, h_{\langle i\rangle}, j\rangle \text{ for } i,j<m_\emptyset\}$$

Since

$$p(0) \in \frac{1}{4}\sigma(1)[h_{\langle i,j\rangle}]$$

for some $i,j<m_\emptyset$, we may find $t\in\max(T'(0))$ such that $p(0)\in\frac{1}{4}\sigma(|o(t)|)[h_{e(t)}]$.

Assume that $T'(n)$ is defined such that there is some $t\in\max(T'(n))$ where $p(n)\in\frac{1}{4}\sigma(|o(t)|)[h_{e(t)}]$. Then let

$$T'(n+1) = T'(n)\cup\{s : s\leq t^\frown\langle p(n), i, h_{e(t)^\frown\langle i\rangle}, j\rangle \text{ for } i,j<m_{e(t)}\}$$

Note that

$$p(n+1) \in \bigcap_{m\leq n+1}\tau(m) = \bigcap_{m\leq n+1}\left(\bigcup_{t\in\max(T(m))}\frac{1}{4}\sigma(2m+1)[h_t]\right)$$

(Need details on why there is some t in $\max(T'(n+1))$ such that $p(n+1)\in\frac{1}{4}\sigma(|o(t)|)[h_{e(t)}]$.)

(Follow this up with choosing an infinite branch $q'\in T'$, and showing that $q=o(q')$ is an attack on σ similar to perfect info result.)

$\square$

Example 41. Let $X = I \times 2$ be the Alexandrov double interval. Then $\mathscr{D} \underset{\text{pre}}{\not\uparrow} Prox_{D,P}(X)$, but $\mathscr{D} \underset{\text{mark}}{\uparrow} Prox_{D,P}(X)$.

Proof. We assume that the uniformity on X is given by entourages

$$D(\epsilon, F) = \{\langle x, 0\rangle, \langle y, 0\rangle : |x - y| < \epsilon\} \cup \{\langle x, 1\rangle, \langle y, 0\rangle : |x - y| < \epsilon \vee x \notin F\}$$

$$\cup \{\langle x, 0\rangle, \langle y, 1\rangle : |x - y| < \epsilon \vee y \notin F\} \cup \{\langle x, 1\rangle, \langle y, 1\rangle : x = y\}$$

That is, points are $D(\epsilon, F)$-close if they are the same point, or the first coordinates are within ϵ of each other while neither second coordinate is in F.

Suppose $\mathscr{D}$ had a predetermined winning strategy $\sigma(n) = D(\epsilon_n, F_n)$. Then $\mathscr{P}$ can choose $x \notin \bigcup_{n<\omega} F_n$, and play $\langle x, 1\rangle$ during even rounds, and $\langle x_{2n+1}, 0\rangle$ where $|x - x_{2n+1}| < \epsilon_{2n}$ during odd rounds, preventing convergence.

However, assume $\mathscr{D}$ uses the Markōv strategy $\sigma(x, n) = D(2^{-n}, \{x\})$. If $\mathscr{P}$ repeats a point of the form $\langle x, 1\rangle$, then since $D(2^{-n}, \{x\})[\langle x, 1\rangle] = \{\langle x, 1\rangle\}$, $\mathscr{P}$ must repeat $\langle x, 1\rangle$ for the rest of the game, and $\mathscr{D}$ wins. Otherwise, $\mathscr{P}$ cannot repeat points played in $I \times \{1\}$, and as the first coordinates form a Cauchy sequence and converge to some z, any open set about $\langle z, 0\rangle$ contains all but finitely many points of $\mathscr{P}$'s sequence, and $\mathscr{D}$ wins. □

Theorem 42. *For any uniformly locally compact space X, $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X) \Leftrightarrow \mathscr{D} \underset{pre}{\uparrow} aProx_{D,P}(X)$*

Proof. Let M be a uniformly locally compact entourage. Let σ witness $\mathscr{D} \underset{\text{pre}}{\uparrow} Prox_{D,P}(X)$ such that $\sigma(n) \subseteq M$ always (so $\overline{\sigma(a)[x]} \subseteq \overline{M[x]}$ is compact), $\sigma(n+1) \subseteq \frac{1}{4}\sigma(n)$.

Let $\tau(n) = \frac{1}{2}\sigma(n)$. If p attacks τ in $aProx_{D,P}(X)$, then

$$p(n+1) \in \tau(n)[p(n)] = \frac{1}{2}\sigma(n)[p(n)]$$

and for

$$x \in \overline{\sigma(n+1)[p(n+1)]} \subseteq \overline{\frac{1}{4}\sigma(n)[p(n+1)]} \subseteq \frac{1}{2}\sigma(n)[p(n+1)]$$

we can conclude $x \in \sigma(n)[p(n)]$. Thus

$$\sigma(n+1)[p(n+1)] \subseteq \overline{\sigma(n+1)[p(n+1)]} \subseteq \sigma(n)[p(n)]$$

Finally, note that p attacks the winning strategy σ in $Prox_{D,P}(X)$, but since the intersection of a chain of nonempty compact sets is nonempty:

$$\bigcap_{n<\omega} \sigma(n)[p(n)] = \bigcap_{n<\omega} \overline{\sigma(n)[p(n)]} \neq \emptyset$$

We conclude that p converges. $\square$

Proposition 43. *If $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$, then X has a G_δ diagonal.*

Proof. If $\mathscr{D} \underset{\text{pre}}{\uparrow} Prox_{D,P}(X)$ with strategy σ, then consider $\langle x, y\rangle \in \bigcap_{n<\omega} \sigma(n)$. It follows that $\langle x, y, x, y, \dots\rangle$ attacks σ, and $\{x, y\} \subseteq \bigcap_{n<\omega} \sigma(n)[x] \cap \bigcap_{n<\omega} \sigma(n)[y] \neq 0$ so it must converge, and $x = y$. Thus $\bigcap_{n<\omega} \sigma(n) = \Delta$ is G_δ. $\square$

Example 44. The Sorgenfrey line S has a G_δ diagonal but $\mathscr{P} \uparrow Prox_{D,P}(S)$.

Corollary 45. *For X with uniformity $\mathbb{D}$ inducing the compact Hausdorff topology τ, the following are equivalent:*

(a) $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$

(b) $\mathscr{D} \underset{pre}{\uparrow} aProx_{D,P}(X)$

(c) X has a G_δ diagonal

(d) $\mathbb{D}$ is metrizable

(e) τ is metrizable

Proof. For compact Hausdorff spaces, it is well known that there is exactly one uniformity inducing the topology. Thus $(d) \Leftrightarrow (e)$. Since X is uniformly locally compact, $(a) \Leftrightarrow (b)$. Also, compact spaces with a G_δ diagonal are metrizable, so $(c) \Rightarrow (e)$. Bell noted $(d) \Rightarrow (a)$ for arbitrary uniform spaces, and the previous proposition shows $(a) \Rightarrow (c)$. $\square$

Theorem 46. *A uniformly locally compact space with a G_δ diagonal is metrizable.*

Proof. Based on several folklore results.

Uniformly locally compact implies the topological sum of σ-compact spaces implies paracompact. Locally compact plus G_δ diagonal implies locally metrizable. Locally metrizable plus paracompact characterizes metrizable. $\square$

Corollary 47. *If X is uniformly locally compact, then $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$ implies X's topology is metrizable.*

Example 48. Let R be the Michael Line. Then $\mathscr{P} \uparrow Prox_{D,P}(X)$.

Proof. During round 0, $\mathscr{P}$ may choose $m(0) = 0$ and $p(0) = 1$, and during round $n+1$, $\mathscr{P}$ may choose $m(n+1) > m(n)$ and $p(n+1) = p(n) + \frac{1}{10^{m(n+1)}}$ such that p is a legal attack.

It follows that p "converges" to $x = \sum_{n<\omega} \frac{1}{10^{m(n)}}$, except x is an irrational number composed of 1s separated by strings of 0s of strictly increasing size. $\square$

Example 49. Let κ be an uncountable regular cardinal with a ladder topology:

- All successor ordinals are isolated.
- Strictly increasing sequences (ladders) $L_\alpha : \omega \to \alpha$ are defined for each limit ordinal α such that L_α converges to α in the order topology, and each limit α is given neighborhoods of the form $\{\alpha\} \cup \{L_\alpha(n) : n \geq m\}$. We assume that all successor ordinals are a part of some ladder.

Then $\mathscr{P} \uparrow Prox_{D,P}(\kappa^*)$ where κ^* is its one-point compactification.

Proof. Entrouages of κ^* are then of the form $D(F, n)$, where $F \in [\kappa^L]^{<\omega}$ and $n < \omega$. $D(F, n)$ partitions κ^* such that ∞'s part is the complement of the ladders leading to points in F. Each of those ladders is then partitioned by isolating the first n rungs of the ladder, and leaving the top of the ladder leading to a point in F as a whole part. (It's possible that the tops of some ladders might overlap, so they must be considered the same part, but this could be prevented by $\mathscr{D}$ by increasing n a sufficient amount to separate all the finite limits in F if desired.)

$\mathscr{P}$'s strategy involves first choosing two disjoint stationary subsets S_0, T_0 of κ^L. During round 0, $\mathscr{D}$'s move partitions ladders leading to limit ordinals in $F_0 \in [\kappa^L]^{<\omega}$. Let $S'_0 = S_0 \setminus F_0$ and $T'_0 = T_0 \setminus F_0$, and observe that both are still stationary sets as only finitely many ordinals were removed.

For $\mathscr{P}$'s initial move, she may apply the pressing down lemma to the sets S'_0, T'_0 and the function $f_i(\alpha) = L_\alpha(i)$ for $i < \omega$ sufficiently large to identify stationary subsets S_1, T_1 of S'_0, T'_0 such that $f_i(\alpha) = s_0$ for $\alpha \in S_1$, $f_i(\alpha) = t_0$ for $\alpha \in T_1$, and s_0, t_0 are not in the range of L_α for $\alpha \in F_0$.

$\mathscr{P}$ chooses s_0 as her initial move.

During round $n+1$, we assume that the disjoint stationary sets S_{n+1}, T_{n+1} were defined in the previous round. $\mathscr{D}$'s move in this round again partitions ladders leading to limit ordinals in $F_{n+1} \in [\kappa^L]^{<\omega}$. Let $S'_{n+1} = S_{n+1} \setminus F_{n+1}$ and $T'_{n+1} = T_{n+1} \setminus F_{n+1}$.

$\mathscr{P}$ then applies the pressing down lemma to the sets S'_{n+1}, T'_{n+1} and the function $f_i(\alpha) = L_\alpha(i)$ for $n < i < \omega$ sufficiently large to identify stationary subsets S_{n+2}, T_{n+2} of S'_{n+1}, T'_{n+1}

such that $f_i(\alpha) = s_{n+1}$ for $\alpha \in S_{n+2}$, $f_i(\alpha) = t_{n+1}$ for $\alpha \in T_{n+2}$, and s_{n+1}, t_{n+1} are not in the range of L_α for $\alpha \in F_{n+1}$.

If $n+1$ is even, $\mathscr{P}$ chooses s_{n+1} as her move; otherwise, she chooses t_{n+1}.

All choices of s_n, t_n by $\mathscr{P}$ were within the partition containing ∞, and no choice was repeated infinitely often, so s_n and t_n must converge. (Need to disprove that either could converge to ∞, or could they? That would happen if $\bigcap_{n<\omega} S_n = \emptyset$ or $\bigcap_{n<\omega} T_n = \emptyset$.)

$\square$