# RELATING GAMES OF MENGER, COUNTABLE FAN TIGHTNESS, AND SELECTIVE SEPARABILITY

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Abstract. By adapting techniques of Arhangelskii, Barman, and Dow, we may equate the existence of perfect-information, Markov, and tactical strategies between two interesting selection games. These results shed some light on Gruenhage's question asking whether all strategically selectively separable spaces are Markov selectively separable.

## 1. Introduction

**Definition 1.** The *selection principle* $S_{fin}(\mathcal{A}, \mathcal{B})$ states that given $A_n \in \mathcal{A}$ for $n < \omega$, there exist $B_n \in [A_n]^{<\omega}$ such that $\bigcup_{n<\omega} B_n \in \mathcal{B}$.

**Definition 2.** The *selection game* $G_{fin}(\mathcal{A}, \mathcal{B})$ is the analogous game to $S_{fin}(\mathcal{A}, \mathcal{B})$, where during each round $n < \omega$, Player I first chooses $A_n \in \mathcal{A}$, and then Player II chooses $B_n \in [A_n]^{<\omega}$. Player II wins in the case that $\bigcup_{n<\omega} B_n \in \mathcal{B}$, and Player I wins otherwise.

This game and property were first formally investigated by Scheepers in "Combinatorics of open covers" [5], which inspired a series of ten sequels with several co-authors. We may quickly observe that if II has a winning strategy for the game $G_{fin}(\mathcal{A}, \mathcal{B})$, then $S_{fin}(\mathcal{A}, \mathcal{B})$ will hold, but the converse need not follow.

The power of this selection principle and game comes from their ability to characterize several properties and games from the literature. Of interest to us are the following.

**Definition 3.** Let $\mathcal{O}_X$ be the collection of open covers for a topological space $X$. Then $S_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger property* for $X$ ($M$ for short), and $G_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger game*.

**Definition 4.** An *$\omega$-cover* $\mathcal{U}$ for a topological space $X$ is an open cover such that for every $F \in [X]^{<\omega}$, there exists some $U \in \mathcal{U}$ such that $F \subseteq U$.

**Definition 5.** Let $\Omega_X$ be the collection of $\omega$-covers for a topological space $X$. Then $S_{fin}(\Omega_X, \Omega_X)$ is the *$\Omega$-Menger property* for $X$ ($\Omega M$ for short), and $G_{fin}(\Omega_X, \Omega_X)$ is the *$\Omega$-Menger game*.

In [4, Theorem 3.9] it was shown that $X$ is $\Omega$-Menger if and only if $X^n$ is Menger for all $n < \omega$.

**Definition 6.** Let $\mathcal{B}_{X,x}$ be the collection of subsets $A \subset X$ where $x \in \operatorname{cl} A$. (Call $A$ a *blade* of $x$.) Then $S_{fin}(\mathcal{B}_{X,x}, \mathcal{B}_{X,x})$ is the *countable fan tightness property* for $X$ at $x$ ($CFT_x$ for short), and $G_{fin}(\mathcal{B}_{X,x}, \mathcal{B}_{X,x})$ is the *countable fan tightness game* for $X$ at $x$.

**Definition 7.** A space $X$ has *countable fan tightness* ($CFT$ for short) if it has countable fan tightness at each point $x \in X$.

**Definition 8.** Let $\mathcal{D}_X$ be the collection of dense subsets of a topological space $X$. (So, $\mathcal{D}_X \subseteq \mathcal{B}_{X,x}$ for all $x \in X$.) Then $S_{fin}(\mathcal{D}_X, \mathcal{B}_{X,x})$ is the *countable dense fan tightness property* for $X$ at $x$ ($CDFT_x$ for short), and $G_{fin}(\mathcal{D}_X, \mathcal{B}_{X,x})$ is the *countable dense fan tightness game* for $X$ at $x$.

**Definition 9.** A space $X$ has *countable dense fan tightness* ($CDFT$ for short) if it has countable dense fan tightness at each point $x \in X$.

The notion of countable fan tightness was first studied by by Arhangel'skii in [1]. A result of that paper showed that for $T_{3\frac{1}{2}}$ spaces $X$, the countable fan tightness of the function space of pointwise convergence $C_p(X)$ is characterized by the $\Omega$-Menger property of $X$.

**Definition 10.** $S_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability property* for $X$ ($SS$ for short), and $G_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability game* for $X$.

Of course, one may easily observe that a selective separable space is separable. In [2] Barman and Dow demonstrated that all separable Frechet spaces are selectively separable. They were also able to produce a space which is selectively separable, but does not allow II a winning strategy in the selective separability game.

The object of this paper is to investigate the game-theoretic properties characterized by the presence of winning *limited information* strategies in these selection games.

**Definition 11.** A *strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function $\sigma$ satisfying $\sigma(\langle A_0, \dots, A_n \rangle) \in [A_n]^{<\omega}$ for $\langle A_0 \dots, A_n \rangle \in \mathcal{A}^{n+1}$. We say this strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(\langle A_0, \dots, A_n \rangle)$ during each round $n < \omega$. If a winning strategy exists, then we write $\text{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$.

**Definition 12.** A *Markov strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function $\sigma$ satisfying $\sigma(A, n) \in [A_n]^{<\omega}$ for $A \in \mathcal{A}$ and $n < \omega$. We say this Markov strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(A_n, n)$ during each round $n < \omega$. If a winning Markov strategy exists, then we write $\text{II} \underset{\text{mark}}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$.

**Notation 13.** If $S_{fin}(\mathcal{A}, \mathcal{B})$ characterizes the property $P$, then we say $\text{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes $P^+$ ("strategically $P$"), and $\text{II} \underset{\text{mark}}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes $P^{+mark}$ ("Markov $P$"). Of course, $P^{+mark} \Rightarrow P^+ \Rightarrow P$.

In this notation, Barman and Dow showed that $SS$ does not imply $SS^+$. We aim to make progress on the following question attributed to Gary Gruenhage:

**Question 14.** *Does $SS^+$ imply $SS^{+mark}$?*

The solution is known to be "yes" in the context of countable spaces [2]. However in general, winning strategies in selection games cannot be improved to be winning Markov strategies. In [3] the author showed that while $M^+$ implies $M^{+mark}$ for second-countable spaces, there exists a simple example of a regular non-second-countable space which is $M^+$ but not $M^{+mark}$.

## 2. $CFT$ and $SS$

We begin by generalizing the following result:

**Lemma 15** (Lem 2.7 of [2])**.** *The following are equivalent for any topological space $X$.*

- *$X$ is $SS$.*
- *$X$ is separable and $CDFT$.*
- *$X$ has a countable dense subset $D$ where $CDFT_x$ holds for all $x \in D$.*

**Lemma 16.** *The following are equivalent for any topological space $X$.*

- *$X$ is $SS$ (resp. $SS^+$, $SS^{+mark}$).*
- *$X$ is separable and $CDFT$ (resp. $CDFT^+$, $CDFT^{+mark}$).*
- *$X$ has a countable dense subset $D$ where $CDFT_x$ (resp. $CDFT_x^+$, $CDFT_x^{+mark}$) holds for all $x \in D$.*

*Proof.* We need only show that the final condition implies the first, and we may assume that $X$ lacks isolated points. Let $D = \{d_i : i < \omega\}$.

Let $\sigma_i$ be a winning strategy witnessing $CDFT_{d_i}^+$ for each $i < \omega$. We define the strategy $\tau$ for the $SS$ game by

$$\tau(\langle D_0, \dots, D_n \rangle) = \bigcup_{i \leq n} \sigma_i(\langle D_i, \dots, D_n \rangle).$$

By $CDFT_{d_i}^+$, we have

$$d_i \in \overline{\bigcup_{i \leq n < \omega} \sigma_i(\langle D_i, \dots, D_n \rangle)} \subseteq \overline{\bigcup_{i \leq n < \omega} \tau(\langle D_0, \dots, D_n \rangle)} \subseteq \overline{\bigcup_{n < \omega} \tau(\langle D_0, \dots, D_n \rangle)}$$

and so it follows that

$$X \subseteq \overline{D} \subseteq \overline{\overline{\bigcup_{n < \omega} \tau(\langle D_0, \dots, D_n \rangle)}} = \overline{\bigcup_{n < \omega} \tau(\langle D_0, \dots, D_n \rangle)}.$$

Therefore $\tau$ witnesses $SS^+$.

The above proof may be easily modified for the Markov case by replacing $\sigma_i(\langle D_i, \dots, D_n \rangle)$ with $\sigma_i(D_n, n)$ and $\tau(\langle D_0, \dots, D_n \rangle)$ with $\tau(D_n, n)$. □

## References


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