ALMOST COMPATIBLE FUNCTIONS AND INFINITE LENGTH GAMES

STEVEN CLONTZ AND ALAN DOW

Abstract. $\mathcal{A}'(\kappa)$ asserts the existence of pairwise almost compatible finite-to-one functions $A \to \omega$ for each countable subset A of κ. The existence of winning 2-Markov strategies in several infinite-length games, including the Menger game on the one-point Lindelöfication $\kappa^\dagger$ of κ, are guaranteed by $\mathcal{A}'(\kappa)$. $\mathcal{A}'(\kappa)$ is implied by the existence of cofinal Kurepa families of size κ, and thus holds for all cardinals less than $\aleph_\omega$. It's consistent that $\mathcal{A}'(\aleph_\omega)$ fails, but there must always be a winning 2-Markov strategy for the second player in the Menger game on $\omega_\omega^\dagger$.

1. Introduction

Definition 1. Two functions f, g are almost compatible, that is, $f \sim g$ when $\{a \in dom\, f \cap dom\, g : f(a) \neq g(a)\}$ is finite.

Marion Scheepers used almost compatible functions in [10] in order to study the existence of limited information strategies on a variation of the meager-nowhere dense game he introduced in [11].

Game 2. Let $Sch^{\cup,\subset}_{C,F}(\kappa)$ denote *Scheepers' strict countable-finite union game* with two players $\mathscr{C}$, $\mathscr{F}$. In round 0, $\mathscr{C}$ chooses $C_0 \in [\kappa]^{\leq\omega}$, followed by $\mathscr{F}$ choosing $F_0 \in [\kappa]^{<\omega}$. In round $n+1$, $\mathscr{C}$ chooses $C_{n+1} \in [\kappa]^{\leq\omega}$ such that $C_{n+1} \supset C_n$, followed by $\mathscr{F}$ choosing $F_{n+1} \in [\kappa]^{<\omega}$.

$\mathscr{F}$ wins the game if $\bigcup_{n<\omega} F_n \supseteq \bigcup_{n<\omega} C_n$; otherwise, $\mathscr{C}$ wins.

Of course, with perfect information this game is trivial: during round n player $\mathscr{F}$ simply chooses n ordinals from each of the n countable sets played by $\mathscr{C}$. However, if $\mathscr{F}$ is limited to using information from the last k moves by $\mathscr{C}$ during each round, the task becomes more difficult. Call such a strategy a *k-tactical strategy* or *k-tactic*; if using the round number is allowed, then the strategy is called a *k-Markov strategy* or a *k-mark*.

Definition 3. The statement $\mathcal{A}(\kappa)$ (given as $S(\kappa, \aleph_0, \omega)$ in [10] and $S(\kappa)$ in [1]) claims that there exist one-to-one functions $f_A : A \to \omega$ for each $A \in [\kappa]^{\leq\aleph_0}$ such that the collection $\{f_A : A \in [\kappa]^{\leq\aleph_0}\}$ is pairwise almost compatible.

In the same paper, Scheepers noted that $\mathcal{A}(\omega_1)$ holds in ZFC, and that it's possible to force $\mathfrak{c}$ to be arbitrarily large while preserving $\mathcal{A}(\mathfrak{c})$. However, $\mathcal{A}(\mathfrak{c}^+)$

[1]The second author acknowledges support provided by NSF-DMS 1501506

always fails. This axiom may be applied to obtain a winning 2-tactic for $\mathscr{F}$ in the countable-finite game.

In [1], Clontz related this game to a game which may be used to characterize the Menger covering property of a topological space.

Game 4. Let $Men_{C,F}(X)$ denote the *Menger game* with players $\mathscr{C}$, $\mathscr{F}$. In round n, $\mathscr{C}$ chooses an open cover $\mathcal{U}_n$, followed by $\mathscr{F}$ choosing a subset F_n of X which may be finitely covered by $\mathcal{U}_n$.

$\mathscr{F}$ wins the game if $X = \bigcup_{n<\omega} F_n$, and $\mathscr{C}$ wins otherwise.

This characterization is slightly different than the typical characterization in which the second player first chooses a specific finite subcollection $\mathcal{F}_n$ of the cover itself and lets $F_n = \bigcup \mathcal{F}_n$, denoted as $G_{fin}(\mathcal{O}, \mathcal{O})$ in [12]. However, it's easily seen that these games are equivalent for perfect information strategies (so both characterize the Menger property in the same way), and this characterization is more convenient for our concerns.

Definition 5. Let $\kappa^\dagger = \kappa \cup \{\infty\}$ where κ is discrete and ∞'s neighborhoods are the co-countable sets containing it.

The relationship between $Sch^{\cup,\subseteq}_{C,F}(\kappa)$ and $Men_{C,F}(\kappa^\dagger)$ is strong; in both games $\mathscr{C}$ essentially chooses a countable subset of κ followed by $\mathscr{F}$ choosing a finite subset of that choice, and it's easy to see the winning perfect information strategy for $\mathscr{F}$ in both games. In addition, it was shown in [1] that when $\mathcal{A}(\kappa)$ holds, $\mathscr{F}$ has a winning 2-Markov strategy in $Men_{C,F}(\kappa^\dagger)$.

One source of motivation is to make progress on the following open question:

Question 6. *Does there exist a topological space X for which $\mathscr{F} \uparrow Men_{C,F}(X)$ but $\mathscr{F} \underset{2\text{-}mark}{\not\uparrow} Men_{C,F}(X)$? (That is, the second player can win the Menger game on X with perfect information but not with 2-Markov information.)*

2. One-to-one and finite-to-one almost compatible functions

We may weaken Scheeper's $\mathcal{A}(\kappa)$ as follows:

Definition 7. The statement $\mathcal{A}'(\kappa)$ weakens $\mathcal{A}(\kappa)$ by only requiring the witnessing almost-compatible functions $f_A : A \to \omega$ to be finite-to-one.

Proposition 8. *$\mathcal{A}(\kappa)$ and $\mathcal{A}'(\kappa)$ need only be witnessed by functions $\{f_A : A \in \mathcal{S}\}$ for some family $\mathcal{S}$ cofinal in $[\kappa]^{\leq\aleph_0}$.*

Proof. For each $A \in [\kappa]^{\leq\aleph_0}$ choose $A' \supseteq A$ from $\mathcal{S}$ and let $g_A = f_{A'} \restriction A$. □

In the next section we will show that $\mathcal{A}'(\kappa)$ is sufficient for the applications to the Scheepers and Menger games. In the meantime, we will demonstrate that $\mathcal{A}'(\kappa)$ is strictly weaker than $\mathcal{A}(\kappa)$.

Recall the following.

Definition 9. A Kurepa family $\mathcal{K} \subseteq [\kappa]^{\aleph_0}$ on κ satisfies that $\mathcal{K} \restriction A = \{K \cap A : K \in \mathcal{K}\}$ is countable for each $A \in [\kappa]^{\aleph_0}$. Let $\mathcal{K}(\kappa)$ be the statement claiming there exists a Kurepa family on κ cofinal in $[\kappa]^{\aleph_0}$.

Theorem 10. $\mathcal{K}(\kappa) \Rightarrow \mathcal{A}'(\kappa)$.

Proof. Let $\mathcal{K} = \{K_\alpha : \alpha < \theta\}$ be a cofinal Kurepa family on κ. We first define $f_\alpha : K_\alpha \to \omega$ for each $\alpha < \theta$.

Suppose we've already defined pairwise almost compatible finite-to-one functions $\{f_\beta : \beta < \alpha\}$. To define f_α, we first recall that $\mathcal{K} \restriction K_\alpha$ is countable, so we may choose $\beta_n < \alpha$ for $n < \omega$ such that $\{K_\beta : \beta < \alpha\} \restriction K_\alpha \setminus \{\emptyset\} = \{K_\alpha \cap K_{\beta_n} : n < \omega\}$. Let $K_\alpha = \{\delta_{i,j} : i \leq \omega, j < w_i\}$ where $w_i \leq \omega$ for each $i \leq \omega$, $K_\alpha \cap \left(K_{\beta_n} \setminus \bigcup_{m<n} K_{\beta_m}\right) = \{\delta_{n,j} : j < w_n\}$, and $K_\alpha \setminus \bigcup_{n<\omega} K_{\beta_n} = \{\delta_{\omega,j} : j < w_\omega\}$. Then let $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ for $n < \omega$ and $f_\alpha(\delta_{\omega,j}) = j$ otherwise.

We should show that f_α is finite-to-one. Let $n < \omega$. Since $f_\alpha(\delta_{m,j}) \geq m$, we only consider the finite cases where $m \leq n$. Since each f_{β_m} is finite-to-one, $f_{\beta_m}(\delta_{m,j}) \leq n$ for only finitely many j. Thus $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$ maps to n for only finitely many j.

We now want to demonstrate that $f_\alpha \sim f_{\beta_n}$ for all $n < \omega$. Note $\delta_{m,j} \in K_{\beta_n}$ implies $m \leq n$. For $m = n$, we have $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ which differs from $f_{\beta_n}(\delta_{n,j})$ for only the finitely many j which are mapped below n by f_{β_n}. For $m < n$ and $\delta_{m,j} \in K_{\beta_n}$,we have $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$ which can only differ from $f_{\beta_n}(\delta_{m,j})$ for only the finitely many j which are mapped below m by f_{β_m} or the finitely many j for which the almost compatible $f_{\beta_n} \sim f_{\beta_m}$ differ.

Finally for any $\beta < \alpha$, we may conclude $f_\alpha \sim f_\beta$ since there is some β_n with $K_\alpha \cap K_\beta = K_\alpha \cap K_{\beta_n}$, $f_\alpha \sim f_{\beta_n}$, and $f_{\beta_n} \sim f_\beta$. □

We now construct a topology on ω_n for each $n < \omega$ which will witness a Kurepa family of size $\aleph_n$. A similar construction was previously shown by Juhász et. al. in [6], and the relationship of Kurepa families and such spaces has also been investigated in a preprint of Nyikos [9].

Proposition 11. *Let X be a T_2 space with a base of countable and compact neighborhoods. Then X is locally metrizable with a base of compact open countable sets.*

Proof. For each point x let K be a countable and compact neighborhood of x, and it follows that it is contained in a countable, open, and locally compact neighborhood W of x, which in turn is zero-dimensional and metrizable. So choose V clopen in W such that $x \in V \subseteq K$; V is a compact open neighborhood of x in X. □

Definition 12. A topological space is said to be *ω-bounded* if each countable subset of the space has compact closure. As in [6] we call a T_2, locally countable, ω-bounded space *splendid*, and let $\mathcal{S}(\kappa)$ represent the claim that there exists a splendid space of cardinality κ.

Proposition 13. *Let X be a T_2 space with cardinality less than $\aleph_\omega$ which is locally countable and ω-bounded. Then the closure operation preserves cardinality and weight.*

Proof. Note that the closure of any countable neighborhood is compact, and any Lindelöf set is countable. This space is locally metrizable and thus first-countable, so cardinality and weight coincide for any subspace. The result is obvious if A is countable; otherwise let $A = \{a_\alpha : \alpha < \omega_{n+1}\}$ and since basic neighborhoods are countable note any limit point of A is a limit point of $A_\beta = \{a_\alpha : \alpha < \beta\}$ for some $\beta < \omega_{n+1}$. Thus $\overline{A} = \bigcup_{\beta<\omega_{n+1}} \overline{A_\beta}$ and by induction $|\overline{A}| = |A|$. $\square$

Lemma 14. *Let X be a T_2 space with cardinality less than $\aleph_\omega$ which is locally countable and locally compact, and such that its closure operation preserves cardinalities. Then X has an ω-bounded extension $\tilde{X}$ with the same properties where $\tilde{X} \setminus X$ has the same cardinality as X.*

Proof. We prove this by induction on n. If $n = 0$, then we can just use the one-point compactification of two copies of X. So suppose $n > 0$ and that $X = \omega_n$ has an appropriate topology. Note that X has a base of countable and compact neighborhoods since the closure operation preserves cardinalities.

For each $\alpha < \omega_n$, γ_α may be chosen such that both the closure of the set α in X and a countable neighborhood of the point α are subsets of γ_α. Note that the set $\{\lambda < \omega_n : \alpha < \lambda \Rightarrow \gamma_\alpha < \lambda\}$ is a cub subset of ω_n containing a cub subset C of limit ordinals. Now for each $\lambda \in C$, the set λ is open as $\alpha < \lambda$ belongs to the neighborhood $\gamma_\alpha \subsetneq \lambda$. Also, if λ has uncountable cofinality, then for $\beta \geq \lambda$ and any countable neighborhood U of β, $U \cap \lambda = U \cap \alpha$ for some $\alpha < \lambda$; thus $U \setminus \overline{\alpha} = U \setminus \lambda$ is a neighborhood of β, showing that λ is clopen.

Let $\tilde{X} = \omega_n \times 2$. By induction on $\lambda \in C$ we will define compatible topologies for $\tilde{X}_\lambda = \omega_n \times \{0\} \cup \lambda \times \{1\}$ such that

- $\omega_n \times \{0\}$ is an open copy of X,
- $\lambda \times 2$ is open, and when $\mathrm{cf}(\lambda) > \omega$ also closed,
- the space has a base of countable and compact neighborhoods, and
- when λ is a successor, for each $\alpha < \lambda$ the closure of $\alpha \times 2$ is an ω-bounded subset of $\lambda \times 2$.

We first consider the case $n = 1$. If λ is a limit in C, then $\tilde{X}_\lambda = \bigcup_{\mu \in C\cap\lambda} \tilde{X}_\mu$ satisfies the induction requirements. Otherwise we choose an increasing sequence of ordinals $\{\alpha_k : k \in \omega\}$ with limit λ such that α_0 is the predecessor of λ in C, or $\alpha_0 = 0$ if λ is the least element of C.

The subspace $\overline{\lambda} \times \{0\} \cup \alpha_0 \times 2$ of X is countable and locally compact; therefore it is metrizable and zero-dimensional. So we may choose increasing sets U_k for $k < \omega$ which are clopen in this topology and satisfy

$$\overline{\alpha_k \times \{0\} \cup \alpha_0 \times 2} = \overline{\alpha_k} \times \{0\} \cup \alpha_0 \times 2 \subseteq U_k \subseteq \lambda \times \{0\} \cup \alpha_0 \times 2$$

Note that U_k is also clopen in $\tilde{X}_{\alpha_0}$ since it is closed in $\overline{\lambda} \times \{0\} \cup \alpha_0 \times 2$ and open in $\lambda \times \{0\} \cup \alpha_0 \times 2$.

We need only describe a base for the points $\langle \alpha, 1\rangle \in (\lambda \setminus \alpha_0) \times \{1\}$. We do so by letting $\langle \alpha, 1\rangle$ be isolated when $\alpha \notin \{\alpha_k : k < \omega\}$, and giving $\langle \alpha_k, 1\rangle$ the open neighborhoods $(U_k \cup ((\alpha_k + 1) \times \{1\})) \setminus K$ for each compact subset K of $U_k \cup (\alpha_k \times \{1\})$; that is, $\langle \alpha_k, 1\rangle$ is the one point compactifying $U_k \cup (\alpha_k \times \{1\})$.

The first two requirements of our inductive hypothesis are obviously satsified. Note points in $\lambda \times 2$ are covered by the compact countable neighborhood $U_k \cup ((\alpha_k + 1) \times \{1\})$ for some $k < \omega$, and for points in $(\omega_n \setminus \lambda) \times \{0\}$ we may use a compact countable neighborhood from X. For the final requirement, note that for $\alpha < \lambda$, we may choose $\alpha < \alpha_k < \lambda$ and note that $\alpha \times 2$ is contained in the compact subset $U_k \cup ((\alpha_k + 1) \times \{1\})$ of $\lambda \times 2$.

For the case $n > 1$, we may assume that the successors in C have uncountable cofinality. We again proceed by induction on $\lambda \in C$. Again when λ is a limit in C, $\tilde{X}_\lambda = \bigcup_{\mu \in C \cap \lambda} \tilde{X}_\mu$ satisfies the given requirements; in particular if $\alpha < \lambda$, then $\alpha < \mu < \lambda$ for some successor $\mu \in C$ with uncountable cofinality. As such, the closure of $\alpha \times 2$ is an ω-bounded subset of the clopen $\mu \times 2$ and therefore also of $\lambda \times 2$. In case λ is not a limit of C, then λ has uncountable cofinality and a predecessor $\mu \in C$. We therefore have that $\lambda \times \{0\}$ is clopen in $\omega_n \times \{0\}$. Since the cardinality of $\lambda \times \{0\} \cup \mu \times 2$ is less than $\aleph_n$, we may simply apply the induction hypothesis to choose an appropriate topology for $\lambda \times 2$.

As a result, $\tilde{X} = \bigcup_{\lambda \in C} \tilde{X}_\lambda$ is ω-bounded as any countable set is contained in some $\alpha \times 2$ for $\alpha < \lambda \in C$. □

Theorem 15. *For each $k < \omega$, there is a T_2, locally countable, ω-bounded topology on ω_k. That is, $\mathcal{S}(\aleph_k)$ for all $k < \omega$.*

Proof. Apply the previous lemma to ω_n with the discrete topology. □

Lemma 16. *The family of compact open sets in a locally countable, ω-bounded topological space X is a Kurepa family cofinal in $[X]^\omega$. That is, $\mathcal{S}(\kappa) \Rightarrow \mathcal{K}(\kappa)$.*

Proof. Let $\mathcal{K}$ collect all compact open subsets of X. Of course, every Lindelöf set in a locally countable space is countable, and the closure of every countable set is a compact countable set; thus $\mathcal{K}$ is cofinal in $[X]^\omega$. It is Kurepa since every countable set is contained in a countable compact open subspace of X; this subspace has a countable base of compact open sets, which closed under finite unions enumerates all compact open subsets of the subspace. □

Corollary 17. *$\mathcal{K}(\aleph_k)$ for all $k < \omega$.*

Alternatively, the previous corollary may be obtained via an observation of Todorcevic communicated by Dow in [3]: if every Kurepa family of size at most κ extends to a cofinal Kurepa family, then the same is true of κ^+.

Nyikos points out in [9] that a cofinal Kurepa family may be used to construct a locally metrizable, ω-bounded, zero-dimensional space with appropriate cardinality, but whether this can be strengthened to locally countable and ω-bounded (as asked in [6]) remains an open question.

Also left open is this extension of the question asked in [9] and [6] on the possible equivalence of $\mathcal{S}(\kappa)$ and $\mathcal{K}(\kappa)$.

Question 18. *May any of the implications in the theorem $\mathcal{S}(\kappa) \Rightarrow \mathcal{K}(\kappa) \Rightarrow \mathcal{A}'(\kappa)$ be reversed?*

Regardless, we have obtained our desired result.

Corollary 19. *$\mathcal{A}'(\aleph_k)$ for all $k < \omega$.*

3. Consistency results

As noted in [3], Jensen's one-gap two-cardinal theorem under $V = L$ introduced in [5] implies that $\mathcal{K}(\kappa)$ holds for all cardinals κ.

Corollary 20 ($V = L$)**.** *$\mathcal{A}'(\kappa)$ for all cardinals κ.*

Weakening to the continuum hypothesis, we see an obvious consequence.

Corollary 21 (CH)**.** *$\mathcal{A}'(\mathfrak{c}^+)$, but $\neg\mathcal{A}(\mathfrak{c}^+)$.*

But CH is not required to have $\mathcal{A}(\aleph_2)$ fail.

The forcing extension of a model M by a poset $\mathbb{P} \in M$ is obtained simply by evaluating all $\mathbb{P}$-names from M by a generic filter G. A set τ is a $\mathbb{P}$-name if τ is a (possibly empty) set of ordered pairs (σ, p) where $p \in \mathbb{P}$ and σ is also itself a $\mathbb{P}$-name. If G is a $\mathbb{P}$-generic filter, then $\mathrm{val}_G(\tau)$ is defined to equal $\{\mathrm{val}_G(\sigma) : (\exists p \in G)\ (\sigma, p) \in \tau\}$.

If $x \in M$, then the canonical $\mathbb{P}$-name, $\check{x}$, is generally, and recursively, taken to be $\{(\check{y}, 1) : y \in x\}$ where 1 is the maximum element of $\mathbb{P}$. However, it will be convenient to consider, when the context is clear, (x, p) (for any $p \in \mathbb{P}$) to be a kind of $\mathbb{P}$-name. In particular if $\tau \subset X \times \mathbb{P}$ (for some fixed $X \in M$), then we may let $\tau[G] = \{x : (\exists p \in G)\ (x, p) \in \tau\}$.

Thus, $\mathrm{val}_G(\tau)$ will denote the recursive evaluation by G and $\tau[G]$ will be defined as above. In fact, if $\tau \in M$ is any set then each of $\mathrm{val}_G(\tau)$ and $\tau[G]$ are well defined. It is a standard convention to use a dotted letter, such as $\dot{x}$, to indicate that we are discussing a $\mathbb{P}$-name.

One says that a condition $p \in \mathbb{P}$ forces a statement φ to hold, denoted $p \Vdash \varphi$, if that statement holds in $M[G]$ for all $\mathbb{P}$-generic filters with $p \in G$. The forcing theorem states that if $M[G] \models \varphi$, then there is some $p \in G$ forcing that φ holds. The following is an immediate consequence of the forcing theorem.

Lemma 22. *If $X \in M$ and $\dot{x}$ is a $\mathbb{P}$-name, then there is a $\tau \subset X \times \mathbb{P}$, such that for any generic G, $\tau[G] = X \cap \mathrm{val}_G(\dot{x})$.*

In other words, the family of subsets of any $X \in M$ in the extension $M[G]$ is equal to $\{\tau[G] : \tau \subset X \times \mathbb{P},\ \ \tau \in M\}$. We will be using the forcing poset $\mathrm{Fn}(\omega_2, 2)$. The elements of this poset are all the finite partial functions from ω_2 into 2 ordered by reverse inclusion. It follows that, for any $\lambda \in \omega_2$, each of $\mathrm{Fn}(\lambda, 2)$ and $\mathrm{Fn}(\omega_2 \setminus \lambda, 2)$ are subposets. For any $\mathrm{Fn}(\omega_2, 2)$-generic filter G, it easily follows that $G_\lambda = G \cap \mathrm{Fn}(\lambda, 2)$ and $G^\lambda = G \cap \mathrm{Fn}(\omega_2 \setminus \lambda, 2)$ are also generic filters. But a much stronger statement is true.

Lemma 23. [7] *Assume that $G \subset \mathrm{Fn}(\omega_2, 2)$ is a generic filter, and let $\lambda \in \omega_2$. Then the final model $M[G]$ is equal to $(M[G_\lambda])[G^\lambda]$ in the sense that G^λ is a $\mathrm{Fn}(\omega_2 \setminus \lambda, 2)$-generic filter over the model $M[G_\lambda]$.*

In addition, for each $X \in M$ and name $\dot{A} \subset X \times \mathrm{Fn}(\omega_2, 2)$, we get that

$$(\dot{A}(G_\lambda))[G^\lambda] = \dot{A}[G] \ \text{ where } \dot{A}(G_\lambda) = \{(x, p \restriction [\lambda, \omega_2)) : (x, p) \in \dot{A} \ \text{ and } \ p \restriction \lambda \in G_\lambda\}$$

With these lemmas in hand we are ready to prove the theorem. The idea of the proof comes from Kunen's result about no ω_2 length mod finite chains of subsets of ω. We consider any family of names of suitable one-to-one functions from countable subsets of ω_2 into ω. We identify a large enough $\lambda \in \omega_2$ so that a pattern has emerged and we pass to the model $M[G_\lambda]$. We then show that this pattern can not continue out to ω_2.

Theorem 24. *There exists a model of ZFC for which $\mathfrak{c} = \aleph_2$ and $\neg\mathcal{A}(\aleph_2)$.*

Proof. We start with a model M of GCH and suppose that G is a $\mathrm{Fn}(\omega_2, 2)$-generic filter. The argument takes place in M. Let $\{\dot{f}_A : A \in [\omega_2]^\omega\}$ be a family of names (in M) such that, for any generic G and each $A \in [\omega_2]^\omega \cap M$, $\dot{f}_A[G]$ is a one-to-one function from A into ω. We also assume that whenever $B \subset A$ are members of $[\omega_2]^\omega$, we have that $\dot{f}_B[G] \subset^* \dot{f}_A[G]$. If we now obtain a contradiction then we will have shown that $\mathcal{A}(\aleph_2)$ fails.

By [2, 1.5], there is a set $H \subset H(\aleph_3)$ such that the family $\{\dot{f}_A : A \in [\omega_2]^\omega\}$ is an element of H, H is an elementary submodel of $H(\aleph_3)$, H has cardinality $\aleph_1$, and $H^\omega \subset H$ (every countable subset of H is an element of H).

Let $\lambda = H \cap \omega_2$ (same as the supremum of $H \cap \omega_2$). Consider the name $\dot{f}_{[\lambda,\lambda+\omega)}$. What is such a name? By Lemma 22, we can assume that it is a set of pairs of the form $((\lambda + k, m), p)$ where $p \in Fn(\omega_2, 2)$ and, of course, $k, m \in \omega$. Furthermore, for each k, m it is enough (see [7, 5.11,5.12]) to take a countable set of such p to get an equivalent (*nice*) name. Given any such nice name $\dot{f}$, let $\mathrm{supp}(\dot{f})$ denote the union of the domains of conditions p appearing in the name.

Now let Y equal $\mathrm{supp}(\dot{f}_{[\lambda,\lambda+\omega)}) \setminus \lambda$. Furthermore, fix any $\mu \in \lambda \subset H$ such that $\mathrm{supp}(\dot{f}_{[\lambda,\lambda+\omega)}) \cap \lambda$ is contained in μ. Let $\delta \in \omega_1$ denote the order type of Y and let $\varphi_{\mu,\lambda}$ be the order-preserving function from $\mu \cup Y$ onto the ordinal $\mu + \delta$. This lifts canonically to an order-preserving bijection $\varphi_{\mu,\lambda} : \mathrm{Fn}(\mu \cup Y, 2) \mapsto \mathrm{Fn}(\mu + \delta, 2)$. We can similarly make sense of the name $\varphi_{\mu,\lambda}(\dot{f}_{[\lambda,\lambda+\omega)})$, call it F_H. Here simply, for each tuple $(\,(\lambda + k, m), p) \in \dot{f}_{[\lambda,\lambda+\omega)}$, we have that $(\,(\mu + k, m), \varphi_{\mu,\lambda}(p))$ is in F_H. Again, let $\varphi_{\mu,\lambda}(\dot{f}_{[\lambda,\lambda+\omega)})$ be interpreted in the above sense as giving F_H (which is an element of H).

Other values replacing $\lambda > \mu$ will result in their own set Y and canonical map $\varphi_{\mu,\lambda}$. Now the object F_H is an element of H, and H believes this statement is true:

$$(\forall \beta \in \omega_2)\ (\exists \lambda \in \omega_2 \setminus \beta)\ \ \mathrm{supp}(\dot{f}_{[\lambda,\lambda+\omega)}) \cap \lambda \subset \mu \ \text{ and } \ F_H = \varphi_{\mu,\lambda}(\dot{f}_{[\lambda,\lambda+\omega)})$$

But now, this means that, not only is there an $\alpha \in H$, $F_H = \varphi_{\mu,\alpha}(\dot{f}_{[\alpha,\alpha+\omega)})$ but also that there is an increasing sequence $\{\alpha_\xi : \xi \in \omega_1\} \subset \lambda$ of such α's satisfying that, for each ξ we have that $\mathrm{supp}(\dot{f}_{[\alpha_\xi,\alpha_\xi+\omega)})$ is contained in $\alpha_{\xi+1}$.

Choose such a sequence. This means that if we let $A = \bigcup_{n>0}[\alpha_n, \alpha_n + \omega)$ we have the name $\dot{f}_A$ in H. This then means that all the $((\beta, m), p)$ appearing in (the nice name) $\dot{f}_A$ have the property that $dom(p)$ is contained in H. There is, also within H, a name $\dot{g}$ satisfying that $\dot{f}_A(\alpha_n + k) = \dot{f}_{[\alpha_n,\alpha_n+\omega)}(\alpha_n + k)$ for all $k > \dot{g}(n)$, or more precisely, $\dot{g} \subset (\omega \times \omega) \times \mathrm{Fn}(\omega_2, 2)$ satisfies that $\dot{g}[G] \in \omega^\omega$ and $\dot{f}_A[G](\alpha_n + k) = \dot{f}_{[\alpha_n,\alpha_n+\omega)}[G](\alpha + k)$ for all $k > \dot{g}[G](n)$.

We now apply Lemma 23 and we are now working in the extension $M[G_\mu]$. We work for a contradiction. Something special has now happened, namely, the supports of the names $\{\dot{f}_{[\alpha_n,\alpha_n+\omega)}(G_\mu) : 0 < n < \omega\}$ are pairwise disjoint and also disjoint from the support of the name $\dot{f}_{[\lambda,\lambda+\omega)}(G_\mu)$. And not only that, these names are pairwise isomorphic (in the way that they all map to F_H).

Since A is disjoint from $[\lambda, \lambda+\omega)$, there must be an integer ℓ together with a condition $q \in Fn(\omega_2 \setminus \mu, 2)$ satisfying that for all $n > \ell$, q forces that

"if $k > \dot{g}(n)$ then $(\dot{f}_{[\alpha_n,\alpha_n+\omega)}(G_\mu))(\alpha_n + k) \neq (\dot{f}_{[\lambda,\lambda+\omega)}(G_\mu))(\lambda + k)$".

Choose $n > \ell$ large enough so that $dom(q) \cap [\alpha_n, \alpha_{n+1})$ is empty. Choose $q_1 < q \restriction \lambda$ (in H) so that

$$\varphi_{\mu,\alpha_n}(q_1 \restriction \mathrm{supp}(\dot{f}_{[\alpha_n,\alpha_n+\omega)}) = \varphi_{\mu,\lambda}(q \restriction \mathrm{supp}(\dot{f}_{[\lambda,\lambda+\omega)})$$

and then (again in H) choose $q_2 < q_1$ so that it both forces a value L on $\ell+\dot{g}(n)$ and subsequently forces a value m on $\dot{f}_{[\alpha_n,\alpha_n+\omega)}(\alpha_n + L+1)$. But now, again calculate

$$q_3 = \varphi^{-1}_{\mu,\lambda} \circ \varphi_{\mu,\alpha_n}(q_2 \restriction \mathrm{supp}(\dot{f}_{[\alpha_n,\alpha_n+\omega)}))$$

and, by the isomorphisms, we have that q_3 forces that $\dot{f}_{[\lambda,\lambda+\omega)}(\lambda + L + 1) = m$.

Technically (or with more care) all of this is taking place in the poset $\mathrm{Fn}(\omega_2\setminus\mu, 2)$ and this means that q_3 and q are with each other. To verify this it suffices to consider $q(\beta) = e$ and to assume that $q_3(\beta)$ is defined. Since $q_3(\beta)$ is defined, we have that there is a $\beta' \in dom(q_2)$ such that $\varphi_{\mu,\lambda}(\beta) = \varphi_{\mu,\alpha_n}(\beta')$, and that $q_3(\beta) = q_2(\beta')$. But, by definition of q_1, $\beta' \in dom(q_1)$ and even that $q_1(\beta') = q(\beta)$. Then, since $q_2 < q_1$, we have that $q_2(\beta') = q_1(\beta') = q(\beta)$. This completes the circle that $q_3(\beta) = q(\beta)$.

Finally, our contradiction is that $q_3 \cup q_2 \cup q$ forces that $k = L + 1$ violates the quoted statement above. □

We are also able to force $\mathcal{A}'(\kappa)$ to fail for every cardinal other than the first ω-many we've already guaranteed.

Theorem 25. *It follows from the existence of a 2-huge cardinal that there is a model of ZFC for which* $\neg\mathcal{A}'(\aleph_\omega)$.

Proof. We will need the model constructed in [8] in which an instance of Chang's conjecture $(\aleph_{\omega+1}, \aleph_\omega)\twoheadrightarrow(\aleph_1, \aleph_0)$ is shown to hold.

We can take as a given (as shown in [8, Theorem 5]) that we may assume that we have a model V of GCH in which there are regular limit cardinals $\kappa < \lambda$ satisfying that $(\lambda^{+\omega+1}, \lambda^{+\omega})\twoheadrightarrow(\kappa^{+\omega+1}, \kappa^{+\omega})$.

What this says is that if L is a countable language with at least one unary relation symbol R and M is a model of L with base set $\lambda^{+\omega+1}$ in which the interpretation of R has cardinality $\lambda^{+\omega}$, then M has an elementary submodel N of cardinality $\kappa^{+\omega+1}$ in which $R \cap N$ has cardinality $\kappa^{+\omega}$ (of course $R \cap N$ is the interpretation of R in N because $N \prec M$).

The interested reader will want to know that it is shown in [8] that if κ is a 2-huge cardinal and j is the 2-huge embedding with critical point κ, then with $\lambda = j(\kappa)$

one has that $(\lambda^{+\omega+1},\lambda^{+\omega})\twoheadrightarrow(\kappa^{+\omega+1},\kappa^{+\omega})$ holds. There is no loss of generality to also assume that GCH holds in this model.

Let $\{h_\xi:\xi\in\lambda^{+\omega+1}\}$ be a scale in $\Pi\{\lambda^{+n+1}:n\in\omega\}$ ordered by the usual mod finite coordinatewise ordering. For convenience we may assume that $h_\xi(n)\geq\lambda^{+n}$ for all ξ and all n. For each integer m the cofinality of the mod finite ordering on $\Pi\{\lambda^{+n+1}:m<n\in\omega\}$ is the same as it is for the entire product $\Pi\{\lambda^{+n+1}:n\in\omega\}$. If P is any poset of cardinality less than λ^{+m} then, in the forcing extension by P, every function in $\Pi\{\lambda^{+n+1}:m<n\in\omega\}$ is bounded above by a ground model function. It therefore follows easily that in the forcing extension by P, the sequence $\{h_\xi:\xi\in\lambda^{+\omega+1}\}$ remains cofinal in $\Pi\{\lambda^{+n+1}:n\in\omega\}$.

The forcing notion $\mathbb{P}_0$ is simply the finite condition collapse of $\kappa^{+\omega}$, i.e. $\mathbb{P}_0=(\kappa^{+\omega})^{<\omega}$. In the forcing extension by $\mathbb{P}_0$, one now has that the ordinal $\kappa^{+\omega+1}$ from V is the first uncountable cardinal $\aleph_1$. Then in this forcing extension we let $\mathbb{P}_1$ be the countable condition Levy collapse, $Lv(\lambda,\omega_2)$, which collapses all cardinals less than λ to have cardinality at most $\aleph_1$. The poset $\mathbb{P}_1$ has cardinality λ. We treat $\mathbb{P}_0*\mathbb{P}_1$ as containing $\mathbb{P}_0$ as a subposet by identifying each $(p_0,1)$ with p_0. After forcing with $\mathbb{P}_0*\mathbb{P}_1$ we will have that ω_1 is the ordinal $\left(\kappa^{+\omega+1}\right)^V$, ω_2 is the ordinal λ, and ω_ω is the ordinal $(\lambda^{+\omega})^V$.

Now we assume that we have an assignment $\dot f_{\dot A}$ of a $\mathbb{P}_0*\mathbb{P}_1$-name of a finite-to-one function from $\dot A$ into ω for each $\mathbb{P}_0*\mathbb{P}_1$-name of a countable subset of $\lambda^{+\omega+1}$. We will obtain a contradiction to the claim of coherence.

Let $\{\dot A_\xi:\xi\in\lambda^{+\omega+1}\}$ be an enumeration of all the nice $\mathbb{P}_0$-names of countable subsets of $\lambda^{+\omega}$. For each $\xi\in\lambda^{+\omega+1}$, let $\dot f_\xi$ be another notation for $\dot f_{\dot A_\xi}$. Since $\mathbb{P}_0$ forces that $\mathbb{P}_1$ is countably closed, the collection of all nice $\mathbb{P}_0$-names will produce all the countable sets in the extension by $\mathbb{P}_0*\mathbb{P}_1$, but $\mathbb{P}_0*\mathbb{P}_1$ can introduce new enumerations of these names. For each $\xi\in\lambda^{+\omega+1}$, there is a minimal ζ_ξ so that $\dot A_{\zeta_\xi}$ is the canonical name for the range of h_ξ. This means that $\dot f_{\zeta_\xi}\circ h_\xi$ is simply the $\mathbb{P}_0*\mathbb{P}_1$-name of a finite-to-one function from ω to ω. For each $\xi\in\lambda^{+\omega+1}$, choose any $p_\xi\in\mathbb{P}_0*\mathbb{P}_1$ so that there is a nice $\mathbb{P}_0$-name, $\dot H_\xi$, that is forced by p_ξ to equal $\dot f_{\zeta_\xi}\circ h_\xi$. Choose $\Lambda\subset\lambda^{+\omega+1}$ of cardinality $\lambda^{+\omega+1}$ and so that there is a pair $p,\dot H$ satisfying that $p_\xi=p$ and $\dot H_\xi=\dot H$ for all $\xi\in\Lambda$. We may assume that p is in a generic filter G.

Let $\{x_\xi:\xi\in\lambda^{+\omega+1}\}$ be any enumeration of $H(\lambda^{+\omega+1})$ such that $\{x_\xi:\xi\in\lambda^{+\omega}\}$ is also equal to $H(\lambda^{+\omega})$. We choose this enumeration in such a way that $x_\xi\in x_\eta$ implies $\xi<\eta$. We use relation symbol R_0 to code (and well order) $(H(\lambda^{+\omega+1}),\in)$ as follows: $(\xi,\eta)\in R_0$ if and only if $x_\xi\in x_\eta$. Let R_1 be a binary relation on $\kappa^{+\omega}$ so that $(\kappa^{+\omega},R_1)$ is isomorphic to $\mathbb{P}_0$. Let R_2 be a binary relation on λ so that $R_2\cap(\kappa^{+\omega}\times\kappa^{+\omega})=R_1$ and (λ,R_2) is isomorphic to $\mathbb{P}_0*\mathbb{P}_1$. Let ψ be the poset isomorphism from (λ,R_2) to $\mathbb{P}_0*\mathbb{P}_1$.

We continue coding. We can code the sequence $\{h_\xi:\xi\in\lambda^{+\omega+1}\}$ as another binary relation R_3 on $\lambda^{+\omega+1}$ where $R_3\cap\left(\{\xi\}\times\lambda^{+\omega+1}\right)=\{(\xi,h_\xi(n)):n\in\omega\}$ for each $\xi\in\lambda^{+\omega+1}$. The relation symbol R_4 can code the sequence $\{\dot A_\xi:\xi\in\lambda^{+\omega+1}\}$ where $(\xi,\alpha,\zeta)\in R_4$ if and only if $(\check\alpha,\psi(\zeta))$ is in the name $\dot A_\xi$. Let R_5 code this collection, i.e. $(\gamma,n,m,\eta)\in R_5$ if and only if $(\widecheck{(n,m)},\psi(\eta))\in\dot H_\gamma$. Also let R_6

code (equal) the set Λ. Finally we use the relation symbol R_7 to similarly code the sequence $\{\dot{f}_\xi : \xi \in \lambda^{+\omega+1}\}$: $(\xi, \alpha, n, \zeta) \in R_7$ if and only if $((\check{\alpha, n}), \psi(\zeta))$ is in the name $\dot{f}_\xi$.

Needless to say, the unary relation symbol R is interpreted as the set $\lambda^{+\omega}$ for the application of $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$. Now we have defined our model M of the language $L = \{\in, R, R_0, \ldots, R_7\}$, and we choose an elementary submodel N witnessing $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$. Of course N is really just a $\kappa^{+\omega+1}$ sized subset of $\lambda^{+\omega+1}$ with the additional property that $N \cap \lambda^{+\omega}$ has cardinality $\kappa^{+\omega}$. In the forcing extension N has cardinality ω_1 and $A = N \cap \lambda^{+\omega}$ is countable.

We will need the following claim from [8]:

Claim. *We may assume that N satisfies that $N \cap \kappa^{+\omega+1}$ is transitive (i.e. an initial segment).*

Proof of Claim: Suppose our originally supplied N fails the conclusion of the claim. We know that $\kappa^{+\omega} \in N$, (via R_1) in which case so is $\kappa^{+\omega+1}$.

Then set $\beta_0 = \sup(N \cap \kappa^{+\omega+1})$ and consider the Skolem closure $Hull(N \cup \beta_0, M)$. A little informally (in that we have to formalize the enumeration of formulas as per Gödel coding) let $\{\varphi_n : n \in \omega\}$ be an enumeration of all formulas in the language L, and let ℓ_n be the minimal integer such that the free variables of φ_n are among $\{v_0, \ldots, v_{\ell_n}\}$. Then, for each tuple $\langle \xi_1, \ldots, \xi_{\ell_n} \rangle$ of elements of $\lambda^{+\omega+1}$, we define $f_n(\xi_1, \ldots, \xi_{\ell_n})$ to be the minimal $\xi_0 \in \lambda^{+\omega+1}$ such that $M \models \varphi_n(\xi_0, \ldots, \xi_{\ell_n})$. If there is no such ξ_0, in other words if $M \models \neg\exists x\ \varphi_n(x, \xi_1, \ldots, \xi_{\ell_n})$, then set $f_n(\xi_1, \ldots, \xi_{\ell_n})$ to be 0. Now $Hull(N \cup \beta_0, M)$ is just the minimal superset X of $N \cup \beta_0$ that satisfies that $f_n[X^{\{1,\ldots,\ell_n\}}] \subset X$ for all n. Since this is simply a large algebra, we can generate all the terms t of the algebraic operations $\{f_n : n \in \omega\}$. It is easily seen that for each $\zeta \in X$, there is a term $t(v_1, \ldots, v_m)$ such that $\zeta = t(\delta_1, \ldots, \delta_m)$ for some sequence $\langle \delta_1, \ldots, \delta_m \rangle$ with each $\delta_i \in N \cup \beta_0$. Assume that $\zeta \in \kappa^{+\omega+1}$. By re-indexing the variables in the term we can assume that there is an $n \leq m$ so that $\delta_i < \beta_0$ for $1 \leq i \leq n$ and $\kappa^{+\omega+1} \leq \delta_i$ for $n < i \leq m$. Let $\vec{a}$ denote the tuple $\langle \delta_{n+1}, \ldots, \delta_m \rangle$. Choose $\eta \in N \cap \kappa^{+\omega+1}$ large enough so that $\{\delta_1, \ldots, \delta_n\}$ is contained in η. Since set-membership in M is coded by R_0 rather than $\in$ we have to argue a little less naturally. Consider the set $s_0(\eta, \vec{a}) = \{t(\gamma_1, \ldots, \gamma_n, \vec{a}) : \{\gamma_1, \cdots, \gamma_n\} \in [\eta]^{\leq n}\}$. Clearly $s_0(\eta, \vec{a})$ is a member of $H(\lambda^{+\omega+1})$. Now define $s_1(\eta, \vec{a})$ to be $\{x_\alpha : \alpha \in s_0(\eta, \vec{a})\}$, and choose the unique $\zeta_1 \in \lambda^{+\omega+1}$ such that $x_{\zeta_1} = s_1(\eta, \vec{a})$. We claim that $\zeta_1 \in N$. Note that $\alpha R_0 \zeta_1$ holds if and only if $\alpha \in s_0(\eta, \vec{a})$, and therefore

$$M \models (\forall\alpha)\,[\alpha R_0 \zeta_1 \text{ iff } (\exists\gamma_1 \in \eta) \cdots (\exists\gamma_n \in \eta)(\alpha = t(\gamma_1, \ldots, \gamma_n, \vec{a}))]\ .$$

By elementarity then we have that $\zeta_1 \in N$, and by similar reasoning the supremum, ζ_0, of $\zeta_1 \cap \kappa^{+\omega+1}$ is also in N. This of course means that $\zeta < \beta_0$. □

We use the elementarity of N to deduce properties of the families $\{\dot{A}_\xi : \xi \in N\}$ and $\{\dot{f}_\xi : \xi \in N\}$. Actually the collection we are most interested in is the family $\{h_\xi : \xi \in \Lambda \cap N\}$.

Now we need a result from Shelah's pcf theory which is proven in Jech [4, 24.9]. Since $\aleph_1 = \mathfrak{c} < \kappa^{+\omega+1}$ there is a function $\langle \varrho_n : n \in \omega \rangle$ in $\Pi_n \lambda^{+\omega}$ such that the sequence $\{h_\xi : \xi \in N\}$ is unbounded mod finite in $\Pi_n \varrho_n$ For each n, $\varrho_n \leq \sup(N \cap \lambda^{+n+2})$. Since $\mathbb{P}_0$ has cardinality $\kappa^{+\omega}$, and so less than $|N| = \kappa^{+\omega+1}$, a standard argument (analogous to the fact that adding a Cohen real does not add a dominating real) shows that the sequence $\{h_\xi : \xi \in \Lambda \cap N\}$ remains unbounded mod finite in $\Pi_n \varrho_n$ (and in $\Pi_n(\varrho_n \cap N)$).

Now pass to the extension by $G \cap \mathbb{P}_0$ and let H be the function $\mathrm{val}_G(\dot{H})$, and we recall that $f_{\zeta_\xi}(h_\xi(n)) = H(n)$ for all $n \in \omega$ and $\xi \in \Lambda$. Now pass to the full extension $V[G]$ and again, since $\mathbb{P}_1$ was forced to be countably closed, the family $\{h_\xi : \xi \in \Lambda \cap N\}$ is still unbounded in $\Pi_n(\varrho_n \cap N)$ (no new elements were added). We let A be the countable set $N \cap \lambda^{+\omega}$, and for each $\xi \in \Lambda \cap N$, there is an n_ξ such that $f_\xi(h_\xi(m)) = f_A(h_\xi(m))$ for all $m > n_\xi$. There is a single n so that $\Lambda_n = \{\xi \in \Lambda \cap N : n_\xi = n\}$ has cardinality ω_1, and thus $\{h_\xi : \xi \in \Lambda_n \cap N\}$ is also unbounded in $\Pi_n(\rho_n \cap N)$. This certainly implies that there is an $m > n$ such that $\{h_\xi(m) : \xi \in \Lambda_n \cap N\}$ is infinite. This completes the proof since $f_A(h_\xi(m)) = H(m)$ for all $\xi \in \Lambda_n \cap N$. $\square$

4. Applications to infinite length games

We introduce three variations of Scheeper's game which we defined in the introduction.

Game 26. Let $Sch^{\cup,\subsetneq}_{C,F}(\kappa)$ denote the *Scheepers countable-finite union game* which proceeds analogously to $Sch^{\cup,\subset}_{C,F}(\kappa)$, except that $\mathscr{C}$'s restriction in round $n+1$ is weakened to $C_{n+1} \supseteq C_n$.

Game 27. Let $Sch^{1,\subsetneq}_{C,F}(\kappa)$ denote the *Scheepers countable-finite initial game* which proceeds analogously to $Sch^{\cup,\subsetneq}_{C,F}(\kappa)$, except that $\mathscr{F}$'s winning condition is weakened to $\bigcup_{n<\omega} F_n \supseteq C_0$.

Game 28. Let $Sch^{\cap}_{C,F}(\kappa)$ denote the *Scheepers countable-finite intersection game* which proceeds analogously to $Sch^{1,\subsetneq}_{C,F}(\kappa)$, except that $\mathscr{C}$ may choose any $C_n \in [\kappa]^{\leq\omega}$ each round, and $\mathscr{F}$'s winning condition is weakened to $\bigcup_{n<\omega} F_n \supseteq \bigcap_{n<\omega} C_n$.

In [1] Clontz extended Scheepers' application of almost-compatible injections to these game variants as well as $Men_{C,F}\left(\kappa^\dagger\right)$. However, when considering Markov strategies, finite-to-one functions suffice.

Theorem 29. *$\mathcal{A}'(\kappa)$ implies the game-theoretic conditions shown in Figure 1.*

Proof. The weaker claim $\mathcal{A}(\kappa) \Rightarrow \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}(\kappa)$ was proven in [1]; however, the strategy only required that the f_A be pairwise almost-compatible and that preimages of finite sets in each f_A are finite, which is also possible assuming $\mathcal{A}'(\kappa)$. Note that the relationships amongst the games were all shown in [1]. $\square$

We include the following proof from [10] for the convenience of the reader.

Theorem 30. $\mathcal{A}(\kappa) \Rightarrow \mathscr{F} \underset{2\text{-}tact}{\uparrow} Sch^{\cup,\subset}_{C,F}(\kappa)$

$$\begin{tikzcd}
& \mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}\left(\kappa^\dagger\right) \arrow[r, leftrightarrow] \arrow[dd] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}\left(\kappa\right) \arrow[dd] \\
\mathcal{A}'(\kappa) \arrow[ur] \arrow[dr] & & \\
& \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{1,\subsetneq}_{C,F}\left(\kappa\right) \arrow[r, leftrightarrow] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cup,\subsetneq}_{C,F}\left(\kappa\right)
\end{tikzcd}$$

FIGURE 1. Diagram of Scheeper/Menger game implications with $\mathcal{A}'(\kappa)$

Proof. Let $\{f_A : A \in [\kappa]^{\leq \aleph_0}\}$ witness $\mathcal{A}(\kappa)$, and define $g_A : A \to \omega$ by $g_A(\alpha) = f_A(\alpha) - |\{\beta \in A : f_A(\beta) < f_A(\alpha)\}|$.

We claim that $\{\alpha \in A : g_A(\alpha) \leq g_B(\alpha)\}$ must be finite as it is bounded above by $\max\{M, f_A(\alpha), f_B(\alpha) : f_A(\alpha) \neq f_B(\alpha)\}$ where $M = f_B(\alpha)$ for some $\alpha \in B \setminus A$. To see this, let $f_A(\alpha) = f_B(\alpha) = N > M$ and assume $f_A(\beta) \neq f_B(\beta)$ implies $f_A(\beta), f_B(\beta) < N$. Then

$$g_A(\alpha) = N - |\{\beta \in A : f_A(\beta) < N\}| > N - |\{\beta \in B : f_B(\beta) < N\}| = g_B(\alpha)$$

with the strictness of the inequality witnessed by $f_B(\alpha) = M < N$ for some $\alpha \in B \setminus A$.

As a result,

$$\sigma(\langle A, B\rangle) = \{\alpha \in A : g_A(\alpha) \leq g_B(\alpha)\}$$

is a legal 2-tactic for $\mathscr{F}$. Let $C = \langle C_0, C_1, \ldots\rangle$ be a strictly increasing sequence of countable sets and $\alpha \in C_n$. Noting that f_A is an injection and not just finite-to-one, $0 \leq g_{C_{n+m}}(\alpha)$ for all $m < \omega$, and it follows that $g_{C_{n+m}}(\alpha) \leq g_{C_{n+m+1}}(\alpha)$ for some $m < \omega$. Therefore $\alpha \in \sigma(\langle C_{n+m}, C_{n+m+1}\rangle)$. $\square$

So it would seem that $\mathcal{A}'(\kappa)$ is sufficient only when considering Markov strategies. (Of course, $\mathcal{A}'(\kappa) \Rightarrow \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cup,\subsetneq}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cup,\subset}_{C,F}(\kappa)$.) We would like to demonstrate that $\mathcal{A}'(\kappa)$ is not necessary.

Theorem 31. *Let α be the limit of increasing ordinals β_n for $n < \omega$. If $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}(\aleph_{\beta_n})$ for all $n < \omega$, then $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}(\aleph_\alpha)$.*

Proof. Let σ_n be a winning 2-mark for $\mathscr{F}$ in $Sch^{\cap}_{C,F}(\aleph_{\beta_n})$. Define the 2-mark σ for $\mathscr{F}$ in $Sch^{\cap}_{C,F}(\aleph_\alpha)$ as follows:

$$\sigma(\langle C\rangle, 0) = \sigma_0(\langle C \cap \aleph_{\beta_0}\rangle, 0)$$

$$\sigma(\langle C, D\rangle, n+1) = \sigma_{n+1}(\langle D \cap \aleph_{\beta_{n+1}}\rangle, 0) \cup \bigcup_{m \leq n} \sigma_m(\langle C \cap \aleph_{\beta_m}, D \cap \aleph_{\beta_m}\rangle, n - m + 1)$$

Let $\langle C_0, C_1, \dots\rangle$ be an attack by $\mathscr{C}$ in $Sch^{\cap}_{C,F}(\aleph_\alpha)$, and $\alpha \in \bigcap_{n<\omega} C_n$. Choose $N < \omega$ with $\alpha < \aleph_{\beta_{N+1}}$. Consider the attack $\langle C_{N+1}\cap\aleph_{\beta_{N+1}}, C_{N+2}\cap\aleph_{\beta_{N+1}}, \dots\rangle$ by $\mathscr{C}$ in $Sch^{\cap}_{C,F}\left(\aleph_{\beta_{N+1}}\right)$. Since σ_{N+1} is a winning 2-mark and $\alpha \in \bigcap_{n<\omega} C_{N+n+1}\cap\aleph_{\beta_{N+1}}$, either $\alpha \in \sigma_{N+1}(\langle C_{N+1}\cap\aleph_{\beta_{N+1}}\rangle, 0)$ and thus $\alpha \in \sigma(\langle C_N, C_{N+1}\rangle, N+1)$, or $\alpha \in \sigma_{N+1}(\langle C_{N+M+1}\cap\aleph_{\beta_{N+1}}, C_{N+M+2}\cap\aleph_{\beta_{N+1}}\rangle, M+1)$ for some $M < \omega$ and thus $\alpha \in \sigma(\langle C_{N+M+1}, C_{N+M+2}\rangle, N+M+2)$. Thus σ is a winning 2-mark. □

Theorem 32. *Let α be the limit of increasing ordinals β_n for $n < \omega$. If $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Sch^{1,\subseteq}_{C,F}(\aleph_{\beta_n})$ for all $n < \omega$, then $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Sch^{1,\subseteq}_{C,F}(\aleph_\alpha)$.*

Proof. The proof proceeds nearly identically to the previous proof: substitute $\alpha \in C_0$ in place of $\alpha \in \bigcap_{n<\omega} C_n$ and proceed. □

Corollary 33. *It is consistent that $\mathcal{A}'(\aleph_\omega)$ fails, but as $\mathcal{A}'(\aleph_k)$ holds in ZFC for all $k < \omega$, both $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Sch^{\cap}_{C,F}(\aleph_\omega)$ and $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Sch^{1,\subseteq}_{C,F}(\aleph_\omega)$ hold in ZFC.*

Note that Question 6 remains unsolved; however, our results have revealed that we cannot hope to find a ZFC counterexample where $X = \kappa^\dagger$. This is because if we also assume $V = L$, it follows that $\mathcal{A}'(\kappa)$ and therefore $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}\left(\kappa^\dagger\right)$ for every cardinal κ.

Question 34. *Is $\mathscr{F} \underset{2\text{-}mark}{\not\uparrow} Sch^{\cap}_{C,F}(\aleph_{\omega+1})$ consistent with ZFC?*

If so, $X = \aleph^\dagger_{\omega+1}$ is consistently a positive answer to Question 6.

References

[1] Steven Clontz. Applications of limited information strategies in menger's game (preprint). 2015.

[2] Alan Dow. An introduction to applications of elementary submodels to topology. *Topology Proc.*, 13(1):17–72, 1988.

[3] Alan Dow. Set theory in topology. In *Recent progress in general topology (Prague, 1991)*, pages 167–197. North-Holland, Amsterdam, 1992.

[4] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[5] R. Björn Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4:229–308; erratum, ibid. 4 (1972), 443, 1972. With a section by Jack Silver.

[6] I. Juhász, Zs. Nagy, and W. Weiss. On countably compact, locally countable spaces. *Period. Math. Hungar.*, 10(2-3):193–206, 1979.

[7] Kenneth Kunen. *Set theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-New York, 1980. An introduction to independence proofs.

[8] Jean-Pierre Levinski, Menachem Magidor, and Saharon Shelah. Chang's conjecture for $\aleph_\omega$. *Israel J. Math.*, 69(2):161–172, 1990.

[9] Peter Nyikos. Generalied kurepa and mad families and topology.

[10] Marion Scheepers. Concerning n-tactics in the countable-finite game. *J. Symbolic Logic*, 56(3):786–794, 1991.

[11] Marion Scheepers. Meager-nowhere dense games. I. n-tactics. *Rocky Mountain J. Math.*, 22(3):1011–1055, 1992.

[12] Marion Scheepers. Combinatorics of open covers. I. Ramsey theory. *Topology Appl.*, 69(1):31–62, 1996.

Department of Mathematics and Statistics, UNC Charlotte, Charlotte, NC 28262

E-mail address: steven.clontz@gmail.com

Department of Mathematics and Statistics, UNC Charlotte, Charlotte, NC 28262

E-mail address: adow@uncc.edu