Scheeper's Meager-NWD Game and the Menger Game

AU Topology Seminar

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October 22, 2013

Abstract

Marion Scheepers designed the Meager-NWD game $Fill^{\subsetneq}_{\mathscr{M}\mathscr{N}}(J)$ in the 80s to study the existence of k-tactics in set-theoretic and topological games.

There are strong similarities between Dr. Scheeper's game and the special case of the Menger game $Cov_{\mathscr{C}\mathscr{F}}(\kappa^{\dagger})$ played upon the one-point "Lindelöfication" of a discrete cardinal κ.

We will explore the relationship between k-tactical stratgies in $Fill^{\subsetneq}_{\mathscr{M}\mathscr{N}}(J)$ and k-Marköv strategies in $Fill^{\subseteq}_{\mathscr{M}\mathscr{N}}(J)$ or $Cov_{\mathscr{C}\mathscr{F}}(\kappa^{\dagger})$, as well as a sentence $S(\kappa, \omega, \omega)$ which is consistent with ZFC.

Menger Game

Game

The two-player Menger Game $Cov_{\mathscr{C},\mathscr{F}}(X)$ proceeds as follows:

- Round n: player $\mathscr{C}$ chooses an open cover $\mathcal{U}_n$ of X
- Round n: player $\mathscr{F}$ chooses finite $\mathcal{F}_n \subseteq \mathcal{U}_n$.

$\mathscr{F}$ wins if $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

- Easy to see that $\mathscr{F}$ can win for any σ-compact space.
- The existence or non-existence of various limited info strategies in this game characterize covering properties of X.

$Cov_{\mathscr{C}\mathscr{F}}(X)$ characterizations

$$
\begin{array}{ccccccc}
\mathscr{F} \uparrow_{\text{mark}} Cov_{\mathscr{C}\mathscr{F}}(X) & \Rightarrow & \mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(X) & \Rightarrow & \mathscr{F} \uparrow Cov_{\mathscr{C}\mathscr{F}}(X) & \Rightarrow & \mathscr{C} \not\uparrow Cov_{\mathscr{C}\mathscr{F}}(X) \\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \\
X \text{ is } \sigma\text{-(rel. compact)} & \Rightarrow & ??? & \Rightarrow & ??? & \Rightarrow & X \text{ is Menger}
\end{array}
$$

- $\uparrow$ denotes a player with a **winning strategy**
- $\uparrow_{\text{mark}}$ denotes a player with a winning **Markröv** strategy (using only the round number and most recent move of opponent)
- $\uparrow_{k\text{-mark}}$ denotes a player with a winning k-**Markröv** strategy (using only the round number and k most recent moves of opponent)

Theorem

Assume $k \geq 2$. $\mathscr{F} \uparrow_{k\text{-mark}} Cov_{\mathscr{C}\mathscr{F}}(X) \Leftrightarrow \mathscr{F} \uparrow_{2\text{-mark}} Cov_{\mathscr{C}\mathscr{F}}(X)$

Theorem

For X second-countable, $\mathscr{F} \uparrow_{\text{mark}} Cov_{\mathscr{C}\mathscr{F}}(X) \Leftrightarrow \mathscr{F} \uparrow Cov_{\mathscr{C}\mathscr{F}}(X)$

$Cov_{\mathscr{C}\mathscr{F}}(X)$ characterizations

Here are a couple properties between σ-compact and Menger:

- Alster
- Hurewicz

An example of a Menger space which doesn't yield a Markov strategy for $\mathscr{F}$ in the Menger game is $\omega_1^\dagger$.

($\kappa^\dagger = \kappa \cup \{\infty\}$ is the one-point "Lindelöfication" of discrete κ.)

Theorem

$\mathscr{F} \not\uparrow_{mark} Cov_{\mathscr{C}\mathscr{F}}(\omega_1^\dagger)$ *but* $\mathscr{F} \uparrow_{2\text{-}mark} Cov_{\mathscr{C}\mathscr{F}}(\omega_1^\dagger)$

What about $Cov_{\mathscr{C}\mathscr{F}}(\kappa^\dagger)$?

- The direct proof of $\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(\omega_1^\dagger)$ uses injective functions $f_\alpha : \alpha \to \omega$ for each $\alpha < \omega_1$ such that for $\alpha < \beta$:

  $$|\{\gamma < \alpha : f_\alpha(\gamma) \neq f_\beta(\gamma)\}| < \omega$$

  (Proof in Kunen's set theory text, used for construction of an Aronszajn tree)
- Would like to extend this idea for $\kappa > \omega_1$ to show $\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(\kappa^\dagger)$...

$Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J)$

Game

The **strict filling game** $Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J)$ on an ideal J proceeds as follows:

- Round 0: player $\mathscr{M}$ chooses $M_0 \in \langle J \rangle$, the σ-completion of J (closure under countable unions)
- Round 0: player $\mathscr{N}$ chooses $N_0 \in J$.
- Round $n+1$: player $\mathscr{M}$ chooses M_{n+1} where $M_n \subsetneq M_{n+1} \in \langle J \rangle$
- Round $n+1$: player $\mathscr{N}$ replies with $N_{n+1} \in J$.

Player $\mathscr{N}$ wins the game if $\bigcup_{n<\omega} N_n \supseteq \bigcup_{n<\omega} M_n$.

- The sets in $\langle J \rangle$ and J are referred to as meager and nowhere-dense sets, respectively.
  - For any topological space, the set of nowhere dense sets J forms an ideal.
  - For every ideal J, there is a topological space where J is the set of nowhere dense sets.
- This game was defined and studied by Marion Scheepers. Here's some facts.

Proposition

$$\mathscr{N} \uparrow Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J)$$

Theorem

$$\mathscr{N} \uparrow_{tact} Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J) \Leftrightarrow J = \langle J \rangle$$

- $\uparrow_{\text{tact}}$ denotes a player with a winning **tactical** strategy (using only the most recent move of opponent)
- $\uparrow_{k\text{-tact}}$ denotes a player with a winning k-**tactical** strategy (using only the k most recent moves of opponent)

Theorem

*Assume $cf(\langle J\rangle) = \omega_1$. Let $J_X = \{N \cap X : N \in J\}$.
$\mathscr{N} \uparrow_{k\text{-tact}} Fill^{\subsetneq}_{\mathscr{M}\mathscr{N}}(J) \Leftrightarrow \mathscr{N} \uparrow_{k\text{-tact}} Fill^{\subsetneq}_{\mathscr{M}\mathscr{N}}(J_X)$ for each $X \in \langle J\rangle \setminus J$*

Proof: $\Rightarrow$ is straight-forward.

Sketch of $\Leftarrow$: Let S_α for $\alpha < \omega_1$ enumerate a cofinal set of $\langle J\rangle$, with $\beta \leq \alpha \Rightarrow S_\beta \subseteq S_\alpha$. Assume the latest move by $\mathscr{M}$ is contained by S_α. There are two types of attacks that $\mathscr{N}$ must defeat.

1. $\mathscr{M}$'s attack may never go outside S_α, so $\mathscr{N}$ can cover according to the strategy for $\mathscr{N} \uparrow_{k\text{-tact}} Fill^{\subsetneq}_{\mathscr{M}\mathscr{N}}(S_\alpha)$.
2. $\mathscr{M}$'s attack may eventually exceed S_α, but by using tree arrangments $<_n$ of ω_1 of finite height approximating $<$, $\mathscr{N}$ can cover according to the *winning perfect information strategy* as though $\mathscr{M}$ had played sets S_β for $\beta \leq_n \alpha$ instead.

Corollary

If $|\bigcup J| \leq \omega_1$ and $|M| \leq \omega$ for $M \in \langle J \rangle$, then $\mathscr{N} \uparrow_{2\text{-tact}} Fill^{\subsetneq}_{\mathscr{M}\mathscr{N}}(J)$.

Proof: Assume $\omega \in \langle J \rangle$ and assume the two latest moves of $\mathscr{M}$ are $M \subsetneq M' \subseteq \omega$. Let $n = \min(M' \setminus M)$, and have $\mathscr{N}$ cover $\{0, \ldots, n\}$. It follows that the generated n must be unbounded for any legal attack by $\mathscr{M}$, making it a winning 2-tactic for $Fill^{\subsetneq}_{\mathscr{M}\mathscr{N}}(J_\omega)$.

Apply the previous theorem to finish the result. $\square$

Countable Finite Game

Game

The special case of $Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J)$ where $J = [\kappa]^{<\omega}$ is the Countable-Finite game $Fill^{\subsetneq}_{\mathscr{C}\,\mathscr{F}}(\kappa)$.

Corollary

$$\mathscr{F} \uparrow_{2\text{-}tact} Fill^{\subsetneq}_{\mathscr{C}\,\mathscr{F}}(\omega_1)$$

So $\mathscr{F} \uparrow_{\text{2-tact}} Fill^{\subsetneq}_{\mathscr{C}\,\mathscr{F}}(\omega_1)$ and $\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\,\mathscr{F}}(\omega_1^\dagger)$. In addition, the basic goal of $\mathscr{F}$ in $Cov_{\mathscr{C}\,\mathscr{F}}(\omega_1^\dagger)$ is similar to the goal of $\mathscr{F}$ in $Fill^{\subsetneq}_{\mathscr{C}\,\mathscr{F}}(\omega_1)$: $\mathscr{F}$ can cover a co-countable neighborhood of ∞ in the initial round, and is trying to cover the countable remainder in the following rounds (most likely using finitely many singletons from $\mathscr{C}$'s covers).

- Question: why does $\mathscr{F}$ need the round number in $Cov_{\mathscr{C},\mathscr{F}}(\omega_1^\dagger)$ and not $Fill^{\subsetneq}_{\mathscr{C},\mathscr{F}}(\omega_1)$?

Proposition

$\mathscr{F} \uparrow_{k\text{-tact}} Cov_{\mathscr{C},\mathscr{F}}(X) \Leftrightarrow X$ is compact

Proof: If X isn't compact, and $\mathscr{C}$ constantly chooses an open cover $\mathcal{U}$ without a finite subcover for X throughout the entire game, then $\mathscr{F}$ only chooses k different finite subcollections of $\mathcal{U}$ by the game's end, which cannot cover X.

If X is compact, $\mathscr{F} \uparrow_{\text{tact}} Cov_{\mathscr{C},\mathscr{F}}(X)$ trivially. □

- Answer: $\mathscr{C}$ cannot choose a constant strategy in $Fill^{\subsetneq}_{\mathscr{C},\mathscr{F}}(\kappa)$, but $\mathscr{C}$ can in $Cov_{\mathscr{C},\mathscr{F}}(\kappa^\dagger)$.

This provides the motivation to change the rules of Scheeper's game to bring it more in line with the Menger game.

Game

The game $Fill^{\subseteq}_{\mathscr{M}\mathscr{N}}(J)$ is identical to $Fill^{\subsetneq}_{\mathscr{M}\mathscr{N}}(J)$, except that $\mathscr{M}$ may choose the same set in successive rounds.

Game

$Fill^{\subseteq}_{\mathscr{C}\mathscr{F}}(\kappa)$ is identical to $Fill^{\subseteq}_{\mathscr{M}\mathscr{N}}([\kappa]^{<\omega})$

It seems reasonable to ask if k-tactics in $Fill^{\subsetneq}_{\mathscr{M}\mathscr{N}}(J)$ correspond to k-Marköv strategies in $Fill^{\subseteq}_{\mathscr{M}\mathscr{N}}(J)$.

Theorem

$$\mathscr{N} \uparrow_{\textit{2-tact}} Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J) \Rightarrow \mathscr{N} \uparrow_{\textit{2-mark}} Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J)$$

Proof: Enumerate the sets in J as A_α for $\alpha < |J|$. For $M \in \langle J \rangle$ and $n < \omega$, let $M + 0 = M$ and $M + n + 1$ be the union of $M + n$ and the least A_α not contained in $M + n$.

Let σ be a winning 2-tactical strategy for N in $Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(\kappa)$, and assume $\sigma(M) \cup \sigma(M') \subseteq \sigma(M, M')$.

We define a 2-Markov strategy τ for F in $Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(\kappa)$ as follows:

$$\tau(M_0, 0) = \sigma(M_0)$$

$$\tau(M_n, M_{n+1}, n+1) = \begin{cases} \sigma(M_n, M_{n+1}) & \text{if } M_n \subsetneq M_{n+1} \\ \bigcup_{m \leq n} \sigma(M_n + m, M_{n+1} + m + 1) & \text{otherwise} \end{cases}$$

- (Essentially, if $\mathscr{M}$ tries to be tricky and not increase the size of her meager set, $\mathscr{N}$ can pretend she added a few extra nowhere dense sets based on the round number.)

Let $M_0 \subseteq M_1 \subseteq \dots$ be an attack by $\mathscr{M}$ against τ. There are two possible cases:

- Assume $M_n = M_N$ for all $n \geq N$.
  The collection produced by σ versus the attack

$$M_N + 0 \subsetneq M_N + 1 \subsetneq \dots$$

  must cover M_N as σ is a winning strategy.
  Let $x \in M_N$. If $x \in \sigma(M_N + 0)$, then x will be covered in round $N + 1$ by

$$\tau(M_N, M_N, N+1) \supseteq \sigma(M_N + 0, M_N + 1) \supseteq \sigma(M_N + 0)$$

  Otherwise, $x \in \sigma(M_N + n, M_N + n + 1)$, and x will be covered in round $N + n + 1$ by

$$\tau(M_N, M_N, N+n+1) \supseteq \sigma(M_N + n, M_N + n + 1)$$

- Otherwise we may find $0 < f(0) < f(1) < \dots$ such that $M_{f(n)} \subsetneq M_{f(n)+1} = M_{f(n+1)}$.
  Then the collection produced by σ versus the attack

$$M_{f(0)} \subsetneq M_{f(1)} \subsetneq M_{f(2)} \dots$$

  must cover $\bigcup_{n<\omega} M_n$ as σ is a winning strategy.
  Let $x \in \bigcup_{n<\omega} M_n$. If $x \in \sigma(M_{f(0)})$, then x will be covered by τ in round $f(0)+1$ by

$$\tau(M_{f(0)}, M_{f(0)+1}, f(0)+1) = \sigma(M_{f(0)}, M_{f(0)+1}) \supseteq \sigma(M_{f(0)})$$

  Otherwise, $x \in \sigma(M_{f(n)}, M_{f(n+1)})$, and x will be covered by τ in round $f(n)+1$ by

$$\tau(M_{f(n)}, M_{f(n)+1}, f(n)+1) = \sigma(M_{f(n)}, M_{f(n)+1}) = \sigma(M_{f(n)}, M_{f(n+1)})$$

Thus τ is a winning strategy.

But the converse need not hold.

Theorem

There is a free ideal J such that $\mathscr{N} \not\uparrow_{\text{2-tact}} Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J)$ but $\mathscr{N} \uparrow_{\text{2-mark}} Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J)$.

Proof: This counterexample was constructed by Scheepers for another purpose, but works for us as well. Assume $\mathbb{R}$ has the usual Euclidean topology.

Choose $A \subseteq \mathbb{R}$ such that $|A| = \omega$ and A is meager but not nowhere dense. Then choose $V \subseteq \mathbb{R}$ such that $|V| = 2^{\omega}$, V is meager, and V is disjoint from A. Assume $A = \{a_n : n < \omega\}$.

Certainly, if J is the collection of nowhere dense subsets of $A \cup V$, then $F \uparrow_{\text{2-mark}} Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J)$. In fact, since $A \cup V$ is meager, $F \uparrow_{\text{pre}} Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J)$ ($\mathscr{F}$ has a **predetermined strategy** using only the round number).

Let σ be a 2-tactical strategy for $\mathscr{N}$ in $Fill^{\subsetneq}_{\mathscr{M}\mathscr{N}}(J)$.

By Cor 28 of Scheepers' "Partition relation for partially ordered sets", for every partition $\{K_n : n < \omega\}$ of the comparable pairs in $[\mathcal{P}(V)]^2$ there is some $n' < \omega$ and sequence $C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq V$ where $\{C_m, C_{m+1}\} \in K_{n'}$ for all $m < \omega$.

Define K_n to be the collection of pairs of sets $\{B, C\}$ such that $B \subsetneq C$ and n is the least integer where $a_n \in A \setminus \sigma(A \cup B, A \cup C)$.

Then σ may be countered by the attack $A \cup C_0, A \cup C_1, \ldots,$ since $a_{n'} \in A \setminus \sigma(A \cup C_m, A \cup C_{m+1})$ for all $m < \omega$ and thus is never covered. $\square$

Question

$\mathscr{N} \uparrow_{\text{2-mark}} Fill^{\subsetneq}_{\mathscr{C}\mathscr{F}}(\kappa) \Rightarrow \mathscr{N} \uparrow_{\text{2-tact}} Fill^{\subsetneq}_{\mathscr{C}\mathscr{F}}(\kappa)$?

$S(\kappa,\omega,\omega)$

Scheepers introduced the sentence $S(\kappa,\omega,\omega)$ (or rather, a sentence equivalent to the one I use below).

Definition

For two functions f, g we say f is **almost compatible** with g ($f\|^* g$) if $|\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(x) \neq g(x)\}| < \omega$.

Definition

$S(\kappa,\omega,\omega)$ is shorthand for the sentence: there exist injective functions $f_A : A \to \omega$ for each $A \in [\kappa]^\omega$ such that $f_A\|^* f_B$ for all $A, B \in [\kappa]^\omega$.

Theorem

$S(\omega_1, \omega, \omega)$

Proof: Use Kunen's f_α mentioned earlier. □

Theorem

$\neg S(\kappa, \omega, \omega)$ *for* $\kappa > 2^\omega$

Proof: Let $A_\alpha = \{\alpha \cdot \omega + n : n < \omega\} \in [\kappa]^\omega$ and $f_{A_\alpha} : A_\alpha \to \omega$ be injective for $\alpha < \kappa$. Since there are $\kappa > |[\omega]^\omega|$ different A_α, there must be α, β where $\mathrm{ran}(f_{A_\alpha}) = \mathrm{ran}(f_{A_\beta})$. Then there is no way to define $f_{A_\alpha \cup A_\beta}$ so that it is almost compatible with both f_{A_α} and f_{A_β}. □

Corollary

$S(\omega_2, \omega, \omega) \Rightarrow \neg CH$

So what about the consistency of $\neg CH + S(\omega_2,\omega,\omega)$? It turns out that's fine (to be shown later).

Theorem

$S(\kappa,\omega,\omega) \Rightarrow \mathscr{F} \uparrow_{2\text{-}tact} Fill^{\subsetneq}_{\mathscr{C}\,\mathscr{F}}(\kappa)$

Proof: Due to Todorcevic. Let $f_A : A \to \omega$ for $A \in [\kappa]^\omega$ witness $S(\kappa,\omega,\omega)$, and let $g_A(\alpha)$ be the number of ordinals "skipped" by f_A below $f_A(\alpha)$, that is, $f_A(\alpha) - |\{\beta \in A : f_A(\beta) < f_A(\alpha)\}|$.

Note that for $A \subsetneq B$, $|\{\alpha \in A : g_A(\alpha) \leq g_B(\alpha)\}| < \omega$ since the difference in f_A and $f_B \restriction A$ is finite, and f_B has to map at least one more ordinal than f_A.

Let $\sigma(C,C') = \{\alpha \in C : g_C(\alpha) \leq g_{C'}(\alpha)\}$. If $C_0 \subsetneq C_1 \subsetneq \dots$ was an attack defeating σ, then let $\alpha \in C_N \setminus \bigcup_{n<\omega} \sigma(C_n, C_{n+1})$.

Observe that $g_{C_N}(\alpha) > g_{C_{N+1}}(\alpha) > g_{C_{N+2}}(\alpha) > \dots$, contradiction.

Theorem

$S(\kappa, \omega, \omega) \Rightarrow \mathscr{F} \uparrow_{2\text{-}mark} Fill^{\subseteq}_{\mathscr{C}\mathscr{F}}(\kappa)$

Proof: Corollary of the previous theorem. Alternatively, $\mathscr{F}$ can use the winning strategy

$$\sigma(C, C', n+1) = f_C^{-1}(\{0, \dots, n-1\}) \cup \{\alpha \in C : f_C(\alpha) \neq f_{C'}(\alpha)\}$$

Back to $Cov_{\mathscr{C}\mathscr{F}}(\kappa^\dagger)$

While a proof $\mathscr{F} \uparrow_{\text{2-mark}} Fill^{\subseteq}_{\mathscr{C}\mathscr{F}}(\kappa) \Rightarrow \mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(\kappa^\dagger)$ has eluded me, the techniques used previously are very useful for dealing with $Cov_{\mathscr{C}\mathscr{F}}(\kappa^\dagger)$ directly.

It will be useful to define a sufficient property for $\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(X)$, which I've called almost-σ-(relatively compact).

Definition

Let $\mathcal{U}$ be a cover of X. We say $C \subseteq X$ is $\mathcal{U}$-compact if there exists a finite subcover of $\mathcal{U}$ which covers C.

We say X is almost-σ-(relatively compact) if there exist functions $r_{\mathcal{V}} : X \to \omega$ for each open cover $\mathcal{V}$ of X such that both of the following sets are $\mathcal{V}$-compact for all open covers $\mathcal{U}$, $\mathcal{V}$ and $n < \omega$:

$$c(\mathcal{V}, n) = \{x \in X : r_{\mathcal{V}}(x) \leq n\}$$

$$p(\mathcal{U}, \mathcal{V}) = \{x \in X : 0 < r_{\mathcal{U}}(x) < r_{\mathcal{V}}(x)\}$$

Proposition

X σ-(relatively compact) $\Rightarrow$ X almost-σ-(relatively compact)

Theorem

If X is almost-σ-(relatively compact), then $\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(X)$.

Proof: Let $\sigma(\mathcal{U}_0, 0)$ cover $c(\mathcal{U}_0, 0)$, and let $\sigma(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1)$ cover both $c(\mathcal{U}_{n+1}, n+1)$ and $p(\mathcal{U}_n, \mathcal{U}_{n+1})$. If $\mathcal{U}_0, \mathcal{U}_1, \dots$ is any play by C, then for each $x \in X$, we note that one of the following must occur:

- $r_{\mathcal{U}_0}(x) = 0$ and thus $x \in c(\mathcal{U}_0, 0)$.
- $r_{\mathcal{U}_0}(x) = N + 1$ for some $N \geq 0$ and:
  - For all $n \leq N$,

$$n + 1 < r_{\mathcal{U}_{n+1}}(x) \leq N + 1$$

and thus $x \in c(\mathcal{U}_{N+1}, r_{\mathcal{U}_{n+1}}(x)) = c(\mathcal{U}_{N+1}, N + 1)$.

- $r_{\mathcal{U}_0}(x) = N + 1$ for some $N \geq 0$ and: (cont.)
  - For some $n \leq N$,

    $$r_{\mathcal{U}_{n+1}}(x) < n + 1 \leq r_{\mathcal{U}_n}(x) < N + 1$$

    and thus $x \in c(\mathcal{U}_{n+1}, r_{\mathcal{U}_{n+1}}(x)) \subseteq c(\mathcal{U}_{n+1}, n + 1)$.
  - For some $n \leq N$,

    $$n + 1 \leq r_{\mathcal{U}_n}(x) < N + 1 < r_{\mathcal{U}_{n+1}}(x)$$

    and thus $x \in p(\mathcal{U}_n, \mathcal{U}_{n+1})$ □

Corollary

X almost-σ-(relatively compact) $\Rightarrow$ X Menger

Question

$\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C},\mathscr{F}}(X) \Rightarrow X$ almost-σ-(relatively compact)? (Or can I slightly adjust the definition to get this result?)

Theorem

If $S(\kappa, \omega, \omega)$, then $\kappa^\dagger$ is almost-σ-(relatively compact).

Proof: Take the injective funcions $f_A : A \to \omega$ witnessing $S(\kappa, \omega, \omega)$. For each cover $\mathcal{V}$ of $\kappa^\dagger$ let $A(\mathcal{V})$ define a set such that $\kappa^\dagger \setminus A(\mathcal{V})$ is in a refinement of $\mathcal{V}$.
Then $r_\mathcal{V}$ defined by

$$r_\mathcal{V}(x) = \begin{cases} 0 & x \in \kappa^\dagger \setminus A(\mathcal{V}) \\ f_{A(\mathcal{V})}(x) + 1 & x \in A(\mathcal{V}) \end{cases}$$

witnesses the property as $c(\mathcal{V}, 0)$ is contained in a single open set in $\mathcal{V}$, $c(\mathcal{V}, n+1)$ is a singleton or empty set, and

$$p(\mathcal{U}, \mathcal{V}) = \{\alpha \in A(\mathcal{U}) \cap A(\mathcal{V}) : f_{A(\mathcal{U})}(\alpha) < f_{A(\mathcal{V})}(\alpha)\}$$

is finite.

Corollary

If $S(\kappa, \omega, \omega)$, then $\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(\kappa^\dagger)$

This result becomes more interesting if we can show $S(\kappa, \omega, \omega)$ is consistent for $\kappa > \omega_1$.

Definition

A finite partial function p from A to B has a domain which is a finite subset of A and a range which is a finite subset of B. Let the set of all finite partial functions from A to B be denoted by $Fn(A, B)$.

Definition

Let $Fn^2(\mathcal{A}, B) \subset Fn(\mathcal{A}, Fn(\bigcup \mathcal{A}, B))$ such that for each $p \in Fn^2(\mathcal{A}, B)$, $p(A) = p_A \in Fn(A, B)$.

Definition

For $\kappa > \omega_1$, let $\mathbb{P}_\kappa \subset Fn^2([\kappa]^\omega, \omega)$ be such that each p_A is injective, and give it the partial order $\leq$ defined by $q \leq p$ if and only if:

- $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$
- For each $A \in \mathrm{dom}(p)$, $q_A \supseteq p_A$
- For each $A, B \in \mathrm{dom}(p)$, if p_A and p_B are not defined for some $\alpha \in A \cap B$, but q_A is, then q_B is also defined for α and $q_A(\alpha) = q_B(x)$. That is, for $\alpha \in A \cap B$

$$\alpha \in \mathrm{dom}(q_A) \setminus (\mathrm{dom}(p_A) \cup \mathrm{dom}(p_B))$$

$$\Downarrow$$

$$\alpha \in \mathrm{dom}(q_B) \text{ and } q_A(x) = q_B(x)$$

Lemma

$\mathbb{P}_\kappa$ has property K (and thus is c.c.c.). That is, let $P \subseteq \mathbb{P}_\kappa$ be uncountable: there is an uncountable $Q \subseteq P$ such that points in Q are pairwise compatible.

Proof: If $|\{\mathrm{dom}(p) : p \in P\}| > \omega$, we will use the Δ-system lemma to find an uncountable $P' \subseteq P$ such that for $p, q \in P'$, $\mathrm{dom}(p) \cap \mathrm{dom}(q) = \mathcal{R}$. Otherwise, we may fix an uncountable $P' \subseteq P$ such that for $p, q \in P'$, $\mathrm{dom}(p) = \mathrm{dom}(q) = \mathcal{R}$.

Similarly, for each $A \in \mathcal{R}$ we may find that $|\{\mathrm{dom}(p_A) : p \in P'\}| > \omega$, and we can use the Δ-system lemma to find an uncountable $P'' \subseteq P'$ where $\mathrm{dom}(p_A) \cap \mathrm{dom}(q_A) = A'$ for all $p, q \in P''$, or otherwise we may find $P'' \subseteq P'$ where $\mathrm{dom}(p_A) = \mathrm{dom}(q_A) = A'$ for all $p, q \in P''$.

Finally, for each $A \in \mathcal{R}$ and $\alpha \in A'$, we may find $n_{A,\alpha}$ such that there are uncountable $p \in P''$ with $p_A(\alpha) = n_{A,\alpha}$, and thus we may choose $Q \subseteq P''$ to be an uncountable collection such that for $p, q \in Q$, $p_A = q_A$ for $A \in \mathcal{R}$.

Then it is easily verified that $p \cup q \in \mathbb{P}_\kappa$ and $p \cup q \leq p, q$ for all $p, q \in Q$. $\square$

Since $\mathbb{P}_\kappa$ is c.c.c.:

Corollary

Any forcing using a $\mathbb{P}_\kappa$-generic filter preserves cardinals and cofinalities.

Corollary

If $cf(\kappa) > \omega$, any forcing using a $\mathbb{P}_\kappa$-generic filter results in $2^\omega \leq \kappa$.

Proposition

For $A \in [\kappa]^\omega$ and $\alpha \in A$, the sets

$$D_A = \{p \in \mathbb{P}_\kappa : A \in dom(p)\}$$

$$D_{A,\alpha} = \{p \in \mathbb{P}_\kappa : A \in dom(p), \alpha \in dom(p_A)\}$$

are dense in $\mathbb{P}_\kappa$.

Theorem

If $cf(\kappa) > \omega$, $S(\kappa, \omega, \omega) + (\kappa = 2^\omega)$ is consistent with ZFC.

Proof: We adapt a forcing argument due to Scheepers (which used a slightly different poset). Let M be a countable transitive submodel of ZFC. Consider the c.c.c. poset $\mathbb{P}_\kappa$ realized in the model M. Let G be a $\mathbb{P}_\kappa$-generic filter over M.

We now work in the smallest model $M[G]$ extending M and containing G.

For each $A \in [\kappa]^\omega$, note $[\kappa]^\omega \cap M$ is cofinal in $[\kappa]^\omega$, so let $A' \supseteq A$ be in $[\kappa]^\omega \cap M$ and let $f_A = \bigcup_{p \in G \cap D_{A'}} p_{A'} \upharpoonright A$. Since G is a $\mathbb{P}_\kappa$-generic filter over M, it is easily verified (considering the dense sets $D_{A,\alpha}$) that f_A is an injective function from A into ω.

In addition, for $A, B \in [\kappa]^\omega \cap M$, let $p \in G \cap D_A \cap D_B$. For all $q \leq p$ it follows that

$$\{\alpha \in \mathrm{dom}(q_A) \cap \mathrm{dom}(q_B) : q_A(\alpha) \neq q_B(\alpha)\} \subseteq \mathrm{dom}(p_A) \cup \mathrm{dom}(p_B)$$

Thus $|\{\alpha \in A \cap B : f_A(\alpha) \neq f_B(\alpha)\}| < \omega$ and $f_A \|^* f_B$ for $A, B \in [\kappa]^\omega \cap M$, and it's immediate that $f_A \|^* f_B$ for $A, B \in [\kappa]^\omega$ as well.

The f_A witness $S(\kappa, \omega, \omega)$. Since $\kappa \geq 2^\omega$ and $S(\kappa, \omega, \omega)$ is a contradiction for $\kappa > 2^\omega$, we know $\kappa = 2^\omega$.

Corollary

For all κ, $\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(\kappa^\dagger)$ is consistent with ZFC.

Question

Is $\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(\omega_2^\dagger)$ a theorem of ZFC?