TACTIC-PROXIMAL COMPACT SPACES ARE STRONG EBERLEIN COMPACT

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Abstract. The author and G. Gruenhage previously showed that J. Bell's proximal game may be used to characterize Corson compactness in compact Hausdorff spaces. Using limited information strategies, the proximal game may also be used to characterize the strong Eberlein compactness property. In doing so, a purely topological characterization of the proximal game is introduced, and several existing results on the proximal game are given analogues considering limited information strategies.

Two papers published in 2014 introduced the *proximal uniform space game* $Bell^{\mathrm{uni}}_{D,P}(X)$ due to Jocelyn Bell. If X is a topological space, and there exists a uniform structure inducing its topology which gives the first player in this game has a winning stategy, then X is said to be a *proximal* space. Bell used this game as a tool in [1] for investigating uniform box products, and the author showed with Gary Gruenhage in [2] that this game characterizes Corson compactness amongst compact Hausdorff spaces, answering a question of Peter Nyikos in [6].

All spaces in this paper are assumed to be $T_{3\frac{1}{2}}$, so that they have a uniform structure inducing the topology on the space. Unlike many game-theoretic topological properties, the game $Bell^{\mathrm{uni}}_{D,P}(X)$ for which the proximal property was defined by is not itself a topological game. However, by considering entourages of the universal uniformity inducing the topology of a space, this original uniform space game may be easily modified to the purely topological games $Bell^{\rightarrow}_{D,P}(X)$, $Bell^{\leftrightarrow}_{D,P}(X)$.

The aim of this paper is to use this topological interpretation of the proximal game to give a new game-theoretic characterization of the strong Eberlein compactness property. Strong Eberlein compacts are Corson compacts, and therefore proximal compact spaces; in fact, it will be shown that strong Eberlein compacts are exactly the compact spaces for which the first player has a *tactical* winning strategy for the proximal game, a strategy which relies on only the most recent move of the opponent.

1. Topologizing $Bell^{\mathrm{UNI}}_{D,P}(X)$

We refer to [2] for definitions, notation, and basic theorems on uniform spaces and the proximal game $Bell^{\mathrm{uni}}_{D,P}(X)$ (denoted as $Prox_{D,P}(X)$ in that paper). In particular recall that:

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Definition 1.1. $\mathscr{P} \uparrow G$ denotes that the player $\mathscr{P}$ has a winning strategy in the ω-length game G.

Game 1.2. Let $Bell^{\text{uni}}_{D,P}(X, \mathbb{D})$ denote the *proximal uniform space game* with players $\mathscr{D}$, $\mathscr{P}$ which proceeds as follows for a space X with uniformity $\mathbb{D}$. In round 0, $\mathscr{D}$ chooses an entourage $D_0 \in \mathbb{D}$, followed by $\mathscr{P}$ choosing a point $p_0 \in X$. In round $n+1$, $\mathscr{D}$ chooses an entourage $D_{n+1} \in \mathbb{D}$, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in D_n[p_n]$.

$\mathscr{D}$ wins in the case that either $\langle p_0, p_1, \dots \rangle$ converges in X, or $\bigcap_{n<\omega} D_n[p_n] = \emptyset$. $\mathscr{P}$ wins otherwise.

Definition 1.3. A uniformizable space X is *proximal* in the case that there exists a uniformity $\mathbb{D}$ for X such that $\mathscr{D} \uparrow Bell^{\text{uni}}_{D,P}(X, \mathbb{D})$.

As it turns out, the search for such a uniformity is trivial.

Definition 1.4. The *universal uniformity* for a uniformizable topology is the union of all uniformities which induce the given topology.

Theorem 1.5 ([7])**.** *The universal uniformity is itself a uniformity compatibile with its given topology.*

Definition 1.6. For a uniformizable space X, a *universal entourage* D is a entourage of the universal uniformity.

Theorem 1.7 ([7])**.** *For every uniformizable space, if D is a neighborhood of the diagonal Δ such that there exist neighborhoods D_n of Δ with $D \supseteq D_0$ and $D_n \supseteq D_{n+1} \circ D_{n+1}$, then D is a universal entourage.*

Definition 1.8. For every entourage D and $n < \omega$, let $\frac{1}{2^n}D$ denote an entourage such that $\frac{1}{1}D = D$ and $\frac{1}{2^{n+1}}D \circ \frac{1}{2^{n+1}}D \subseteq \frac{1}{2^n}D$.

Definition 1.9. An *open symmetric entourage* D is a entourage which is open in the product topology induced by the uniformity and where $D = D^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in D\}$.

Theorem 1.10. *For every entourage D, there exists an open symmetric entourage $U \subseteq D$.*

Due to this theorem, we will simply use the word *entourage* to refer to open symmetric universal entourages. Note that if D is an entourage, then $D[x] = \{y : \langle x, y\rangle \in D\}$ is an open neighborhood of x. One may consider $D[x]$ to be an entourage-"ball" about x, generalizing the notion of an ϵ-ball given by a metric structure.

In the case that the space is paracompact, entourages are even more easily found.

Theorem 1.11 ([7])**.** *Every open neighborhood of the diagonal is a universal entourage for paracompact uniformizable spaces.*

1.1. **Using universal entourages to characterize the proximal property.**

The natural adaptation of the original uniform space game $Bell^{\text{uni}}_{D,P}(X)$ to a topological game requires the use of the universal uniformity on X.

Game 1.12. Let $Bell^{\rightarrow,\star}_{D,P}(X)$ denote the *hard Bell convergence game* with players $\mathscr{D}$, $\mathscr{P}$ which proceeds as follows for a uniformizable space X. In round 0, $\mathscr{D}$ chooses an entourage D_0, followed by $\mathscr{P}$ choosing a point $p_0 \in X$. In round $n+1$, $\mathscr{D}$ chooses an entourage D_{n+1}, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in D_n[p_n]$.

$\mathscr{D}$ wins in the case that either $\langle p_0, p_1, \dots \rangle$ converges in X, or $\bigcap_{n<\omega} D_n[p_n] = \emptyset$. $\mathscr{P}$ wins otherwise.

This game is considered "hard" due to the requirement that $\mathscr{D}$ keep track of the history of the game to ensure that successive moves refine previous moves. This record-keeping may be eliminated by requiring that $\mathscr{P}$ respect all moves made by $\mathscr{D}$ rather than only the most recent move.

Game 1.13. Let $Bell^{\rightarrow}_{D,P}(X)$ denote the *Bell convergence game* with players $\mathscr{D}$, $\mathscr{P}$ which proceeds analogously to $Bell^{\rightarrow,\star}_{D,P}(X)$, except for the following. Let $E_n = \bigcap_{m\leq n} D_n$, where D_n is the entourage played by $\mathscr{D}$ in round n. Then $\mathscr{P}$ must ensure that $p_{n+1} \in E_n[p_n]$, and $\mathscr{D}$ wins when either $\langle p_0, p_1, \dots \rangle$ converges in X or $\bigcap_{n<\omega} E_n[p_n] = \emptyset$.

These games are all essentially equivalent with respect to perfect information for $\mathscr{D}$.

Theorem 1.14. *$\mathscr{D} \uparrow Bell^{\rightarrow,\star}_{D,P}(X)$ if and only if $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$ if and only if X is proximal.*

Proof. If $\mathscr{D} \uparrow Bell^{\rightarrow,\star}_{D,P}(X)$, then we immediately see that $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$. If σ is a winning strategy for $\mathscr{D}$ in $Bell^{\rightarrow}_{D,P}(X)$, then τ defined by $\tau(s) = \bigcap_{t\leq s} \sigma(t)$ is easily seen to be a winning strategy for $\mathscr{D}$ in $Bell^{\rightarrow,\star}_{D,P}(X)$.

If $\mathscr{D} \uparrow Bell^{\rightarrow,\star}_{D,P}(X)$, then $\mathscr{D} \uparrow Bell^{\mathrm{uni}}_{D,P}(X)$ for the universal uniformity, showing X is proximal. Finally, if X is proximal, then there exists a winning strategy σ for $Bell^{\mathrm{uni}}_{D,P}(X)$ for a uniformity inducing the topology on X. Then a winning strategy for $\mathscr{D}$ in $Bell^{\rightarrow,\star}_{D,P}(X)$ may be constructed by converting every entourage in this uniformity into a smaller open symmetric universal entourage. $\square$

The secondary winning condition in $Bell^{\rightarrow}_{D,P}(X)$ allows for a space to be incomplete: $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(\mathbb{Q})$ by playing $E_{1/2^n} = \{\langle x, y\rangle : d(x,y) < \frac{1}{2^n}\}$ in each round. This forces $\mathscr{P}$'s sequence to be Cauchy, and thus it either converges to a rational, or the sets $E_{1/2^n}[x_n]$ will have empty intersection (where the irrational point of convergence would be). Uniformly locally compact spaces (and in particular, compact spaces) lack such holes, so it will be convenient to eliminate this technicality when it is irrelevant.

Definition 1.15. Let $Bell^{\rightarrow}_{D,P}(X)$ denote the *absolute Bell convergence game* which proceeds analogously to $Bell^{\rightarrow}_{D,P}(X)$, except that $\mathscr{D}$ must always ensure that $\langle p_0, p_1, \dots \rangle$ converges in X in order to win.

Definition 1.16. A uniformizable space X is *absolutely proximal* if $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$.

As was shown in [2]:

Definition 1.17. A uniformizable space X is *uniformly locally compact* if there exists an entourage D such that $\overline{D[x]}$ is compact for all x.

Theorem 1.18. *If X is a uniformly locally compact space, then $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$ if and only if $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$.*

So absolutely proximal compacts are simply proximal compacts (and therefore Corson compacts).

2. Limited information analogues

First recall the definitions of the following limited information strategies.

Definition 2.1. A *k-tactical strategy* or *k-tactic* is a strategy which considers only the most recent move of the opponent.

If $\mathscr{P}$ has a winning k-tactic for a game G, we write $\mathscr{P} \underset{k\text{-tact}}{\uparrow} G$. If omitted, assume $k = 1$.

Definition 2.2. A *k-Markov strategy* or *k-mark* is a strategy which considers only the round number and most recent move of the opponent.

If $\mathscr{P}$ has a winning k-mark for a game G, we write $\mathscr{P} \underset{k\text{-mark}}{\uparrow} G$. If omitted, assume $k = 1$.

Limited information strategies may be used to strengthen game-theoretic topological properties.

Definition 2.3. A uniformizable space X is *(absolutely) tactic-proximal* if $\mathscr{D} \underset{\text{tact}}{\uparrow} Bell^{\rightarrow}_{D,P}(X)$ ($\mathscr{D} \underset{\text{tact}}{\uparrow} Bell^{\rightarrow}_{D,P}(X)$).

Definition 2.4. A uniformizable space X is *(absolutely) Markov-proximal* if $\mathscr{D} \underset{\text{mark}}{\uparrow} Bell^{\rightarrow}_{D,P}(X)$ ($\mathscr{D} \underset{\text{mark}}{\uparrow} Bell^{\rightarrow}_{D,P}(X)$).

As in Theorem 1.18, the "absolutely" is redundant in the above definitions when X is uniformly locally compact, so the absolute Bell convergence game will be used for convenience in the context of compact spaces.

Some results of Bell may be generalized to hold for such limited information strategies. The proofs of the following propositions are straight forward.

Proposition 2.5. *Let X be a uniformizable space and H be a closed subset of X. If $k < \omega$, then*

- $\mathscr{D} \underset{k\text{-tact}}{\uparrow} Bell^{\rightarrow}_{D,P}(X) \Rightarrow \mathscr{D} \underset{k\text{-tact}}{\uparrow} Bell^{\rightarrow}_{D,P}(H)$
- $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell^{\rightarrow}_{D,P}(X) \Rightarrow \mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell^{\rightarrow}_{D,P}(H)$
- $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X) \Rightarrow \mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(H)$

Proposition 2.6. *Let X be a uniformizable space and H be a closed subset of X. If $k < \omega$, then*

- $\mathscr{D} \underset{k\text{-}tact}{\uparrow} Bell^{\rightarrow}_{D,P}(X) \Rightarrow \mathscr{D} \underset{k\text{-}tact}{\uparrow} Bell^{\rightarrow}_{D,P}(H)$
- $\mathscr{D} \underset{k\text{-}mark}{\uparrow} Bell^{\rightarrow}_{D,P}(X) \Rightarrow \mathscr{D} \underset{k\text{-}mark}{\uparrow} Bell^{\rightarrow}_{D,P}(H)$
- $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X) \Rightarrow \mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(H)$

Less obvious is the following.

Definition 2.7. Let X_α be a topological space for $\alpha < \kappa$ and $z \in \prod_{\alpha<\kappa} X_\alpha$. The Σ-*product* $\sum^z_{\alpha<\kappa} X_\alpha$ with base point z is given by

$$\sum^z_{\alpha<\kappa} X_\alpha = \left\{ x \in \prod_{\alpha<\kappa} X_\alpha : |\{\alpha < \kappa : x(\alpha) \neq z(\alpha)\}| \leq \omega \right\}$$

When $X_\alpha = X$ and $z(\alpha) = 0$ for all $\alpha < \kappa$, then we write $\Sigma X^\kappa = \sum^z_{\alpha<\kappa} X$.

Theorem 2.8 ([1]). *If X_α is proximal for all $\alpha < \kappa$, then $\sum^z_{\alpha<\kappa} X_\alpha$ is proximal for any base point z.*

2.1. **$\Sigma^\star$- and σ-Products.** We may consider smaller subspaces of X^κ than given by ΣX^κ.

Definition 2.9. Let X be a metrizable space with compatible metric d, and let $z \in X^\kappa$. The $\Sigma^\star$-*product* $\sum\!\!\star^z_{\alpha<\kappa} X$ with base point z is given by

$$\sum\!\!\star^z_{\alpha<\kappa} X = \left\{ x \in \prod_{\alpha<\kappa} X : n < \omega \Rightarrow \left| \left\{ \alpha < \kappa : d(x(\alpha), z(\alpha)) > \frac{1}{2^n} \right\} \right| < \omega \right\}$$

When $z(\alpha) = 0$ for all $\alpha < \kappa$, then we write $\Sigma^\star X^\kappa = \sum\!\!\star^z_{\alpha<\kappa} X$.

Definition 2.10. Let X_α be a topological space for $\alpha < \kappa$ and $z \in \prod_{\alpha<\kappa} X_\alpha$. The σ-*product* $\boldsymbol{\sigma}^z_{\alpha<\kappa} X_\alpha$ with base point z is given by

$$\underset{\alpha<\kappa}{\overset{z}{\boldsymbol{\sigma}}} X_\alpha = \left\{ x \in \prod_{\alpha<\kappa} X_\alpha : |\{\alpha < \kappa : x(\alpha) \neq z(\alpha)\}| < \omega \right\}$$

When $X_\alpha = X$ and $z(\alpha) = 0$ for all $\alpha < \kappa$, then we write $\sigma X^\kappa = \boldsymbol{\sigma}^z_{\alpha<\kappa} X$.

Of course,

$$\sigma X^\kappa \subseteq \Sigma^\star X^\kappa \subseteq \Sigma X^\kappa \subseteq X^\kappa$$

for metrizable X, and

$$\underset{\alpha<\kappa}{\overset{z}{\boldsymbol{\sigma}}} X_\alpha \subseteq \sum^z_{\alpha<\kappa} X_\alpha \subseteq \prod_{\alpha<\kappa} X_\alpha$$

for all spaces X_α and base points z.

Just as Σ products preserve winning perfect-information strategies, we will show that these product subspaces preserve certain winning limited-information strategies.

Definition 2.11. For a metric space $\langle X, d\rangle$, let $E_\epsilon = \{\langle x, y\rangle : d(x, y) < \epsilon\}$. Note that E_ϵ is an (open symmetric) entrouage on X.

Proposition 2.12. *For any metrizable space X, $\mathscr{D} \underset{0\text{-}mark}{\uparrow} Bell^{\rightarrow}_{D,P}(X)$. For any completely metrizable space X, $\mathscr{D} \underset{0\text{-}mark}{\uparrow} Bell^{\rightarrow}_{D,P}(X)$.*

Proof. Essentially shown by Bell in her paper: in either case, $\mathscr{D}$ chooses $E_{1/2^n}$ during round n, forcing legal attacks by $\mathscr{P}$ to be Cauchy. $\square$

As an aside, this next proposition may be proved similarly using $E_{d(x,y)/2}$.

Proposition 2.13. *For any metrizable space X, $\mathscr{D} \underset{2\text{-}tact}{\uparrow} Bell_{\overrightarrow{D},P}(X)$. For any completely metrizable space X, $\mathscr{D} \underset{2\text{-}tact}{\uparrow} Bell_{\overrightarrow{D},P}(X)$.*

We will exploit the last move of the opponent to obtain winning Markov strategies under $\Sigma^\star$ products.

Proposition 2.14. *If X_α is a uniformizable space for $\alpha < \kappa$, and D_α is an entourage of X_α for $\alpha \in F \in [\kappa]^{<\omega}$, then*

$$P(\{\langle \alpha, D_\alpha\rangle : \alpha \in F\}) = \left\{\langle x, y\rangle \in \left(\prod_{\alpha<\kappa} X_\alpha\right)^2 : \alpha \in F \Rightarrow \langle x(\alpha), y(\alpha)\rangle \in D_\alpha\right\}$$

is an entourage of $\prod_{\alpha<\kappa} X_\alpha$.

Theorem 2.15. *For any metrizable space X and $z \in X^\kappa$, $\mathscr{D} \underset{mark}{\uparrow} Bell_{\overrightarrow{D},P}\left(\sum\nolimits^{\star z}_{\alpha<\kappa} X\right)$.*

Proof. For $x \in \sum\nolimits^{\star z}_{\alpha<\kappa} X$, let

$$\mathrm{supp}_n(x) = \left\{\alpha < \kappa : d(x(\alpha), z(\alpha)) > \frac{1}{2^n}\right\} \in \kappa^{<\omega}$$

where d is a metric compatible with the topology on X.

Define a strategy τ for $\mathscr{D}$ by

$$\tau(\emptyset, 0) = \left(\sum^{\star z}_{\alpha<\kappa} X\right)^2$$

$$\tau(\langle p\rangle, n+1) = \left(\sum^{\star z}_{\alpha<\kappa} X\right)^2 \cap P(\{\langle \alpha, E_{1/2^{n+1}}\rangle : \alpha \in \mathrm{supp}_n(p)\})$$

Let $a : \omega \to \sum\nolimits^{\star z}_{\alpha<\kappa} X$ be a legal attack by $\mathscr{P}$ against τ, so

$$\langle a(n+1), a(n+2)\rangle \in \bigcap_{m\leq n} \tau(\langle a(m)\rangle, m+1)$$

$$= \left(\sum^{\star z}_{\alpha<\kappa} X\right)^2 \cap \bigcap_{m\leq n} P(\{\langle \alpha, E_{1/2^{m+1}}\rangle : \alpha \in \mathrm{supp}_m(a(m))\})$$

and let $\alpha < \kappa$.

If $d(a(m+1)(\alpha), z(\alpha)) \leq \frac{1}{2^m}$ for all but finite $m < \omega$, then $\lim_{n<\omega} a(n)(\alpha) = z(\alpha)$. Otherwise, we have $\alpha \in \mathrm{supp}_m(a(m))$ for infinitely many $m < \omega$.

If $\bigcap_{n<\omega} \tau(\langle a(n)\rangle, n+1)[a(n+1)] = \emptyset$, then $\mathscr{D}$ has already won. Otherwise there exists a nonstrictly increasing unbounded function $f \in \omega^\omega$ with $\bigcap_{n<\omega} E_{1/2^{f(n)+1}}[a(n+$

$1)(\alpha)] \neq \emptyset$. Since the diameter of these sets approaches 0, the intersection is a singleton x; furthermore, the α coordinate of a must then converge to x. We conclude that $\mathscr{D} \underset{\text{mark}}{\uparrow} Bell_{D,P}^{\rightarrow}\left(\sum\nolimits^{\star z}_{\alpha<\kappa} X\right)$. $\square$

A similar result holds for tactics and σ-products.

Theorem 2.16. *Let X_α be a uniformizable space for $\alpha < \kappa$ and $z \in \prod_{\alpha<\kappa} X_\alpha$. If $\mathscr{D} \underset{tact}{\uparrow} Bell_{D,P}^{\rightarrow}(X_\alpha)$ for all $\alpha < \kappa$, then $\mathscr{D} \underset{tact}{\uparrow} Bell_{D,P}^{\rightarrow}\left(\sigma^{z}_{\alpha<\kappa} X_\alpha\right)$.*

Proof. For $x \in \sigma^{z}_{\alpha<\kappa} X_\alpha$, let

$$\operatorname{supp}(x) = \{\alpha < \kappa : x(\alpha) \neq z(\alpha)\} \in \kappa^{<\omega}$$

and let τ_α be a winning strategy for $\mathscr{D}$ in $Bell_{D,P}^{\rightarrow}(X_\alpha)$ for $\alpha < \kappa$.

Define a strategy τ for $\mathscr{D}$ by

$$\tau(\emptyset) = \left(\sigma^{z}_{\alpha<\kappa} X\right)^2$$

$$\tau(\langle p\rangle) = \left(\sigma^{z}_{\alpha<\kappa} X\right)^2 \cap P(\{\langle \alpha, \tau_\alpha(p(\alpha))\rangle : \alpha \in \operatorname{supp}(p)\})$$

Let $a : \omega \to \sigma^{z}_{\alpha<\kappa} X$ be a legal attack by $\mathscr{P}$ against τ, so

$$\langle a(n+1), a(n+2)\rangle \in \bigcap_{m\leq n} \tau(\langle a(m)\rangle)$$

$$= \left(\sigma^{z}_{\alpha<\kappa} X\right)^2 \cap \bigcap_{m\leq n} P(\{\langle \alpha, \tau_\alpha(a(m)(\alpha))\rangle : \alpha \in \operatorname{supp}(a(m))\})$$

and let $\alpha < \kappa$.

If $a(m+1)(\alpha) = z(\alpha)$ for all but finite $m < \omega$, then $\lim_{n<\omega} a(n)(\alpha) = z(\alpha)$. Otherwise, we have $\alpha \in \operatorname{supp}(a(m))$ for infinitely many $m < \omega$.

TODO: wrap this up unless it's wrong (it's not needed to get the main result) $\square$

2.2. **Eberlein and Strong Eberlein Compacts.**
We recall some convenient definitions for a few strengthenings of compactness.

Definition 2.17. A *Corson compact* space is a compact space which may be embedded in $\Sigma\mathbb{R}^\kappa$.

Definition 2.18. An *Eberlein compact* space is a compact space which may be embedded in $\Sigma^\star\mathbb{R}^\kappa$.

Definition 2.19. A *strong Eberlein compact* space is a compact space which may be embedded in $\sigma 2^\kappa$.

Obviously, strong Eberlein compacts are Eberlein compact, and Eberlein compacts are Corson compact. Nyikos observed in [5] that as the Σ-product of proximal spaces are proximal and the closed subspaces of proximal spaces are proximal:

Corollary 2.20. *Corson compacts are proximal.*

By the previous section we may now obtain analogues of this observation.

Corollary 2.21. *Eberlein compacts are Markov-proximal. Strong Eberlein compacts are tactic-proximal.*

The author showed with Gary Gruenhage in [2] that Nyikos's observation can actually be reversed. The result required an earlier game characterization of Corson compactness due to Gruenhage.

Game 2.22. Let $Gru_{O,P}^{\rightarrow}(X,S)$ denote *Gruenhage's convergence game* with players $\mathscr{O}$, $\mathscr{P}$ which proceeds as follows for a space X and a set S. In round n, $\mathscr{O}$ chooses an open set $U_n \subseteq X^2$ containing S, followed by $\mathscr{P}$ choosing a point $p_n \in \bigcap_{m \leq n} U_m$.

$\mathscr{O}$ wins in the case that $\langle p_0, p_1, \dots \rangle$ converges to the set S; that is, any open set containing S contains all but finite p_n. $\mathscr{P}$ wins otherwise.

When $S = \{x\}$, we abuse notation and write simply $Gru_{O,P}^{\rightarrow}(X,x)$.

Corollary 2.23 ([3],[4])**.** *A compact space X is Corson compact if and only if $\mathscr{O} \uparrow Gru_{O,P}^{\rightarrow}\left(X^2,\Delta\right)$. A compact space X is Eberlein compact if and only if $\mathscr{O} \underset{mark}{\uparrow} Gru_{O,P}^{\rightarrow}\left(X^2,\Delta\right)$.*

In particular note that Corson compactness is characterized by both $\mathscr{O} \uparrow Gru_{O,P}^{\rightarrow}\left(X^2,\Delta\right)$ and $\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$. We will see that these games are not equivalent with respect to limited information strategies, but the following results are useful.

Theorem 2.24. *Let $k < \omega$. For all $x \in X$:*

- $\mathscr{D} \underset{2k\text{-}tact}{\uparrow} Bell_{D,P}^{\rightarrow}(X) \Rightarrow \mathscr{O} \underset{k\text{-}tact}{\uparrow} Gru_{O,P}^{\rightarrow}(X,x)$
- $\mathscr{D} \underset{2k\text{-}mark}{\uparrow} Bell_{D,P}^{\rightarrow}(X) \Rightarrow \mathscr{O} \underset{k\text{-}mark}{\uparrow} Gru_{O,P}^{\rightarrow}(X,x)$
- $\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(X) \Rightarrow \mathscr{O} \uparrow Gru_{O,P}^{\rightarrow}(X,x)$

Proof. The perfect-information result was proven in Bell's original paper [1]. We proceed by proving the Markov-information result, and note that the tactical-information result may be proven by simply dropping usage of the round number from the given proof.

Let σ be a winning $2k$-mark for $\mathscr{D}$ in $Bell_{D,P}^{\rightarrow}(X)$; without loss of generality, assume that $\sigma(t,n) \subseteq \sigma(s,m)$ whenever $n \geq m$ and s is a permutation of a subsequence of t. We define the k-mark τ for $\mathscr{O}$ in $Gru_{O,P}^{\rightarrow}(X,x)$ such that

$$\tau(t,n) = \sigma\Big(\langle t(0), \dots, x, t(|t|-1), x\rangle, 2n+1\Big)[x]$$

Let p be a legal attack against τ. Consider the attack q against the winning $2k$-mark σ such that $q(2n) = x$ and $q(2n+1) = p(n)$. Let D_n be the entourage played according to σ in round n versus q, and $E_n = \bigcap_{m \leq n} D_m$.

□

The presence of a Cantor set determines the success of $\mathscr{D}$'s tactial strategies in $Bell_{D,P}^{\rightarrow}(X)$.

Lemma 2.25. *Tactic-proximal spaces cannot contain a copy of the Cantor set.*

Proof. The result will follow once we show $\mathscr{D} \underset{\text{tact}}{\not\uparrow} Bell_{D,P}^{\rightarrow}(2^\omega)$. Let σ be a tactic for $\mathscr{D}$ in $Bell_{D,P}^{\rightarrow}(2^\omega)$ and let $D_k = \{\langle f, g\rangle : f \upharpoonright k = g \upharpoonright k\}$. Since $\{D_k : k < \omega\}$ is a base for the universal uniformity on 2^ω (namely, all open neighborhoods of the diagonal), we may fix $k(f) < \omega$ for each $f \in 2^\omega$ such that $D_{k(f)} \subseteq \sigma(\langle f\rangle)$.

Then there exists $k < \omega$ such that $\{f : k(f) = k\}$ is uncountable, and therefore there exist distinct $f, g \in 2^\omega$ such that $k = k(f) = k(g)$ and $f \upharpoonright k = g \upharpoonright k$. Then $p : \omega \to 2^\omega$ defined by $p(2n) = f$ and $p(2n+1) = g$ is an attack against σ which obviously doesn't converge. This attack is legal since $f \in D_k[g] \subseteq \sigma(\langle g\rangle)[g]$ and $g \in D_k[f] \subseteq \sigma(\langle f\rangle)[f]$, so σ is not a winning tactic. □

Lemma 2.26. *Every non-scattered Corson compact space contains a homeomorphic copy of the Cantor set.*

Proof. Every non-scattered space contains a closed subspace without isolated points. Let X be such a subspace, and assume that this Corson compact is embedded in $\Sigma\mathbb{R}^\kappa$. Let $B_{\alpha,\epsilon}(x) = \{y : d(x(\alpha), y(\alpha)) < \epsilon\}$. For each $x \in X$ and $n < \omega$, let $\beta(x, n) < \kappa$ be defined such that $\operatorname{supp}(x) = \{\beta(x, n) : n < \omega\}$.

Choose an arbitrary $x_\emptyset \in X$ and $\epsilon_0 > 0$, and and let $A_0 = \emptyset$.

Suppose then that for some $n < \omega$, $x_s \in X$ is defined for all $s \in 2^n$, and $\epsilon_n > 0$ and $A_n \in [\kappa]^{<\omega}$ are defined. Since each x_s is not isolated in X, let U_s be the open set

$$U_s = X \cap \bigcap_{\alpha \in A_{|s|}} B_{\alpha,\epsilon_{|s|}}(x_s)$$

and choose $x_{s^\frown\langle 0\rangle}, x_{s^\frown\langle 1\rangle} \in U_s$ distinct. Then let $\alpha_s < \kappa$ such that $x_{s^\frown\langle 0\rangle}(\alpha_s) \neq x_{s^\frown\langle 1\rangle}(\alpha_s)$. Let

$$A_{n+1} = \{\alpha_s : s \in 2^{\leq n}\} \cup \{\beta(x_s, i) : s \in 2^{\leq n}, i \leq n\}$$

Then choose $0 < \epsilon_{n+1} < \frac{1}{2}\epsilon_n$ such that

$$B_{\alpha_s,\epsilon_{n+1}}(x_s{}^\frown\langle 0\rangle) \cap B_{\alpha_s,\epsilon_{n+1}}(x_s{}^\frown\langle 1\rangle) = \emptyset$$

and

$$\overline{\bigcap_{\alpha \in A_{n+1}} B_{\alpha,\epsilon_{n+1}}(x_s{}^\frown\langle 0\rangle)} \cup \overline{\bigcap_{\alpha \in A_{n+1}} B_{\alpha,\epsilon_{n+1}}(x_s{}^\frown\langle 1\rangle)} \subseteq \bigcap_{\alpha \in A_n} B_{\alpha,\epsilon_n}(x_s)$$

for all $s \in 2^n$.

Let $x_f = \lim_{n<\omega} x_{f\upharpoonright n} \in X$ for each $f \in 2^\omega$. We claim $C = \{x_f : f \in 2^\omega\}$ is a copy of the Cantor set. This will follow if we can show that $\{U_s : s \in 2^{<\omega}\}$ is a base for C, since it has the structure of the Cantor tree.

Consider x_f for some $f \in 2^\omega$, and a subbasic open ball $B_{\alpha,\epsilon}(x_f)$. Observe that $x_f \in \bigcap_{n<\omega} U_{f\upharpoonright n}$ since $x_{f\upharpoonright n} \in U_{f\upharpoonright m}$ for all $m < n < \omega$.

If $\alpha \in \{\beta(x_s, n) : s \in 2^{<\omega}, n < \omega\}$, choose $k < \omega$ with $\alpha \in A_k$. Then choose $l < \omega$ such that $\epsilon_l < \epsilon$. Then $U_{f\upharpoonright(l+k)} \subseteq B_{\alpha,\epsilon}(x_f)$.

Otherwise, $x_s(\alpha) = 0$ for all $s \in 2^{<\omega}$, so $x_g(\alpha) = 0$ for all $g \in 2^\omega$ and therefore $C \subseteq B_{\alpha,\epsilon}(x_f)$. □

A new game characterization of strong Eberlein compactness follows from the above and an earlier characterization by Gruenhage.

Theorem 2.27. *For compact spaces X, X is strong Eberlein compact if and only if X is scattered and $\mathscr{O} \uparrow Gru_{O,P}^{\rightarrow}(X,x)$ for all $x \in X$.*

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