

# APPLICATIONS OF LIMITED INFORMATION STRATEGIES IN Menger's GAME

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ABSTRACT. I need an abstract

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## 1. THE Menger PROPERTY AND GAME

Recall the following definition.

**Definition 1.1.** A space  $X$  is Menger if for every sequence  $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$  of open covers of  $X$  there exists a sequence  $\langle \mathcal{F}_0, \mathcal{F}_1, \dots \rangle$  such that  $\mathcal{F}_n \subseteq \mathcal{U}_n$ ,  $|\mathcal{F}_n| < \omega$ , and  $\bigcup_{n < \omega} \mathcal{F}_n$  is a cover of  $X$ .

Note that many authors refer to this property as  $S_{fin}(\mathcal{O}, \mathcal{O})$ , where  $\mathcal{O}$  is the collection of open covers of  $X$ , and  $S_{fin}(A, B)$  denotes the selection property such that for each sequence in  $A^\omega$ , there are finite subsets of each entry for which the union of these subsets belongs in  $B$ .

**Proposition 1.2.**  $X$  is  $\sigma$ -compact  $\Rightarrow X$  is Menger  $\Rightarrow X$  is Lindelöf.

None of these implications may be reversed; the irrationals are a simple example of a Lindelöf space which is not Menger, and we'll see several examples of Menger spaces which are not  $\sigma$ -compact.

It will be convenient to consider subsets of  $X$  rather than subsets of the open covers.

**Definition 1.3.** For each cover  $\mathcal{U}$  of  $X$ ,  $S \subseteq X$  is  $\mathcal{U}$ -finite if there exists a finite subcollection of  $\mathcal{U}$  which covers  $S$ .

Of course, a compact space is  $\mathcal{U}$ -finite for all open covers  $\mathcal{U}$ .

**Proposition 1.4.** A space  $X$  is Menger if and only if for every sequence  $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$  of open covers of  $X$  there exists a sequence  $\langle F_0, F_1, \dots \rangle$  such that  $F_n \subseteq X$ ,  $F_n$  is  $\mathcal{U}_n$ -finite, and  $X = \bigcup_{n < \omega} F_n$ .

This is the characterization we will use in this paper. A game version of the Menger property is also often considered.

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**Game 1.5.** Let  $Men_{C,F}(X)$  denote the *Menger game* with players  $\mathcal{C}$ ,  $\mathcal{F}$ . In round  $n$ ,  $\mathcal{C}$  chooses an open cover  $\mathcal{U}_n$ , followed by  $\mathcal{F}$  choosing a  $\mathcal{U}_n$ -finite subset  $F_n$  of  $X$ .

$\mathcal{F}$  wins the game if  $X = \bigcup_{n < \omega} F_n$ , and  $\mathcal{C}$  wins otherwise.

As with the Menger property, authors usually characterize this game using finite subsets  $\mathcal{F}_n$  of  $\mathcal{U}_n$  instead (and often refer to it as  $G(\mathcal{O}, \mathcal{O})$  as with the selection property above). This is obviously equivalent in the case of perfect information, and is also equivalent in the case of limited information, provided  $\mathcal{F}$  knows  $\mathcal{U}_n$  during round  $n$ . However, we make this change as we will investigate 0-Markov strategies which consider no moves of the opponent, and instead rely only on the current round number.

This game may be used to characterize the Menger property.

**Definition 1.6.** If  $\mathcal{A}$  has a winning strategy for a game  $G$  (which defeats every possible counterattack by her opponent), then we write  $\mathcal{A} \uparrow G$ .

**Theorem 1.7** (Hurewicz [1]). *A space  $X$  is Menger if and only if  $\mathcal{C} \not\uparrow Men_{C,F}(X)$ .*

## 2. LIMITED INFORMATION STRATEGIES

Recall the following definitions.

**Definition 2.1.** A  $k$ -tactical strategy for a game  $G$  with moveset  $M$  is a function  $\sigma : M^{\leq k} \rightarrow M$ ; intuitively, it is a strategy which only considers the previous  $k$  moves of the opponent. If a winning  $k$ -tactical strategy exists for  $\mathcal{P}$  in the game  $G$ , then we write  $\mathcal{P} \underset{k\text{-tact}}{\uparrow} G$ .

**Definition 2.2.** A  $k$ -Markov strategy for a game  $G$  with moveset  $M$  is a function  $\sigma : M^{\leq k} \times \omega \rightarrow M$ ; intuitively, it is a strategy which only considers the previous  $k$  moves of the opponent and the round number. If a winning  $k$ -Markov strategy exists for  $\mathcal{P}$  in the game  $G$ , then we write  $\mathcal{P} \underset{k\text{-mark}}{\uparrow} G$ .

We will call  $k$ -tactical strategies “ $k$ -tactics” and  $k$ -Markov strategies “ $k$ -marks”. If the  $k$  is omitted then it is assumed that  $k = 1$ . In addition, note that some authors refer to tactics as *stationary strategies*.

Tactics will be of interest in a game discussed later; proving the following is an easy exercise.

**Proposition 2.3.**  *$X$  is compact if and only if  $\mathcal{F} \underset{tact}{\uparrow} Men_{C,F}(X)$  if and only if  $\mathcal{F} \underset{k+1\text{-tact}}{\uparrow} Men_{C,F}(X)$  for some  $k < \omega$ .*

Essentially, because  $\mathcal{C}$  may repeat the same finite sequence of open covers,  $\mathcal{F}$  needs to be seeded with knowledge of the round number to prevent being trapped in a loop.

If  $\mathcal{F}$ ’s memory of  $\mathcal{C}$ ’s past moves is bounded, then there is no need to consider more than the two most recent moves. The intuitive reason is that  $\mathcal{C}$  could simply play the same cover repeatedly until  $\mathcal{F}$ ’s memory is exhausted, in which case  $\mathcal{F}$  would only ever see the change from one cover to another.

**Theorem 2.4.** For each  $k < \omega$ ,  $F \stackrel{(k+2)\text{-mark}}{\uparrow} Men_{C,F}(X)$  if and only if  $F \stackrel{2\text{-mark}}{\uparrow} Men_{C,F}(X)$ .

*Proof.* Let  $\sigma$  be a winning  $(k+2)$ -mark. We define the 2-mark  $\tau$  as follows:

$$\begin{aligned}\tau(\langle \mathcal{U} \rangle, 0) &= \bigcup_{m < k+2} \sigma(\underbrace{\langle \mathcal{U}, \dots, \mathcal{U} \rangle}_{m+1}, m) \\ \tau(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) &= \bigcup_{m < k+2} \sigma(\underbrace{\langle \mathcal{U}, \dots, \mathcal{U} \rangle}_{k+1-m}, \underbrace{\langle \mathcal{V}, \dots, \mathcal{V} \rangle}_{m+1}, (n+1)(k+2) + m)\end{aligned}$$

Let  $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$  be an attack by  $\mathcal{C}$  against  $\tau$ . Then consider the attack

$$\langle \underbrace{\mathcal{U}_0, \dots, \mathcal{U}_0}_{k+2}, \underbrace{\mathcal{U}_1, \dots, \mathcal{U}_1}_{k+2}, \dots \rangle$$

by  $\mathcal{C}$  against  $\sigma$ . Since  $\sigma$  is a winning  $(k+2)$ -mark,

$$\begin{aligned}X &= \bigcup_{m < k+2} \sigma(\underbrace{\langle \mathcal{U}_0, \dots, \mathcal{U}_0 \rangle}_{m+1}, m) \cup \bigcup_{n < \omega} \bigcup_{m < k+2} \sigma(\underbrace{\langle \mathcal{U}_n, \dots, \mathcal{U}_n \rangle}_{k+1-m}, \underbrace{\langle \mathcal{U}_{n+1}, \dots, \mathcal{U}_{n+1} \rangle}_{m+1}, (n+1)(k+2) + m) \\ &= \tau(\langle \mathcal{U}_0 \rangle, 0) \cup \bigcup_{n < \omega} \tau(\langle \mathcal{U}_n, \mathcal{U}_{n+1} \rangle, n+1)\end{aligned}$$

Thus  $\tau$  is a winning 2-mark.  $\square$

A natural question arises: is there an example of a space  $X$  for which  $\mathcal{F} \stackrel{2\text{-mark}}{\uparrow} Men_{C,F}(X)$  but  $\mathcal{F} \not\stackrel{\text{mark}}{\uparrow} Men_{C,F}(X)$ ? We quickly see that perhaps the simplest example of a Lindelöf non- $\sigma$ -compact space has this property.

**Definition 2.5.** For any cardinal  $\kappa$ , let  $\kappa^\dagger = \kappa \cup \{\infty\}$  denote the *one-point Lindelöf-ification* of discrete  $\kappa$ , where points in  $\kappa$  are isolated, and the neighborhoods of  $\infty$  are the co-countable sets containing it.

**Proposition 2.6.**  $\mathcal{F} \not\stackrel{\text{mark}}{\uparrow} Men_{C,F}(\omega_1^\dagger)$ ; in fact,  $\omega_1^\dagger$  is not  $\sigma$ -compact.

The greatest advantage of a strategy which has knowledge of two or more previous moves of the opponent, versus only knowledge of the most recent move, is the ability to react to changes from one round to the next. It's this ability to react that will give  $\mathcal{F}$  her winning 2-Markov strategy in the Menger game on  $\omega_1^\dagger$ . This may be proven directly; however, we will postpone giving a proof so that we may take advantage of an equivalent game characterization of this property.

### 3. SCHEEPERS' FILLING GAMES

We now turn to a related game whose  $k$ -tactics were studied by Marion Scheepers in [3].

**Game 3.1.** Let  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$  denote *Scheepers' strict union filling game* with two players  $\mathcal{C}$ ,  $\mathcal{F}$ . In round 0,  $\mathcal{C}$  chooses  $C_0 \in [\kappa]^{\leq \omega}$ , followed by  $\mathcal{F}$  choosing  $F_0 \in [\kappa]^{< \omega}$ . In round  $n+1$ ,  $\mathcal{C}$  chooses  $C_{n+1} \in [\kappa]^{\leq \omega}$  such that  $C_{n+1} \supset C_n$ , followed by  $\mathcal{F}$  choosing  $F_{n+1} \in [\kappa]^{< \omega}$ .

$\mathcal{F}$  wins the game if  $\bigcup_{n < \omega} F_n \supseteq \bigcup_{n < \omega} C_n$ ; otherwise,  $\mathcal{C}$  wins.

In  $Men_{C,F}(\kappa^\dagger)$ ,  $\mathcal{C}$  essentially chooses a countable set to not include in her neighborhood of  $\infty$ , followed by  $\mathcal{F}$  choosing a finite subset of this complement to cover during each round. Thus,  $\mathcal{F}$  need only be concerned with the *intersection* of the countable sets chosen by  $\mathcal{C}$  in  $Men_{C,F}(\kappa^\dagger)$ , rather than the union as in  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$ .

Another difference between these games: Scheepers required that  $\mathcal{C}$  always choose strictly growing countable sets. If the goal is to study tactics, then  $\mathcal{C}$  cannot be allowed to trap  $\mathcal{F}$  in a loop by repeating the same moves. But by eliminating this requirement, the study can then turn to Markov strategies, bringing the game further in line with the Menger game played upon  $\kappa^\dagger$ .

We introduce a few games to make the relationship between Scheepers's  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$  and  $Men_{C,F}(\kappa^\dagger)$  more precise.

**Game 3.2.** Let  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$  denote the *union filling game* which proceeds analogously to  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$ , except that  $\mathcal{C}$ 's restriction in round  $n+1$  is reduced to  $C_{n+1} \supseteq C_n$ .

**Game 3.3.** Let  $Sch_{C,F}^{1,\subseteq}(\kappa)$  denote the *initial filling game* which proceeds analogously to  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$ , except that  $\mathcal{F}$  wins whenever  $\bigcup_{n < \omega} F_n \supseteq C_0$ .

**Game 3.4.** Let  $Sch_{C,F}^\cap(\kappa)$  denote the *intersection filling game* which proceeds analogously to  $Sch_{C,F}^{1,\subseteq}(\kappa)$ , except that  $\mathcal{C}$  may choose any  $C_n \in [\kappa]^{\leq \omega}$  each round, and  $\mathcal{F}$  wins whenever  $\bigcup_{n < \omega} F_n \supseteq \bigcap_{n < \omega} C_n$ .

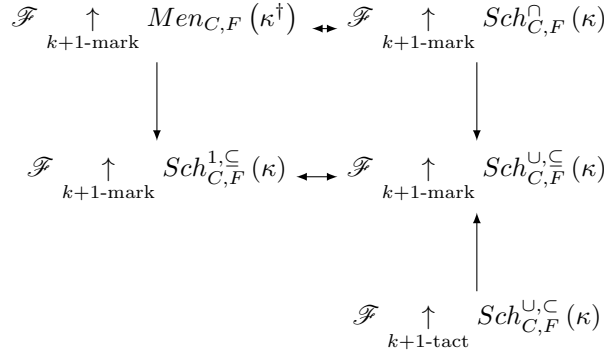


FIGURE 1. Diagram of Scheepers/Menger game implications

**Theorem 3.5.** For any cardinal  $\kappa \geq \omega$  and integer  $k < \omega$ , Figure 1 holds.

*Proof.*  $\mathcal{F} \uparrow_{k+1\text{-mark}} Men_{C,F}(\kappa^\dagger) \Rightarrow \mathcal{F} \uparrow_{k+1\text{-mark}} Sch_{C,F}^\cap(\kappa)$ : Let  $\sigma$  be a winning  $k$ -mark for  $\mathcal{F}$  in  $Men_{C,F}(\kappa^\dagger)$ . Let  $\mathcal{U}(C)$  (resp.  $\mathcal{U}(s)$ ) convert each countable subset  $C$  of  $\kappa$  (resp. finite sequence  $s$  of such subsets) into the open cover  $[C]^1 \cup \{\kappa^\dagger \setminus C\}$  (resp. finite sequence of such open covers). Then  $\tau$  defined by

$$\tau(s^\frown \langle C \rangle, n) = C \cap \sigma(\mathcal{U}(s^\frown \langle C \rangle), n)$$

is a winning  $(k+1)$ -mark for  $\mathcal{F}$  in  $Sch_{C,F}^\cap(\kappa)$ .

$\mathcal{F} \uparrow_{k+1\text{-mark}} Sch_{C,F}^\cap(\kappa) \Rightarrow \mathcal{F} \uparrow_{k+1\text{-mark}} Men_{C,F}(\kappa^\dagger)$ : Let  $\sigma$  be a winning  $k$ -mark for  $\mathcal{F}$  in  $Sch_{C,F}^\cap(\kappa)$ . Let  $C(\mathcal{U})$  (resp.  $C(s)$ ) convert each open cover  $\mathcal{U}$  of  $\kappa^\dagger$  (resp. finite sequence  $s$  of such covers) into a countable set  $C$  which is the complement of some neighborhood of  $\infty$  in  $\mathcal{U}$  (resp. finite sequence of such countable sets). Then  $\tau$  defined by

$$\tau(s^\frown \langle \mathcal{U} \rangle, n) = (\kappa^\dagger \setminus C(\mathcal{U})) \cup \sigma(C(s^\frown \langle \mathcal{U} \rangle), n)$$

is a winning  $(k+1)$ -mark for  $\mathcal{F}$  in  $Men_{C,F}(\kappa^\dagger)$ .

$\mathcal{F} \uparrow_{k+1\text{-mark}} Sch_{C,F}^\cap(\kappa) \Rightarrow \mathcal{F} \uparrow_{k+1\text{-mark}} Sch_{C,F}^{1,\subseteq}(\kappa)$ : Let  $\sigma$  be a winning  $k$ -mark for  $\mathcal{F}$  in  $Sch_{C,F}^\cap(\kappa)$ .  $\sigma$  is also a winning  $k$ -mark for  $\mathcal{F}$  in  $Sch_{C,F}^{1,\subseteq}(\kappa)$ .

$\mathcal{F} \uparrow_{k+1\text{-mark}} Sch_{C,F}^{1,\subseteq}(\kappa) \Rightarrow \mathcal{F} \uparrow_{k+1\text{-mark}} Sch_{C,F}^{\cup,\subseteq}(\kappa)$ : Let  $\sigma$  be a winning  $k$ -mark for  $\mathcal{F}$  in  $Sch_{C,F}^{1,\subseteq}(\kappa)$ . For each finite sequence  $s$ , let  $t \preceq s$  mean  $t$  is a final subsequence of  $s$ . Then  $\tau$  defined by

$$\tau(s^\frown \langle C \rangle, n) = \bigcup_{t \preceq s, m \leq n} \sigma(t^\frown \langle C \rangle, m)$$

is a winning  $(k+1)$ -mark for  $\mathcal{F}$  in  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$ .

$\mathcal{F} \uparrow_{k+1\text{-mark}} Sch_{C,F}^{\cup,\subseteq}(\kappa) \Rightarrow \mathcal{F} \uparrow_{k+1\text{-mark}} Sch_{C,F}^{1,\subseteq}(\kappa)$ : Let  $\sigma$  be a winning  $k$ -mark for  $\mathcal{F}$  in  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$ .  $\sigma$  is also a winning  $(k+1)$ -mark for  $\mathcal{F}$  in  $Sch_{C,F}^{1,\subseteq}(\kappa)$ .

$\mathcal{F} \uparrow_{k+1\text{-tact}} Sch_{C,F}^{\cup,\subseteq}(\kappa) \Rightarrow \mathcal{F} \uparrow_{k+1\text{-mark}} Sch_{C,F}^{\cup,\subseteq}(\kappa)$ : Let  $\sigma$  be a winning  $k$ -tactic for  $\mathcal{F}$  in  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$ . For each countable subset  $C$  of  $\kappa$ , let  $C + n$  be the union of  $C$  with the  $n$  least ordinals in  $\kappa \setminus C$ . Then  $\tau$  defined by

$$\tau(\langle C_0, \dots, C_i \rangle, n) = \sigma(\langle C_0 + (n-i), \dots, C_i + n \rangle)$$

is a winning  $(k+1)$ -mark for  $\mathcal{F}$  in  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$ . □

While we have not shown a direct implication between the Menger game and Scheeper's original filling game, Scheepers introduced the statement  $S(\kappa)$  relating to the almost-compatibility of functions from countable subsets of  $\kappa$  into  $\omega$  which may be applied to both.

**Definition 3.6.** For two functions  $f, g$  we say  $f$  is  $\mu$ -almost compatible with  $g$  ( $f \parallel_\mu^* g$ ) if  $|\{x \in \text{dom}(f) \cap \text{dom}(g) : f(x) \neq g(x)\}| < \mu$ . If  $\mu = \omega$  then we say  $f, g$  are almost compatible ( $f \parallel^* g$ ).

**Definition 3.7.**  $S(\kappa)$  states that there exist functions  $f_A : A \rightarrow \omega$  for each  $A \in [\kappa]^{\leq \omega}$  such that  $|\{\alpha \in A : f_A(\alpha) \leq n\}| < \omega$  for all  $n < \omega$  and  $f_A \parallel^* f_B$  for all  $A, B \in [\kappa]^\omega$ .<sup>1</sup>

Scheepers went on to show that  $S(\kappa)$  implies  $\mathcal{F} \xrightarrow[2\text{-tact}]{\uparrow} Sch_{C,F}^{\cup, \subseteq}(\kappa)$ . This proof, along with the following facts, give us inspiration for finding a winning 2-Markov strategy in the Menger game played on  $\kappa^\dagger$ .

**Theorem 3.8.**  $S(\omega_1)$  and  $\kappa > 2^\omega \Rightarrow \neg S(\kappa)$  are theorems of ZFC.  $S(2^\omega)$  is a theorem of ZFC + CH and consistent with ZFC +  $\neg CH$ .

*Proof.* For  $S(\omega_1)$ , look at pg. 70 of [2]; this of course implies  $S(2^\omega)$  under CH.  $\neg S((2^\omega)^+)$  is shown by a cardinality argument in [3]. The consistency result under ZFC +  $\neg CH$  is a lemma for the main theorem in [3].  $\square$

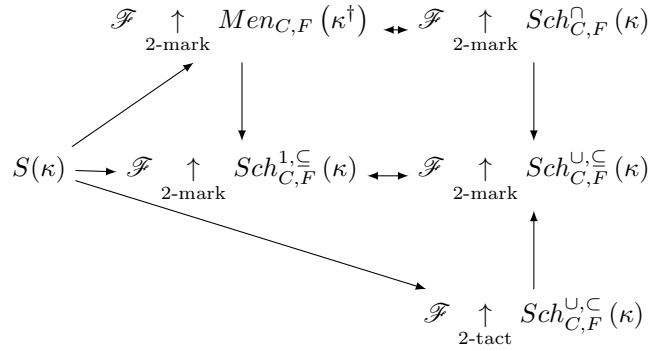


FIGURE 2. Diagram of Filling/Menger game implications with  $S(\kappa)$

**Theorem 3.9.**  $S(\kappa)$  implies the game-theoretic results in Figure 2.

*Proof.* Since  $S(\kappa) \Rightarrow \mathcal{F} \xrightarrow[2\text{-tact}]{\uparrow} Sch_{C,F}^{\cup, \subseteq}(\kappa)$  was a main result of [3], it is sufficient to show that  $S(\kappa) \Rightarrow \mathcal{F} \xrightarrow[2\text{-mark}]{\uparrow} Sch_{C,F}^\cap(\kappa)$ .

Let  $f_A$  for  $A \in [\kappa]^{\leq \omega}$  witness  $S(\kappa)$ . We define the 2-mark  $\sigma$  as follows:

$$\sigma(\langle A \rangle, 0) = \{\alpha \in A : f_A(\alpha) \leq 0\}$$

$$\sigma(\langle A, B \rangle, n+1) = \{\alpha \in A \cap B : f_B(\alpha) \leq n+1 \text{ or } f_A(\alpha) \neq f_B(\alpha)\}$$

For any attack  $\langle A_0, A_1, \dots \rangle$  by  $\mathcal{C}$  and  $\alpha \in \bigcap_{n < \omega} A_n$ , either  $f_{A_n}(\alpha)$  is constant for all  $n$ , or  $f_{A_n}(\alpha) \neq f_{A_{n+1}}(\alpha)$  for some  $n$ ; either way,  $\alpha$  is covered.  $\square$

**Corollary 3.10.**  $\mathcal{F} \xrightarrow[2\text{-mark}]{\uparrow} Men_{C,F}(\omega_1^\dagger)$ .

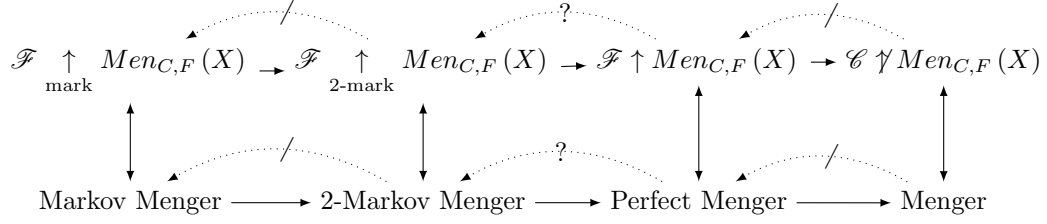


FIGURE 3. Diagram of covering properties related to the Menger game

#### 4. MENDER GAME DERIVED COVERING PROPERTIES

Limited information strategies for the Menger game naturally define a spectrum of covering properties, see Figure 3. However, we do not know if the middle two properties are actually distinct.

**Question 4.1.** Does there exist a space  $X$  such that  $\mathcal{F} \uparrow \text{Men}_{C,F}(X)$  but  $\mathcal{F} \nmid_{\text{2-mark}} \text{Men}_{C,F}(X)$ ?

Note that while it's consistent that  $\mathcal{F} \uparrow_{\text{2-mark}} \text{Men}_{C,F}((2^\omega)^\dagger)$ ,  $\kappa^\dagger$  for  $\kappa > 2^\omega$  is a candidate to answer the above question.

We are also interested in non-game-theoretic characterizations of these covering properties. It has been known for some time that for metrizable spaces, winning Menger spaces are exactly the  $\sigma$ -compact spaces, shown first by Telgarsky in [5] and later directly by Scheepers in [4].

In the interest of generality, we will first characterize the Markov Menger spaces without any separation axioms.

**Definition 4.2.** A subset  $Y$  of  $X$  is *relatively compact* to  $X$  if for every open cover of  $X$ , there exists a finite subcollection which covers  $Y$ .

For example, any bounded subset of Euclidean space is relatively compact whether it is closed or not. Actually, relative compactness can be thought of as an analogue of boundedness for regular spaces.

**Proposition 4.3.** *For regular spaces,  $Y$  is relatively compact to  $X$  if and only if  $\overline{Y}$  is compact in  $X$ .*<sup>2</sup>

*Proof.* For any space, any subset of a compact set is relatively compact.

Assume  $Y$  is relatively compact, let  $\mathcal{U}$  be an open cover of  $\overline{Y}$ , and define  $x \in V_x \subseteq \overline{V_x} \subseteq U_x \in \mathcal{U}$  for each  $x \in X$ . Then if we take a subcollection  $\mathcal{F} = \{V_{x_i} : i < n\}$

<sup>1</sup>This is equivalent to the original characterization given in [3]: there exist injections  $g_A : A \rightarrow \omega$  such that  $g_A \parallel^* g_B$  for all  $A, B \in [\kappa]^\omega$  and  $A \subset B$ .

<sup>2</sup>It should be noted that some authors define relative compactness in this way, but such a definition creates pathological implications for non-regular spaces. For example, the singleton containing the particular point of an infinite space with the particular point topology would not be relatively compact since its closure is not compact, even though it is finite.

covering  $Y$  by relative compactness, then  $\{U_{x_i} : i < n\}$  is a finite subcollection of  $\mathcal{U}$  covering  $\overline{Y}$ , showing compactness.  $\square$

We now begin the process of factoring out Scheeper's proof to reveal the limited information implications at work.

**Lemma 4.4.** *Let  $\sigma(\mathcal{U}, n)$  be a Markov strategy for  $F$  in  $\text{Men}_{C,F}(X)$ , and  $\mathfrak{C}$  collect all open covers of  $X$ . Then the set*

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \sigma(\mathcal{U}, n)$$

*is relatively compact to  $X$ . If  $\sigma$  is a winning Markov strategy, then  $\bigcup_{n < \omega} R_n = X$ .*

*Proof.* First, for every open cover  $\mathcal{U} \in \mathfrak{C}$ ,  $R_n \subseteq \sigma(\mathcal{U}, n)$  is covered by a finite subcollection of  $\mathcal{U}$ .

Suppose that  $x \notin R_n$  for any  $n < \omega$ . Then for each  $n$ , pick  $\mathcal{U}_n \in \mathfrak{C}$  such that  $x \notin \sigma(\mathcal{U}_n, n)$ . Then  $\mathcal{C}$  may counter  $\sigma$  with the attack  $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ .  $\square$

**Definition 4.5.** A  $\sigma$ -relatively-compact space is the countable union of relatively compact subsets.

**Corollary 4.6.** *The following are equivalent:*

- $X$  is  $\sigma$ -relatively-compact
- $\mathcal{F} \uparrow_{pre} \text{Men}_{C,F}(X)$
- $\mathcal{F} \uparrow_{mark} \text{Men}_{C,F}(X)$

*Proof.* If  $X = \bigcup_{n < \omega} R_n$  for  $R_n$  relatively compact, then  $\sigma(n) = R_n$  is a winning predetermined strategy, which yields a winning Markov strategy. The previous lemma finishes the proof.  $\square$

**Corollary 4.7.** *Let  $X$  be a regular space. The following are equivalent:*

- $X$  is  $\sigma$ -compact
- $X$  is  $\sigma$ -relatively-compact
- $\mathcal{F} \uparrow_{pre} \text{Men}_{C,F}(X)$
- $\mathcal{F} \uparrow_{mark} \text{Men}_{C,F}(X)$

For Lindelöf spaces, metrizability is characterized by regularity and second-countability, the latter of which was essentially used by Scheepers in this way:

**Lemma 4.8.** *Let  $X$  be a second-countable space.  $\mathcal{F} \uparrow \text{Men}_{C,F}(X)$  if and only if  $\mathcal{F} \uparrow_{mark} \text{Men}_{C,F}(X)$ .*

*Proof.* Let  $\sigma$  be a strategy for  $\mathcal{F}$ , and note that it's sufficient to consider playthroughs with only basic open covers.

So if  $\mathcal{U}_t$  is a basic open cover for  $t < s \in \omega^{<\omega}$ , and  $\mathcal{V}$  is any basic open cover, we may choose a finite subcollection  $\mathcal{F}(s, \mathcal{V})$  of  $\mathcal{V}$  such that

$$\sigma(\langle \mathcal{U}_{s \upharpoonright 1}, \dots, \mathcal{U}_s, \mathcal{V} \rangle) \subseteq \bigcup \mathcal{F}(s, \mathcal{V})$$



Note that there are only countably-many finite collections of basic open sets. Thus we may choose basic open covers  $\mathcal{U}_{s \smallfrown \langle n \rangle}$  for  $n < \omega$  such that for any basic open cover  $\mathcal{V}$ , there exists  $n < \omega$  where  $\mathcal{F}(s, \mathcal{V}) = \mathcal{F}(s, \mathcal{U}_{s \smallfrown \langle n \rangle})$ .

Let  $t : \omega \rightarrow \omega^{<\omega}$  be a bijection. We define the Marköv strategy  $\tau$  as follows:

$$\tau(\langle \mathcal{V} \rangle, n) = \bigcup \mathcal{F}(t(n), \mathcal{V})$$

Suppose there exists a counter-attack  $\langle \mathcal{V}_0, \mathcal{V}_1, \dots \rangle$  of basic open covers which defeats  $\tau$ . Then there exists  $f : \omega \rightarrow \omega$  such that, letting  $t(m_n) = f \upharpoonright n$ :

$$\begin{aligned} x &\notin \tau(\langle \mathcal{V}_{m_n} \rangle, m_n) \\ &= \bigcup \mathcal{F}(f \upharpoonright n, \mathcal{V}_{m_n}) \\ &= \bigcup \mathcal{F}(f \upharpoonright n, \mathcal{U}_{f \upharpoonright (n+1)}) \\ &\supseteq \sigma(\langle \mathcal{U}_{f \upharpoonright 1}, \dots, \mathcal{U}_{f \upharpoonright (n+1)} \rangle) \end{aligned}$$

Thus  $\langle \mathcal{U}_{f \upharpoonright 1}, \mathcal{U}_{f \upharpoonright 2}, \dots \rangle$  is a successful counter-attack by  $\mathcal{C}$  against the perfect information strategy  $\sigma$ .  $\square$

**Corollary 4.9.** *Let  $X$  be a second-countable space. The following are equivalent:*

- $X$  is  $\sigma$ -relatively-compact
- $F \uparrow_{pre} Men_{C,F}(X)$
- $F \uparrow_{mark} Men_{C,F}(X)$
- $F \uparrow Men_{C,F}(X)$

**Corollary 4.10.** *Let  $X$  be a metrizable space. The following are equivalent:*

- $X$  is  $\sigma$ -compact
- $X$  is  $\sigma$ -relatively-compact
- $F \uparrow_{pre} Men_{C,F}(X)$
- $F \uparrow_{mark} Men_{C,F}(X)$
- $F \uparrow Men_{C,F}(X)$

*Proof.* Each property implies Lindelöf, so  $X$  may be assumed to be regular and second-countable.  $\square$

## 5. ROBUSTLY LINDELÖF

To help describe  $\mathcal{F} \uparrow_{2\text{-mark}} Men_{C,F}(X)$  topologically, we introduce a subset variant of the Menger game and a related covering property.

**Game 5.1.** Let  $Men_{C,F}(X, Y)$  denote the *Menger subspace game* which proceeds analogously to the Menger game, except that  $\mathcal{F}$  wins whenever  $\bigcup_{n < \omega} \mathcal{F}_n$  is a cover for  $Y \subseteq X$ .

Note of course that  $Men_{C,F}(X, X) = Men_{C,F}(X)$ .

**Definition 5.2.** A subset  $Y$  of  $X$  is *relatively robustly Menger* if there exist functions  $r_{\mathcal{V}} : Y \rightarrow \omega$  for each open cover  $\mathcal{V}$  of  $X$  such that for all open covers  $\mathcal{U}, \mathcal{V}$  and numbers  $n < \omega$ , the following sets are  $\mathcal{V}$ -finite:

$$c(\mathcal{V}, n) = \{x \in Y : r_{\mathcal{V}}(x) \leq n\}$$

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{x \in Y : n < r_{\mathcal{U}}(x) < r_{\mathcal{V}}(x)\}$$

**Definition 5.3.** A space  $X$  is *robustly Menger* if it is relatively robustly Menger to itself.

**Proposition 5.4.** All  $\sigma$ -relatively-compact spaces are robustly Menger.

*Proof.* If  $X = \bigcup_{n < \omega} R_n$ , then for all  $\mathcal{U}$ , let  $r_{\mathcal{U}}(x)$  be the least  $n$  such that  $x \in R_n$ . Then  $c(\mathcal{V}, n) = \bigcup_{m \leq n} R_m$  and  $p(\mathcal{U}, \mathcal{V}) = \emptyset$ .  $\square$

**Theorem 5.5.** If  $Y \subseteq X$  is relatively robustly Menger, then  $\mathcal{F} \uparrow_{2\text{-mark}} \text{Men}_{C,F}(X, Y)$ .

*Proof.* We define the Markov strategy  $\sigma$  as follows. Let  $\sigma(\langle \mathcal{U} \rangle, 0) = c(\mathcal{U}, 0)$ , and let  $\sigma(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) = c(\mathcal{V}, n+1) \cup p(\mathcal{U}, \mathcal{V}, n+1)$ .

For any attack  $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$  by  $\mathcal{C}$  and  $x \in Y$ , one of the following must occur:

- $r_{\mathcal{U}_0}(x) = 0$  and thus  $x \in c(\mathcal{U}_0, 0) \subseteq \sigma(\langle \mathcal{U}_0 \rangle, 0)$ .
- $r_{\mathcal{U}_0}(x) = N+1$  for some  $N \geq 0$  and:
  - For all  $n \leq N$ ,

$$r_{\mathcal{U}_{n+1}}(x) \leq N+1$$

$$\text{and thus } x \in c(\mathcal{U}_{N+1}, N+1) \subseteq \sigma(\langle \mathcal{U}_N, \mathcal{U}_{N+1} \rangle, N+1).$$

- For some  $n \leq N$ ,

$$r_{\mathcal{U}_n}(x) \leq n$$

$$\text{and thus } x \in c(\mathcal{U}_{n+1}, n+1) \subseteq \sigma(\langle \mathcal{U}_n, \mathcal{U}_{n+1} \rangle, n+1).$$

- For some  $n \leq N$ ,

$$n < r_{\mathcal{U}_n}(x) \leq N+1 < r_{\mathcal{U}_{n+1}}(x)$$

$$\text{and thus } x \in p(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1) \subseteq \sigma(\langle \mathcal{U}_n, \mathcal{U}_{n+1} \rangle, n+1)$$

$\square$

**Theorem 5.6.**  $S(\kappa)$  implies  $\kappa^\dagger$  is robustly Menger, and thus  $\mathcal{F} \uparrow_{2\text{-mark}} \text{Men}_{C,F}(\kappa^\dagger)$ .

*Proof.* Let  $f_A$  for  $A \in [\kappa]^{<\omega}$  witness  $S(\kappa)$  and fix  $A(\mathcal{U}) \in [\kappa]^{<\omega}$  for each open cover  $\mathcal{U}$  such that  $\kappa^\dagger \setminus A(\mathcal{U})$  is contained in some element of  $\mathcal{U}$ . Then let  $r_{\mathcal{U}}(x) = 0$  for  $x \in \kappa^\dagger \setminus A(\mathcal{U})$ , and  $r_{\mathcal{U}}(\alpha) = f_{A(\mathcal{U})}(\alpha)$  for  $\alpha \in A(\mathcal{U})$ .

It follows that

$$c(\mathcal{U}, n) = (\kappa^\dagger \setminus A(\mathcal{U})) \cup \{\alpha \in A(\mathcal{U}) : f_{A(\mathcal{U})}(\alpha) \leq n\}$$

is  $\mathcal{U}$ -finite,  $\bigcup_{n < \omega} c(\mathcal{U}, n) = X$ , and

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{\alpha \in A(\mathcal{U}) \cap A(\mathcal{V}) : n < f_{A(\mathcal{U})}(\alpha) < f_{A(\mathcal{V})}(\alpha)\}$$

is finite.  $\square$

We may also consider common (non-regular) counterexamples which are finer than the usual Euclidean line.

**Definition 5.7.** Let  $R_{\mathbb{Q}}$  be the real line with the topology generated by open intervals with or without the rationals removed.

**Theorem 5.8.**  $R_{\mathbb{Q}}$  is non-regular and non- $\sigma$ -compact, but is second-countable and  $\sigma$ -relatively-compact.

*Proof.* Compact sets in  $R_{\mathbb{Q}}$  can be shown to not contain open intervals, and thus are nowhere dense in nonmeager  $\mathbb{R}$ , so  $R_{\mathbb{Q}}$  is not  $\sigma$ -compact. The usual base of intervals with rational endpoints (with or without rationals removed) witnesses second-countability.

To see that  $R_{\mathbb{Q}}$  is  $\sigma$ -relatively compact, consider  $[a, b] \setminus \mathbb{Q}$ . Let  $\mathcal{U}$  be a cover of  $R_{\mathbb{Q}}$ , and let  $\mathcal{U}'$  fill in the missing rationals for any open set in  $\mathcal{U}$ . There is a finite subcover  $\mathcal{V}' \subseteq \mathcal{U}'$  for  $[a, b]$  since  $\mathcal{U}'$  contains open sets from the Euclidean topology. Let  $\mathcal{V} = \{V \setminus \mathbb{Q} : V \in \mathcal{V}'\}$ : this is a finite refinement of  $\mathcal{U}$  covering  $[a, b] \setminus \mathbb{Q}$ , so  $[a, b] \setminus \mathbb{Q}$  is relatively compact. It follows then that  $R_{\mathbb{Q}} \setminus \mathbb{Q}$  is  $\sigma$ -relatively-compact, and since  $\mathbb{Q}$  is countable,  $R_{\mathbb{Q}}$  is  $\sigma$ -relatively-compact. Non-regularity follows since regular and  $\sigma$ -relatively-compact implies  $\sigma$ -compact.  $\square$

**Definition 5.9.** Let  $R_{\omega}$  be the real line with the topology generated by open intervals with countably many points removed.

**Theorem 5.10.**  $R_{\omega}$  is non-regular, non-second-countable, and non- $\sigma$ -relatively-compact, but  $\mathcal{F} \uparrow \text{Men}_{C,F}(R_{\omega})$ .

*Proof.* The closure of any open set is its closure in the usual Euclidean topology, so  $R_{\omega}$  is not regular. If  $S \supseteq \{s_n : n < \omega\}$  for  $s_n$  discrete, then  $U_m = R_{\omega} \setminus \{s_n : m < n < \omega\}$  yields an infinite cover  $\{U_m : m < \omega\}$  with no finite subcollection covering  $S$ , showing that all relatively compact sets are finite, and  $R_{\omega}$  is not  $\sigma$ -relatively-compact.

Define the winning strategy  $\sigma$  for  $\mathcal{F}$  in  $\text{Men}_{C,F}(R_{\omega})$  as follows: let  $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{2n}) = [-n, n] \setminus C_n$  for some countable  $C_n = \{c_{n,m} : m < \omega\}$ , and let  $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{2n+1}) = \{c_{i,j} : i, j < n\}$ . Non-second-countable follows since second-countable and  $\mathcal{F} \uparrow \text{Men}_{C,F}(X)$  implies  $\sigma$ -relatively-compact.  $\square$

We will soon see that, assuming  $S(2^{\omega})$ ,  $\mathcal{F}$  has a winning 2-Markov strategy for  $\text{Men}_{C,F}(R_{\omega})$  as well.

**Proposition 5.11.** Let  $\uparrow$  be either  $\uparrow_{\text{limit}}$  or  $\uparrow_{k\text{-mark}}$ . If  $X = \bigcup_{i < \omega} X_i$  and  $\mathcal{F} \uparrow_{\text{limit}} \text{Men}_{C,F}(X, X_i)$  for  $i < \omega$ , then  $\mathcal{F} \uparrow_{\text{limit}} \text{Men}_{C,F}(X)$ .

*Proof.* Let  $L$  be the  $k$ -Markov fog-of-war  $\mu_k$  (resp. the identity), and let  $\sigma_i$  be a  $k$ -Markov strategy (resp. perfect information strategy) for  $\mathcal{F}$  in  $\text{Men}_{C,F}(X, X_i)$ .

We define the  $k$ -Markov strategy (resp. perfect information strategy)  $\sigma$  for  $\text{Men}_{C,F}(X)$  as follows:

$$\sigma \circ L(\langle \mathcal{U}_0, \dots, \mathcal{U}_n \rangle) = \bigcup_{i \leq n} \sigma_i \circ L(\langle \mathcal{U}_i, \dots, \mathcal{U}_n \rangle)$$

Let  $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$  be a successful counter-attack by  $\mathcal{C}$  against  $\sigma$ . Then there exists  $x \in X_i$  for some  $i < \omega$  such that  $x$  is not covered by  $\bigcup_{n < \omega} \sigma \circ L(\langle \mathcal{U}_0, \dots, \mathcal{U}_n \rangle)$ . It follows that  $x$  is not covered by  $\bigcup_{n < \omega} \sigma_i \circ L(\langle \mathcal{U}_i, \dots, \mathcal{U}_{i+n} \rangle)$ , and  $\langle \mathcal{U}_i, \mathcal{U}_{i+1}, \dots \rangle$  is a successful counter-attack by  $\mathcal{C}$  against  $\sigma_i$ .  $\square$

**Theorem 5.12.** *If  $S(2^\omega)$ , then  $\mathcal{F} \uparrow_{2\text{-mark}} \text{Men}_{C,F}(R_\omega)$ .*

*Proof.* It's sufficient to show that  $[0, 1] \subseteq R_\omega$  is relatively robustly Menger. Let  $f_A$  witness  $S(2^\omega)$  for  $A \in [[a, b]]^{\leq \omega}$ . For each open cover  $\mathcal{U}$ , let  $A_{\mathcal{U}}$  be such that  $[0, 1] \setminus A_{\mathcal{U}}$  is  $\mathcal{U}$ -finite. Let  $r_{\mathcal{U}}(x) = 0$  if  $x \in [0, 1] \setminus A_{\mathcal{U}}$  and  $r_{\mathcal{U}}(x) = f_{A_{\mathcal{U}}}(x)$  otherwise.

It follows then that

$$c(\mathcal{U}, n) = [0, 1] \setminus \{x \in A_{\mathcal{U}} : f_{A_{\mathcal{U}}}(x) > n\}$$

is  $\mathcal{U}$ -finite and

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{x \in A_{\mathcal{U}} \cap A_{\mathcal{V}} : n < f_{A_{\mathcal{U}}}(x) < f_{A_{\mathcal{V}}}(x)\}$$

is finite.  $\square$

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