Mahavier Products, Idempotent Relations, and Condition Γ

Steven Clontz and Scott Varagona

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Abstract

Clearly, a generalized inverse limit of metrizable spaces indexed by $\mathbb{N}$ is metrizable, as it is a subspace of a countable product of metrizable spaces. The authors previously showed that all idempotent, upper semi-continuous, surjective, continuum-valued bonding functions on $[0, 1]$ (besides the identity) satisfy a certain Condition Γ; it follows that only in trivial cases can a generalized inverse limit of copies of $[0, 1]$ indexed by an uncountable ordinal be metrizable. The authors show that Condition Γ is in fact guaranteed by much weaker criteria, proving a more general metrizability theorem for certain Mahavier Products.

1 Introduction

In the spirit of the celebrated work by Mahavier [9] and Ingram & Mahavier [7], who first generalized traditional inverse limits to those with set-valued bonding functions, researchers have sought more ways to generalize inverse limits. One route proposed in [8] would be to index the factor spaces of an inverse limit not necessarily by the natural numbers, but rather, by some other directed set. However, a poorly-behaved directed set can cause an inverse limit to be empty [8]; therefore, most research in this area has involved inverse limits with totally ordered index sets.

Various results in recent years have shown that investigating totally ordered index sets besides the natural numbers is fertile ground for new work. Ingram and Mahavier laid the foundation in [8] by proving some basic connectedness theorems. Notably, Patrick Vernon in [11] studied inverse limits on $[0, 1]$ with set-valued functions indexed by the integers, and he showed that such an inverse limit with a single bonding function could be homeomorphic to a 2-cell—a striking result, considering Van Nall had shown in [10] that this could never happen for an inverse limit indexed by the natural numbers. Later, the notion of a generalized inverse limit was further generalized to the notion of a Mahavier product, which only required the index set to be a preordered set. Thus, the recent interest in Mahavier products (e.g., Greenwood & Kennedy [4] and Charatonik & Roe [2]) has helped bring "alternate" index sets further into the mainstream. In particular, in [2], Charatonik and Roe proved theorems about Mahavier products indexed by arbitrary totally ordered sets, and in the process introduced some helpful terminology in the study of generalized inverse limits. Therefore, given this context, it is natural to consider generalized inverse limits (or Mahavier products) indexed by ordinals.

In [3] the authors studied the special case of a generalized inverse limit of copies of $[0, 1]$ with a single continuum-valued upper semi-continuous idempotent bonding function.

They proved that when f is a surjection besides the identity, the graph of f must contain two distinct points $\langle x, x\rangle, \langle y, y\rangle$ on the diagonal, in addition to a third point $\langle x, y\rangle$. This Condition Γ is sufficient to guarantee that whenever the inverse limit is indexed by an ordinal α, the inverse limit contains a copy of $\alpha+1$. It therefore follows that such an inverse limit is a metric continuum if and only if α is countable. The authors suspected, however, that this was merely a special case of a more general trend. To this end, it will be shown that to guarantee Condition Γ, $[0,1]$ may be replaced with any weakly countably compact space, f need not be continuum-valued, and the surjectivity of f may be replaced with a weaker assumption to prevent trivialities. Applying this result, we also prove metrizability theorems for certain Mahavier products indexed by ordinals.

2 Definitions and Conventions

Except when otherwise stated, we assume that all topological spaces are Hausdorff. By convention, all natural numbers are ordinals, e.g. $4 = \{0, 1, 2, 3\}$. Let B^A denote the set of functions from A to B; in particular, X^2 is the usual square of ordered pairs $\{\langle x, y\rangle : x, y \in X\}$ (each pair is a function from $\{0, 1\}$ to X) and 2^X is the Cantor set of functions from X to $\{0, 1\}$. Let $F(X)$ denote the non-empty closed subsets of X.

Given a relation $R \subseteq X \times Y$, let $R(x) = \{y : xRy\} = \{y : \langle x, y\rangle \in R\}$; we often will use letters f, g to define relations and treat them as set-valued functions in this way. Given relations or set-valued functions f, g and a set A, let $f(A) = \{f(x) : x \in A\}$ and $(f \circ g)(x) = f(g(x)) = \bigcup_{y\in g(x)} f(y) = \{z : \exists y(z \in f(y), y \in g(x))\}$. If $f : X \to F(X)$ or $f \subseteq X^2$, let $f^2 = f \circ f$.

Definition 1. Suppose $\langle P, \leq\rangle$ is a directed set ($\leq$ is transitive and reflexive, and each pair of points shares a common upper bound) and for each $p \in P$, X_p is a space. Let $\Pi = \prod_{p\in P} X_p$; we will use boldface letters such as $\mathbf{x} \in \Pi$ to denote sequences in Π. Suppose further that for each $p \leq q \in P$, there is a set-valued *bonding function* $f_{p,q} : X_q \to F(X_p)$ such that for $p \leq q \leq r$, $f_{p,r} = f_{p,q} \circ f_{q,r}$ with $f_{p,p} : X_p \to F(X_p)$ defined by $f_{p,p}(x) = \{x\}$. Then the *generalized inverse limit* $\varprojlim \langle X_p, f_{p,q}, P\rangle \subseteq \Pi$ is given by:

$$\varprojlim \langle X_p, f_{p,q}, P\rangle = \{\mathbf{x} \in \Pi : p \leq q \Rightarrow \mathbf{x}(p) \in f_{p,q}(\mathbf{x}(q))\}.$$

The preceding definition is based upon the one given in [8]. Much of the literature assumes that P is a total order, often simply $\omega = \mathbb{N} = \{0, 1, 2, \dots\}$, but allowing for other orders such as $\mathbb{Z}$ enables the construction of many interesting examples unattainable with simply ω [11].

The study of generalized inverse limits typically focuses on upper semi-continuous bonding functions $f_{p,q} : X_q \to F(X_p)$, which for compact spaces may be characterized as those that map points to non-empty sets and whose graphs are closed in $X_q \times X_p$. In fact, it will be convenient to simply consider this graph itself as a relation. This is exactly the approach taken with Mahavier products in [2], [4].

Definition 2. Take the assumptions of the previous definition, but let $\langle P, \leq\rangle$ be a preordered set ($\leq$ is transitive and reflexive) and let $f_{p,q} \subseteq X_q \times X_p$ such that $f_{p,r} \subseteq f_{p,q} \circ f_{q,r}$. Then the *Mahavier product* $\mathbf{M}\langle X_p, f_{p,q}, P\rangle \subseteq \Pi$ is given by:

$$\mathbf{M}\langle X_p, f_{p,q}, P\rangle = \{\mathbf{x} \in \Pi : p \leq q \Rightarrow \mathbf{x}(p) \in f_{p,q}(\mathbf{x}(q))\}$$

Note that the condition $f_{p,r} \subseteq f_{p,q} \circ f_{q,r}$ has been weakened from equality, $f_{p,q}$ is an arbitrary relation from X_q to X_p, and the directed set is now only preordered, as is done in [2].

Definition 3. A Mahavier product $\mathbf{M}\langle X_p, f_{p,q}, P\rangle$ is *exact* whenever $f_{p,r} = f_{p,q} \circ f_{q,r}$ for all $p \leq q \leq r$.

Definition 4. A relation $f \subseteq X \times Y$ is said to be *left-total* (also called *full* [2]) if $\forall x \in X$ $\exists y \in Y$ such that $\langle x, y\rangle \in f$; that is, $f(x) \neq \emptyset$ for all $x \in X$.

Note that we will refrain from using the term "full" to describe such relations, as "full" bonding functions are defined in another sense in [5].

Observation 5. *Exact Mahavier products bonded by closed-valued left-total relations indexed by a directed set are generalized inverse limits.*

Definition 6. A left-total relation that is a closed subset of $X \times Y$ is called *USC*.

Here USC stands for upper semi-continuous, as in the case that X, Y are compact, this property may be characterized as follows: for every $x \in X$ and open set $V \supseteq f(x)$, there exists an open neighborhood U of x such that $f(u) \subseteq V$ for all $u \in U$ [7]. Put another way, the set-valued function $f : X \to F(Y)$ is continuous where $F(Y)$ is given the upper Vietoris topology.

Definition 7. A relation $f \subseteq X \times Y$ is *surjective* if, for all $y \in Y$, there exists $x \in X$ such that $\langle x, y\rangle \in f$.

Definition 8. An *idempotent* relation on $f \subseteq X^2$ is one that satisfies $f^2 = f$.

Note that transitivity may be characterized by $f^2 \subseteq f$, so all idempotent relations are transitive. Put another way, an idempotent relation $R \subseteq X^2$ satisfies xRz if and only if $xRyRz$ for some $y \in X$; transitivity is the backwards implication. Assuming reflexivity, idempotence and transitivity are equivalent: let $y = z$, so $xRz \Leftrightarrow xRzRz \Leftrightarrow xRyRz$. Note also that the inverse of an idempotent relation is idempotent.

The usual strict linear order $<$ on $\mathbb{Z}$ is an example of a transitive yet non-idempotent relation. Likewise, the strict linear order on any dense subset of $\mathbb{R}$ is a non-reflexive idempotent relation.

Idempotent relations are of significant interest when studying exact Mahavier products of many copies of the same topological space.

Observation 9. *If $X_p = X$ and $f_{p,q} = f$ for all $p < q$, and there exist $p, q, r \in P$ such that $p < q < r$, then the bonding relation f in an exact Mahavier product $\mathbf{M}\langle X, f, P\rangle$ must be idempotent.*

Note that $f_{p,p} \neq f$ unless f is the identity, but for simplicity we still simply write $\mathbf{M}\langle X, f, P\rangle$.

Definition 10. For convenience, we call a USC idempotent surjective relation $f \subseteq X^2$ a *V-relation*.

Examples of V-relations are given in Figure 1 (the designated points illustrate the definition of Condition Γ, given in Section 4). The next section will outline how to construct and identify V-relations.

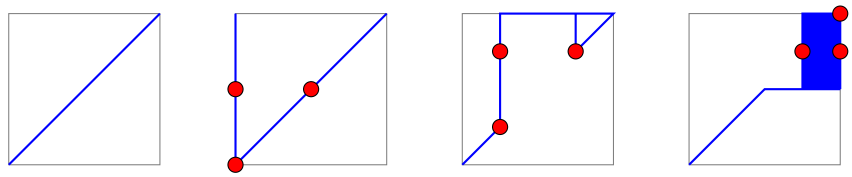

Figure 1: Illustrating Condition Γ for V-relations on $[0, 1]$

3 Constructing V-relations

The following propositions not only give necessary and/or sufficient conditions for a relation f to be a V-relation, but also give the reader a few tools for constructing simple V-relations from scratch. The first proposition is a useful recharacterization of idempotence.

Proposition 11. *Let X be a space and let $f \subseteq X^2$ be USC and surjective. Then f is a V-relation iff whenever A is the image of some $x \in X$, then $f(A) = A$.*

As an obvious consequence of Proposition 11, if $f \subseteq X^2$ is a V-relation and for some $x \in X$, $f(x) = \{y\}$, then $f(y) = \{y\}$.

Proposition 12. *Let $f \subseteq [0, 1]^2$ be USC. If f satisfies at least one of the following conditions, then f is a V-relation.*

1. *For each $x \in [0, 1]$, $f(x) = \{x, 1 - x\}$.*
2. *The relation f is surjective, and for each $x \in [0, 1]$, $f(x) = [0, x]$ or $f(x) \subseteq \{0, x\}$.*
3. *The relation f is surjective, and for each $x \in [0, 1]$, $f(x) = [x, 1]$ or $f(x) \subseteq \{x, 1\}$.*
4. *For some non-empty $A, B \subseteq [0, 1]$ with $A \cap B = \emptyset$, we have $f(a) = [0, 1]$ for each $a \in A$ and $f(x) = B$ for each $x \in [0, 1] \setminus A$.*
5. *The relation f is surjective, and there exists $b \in [0, 1]$ with $b \in f(b) = B$ so that for all $x \in [0, 1]$, either $f(x) = \{x\}$, $f(x) = B$, or $f(x) = \{y\}$ for some $y \in B$ satisfying $f(y) = \{y\}$.*

Proof. Each of the five conditions implies that f is surjective, so it remains to verify that each condition implies that f is idempotent. Ingram already observed that condition 1 implies f is idempotent in [6]. For condition 2, let $x \in [0, 1]$. We note that $f(y) \subseteq [0, x]$ for every $y \leq x$. Therefore, if $f(x) = [0, x]$, then clearly $f^2(x) = [0, x] = f(x)$. On the other hand, if $f(x) = \{0, x\}$, then $f^2(x) = f(0) \cup f(x) = \{0\} \cup \{0, x\} = f(x)$; the remaining cases are obvious. The details for condition 3 are similarly straightforward and are left to the reader.

For condition 4, we note that when $x \in A$, $f(x) = [0, 1]$ (so of course $f^2(x) = [0, 1]$), whereas if $x \in [0, 1] \setminus A$, then $f(x) = B$, so that $f^2(x) = f(B)$. However, when $b \in B$, since $b \notin A$, it follows that $f(b) = B$; thus, $f^2(x) = B$, and we conclude that f is idempotent.

To prove that condition 5 implies f is idempotent, let $x \in [0, 1]$. If $f(x) = \{x\}$ then clearly $f^2(x) = f(x)$. If $f(x) = \{y\}$ for some $y \in B$ with $f(y) = \{y\}$, then $f^2(x) = f(y) = \{y\} = f(x)$. Finally, if $f(x) = B$, then $f^2(x) = f(B)$; since $f(b) \subseteq B$ for each $b \in B$, and there is some $b \in B$ that satisfies $f(b) = B$, it follows that $f(B) = B$. Thus, $f^2(x) = f(x)$ in each case. $\square$

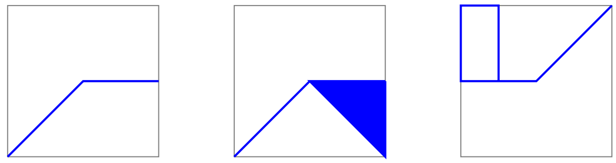

Figure 2: Trivial idempotent relations on $[0,1]$ besides ι

Note that Proposition 12 implies that each of the relations pictured in Figure 1 is a V-relation. (Of course, the sufficient conditions given in Proposition 12 are by no means exhaustive.)

4 Condition Γ

Definition 13. A relation $f \subseteq X^2$ satisfies *Condition* Γ if there exist distinct $x, y \in X$ such that $\langle x, x\rangle, \langle x, y\rangle, \langle y, y\rangle \in f$.

Note that in Figure 1, every example given besides the identity satisfies Condition Γ. In fact, this section will show that every V-relation besides the identity on a weakly countably compact space satisfies Condition Γ.

Definition 14. Let $\iota \subseteq X^2$ be the diagonal $\iota = \{\langle x, x\rangle : x \in X\}$, i.e. the identity relation.

Definition 15. For $f \subseteq X \times Y$ and $A \subseteq X$, let $f \restriction A = f \cap (A \times Y)$.

Proposition 16. *For f transitive, $f \restriction f(x) = f \cap (f(x))^2$.*

Definition 17. A relation is said to be *trivial* if for all $x \in X$, $f \restriction f(x) = \iota \restriction f(x)$.

Any trivial relation is idempotent, and of course ι is trivial, but there exist trivial idempotent relations besides the identity. For example, let $t \subseteq 3^2 = \{0, 1, 2\}^2$ be defined by

$$t = \{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 2, 0\rangle, \langle 2, 1\rangle\}.$$

It follows that $t \restriction f(0) = t \restriction \{0\} = \iota \restriction \{0\}$, $t \restriction f(1) = t \restriction \{1\} = \iota \restriction \{1\}$, and $t \restriction f(2) = t \restriction \{0, 1\} = \iota \restriction \{0, 1\}$. Figure 2 shows examples of trivial relations defined on $[0, 1]$; note that these do not satisfy Condition Γ.

Observing that none of the examples in Figure 2 are surjective, the following proposition will allow us to ignore such exceptional cases when considering V-relations besides the identity.

Proposition 18. *Let f be a transitive surjective relation on X. Then the following are equivalent.*

a) $f = \iota$

b) $|f(x)| = 1$ *for all* $x \in X$

c) f *is trivial*

Proof. (a) implies (b) trivially, so assume (b). For an arbitrary $x \in X$, there exists some $y \in X$ such that $f(x) = \{y\}$. Thus $f^2(x) = \{y\} = f(y)$ by transitivity, so it follows that $f \restriction \{y\} = \{\langle y, y\rangle\} = \iota \restriction \{y\}$, showing (c).

Finally, assuming (c), let $y \in X$ and by surjectivity choose $x \in X$ with $y \in f(x)$. By triviality, $f(y) = (f \restriction f(x))(y) = (\iota \restriction f(x))(y) = \iota(y)$, showing (a). □

Corollary 19. *ι is the only trivial V-relation.*

We now aim to exploit the properties of non-trivial relations to produce Condition Γ in V-relations besides ι. To do this, we require the following lemmas.

Definition 20. A weakly countably compact space is a space such that every infinite subset has a limit point.

Assuming spaces are T_2, this property is equivalent to countable compactness: every countable open cover has a finite subcover. But we will need only this weaker characterization, and the remainder of this section does not assume any separation axioms for the spaces under consideration.

Lemma 21. *Let X be weakly countably compact. Every USC idempotent relation $f \subseteq X^2$ besides ι contains two points $\langle y, y\rangle, \langle x, y\rangle$ for some distinct $x, y \in X$.*

Proof. Note first that if $\iota \subseteq f$, then since $\iota \neq f$ the lemma follows immediately. So let $x_0 \in X$ be a point where $\langle x_0, x_0\rangle \notin f$.

Suppose x_i is defined for $i \leq n$ such that $\langle x_i, x_j\rangle \in f$ if and only if $i < j$. Since $\{x_0, \ldots, x_n\} \cap f(x_n) = \emptyset$, we may choose $x_{n+1} \notin \{x_0, \ldots, x_n\}$ such that $\langle x_n, x_{n+1}\rangle \in f$. Note that by idempotence, $\langle x_{n+1}, x_i\rangle \notin f$ for $i \leq n$ since $x_i \notin f(x_n) = f^2(x_n) \supseteq f(x_{n+1})$. Similarly, $\langle x_i, x_{n+1}\rangle \in f$ for $i < n$ since $\langle x_i, x_n\rangle \in f$ and $\langle x_n, x_{n+1}\rangle \in f$. If $\langle x_{n+1}, x_{n+1}\rangle \in f$, then the lemma is satisfied by $x = x_n$ and $y = x_{n+1}$.

If not, we have recursively constructed an infinite set $\{x_n : n < \omega\}$. Since X is a weakly countably compact space, $\{x_n : n < \omega\}$ has a limit point x_ω. Since $\{\langle x_n, x_{n+1}\rangle : n < \omega\} \subseteq f$, it follows that $\langle x_\omega, x_\omega\rangle$ is a limit point of f. Similarly, $\{\langle x_0, x_n\rangle : 0 < n < \omega\} \subseteq f$, so it follows that $\langle x_0, x_\omega\rangle$ is also a limit point of f. Since f is closed, these limit points belong to f. Therefore the lemma is witnessed by $x = x_0$ and $y = x_\omega$. □

Lemma 22. *Let $f \subseteq X^2$ be idempotent and left-total, and let $x_0 \in X$. Then $f \restriction f(x_0)$ and its inverse are idempotent, left-total, surjective relations on $f(x_0)$.*

Proof. Let $g = f \restriction f(x_0)$. For each $x \in f(x_0)$, $f(x) \subseteq f^2(x_0) = f(x_0)$. Thus $g(x) = f(x) \neq \emptyset$ and $g^2(x) = f^2(x) = f(x) = g(x)$, showing that g is idempotent and left-total. It is also surjective: for $y \in f(x_0)$, $y \in f^2(x_0)$, so there exists some $x \in f(x_0)$ such that $g(x) = f(x) = y$. Since g is left-total, surjective, and idempotent, so is g^{-1}. □

Lemma 23. *Let $f \subseteq X^2$ be idempotent and left-total, and let $x_0 \in X$ witness that f is non-trivial. Then $f \restriction f(x_0)$ and its inverse are idempotent, left-total, surjective, non-trivial relations on $f(x_0)$.*

Proof. By the previous lemma, $g = f \restriction f(x_0)$ and g^{-1} are idempotent, left-total, and surjective. By non-triviality, $g \neq \iota \restriction f(x_0)$, so $g^{-1} \neq \iota \restriction f(x_0)$ too. But since $\iota \restriction f(x_0)$ is the only surjective idempotent trivial relation on $f(x_0)$, neither g nor g^{-1} are trivial. □

Theorem 24. *Let X be weakly countably compact, and let $f \subseteq X^2$ be an idempotent USC relation. Then the following are equivalent.*

a) f satisfies Condition Γ

b) f contains points $\langle x,x\rangle, \langle x,y\rangle$ for some distinct $x,y \in X$

c) f is non-trivial

Proof. (a) implies (b) trivially. Assuming (b), it follows that $f \restriction f(x) \neq \iota \restriction f(x)$ since $\langle x,y\rangle \in f \restriction f(x)$, showing (c).

Assuming (c), we may apply Lemma 23 to choose x_0 such that $g = f \restriction f(x_0)$ and g^{-1} are idempotent, USC, surjective, non-trivial relations. Then Lemma 21 allows us to conclude that g^{-1} contains points $\langle x,x\rangle, \langle y,x\rangle$ for some distinct $x,y \in X$, so (b) is satisfied.

Assume (b) such that $\langle y,y\rangle \notin f$. Note then that $\langle y,x\rangle \notin f$ as otherwise $\langle y,x\rangle, \langle x,y\rangle \in f$ would imply $\langle y,y\rangle \in f$. Let $z_0 = x$ and $z_1 = y$.

Suppose z_i is defined for $i \leq n+1$ such that $\langle z_i, z_j\rangle \in f$ if and only if $i < j$ or $i = j = 0$. Since $\{z_0, \ldots, z_{n+1}\} \cap f(z_{n+1}) = \emptyset$, we may choose z_{n+2} distinct from z_i for $i \leq n+1$ such that $\langle z_{n+1}, z_{n+2}\rangle \in f$. Note that by idempotence, $\langle z_{n+2}, z_i\rangle \notin f$ for $i \leq n+1$ since $z_i \notin f(z_{n+1}) = f^2(z_{n+1}) \supseteq f(z_{n+2})$. Similarly, $\langle z_i, z_{n+2}\rangle \in f$ for $i < n+1$ since $\langle z_i, z_{n+1}\rangle \in f$ and $\langle z_{n+1}, z_{n+2}\rangle \in f$. If $\langle z_{n+2}, z_{n+2}\rangle \in f$, then Condition Γ is witnessed by $\langle z_0, z_0\rangle, \langle z_0, z_{n+2}\rangle, \langle z_{n+2}, z_{n+2}\rangle$.

If not, we have recursively constructed an infinite set $\{z_n : n < \omega\}$. Since X is a weakly countably compact space, $\{z_n : n < \omega\}$ has a limit point z_ω. Since $\{\langle z_n, z_{n+1}\rangle : n < \omega\} \subseteq f$, it follows that $\langle z_\omega, z_\omega\rangle$ is a limit point of f. Similarly, $\{\langle z_0, z_n\rangle : n < \omega\} \subseteq f$, so it follows that $\langle z_0, z_\omega\rangle$ is also a limit point of f. Since f is closed, these limit points belong to f. Therefore Condition Γ is witnessed by $\langle z_0, z_0\rangle, \langle z_0, z_\omega\rangle, \langle z_\omega, z_\omega\rangle$, showing (a). □

Corollary 25. *Let X be weakly countably compact, and let $f \neq \iota$ be a V-relation on X. Then f satisfies Condition Γ.*

It is worth noting that the strict linear order $<$ on $\mathbb{Q}$ with the discrete topology is an example of a non-trivial idempotent USC relation on a space that is not weakly countably compact that does not satisfy Condition Γ. Likewise, the strict lexicographic order $<$ on the long ray $\omega_1 \times [0,1]$ with the topology induced by this linear order is an example of a non-trivial idempotent left-total non-USC relation on a Hausdorff countably compact space that does not satisfy Condition Γ.

5 Applications

Let $\alpha = \{\beta : \beta < \alpha\}$ be an ordinal with its usual linear order. As noted in [3], it is well-known and easy to see that $\alpha + 1$ as a totally ordered topological space is metrizable if and only if α is countable: if α is uncountable, then the first uncountable ordinal ω_1 is a point of non-first-countability in $\alpha + 1$; if α is countable, then $\alpha + 1$ is regular and second-countable.

Theorem 26. *Let X be a T_1 topological space, let α be an uncountable ordinal, and let f be a relation on X (with $f \subseteq f \circ f$) satisfying Condition Γ. Then the Mahavier product $\mathbf{M}\langle X, f, \alpha\rangle$ contains a copy of $\alpha + 1$; therefore, $\mathbf{M}\langle X, f, \alpha\rangle$ cannot be metrizable.*

Proof. Let $\langle x,x\rangle, \langle x,y\rangle, \langle y,y\rangle \in f$ for distinct $x,y \in X$. For $\gamma \leq \alpha$, define $\mathbf{x}_\gamma \in \mathbf{M}\langle X, f, \alpha\rangle$ by $\mathbf{x}_\gamma(\beta) = y$ for $\beta < \gamma$ and $\mathbf{x}_\gamma(\beta) = x$ for $\gamma \leq \beta < \alpha$. That is, $\mathbf{x}_\gamma$ is defined such that γ is the least ordinal such that $\mathbf{x}_\gamma(\gamma) = x$.

We will show that the map $\gamma \mapsto \mathbf{x}_\gamma$ is a homeomorphism from $\alpha + 1$ to $A = \{\mathbf{x}_\gamma : \gamma \leq \alpha\} \subseteq \mathbf{M}\langle X, f, \alpha\rangle$. This may be accomplished by comparing subbases: for $\beta \leq \alpha$, the subbasic open set $[0, \beta) \subseteq \alpha + 1$ maps to the set $\{\mathbf{x} \in A : \mathbf{x}(\beta) = x\} \subseteq A$, which equals the open set $A \cap \prod_{\gamma<\alpha} U_\gamma$ where $U_\beta = X \setminus \{y\}$ and $U_\gamma = X$ otherwise. Similarly, the subbasic open set $(\beta, \alpha] \subseteq \alpha + 1$ maps to the set $\{\mathbf{x} \in A : \mathbf{x}(\beta) = y\} \subseteq A$, which equals the open set $A \cap \prod_{\gamma<\alpha} U_\gamma$ where $U_\beta = X \setminus \{x\}$ and $U_\gamma = X$ otherwise. And since $A \cap \prod_{\gamma<\alpha} U_\gamma$ describes every subbasic open subset of A for $U_\beta \in \{\emptyset, X \setminus \{x\}, X \setminus \{y\}, X\}$ and $U_\gamma = X$ otherwise, it follows that this map is indeed a homeomorphism. □

Noting that in metrizable spaces, compactness is characterized by weak countable compactness, we have the following corollary.

Corollary 27. *Let X be any compact metrizable space, let α be an ordinal, and let $f \neq \iota$ be a V-relation on X. Then the exact Mahavier product $\mathbf{M}\langle X, f, \alpha\rangle$ contains a copy of $\alpha + 1$; therefore, $\mathbf{M}\langle X, f, \alpha\rangle$ is metrizable if and only if α is countable.*

As noted in [3], a bit more can be said.

Definition 28. The Σ-*product of reals* $\Sigma\mathbb{R}^\kappa$ for a cardinal κ is given by

$$\{\mathbf{x} \in \mathbb{R}^\kappa : |\{\alpha < \kappa : \mathbf{x}(\alpha) \neq 0\}| \leq \aleph_0\}$$

Note $\Sigma\mathbb{R}^\omega = \mathbb{R}^\omega$, and since every compact metrizable space embeds in $[0, 1]^\omega$, it follows that every compact metrizable space embeds in a Σ-product of reals.

Compact subspaces of $\Sigma\mathbb{R}^\kappa$ are known as Corson compacts: see e.g. [1] for an investigation into the applications of Corson compacts in functional analysis.

Corollary 29. *Let X be a T_1 topological space, let α be an uncountable ordinal, and let f be a relation on X (with $f \subseteq f \circ f$) satisfying Condition Γ. Then the Mahavier product $\mathbf{M}\langle X, f, \alpha\rangle$ cannot be embedded in a Σ-product of reals.*

Proof. The Mahavier product $\mathbf{M}\langle X, f, \alpha\rangle$ contains a copy of $\alpha + 1$, which cannot be embedded into a Σ-product of reals. □

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References

[1] Argyros, S., Mercourakis, S., & Negrepontis, S., *Functional-analytic properties of Corson-compact spaces*, Studia Math. vol. 89 (1988), no. 3, 197–229.

[2] Wlodzimierz J. Charatonik, Robert P. Roe, *On Mahavier Products*, Topology and its Applications, vol. 166 (2014), 92-97.

[3] Steven Clontz, Scott Varagona, *Destruction of Metrizability in Generalized Inverse Limits*, Topology Proceedings, vol. 48 (2016), pp. 289-297.

[4] Sina Greenwood, Judy Kennedy, *Connectedness and Ingram-Mahavier products*, Topology and its Applications, vol. 166 (2014), 1-9.

[5] Sina Greenwood, Judy Kennedy, Michael Lockyer, *Connectedness and inverse limits with set-valued functions on intervals*, Topology and its Applications, vol. 221 (2017), 69-90.

[6] W. T. Ingram, *An introduction to inverse limits with set-valued functions*, Springer Briefs in Mathematics, 2012.

[7] W. T. Ingram, William S. Mahavier, *Inverse limits of upper semi-continuous set valued functions*, Houston Journal of Mathematics, vol. 32 (2006) no. 1, 119-130.

[8] W. T. Ingram, William S. Mahavier, *Inverse Limits: from Continua to Chaos*, Springer, Developments in Mathematics (vol. 25), 2012.

[9] William S. Mahavier, *Inverse Limits with Subsets of* $[0,1] \times [0,1]$, Topology and its Applications, vol. 141 (2004), no. 1-3, 225-231.

[10] Van Nall, *Inverse limits with set valued functions*, Houston Journal of Mathematics, vol. 37 (2011), no. 4, 1323-1332.

[11] Patrick Vernon, *Inverse limits of set-valued functions indexed by the integers*, Topology and its Applications, vol. 171 (2014), 35-40.