On constructing permutation-fair dice

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By convention, $n = \{0, 1, \dots, n-1\}$ for each natural number n.

Definition 1. A function $F : n \to S$ is called a **formula** for a set S of dice, where $F(k)$ returns which die in the set S has a face with the number $k < n$.

For example, the formula $F : 4 \to \{A, B\}$ defined by $ABBA$ represents an assignment of values to the faces of dice in the set $\{A, B\}$. In particular A has two sides $\{0, 3\}$ while die B has sides $\{1, 2\}$.

Definition 2. Given a formula $F : n \to S$ and $T \subseteq S$, let $F_T : m \to T$ be the **subformula** obtained by removing the dice in $S \setminus T$ from the sequence.

So for $F : 7 \to \{A, B, C\}$ defined by $ABCACBA$, $F_{\{A,C\}}$ is defined by $ACACA$.

Definition 3. Given a sequence $F : n \to S$ and $m \leq n$, let $F \restriction m : m \to S$ be the **restriction sequence** where $(F \restriction m)(k) = F(k)$ for $k < m$.

So for $F : 7 \to \{A, B, C\}$ defined by $ABCACBA$, $F \restriction 4$ is defined by $ABCA$.

Definition 4. A **sample** for a formula $F : n \to S$ is a function $s : S \to n$ such that $F(s(A)) = A$ for all $A \in S$. Let $\sigma(F)$ collect all samples for the formula F.

Definition 5. The **result** of a sample $s : S \to n$ is a permutation $r_s : |S| \to S$ of S such that $i \leq j$ if and only if $f(r_s(i)) \leq f(r_s(j))$.

So for the formula $ABCACBA$, a sample might be defined by $f(A) = 3, f(B) = 2, f(C) = 4$, yielding the result BAC. Note that the highest-valued die in the sample s is given by $r_s(|S| - 1)$.

Definition 6. A formula $F : n \to S$ is called **permutation-fair** if for all permutations p, q of S, $|\{s \in \sigma(F) : r_s = p\}| = |\{s \in \sigma(F) : r_s = q\}|$.

Definition 7. A formula $F : n \to S$ is called **go-first-fair** if for all $A, B \in S$, $|\{s \in \sigma(F) : r_s(|S| - 1) = A\}| = |\{s \in \sigma(F) : r_s(|S| - 1) = B\}|$.

Proposition 8. *Suppose $F : n \to S$ is a permutation-fair (resp. go-first-fair) formula. Then F_T is permutation-fair (resp. go-first-fair) for all $T \subseteq S$.*

Theorem 9. *Suppose $F : n \to S \cup \{X\}$ is a go-first-fair formula such that F_S is permutation fair, and for each $m \leq n$ where $f(m) = X$, $(F \upharpoonright m)_S$ is permutation-fair. Then F is permutation-fair.*

Proof. Since go-first-fair implies permutation-fair in the base case $|S| = 0$, assume the theorem holds when $|S| \leq k$, and let $|S| = k + 1$.

For each $T \subsetneq S$, we note that $F_{T\cup\{X\}}$ is a go-first-fair formula such that for each $m \leq |F_{T\cup\{X\}}|$ where $F_{T\cup\{X\}}(m) = X$, $F_{T\cup\{X\}} \upharpoonright m = F \upharpoonright m'$ for some $m \leq m' < n$ and $F(m') = X$. Thus $(F_{T\cup\{X\}} \upharpoonright m)_T = (F \upharpoonright m')_T = ((F \upharpoonright m')_S)_T$ is permutation-fair, and since $|T| \leq k$, $F_{T\cup\{X\}}$ is permutation-fair.

So let p, q be permutations of $S \cup \{X\}$; we aim to show that $|\{s \in \sigma(F) : r_s = p\}| = |\{s \in \sigma(F) : r_s = q\}|$.

First, suppose $p(|S|) = q(|S|) = X$. Thus

$$|\{s \in \sigma(F) : r_s = p\}| = \sum \left\{ \left| \{s \in \sigma((F \upharpoonright m)_S) : r_s = p \upharpoonright |S|\} \right| : f(m) = X \right\}$$

and likewise for q. Since $(F \upharpoonright m)_S$ is permutation-fair,

$$|\{s \in \sigma((F \upharpoonright m)_S) : r_s = p \upharpoonright |S|\}| = |\{s \in \sigma((F \upharpoonright m)_S) : r_s = q \upharpoonright |S|\}|$$

so the result follows in this case.

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