

# PREDETERMINED PROXIMAL SPACES ARE METRIZABLE

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ABSTRACT. TODO

We take the following from Willard's text.

**Definition 0.1.** A *normal covering sequence* for a space  $X$  is a sequence  $\{\mathcal{U}_n : n < \omega\}$  of open covers such that  $\mathcal{U}_{n+1}$  star-refines  $\mathcal{U}_n$ . Such a sequence is *compatible* with  $X$  if  $\{St(x, \mathcal{U}_n) : n < \omega\}$  is a local base at each  $x \in X$ .

**Theorem 0.2.** A space  $X$  is *psuedometrizable* if and only if it has a compatible normal covering sequence.

For convenience, we will recast these results in terms of entourages.

**Definition 0.3.** An *entourage sequence* for a space  $X$   $\{D_n : n < \omega\}$  is *compatible* with  $X$  if  $\{D_n[x] : n < \omega\}$  is a local base at each  $x \in X$ .

**Theorem 0.4.** A space  $X$  is *psuedometrizable* if and only if it has a compatible entourage sequence.

*Proof.* Let  $d$  be a psuedometric generating  $X$ ; then  $\{D_n : n < \omega\}$  given by  $D_n = \{\langle x, y \rangle : d(x, y) < 2^{-n}\}$  is a entourage sequence, which is compatible since  $D_n[x] = B_{2^{-n}}(x)$ .

On the other hand, given a compatible entourage sequence  $\{D_n : n < \omega\}$ , let  $E_0 = D_0$ ,  $E_{n+1} = \frac{1}{4}D_n \cap \frac{1}{4}E_n$ , and  $\mathcal{U}_n = \{E_{n+1}[x] : x \in X\}$ . Fix  $x \in X$  and consider  $St(x, \mathcal{U}_n) \subseteq St(E_{n+1}[x], \mathcal{U}_n) = \bigcup \{E_{n+1}[y] \in \mathcal{U}_n : E_{n+1}[x] \cap E_{n+1}[y] \neq \emptyset\}$ .

Let  $z \in St(E_{n+1}, \mathcal{U}_n)$ , so  $z \in E_{n+1}[y]$  for some  $y \in X$  where  $w \in E_{n+1}[x] \cap E_{n+1}[y]$  for some  $w \in X$ . It follows that  $\langle z, y \rangle, \langle y, w \rangle, \langle w, x \rangle \in E_{n+1} = \frac{1}{4}D_n \cap \frac{1}{4}E_n$ ; therefore  $\langle z, x \rangle \in D_n \cap E_n$  and  $z \in D_n[x] \cap E_n[x]$ . Therefore  $St(x, \mathcal{U}_n) \subseteq St(E_{n+1}[x], \mathcal{U}_n) \subseteq D_n[x] \cap E_n[x]$ .

We can then observe that  $\mathcal{U}_{n+1}$  star refines  $\mathcal{U}_n$  since for each  $U \in \mathcal{U}_{n+1}$ ,  $U = E_{n+2}[x]$  for some  $x \in X$  and  $St(E_{n+2}[x], \mathcal{U}_{n+1}) \subseteq E_{n+1}[x] \in \mathcal{U}_n$ , making  $\{\mathcal{U}_n : n < \omega\}$  a normal covering sequence. Finally, the sequence is compatible since for  $x \in X$ ,  $St(x, \mathcal{U}_n) \subseteq D_n[x]$  and  $\{D_n[x] : n < \omega\}$  is a local base at  $x$ .  $\square$

**Theorem 0.5.** A space  $X$  is *psuedometrizable* if and only if  $I \uparrow_{pre} Bell_{D,P}^{\rightarrow, \emptyset}(X)$ .

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*Proof.* Suppose  $X$  is psuedometrizable by  $d$ ; then let  $\sigma$  be the predetermined strategy for  $Bell_{D,P}^{\rightarrow,\emptyset}(X)$  defined by  $\sigma(n) = \{\langle x, y \rangle : d(x, y) < 2^{-n}\}$ . For any legal attack  $\alpha$  against  $\sigma$ ,  $\alpha(n+1) \in \sigma(n)[\alpha(n)]$ . It follows that if  $x \in \bigcap_{n < \omega} \sigma(n)[\alpha(n)]$  and  $\epsilon > 0$ , we may choose  $N < \omega$  such that  $2^{-N} < \epsilon$ . Therefore  $d(x, \alpha(n)) < 2^{-n} \leq 2^{-N} < \epsilon$  for all  $n \geq N$ , showing  $\alpha$  converges to  $x$ . Thus  $\sigma$  is a winning strategy.

Now let  $\sigma$  be any predetermined winning strategy satisfying  $\sigma(n) \subseteq \sigma(m)$  for all  $n \geq m$ , and suppose  $\{\frac{1}{2^{n+1}}\sigma(n)[x] : n < \omega\}$  is not a local base at some  $x \in X$ . Then we may pick an entourage  $D$  such that  $\frac{1}{2^{n+1}}\sigma(n)[x] \not\subseteq D[x]$  for all  $n < \omega$ . So choose  $\alpha(n) \in \frac{1}{2^{n+1}}\sigma(n)[x] \setminus D[x]$ .

Observe that  $\langle \alpha(n), x \rangle \in \frac{1}{2^{n+1}}\sigma(n)$  and  $\langle \alpha(n+1), x \rangle \in \frac{1}{2^{n+2}}\sigma(n+1) \subseteq \frac{1}{2^{n+1}}\sigma(n)$ . It follows that  $\langle \alpha(n), \alpha(n+1) \rangle \in \frac{1}{2^n}\sigma(n) \subseteq \sigma(n)$ , witnessing that  $\alpha(n+1) \in \sigma(n)[\alpha(n)]$ , that is,  $\alpha$  is a legal counterattack to  $\sigma$ . Since  $x \in \frac{1}{2^{n+1}}\sigma(n)[\alpha(n)] \subseteq \sigma(n)[\alpha(n)]$  for all  $n < \omega$ ,  $\sigma$  can only win for I if  $\alpha$  converges. But  $\alpha(n) \notin D[x]$  for all  $n < \omega$ , so  $\alpha$  fails to converge as well. Thus  $\sigma$  is not a winning strategy.

As a result, if  $\sigma$  is a winning predetermined strategy, we have that  $\{\frac{1}{2^{n+1}}\sigma(n)[x] : n < \omega\}$  is a local base at each  $x \in X$ . This shows that  $\{\frac{1}{2^{n+1}}\sigma(n) : n < \omega\}$  is a compatible entourage sequence; therefore by the previous lemma,  $X$  is psuedometrizable.  $\square$

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