

Definition 1. X is **Menger** if for all open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ there exist finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\bigcup_{n < \omega} \mathcal{F}_n$ is a cover of X .

Proposition 2. $\sigma\text{-compact} \Rightarrow \text{Menger} \Rightarrow \text{Lindelof}$

Definition 3. In the two-player game $\text{Cov}_{C,S}(X)$ player C chooses open covers \mathcal{U}_n of X , followed by player S choosing a finite subcollection $\mathcal{F}_n \subseteq \mathcal{U}_n$. S wins if $\bigcup_{n < \omega} \mathcal{F}_n$ is a cover of X .

Theorem 4. X is Menger if and only if $S \uparrow \text{Cov}_{C,S}(X)$.

Proof. First, suppose X wasn't Menger. Then there would exist open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ of X such that for any choice of finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$, $\bigcup_{n < \omega} \mathcal{F}_n$ isn't a cover of X . Thus $C \uparrow_{\text{pre}} \text{Cov}_{C,S}(X) \Rightarrow S \not\uparrow \text{Cov}_{C,S}(X)$.

The other direction is based upon Gruenhage's topological game presentation. Assume X is Menger, and consider a strategy for C in $\text{Cov}_{C,S}(X)$.

Since X is Lindelof, we can assume C plays only countable covers of X . Then, since S is choosing finite subsets, we may assume S chooses some initial segment of the countable cover. In turn, we can assume C plays an increasing open cover $\{U_0, U_1, \dots\}$ where $U_n \subseteq U_{n+1}$. And in that case, it's sufficient to assume S simply chooses a singleton subset of each cover. And finally, since choices made by S are already covered, we can assume that every open set in a cover played by C covers the sets chosen by S previously.

As a result, we have the following figure of a tree of plays which I need to draw:

(Insert figure here.)

Note that for $a, b \in \omega^{<\omega}$ and $m \leq n$, we know:

- (a) $U_{a \smallfrown m} \subseteq U_{a \smallfrown n}$
(for example, $U_{1627} \subseteq U_{1629}$ - increasing the final digit yields supersets)
- (b) $U_a \subseteq U_{a \smallfrown b}$
(for example, $U_{1627} \subseteq U_{162789}$ - appending any sequence to the end yields supersets)
- (c) $U_{a \smallfrown m} \subseteq U_{a \smallfrown n} \subseteq U_{a \smallfrown n \smallfrown b} \subseteq U_{a \smallfrown n \smallfrown b \smallfrown m}$
(for example: $U_{1627} \subseteq U_{1629283287}$ - injecting a subsequence with initial number larger than the original's final number, prior to the final number, yields supersets)

We may observe that if S can find an $f : \omega \rightarrow \omega$ such that $\bigcup_{n < \omega} U_{f \upharpoonright (n+1)} = X$, she can use $\{U_{f \upharpoonright 0}\}, \{U_{f \upharpoonright 1}\}, \dots$ to counter C 's strategy.

Let $V_k^n = \bigcap_{a \in \omega^{\leq n}} U_{a \smallfrown k}$. We claim that (1) V_k^n is open, (2) $\mathcal{V}^n = \{V_0^n, V_1^n, \dots\}$ is increasing, and (3) \mathcal{V}^n is a cover. Proofs:

1. Since due to (c) for each $b \in \omega^{\leq n} \setminus k^{\leq n}$, there is an $a \in k^{\leq n}$ with $U_{a \smallfrown k} \subseteq U_{b \smallfrown k}$:

$$V_k^n = \bigcap_{a \in \omega^{\leq n}} U_{a \smallfrown k} = \bigcap_{a \in k^{\leq n}} U_{a \smallfrown k} \cap \bigcap_{b \in \omega^{\leq n} \setminus k^{\leq n}} U_{b \smallfrown k} = \bigcap_{a \in k^{\leq n}} U_{a \smallfrown k}$$

making V_k^n a finite intersection of open sets.

2. We show $V_k^0 \subseteq V_{k+1}^0$:

$$V_k^0 = U_k \subseteq U_{k+1} = V_{k+1}^0$$

and then assume $V_k^n \subseteq V_{k+1}^n$:

$$V_k^{n+1} = \bigcap_{a \in \omega^{\leq n+1}} U_{a \smallfrown k} = V_k^n \cap \bigcap_{a \in \omega^{n+1}} U_{a \smallfrown k} \subseteq V_{k+1}^n \cap \bigcap_{a \in \omega^{n+1}} U_{a \smallfrown (k+1)} = V_{k+1}^{n+1}$$

3. We easily see that $\mathcal{V}^0 = \{U_0, U_1, \dots\}$ is a cover, and then assume \mathcal{V}^n is a cover.

Let $x \in X$ and pick $l < \omega$ such that $x \in V_l^n$. For $a \in l^{n+1}$ choose l_a such that $x \in U_{a \smallfrown l_a}$, giving

$$x \in \bigcap_{a \in l^{n+1}} U_{a \smallfrown l_a}$$

We will assume $k > l, l_a$ for all $a \in l^{\leq n+1}$.

For any $a \in k^{n+1} \setminus l^{n+1}$ note that $a = b \smallfrown c$ where $b \in l^{\leq n}$ and c begins with a number l or greater:

$$V_l^n \subseteq U_{b \smallfrown l} \subseteq U_{b \smallfrown c} \subseteq U_{b \smallfrown c \smallfrown l_a} = U_{a \smallfrown l_a}$$

Thus:

$$\begin{aligned} x &\in V_l^n \cap \left(\bigcap_{a \in l^{n+1}} U_{a \smallfrown l_a} \right) \\ &= V_l^n \cap \left(\bigcap_{a \in k^{n+1} \setminus l^{n+1}} U_{a \smallfrown l_a} \right) \cap \left(\bigcap_{a \in l^{n+1}} U_{a \smallfrown l_a} \right) \\ &= V_l^n \cap \left(\bigcap_{a \in k^{n+1}} U_{a \smallfrown l_a} \right) \\ &\subseteq V_k^n \cap \left(\bigcap_{a \in k^{n+1}} U_{a \smallfrown k} \right) \\ &= V_k^{n+1} \end{aligned}$$

Finally, apply Menger to \mathcal{V}^n , resulting in the cover $\{V_{f(0)}^0, V_{f(1)}^1, \dots\}$, noting

$$X = \bigcup_{n < \omega} V_{f(n)}^n \subseteq \bigcup_{n < \omega} U_{(f \upharpoonright n) \cap f(n)} = \bigcup_{n < \omega} U_{f \upharpoonright (n+1)}$$

□