SELECTION GAMES AND ARHANGELSKII'S CONVERGENCE PRINCIPLES

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Abstract. We prove the things.

1. Clontz results

Definition 1. Say a collection $\mathcal{A}$ is *sequence-like* if it satisfies the following for each $A \in \mathcal{A}$.

- $|A| \geq \aleph_0$.
- If $A' \subseteq A$ and $|A'| \geq \aleph_0$, then $A' \in \mathcal{A}$.

Definition 2. Let Γ_X be the collection of open γ-*covers* $\mathcal{U}$ of X, that is, infinite open covers of X such that for each $x \in X$, $\{U \in \mathcal{U} : x \in U\}$ is cofinite in $\mathcal{U}$.

Definition 3. Let $\Gamma_{X,x}$ be the collection of non-trivial sequences $S \subseteq X$ converging to x, that is, infinite subsets of X such that for each neighborhood U of x, $S \cap U$ is cofinite in S.

It follows that $\Gamma_X, \Gamma_{X,x}$ are both sequence-like. We also require the following.

Definition 4. Say a collection $\mathcal{A}$ is *almost-sequence-like* if for each $A \in \mathcal{A}$, there is $A' \subseteq A$ such that:

- $|A'| = \aleph_0$.
- If A'' is a cofinite subset of A', then $A'' \in \mathcal{A}$.

So all sequence-like sets are almost-sequence-like.

Theorem 5. *Let $\mathcal{B}$ be sequence-like. Then $\alpha_1(\mathcal{A}, \mathcal{B})$ holds if and only if* $\mathrm{I} \underset{pre}{\not\uparrow} G_{cf}(\mathcal{A}, \mathcal{B})$.

Proof. We first assume $\alpha_1(\mathcal{A}, \mathcal{B})$ and let $A_n \in \mathcal{A}$ for $n < \omega$ define a predetermined strategy for I. By $\alpha_1(\mathcal{A}, \mathcal{B})$, we immediately obtain $B \in \mathcal{B}$ such that $|A_n \setminus B| < \aleph_0$. Thus $B_n = A_n \cap B$ is a cofinite choice from A_n, and $B' = \bigcup\{B_n : n < \omega\}$ is an infinite subset of B, so $B' \in \mathcal{B}$. Thus II may defeat I by choosing $B_n \subseteq A_n$ each round, witnessing $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_{cf}(\mathcal{A}, \mathcal{B})$.

On the other hand, let $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_{cf}(\mathcal{A}, \mathcal{B})$. Given $A_n \in \mathcal{A}$ for $n < \omega$, we note that II may choose a cofinite subset $B_n \subseteq A_n$ such that $B = \bigcup\{B_n : n < \omega\} \in \mathcal{B}$. Then B witnesses $\alpha_1(\mathcal{A}, \mathcal{B})$ since $|A_n \setminus B| \leq |A_n \setminus B_n| \leq \aleph_0$. $\square$

Theorem 6. *Let $\mathcal{A}$ be almost-sequence-like and $\mathcal{B}$ be sequence-like. Then $\alpha_2(\mathcal{A}, \mathcal{B})$ holds if and only if* $\mathrm{I} \underset{pre}{\not\uparrow} G_1(\mathcal{A}, \mathcal{B})$.

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Proof. We first assume $\alpha_2(\mathcal{A},\mathcal{B})$ and let $A_n \in \mathcal{A}$ for $n<\omega$ define a predetermined strategy for $\mathscr{I}$. We may apply $\alpha_2(\mathcal{A},\mathcal{B})$ to choose $B \in \mathcal{B}$ such that $|A_n \cap B| \geq \aleph_0$. We may then choose $a_n \in (A_n \cap B) \setminus \{a_i : i < n\}$ for each $n<\omega$. It follows that $B' = \{a_n : n < \omega\} \in \mathcal{B}$ since B' is an infinite subset of $B \in \mathcal{B}$; therefore A_n does not define a winning predetermined strategy for I.

Now suppose I $\underset{\text{pre}}{\not\uparrow}$ $G_1(\mathcal{A},\mathcal{B})$. Given $A_n \in \mathcal{A}$ for $n<\omega$, first choose $A'_n \in \mathcal{A}$ such that $A'_n = \{a_{n,j} : j < \omega\} \subseteq A_n$, $j<k$ implies $a_{n,j} \neq a_{n,k}$, and $A_{n,m} = \{a_{n,j} : m \leq j < \omega\} \in \mathcal{A}$. Finally choose some $\theta : \omega \to \omega$ such that $|\theta^{\leftarrow}(n)| = \aleph_0$ for each $n<\omega$.

Since playing $A_{\theta(m),m}$ during round m does not define a winning strategy for I in $G_1(\mathcal{A},\mathcal{B})$, II may choose $x_m \in A_{\theta(m),m}$ such that $B = \{x_m : m < \omega\} \in \mathcal{B}$. Choose $i_m < \omega$ for each $m<\omega$ such that $x_m = a_{\theta(m),i_m}$, noting $i_m \geq m$. It follows that $A_n \cap B \supseteq \{a_{\theta(m),i_m} : m \in \theta^{\leftarrow}(n)\}$. Since for each $m \in \theta^{\leftarrow}(n)$ there exists $M \in \theta^{\leftarrow}(n)$ such that $m \leq i_m < M \leq i_M$, and therefore $a_{\theta(m),i_m} \neq a_{\theta(m),i_M} = a_{\theta(M),i_M}$, we have shown that $A_n \cap B$ is infinite. Thus B witnesses $\alpha_2(\mathcal{A},\mathcal{B})$. $\square$

Theorem 7. *Let $\mathcal{A}$ be almost-sequence-like and $\mathcal{B}$ be sequence-like. Then $\alpha_4(\mathcal{A},\mathcal{B})$ holds if and only if* I $\underset{pre}{\not\uparrow}$ $G_{<2}(\mathcal{A},\mathcal{B})$ *if and only if* I $\underset{pre}{\not\uparrow}$ $G_{fin}(\mathcal{A},\mathcal{B})$.

Proof. We first assume $\alpha_4(\mathcal{A},\mathcal{B})$ and let $A_n \in \mathcal{A}$ for $n<\omega$ define a predetermined strategy for I in $G_{<2}(\mathcal{A},\mathcal{B})$. We then may choose $A'_n \in \mathcal{A}$ where $A'_n = \{a_{n,j} : j < \omega\} \subseteq A_n$, $j<k$ implies $a_{n,j} \neq a_{n,k}$, and $A''_n = A'_n \setminus \{a_{i,j} : i,j < n\} \in \mathcal{A}$.

By applying $\alpha_4(\mathcal{A},\mathcal{B})$ to A''_n, we obtain $B \in \mathcal{B}$ such that $A''_n \cap B \neq \emptyset$ for infintely-many $n<\omega$. We then let $F_n = \emptyset$ when $A''_n \cap B = \emptyset$, and $F_n = \{x_n\}$ for some $x_n \in A''_n \cap B$ otherwise. Then we will have that $B' = \bigcup\{F_n : n < \omega\} \subseteq B$ belongs to $\mathcal{B}$ once we show that B' is infinite. To see this, for $m \leq n < \omega$ note that either F_m is empty (and we let $j_m = 0$) or $F_m = \{a_{m,j_m}\}$ for some $j_m \geq m$; choose $N < \omega$ such that $j_m < N$ for all $m \leq n$ and $F_N = \{x_N\}$. Thus $F_m \neq F_N$ for all $m \leq n$ since $x_N \notin \{a_{i,j} : i,j < N\}$. Thus II may defeat the predetermined strategy A_n by playing F_n each round.

Since I $\underset{\text{pre}}{\not\uparrow}$ $G_{<2}(\mathcal{A},\mathcal{B})$ immediately implies I $\underset{\text{pre}}{\not\uparrow}$ $G_{fin}(\mathcal{A},\mathcal{B})$, we assume the latter. Given $A_n \in \mathcal{A}$ for $n<\omega$, we note this defines a (non-winning) predetermined strategy for I, so II may choose $F_n \in [A_n]^{<\aleph_0}$ such that $B = \bigcup\{F_n : n < \omega\} \in \mathcal{B}$. Since B is infinite, we note $F_n \neq \emptyset$ for infinitely-many $n<\omega$. Thus B witnesses $\alpha_4(\mathcal{A},\mathcal{B})$ since $A_n \cap B \supseteq F_n \neq \emptyset$ for infinitely-many $n<\omega$. $\square$

Theorem 8. *Let $\mathcal{B}$ be sequence-like. Then* I $\underset{pre}{\uparrow}$ $G_{<2}(\mathcal{A},\mathcal{B})$ *if and only if* I $\underset{pre}{\uparrow}$ $G_{fin}(\mathcal{A},\mathcal{B})$.

Proof. Assume $\bigcup\mathcal{A}$ is well-ordered. Given a winning predetermined strategy A_n for I in $G_{<2}(\mathcal{A},\mathcal{B})$, consider $F_n \in [A_n]^{<\aleph_0}$. We set

$$F^*_n = \begin{cases} \emptyset & \text{if } F_n \setminus \bigcup\{F_m : m < n\} = \emptyset \\ \{\min(F_n \setminus \bigcup\{F_m : m < n\})\} & \text{otherwise} \end{cases}$$

Since $|F_n| < 2$, we have that $\bigcup\{F^*_n : n < \omega\} \notin \mathcal{B}$. In the case that $\bigcup\{F^*_n : n < \omega\}$ is finite, we immediately see that $\bigcup\{F_n : n < \omega\}$ is also finite and therefore not in $\mathcal{B}$. Otherwise $\bigcup\{F^*_n : n < \omega\} \notin \mathcal{B}$ is an infinite subset of $\bigcup\{F_n : n < \omega\}$, and thus $\bigcup\{F_n : n < \omega\} \notin \mathcal{B}$ too. Therefore A_n is a winning predetermined strategy for I in $G_{fin}(\mathcal{A},\mathcal{B})$ as well. $\square$

Theorem 9. *Let $\mathcal{B}$ be sequence-like. Then* $\mathrm{I} \uparrow G_{<2}(\mathcal{A},\mathcal{B})$ *if and only if* $\mathrm{I} \uparrow G_{fin}(\mathcal{A},\mathcal{B})$.

Proof. Assume $\bigcup\mathcal{A}$ is well-ordered. Suppose $\mathrm{I} \uparrow G_{<2}(\mathcal{A},\mathcal{B})$ is witnessed by the strategy σ. Let $\langle\rangle^\star = \langle\rangle$, and for $s^\frown\langle F\rangle \in ([\bigcup\mathcal{A}]^{<\aleph_0})^{<\omega} \setminus \{\langle\rangle\}$ let

$$(s^\frown\langle F\rangle)^\star = \begin{cases} s^{\star\frown}\langle\emptyset\rangle & \text{if } F\setminus\bigcup\mathrm{range}(s)=\emptyset \\ s^{\star\frown}\langle\{\min(F\setminus\bigcup\mathrm{range}(s))\}\rangle & \text{otherwise}\end{cases}$$

We then define the strategy τ for I in $G_{fin}(\mathcal{A},\mathcal{B})$ by $\tau(s)=\sigma(s^\star)$. Then given any counterattack $\alpha \in ([\bigcup\mathcal{A}]^{<\aleph_0})^\omega$ by II played against τ, we note that $\alpha^* = \bigcup\{(\alpha\restriction n)^* : n<\omega\}$ is a counterattack to σ, and thus loses. This means $B = \bigcup\mathrm{range}(\alpha^*) \notin \mathcal{B}$.

We consider two cases. The first is the case that $\bigcup\mathrm{range}(\alpha^*)$ is finite. Noting that $\alpha^*(m)\cap\alpha^*(n)=\emptyset$ whenever $m\neq n$, there exists $N<\omega$ such that $\alpha^*(n)=\emptyset$ for all $n>N$. As a result, $\bigcup\mathrm{range}(\alpha)=\bigcup\mathrm{range}(\alpha\restriction n)$, and thus $\bigcup\mathrm{range}(\alpha)$ is finite, and therefore not in $\mathcal{B}$.

In the other case, $\bigcup\mathrm{range}(\alpha^*) \notin \mathcal{B}$ is an infinite subset of $\bigcup\mathrm{range}(\alpha)$, and therefore $\bigcup\mathrm{range}(\alpha)\notin\mathcal{B}$ as well. Thus we have shown that τ is a winning strategy for I in $G_{fin}(\mathcal{A},\mathcal{B})$. □

Theorem 10. *Let $\mathcal{B}$ be sequence-like. Then* $\mathrm{II} \uparrow G_{<2}(\mathcal{A},\mathcal{B})$ *if and only if* $\mathrm{II} \uparrow G_{fin}(\mathcal{A},\mathcal{B})$.

Theorem 11. *Let $\mathcal{B}$ be sequence-like. Then* $\mathrm{II} \underset{mark}{\uparrow} G_{<2}(\mathcal{A},\mathcal{B})$ *if and only if* $\mathrm{II} \underset{mark}{\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$.

Theorem 12. *Let $\mathcal{A}$ be almost-sequence-like and $\mathcal{B}$ be sequence-like. Then* $\mathrm{I} \uparrow G_{fin}(\mathcal{A},\mathcal{B})$ *if and only if* $\mathrm{I} \underset{pre}{\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$.

Proof. We assume $\mathrm{I}\uparrow G_{fin}(\mathcal{A},\mathcal{B})$. As $\mathcal{A}$ is almost-sequence-like, there is a strategy σ witnessing $\mathrm{I}\uparrow G_{fin}(\mathcal{A},\mathcal{B})$ where $|\sigma(s)|=\aleph_0$ and $\sigma(s)\cap\bigcup\mathrm{range}(s)=\emptyset$ (that is, σ never replays the choices of II) for all partial plays s by II.

For each $s\in\omega^{<\omega}$, suppose $F_{s\restriction m}\in[\bigcup A]^{<\aleph_0}$ is defined for each $0<m\leq|s|$. Then let $s^\star : |s| \to [\bigcup\mathcal{A}]^{<\aleph_0}$ be defined by $s^\star(m)=F_{s\restriction m+1}$, and define $\tau' : \omega^{<\omega}\to\mathcal{A}$ by $\tau'(s)=\sigma(s^\star)$. Finally, set $[\sigma(s^\star)]^{<\aleph_0} = \{F_{s^\frown\langle n\rangle} : n<\omega\}$, and for some bijection $b:\omega^{<\omega}\to\omega$ let $\tau(n)=\tau'(b(n))$ be a predetermined strategy for I in $G_{fin}(\mathcal{A},\mathcal{B})$.

Suppose α is a counterattack by II against τ, so

$$\alpha(n)\in[\tau(n)]^{<\aleph_0}=[\tau'(b(n))]^{<\aleph_0}=[\sigma(b(n)^\star)]^{<\aleph_0}$$

It follows that $\alpha(n)=F_{b(n)^\frown\langle m\rangle}$ for some $m<\omega$. In particular, there is some infinite subset $W\subseteq\omega$ and $f\in\omega^\omega$ such that $\{\alpha(n):n\in W\}=\{F_{f\restriction n+1} : n<\omega\}$. Note here that $(f\restriction n+1)^\star = (f\restriction n)^{\star\frown}\langle F_{f\restriction n+1}\rangle$. This shows that $F_{f\restriction n+1}\in[\sigma((f\restriction n)^\star)]^{<\aleph_0}$ is an attempt by II to defeat σ, which fails. Thus $\bigcup\{F_{f\restriction n+1} : n<\omega\}=\bigcup\{\alpha(n):n\in W\}\notin\mathcal{B}$, and since this set is infinite (as σ prevents II from repeating choices) we have $\bigcup\{\alpha(n):n<\omega\}\notin\mathcal{B}$ too. Therefore τ is winning. □

Proposition 13. *Let $\mathcal{B}$ be sequence-like, $\mathcal{A}\subseteq\mathcal{B}$, and* $\mathrm{I} \underset{pre}{\not\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$. *Then $\mathcal{A}$ is almost-sequence-like.*

Proof. Let $A \in \mathcal{A}$, and for all $n < \omega$ let $A_n = A$. Then A_n is not a winning predetermined strategy for I, so II may choose finite subsets $B_n \subseteq A_n$ such that $A'_n \bigcup\{B_n : n < \omega\} \in \mathcal{B} \subseteq \mathcal{A}$.

It follows that $|A'_n| = \aleph_0$, and for any infinite subset $A''_n \subseteq A'_n$ (in particular, any cofinite subset), $A''_n \in \mathcal{B} \subseteq \mathcal{A}$. Thus $\mathcal{A}$ is almost-sequence-like. □

Corollary 14. *Let $\mathcal{B}$ be sequence-like and $\mathcal{A} \subseteq \mathcal{B}$. Then* $\mathrm{I} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ *if and only if* $\mathrm{I} \underset{pre}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$.

Proof. Assuming $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$, we have $\mathrm{I} \not\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ by Proposition 13 and Theorem 12. □

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