PROXIMAL COMPACT SPACES ARE CORSON COMPACT

STEVEN CLONTZ AND GARY GRUENHAGE

Abstract. J. Bell defined a topological space X to be *proximal* if X has a compatible uniformity with respect to which the first player has a winning strategy in a certain ω-length game. As noted by P.J. Nyikos, it follows easily from Bell's results that Corson compact spaces are proximal. We answer a question of Nyikos by showing that a compact space is proximal iff it is Corson compact.

A common generalization of metric spaces is the idea of a *uniform space*. A uniform space is a determined by a collection of supersets of the diagonal in the square (called "entourages") satisfying certain conditions. A uniformity induces in a natural way a topology on X, called the *uniform topology*. A topological space X is *uniformizable* if there is a uniformity on X which generates its topology. Any completely regular space is uniformizable. See the next section for more complete definitions of these concepts and some of their basic properties.

Jocelyn Bell introduced the concept of proximal spaces in her doctoral dissertation while working on some uniform box product problems due to her advisor Scott Williams. Proximal spaces are defined to be the spaces X for which there is a compatible uniformity on X such that the entourage picker has a winning strategy in a certain ω-length game. Every metric space is easily seen to be proximal. Bell has shown that proximal spaces are collectionwise normal, countably paracompact, and have strong preservation properties, particularly, closed subpaces and Σ-products of proximal spaces are proximal [1].

In [5], Peter Nyikos observed that Bell's results imply that compact subspaces of the Σ-product of real lines, known as Corson compacts, must be proximal. He asked the natural question as to whether any proximal compact must then be Corson compact. Using a characterization of Corson compact due to the second author in [4], we answer that question in the affirmative.

In this paper, all topological spaces are assumed to be completely regular.

1. Definitions and Properites of Uniform Spaces

We relate some definitions and properties of uniform spaces.

Definition 1.1. A *uniform space* is a pair $\langle X, \mathcal{D}\rangle$ where X is a set, and $\mathcal{D}$ is a uniformity. A *uniformity* is a filter on subsets of X^2, called *entourages*, such that for each entourage $D \in \mathcal{D}$:

- D is reflexive, i.e., the diagonal $\Delta = \{\langle x, x\rangle : x \in X\} \subseteq D$.
- Its inverse $D^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in D\} \in \mathcal{D}$.

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- There exists $\frac{1}{2}D \in \mathcal{D}$ such that

$$2\left(\frac{1}{2}D\right) = \frac{1}{2}D \circ \frac{1}{2}D = \left\{\langle x, z\rangle : \exists y \left(\langle x, y\rangle, \langle y, z\rangle \in \frac{1}{2}D\right)\right\} \subseteq D.$$

Definition 1.2. The *uniform topology* induced by a uniformity declares a set U to be open if for every $x \in U$, there is some $D \in \mathcal{D}$ with $x \in D[x] = \{y : \langle x, y\rangle \in D\} \subseteq U$. A topological space X is said to be *uniformizable* if there is a uniformity D on X which induces its topology.

We now list some basic results about uniformities and uniform spaces. One may see [2], for example, for proofs.

- Every uniform topology is completely regular, and every completely regular space is uniformizable.
- For every entourage D, there is an open symmetric entourage $E \subseteq D$. That is, $\langle x, y\rangle \in E \Leftrightarrow \langle y, x\rangle \in E$, and E is open in X^2 with the usual product topology induced by the uniform topology on X.
- If D is an open entourage, then for all $x \in X$, $D[x]$ is an open neighborhood of x.

We will make frequent use of $\frac{1}{2}D$ in this paper. Of course, $\frac{1}{2}D$ is not unique, but for convenience we will assume that for each $D \in \mathcal{D}$, a unique $\frac{1}{2}D$ has been chosen, which by the second item above may be assumed to be open and symmetric. Then we define $\frac{1}{4}D = \frac{1}{2}(\frac{1}{2}D)$, $\frac{1}{8}D = \frac{1}{2}(\frac{1}{4}D)$, and so on.

We quickly note a couple properites of $\frac{1}{2}D$.

Proposition 1.3. *If $x \in \frac{1}{2}D[y]$ and $y \in \frac{1}{2}D[z]$, then $x \in D[z]$*

Proof. Directly from the definition of $\frac{1}{2}D$. Note that the same result holds if we assumed instead that $y \in \frac{1}{2}D[x]$ or $z \in \frac{1}{2}D[y]$, since $\frac{1}{2}D$ is assumed to be symmetric. □

Proposition 1.4. *If X is a uniform space, then for all $x \in X$ and symmetric entourages D:*

$$\frac{1}{2}D[x] \subseteq \overline{\frac{1}{2}D[x]} \subseteq D[x]$$

Proof. If y is a limit point of $\frac{1}{2}D[x]$, then $\frac{1}{2}D[y]$ must intersect $\frac{1}{2}D[x]$ at some z. It follows then that $y \in D[x]$. □

1.1. **The Proximal Game.** The theory of proximal uniform spaces relies on the following ω-length game:

Definition 1.5. The **proximal game** $Prox_{D,P}(X)$ of length ω played on a uniform space X with two players $\mathscr{D}$, $\mathscr{P}$ proceeds as follows:

- In the initial round 0, $\mathscr{D}$ chooses an open symmetric entourage D_0, followed by $\mathscr{P}$ choosing a point $p_0 \in X$.
- In round $n + 1$, $\mathscr{D}$ chooses an open symmetric entourage $D_{n+1} \subseteq D_n$, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in D_n[p_n]$.

At the conclusion of the game, $\mathscr{D}$ wins if either $\bigcap_{n<\omega} D_n[p_n] = \emptyset$ or $\langle p_0, p_1, \ldots\rangle$ converges, and $\mathscr{P}$ wins otherwise.

The reader familiar with proximal spaces may note that this definition is different than the original formulation of the game (see [1]); however, it is easily verified that $\mathscr{D}$ has a winning strategy in the original game if and only if $\mathscr{D}$ has a winning strategy in our version. Also, if we wish we may assume that if σ is a winning strategy for $\mathscr{D}$, then $D_{n+1} = \sigma(x_0, x_1, ..., x_n)$ is contained in something smaller than D_n, e.g., $\frac{1}{4}D_n$. That is because a sequence of legal moves by $\mathscr{P}$ with $\mathscr{D}$ using the refined strategy is also a legal sequence of moves with $\mathscr{D}$ using the original strategy.

Definition 1.6. If a player $\mathscr{P}$ has a winning strategy in a game G, we write $\mathscr{P} \uparrow G$. Otherwise, we write $\mathscr{P} \not\uparrow G$.

Definition 1.7. A uniform space $(X, \mathcal{D})$ is said to be *proximal* if and only if $\mathscr{D} \uparrow Prox_{D,P}(X)$. A topological space is *proximal* iff it admits a compatible uniformity (i.e., one which induces its topology) which is proximal.

2. Proximal Games and W Games

The second author introduced the following game in [3].

Definition 2.1. The **W-convergence game** $Con_{O,P}(X, H)$ of length ω played on a topological space X and set $H \subseteq X$ with two players $\mathscr{O}$, $\mathscr{P}$ proceeds as follows:

- In the initial round 0, $\mathscr{O}$ chooses an open neighborhood O_0 of H, followed by $\mathscr{P}$ choosing a point $p_0 \in O_0$.
- In round $n+1$, $\mathscr{O}$ chooses an open neigborhood $O_{n+1} \subseteq O_n$ of H, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in O_{n+1}$.

At the conclusion of the game, $\mathscr{O}$ wins if $\langle p_0, p_1, \ldots \rangle$ converges to H (every open neighborhood of H contains all but finitely many points of the sequence), and $\mathscr{P}$ wins otherwise.

In the case of $H = \{x\}$, we abuse notation and write $Con_{O,P}(X, x)$.

Definition 2.2. A topological space X is said to be a W*-space* if and only if for every $x \in X$, $\mathscr{O} \uparrow Con_{O,P}(X, x)$.

We find it useful to consider a (seemingly) weaker version of this game.

Definition 2.3. The **W-clustering game** $Clus_{O,P}(X, H)$ proceeds identically to $Con_{O,P}(X, H)$, except that $\mathscr{O}$ need only force p to cluster at H (every open neighborhood of H contains infinitely many points of the sequence).

Theorem 2.4. *$\mathscr{O} \uparrow Con_{O,P}(X, H)$ if and only if $\mathscr{O} \uparrow Clus_{O,P}(X, H)$*

Proof. Shown for $H = \{x\}$ in [3], but the analogous proof works for arbitrary H. The forward implication is immediate. Let σ be a strategy for $\mathscr{O}$ witnessing $\mathscr{O} \uparrow Clus_{O,P}(X, H)$. Note σ is a function whose domain is all possible partial attacks by $\mathscr{P}$ (finite sequences of points in X), and whose range is the open neighborhoods of H, such that for any legal attack $p = \langle p_0, p_1, \ldots \rangle$ against σ, it follows that p must cluster at H.

For every partial attack a, let $S(a)$ contain all subsequences of a. We then define $\tau(p \restriction n) = \bigcap_{b \in S(p \restriction n)} \sigma(b)$. If p attacks the strategy τ for $Con_{O,P}(X, H)$, then it also is a legal attack against σ in $Clus_{O,P}(X, H)$, so p clusters at H.

But not only that: let q be a subsequence of p. If $q \restriction n$ is a legal partial attack against σ and $q \restriction n$ is a subsequence of $p \restriction m$ with $q(n) = p(m)$, then since

$q(n) = p(m) \in \tau(p \restriction m) \subseteq \sigma(q \restriction n)$, $q \restriction n+1$ is a legal partial attack against σ as well. Thus q is a legal attack against σ in $Clus_{O,P}(X, H)$, so q clusters at H.

Since no infinite subsequence of p can be completely missed by an open neighborhood of H, it follows that p converges to H. □

Theorem 2.5. [1] *All proximal spaces are W-spaces.*

The idea of Bell's proof of the above result is to consider the result of an attack on the winning strategy for $\mathscr{D}$ in $Prox_{D,P}(X)$ which alternates between points chosen by $\mathscr{P}$ in $Con_{O,P}(X, x)$ and the particular point x itself. Since the intersection $\bigcap_{n<\omega} D_n[p_n]$ must contain x, the sequence must converge, and since the sequence contains x infinitely often, it must converge to x.

While there is not much trouble in ensuring the nonempty intersection of $\bigcap_{n<\omega} D_n[p_n]$ for this particular proof, we turn to a stronger version of $Prox_{D,P}(X)$, also introduced by Bell, which avoids the issue entirely.

Definition 2.6. The *absolutely proximal game* proceeds identially to $Prox_{D,P}(X)$, except that $\mathscr{D}$ may only win in the case that $\langle p_0, p_1, \dots \rangle$ converges.

Obviously, all absolutely proximal spaces are proximal. We are interested in when we have equivalence.

Definition 2.7. A uniform space is *uniformly locally compact* if there exists an open symmetric entourage L such that $\overline{L[x]}$ is a compact neighborhood of x for all $x \in X$. A topological space is uniformly locally compact if it admits a compatible uniformly locally compact uniformity.

Obviously every compact space is uniformly locally compact, but not every locally compact space is uniformly locally compact. For example, the space of countable ordinals is locally compact, but does not admit a compatible uniformly locally compact uniformity.

Theorem 2.8. *A uniformly locally compact space X is proximal if and only if it is absolutely proximal.*

Proof. Let L be a uniformly locally compact entourage. Let σ be a strategy for $\mathscr{D}$ witnessing $\mathscr{D} \uparrow Prox_{D,P}(X)$. Without loss of generality, we may assume such that $\sigma(p \restriction n) \subseteq L$ for all partial attacks $p \restriction n$ (so $\overline{\sigma(p \restriction n)[x]} \subseteq \overline{L[x]}$ is compact), and that $n > m$ implies $\sigma(p \restriction n) \subset \frac{1}{4}\sigma(p \restriction m)$.

Let $\tau(p \restriction n) = \frac{1}{2}\sigma(p \restriction n)$. If p attacks τ, then

$$p(n+1) \in \tau(p \restriction n)[p(n)] = \frac{1}{2}\sigma(p \restriction n)[p(n)]$$

and for

$$x \in \overline{\sigma(p \restriction (n+1))[p(n+1)]} \subseteq \overline{\frac{1}{4}\sigma(p \restriction n)[p(n+1)]} \subseteq \frac{1}{2}\sigma(p \restriction n)[p(n+1)]$$

we can conclude $x \in \sigma(p \restriction n)[p(n)]$. Thus

$$\sigma(p \restriction (n+1))[p(n+1)] \subseteq \overline{\sigma(p \restriction (n+1))[p(n+1)]} \subseteq \sigma(p \restriction n)[p(n)]$$

Finally, note that since τ yields refinements of σ, then p attacks the winning strategy σ in $Prox_{D,P}(X)$, but since the intersection of a descending chain of nonempty compact sets is nonempty, we have

$$\bigcap_{n<\omega} \sigma(p \upharpoonright n)[p(n)] = \bigcap_{n<\omega} \overline{\sigma(p \upharpoonright n)[p(n)]} \neq \emptyset.$$

We conclude that p converges. □

3. Corson Compacts and Proximal Compacts

We recall the definition of Corson compact.

Definition 3.1. A space is said to be *Corson compact* if and only if it is homeomorphic to a compact set within the Σ-product $\Sigma\mathbb{R}^\kappa$ of κ-many real lines, that is:

$$\Sigma\mathbb{R}^\kappa = \{x \in \mathbb{R}^\kappa : |\{\alpha : x(\alpha) \neq 0\}| < \omega\}$$

Since proximal spaces are closed under closed subsets and Σ-products, it follows (as noted by Nyikos) that every Corson compact is proximal.

However, the given characterization of Corson compact is less useful when proving the other direction. Instead we use the following game characterization due to the second author.

Theorem 3.2. [4] *A space X is Corson compact if and only if X is compact and $\mathscr{O} \uparrow Con_{O,P}(X^2, \Delta)$.*

The following contains the meat of our proof of the title result.

Theorem 3.3. *For any absolutely proximal space X, $\mathscr{O} \uparrow Con_{O,P}(X, H)$ for all compact $H \subseteq X$.*

Proof. Let σ be a winning strategy for $\mathscr{D}$ in the absolutely proximal game such that $p \supsetneq q$ implies $\sigma(p) \subset \frac{1}{4}\sigma(q)$. For any sequence $t = \langle t_0, t_1, \ldots \rangle$, let $o(t)$ be the subsequence of t consisting of its odd-indexed terms.

For certain finite sequences s in X, we will define by induction a tree $T(s)$ of finite height such that $t \supset s$ implies $T(s)$ is a subtree of $T(t)$, and if u is a node of $T(s)$ having successors in $T(s)$, then u has no new successors in $T(t)$. We let $\max(T)$ denote the nodes of T that have no successors in T.

First we define $T(\emptyset)$.

- Let $\emptyset \in T(\emptyset)$.
- Choose $m_\emptyset < \omega$, $h_{\emptyset,i} \in H$ for $i < m_\emptyset$, and $h_{\emptyset,i,j} \in H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$ for $i, j < m_\emptyset$ such that

  $$\{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}] : i < m_\emptyset\}$$

  is a cover for H and such that for each $i < m_\emptyset$

  $$\{\frac{1}{4}\sigma(\langle h_{\emptyset,i}\rangle)[h_{\emptyset,i,j}] : j < m_\emptyset\}$$

  is a cover for $H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$.
- Let $\langle i \rangle, \langle i, h_{\emptyset,i}\rangle$, and $\langle i, h_{\emptyset,i}, j\rangle$ be in $T(\emptyset)$ for $i, j < m_\emptyset$.

This ends the construction of $T(\emptyset)$. We remark that the indexings $i \mapsto h_{\emptyset,i}$ and $(i,j) \mapsto h_{\emptyset,i,j}$ need not be one-to-one; repetition of points is allowed.

Now suppose $T(a)$ is defined. We then define $T(a^\frown\langle x\rangle)$, etc., for

$$x \in \bigcup_{s^\frown\langle i,h_{s,i},j\rangle\in\max(T(a))} \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

as follows:

- Let $T(a) \subseteq T(a^\frown\langle x\rangle)$.
- Choose $t = s^\frown\langle i, h_{s,i}, j, x\rangle$ such that $s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(a))$ and $x \in \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$. Then $s^\frown\langle i, h_{s,i}, j\rangle$ is the only node of $T(a)$ that will be extended in this step.
- Note that, assuming $o(s)^\frown\langle h_{s,i}\rangle$ is a legal partial attack against σ, then

  $$x \in \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}] \subseteq \frac{1}{4}\sigma(o(s))[h_{s,i,j}]$$

  and

  $$h_{s,i,j} \in \overline{\frac{1}{4}\sigma(o(s))[h_{s,i}]} \subseteq \frac{1}{2}\sigma(o(s))[h_{s,i}]$$

  implies

  $$x \in \sigma(o(s))[h_{s,i}]$$

  and thus $o(s)^\frown\langle h_{s,i}, x\rangle = o(t)$ is a legal partial attack against σ.
- Choose $m_t < \omega$, $h_{t,k} \in H \cap \overline{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]}$ for $k < m_t$, and $h_{t,k,l} \in H \cap \overline{\frac{1}{4}\sigma(o(t))[h_{t,k}]}$ for $k, l < m_t$ such that

  $$\{\frac{1}{4}\sigma(o(t))[h_{t,k}] : k < m_t\}$$

  is a cover for $H \cap \overline{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]}$ and such that for each $k < m_t$

  $$\{\frac{1}{4}\sigma(o(t)^\frown\langle h_{t,k}\rangle)[h_{t,k,l}] : l < m_t\}$$

  is a cover for $H \cap \overline{\frac{1}{4}\sigma(o(t))[h_{t,k}]}$.
- Note that, assuming $o(t)$ is a legal partial attack against σ, then

  $$h_{t,k} \in \overline{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]} \subseteq \frac{1}{2}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

  and

  $$x \in \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

  implies

  $$h_{t,k} \in \sigma(o(s)^\frown\langle h_{s,i}\rangle)[x]$$

  and thus $o(t)^\frown\langle h_{t,k}\rangle$ is a legal partial attack against σ.
- For each $k, l < m_t$, put $t^\frown\langle k, h_{t,k}, l\rangle$ and all of its initial segments in $T(a^\frown\langle x\rangle)$. This completes the construction of $T(a^\frown\langle x\rangle)$.

- Note that assuming

  $$\{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}] : s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(a))\}$$

  covers H, then since

$$\{\frac{1}{4}\sigma(o(t)^\frown\langle h_{t,k}\rangle)[h_{t,k,l}] : s^\frown\langle i, h_{s,i}, j, x, k, h_{t,k}, l\rangle \in \max(T(a^\frown\langle x\rangle)) \setminus \max(T(a))\}$$

covers $H \cap \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$, we have that

$$\{\frac{1}{4}\sigma(o(t)^\frown\langle h_{t,k}\rangle)[h_{t,k,l}] : t^\frown\langle k, h_{t,k}, l\rangle \in \max(T(a^\frown\langle x\rangle))\}$$

covers H.

Hence we may define the strategy τ for $\mathscr{O}$ in $Clus_{O,P}(X,H)$ such that:

$$\tau(p \restriction n) = \bigcup_{s^\frown\langle i,h_{s,i},j\rangle \in \max(T(p\restriction n))} \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

If p attacks τ, then it follows that $T(p \restriction n)$ is defined for all $n < \omega$, so let $T(p) = \bigcup_{n<\omega} T(p \restriction n)$. We note $T(p)$ is an infinite tree with finite levels:

- $\emptyset$ has exactly $m_\emptyset$ successors $\langle i\rangle$.
- $s^\frown\langle i\rangle$ has exactly one successor $s^\frown\langle i, h_{s,i}\rangle$
- $s^\frown\langle i, h_{s,i}\rangle$ has exactly m_s successors $s^\frown\langle i, h_{s,i}, j\rangle$
- $s^\frown\langle i, h_{s,i}, j\rangle$ has either no successors or exactly one successor $s^\frown\langle i, h_{s,i}, j, x\rangle$
- $t = s^\frown\langle i, h_{s,i}, j, x\rangle$ has exactly m_t successors $t^\frown\langle k\rangle$

Hence $T(p)$ has an infinite branch $q' = \langle i_0, h_0, j_0, x_0, i_1, h_1, j_1, x_1, \ldots\rangle$. Let $q = o(q') = \langle h_0, x_0, h_1, x_1, \ldots\rangle$. Note that by the construction of $T(p)$, q is an attack on the winning strategy σ in the absolutely proximal game, so it must converge. Since every other term of q is in H, it must converge to H. Then since $o(q)$ is a subsequence of p, p must cluster at H.

Since $\mathscr{O} \uparrow Clus_{O,P}(X,H)$ if and only if $\mathscr{O} \uparrow Con_{O,P}(X,H)$, the result follows. □

The equivalency result follows as a quick corollary.

Corollary 3.4. *For any compact space X, X is proximal if and only if X is Corson compact.*

Proof. The reverse implication was noted earlier. If X is compact proximal, then so is X^2. By the previous theorem, $\mathscr{O} \uparrow Con_{O,P}(X^2, \Delta)$, which is a characterization of Corson compact. □

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Department of Mathematics, Auburn University, Auburn, AL 36830
E-mail address: steven.clontz@auburn.edu
URL: www.stevenclontz.com

Department of Mathematics, Auburn University, Auburn, AL 36830
E-mail address: gruengf@auburn.edu
URL: www.auburn.edu/∼gruengf