Menger

Definition 1. X is **Menger** if for all open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ there exist finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Proposition 2. *σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelof*

Definition 3. In the two-player game $Cov_{C,F}(X)$ player C chooses open covers $\mathcal{U}_n$ of X, followed by player F choosing a finite subcollection $\mathcal{F}_n \subseteq \mathcal{U}_n$. F wins if $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Theorem 4. *X is Menger if and only if $C \not\uparrow Cov_{C,F}(X)$.*

Proof. Result due to (???)

First, suppose X wasn't Menger. Then there would exist open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ of X such that for any choice of finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$, $\bigcup_{n<\omega} \mathcal{F}_n$ isn't a cover of X. Thus $C \uparrow_{\text{pre}} Cov_{C,F}(X) \Rightarrow S \not\uparrow Cov_{C,F}(X)$.

The other direction is based upon Gruenhage's topological game presentation. Assume X is Menger, and consider a strategy for C in $Cov_{C,F}(X)$.

Since X is Lindelof, we can assume C plays only countable covers of X. Then, since F is choosing finite subsets, we may assume F chooses some initial segement of the countable cover. In turn, we can assume C plays an increasing open cover $\{U_0, U_1, \dots\}$ where $U_n \subseteq U_{n+1}$. And in that case, it's suffient to assume F simply chooses a singleton subset of each cover. And finally, since choices made by F are already covered, we can assume that every open set in a cover played by C covers the sets chosen by F previously.

As a result, we have the following figure of a tree of plays which I need to draw:

(Insert figure here.)

Note that for $a, b \in \omega^{<\omega}$ and $m \leq n$, we know:

(a) $U_{a^\frown m} \subseteq U_{a^\frown n}$
(for example, $U_{1627} \subseteq U_{1629}$ - increasing the final digit yields supersets)

(b) $U_a \subseteq U_{a^\frown b}$
(for example, $U_{1627} \subseteq U_{162789}$ - appending any sequence to the end yields supersets)

(c) $U_{a^\frown m} \subseteq U_{a^\frown n} \subseteq U_{a^\frown n^\frown b} \subseteq U_{a^\frown n^\frown b^\frown m}$
(for example: $U_{1627} \subseteq U_{1629283287}$ - injecting a subseqence with initial number larger than the original's final number, prior to the final number, yields supersets)

We may observe that if F can find an $f : \omega \to \omega$ such that $\bigcup_{n<\omega} U_{f\restriction(n+1)} = X$, she can use $\{U_{f\restriction 0}\}, \{U_{f\restriction 1}\}, \dots$ to counter C's strategy.

Let $V^n_k = \bigcap_{a \in \omega^{\leq n}} U_{a^\frown k}$. We claim that (1) V^n_k is open, (2) $\mathcal{V}^n = \{V^n_0, V^n_1, \dots\}$ is increasing, and (3) $\mathcal{V}^n$ is a cover. Proofs:

1. Since due to (c) for each $b \in \omega^{\leq n} \setminus k^{\leq n}$, there is an $a \in k^{\leq n}$ with $U_{a^\frown k} \subseteq U_{b^\frown k}$:
$$V^n_k = \bigcap_{a \in \omega^{\leq n}} U_{a^\frown k} = \bigcap_{a \in k^{\leq n}} U_{a^\frown k} \cap \bigcap_{b \in \omega^{\leq n} \setminus k^{\leq n}} U_{b^\frown k} = \bigcap_{a \in k^{\leq n}} U_{a^\frown k}$$
making V^n_k a finite intersection of open sets.

2. We show $V^0_k \subseteq V^0_{k+1}$:
$$V^0_k = U_k \subseteq U_{k+1} = V^0_{k+1}$$
and then assume $V^n_k \subseteq V^n_{k+1}$:
$$V^{n+1}_k = \bigcap_{a \in \omega^{\leq n+1}} U_{a^\frown k} = V^n_k \cap \bigcap_{a \in \omega^{n+1}} U_{a^\frown k} \subseteq V^n_{k+1} \cap \bigcap_{a \in \omega^{n+1}} U_{a^\frown (k+1)} = V^{n+1}_{k+1}$$

3. We easily see that $\mathcal{V}^0 = \{U_0, U_1, \dots\}$ is a cover, and then assume $\mathcal{V}^n$ is a cover.
Let $x \in X$ and pick $l < \omega$ such that $x \in V^n_l$. For $a \in l^{n+1}$ choose l_a such that $x \in U_{a^\frown l_a}$, giving
$$x \in \bigcap_{a \in l^{n+1}} U_{a^\frown l_a}$$
We will assume $k > l, l_a$ for all $a \in l^{\leq n+1}$.
For any $a \in k^{n+1} \setminus l^{n+1}$ note that $a = b^\frown c$ where $b \in l^{\leq n}$ and c begins with a number l or greater:
$$V^n_l \subseteq U_{b^\frown l} \subseteq U_{b^\frown c} \subseteq U_{b^\frown c^\frown l_a} = U_{a^\frown l_a}$$
Thus:
$$\begin{aligned} x &\in V^n_l \cap \left(\bigcap_{a \in l^{n+1}} U_{a^\frown l_a} \right) \\ &= V^n_l \cap \left(\bigcap_{a \in k^{n+1} \setminus l^{n+1}} U_{a^\frown l_a} \right) \cap \left(\bigcap_{a \in l^{n+1}} U_{a^\frown l_a} \right) \\ &= V^n_l \cap \left(\bigcap_{a \in k^{n+1}} U_{a^\frown l_a} \right) \\ &\subseteq V^n_k \cap \left(\bigcap_{a \in k^{n+1}} U_{a^\frown k} \right) \\ &= V^{n+1}_k \end{aligned}$$

Finally, apply Menger to $\mathcal{V}^n$, resulting in the cover $\{V^0_{f(0)}, V^1_{f(1)}, \dots\}$, noting

$$X = \bigcup_{n<\omega} V^n_{f(n)} \subseteq \bigcup_{n<\omega} U_{(f\restriction n)^\frown f(n)} = \bigcup_{n<\omega} U_{f\restriction(n+1)}$$

□

Proposition 5. *X is compact if and only if $F \uparrow_{tact} Cov_{C,F}(X)$ if and only if $F \uparrow_{k\text{-}tact} Cov_{C,F}(X)$*

Proof. Assume X is compact. For each open cover played by C, pick a finite subcover, and this yields a winning tactical strategy.

Assume F has a winning k-tactical strategy. For any open cover, have C play only it during the entire game. F's only choice must be a finite subcover. □

Proposition 6. *If X is σ-compact then $F \uparrow_{mark} Cov_{C,F}(X)$*

Proof. Let $X = \bigcup_{n<\omega} X_n$ for compact X_n. On round n, F picks the finite subcover of C's open cover of X_n. □

For Menger's game, there is no useful distinction between a k-Markov strategy for F, and a 2-Markov strategy.

Theorem 7. *For any topological space X and all $k \geq 2$, $F \uparrow_{k\text{-}mark} Cov_{C,F}(X)$ if and only if $F \uparrow_{2\text{-}mark} Cov_{C,F}(X)$.*

Proof. Assume $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{k-1}, n)$ is a winning k-Markov strategy. Define the 2-Markov strategy $\tau(\mathcal{U}, \mathcal{V}, n)$ so that it contains $\sigma(\mathcal{W}_0, \dots, \mathcal{W}_{k-2}, \mathcal{V}, m)$ for the following conditions on $\mathcal{W}_0, \dots, \mathcal{W}_{k-2}, m$:

- Each $\mathcal{W}_i \in \{\mathcal{U}, \mathcal{V}\}$
- $m \leq (n+1)k$; in particular, for $i < k$,

$$\sigma(\mathcal{W}_0, \dots, \mathcal{W}_{k-2}, \mathcal{V}, (n+1)k + i) \subseteq \tau(\mathcal{U}, \mathcal{V}, n+1)$$

Considering an arbitrary play $\mathcal{U}_0, \mathcal{U}_1, \dots$ by C versus τ, we note that σ defeats the play

$$\underbrace{\mathcal{U}_0, \mathcal{U}_0, \dots, \mathcal{U}_0}_{k}, \underbrace{\mathcal{U}_1, \mathcal{U}_1, \dots, \mathcal{U}_1}_{k} \dots$$

So we have that

$$\bigcup_{i<k, n<\omega} \sigma(\underbrace{\mathcal{U}_n, \dots, \mathcal{U}_n}_{k-i-1}, \underbrace{\mathcal{U}_{n+1}, \dots, \mathcal{U}_{n+1}}_{i+1}, (n+1)k + i)$$

is a cover for X, and as

$$\sigma(\underbrace{\mathcal{U}_n, \dots, \mathcal{U}_n}_{k-i-1}, \underbrace{\mathcal{U}_{n+1}, \dots, \mathcal{U}_{n+1}}_{i+1}, (n+1)k+i) \subseteq \tau(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1)$$

τ defeats the play $\mathcal{U}_0, \mathcal{U}_1, \dots$. □

But there are spaces for which there is no Markov strategy, but there is a 2-Markov strategy.

In a question I posed to G, he answered:

Lemma 8. *For all functions $\tau : \omega_1 \times \omega \to [\omega_1]^{<\omega}$, there exists a sequence $\alpha_0, \alpha_1, \dots < \omega_1$ such that $\{\tau(\alpha_n, n) : n < \omega\}$ is not a cover for $\{\beta : \forall n < \omega(\beta < \alpha_n)\}$.*

Proof. Let $P_n = \{\beta : \beta < \alpha \Rightarrow \beta \in \tau(\alpha, n)\}$. Observe that each P_n is finite; else there is some α larger than every member of some countably infinite $P_n^* \subseteq P_n$ such that $P_n^* \subseteq \tau(\alpha, n)$.

Choose $\beta \notin \bigcup_{n<\omega} P_n$. Then for each $n < \omega$, pick $\alpha_n > \beta$ such that $\beta \notin \tau(\alpha_n, n)$. □

Note that the one-point Lindelöfication of discrete ω_1, $\omega_1^\dagger$, is not σ-compact. With the above lemma, we may see that:

Example 9. $F \uparrow Cov_{C,F}(\omega_1^\dagger)$ but $F \not\uparrow_{\text{mark}} Cov_{C,F}(\omega_1^\dagger)$.

Proof. First, we see F has a simple perfect information strategy: in response to the initial cover of $\omega_1^\dagger$, F chooses a co-countable neighborhood of ∞. On successive turns she may pick a single set from C's covers to cover the countable remainder.

Now, suppose that $\sigma(\mathcal{U}, n)$ was a winning Markov strategy and aim for a contradiction. Consider the covers

$$\mathcal{U}(\alpha) = \{[\alpha, \omega_1) \cup \{\infty\}\} \cup \{\{\beta\} : \beta < \alpha\}$$

and define $\tau(\alpha, n)$ to be the union of singletons chosen by $\sigma(\mathcal{U}(\alpha), n)$.

Using the sequence $\alpha_0, \alpha_1, \dots < \omega_1$ from the previous lemma, we consider the play $\mathcal{U}(\alpha_0), \mathcal{U}(\alpha_1), \dots$.

As σ was a winning strategy, $\{\sigma(\mathcal{U}(\alpha_n), n) : n < \omega\}$ must cover $\omega_1^\dagger$, and thus $\{\tau(\alpha_n, n) : n < \omega\}$ must cover $\{\beta : \forall n < \omega(\beta < \alpha_n)\}$, contradiction. □

Telgarski showed in "On Games of Topsoe" that a metrizable space is σ-compact if and only if there exists a winning strategy for F in the Menger game, and Scheepers gave a more direct proof later. We generalize Scheeper's proof to handle a number of cases.

Definition 10. A set $R \subseteq X$ is relatively compact to the topological space X if for every open cover of the entire space X, there is a finite subcover of the set R.

Proposition 11. *If X is regular, then Y is relatively compact if and only if $\overline{Y}$ is compact.*

Proof. The reverse implication is trivial.

Assume Y is relatively compact, let $\mathcal{U}$ be an open cover of $\overline{Y}$, and define $x \in V_x \subseteq \overline{V_x} \subseteq U_x \in \mathcal{U}$ for each $x \in X$. Then if we take a cover $\mathcal{F} = \{V_{x_i} : i < n\}$ of Y by relative compactness, then $\{U_{x_i} : i < n\}$ is a finite cover of $\overline{Y}$, showing compactness. □

Lemma 12. *Let $\sigma(\mathcal{U}, n)$ be a winning Markov strategy for F in $Cov_{C,F}(X)$, and $\mathfrak{C}$ collect all open covers of X. Then for*

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

it follows that R_n is relatively compact to X, and $\bigcup_{n<\omega} R_n = X$.

Proof. First, we see that $\sigma(\mathcal{U}, n)$ witnesses the relative compactness of R_n. Suppose that $x \notin R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$ for any $n < \omega$. Then for each n, pick $\mathcal{U}_n \in \mathfrak{C}$ such that $x \notin \bigcup \sigma(\mathcal{U}_n, n)$. Then σ does not defeat the play $\mathcal{U}_0, \mathcal{U}_1, \ldots$ □

Theorem 13. *A space X is σ-(relatively compact) if and only if $F \uparrow_{mark} Cov_{C,F}(X)$.*

Proof. For the forward implication, let $X = \bigcup_{n<\omega} R_n$ for R_n relatively compact, and define $\sigma(\mathcal{U}, n)$ to be a finite subcover of R_n. The previous lemma proves the other direction. □

Corollary 14. *For regular spaces X, the following are equivalent:*

(a) X is σ-compact

(b) X is σ-(relatively compact)

(c) $F \uparrow_{mark} Cov_{C,F}(X)$

Theorem 15. *For second-countable X, the following are equivalent:*

(a) X is σ-(relatively compact)

(b) $F \uparrow Cov_{C,F}(X)$

(c) $F \uparrow_{mark} Cov_{C,F}(X)$

Proof. . We need only show $(b) \Rightarrow (a)$, so let $\sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{n-1})$ be a winning strategy for F, and observe that since X is second-countable, we may assume all covers are countable. Let $\mathfrak{C}$ be the collection of all countable covers of X. We define $R_s, \mathcal{U}_s$ for $s \in \omega^{<\omega}$ as follows:

- $R_\emptyset = \bigcap_{\mathcal{U} \in \mathfrak{C}} \left(\bigcup \sigma(\mathcal{U}) \right)$
- Note that there are only countably many distinct finite subsets $\sigma(\mathcal{U})$ of the countable collection $\mathcal{U}$. Thus define each $\mathcal{U}_{\langle n \rangle}$ so that

$$R_\emptyset = \bigcap_{n<\omega} \left(\bigcup \sigma(\mathcal{U}_{\langle n \rangle}) \right)$$

- $R_s = \bigcap_{\mathcal{U} \in \mathfrak{C}} \left(\bigcup \sigma(\mathcal{U}_{s\restriction 1}, \mathcal{U}_{s\restriction 2}, \dots, \mathcal{U}_s, \mathcal{U}) \right)$
- Again, note that there are only countably many distinct finite subsets $\sigma(\mathcal{U}_{s\restriction 1}, \mathcal{U}_{s\restriction 2}, \dots, \mathcal{U}_s, \mathcal{U})$ of the countable collection $\mathcal{U}$. Thus define each $\mathcal{U}_{s^\frown\langle n \rangle}$ so that

$$R_s = \bigcap_{n<\omega} \left(\bigcup \sigma(\mathcal{U}_{s\restriction 1}, \mathcal{U}_{s\restriction 2}, \dots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle n \rangle}) \right)$$

We quickly confirm that each R_s is relatively compact as for each open cover $\mathcal{U}$ of X we have the finite subcover $\sigma(\mathcal{U}_{s\restriction 1}, \mathcal{U}_{s\restriction 2}, \dots, \mathcal{U}_s, \mathcal{U})$ of R_s.

Finally, we claim that $X = \bigcup_{s \in \omega^{<\omega}} R_s$. If not, let x be missed by every R_s, and define $f \in \omega^\omega$ such that $x \notin \bigcup \sigma(\mathcal{U}_{f\restriction 1}, \dots, \mathcal{U}_{f\restriction n})$ for any n. Then $\mathcal{U}_{f\restriction 1}, \mathcal{U}_{f\restriction 2}, \dots$ is a counter to the winning strategy σ, a contradiction. □

Corollary 16. *For metric spaces X, the following are equivalent:*

(a) X is σ-compact

(b) X is σ-(relatively compact)

(c) $F \uparrow Cov_{C,F}(X)$

(d) $F \uparrow_{mark} Cov_{C,F}(X)$

Example 17. Let R be given the topology from example 63 from Counterexamples in Topology, the topology generated by open intervals with countable sets removed. This space is non-regular, non-σ-compact, and Lindelöf. It is also Menger as $F \uparrow Cov_{C,F}(R)$, but $F \not\uparrow_{\text{mark}} Cov_{C,F}(R)$.

Proof. From Counterexamples: The irrationals are open, but contain no closed neighborhood, showing non-regular. Compact subsets are exactly finite subsets, showing non-σ-compact.

Take open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$. Define $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{2n})$ to be a finite subcover of $[-n, n] \setminus C_n$ for some countable $C_n = \{c_{n,0}, c_{n,1}, \dots\}$. For $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{2n+1})$, use any subcover of $\{c_{i,j} : i, j < n\}$. It is easily seen that σ is a winning perfect information strategy.

There cannot be a winning Markov strategy $\sigma(\mathcal{U}, n)$, however. Define

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

where $\mathfrak{C}$ is the collection of open covers of R. For any $x_0, x_1, \dots \in R$, we may define the open cover $\mathcal{U} = \{R \setminus \{x_i : i \neq n\} : n < \omega\}$, and observe that $\bigcup \sigma(\mathcal{U}, n) \supseteq R_n$ contains only finitely many x_i. Thus R_n is finite, but since the previous lemma requires $\bigcup_{n<\omega} R_n = R$ if σ is a winning strategy, there exists a counter to σ. $\square$

Example 18. Let R be given the topology from example 67 from Counterexamples in Topology, the topology generated by open intervals with or without the rationals removed. This space is non-regular, non-σ-compact, and Lindelöf.

This space is an example of non-σ-compact but $F \uparrow_{\text{mark}} Cov_{C,F}(R)$ (and is thus also Menger).

Proof. From Counterexamples: The rationals are closed, but the closure of any open neighborhood is the whole real line, so they cannot be separated from any irrational point. Compact sets in this topology are nowhere dense in the Euclidean topology, so there cannot be countably many which union to the whole space. $\{(a, b) \setminus D : a, b \in \mathbb{Q}, D \in \{\emptyset, \mathbb{Q}\}\}$ is a countable base for the space, and second-countability implies Lindelöf.

To see $F \uparrow_{\text{mark}} Cov_{C,F}(R)$, we define $\sigma(\mathcal{U}_{2n}, 2n)$ to be a finite cover of $[-n, n] \setminus \mathbb{Q}$, and $\sigma(\mathcal{U}_{2n+1}, 2n + 1)$ to be a finite cover of $\{q_n\}$ for each $q_n \in \mathbb{Q}$. $\square$

We define a new property "almost-σ-(relatively compact)" to describe a sufficient condition for $F \uparrow_{\text{2-mark}} Cov_{C,F}(X)$.

Definition 19. Let $\mathcal{U}$ be a cover of X. We say $C \subseteq X$ is $\mathcal{U}$-compact if there exists a finite subcover of $\mathcal{U}$ which covers C.

Let $\mathfrak{C}$ collect all the open covers of X. We say X is almost-σ-(relatively compact) if there exists a function $f : \mathfrak{C} \times \omega \to \mathcal{P}(X)$ such that:

- $f(\mathcal{V}, n)$ is $\mathcal{V}$-compact
- $f(\mathcal{V}, n) \cap f(\mathcal{V}, n + 1) = \emptyset$
- $\bigcup_{n<\omega} f(\mathcal{V}, n) = X$
- The set

  $$g(\mathcal{U}, \mathcal{V}, n + 1) = \bigcup_{p \geq n} \Big(f(\mathcal{U}, p + 1) \setminus f(\mathcal{V}, p + 1) \Big) \setminus \bigcup_{m \leq n} f(\mathcal{U}, m + 1)$$

  is $\mathcal{V}$-compact

Obviously σ-(relatively compact) implies almost-σ-(relatively compact) implies Lindelöf. We shall see that the non-σ-(relatively compact) space $\omega_1^\dagger$ is almost-σ-(relatively compact).

Definition 20. For two functions f, g we say f is **μ-almost compatible** with g ($f \perp_\mu^* g$) if $|\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(x) \neq g(x)\}| < \mu$. If $\mu = \omega$ then we say f, g are **almost compatible** ($f \perp^* g$).

Lemma 21. *For each $\alpha < \omega_1$, there exist injective functions $f_\alpha : \alpha \to \omega$ such that if $\alpha < \beta$, then*

$$f_\alpha \perp^* f_\beta$$

that is, f_α and $f_\beta \upharpoonright \alpha$ agree on all but finitely many ordinals. In addition, the range of each f_α is co-infinite.

Proof. Taken from Kunen (used for the construction of an ω_1-Aronszajn tree).

We begin with the empty function $f_0 : 0 \to \omega_1$ which satisfies the hypothesis, and assume f_α is defined by induction. Let $f_{\alpha+1} = f_\alpha \cup \{\langle \alpha, n \rangle\}$ where n is not defined for f_α, and this satisfies the hypothesis.

Finally, suppose γ is the limit of $\alpha_0, \alpha_1, \dots,$ and f_α is defined for $\alpha < \gamma$. Let $g_0 = f_{\alpha_0}$, and assuming $g_n \perp^* f_{\alpha_n}$ with coinfinite range, define $g_{n+1} : \alpha_{n+1} \to \omega$ so that $g_{n+1} \upharpoonright \alpha_n = g_n$ and $g_{n+1} \upharpoonright (\alpha_{n+1} \setminus \alpha_n) \perp^* f_{\alpha_{n+1}}$ with coinfinite range. Then $g = \bigcup_{n<\omega} g_n$ is an injective function from $\gamma \to \omega$ and $g \perp^* f_\alpha$ for $\alpha < \gamma$, but the range need not be coinfinite. So let

$$f_\gamma(\beta) = \begin{cases} g(\alpha_{2n}) & \beta = \alpha_n \\ g(\beta) & \text{otherwise} \end{cases}$$

which frees up $\{g(\alpha_{2n+1}) : n < \omega\}$ from the range of f_γ, and allows $f_\gamma \perp^* f_\alpha$. $\square$

Theorem 22. *The one-point Lindelöfication of the uncountable discrete space, $\omega_1^\dagger$, is almost-σ-(relatively compact).*

Proof. Take the injective funcions f_α from Kunen's lemma such that $f_\alpha \perp^* f_\beta$. For each open cover $\mathcal{V}$ of $\omega_1^\dagger$ let $\gamma(\mathcal{V})$ identify the least ordinal such that $[\gamma(\mathcal{V}), \omega_1) \cup \{\infty\}$ is in a refinement of $\mathcal{V}$. Then f defined by $f(\mathcal{V}, 0) = [\gamma(\mathcal{V}), \omega_1) \cup \{\infty\}$ and $f(\mathcal{V}, n+1) = f_{\gamma(\mathcal{V})}^{-1}(\{n\})$ is easily seen to witness the property. $\square$

Theorem 23. *If X is almost-σ-(relatively compact), then $F \uparrow_{2\text{-}mark} Cov_{C,F}(X)$.*

Proof. Let $\sigma(\mathcal{U}_0, 0)$ cover $f(\mathcal{U}_0, 0)$, and let $\sigma(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1)$ cover $\bigcup_{m \leq n+1} f(\mathcal{U}_{n+1}, m)$ and $g(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1)$. If $\mathcal{U}_0, \mathcal{U}_1, \dots$ is any play by C, then for each $x \in X$, we note that one of the following must occur:

- $x \in f(\mathcal{U}_0, 0)$ and is covered in the initial round.

- $x \in f(\mathcal{U}_0, P+1)$ for some $P \geq 0$ and:
  - $x \in \bigcup_{m \leq n \leq P} f(\mathcal{U}_{n+1}, m+1)$ and is covered by the strategy by round $P+1$.
  - $x \in \bigcap_{n \leq P} f(\mathcal{U}_{n+1}, P+1)$ and is covered by the strategy during round $P+1$.
  - For some $n \leq P$,

    $$x \in \Big(f(\mathcal{U}_n, P+1) \setminus f(\mathcal{U}_{n+1}, P+1)\Big) \setminus \bigcup_{m \leq n} f(\mathcal{U}_{n+1}, m+1) \subseteq g(\mathcal{U}_m, \mathcal{U}_{m+1}, m+1)$$

    and is covered during round $n+1$.

□

Corollary 24. $F \uparrow_{2\text{-}mark} Cov_{C,F}(\omega_1^\dagger)$

Definition 25. The statement $S(\kappa, \mu, \lambda)$ due to Scheepers is shorthand for the following: there exist injective functions $f_A : A \to \lambda$ for each $A \in [\kappa]^\mu$ such that $f_A \perp^*_\mu f_B$ for all $A, B \in [\kappa]^\mu$.

Theorem 26. *$S(\kappa, \omega, \omega)$ implies $\kappa^\dagger$ is almost-σ-(relatively compact).*

Proof. Take the injective funcions $f_A : A \to \omega$ witnessing $S(\kappa, \omega, \omega)$. For each cover $\mathcal{V}$ of $\kappa^\dagger$ let $A(\mathcal{V})$ define a set such that $\kappa^\dagger \setminus A(\mathcal{V})$ is in a refinement of $\mathcal{V}$. Then define f by $f(\mathcal{V}, 0) = \kappa^\dagger \setminus A(\mathcal{V})$ and $f(\mathcal{V}, n+1) = f^{-1}_{A(\mathcal{V})}(\{n\})$.

- $f(\mathcal{V}, n)$ is $\mathcal{V}$-compact by definition.
- $f(\mathcal{V}, n) \cap f(\mathcal{V}, n+1) = \emptyset$ by definition.
- $\bigcup_{n<\omega} f(\mathcal{V}, n) = X$ by definition.
- Note that as $f(\mathcal{U}, p+1) = f^{-1}_{A(\mathcal{U})}(\{p\})$, $f(\mathcal{V}, p+1) = f^{-1}_{A(\mathcal{V})}(\{p\})$, and $f_{A(\mathcal{U})} \perp^* f_{A(\mathcal{V})}$, the set

  $$\left(\bigcup_{p<\omega} \Big(f(\mathcal{U}, p+1) \setminus f(\mathcal{V}, p+1)\Big) \setminus A(\mathcal{U})\right) \setminus A(\mathcal{V})$$

  is finite.

  Thus the set

  $$g(\mathcal{U}, \mathcal{V}, n+1) = \bigcup_{p \geq n} \Big(f(\mathcal{U}, p+1) \setminus f(\mathcal{V}, p+1)\Big) \setminus \bigcup_{m \leq n} f(\mathcal{U}, m+1)$$

  is $\mathcal{V}$-compact.

□

Corollary 27. *$S(\kappa, \omega, \omega)$ implies $F \uparrow_{2\text{-}mark} Cov_{C,F}(\kappa^\dagger)$.*

Definition 28. A finite partial function p from A to B has a domain which is a finite subset of A and a range which is a finite subset of B. Let the set of all finite partial functions from A to B be denoted by $Fn(A, B)$.

Then let $Fn^2(\mathcal{A}, B) \subset Fn(\mathcal{A}, Fn(\bigcup \mathcal{A}, B))$ such that for each $p \in Fn^2(\mathcal{A}, B)$, $p(A) = p_A \in Fn(A, B)$.

Definition 29. Let $\mathbb{P}_\kappa \subset Fn^2([\kappa]^\omega, \omega)$ be such that each p_A is injective, and give it the partial order $\leq$ defined by $q \leq p$ if and only if:

- $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$
- For each $A \in \mathrm{dom}(p)$, $q_A \supseteq p_A$
- For each $A, B \in \mathrm{dom}(p)$, if p_A and p_B are not defined for some $\alpha \in A \cap B$, but q_A is, then q_B is also defined for α and $q_A(\alpha) = q_B(x)$. That is,

$$\alpha \in \mathrm{dom}(q_A) \setminus (\mathrm{dom}(p_A) \cup \mathrm{dom}(p_B)) \Rightarrow \alpha \in \mathrm{dom}(q_B) \text{ and } q_A(x) = q_B(x)$$

Lemma 30. *$\mathbb{P}_\kappa$ has property K (and thus is c.c.c.). That is, let $P \subseteq \mathbb{P}_\kappa$ be uncountable: there is an uncountable $Q \subseteq P$ such that points in Q are pairwise compatible.*

Proof. If $|\{\mathrm{dom}(p) : p \in P\}| > \omega$, we will use the Δ-system lemma to find an uncountable $P' \subseteq P$ such that for $p, q \in P'$, $\mathrm{dom}(p) \cap \mathrm{dom}(q) = \mathcal{R}$. Otherwise, we may fix an uncountable $P' \subseteq P$ such that for $p, q \in P'$, $\mathrm{dom}(p) = \mathrm{dom}(q) = \mathcal{R}$.

Similarly, for each $A \in \mathcal{R}$ we may find that $|\{\mathrm{dom}(p_A) : p \in P'\}| > \omega$, and we can use the Δ-system lemma to find an uncountable $P'' \subseteq P'$ where $\mathrm{dom}(p_A) \cap \mathrm{dom}(q_A) = A'$ for all $p, q \in P''$, or otherwise we may find $P'' \subseteq P'$ where $\mathrm{dom}(p_A) = \mathrm{dom}(q_A) = A'$ for all $p, q \in P''$.

Finally, for each $A \in \mathcal{R}$ and $\alpha \in A'$, we may find $n_{A,\alpha}$ such that there are uncountable $p \in P''$ with $p_A(\alpha) = n_{A,\alpha}$, and thus we may choose $Q \subseteq P''$ to be an uncountable collection such that for $p, q \in Q$, $p_A = q_A$ for $A \in \mathcal{R}$.

Then it is easily verified that $p \cup q \in \mathbb{P}_\kappa$ and $p \cup q \leq p, q$ for all $p, q \in Q$. □

Proposition 31. *For $A \in [\kappa]^\omega$ and $\alpha \in A$, the sets*

$$D_A = \{p \in \mathbb{P}_\kappa : A \in dom(p)\}$$

$$D_{A,\alpha} = \{p \in \mathbb{P}_\kappa : A \in dom(p), \alpha \in dom(p_A)\}$$

are dense in $\mathbb{P}_\kappa$.

Proof. Let $A \in [\kappa]^\omega$ and $p \in \mathbb{P}_\kappa$. Either $p' = p \in D_A$, or $p' = p \cup \{\langle A, \emptyset\rangle\} \in D_A$ with $p' \leq p$.

Let $\alpha \in A$. Either $p'' = p' \in D_{A,\alpha}$, or we may find $n < \omega$ not in the range of p'_A, and then $p'' = p' \setminus \{\langle A, p'_A\rangle\} \cup \{\langle A, p'_A \cup \{\langle \alpha, n\rangle\}\rangle\} \in D_{A,\alpha}$ with $p'' \leq p' \leq p$. □

Theorem 32. *$S(\kappa, \omega, \omega)$ is consistent with ZFC for any cardinal κ.*

Proof. If $\kappa \leq \omega_1$, then $S(\kappa, \omega, \omega)$ is a theorem of ZFC - use Lemma 21.

Assume $\kappa > \omega_1$; we adapt a forcing argument due to Scheepers. Let M be a countable transitive submodel of ZFC. Consider the c.c.c. poset $\mathbb{P}_\kappa$ realized in the model M. Let G be a $\mathbb{P}_\kappa$-generic filter over M.

We now work in the smallest model $M[G]$ extending M and containing G. Observe that by (Kunen), $M[G]$ preserves cofinalities and cardinals.

For each $A \in [\kappa]^\omega$, note $[\kappa]^\omega \cap M$ is cofinal in $[\kappa]^\omega$, so let $A' \supseteq A$ be in $[\kappa]^\omega \cap M$ and let $f_A = \bigcup_{p \in G} p_{A'} \restriction A$. Since G is a $\mathbb{P}_\kappa$-generic filter over M, it is easily verified (considering the dense sets $D_{A,\alpha}$) that f_A is an injective function from A into ω.

In addition, for $A, B \in [\kappa]^\omega \cap M$, let $p \in G \cap D_A \cap D_B$. For all $q \leq p$ it follows that $\{\alpha \in \mathrm{dom}(q_A) \cap \mathrm{dom}(q_B) : q_A(\alpha) \neq q_B(\alpha)\} \subseteq \mathrm{dom}(p_A) \cup \mathrm{dom}(p_B)$. Thus $|\{\alpha \in A \cap B : f_A(\alpha) \neq f_B(\alpha)\}| < \omega$ and $f_A \perp^* f_B$ for $A, B \in [\kappa]^\omega \cap M$, and it's immediate that $f_A \perp^* f_B$ for $A, B \in [\kappa]^\omega$ as well.

The f_A witness $S(\kappa, \omega, \omega)$ relativized to $M[G]$. □

As a result of this forcing for $\kappa > \omega_1$, CH cannot hold in $M[G]$ (Scheepers points out that $\omega_1 < \kappa \leq 2^\omega$). It is an open question whether $S(\kappa, \omega, \omega)$ is a theorem of ZFC+CH or ZFC. Nonetheless,

Corollary 33. *$F \uparrow_{2\text{-}mark} Cov_{C,F}(\kappa^\dagger)$ is consistent with ZFC.*

Filling Games

Definition 34. The **filling game** $Fill_{M,N}^{\subseteq}(J)$ on an ideal J proceeds as follows: player M chooses $M_0 \in \langle J \rangle$, the σ-completion of J, in the initial round, followed by N choosing $N_0 \in J$. In round $n+1$, player M chooses M_{n+1} where $M_n \subseteq M_{n+1} \in \langle J \rangle$, and player N replies with $N_{n+1} \in J$. Player N wins the game if $\bigcup_{n<\omega} N_n \supseteq \bigcup_{n<\omega} M_n$. (The sets in J and $\langle J \rangle$ are thought of as nowhere-dense and meager sets, respectively.)

The **strict filling game** $Fill_{M,N}^{\subsetneq}(J)$ proceeds analogously, with the added requirement that $M_n \subsetneq M_{n+1}$. This game has been studied by Scheepers.

Theorem 35. $N \uparrow_{2\text{-}tact} Fill_{M,N}^{\subsetneq}(J) \Rightarrow N \uparrow_{2\text{-}mark} Fill_{M,N}^{\subseteq}(J)$

Proof. Enumerate the sets in J as A_α for $\alpha < |J|$. For $M \in \langle J \rangle$ and $n < \omega$, let $M + 0 = M$ and $M + n + 1$ be the union of $M + n$ and the least A_α not contained in $M + n$.

Let σ be a winning 2-tactical strategy for N in $Fill_{M,N}^{\subsetneq}(\kappa)$, and assume $\sigma(M) \cup \sigma(M') \subseteq \sigma(M, M')$.

We define a 2-Markov strategy τ for F in $Fill_{M,N}^{\subseteq}(\kappa)$ as follows:

$$\tau(M_0, 0) = \sigma(M_0)$$

$$\tau(M_n, M_{n+1}, n+1) = \begin{cases} \sigma(M_n, M_{n+1}) & \text{if } M_n \subsetneq M_{n+1} \\ \bigcup_{m \leq n} \sigma(M_n + m, M_{n+1} + m + 1) & \text{otherwise} \end{cases}$$

Let $M_0 \subseteq M_1 \subseteq \dots$ be an attack by C against τ. There are two possible cases:

- Assume $M_n = M_N$ for all $n \geq N$.

  The collection produced by σ versus the attack

  $$M_N + 0 \subsetneq M_N + 1 \subsetneq \dots$$

  must cover M_N as σ is a winning strategy.

  Let $x \in M_N$. If $x \in \sigma(M_N + 0)$, then x will be covered in round $N + 1$ by

  $$\tau(M_N, M_N, N+1) \supseteq \sigma(M_N + 0, M_N + 1) \supseteq \sigma(M_N + 0)$$

  Otherwise, $x \in \sigma(M_N + n, M_N + n + 1)$, and x will be covered in round $N + n + 1$ by

  $$\tau(M_N, M_N, N+n+1) \supseteq \sigma(M_N + n, M_N + n + 1)$$

- Otherwise we may find $0 < f(0) < f(1) < \dots$ such that $M_{f(n)} \subsetneq M_{f(n)+1} = M_{f(n+1)}$. Then the collection produced by σ versus the attack

  $$M_{f(0)} \subsetneq M_{f(1)} \subsetneq M_{f(2)} \dots$$

  must cover $\bigcup_{n<\omega} M_n$ as σ is a winning strategy.

  Let $x \in \bigcup_{n<\omega} M_n$. If $x \in \sigma(M_{f(0)})$, then x will be covered by τ in round $f(0)+1$ by

  $$\tau(M_{f(0)}, M_{f(0)+1}, f(0)+1) = \sigma(M_{f(0)}, M_{f(0)+1}) \supseteq \sigma(M_{f(0)})$$

  Otherwise, $x \in \sigma(M_{f(n)}, M_{f(n+1)})$, and x will be covered by τ in round $f(n)+1$ by

  $$\tau(M_{f(n)}, M_{f(n)+1}, f(n)+1) = \sigma(M_{f(n)}, M_{f(n)+1}) = \sigma(M_{f(n)}, M_{f(n+1)})$$

Thus τ is a winning strategy. □

Example 36. There is a free ideal J such that $N \not\uparrow_{\text{2-tact}} Fill_{M,N}^{\subsetneq}(J)$ but $N \uparrow_{\text{2-mark}} Fill_{M,N}^{\subsetneq}(J)$.

Proof. Result based on "Meager nowhere dense games" Prop 9 by Scheepers. Assume $\mathbb{R}$ has the usual Euclidean topology.

Choose $A \subseteq \mathbb{R}$ such that $|A| = \omega$ and A is meager but not nowhere dense. Then choose $V \subseteq \mathbb{R}$ such that $|V| = 2^\omega$, V is meager, and V is disjoint from A. Assume $A = \{a_n : n < \omega\}$.

Certainly, if J is the collection of nowhere dense subsets of $A \cup V$, then $F \uparrow_{\text{2-mark}} Fill_{M,N}^{\subseteq}(J)$. In fact, since $A \cup V$ is meager, $F \uparrow_{\text{pre}} Fill_{M,N}^{\subseteq}(J)$.

By Prop 9 in Scheeper's paper, $F \not\uparrow_{\text{2-tact}} Fill_{M,N}^{\subsetneq}(J)$ immediately. A proof follows: let σ be a 2-tactical strategy such that $\sigma(M) \subseteq \sigma(M, M')$.

We may define K_n to be the collection of pairs of comparable sets $\{B, C\}$ such that $B \subsetneq C$ and n is the least integer where $a_n \in A \setminus \sigma(A \cup B, A \cup C)$.

By Cor 28 of Scheeper's "A partition relation for partially ordered sets", for every partition $\{K_n : n < \omega\}$ of the comparable pairs in $[\mathcal{P}(V)]^2$ there is some $n' < \omega$ and branch $C_0 \subsetneq C_1 \subsetneq \dots \subsetneq V$ where $\{C_m, C_{m+1}\} \in K_{n'}$ for all $m < \omega$.

Then σ may be countered by the attack $A \cup C_0, A \cup C_1, \dots$, since $a_{n'} \in A \setminus \sigma(A \cup C_m, A \cup C_{m+1})$ for all $m < \omega$ and thus is never covered. □

Rothberger

Definition 37. X is **Rothberger** if for all open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ there exist open sets $U_n \in \mathcal{U}_n$ such that $\{U_n : n < \omega\}$ is a cover of X.

Proposition 38. *Rothberger* $\Rightarrow$ *Menger*

Definition 39. In the two-player game $Cov_{C,S}(X)$ player C chooses open covers $\mathcal{U}_n$ of X, followed by player S choosing an open set $U_n \in \mathcal{U}_n$. S wins if $\{U_n : n < \omega\}$ is a cover of X.

Theorem 40. *X is Rothberger if and only if $C \not\uparrow Cov_{C,S}(X)$.*

Proof. Due to Pawlikowski $\square$

Definition 41. A space X is scattered if every subspace contains an isolated point. By convention, $X = \bigcup_{\alpha < \text{rank}(X)} X^\alpha$ where X^α is the set of isolated points of $X \setminus \bigcup_{\beta<\alpha} X^\beta$.

Proposition 42. *A space X is scattered if and only if every closed subspace contains an isolated point.*

Proposition 43. *The rank of a compact scattered T_1 space is a successor ordinal, and $X^{rank(X)-1}$ is finite.*

Proof. Suppose that the rank of X was a limit ordinal β. Then by choosing $\beta_n \to \beta$, we may pick a point $x_n \in X^{\beta_n}$, and $\{x_n : n < \omega\}$ may be shown to be a closed discrete subspace of X.

It's easily seen that $X^{\text{rank}(X)-1}$ must be finite - it is a closed discrete subspace of compact X. $\square$

Theorem 44. *The following are equivalent for compact T_2 X:*

(a) *X is Rothberger*

(b) *X is scattered*

(c) $S \uparrow Cov_{C,S}(X)$

(d) $C \not\uparrow Cov_{C,S}(X)$

Proof. To show $(a) \Rightarrow (b)$, we use Aurichi's proof in *D-Spaces*: suppose X has a closed subspace without isolated points. Then it is compact and contains a closed copy of the Cantor set, which is not Rothberger, contradiction.

To show $(b) \Rightarrow (c)$, if X is scattered, suppose during a particular round n, player S observes that the uncovered subspace $Y \subseteq X$ is nonempty. Then as Y is also compact scattered, select one of the finite points in $Y^{\mathrm{rank}(Y)-1}$, label it x_n, and choose an open set containing x_n from the given cover.

We claim that if S follows this strategy, player S will observe that the uncovered subspace $Y \subseteq X$ is empty during some round. If not, consider the x_n chosen by Y by the end of the game - the rank of each point within X is nonincreasing, and does not contain a constant final sequence, contradiction.

Of course, $(c) \Rightarrow (a)$ is trivial, and $(a) \Leftrightarrow (d)$. □

Definition 45. In the two-player game $Cov_{P,O}(X)$ player P chooses points $x_n \in X$, followed by player O choosing an open neighborhood U_n of x_n. P wins if $\{U_n : n < \omega\}$ is a cover of X.

Theorem 46. *$Cov_{P,O}(X)$ is "perfect information equivalent" to $Cov_{C,S}(X)$. That is:*

- *$P \uparrow Cov_{P,O}(X)$ if and only if $S \uparrow Cov_{C,S}(X)$*
- *$O \uparrow Cov_{P,O}(X)$ if and only if $C \uparrow Cov_{C,S}(X)$.*

Proof. Due to Galvin.

- Let σ be a strategy for S in $Cov_{C,S}(X)$.

  Let $n < \omega$, and consider open covers $\mathcal{U}_m$ for each $m < n$. Suppose that for each $x \in X$, there was an open neighborhood U_x of x where for every open cover $\mathcal{U}$, $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{n-1}, \mathcal{U}) \neq U_x$. The open cover $\{U_x : x \in X\}$ demonstrates the contradiction.

  We define a strategy for P in $Cov_{P,O}(X)$ as follows: during round n, P chooses a point x_n for which every open neighborhood is of the form $U_n = \sigma(\mathcal{U}_0, \dots, \mathcal{U}_{n-1}, \mathcal{U}_n)$ for some open cover $\mathcal{U}_0$.

  If σ was a winning strategy for S in $Cov_{C,S}(X)$, then the open sets chosen by O in response to P's strategy for $Cov_{P,O}(X)$ are of the form $\{\sigma(\mathcal{U}_0), \sigma(\mathcal{U}_0, \mathcal{U}_1), \dots\}$ and are an open cover of X.

- Let σ be a strategy for P in $Cov_{P,O}(X)$.

  We define a strategy for S in $Cov_{C,S}(X)$ as follows: during round n, if S has chosen $U_0, \dots, U_{n-1}$ in previous rounds, S chooses any open set in C's latest cover containing the point $\sigma(U_0, \dots, U_{n-1})$. If σ was a winning strategy for P, then for any open sets $U_0, U_1, \dots$ containing $\sigma(\cdot), \sigma(U_0), \dots$, the collection $\{U_0, U_1, \dots\}$ is a cover for X.

- Let σ be a strategy for C in $Cov_{C,S}(X)$.

  We define a strategy for O in $Cov_{P,O}(X)$ as follows: during round n, if O has chosen $U_0, \dots, U_{n-1}$ in previous rounds, O chooses an open set from the cover $\sigma(U_0, \dots, U_{n-1})$

containing the point chosen by P that round. If σ was a winning strategy for C, then for any open sets $U_0, U_1, \dots$ from the covers $\sigma(\cdot), \sigma(U_0), \dots$, the collection $\{U_0, U_1, \dots\}$ is not a cover for X.

- Let σ be a strategy for O in $Cov_{P,O}(X)$.

  We define a strategy for C in $Cov_{C,S}(X)$ as follows: during round 0, C chooses $\mathcal{U}_0 = \{\sigma(x) : x \in X\}$. In response, S chooses some $\sigma(x_0)$. During round $n+1$, if S has chosen $\sigma(x_0), \dots, \sigma(x_0, \dots, x_n)$ in previous rounds, C chooses $\mathcal{U}_{n+1} = \{\sigma(x_0, \dots, x_n, x) : x \in X\}$. If σ was a winning strategy for O, then $\{\sigma(x_0), \sigma(x_0, x_1), \dots\}$ is not a cover for X.

□

A similar theorem exists for limited information strategies.

Theorem 47.
- *$P \uparrow_{pre} Cov_{P,O}(X)$ if and only if $S \uparrow_{mark} Cov_{C,S}(X)$*
- *$O \uparrow_{mark} Cov_{P,O}(X)$ if and only if $C \uparrow_{pre} Cov_{C,S}(X)$.*

Proof.
- Let $\sigma(\mathcal{U}_n, n)$ be a Markov strategy for S in $Cov_{C,S}(X)$.

  Let $n < \omega$. Suppose that for each $x \in X$, there was an open neighborhood U_x of x where for every open cover $\mathcal{U}$, $\sigma(\mathcal{U}, n) \neq U_x$. The open cover $\{U_x : x \in X\}$ demonstrates the contradiction.

  We define a predetermined strategy for P in $Cov_{P,O}(X)$ as follows: during round n, P chooses a point x_n for which every open neighborhood is of the form $U_n = \sigma(\mathcal{U}, n)$ for some open cover $\mathcal{U}$.

  If σ was a winning strategy for S in $Cov_{C,S}(X)$, then the open sets chosen by O in response to P's strategy for $Cov_{P,O}(X)$ are of the form $\{\sigma(\mathcal{U}_0, 0), \sigma(\mathcal{U}_1, 1), \dots\}$ and are an open cover of X.

- Let $\sigma(n)$ be a predetermined strategy for P in $Cov_{P,O}(X)$.

  We define a Markov strategy for S in $Cov_{C,S}(X)$ as follows: during round n, S chooses any open set in C's cover containing the point $\sigma(n)$. If σ was a winning strategy for P, then for any open sets $U_0, U_1, \dots$ containing $\sigma(0), \sigma(1), \dots$, the collection $\{U_0, U_1, \dots\}$ is a cover for X.

- Let $\sigma(n)$ be a predetermined strategy for C in $Cov_{C,S}(X)$.

  We define a Markov strategy for O in $Cov_{P,O}(X)$ as follows: during round n, O chooses an open set from the cover $\sigma(n)$ containing the point chosen by P that round. If σ was a winning strategy for C, then for any open sets $U_0, U_1, \dots$ from the covers $\sigma(0), \sigma(1), \dots$, the collection $\{U_0, U_1, \dots\}$ is not a cover for X.

- Let $\sigma(x_n, n)$ be a Markov strategy for O in $Cov_{P,O}(X)$.

  We define a predetermined strategy for C in $Cov_{C,S}(X)$ as follows: during round n, C chooses $\mathcal{U}_n = \{\sigma(x, n) : x \in X\}$. If σ was a winning strategy for O, we observe any play $\{\sigma(x_0, 0), \sigma(x_1, 1), \dots\}$ by S is not a cover for X.

$\square$

Definition 48. Let $\mathcal{N}(x)$ be the **neighborhood network** of x, that is, the collection of all neighborhoods of x.

Definition 49. A topological space X is **almost countable** if there exist $x_n \in X$ for each $n < \omega$ such that $X = \bigcup_{n<\omega} \bigcap \mathcal{N}(x_n)$.

Theorem 50. *For any space X, the following are equivalent:*

- $S \uparrow_{mark} Cov_{C,S}(X)$
- $P \uparrow_{pre} Cov_{P,O}(X)$
- *X is almost countable*

Proof. If there exist $x_n \in X$ for each $n < \omega$ such that $X = \bigcup_{n<\omega} \bigcap \mathcal{N}(x_n)$, then let $\sigma(n) = x_n$ be a predetermined strategy for P in $Cov_{P,O}(X)$. For any neighborhoods O_n of x_n chosen by O to attack σ, note that $\bigcup_{n<\omega} O_n \supseteq \bigcup_{n<\omega} \bigcap \mathcal{N}(x_n) = X$ results in a win for P.

Likewise, if for each sequence $x_n \in X$ there is $x \in X$ with $x \notin \bigcup_{n<\omega} \bigcap \mathcal{N}(x_n)$, then for a fixed strategy $\sigma(n)$ for P, O may counter σ by choosing $O_n \in \mathcal{N}(x_n)$ which misses x during each round, causing σ to lose. $\square$

Theorem 51. *For any T_1 space X, the following are equivalent:*

- $S \uparrow_{mark} Cov_{C,S}(X)$
- $P \uparrow_{pre} Cov_{P,O}(X)$
- *X is almost countable*
- $|X| \leq \omega$

Proof. If $|X| = \omega$ then the winning Markov strategy is obvious, so let $\sigma(\mathcal{U}, n)$ be a Markov strategy.

Let $n < \omega$. Suppose that for each $x \in X$, there was an open neighborhood U_x of x where for every open cover $\mathcal{U}$, $\sigma(\mathcal{U}, n) \neq U_x$. The open cover $\{U_x : x \in X\}$ demonstrates the contradiction.

So let $x_n \in X$ be chosen such that for each open neighborhood U of x_n, there is an open cover $\mathcal{U}$ such that $\sigma(\mathcal{U}, n) = U$. Then if $x \neq \{x_n : n < \omega\}$, C may counter σ as follows: during round n, choose U_n which contains x_n but not x, and then choose $\mathcal{U}_n$ such that $\sigma(\mathcal{U}_n, n) = U_n$. □

Example 52. Let $X = \omega_1 \cup \{\infty\}$ be a "weak Lindelöfication" of discrete ω_1 such that open neighborhoods of ∞ contain $\omega_1 \setminus \omega$. This space is T_0 but not T_1, and note that $S \uparrow_{\text{mark}} Cov_{C,S}(X)$ and $|X| > \omega$.

Theorem 53. *The following are equivalent for points-G_δ X:*

(a) $S \uparrow Cov_{C,S}(X)$

(b) $P \uparrow Cov_{P,O}(X)$

(c) $S \uparrow_{k\text{-}mark} Cov_{C,S}(X)$ *for some* $k \geq 1$

(d) $S \uparrow_{mark} Cov_{C,S}(X)$

(e) $P \uparrow_{pre} Cov_{P,O}(X)$

(f) X *is almost countable*

(g) $|X| \leq \omega$

Proof. Due to Galvin. Let σ be a strategy for S in $Cov_{C,S}(X)$.

Let $G_{x,m}$ designate open sets such that $\{x\} = \bigcap_{m<\omega} G_{x,m}$ for all $x \in X$.

Let $n < \omega$, $s \in \omega^n$, and consider open covers $\mathcal{U}_t$ for each $t \leq s$. Suppose that for each $x \in X$, there was an open neighborhood U_x of x where for every open cover $\mathcal{U}$, $\sigma(\mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{U}) \neq U_x$. The open cover $\{U_x : x \in X\}$ demonstrates the contradiction.

Thus C may find points x_s such that for each $m < \omega$, there exists an open cover $\mathcal{U}_{s^\frown\langle m\rangle}$ where $\sigma(\mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle m\rangle}) = G_{x_s,m}$. Then if $x \neq \{x_s : s \in \omega^{<\omega}\}$, C may counter σ as follows: during round n, choose $f(n)$ so that $x \notin G_{x_{f\restriction n}, f(n)}$, and then choose $\mathcal{U}_{f\restriction n^\frown\langle f(n)\rangle}$ such that $\sigma(\mathcal{U}_{f\restriction 1}, \dots, \mathcal{U}_{f\restriction n}, \mathcal{U}_{f\restriction n^\frown\langle f(n)\rangle}) = G_{x_{f\restriction n}, f(n)}$. □

Corollary 54. *The following are equivalent for compact points-G_δ X:*

(a) $S \uparrow Cov_{C,S}(X)$

(b) $P \uparrow Cov_{P,O}(X)$

(c) $S \uparrow_{k\text{-}mark} Cov_{C,S}(X)$ *for some* $k \geq 1$

(d) $S \uparrow_{mark} Cov_{C,S}(X)$

(e) $P \uparrow_{pre} Cov_{P,O}(X)$

(f) X *is almost countable*

(g) $|X| \leq \omega$

(h) $C \not\uparrow Cov_{C,S}(X)$

(i) $O \not\uparrow Cov_{P,O}(X)$

(j) X *is Rothberger*

(k) X *is scattered*

Definition 55. The game $Rec^m_{F,S}(\kappa)$ proceeds as follows: during round 0, player F chooses $F_0 \in [\kappa]^m$, followed by player S choosing $x_0 \in F_0 \cup \{\infty\}$. During round $n+1$, F chooses $F_{n+1} \in [\kappa]^{m^{n+2}}$ such that $F_{n+1} \supset F_n$, followed by S choosing $x_{n+1} \in F_{n+1} \cup \{\infty\}$.

S wins the game if $\{x_n : n < \omega\} \supseteq F_0 \cup \{\infty\}$, and F wins otherwise.

Proposition 56. $S \uparrow_{limit} Cov_{C,S}(\kappa^\dagger) \Rightarrow S \uparrow_{limit} Rec^m_{F,S}(\kappa)$

Proof. Let σ be a limited information strategy for S in $Cov_{C,S}(\kappa^\dagger)$.

Suppose $C(\cdot)$ converts any finite set G played by F in $Rec^m_{F,S}(\kappa)$ into the open cover $\mathcal{U}_G = [G]^1 \cup \{\kappa^* \setminus G\}$. Then we may define a strategy τ using the same type of information as σ by setting $\tau(\cdot) \in \sigma(C(\cdot))$.

Suppose that the attack $F_0, F_1, \dots$ countered τ. Let x_n be the point given by τ during round n, and choose $\alpha \in (F_0 \cup \{\infty\}) \setminus \{x_n : n < \omega\}$.

If $\alpha = \infty$, then σ may be countered by the attack $\mathcal{U}_{F_0}, \mathcal{U}_{F_1}, \dots$ since no neighborhood of ∞ is ever chosen.

Similarly, if $\alpha \in F_0$, then σ may also be countered by the attack $\mathcal{U}_{F_0}, \mathcal{U}_{F_1}, \dots$ since the singleton $\{\alpha\}$ is never chosen. $\square$

Proposition 57. $S \uparrow_{k\text{-}mark} Rec^m_{F,S}(\kappa) \Leftrightarrow S \uparrow_{k\text{-}tact} Rec^m_{F,S}(\kappa)$

Proof. The round number is deterimined by the size of the sets played by C. $\square$