PREDETERMINED PROXIMAL SPACES ARE METRIZABLE

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ABSTRACT. TODO

1. Predetermined Proximal

We take the following from Willard's text.

Definition 1.1. A *normal covering sequence* for a space X is a sequence $\{\mathcal{U}_n : n < \omega\}$ of open covers such that $\mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$. Such a sequence is *compatible* with X if $\{St(x,\mathcal{U}_n) : n < \omega\}$ is a local base at each $x \in X$.

Theorem 1.2. *A space X is psuedometrizable if and only if it has a compatible normal covering sequence.*

For convenience, we will recast these results in terms of entourages.

Definition 1.3. An *entourage sequence* for a space X $\{D_n : n < \omega\}$ is *compatible* with X if $\{D_n[x] : n < \omega\}$ is a local base at each $x \in X$.

Theorem 1.4. *A space X is psuedometrizable if and only if it has a compatible entourage sequence.*

Proof. Let d be a psuedometric generating X; then $\{D_n : n < \omega\}$ given by $D_n = \{\langle x, y\rangle : d(x, y) < 2^{-n}\}$ is a entourage sequence, which is compatible since $D_n[x] = B_{2^{-n}}(x)$.

On the other hand, given a compatible entourage sequence $\{D_n : n < \omega\}$, let $E_0 = D_0$, $E_{n+1} = \frac{1}{4}D_n \cap \frac{1}{4}E_n$, and $\mathcal{U}_n = \{E_{n+1}[x] : x \in X\}$. Fix $x \in X$ and consider $St(x,\mathcal{U}_n) \subseteq St(E_{n+1}[x],\mathcal{U}_n) = \bigcup \{E_{n+1}[y] \in \mathcal{U}_n : E_{n+1}[x] \cap E_{n+1}[y] \neq \emptyset\}$.

Let $z \in St(E_{n+1},\mathcal{U}_n)$, so $z \in E_{n+1}[y]$ for some $y \in X$ where $w \in E_{n+1}[x] \cap E_{n+1}[y]$ for some $w \in X$. It follows that $\langle z, y\rangle , \langle y, w\rangle , \langle w, x\rangle \in E_{n+1} = \frac{1}{4}D_n \cap \frac{1}{4}E_n$; therefore $\langle z, x\rangle \in D_n \cap E_n$ and $z \in D_n[x] \cap E_n[x]$. Therefore $St(x,\mathcal{U}_n) \subseteq St(E_{n+1}[x],\mathcal{U}_n) \subseteq D_n[x] \cap E_n[x]$.

We can then observe that $\mathcal{U}_{n+1}$ star refines $\mathcal{U}_n$ since for each $U \in \mathcal{U}_{n+1}$, $U = E_{n+2}[x]$ for some $x \in X$ and $St(E_{n+2}[x],\mathcal{U}_{n+1}) \subseteq E_{n+1}[x] \in \mathcal{U}_n$, making $\{\mathcal{U}_n : n < \omega\}$ a normal covering sequence. Finally, the sequence is compatible since for $x \in X$, $St(x,\mathcal{U}_n) \subseteq D_n[x]$ and $\{D_n[x] : n < \omega\}$ is a local base at x. $\square$

Theorem 1.5. *A space X is psuedometrizable if and only if* $\mathrm{I} \underset{pre}{\uparrow} Bell_{D,P}^{\rightarrow,\emptyset}(X)$.

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Proof. Suppose X is psuedometrizable by d; then let σ be the predetermined strategy for $Bell_{D,P}^{\rightarrow,\emptyset}(X)$ defined by $\sigma(n) = \{\langle x, y\rangle : d(x, y) < 2^{-n}\}$. For any legal attack α against σ, $\alpha(n+1) \in \sigma(n)[\alpha(n)]$. It follows that if $x \in \bigcap_{n<\omega} \sigma(n)[\alpha(n)]$ and $\epsilon > 0$, we may choose $N < \omega$ such that $2^{-N} < \epsilon$. Therefore $d(x, \alpha(n)) < 2^{-n} \leq 2^{-N} < \epsilon$ for all $n \geq N$, showing α converges to x. Thus σ is a winning strategy.

Now let σ be any predetermined winning strategy satisfying $\sigma(n) \subseteq \sigma(m)$ for all $n \geq m$, and suppose $\left\{\frac{1}{2^{n+1}}\sigma(n)[x] : n < \omega\right\}$ is not a local base at some $x \in X$. Then we may pick an entourage D such that $\frac{1}{2^{n+1}}\sigma(n)[x] \not\subseteq D[x]$ for all $n < \omega$. So choose $\alpha(n) \in \frac{1}{2^{n+1}}\sigma(n)[x] \setminus D[x]$.

Observe that $\langle \alpha(n), x\rangle \in \frac{1}{2^{n+1}}\sigma(n)$ and $\langle \alpha(n+1), x\rangle \in \frac{1}{2^{n+2}}\sigma(n+1) \subseteq \frac{1}{2^{n+1}}\sigma(n)$. It follows that $\langle \alpha(n), \alpha(n+1)\rangle \in \frac{1}{2^n}\sigma(n) \subseteq \sigma(n)$, witnessing that $\alpha(n+1) \in \sigma(n)[\alpha(n)]$, that is, α is a legal counterattack to σ. Since $x \in \frac{1}{2^{n+1}}\sigma(n)[\alpha(n)] \subseteq \sigma(n)[\alpha(n)]$ for all $n < \omega$, σ can only win for I if α converges. But $\alpha(n) \notin D[x]$ for all $n < \omega$, so α fails to converge as well. Thus σ is not a winning strategy.

As a result, if σ is a winning predetermined strategy, we have that $\left\{\frac{1}{2^{n+1}}\sigma(n)[x] : n < \omega\right\}$ is a local base at each $x \in X$. This shows that $\left\{\frac{1}{2^{n+1}}\sigma(n) : n < \omega\right\}$ is a compatible entourage sequence; therefore by the previous lemma, X is psuedometrizable. $\square$

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