

Definition 1. Let a V-map be a u.s.c. idempotent surjection.

Definition 2. For any LOS $\langle L, \leq \rangle$, let \check{L} be the collection of leftward subsets of L (subsets for which $b \in L, a \leq b \Rightarrow a \in L$) linearly ordered by \subseteq , and let \hat{L} be the collection of left-closed subsets of L (leftward subsets which are closed) linearly ordered by \subseteq .

Proposition 3. \check{L}, \hat{L} are compact.

Proof. Each subset S has an infimum $\cap S$ and a supremum $\cup S$ (or $\text{cl}(\cap S)$). \square

Theorem 4. \check{L} is second-countable if and only if L is countable.

Proof. If L is uncountable, let \mathcal{B} be a base for \check{L} . For each $l \in L$, there is some neighborhood $B_l \in \mathcal{B}$ of (\leftarrow, l) which is a subset of $[\emptyset, (\leftarrow, l)) = [\emptyset, (\leftarrow, l)]$. Each B_l is unique, so \check{L} is not second-countable.

If L is countable, then note that the union of $\mathcal{S} = \{((\leftarrow, l), L) : l \in L\}$ and $\mathcal{T} = \{[\emptyset, (\leftarrow, l)) : l \in L\}$ generates a countable base for \check{L} . \square

Corollary 5. \check{L} is metrizable if and only if L is countable.

Note that \check{L} is not a “compactification” as L does not necessarily embed as a dense subspace of \check{L} : if $L = I$, we might attempt to embed $t \mapsto [0, t)$, but then note that the subspace topology induces the reverse Sorgenfrey interval as $([0, s), [0, t)) = ([0, s), [0, t))$ is open. However \hat{L} is the typical way of compactifying a linearly ordered space L , provided L lacks a least element (otherwise the empty set is an [easily removable] isolated point in \hat{L}). Note that we **always** assume that $\emptyset \in \hat{L}$:

Example 6. $\hat{I} \cong \{-\infty\} \cup I$ where $\emptyset \mapsto -\infty$ and $[0, t] \mapsto t$.

Example 7. For limit ordinals α , $\hat{\alpha} \cong \alpha + 1$, and for all other infinite ordinals, $\hat{\alpha} \cong \alpha$. (The addition of a new least isolated point is of course irrelevant).

Definition 8. For any compact LOTS K with minimum 0 and maximum 1, let γ be the V-map on K where $\gamma(0) = K$ and $\gamma(t) = \{1\}$ for $t > 0$.

Theorem 9. $X = \varprojlim \{2, \gamma, L\} \cong \check{L}$

Proof. We start by placing an order on X . Let $\vec{x} < \vec{y}$ if there exists $a \in L$ with $\vec{x}(a) = 0, \vec{y}(a) = 1$. We claim this is a total order inducing the topology on X .

We first observe that if $\vec{x}(b) = 1$, then for all $a \leq b$, $\vec{x}(a) \in \gamma(1) = \{1\}$. If $\vec{x} \neq \vec{y}$, then assume without loss of generality that $\vec{x}(a) = 0, \vec{y}(a) = 1$, so $\vec{x} < \vec{y}$. Also, whenever $\vec{x}(b) = 1$, we have that $b < a$, so $\vec{y}(b) = 1$, preventing $\vec{y} < \vec{x}$. Finally if $\vec{x} < \vec{y}$ and $\vec{y} < \vec{z}$, take a, b with $\vec{x}(a) = 0, \vec{y}(a) = 1, \vec{y}(b) = 0, \vec{z}(b) = 1$. It follows that $a < b$ so $\vec{z}(a) = 1$ and $\vec{x} < \vec{z}$.

Consider the basic open set $B(\vec{x}, F)$ for a finite set $F \in [L]^{<\omega}$ about the sequence $\vec{x} \in X$ which contains all sequences \vec{y} agreeing with \vec{x} on F . If $\vec{x}(a) = 1$ for all $a \in F$, then let $\vec{w} \in X$ be 0 on the maximum of F , and 1 for anything less. It follows that $B(\vec{x}, F) = (\vec{w}, \rightarrow)$. If $\vec{x}(a) = 0$ for all $a \in F$, then let $\vec{y} \in X$ be 1 on the minimum of F , and 0 for anything greater. It follows that $B(\vec{x}, F) = (\leftarrow, \vec{y})$. Finally if $\vec{x}(a) = 1$ and $\vec{x}(b) = 0$ for $a < b$ in F and nothing between a, b is in F , then let $\vec{w} \in X$ be 0 on a and 1 for anything less, and let $\vec{y} \in X$ be 1 on b and 0 for anything greater. It follows that $B(\vec{x}, F) = (\vec{w}, \vec{y})$.

Let ϕ evaluate each $\vec{x} \in X \subseteq 2^L$ as the characteristic function for a subset of L . It's easy to see that ϕ is an order isomorphism between $\langle X, \leq \rangle$ and $\langle \check{L}, \subseteq \rangle$. \square

Corollary 10. *If f is a non-trivial V-map satisfying condition Γ , then $\varprojlim\{X, f, L\}$ is metrizable if and only if X is metrizable and L is countable.*

Proof. If X is metrizable and L is countable, then of course $\varprojlim\{X, f, L\} \subseteq X^L$ is metrizable. If X is nonmetrizable, then $\varprojlim\{X, f, L\}$ cannot be metrizable as it contains a copy of X . Finally, if L is uncountable, then $\varprojlim\{X, f, L\}$ cannot be metrizable as it contains a copy of $\varprojlim\{2, \gamma, L\} \cong \check{L}$. \square

Corollary 11. $\varprojlim\{2, \gamma, \alpha\} \cong \alpha + 1$ for every ordinal α .

Proof. Since $\check{\alpha} = \alpha + 1$ (actually equals, not just homeomorphic!), we get $\varprojlim^*\{2, \gamma, \alpha\} \cong \check{\alpha} = \alpha + 1$ for free. Note that C and Varagona used this in (TODO create citation) to break metrizability in uncountable-ordinal-indexed inverse limits (for any V-map there exists a two-point set 2 such that $f \restriction 2 \supseteq \gamma$, that is, “ f has condition Γ ”). \square

We may generalize theorem 8 as follows:

Theorem 12. *If M is a LOTS with minimum 0 and maximum 1, then $\varprojlim\{M, \gamma, L\} \cong \hat{L} \times_{\text{lex}} M / \sim$, where $\langle (\leftarrow, l_0], 1 \rangle \sim \langle (\leftarrow, l_1], 0 \rangle$ if $l_0 < l_1$ and $(l_0, l_1) = \emptyset$, and where $\langle A, m \rangle \sim \langle A, m' \rangle$ if $A \in \hat{L} \setminus L$.*

Proof. Let $\rho(\vec{x}) = \text{cl}\{l \in L : \vec{x}(l) > 0\}$, $v(\vec{0}) = 0$, and $v(\vec{x}) = \min\{\vec{x}(l) : l \in \rho(\vec{x})\}$ otherwise. Say $\vec{x} < \vec{y}$ if $\rho(\vec{x}) \subsetneq \rho(\vec{y})$ or both $\rho(\vec{x}) = \rho(\vec{y})$ and $v(\vec{x}) < v(\vec{y})$. The reader may verify that this is a linear order on $\varprojlim\{M, \gamma, L\}$, and $\theta(\vec{x}) = \langle \rho(\vec{x}), v(\vec{x}) \rangle \in \hat{L} \times_{\text{lex}} M / \sim$ preserves order. For each left-closed set A and $m \in M$, let $\vec{x}_{A,m}(l) = 1$ for $l \in A$ unless l is the supremum element of A , $\vec{x}_{A,m}(l) = m$ if l is the supremum of A , and $\vec{x}_{A,m}(l) = 0$ for $l \notin A$. To complete the proof, we should demonstrate that the linear order we defined induces the topology of the inverse limit, and that θ is a surjection.

A basic open set in $\varprojlim\{M, \gamma, L\} \subseteq L^m$ is of the form $\llbracket U, F \rrbracket$ where $U(l)$ is an open interval in M for each $l \in F \in [L]^{<\omega}$, and $\llbracket U, F \rrbracket = \{\vec{x} : l \in F \Rightarrow \vec{x}(l) \in U(l)\}$. If we assume that $\llbracket U, F \rrbracket$ is non-empty, one of the following must hold:

- $U[l_0] = (a, b)$ for some $l_0 \in F$. Then $\llbracket U, F \rrbracket = \llbracket U, \{l_0\} \rrbracket$, and note that $\llbracket U, \{l_0\} \rrbracket = (\vec{x}_{(\leftarrow, l_0], a}, \vec{x}_{(\leftarrow, l_0], b})$.
- $U(l_0) = (a, 1]$ and $U(l_1) = [0, b)$ for some $l_0 < l_1 \in L$ and $\llbracket U, F \rrbracket = \llbracket U, \{l_0, l_1\} \rrbracket$. Then $\llbracket U, \{l_0, l_1\} \rrbracket = (\vec{x}_{l_0, a}, \vec{x}_{l_1, b})$.

In the other direction, consider $\vec{y} \in (\vec{x}, \vec{z})$.

- In the case that $l_0 \in \rho(\vec{y}) \setminus \rho(\vec{x})$ and $l_1 \in \rho(\vec{z}) \setminus \rho(\vec{y})$, let $U(l_0) = (0, 1]$, $U(l_1) = [0, v(\vec{z}))$ and note $\vec{y} \in \llbracket U, \{l_0, l_1\} \rrbracket \subseteq (\vec{x}, \vec{z})$.
- In the case that $l_0 \in \rho(\vec{y}) \setminus \rho(\vec{x})$, $\rho(\vec{y}) = \rho(\vec{z})$, and $v(\vec{y}) < v(\vec{z})$, it follows that $\rho(\vec{y}) = \rho(\vec{z}) = (\leftarrow, l_1]$, so let $U(l_0) = (0, 1]$, $U(l_1) = [0, v(\vec{z}))$ and note $\vec{y} \in \llbracket U, \{l_0, l_1\} \rrbracket \subseteq (\vec{x}, \vec{z})$.
- In the case that $\rho(\vec{x}) = \rho(\vec{y})$, $v(\vec{x}) < v(\vec{y})$, and $l_1 \in \rho(\vec{z}) \setminus \rho(\vec{y})$, it follows that $\rho(\vec{x}) = \rho(\vec{y}) = (\leftarrow, l_0]$, so let $U(l_0) = (v(\vec{x}), 1]$, $U(l_1) = [0, v(\vec{z}))$ and note $\vec{y} \in \llbracket U, \{l_0, l_1\} \rrbracket \subseteq (\vec{x}, \vec{z})$.
- In the case that $\rho(\vec{x}) = \rho(\vec{y}) = \rho(\vec{z})$ and $v(\vec{x}) < v(\vec{y}) < v(\vec{z})$, it follows that $\rho(\vec{x}) = \rho(\vec{y}) = \rho(\vec{z}) = (\leftarrow, l_0]$, so let $U(l_0) = (v(\vec{x}), v(\vec{z}))$ and note $\vec{y} \in \llbracket U, \{l_0\} \rrbracket = (\vec{x}, \vec{z})$.

We conclude by showing that θ is a surjection. If $B \in \hat{L} \setminus L$ and $m \in M$, consider $\langle B, m \rangle$. B lacks a supremum in L , so $\vec{x}_{B,0}(l) = 1$ for $l \in B$ and $\vec{x}_{B,0}(l) = 0$ otherwise. So $\theta(\vec{x}_{B,0}) = \langle \text{cl} B, 1 \rangle = \langle B, 1 \rangle \sim \langle B, m \rangle$ for all $m \in M$. Otherwise, $B = (\leftarrow, l_1]$ for some $l_1 \in L$. Let $m > 0$. Then $\theta(\vec{x}_{(\leftarrow, l_1], m}) = \langle \text{cl}(\leftarrow, l_1], v(\vec{x}_{(\leftarrow, l_1], m}) \rangle = \langle (\leftarrow, l_1], m \rangle$. Finally, we want to map onto $\langle (\leftarrow, l_1], 0 \rangle$. If there exists $l_0 < l_1$ with $(l_0, l_1) = \emptyset$, then $\theta(\vec{x}_{(\leftarrow, l_1], 0}) = \theta(\vec{x}_{(\leftarrow, l_0], 1}) = \langle (\leftarrow, l_0], 1 \rangle$ will suffice. Otherwise, $\theta(\vec{x}_{(\leftarrow, l_1], 0}) = \langle \text{cl}(\leftarrow, l_1), v(\vec{x}_{(\leftarrow, l_1], 0}) \rangle = \langle (\leftarrow, l_1], 0 \rangle$. \square

Here are some applications:

Example 13. $\varprojlim \{2, \gamma, I\} \cong (\hat{I} \setminus \emptyset) \times_{\text{lex}} 2 \cong I \times_{\text{lex}} 2 \cong \check{I}$ (of course, this could be found quicker with theorem 8).

Example 14. $\varprojlim \{I, \gamma, I\} \cong (\hat{I} \setminus \emptyset) \times_{\text{lex}} I \cong I \times_{\text{lex}} I$.

Example 15. For infinite ordinals α , $\varprojlim \{I, \gamma, \alpha\} \cong (\alpha \times_{\text{lex}} [0, 1)) \cup \{\infty\}$. In particular, $\alpha = \kappa$ for an infinite cardinal κ gives the closed long ray of length κ .

Definition 16. For any M containing a point 0 , let ν be the V-map on M where $\nu(0) = M$ and $\nu(t) = \{t\}$ for $t > 0$.

Note for $M = 2$ that $\nu = \gamma$.

Corollary 17. $\varprojlim\{2, \nu, L\} \cong \check{L}$.

Lemma 18. If M is T_2 , then $\varprojlim\{M, \nu, L\} \setminus \{\vec{0}\} \cong (\check{L} \setminus \{\emptyset\}) \times (M \setminus \{0\})$ with the usual product topology.

Proof. Each point in $\varprojlim\{M, \nu, L\} \setminus \{\vec{0}\}$ is of the form $\vec{x}_{C,m}$ where $C \in \check{L} \setminus \{\emptyset\}$ and $m \in M \setminus \{0\}$ defined by $\vec{x}_{C,m}(l) = m$ for $l \in C$ and $x_{C,m}(l) = 0$ otherwise.

We claim that the bijection $\theta(\vec{x}_{C,m}) = \langle C, m \rangle$ is a homeomorphism. Note that basic open sets of $\varprojlim\{M, \nu, L\}$ are of the form $\llbracket U, F \rrbracket$ where $U(l)$ is an open subset of M for each $l \in F \in [L]^{<\omega}$.

Consider the point $\langle C, m \rangle$ in the basic open set $V \times W$ in $(\check{L} \setminus \{\emptyset\}) \times (M \setminus \{0\})$. Note that V is either of the form $(A, L]$ or (A, B) , and we may choose $l_0 \in C \setminus A$. We also may assume that W misses an open neighborhood Z of 0 as M is T_2 .

In the case that $V = (A, L]$ we let $U(l_0) = W$. Then since $l_0 \in C$ and $m \in W = U(l_0)$, it follows that $\vec{x}_{C,m} \in \llbracket U, \{l_0\} \rrbracket$. For any $\vec{x}_{D,n} \in \llbracket U, \{l_0\} \rrbracket$ we have that $\vec{x}_{D,n}(l_0) = n \in W$; in particular, it's nonzero. So $A \subsetneq (\leftarrow, l_0] \subseteq D$, putting $D \in (A, L] = V$. Thus $\theta(\vec{x}_{D,n}) = \langle D, n \rangle \in V \times W$.

In the case that $V = (A, B)$, we may also choose $l_1 \in B \setminus C$. We again let $U(l_0) = W$, and we also let $U(l_1) = Z$. Then as $\vec{x}_{C,m}(l_0) = m \in W = U(l_0)$ and $\vec{x}_{C,m}(l_1) = 0 \in Z = U(l_1)$, we have shown $\vec{x}_{C,m} \in \llbracket U, \{l_0, l_1\} \rrbracket$. For any $\vec{x}_{D,n} \in \llbracket U, \{l_0, l_1\} \rrbracket$, $\vec{x}_{D,n}(l_0) \in W$ and $\vec{x}_{D,n}(l_1) \in Z$. This shows that $\vec{x}_{D,n}(l_0) = n \in W$ and $\vec{x}_{D,n}(l_1) = 0$, so $A \subsetneq (\leftarrow, l_0] \subseteq D \subsetneq (\leftarrow, l_1] \subseteq B$, putting $D \in (A, B) = V$. Thus $\theta(\vec{x}_{D,n}) = \langle D, n \rangle \in V \times W$.

On the other hand, consider the point $\vec{x}_{C,m}$ in the basic open set $\llbracket U, F \rrbracket$ of $\varprojlim\{M, \nu, L\} \setminus \{\vec{0}\}$. It follows that $m \in U(l)$ for all $l \in F \cap C$, and $0 \in U(l)$ for all $l \in F \setminus C$.

If $F \subseteq C$, then let l_0 be the maximum element of F and $U' = \bigcap_{l \in F} U(l)$. Note $\langle C, m \rangle \in ((\leftarrow, l_0), L] \times U'$. So let $\langle D, n \rangle \in ((\leftarrow, l_0), L] \times U'$. Since $l_0 \in D$, $\vec{x}_{D,n}(l_0) = n \in U'$. Since $n \neq 0$, we have $\vec{x}_{D,n}(l) = n \in U' \subseteq U(l)$ for all $l \in F$, so $\vec{x}_{D,n} \in \llbracket U, F \rrbracket$.

If $F \cap C = \emptyset$, then let l_1 be the minimum element of $F \setminus C$. Note $\langle C, m \rangle \in (\emptyset, (\leftarrow, l_1]) \times (M \setminus \{0\})$. So let $\langle D, n \rangle \in (\emptyset, (\leftarrow, l_1]) \times (M \setminus \{0\})$. Since $l_1 \notin D$, $\vec{x}_{D,n}(l) = 0 \in U(l)$ for all $l \in F$, giving $\vec{x}_{D,n} \in \llbracket U, F \rrbracket$.

Otherwise, let l_0 be the maximum element of $F \cap C$ and l_1 be the minimum element of $F \setminus C$. Let $U' \subseteq \bigcap_{l \in C \cap F} U(l)$ and $U'' \subseteq \bigcap_{l \in F \setminus C} U(l)$. Note $\langle C, m \rangle \in ((\leftarrow, l_0), (\leftarrow, l_1]) \times U'$. So let $\langle D, n \rangle \in ((\leftarrow, l_0), (\leftarrow, l_1]) \times U'$. Since $l_0 \in D$ and $l_1 \notin D$, we have $\vec{x}_{D,n}(l_0) = n \in U'$

and $\vec{x}_{D,n}(l_1) = 0 \in U''$. Furthermore, $\vec{x}_{D,n}(l) = n \in U' \subseteq U(l)$ for all $l \in F \cap C$, and $\vec{x}_{D,n}(l) = 0 \in U'' \subseteq U(l)$ for all $l \in F \setminus C$, so $\vec{x}_{D,n} \in \llbracket U, F \rrbracket$. \square

Theorem 19. *If M is T_2 , then $\varprojlim \{M, \nu, L\} \cong (\check{L} \times M) / \sim$ with the usual product topology, where \sim identifies every point in $\{\emptyset\} \times M \cup \check{L} \times \{0\}$.*

Proof. We will extend θ as defined in the previous lemma so that $\theta(\vec{0})$ is sent to the points identified in $\{\emptyset\} \times M \cup \check{L} \times \{0\}$.

Consider the basic open neighborhood $\llbracket U, F \rrbracket$ of $\vec{0}$ in $\varprojlim \{M, \nu, L\}$. Let l_1 be the least element of F , and $U' = \bigcap_{l \in F} U(l)$. Note that $\{\emptyset\} \times M \cup \check{L} \times \{0\} \subseteq [\emptyset, (\leftarrow, l_1)) \times M \cup \check{L} \times U'$. So let $\vec{x}_{C,m} \in [\emptyset, (\leftarrow, l_1)) \times M$, and note that as $l_1 \notin C$, $\vec{x}_{C,m}(l) = 0 \in U(l)$ for all $l \in F$, and thus $\vec{x}_{C,m} \in \llbracket U, F \rrbracket$. Likewise if we let $\vec{x}_{C,m} \in \check{L} \times U'$, we have that $\vec{x}_{C,m}(l) \in U' \subseteq U(l)$ for all $l \in L$, and thus $\vec{x}_{C,m} \in \llbracket U, F \rrbracket$.

Now in $(\check{L} \times M) / \sim$ consider the basic open set formed by the union of $V \times M$ containing $\{\emptyset\} \times M$ and $\check{L} \times W$ containing $\check{L} \times \{0\}$. $V = [\emptyset, B)$ for some nonempty leftward set B , so choose $l_0 \in B$ and note that as $(\leftarrow, l_0] \subseteq B$ we have $[\emptyset, (\leftarrow, l_0)) \subseteq [\emptyset, B) = V$. So let $U(l_0) = W$ and note $\vec{0} \in \llbracket U, \{l_0\} \rrbracket$. For any $\vec{x}_{C,m} \in \llbracket U, \{l_0\} \rrbracket$, we have two cases.

If $l_0 \in C$, then $\vec{x}_{C,m}(l) = m \in U(l_0) = W$ for all $l \in C$, and $\vec{x}_{C,m}(l) = 0 \in W$ otherwise. Thus $\langle C, m \rangle \in \check{L} \times W$.

Otherwise $l_0 \notin C$, and thus $C \subsetneq (\leftarrow, l_0]$. Then as $C \in [\emptyset, (\leftarrow, l_0)) \subseteq V$, we have $\langle C, m \rangle \in V \times M$. \square

We introduce an alternate definition of an arbitrarily indexed inverse limit.

Definition 20. Let $\varprojlim^* \{X, f, L\} \subseteq \varprojlim \{X, f, L\}$ satisfy that $\vec{x}(a) = \lim_{t \rightarrow a} \vec{x}(t)$ for all $a \in L$ (for any open neighborhood U of $\vec{x}(a)$ there is $b < a$ where $\vec{x}(t) \in U$ for all $t \in (b, a]$).

Theorem 21. $Y = \varprojlim^* \{2, \gamma, L\} \cong \hat{L}$.

Proof. Consider Y as a subspace of $X = \varprojlim \{2, \gamma, L\}$ with the linear order described above. We claim that if ϕ is the characteristic function for a subset of L , then ϕ is an order isomorphism between $\langle Y, \leq \rangle$ and $\langle \hat{L}, \subseteq \rangle$.

Let A be a left-closed subset of L . Let $\vec{x}(a) = 1$ when $a \in A$ and $\vec{x}(a) = 0$ otherwise. Then $\vec{x} \in Y$ and $\phi(\vec{x}) = A$.

Let $\vec{x}, \vec{y} \in Y$. If $\phi(\vec{x}) = \phi(\vec{y}) = A$, then A is a left-closed set where $\vec{x}(a) = \vec{y}(a) = 1$ for $a \in A$ and $\vec{x}(a) = \vec{y}(a) = 0$ otherwise, so $\vec{x} = \vec{y}$.

Finally let $\vec{x} < \vec{y}$, so there exists $a \in L$ with $\vec{x}(a) = 0$, $\vec{y}(a) = 1$. Then $\phi(\vec{x}) \subseteq (\leftarrow, a) \subseteq \phi(\vec{y})$. Thus ϕ preserves order. \square

Corollary 22. $\varprojlim^* \{2, \gamma, \alpha\} \cong \alpha + 1$ for every infinite limit or finite ordinal α .

Proof. If α is finite, then of course all (leftward) sets are closed and we get $\hat{\alpha} = \check{\alpha} = \alpha + 1$ for free. Otherwise, as observed previously $\hat{\alpha}$ is homeomorphic to its usual compactification $\alpha + 1$ for limit ordinals. \square

In fact, $\hat{\alpha} = \alpha + 1 \setminus L(\alpha)$ where $L(\alpha)$ is the collection of all limit ordinals less than α , which also shows $\hat{\alpha} \cong \alpha$ for infinite successor ordinals α .

References