

# ALMOST COMPATIBLE FUNCTIONS AND INFINITE LENGTH GAMES

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ABSTRACT.  $\mathcal{A}'(\kappa)$  asserts the existence of pairwise almost compatible finite-to-one functions  $A \rightarrow \omega$  for each countable subset  $A$  of  $\kappa$ . The existence of winning 2-Markov strategies in several infinite-length games, including the Menger game on the one-point Lindelöfication  $\kappa^\dagger$  of  $\kappa$ , are guaranteed by  $\mathcal{A}'(\kappa)$ .  $\mathcal{A}'(\kappa)$  is implied by the existence of cofinal Kurepa families of size  $\kappa$ , and thus holds for all cardinals less than  $\aleph_\omega$ . It's consistent that  $\mathcal{A}'(\aleph_\omega)$  fails, but there must always be a winning 2-Markov strategy for the second player in the Menger game on  $\omega_\omega^\dagger$ .

## 1. INTRODUCTION

**Definition 1.** Two functions  $f, g$  are almost compatible, that is,  $f \sim g$  when  $\{a \in \text{dom } f \cap \text{dom } g : f(a) \neq g(a)\}$  is finite.

Marion Scheepers used almost compatible functions in [10] in order to study the existence of limited information strategies on a variation of the meager-nowhere dense game he introduced in [11].

**Game 2.** Let  $Sch_{C,F}^{\cup, \subset}(\kappa)$  denote *Scheepers' strict countable-finite union game* with two players  $\mathcal{C}, \mathcal{F}$ . In round 0,  $\mathcal{C}$  chooses  $C_0 \in [\kappa]^{\leq \omega}$ , followed by  $\mathcal{F}$  choosing  $F_0 \in [\kappa]^{< \omega}$ . In round  $n + 1$ ,  $\mathcal{C}$  chooses  $C_{n+1} \in [\kappa]^{\leq \omega}$  such that  $C_{n+1} \supset C_n$ , followed by  $\mathcal{F}$  choosing  $F_{n+1} \in [\kappa]^{< \omega}$ .

$\mathcal{F}$  wins the game if  $\bigcup_{n < \omega} F_n \supseteq \bigcup_{n < \omega} C_n$ ; otherwise,  $\mathcal{C}$  wins.

Of course, with perfect information this game is trivial: during round  $n$  player  $\mathcal{F}$  simply chooses  $n$  ordinals from each of the  $n$  countable sets played by  $\mathcal{C}$ . However, if  $\mathcal{F}$  is limited to using information from the last  $k$  moves by  $\mathcal{C}$  during each round, the task becomes more difficult. Call such a strategy a *k-tactical strategy* or *k-tactic*; if using the round number is allowed, then the strategy is called a *k-Markov strategy* or a *k-mark*.

**Definition 3.** The statement  $\mathcal{A}(\kappa)$  (given as  $S(\kappa, \aleph_0, \omega)$  in [10]) claims that there exist one-to-one functions  $f_A : A \rightarrow \omega$  for each  $A \in [\kappa]^{\leq \aleph_0}$  such that the collection  $\{f_A : A \in [\kappa]^{\leq \aleph_0}\}$  is pairwise almost compatible.

In [10], Scheepers noted that  $\mathcal{A}(\omega_1)$  holds in *ZFC*, and that it's possible to force  $\mathfrak{c}$  to be arbitrarily large while preserving  $\mathcal{A}(\mathfrak{c})$ . However,  $\mathcal{A}(\mathfrak{c}^+)$  always fails. This

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axiom may be applied to obtain a winning 2-tactic for  $\mathcal{F}$  in the countable-finite game.

In [1], Clontz related this game to a game which may be used to characterize the Menger covering property of a topological space.

**Game 4.** Let  $Men_{C,F}(X)$  denote the *Menger game* with players  $\mathcal{C}$ ,  $\mathcal{F}$ . In round  $n$ ,  $\mathcal{C}$  chooses an open cover  $\mathcal{U}_n$ , followed by  $\mathcal{F}$  choosing a subset  $F_n$  of  $X$  which may be finitely covered by  $\mathcal{U}_n$ .

$\mathcal{F}$  wins the game if  $X = \bigcup_{n < \omega} F_n$ , and  $\mathcal{C}$  wins otherwise.

This characterization is slightly different than the typical characterization in which the second player first chooses a specific finite subcollection  $\mathcal{F}_n$  of the cover itself and lets  $F_n = \bigcup \mathcal{F}_n$ , denoted as  $G_{fin}(\mathcal{O}, \mathcal{O})$  in [12]. However, it's easily seen that these games are equivalent for perfect information strategies (so both characterize the Menger property in the same way), and this characterization is more convenient for our concerns.

**Definition 5.** Let  $\kappa^\dagger = \kappa \cup \{\infty\}$  where  $\kappa$  is discrete and  $\infty$ 's neighborhoods are the co-countable sets containing it.

The relationship between  $Sch_{C,F}^{\cup, \subseteq}(\kappa)$  and  $Men_{C,F}(\kappa^\dagger)$  is strong; in both games  $\mathcal{C}$  essentially chooses a countable subset of  $\kappa$  followed by  $\mathcal{F}$  choosing a finite subset of that choice, and it's easy to see the winning perfect information strategy for  $\mathcal{F}$  in both games. In addition, it was shown in [1] that when  $\mathcal{A}(\kappa)$  holds,  $\mathcal{F}$  has a winning 2-Markov strategy in  $Men_{C,F}(\kappa^\dagger)$ .

One source of motivation is to make progress on the following open question:

**Question 6.** *Does there exist a topological space  $X$  for which  $\mathcal{F} \uparrow Men_{C,F}(X)$  but  $\mathcal{F} \not\uparrow_{2\text{-mark}} Men_{C,F}(X)$ ? (That is, the second player can win the Menger game on  $X$  with perfect information but not with 2-Markov information.)*

## 2. ONE-TO-ONE AND FINITE-TO-ONE ALMOST COMPATIBLE FUNCTIONS

We may weaken Scheeper's  $\mathcal{A}(\kappa)$  as follows:

**Definition 7.** The statement  $\mathcal{A}'(\kappa)$  weakens  $\mathcal{A}(\kappa)$  by only requiring the witnessing almost-compatible functions  $f_A : A \rightarrow \omega$  to be finite-to-one.

**Proposition 8.**  $\mathcal{A}(\kappa)$  and  $\mathcal{A}'(\kappa)$  need only be witnessed by functions  $\{f_A : A \in \mathcal{S}\}$  for some family  $\mathcal{S}$  cofinal in  $[\kappa]^{\leq \aleph_0}$ .

*Proof.* For each  $A \in [\kappa]^{\leq \aleph_0}$  choose  $A' \supseteq A$  from  $\mathcal{S}$  and let  $g_A = f_{A'} \upharpoonright A$ . □

In the final section we will show that  $\mathcal{A}'(\kappa)$  is sufficient for many applications to the Scheepers and Menger games. In the meantime, we will demonstrate that  $\mathcal{A}'(\kappa)$  is strictly weaker than  $\mathcal{A}(\kappa)$ .

Recall the following.

**Definition 9.** A Kurepa family  $\mathcal{K} \subseteq [\kappa]^{\aleph_0}$  on  $\kappa$  satisfies that  $\mathcal{K} \upharpoonright A = \{K \cap A : K \in \mathcal{K}\}$  is countable for each  $A \in [\kappa]^{\aleph_0}$ . Let  $\mathcal{K}(\kappa)$  be the statement claiming there exists a Kurepa family on  $\kappa$  cofinal in  $[\kappa]^{\aleph_0}$ .

**Theorem 10.**  $\mathcal{K}(\kappa) \Rightarrow \mathcal{A}'(\kappa)$ .

*Proof.* Let  $\mathcal{K} = \{K_\alpha : \alpha < \theta\}$  be a cofinal Kurepa family on  $\kappa$ . We first define  $f_\alpha : K_\alpha \rightarrow \omega$  for each  $\alpha < \theta$ .

Suppose we've already defined pairwise almost compatible finite-to-one functions  $\{f_\beta : \beta < \alpha\}$ . To define  $f_\alpha$ , we first recall that  $\mathcal{K} \upharpoonright K_\alpha$  is countable, so we may choose  $\beta_n < \alpha$  for  $n < \omega$  such that  $\{K_\beta : \beta < \alpha\} \upharpoonright K_\alpha \setminus \{\emptyset\} = \{K_\alpha \cap K_{\beta_n} : n < \omega\}$ . Let  $K_\alpha = \{\delta_{i,j} : i \leq \omega, j < w_i\}$  where  $w_i \leq \omega$  for each  $i \leq \omega$ ,  $K_\alpha \cap (K_{\beta_n} \setminus \bigcup_{m < n} K_{\beta_m}) = \{\delta_{n,j} : j < w_n\}$ , and  $K_\alpha \setminus \bigcup_{n < \omega} K_{\beta_n} = \{\delta_{\omega,j} : j < w_\omega\}$ . Then let  $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$  for  $n < \omega$  and  $f_\alpha(\delta_{\omega,j}) = j$  otherwise.

We should show that  $f_\alpha$  is finite-to-one. Let  $n < \omega$ . Since  $f_\alpha(\delta_{m,j}) \geq m$ , we only consider the finite cases where  $m \leq n$ . Since each  $f_{\beta_m}$  is finite-to-one,  $f_{\beta_m}(\delta_{m,j}) \leq n$  for only finitely many  $j$ . Thus  $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$  maps to  $n$  for only finitely many  $j$ .

We now want to demonstrate that  $f_\alpha \sim f_{\beta_n}$  for all  $n < \omega$ . Note  $\delta_{m,j} \in K_{\beta_n}$  implies  $m \leq n$ . For  $m = n$ , we have  $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$  which differs from  $f_{\beta_n}(\delta_{n,j})$  for only the finitely many  $j$  which are mapped below  $n$  by  $f_{\beta_n}$ . For  $m < n$  and  $\delta_{m,j} \in K_{\beta_n}$ , we have  $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$  which can only differ from  $f_{\beta_n}(\delta_{m,j})$  for only the finitely many  $j$  which are mapped below  $m$  by  $f_{\beta_m}$  or the finitely many  $j$  for which the almost compatible  $f_{\beta_n} \sim f_{\beta_m}$  differ.

Finally for any  $\beta < \alpha$ , we may conclude  $f_\alpha \sim f_\beta$  since there is some  $\beta_n$  with  $K_\alpha \cap K_\beta = K_\alpha \cap K_{\beta_n}$ ,  $f_\alpha \sim f_{\beta_n}$ , and  $f_{\beta_n} \sim f_\beta$ .  $\square$

We now construct a topology on  $\omega_n$  for each  $n < \omega$  which will witness a Kurepa family of size  $\aleph_n$ , exploiting a theorem of Juhász et. al. in [6].

**Definition 11.** A topological space is said to be  $\omega$ -bounded if each countable subset of the space has compact closure. As in [6] we call a  $T_2$ , locally countable,  $\omega$ -bounded space *splendid*, and let  $\mathcal{S}(\kappa)$  represent the claim that there exists a splendid space of cardinality  $\kappa$ .

**Theorem 12** ([6]).  $\mathcal{S}(\aleph_k)$  for  $k < \omega$ .

**Lemma 13.** *The family of compact open sets in a locally countable,  $\omega$ -bounded topological space  $X$  is a Kurepa family cofinal in  $[X]^\omega$ . That is,  $\mathcal{S}(\kappa) \Rightarrow \mathcal{K}(\kappa)$ .*

*Proof.* Let  $\mathcal{K}$  collect all compact open subsets of  $X$ . Of course, every Lindelöf set in a locally countable space is countable, and the closure of every countable set is a compact countable set; thus  $\mathcal{K}$  is cofinal in  $[X]^\omega$ . It is Kurepa since every countable set is contained in a countable compact open subspace of  $X$ ; this subspace has a countable base of compact open sets, which closed under finite unions enumerates all compact open subsets of the subspace.  $\square$

**Corollary 14.**  $\mathcal{K}(\aleph_k)$  for all  $k < \omega$ .

Alternatively, the previous corollary may be obtained via an observation of Todorćević communicated by Dow in [3]: if every Kurepa family of size at most  $\kappa$  extends to a cofinal Kurepa family, then the same is true of  $\kappa^+$ .

Nyikos points out in [9] that a cofinal Kurepa family may be used to construct a locally metrizable,  $\omega$ -bounded, zero-dimensional space with appropriate cardinality, but whether this can be strengthened to locally countable and  $\omega$ -bounded (as asked in [6]) remains an open question.

Also left open is this extension of the question asked in [9] and [6] on the possible equivalence of  $\mathcal{S}(\kappa)$  and  $\mathcal{K}(\kappa)$ .

**Question 15.** *May any of the implications in the theorem  $\mathcal{S}(\kappa) \Rightarrow \mathcal{K}(\kappa) \Rightarrow \mathcal{A}'(\kappa)$  be reversed?*

Regardless, we have obtained our desired result.

**Corollary 16.**  $\mathcal{A}'(\aleph_k)$  for all  $k < \omega$ .

### 3. CONSISTENCY RESULTS

As noted in [3], Jensen's one-gap two-cardinal theorem under  $V = L$  introduced in [5] implies that  $\mathcal{K}(\kappa)$  holds for all cardinals  $\kappa$ .

**Corollary 17** ( $V = L$ ).  $\mathcal{A}'(\kappa)$  for all cardinals  $\kappa$ .

As noted earlier, Scheepers proved in [10] that  $\neg\mathcal{A}(\mathfrak{c}^+)$  is a theorem of ZFC, showing  $\mathcal{A}'(\kappa)$  is not equivalent to  $\mathcal{A}(\kappa)$ .

We now demonstrate that  $CH$  is not required to have  $\mathcal{A}(\aleph_2)$  fail.

The forcing extension of a model  $M$  by a poset  $\mathbb{P} \in M$  is obtained simply by evaluating all  $\mathbb{P}$ -names from  $M$  by a generic filter  $G$ . A set  $\tau$  is a  $\mathbb{P}$ -name if  $\tau$  is a (possibly empty) set of ordered pairs  $(\sigma, p)$  where  $p \in \mathbb{P}$  and  $\sigma$  is also itself a  $\mathbb{P}$ -name. If  $G$  is a  $\mathbb{P}$ -generic filter, then  $\text{val}_G(\tau)$  is defined to equal  $\{\text{val}_G(\sigma) : (\exists p \in G) (\sigma, p) \in \tau\}$ .

If  $x \in M$ , then the canonical  $\mathbb{P}$ -name,  $\check{x}$ , is generally, and recursively, taken to be  $\{(\check{y}, 1) : y \in x\}$  where 1 is the maximum element of  $\mathbb{P}$ . However, it will be convenient to consider, when the context is clear,  $(x, p)$  (for any  $p \in \mathbb{P}$ ) to be a kind of  $\mathbb{P}$ -name. In particular if  $\tau \subset X \times \mathbb{P}$  (for some fixed  $X \in M$ ), then we may let  $\tau[G] = \{x : (\exists p \in G) (x, p) \in \tau\}$ .

Thus,  $\text{val}_G(\tau)$  will denote the recursive evaluation by  $G$  and  $\tau[G]$  will be defined as above. In fact, if  $\tau \in M$  is any set then each of  $\text{val}_G(\tau)$  and  $\tau[G]$  are well defined. It is a standard convention to use a dotted letter, such as  $\dot{x}$ , to indicate that we are discussing a  $\mathbb{P}$ -name.

One says that a condition  $p \in \mathbb{P}$  forces a statement  $\varphi$  to hold, denoted  $p \Vdash \varphi$ , if that statement holds in  $M[G]$  for all  $\mathbb{P}$ -generic filters with  $p \in G$ . The forcing theorem states that if  $M[G] \models \varphi$ , then there is some  $p \in G$  forcing that  $\varphi$  holds. The following is an immediate consequence of the forcing theorem.

**Lemma 18.** *If  $X \in M$  and  $\dot{x}$  is a  $\mathbb{P}$ -name, then there is a  $\tau \subset X \times \mathbb{P}$ , such that for any generic  $G$ ,  $\tau[G] = X \cap \text{val}_G(\dot{x})$ .*

In other words, the family of subsets of any  $X \in M$  in the extension  $M[G]$  is equal to  $\{\tau[G] : \tau \subset X \times \mathbb{P}, \tau \in M\}$ . We will be using the forcing poset

$\text{Fn}(\omega_2, 2)$ . The elements of this poset are all the finite partial functions from  $\omega_2$  into 2 ordered by reverse inclusion. It follows that, for any  $\lambda \in \omega_2$ , each of  $\text{Fn}(\lambda, 2)$  and  $\text{Fn}(\omega_2 \setminus \lambda, 2)$  are subposets. For any  $\text{Fn}(\omega_2, 2)$ -generic filter  $G$ , it easily follows that  $G_\lambda = G \cap \text{Fn}(\lambda, 2)$  and  $G^\lambda = G \cap \text{Fn}(\omega_2 \setminus \lambda, 2)$  are also generic filters. But a much stronger statement is true.

**Lemma 19.** [7] *Assume that  $G \subset \text{Fn}(\omega_2, 2)$  is a generic filter, and let  $\lambda \in \omega_2$ . Then the final model  $M[G]$  is equal to  $(M[G_\lambda])[G^\lambda]$  in the sense that  $G^\lambda$  is a  $\text{Fn}(\omega_2 \setminus \lambda, 2)$ -generic filter over the model  $M[G_\lambda]$ .*

*In addition, for each  $X \in M$  and name  $\dot{A} \subset X \times \text{Fn}(\omega_2, 2)$ , we get that*

$$(\dot{A}(G_\lambda))[G^\lambda] = \dot{A}[G] \quad \text{where } \dot{A}(G_\lambda) = \{(x, p \restriction [\lambda, \omega_2)) : (x, p) \in \dot{A} \text{ and } p \restriction \lambda \in G_\lambda\}$$

With these lemmas in hand we are ready to prove the theorem. The idea of the proof comes from Kunen's result about no  $\omega_2$  length mod finite chains of subsets of  $\omega$ . We consider any family of names of suitable one-to-one functions from countable subsets of  $\omega_2$  into  $\omega$ . We identify a large enough  $\lambda \in \omega_2$  so that a pattern has emerged and we pass to the model  $M[G_\lambda]$ . We then show that this pattern can not continue out to  $\omega_2$ .

**Theorem 20.** *There exists a model of ZFC for which  $\mathfrak{c} = \aleph_2$  and  $\neg \mathcal{A}(\aleph_2)$ .*

*Proof.* We start with a model  $M$  of GCH and suppose that  $G$  is a  $\text{Fn}(\omega_2, 2)$ -generic filter. The argument takes place in  $M$ . Let  $\{\dot{f}_A : A \in [\omega_2]^\omega\}$  be a family of names (in  $M$ ) such that, for any generic  $G$  and each  $A \in [\omega_2]^\omega \cap M$ ,  $\dot{f}_A[G]$  is a one-to-one function from  $A$  into  $\omega$ . We also assume that whenever  $B \subset A$  are members of  $[\omega_2]^\omega$ , we have that  $\dot{f}_B[G] \subset^* \dot{f}_A[G]$ . If we now obtain a contradiction then we will have shown that  $\mathcal{A}(\aleph_2)$  fails.

By [2, 1.5], there is a set  $H \subset H(\aleph_3)$  such that the family  $\{\dot{f}_A : A \in [\omega_2]^\omega\}$  is an element of  $H$ ,  $H$  is an elementary submodel of  $H(\aleph_3)$ ,  $H$  has cardinality  $\aleph_1$ , and  $H^\omega \subset H$  (every countable subset of  $H$  is an element of  $H$ ).

Let  $\lambda = H \cap \omega_2$  (same as the supremum of  $H \cap \omega_2$ ). Consider the name  $\dot{f}_{[\lambda, \lambda + \omega)}$ . What is such a name? By Lemma 18, we can assume that it is a set of pairs of the form  $((\lambda + k, m), p)$  where  $p \in \text{Fn}(\omega_2, 2)$  and, of course,  $k, m \in \omega$ . Furthermore, for each  $k, m$  it is enough (see [7, 5.11, 5.12]) to take a countable set of such  $p$  to get an equivalent (*nice*) name. Given any such nice name  $\dot{f}$ , let  $\text{supp}(\dot{f})$  denote the union of the domains of conditions  $p$  appearing in the name.

Now let  $Y$  equal  $\text{supp}(\dot{f}_{[\lambda, \lambda + \omega)}) \setminus \lambda$ . Furthermore, fix any  $\mu \in \lambda \subset H$  such that  $\text{supp}(\dot{f}_{[\lambda, \lambda + \omega)}) \cap \lambda$  is contained in  $\mu$ . Let  $\delta \in \omega_1$  denote the order type of  $Y$  and let  $\varphi_{\mu, \lambda}$  be the order-preserving function from  $\mu \cup Y$  onto the ordinal  $\mu + \delta$ . This lifts canonically to an order-preserving bijection  $\varphi_{\mu, \lambda} : \text{Fn}(\mu \cup Y, 2) \rightarrow \text{Fn}(\mu + \delta, 2)$ . We can similarly make sense of the name  $\varphi_{\mu, \lambda}(\dot{f}_{[\lambda, \lambda + \omega)})$ , call it  $F_H$ . Here simply, for each tuple  $((\lambda + k, m), p) \in \dot{f}_{[\lambda, \lambda + \omega)}$ , we have that  $((\mu + k, m), \varphi_{\mu, \lambda}(p))$  is in  $F_H$ . Again, let  $\varphi_{\mu, \lambda}(\dot{f}_{[\lambda, \lambda + \omega)})$  be interpreted in the above sense as giving  $F_H$  (which is an element of  $H$ ).

Other values replacing  $\lambda > \mu$  will result in their own set  $Y$  and canonical map  $\varphi_{\mu,\lambda}$ . Now the object  $F_H$  is an element of  $H$ , and  $H$  believes this statement is true:

$$(\forall \beta \in \omega_2) (\exists \lambda \in \omega_2 \setminus \beta) \text{ supp}(\dot{f}_{[\lambda, \lambda+\omega)}) \cap \lambda \subset \mu \text{ and } F_H = \varphi_{\mu,\lambda}(\dot{f}_{[\lambda, \lambda+\omega)})$$

But now, this means that, not only is there an  $\alpha \in H$ ,  $F_H = \varphi_{\mu,\alpha}(\dot{f}_{[\alpha, \alpha+\omega)})$  but also that there is an increasing sequence  $\{\alpha_\xi : \xi \in \omega_1\} \subset \lambda$  of such  $\alpha$ 's satisfying that, for each  $\xi$  we have that  $\text{supp}(\dot{f}_{[\alpha_\xi, \alpha_\xi+\omega)})$  is contained in  $\alpha_{\xi+1}$ .

Choose such a sequence. This means that if we let  $A = \bigcup_{n>0} [\alpha_n, \alpha_n + \omega)$  we have the name  $\dot{f}_A$  in  $H$ . This then means that all the  $((\beta, m), p)$  appearing in (the nice name)  $\dot{f}_A$  have the property that  $\text{dom}(p)$  is contained in  $H$ . There is, also within  $H$ , a name  $\dot{g}$  satisfying that  $\dot{f}_A(\alpha_n + k) = \dot{f}_{[\alpha_n, \alpha_n+\omega)}(\alpha_n + k)$  for all  $k > \dot{g}(n)$ , or more precisely,  $\dot{g} \subset (\omega \times \omega) \times \text{Fn}(\omega_2, 2)$  satisfies that  $\dot{g}[G] \in \omega^\omega$  and  $\dot{f}_A[G](\alpha_n + k) = \dot{f}_{[\alpha_n, \alpha_n+\omega)}[G](\alpha_n + k)$  for all  $k > \dot{g}[G](n)$ .

We now apply Lemma 19 and we are now working in the extension  $M[G_\mu]$ . We work for a contradiction. Something special has now happened, namely, the supports of the names  $\{\dot{f}_{[\alpha_n, \alpha_n+\omega)}(G_\mu) : 0 < n < \omega\}$  are pairwise disjoint and also disjoint from the support of the name  $\dot{f}_{[\lambda, \lambda+\omega)}(G_\mu)$ . And not only that, these names are pairwise isomorphic (in the way that they all map to  $F_H$ ).

Since  $A$  is disjoint from  $[\lambda, \lambda + \omega)$ , there must be an integer  $\ell$  together with a condition  $q \in \text{Fn}(\omega_2 \setminus \mu, 2)$  satisfying that for all  $n > \ell$ ,  $q$  forces that

$$\text{“if } k > \dot{g}(n) \text{ then } (\dot{f}_{[\alpha_n, \alpha_n+\omega)}(G_\mu))(\alpha_n + k) \neq (\dot{f}_{[\lambda, \lambda+\omega)}(G_\mu))(\lambda + k)\text{”}.$$

Choose  $n > \ell$  large enough so that  $\text{dom}(q) \cap [\alpha_n, \alpha_{n+1})$  is empty. Choose  $q_1 < q \restriction \lambda$  (in  $H$ ) so that

$$\varphi_{\mu,\alpha_n}(q_1 \restriction \text{supp}(\dot{f}_{[\alpha_n, \alpha_n+\omega)})) = \varphi_{\mu,\lambda}(q \restriction \text{supp}(\dot{f}_{[\lambda, \lambda+\omega)}))$$

and then (again in  $H$ ) choose  $q_2 < q_1$  so that it both forces a value  $L$  on  $\ell + \dot{g}(n)$  and subsequently forces a value  $m$  on  $\dot{f}_{[\alpha_n, \alpha_n+\omega)}(\alpha_n + L + 1)$ . But now, again calculate

$$q_3 = \varphi_{\mu,\lambda}^{-1} \circ \varphi_{\mu,\alpha_n}(q_2 \restriction \text{supp}(\dot{f}_{[\alpha_n, \alpha_n+\omega)}))$$

and, by the isomorphisms, we have that  $q_3$  forces that  $\dot{f}_{[\lambda, \lambda+\omega)}(\lambda + L + 1) = m$ .

Technically (or with more care) all of this is taking place in the poset  $\text{Fn}(\omega_2 \setminus \mu, 2)$  and this means that  $q_3$  and  $q$  are with each other. To verify this it suffices to consider  $q(\beta) = e$  and to assume that  $q_3(\beta)$  is defined. Since  $q_3(\beta)$  is defined, we have that there is a  $\beta' \in \text{dom}(q_2)$  such that  $\varphi_{\mu,\lambda}(\beta) = \varphi_{\mu,\alpha_n}(\beta')$ , and that  $q_3(\beta) = q_2(\beta')$ . But, by definition of  $q_1$ ,  $\beta' \in \text{dom}(q_1)$  and even that  $q_1(\beta') = q(\beta)$ . Then, since  $q_2 < q_1$ , we have that  $q_2(\beta') = q_1(\beta') = q(\beta)$ . This completes the circle that  $q_3(\beta) = q(\beta)$ .

Finally, our contradiction is that  $q_3 \cup q_2 \cup q$  forces that  $k = L + 1$  violates the quoted statement above.  $\square$

We are also able to force  $\mathcal{A}'(\kappa)$  to fail for every cardinal other than the first  $\omega$ -many we've already guaranteed.

**Theorem 21.** *It follows from the existence of a 2-huge cardinal that there is a model of ZFC for which  $\neg \mathcal{A}'(\aleph_\omega)$ .*

*Proof.* We will need the model constructed in [8] in which an instance of Chang's conjecture  $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$  is shown to hold.

We can take as a given (as shown in [8, Theorem 5]) that we may assume that we have a model  $V$  of GCH in which there are regular limit cardinals  $\kappa < \lambda$  satisfying that  $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$ .

What this says is that if  $L$  is a countable language with at least one unary relation symbol  $R$  and  $M$  is a model of  $L$  with base set  $\lambda^{+\omega+1}$  in which the interpretation of  $R$  has cardinality  $\lambda^{+\omega}$ , then  $M$  has an elementary submodel  $N$  of cardinality  $\kappa^{+\omega+1}$  in which  $R \cap N$  has cardinality  $\kappa^{+\omega}$  (of course  $R \cap N$  is the interpretation of  $R$  in  $N$  because  $N \prec M$ ).

The interested reader will want to know that it is shown in [8] that if  $\kappa$  is a 2-huge cardinal and  $j$  is the 2-huge embedding with critical point  $\kappa$ , then with  $\lambda = j(\kappa)$  one has that  $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$  holds. There is no loss of generality to also assume that GCH holds in this model.

Let  $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$  be a scale in  $\Pi\{\lambda^{+n+1} : n \in \omega\}$  ordered by the usual mod finite coordinatewise ordering. For convenience we may assume that  $h_\xi(n) \geq \lambda^{+n}$  for all  $\xi$  and all  $n$ . For each integer  $m$  the cofinality of the mod finite ordering on  $\Pi\{\lambda^{+n+1} : m < n \in \omega\}$  is the same as it is for the entire product  $\Pi\{\lambda^{+n+1} : n \in \omega\}$ . If  $P$  is any poset of cardinality less than  $\lambda^{+m}$  then, in the forcing extension by  $P$ , every function in  $\Pi\{\lambda^{+n+1} : m < n \in \omega\}$  is bounded above by a ground model function. It therefore follows easily that in the forcing extension by  $P$ , the sequence  $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$  remains cofinal in  $\Pi\{\lambda^{+n+1} : n \in \omega\}$ .

The forcing notion  $\mathbb{P}_0$  is simply the finite condition collapse of  $\kappa^{+\omega}$ , i.e.  $\mathbb{P}_0 = (\kappa^{+\omega})^{<\omega}$ . In the forcing extension by  $\mathbb{P}_0$ , one now has that the ordinal  $\kappa^{+\omega+1}$  from  $V$  is the first uncountable cardinal  $\aleph_1$ . Then in this forcing extension we let  $\mathbb{P}_1$  be the countable condition Levy collapse,  $Lv(\lambda, \omega_2)$ , which collapses all cardinals less than  $\lambda$  to have cardinality at most  $\aleph_1$ . The poset  $\mathbb{P}_1$  has cardinality  $\lambda$ . We treat  $\mathbb{P}_0 * \mathbb{P}_1$  as containing  $\mathbb{P}_0$  as a subposet by identifying each  $(p_0, 1)$  with  $p_0$ . After forcing with  $\mathbb{P}_0 * \mathbb{P}_1$  we will have that  $\omega_1$  is the ordinal  $(\kappa^{+\omega+1})^V$ ,  $\omega_2$  is the ordinal  $\lambda$ , and  $\omega_\omega$  is the ordinal  $(\lambda^{+\omega})^V$ .

Now we assume that we have an assignment  $\dot{f}_{\dot{A}}$  of a  $\mathbb{P}_0 * \mathbb{P}_1$ -name of a finite-to-one function from  $\dot{A}$  into  $\omega$  for each  $\mathbb{P}_0 * \mathbb{P}_1$ -name of a countable subset of  $\lambda^{+\omega+1}$ . We will obtain a contradiction to the claim of coherence.

Let  $\{\dot{A}_\xi : \xi \in \lambda^{+\omega+1}\}$  be an enumeration of all the nice  $\mathbb{P}_0$ -names of countable subsets of  $\lambda^{+\omega}$ . For each  $\xi \in \lambda^{+\omega+1}$ , let  $\dot{f}_\xi$  be another notation for  $\dot{f}_{\dot{A}_\xi}$ . Since  $\mathbb{P}_0$  forces that  $\mathbb{P}_1$  is countably closed, the collection of all nice  $\mathbb{P}_0$ -names will produce all the countable sets in the extension by  $\mathbb{P}_0 * \mathbb{P}_1$ , but  $\mathbb{P}_0 * \mathbb{P}_1$  can introduce new enumerations of these names. For each  $\xi \in \lambda^{+\omega+1}$ , there is a minimal  $\zeta_\xi$  so that  $\dot{A}_{\zeta_\xi}$  is the canonical name for the range of  $h_\xi$ . This means that  $\dot{f}_{\zeta_\xi} \circ h_\xi$  is simply the  $\mathbb{P}_0 * \mathbb{P}_1$ -name of a finite-to-one function from  $\omega$  to  $\omega$ . For each  $\xi \in \lambda^{+\omega+1}$ , choose any  $p_\xi \in \mathbb{P}_0 * \mathbb{P}_1$  so that there is a nice  $\mathbb{P}_0$ -name,  $\dot{H}_\xi$ , that is forced by  $p_\xi$  to equal  $\dot{f}_{\zeta_\xi} \circ h_\xi$ . Choose  $\Lambda \subset \lambda^{+\omega+1}$  of cardinality  $\lambda^{+\omega+1}$  and so that there is a pair  $p, \dot{H}$  satisfying that  $p_\xi = p$  and  $\dot{H}_\xi = \dot{H}$  for all  $\xi \in \Lambda$ . We may assume that  $p$  is in a generic filter  $G$ .

Let  $\{x_\xi : \xi \in \lambda^{+\omega+1}\}$  be any enumeration of  $H(\lambda^{+\omega+1})$  such that  $\{x_\xi : \xi \in \lambda^{+\omega}\}$  is also equal to  $H(\lambda^{+\omega})$ . We choose this enumeration in such a way that  $x_\xi \in x_\eta$  implies  $\xi < \eta$ . We use relation symbol  $R_0$  to code (and well order)  $(H(\lambda^{+\omega+1}), \in)$  as follows:  $(\xi, \eta) \in R_0$  if and only if  $x_\xi \in x_\eta$ . Let  $R_1$  be a binary relation on  $\kappa^{+\omega}$  so that  $(\kappa^{+\omega}, R_1)$  is isomorphic to  $\mathbb{P}_0$ . Let  $R_2$  be a binary relation on  $\lambda$  so that  $R_2 \cap (\kappa^{+\omega} \times \kappa^{+\omega}) = R_1$  and  $(\lambda, R_2)$  is isomorphic to  $\mathbb{P}_0 * \mathbb{P}_1$ . Let  $\psi$  be the poset isomorphism from  $(\lambda, R_2)$  to  $\mathbb{P}_0 * \mathbb{P}_1$ .

We continue coding. We can code the sequence  $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$  as another binary relation  $R_3$  on  $\lambda^{+\omega+1}$  where  $R_3 \cap (\{\xi\} \times \lambda^{+\omega+1}) = \{(\xi, h_\xi(n)) : n \in \omega\}$  for each  $\xi \in \lambda^{+\omega+1}$ . The relation symbol  $R_4$  can code the sequence  $\{\dot{A}_\xi : \xi \in \lambda^{+\omega+1}\}$  where  $(\xi, \alpha, \zeta) \in R_4$  if and only if  $(\check{\alpha}, \psi(\zeta))$  is in the name  $\dot{A}_\xi$ . Let  $R_5$  code this collection, i.e.  $(\gamma, n, m, \eta) \in R_5$  if and only if  $((n, m), \psi(\eta)) \in \dot{H}_\gamma$ . Also let  $R_6$  code (equal) the set  $\Lambda$ . Finally we use the relation symbol  $R_7$  to similarly code the sequence  $\{\dot{f}_\xi : \xi \in \lambda^{+\omega+1}\}$ :  $(\xi, \alpha, n, \zeta) \in R_7$  if and only if  $((\alpha, n), \psi(\zeta))$  is in the name  $\dot{f}_\xi$ .

Needless to say, the unary relation symbol  $R$  is interpreted as the set  $\lambda^{+\omega}$  for the application of  $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$ . Now we have defined our model  $M$  of the language  $L = \{\in, R, R_0, \dots, R_7\}$ , and we choose an elementary submodel  $N$  witnessing  $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$ . Of course  $N$  is really just a  $\kappa^{+\omega+1}$  sized subset of  $\lambda^{+\omega+1}$  with the additional property that  $N \cap \lambda^{+\omega}$  has cardinality  $\kappa^{+\omega}$ . In the forcing extension  $N$  has cardinality  $\omega_1$  and  $A = N \cap \lambda^{+\omega}$  is countable.

We will need the following claim from [8]:

**Claim.** *We may assume that  $N$  satisfies that  $N \cap \kappa^{+\omega+1}$  is transitive (i.e. an initial segment).*

*Proof of Claim:* Suppose our originally supplied  $N$  fails the conclusion of the claim. We know that  $\kappa^{+\omega} \in N$ , (via  $R_1$ ) in which case so is  $\kappa^{+\omega+1}$ .

Then set  $\beta_0 = \sup(N \cap \kappa^{+\omega+1})$  and consider the Skolem closure  $Hull(N \cup \beta_0, M)$ . A little informally (in that we have to formalize the enumeration of formulas as per Gödel coding) let  $\{\varphi_n : n \in \omega\}$  be an enumeration of all formulas in the language  $L$ , and let  $\ell_n$  be the minimal integer such that the free variables of  $\varphi_n$  are among  $\{v_0, \dots, v_{\ell_n}\}$ . Then, for each tuple  $\langle \xi_1, \dots, \xi_{\ell_n} \rangle$  of elements of  $\lambda^{+\omega+1}$ , we define  $f_n(\xi_1, \dots, \xi_{\ell_n})$  to be the minimal  $\xi_0 \in \lambda^{+\omega+1}$  such that  $M \models \varphi_n(\xi_0, \dots, \xi_{\ell_n})$ . If there is no such  $\xi_0$ , in other words if  $M \models \neg \exists x \varphi_n(x, \xi_1, \dots, \xi_{\ell_n})$ , then set  $f_n(\xi_1, \dots, \xi_{\ell_n})$  to be 0. Now  $Hull(N \cup \beta_0, M)$  is just the minimal superset  $X$  of  $N \cup \beta_0$  that satisfies that  $f_n[X^{\{1, \dots, \ell_n\}}] \subset X$  for all  $n$ . Since this is simply a large algebra, we can generate all the terms  $t$  of the algebraic operations  $\{f_n : n \in \omega\}$ . It is easily seen that for each  $\zeta \in X$ , there is a term  $t(v_1, \dots, v_m)$  such that  $\zeta = t(\delta_1, \dots, \delta_m)$  for some sequence  $\langle \delta_1, \dots, \delta_m \rangle$  with each  $\delta_i \in N \cup \beta_0$ . Assume that  $\zeta \in \kappa^{+\omega+1}$ . By re-indexing the variables in the term we can assume that there is an  $n \leq m$  so that  $\delta_i < \beta_0$  for  $1 \leq i \leq n$  and  $\kappa^{+\omega+1} \leq \delta_i$  for  $n < i \leq m$ . Let  $\vec{a}$  denote the tuple  $\langle \delta_{n+1}, \dots, \delta_m \rangle$ . Choose  $\eta \in N \cap \kappa^{+\omega+1}$  large enough so that  $\{\delta_1, \dots, \delta_n\}$  is contained in  $\eta$ . Since set-membership in  $M$  is coded by  $R_0$  rather than  $\in$  we have to argue a little less naturally. Consider the set  $s_0(\eta, \vec{a}) = \{t(\gamma_1, \dots, \gamma_n, \vec{a}) : \{\gamma_1, \dots, \gamma_n\} \in [\eta]^{\leq n}\}$ . Clearly  $s_0(\eta, \vec{a})$  is a member of  $H(\lambda^{+\omega+1})$ . Now define  $s_1(\eta, \vec{a})$  to be  $\{x_\alpha : \alpha \in s_0(\eta, \vec{a})\}$ , and choose the unique  $\zeta_1 \in \lambda^{+\omega+1}$



such that  $x_{\zeta_1} = s_1(\eta, \vec{a})$ . We claim that  $\zeta_1 \in N$ . Note that  $\alpha R_0 \zeta_1$  holds if and only if  $\alpha \in s_0(\eta, \vec{a})$ , and therefore

$$M \models (\forall \alpha) [\alpha R_0 \zeta_1 \text{ iff } (\exists \gamma_1 \in \eta) \cdots (\exists \gamma_n \in \eta)(\alpha = t(\gamma_1, \dots, \gamma_n, \vec{a}))] .$$

By elementarity then we have that  $\zeta_1 \in N$ , and by similar reasoning the supremum,  $\zeta_0$ , of  $\zeta_1 \cap \kappa^{+\omega+1}$  is also in  $N$ . This of course means that  $\zeta < \beta_0$ .  $\square$

We use the elementarity of  $N$  to deduce properties of the families  $\{\dot{A}_\xi : \xi \in N\}$  and  $\{\dot{f}_\xi : \xi \in N\}$ . Actually the collection we are most interested in is the family  $\{h_\xi : \xi \in \Lambda \cap N\}$ .

Now we need a result from Shelah's pcf theory which is proven in Jech [4, 24.9]. Since  $\aleph_1 = \mathfrak{c} < \kappa^{+\omega+1}$  there is a function  $\langle \varrho_n : n \in \omega \rangle$  in  $\Pi_n \lambda^{+\omega}$  such that the sequence  $\{h_\xi : \xi \in N\}$  is unbounded mod finite in  $\Pi_n \varrho_n$ . For each  $n$ ,  $\varrho_n \leq \sup(N \cap \lambda^{+n+2})$ . Since  $\mathbb{P}_0$  has cardinality  $\kappa^{+\omega}$ , and so less than  $|N| = \kappa^{+\omega+1}$ , a standard argument (analogous to the fact that adding a Cohen real does not add a dominating real) shows that the sequence  $\{h_\xi : \xi \in \Lambda \cap N\}$  remains unbounded mod finite in  $\Pi_n \varrho_n$  (and in  $\Pi_n(\varrho_n \cap N)$ ).

Now pass to the extension by  $G \cap \mathbb{P}_0$  and let  $H$  be the function  $\text{val}_G(\dot{H})$ , and we recall that  $f_{\zeta_\xi}(h_\xi(n)) = H(n)$  for all  $n \in \omega$  and  $\xi \in \Lambda$ . Now pass to the full extension  $V[G]$  and again, since  $\mathbb{P}_1$  was forced to be countably closed, the family  $\{h_\xi : \xi \in \Lambda \cap N\}$  is still unbounded in  $\Pi_n(\varrho_n \cap N)$  (no new elements were added). We let  $A$  be the countable set  $N \cap \lambda^{+\omega}$ , and for each  $\xi \in \Lambda \cap N$ , there is an  $n_\xi$  such that  $f_\xi(h_\xi(m)) = f_A(h_\xi(m))$  for all  $m > n_\xi$ . There is a single  $n$  so that  $\Lambda_n = \{\xi \in \Lambda \cap N : n_\xi = n\}$  has cardinality  $\omega_1$ , and thus  $\{h_\xi : \xi \in \Lambda_n \cap N\}$  is also unbounded in  $\Pi_n(\varrho_n \cap N)$ . This certainly implies that there is an  $m > n$  such that  $\{h_\xi(m) : \xi \in \Lambda_n \cap N\}$  is infinite. This completes the proof since  $f_A(h_\xi(m)) = H(m)$  for all  $\xi \in \Lambda_n \cap N$ .  $\square$

#### 4. APPLICATIONS TO INFINITE LENGTH GAMES

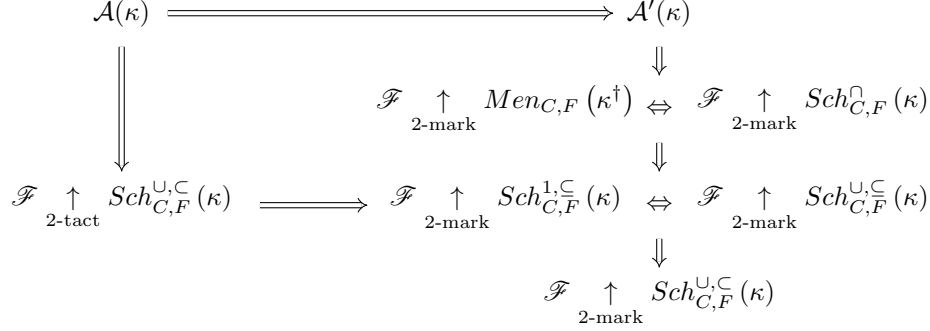
We introduce three variations of Scheeper's game which we defined in the introduction.

**Game 22.** Let  $Sch_{C,F}^{\cup, \subseteq}(\kappa)$  denote the *Scheepers countable-finite union game* which proceeds analogously to  $Sch_{C,F}^{\cup, \subseteq}(\kappa)$ , except that  $\mathcal{C}$ 's restriction in round  $n+1$  is weakened to  $C_{n+1} \supseteq C_n$ .

**Game 23.** Let  $Sch_{C,F}^{1, \subseteq}(\kappa)$  denote the *Scheepers countable-finite initial game* which proceeds analogously to  $Sch_{C,F}^{\cup, \subseteq}(\kappa)$ , except that  $\mathcal{F}$ 's winning condition is weakened to  $\bigcup_{n < \omega} F_n \supseteq C_0$ .

**Game 24.** Let  $Sch_{C,F}^\cap(\kappa)$  denote the *Scheepers countable-finite intersection game* which proceeds analogously to  $Sch_{C,F}^{1, \subseteq}(\kappa)$ , except that  $\mathcal{C}$  may choose any  $C_n \in [\kappa]^{\leq \omega}$  each round, and  $\mathcal{F}$ 's winning condition is weakened to  $\bigcup_{n < \omega} F_n \supseteq \bigcap_{n < \omega} C_n$ .

In [1] Clontz extended Scheepers' application of almost-compatible injections to these game variants as well as  $Men_{C,F}(\kappa^\dagger)$ . However, when considering Markov strategies, finite-to-one functions suffice.

FIGURE 1. Diagram of Scheeper/Menger game implications with  $\mathcal{A}(\kappa), \mathcal{A}'(\kappa)$ 

**Theorem 25.** *Figure 1.*

*Proof.*  $\mathcal{A}(\kappa) \Rightarrow \mathcal{F} \uparrow_{2\text{-tact}} Sch_{C,F}^{\cup,\subseteq}(\kappa)$  was shown by Scheepers in [10] (also, see the following theorem). Most of the other results in the figure were proven in [1], with the exception that  $\mathcal{A}'(\kappa)$  was not considered at the time. The following proof that  $\mathcal{A}'(\kappa) \Rightarrow \mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^\cap(\kappa)$  is a trivial modification of the proof presented in [1] assuming  $\mathcal{A}(\kappa)$ , but as that paper is under review at the time of this writing, we provide it here.

Let  $f_A$  for  $A \in [\kappa]^{\leq \omega}$  witness  $\mathcal{A}'(\kappa)$ . We define a 2-mark  $\sigma$  for  $Sch_{C,F}^\cap(\kappa)$  as follows:

$$\sigma(\langle A \rangle, 0) = \{\alpha \in A : f_A(\alpha) = 0\}$$

$$\sigma(\langle A, B \rangle, n+1) = \{\alpha \in A \cap B : f_B(\alpha) \leq n+1 \text{ or } f_A(\alpha) \neq f_B(\alpha)\}$$

For any attack  $\langle A_0, A_1, \dots \rangle$  by  $\mathcal{C}$  and  $\alpha \in \bigcap_{n < \omega} A_n$ , either  $f_{A_n}(\alpha)$  is constant for all  $n$ , or  $f_{A_n}(\alpha) \neq f_{A_{n+1}}(\alpha)$  for some  $n$ ; either way,  $\alpha$  is covered.  $\square$

We include the following proof from [10] to point out why  $\mathcal{A}'(\kappa)$  seems insufficient for providing  $\mathcal{F}$  a winning 2-tactic in  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$ , despite that it witnesses a winning 2-mark.

**Theorem 26** ([10]).  $\mathcal{A}(\kappa) \Rightarrow \mathcal{F} \uparrow_{2\text{-tact}} Sch_{C,F}^{\cup,\subseteq}(\kappa)$

*Proof.* Let  $\{f_A : A \in [\kappa]^{\leq \aleph_0}\}$  witness  $\mathcal{A}(\kappa)$ , and define  $g_A : A \rightarrow \omega$  by  $g_A(\alpha) = f_A(\alpha) - |\{\beta \in A : f_A(\beta) < f_A(\alpha)\}|$ .

We claim that  $\{\alpha \in A : g_A(\alpha) \leq g_B(\alpha)\}$  must be finite as it is bounded above by  $\max\{M, f_A(\alpha), f_B(\alpha) : f_A(\alpha) \neq f_B(\alpha)\}$  where  $M = f_B(\alpha)$  for some  $\alpha \in B \setminus A$ . To see this, let  $f_A(\alpha) = f_B(\alpha) = N > M$  and assume  $f_A(\beta) \neq f_B(\beta)$  implies  $f_A(\beta), f_B(\beta) < N$ . Then

$$g_A(\alpha) = N - |\{\beta \in A : f_A(\beta) < N\}| > N - |\{\beta \in B : f_B(\beta) < N\}| = g_B(\alpha)$$

with the strictness of the inequality witnessed by  $f_B(\alpha) = M < N$  for some  $\alpha \in B \setminus A$ .

As a result,

$$\sigma(\langle A, B \rangle) = \{\alpha \in A : g_A(\alpha) \leq g_B(\alpha)\}$$

is a legal 2-tactic for  $\mathcal{F}$ . Let  $C = \langle C_0, C_1, \dots \rangle$  be a strictly increasing sequence of countable sets and  $\alpha \in C_n$ .

Noting that  $f_A$  is an injection (not just finite-to-one),  $0 \leq g_{C_{n+m}}(\alpha)$  for all  $m < \omega$ , and it follows that  $g_{C_{n+m}}(\alpha) \leq g_{C_{n+m+1}}(\alpha)$  for some  $m < \omega$ . Therefore  $\alpha \in \sigma(\langle C_{n+m}, C_{n+m+1} \rangle)$ .  $\square$

While the above proof cannot be trivially modified to utilize the finite-to-one functions witnessed by  $\mathcal{A}'(\kappa)$  in constructing a winning 2-tactical strategy for  $Sch_{C,F}^{\cup, \subseteq}(\kappa)$ , whether  $\mathcal{A}'(\kappa)$  is sufficient for  $\mathcal{F} \uparrow_{2\text{-tact}} Sch_{C,F}^{\cup, \subseteq}(\kappa)$  after all does remain open:

**Question 27.** *May the previous theorem be improved by replacing  $\mathcal{A}(\kappa)$  with  $\mathcal{A}'(\kappa)$ ?*

We would like to demonstrate that  $\mathcal{A}'(\kappa)$  is not necessary for constructing winning 2-Markov strategies in  $Sch_{C,F}^{\cap}(\kappa)$ .

**Theorem 28.** *Let  $\alpha$  be the limit of increasing ordinals  $\beta_n$  for  $n < \omega$ . If  $\mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^{\cap}(\aleph_{\beta_n})$  for all  $n < \omega$ , then  $\mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^{\cap}(\aleph_{\alpha})$ .*

*Proof.* Let  $\sigma_n$  be a winning 2-mark for  $\mathcal{F}$  in  $Sch_{C,F}^{\cap}(\aleph_{\beta_n})$ . Define the 2-mark  $\sigma$  for  $\mathcal{F}$  in  $Sch_{C,F}^{\cap}(\aleph_{\alpha})$  as follows:

$$\begin{aligned} \sigma(\langle C \rangle, 0) &= \sigma_0(\langle C \cap \aleph_{\beta_0} \rangle, 0) \\ \sigma(\langle C, D \rangle, n+1) &= \sigma_{n+1}(\langle D \cap \aleph_{\beta_{n+1}} \rangle, 0) \cup \bigcup_{m \leq n} \sigma_m(\langle C \cap \aleph_{\beta_m}, D \cap \aleph_{\beta_m} \rangle, n-m+1) \end{aligned}$$

Let  $\langle C_0, C_1, \dots \rangle$  be an attack by  $\mathcal{C}$  in  $Sch_{C,F}^{\cap}(\aleph_{\alpha})$ , and  $\alpha \in \bigcap_{n < \omega} C_n$ . Choose  $N < \omega$  with  $\alpha < \aleph_{\beta_{N+1}}$ . Consider the attack  $\langle C_{N+1} \cap \aleph_{\beta_{N+1}}, C_{N+2} \cap \aleph_{\beta_{N+1}}, \dots \rangle$  by  $\mathcal{C}$  in  $Sch_{C,F}^{\cap}(\aleph_{\beta_{N+1}})$ . Since  $\sigma_{N+1}$  is a winning 2-mark and  $\alpha \in \bigcap_{n < \omega} C_{N+n+1} \cap \aleph_{\beta_{N+1}}$ , either  $\alpha \in \sigma_{N+1}(\langle C_{N+1} \cap \aleph_{\beta_{N+1}} \rangle, 0)$  and thus  $\alpha \in \sigma(\langle C_N, C_{N+1} \rangle, N+1)$ , or  $\alpha \in \sigma_{N+1}(\langle C_{N+M+1} \cap \aleph_{\beta_{N+1}}, C_{N+M+2} \cap \aleph_{\beta_{N+1}} \rangle, M+1)$  for some  $M < \omega$  and thus  $\alpha \in \sigma(\langle C_{N+M+1}, C_{N+M+2} \rangle, N+M+2)$ . Thus  $\sigma$  is a winning 2-mark.  $\square$

**Theorem 29.** *Let  $\alpha$  be the limit of increasing ordinals  $\beta_n$  for  $n < \omega$ . If  $\mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^{1, \subseteq}(\aleph_{\beta_n})$  for all  $n < \omega$ , then  $\mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^{1, \subseteq}(\aleph_{\alpha})$ .*

*Proof.* The proof proceeds nearly identically to the previous proof: substitute  $\alpha \in C_0$  in place of  $\alpha \in \bigcap_{n < \omega} C_n$  and proceed.  $\square$

**Corollary 30.** *It is consistent that  $\mathcal{A}'(\aleph_{\omega})$  fails, but as  $\mathcal{A}'(\aleph_k)$  holds in ZFC for all  $k < \omega$ , both  $\mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^{\cap}(\aleph_{\omega})$  and  $\mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^{1, \subseteq}(\aleph_{\omega})$  hold in ZFC.*

We conclude by returning our attention to Question 6, which asks whether there exists a space for which the second player  $\mathcal{F}$  in the game  $Men_{C,F}(X)$  has a winning strategy without a winning 2-mark.

**Question 31.** Does  $\mathcal{F} \overset{2\text{-mark}}{\uparrow} Sch_{C,F}^\cap(\kappa)$  hold for all cardinals  $\kappa$  in  $ZFC$ ?

If not, the model producing  $\kappa > \aleph_\omega$  where  $\mathcal{F} \not\overset{2\text{-mark}}{\uparrow} Sch_{C,F}^\cap(\kappa)$  yields a positive answer to Question 6:  $X = \kappa^\dagger$ . On the other hand, under  $V = L$  Corollary 17 shows that  $\mathcal{A}'(\kappa)$  and therefore  $\mathcal{F} \overset{2\text{-mark}}{\uparrow} Men_{C,F}(\kappa^\dagger)$  for every cardinal  $\kappa$ , so a more exotic example than  $X = \kappa^\dagger$  would be required to answer Question 6 in  $ZFC$ .

Solving the following weaker question would not answer Question 6 by itself, but a solution would be interesting nonetheless.

**Question 32.** Does  $\mathcal{F} \overset{2\text{-mark}}{\uparrow} Sch_{C,F}^{\cup,\subseteq}(\kappa)$  hold for all cardinals  $\kappa$  in  $ZFC$ ?

Although, it's still open whether the previous two questions are even distinct.

**Question 33.** Can a winning 2-Markov strategy in  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$  be used to construct a winning 2-Markov strategy in  $Sch_{C,F}^\cap(\kappa)$ ?

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