DUAL SELECTION GAMES

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Abstract. Often, a given selection game studied in the literature has a known dual game. In dual games, a winning strategy for a player in either game may be used to create a winning strategy for the opponent in the dual. For example, the Rothberger selection game involving open covers is dual to the point-open game. This extends to a general theorem: if $\{\text{range}(f) : f \in \mathbf{C}(\mathcal{R})\}$ is coinitial in $\mathcal{A}$ with respect to $\subseteq$, where $\mathbf{C}(\mathcal{R}) = \{f \in (\bigcup \mathcal{R})^{\mathcal{R}} : R \in \mathcal{R} \Rightarrow f(R) \in R\}$ collects the choice functions on the set $\mathcal{R}$, then $G_1(\mathcal{A}, \mathcal{B})$ and $G_1(\mathcal{R}, \neg\mathcal{B})$ are dual selection games.

1. Introduction

Definition 1. The *selection game* $G_1(\mathcal{A}, \mathcal{B})$ is an ω-length game involving Players I and II. During round n, I chooses $A_n \in \mathcal{A}$, followed by II choosing $B_n \in A_n$. Player II wins in the case that $\{B_n : n < \omega\} \in \mathcal{B}$, and Player I wins otherwise.

For brevity, let

$$G_1(\mathcal{A}, \neg\mathcal{B}) = G_1(\mathcal{A}, \mathcal{P}\left(\bigcup \mathcal{A}\right) \setminus \mathcal{B}).$$

That is, II wins in the case that $\{B_n : n < \omega\} \notin \mathcal{B}$, and I wins otherwise.

Definition 2. For a set X, let $\mathbf{C}(X) = \{f \in (\bigcup X)^X : x \in X \Rightarrow f(x) \in x\}$ be the collection of all choice functions on X.

Definition 3. Write $X \preceq Y$ if X is coinitial in Y with respect to $\subseteq$; that is, $X \subseteq Y$, and for all $y \in Y$, there exists $x \in X$ such that $x \subseteq y$.

In the context of selection games, we will say $\mathcal{A}'$ is a *selection basis* for $\mathcal{A}$ when $\mathcal{A}' \preceq \mathcal{A}$.

Definition 4. The set $\mathcal{R}$ is said to be a *reflection* of the set $\mathcal{A}$ if

$$\{\text{range}(f) : f \in \mathbf{C}(\mathcal{R})\}$$

is a selection basis for $\mathcal{A}$.

As we will see, reflections of selection sets are used frequently (but implicitly) throughout the literature to define dual selection games.

2. Main Results

The following four theorems demonstrate that reflections characterize dual selection games for both perfect information strategies and certain limited information strategies.

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Definition 5. A pair of games $G(X), H(X)$ are *Markov information dual* if both of the following hold.

- $I \underset{\text{pre}}{\uparrow} G(X)$ if and only if $II \underset{\text{mark}}{\uparrow} H(X)$.
- $II \underset{\text{mark}}{\uparrow} G(X)$ if and only if $I \underset{\text{pre}}{\uparrow} H(X)$.

Theorem 6. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*
Then $\mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$ *if and only if* $\mathrm{II} \underset{mark}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let σ witness $\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. Since $\sigma(n) \in \mathcal{A}$, $\text{range}(f_n) \subseteq \sigma(n)$ for some $f_n \in \mathbf{C}(\mathcal{R})$. So let $\tau(R, n) = f_n(R)$ for all $R \in \mathcal{R}$ and $n < \omega$. Suppose $R_n \in \mathcal{R}$ for all $n < \omega$. Note that since σ is winning and $\tau(R_n, n) = f_n(R_n) \in \text{range}(f_n) \subseteq \sigma(n)$, $\{\tau(R_n, n) : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\mathrm{II} \underset{\text{mark}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Now let σ witness $\mathrm{II} \underset{\text{mark}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$. Let $f_n \in \mathbf{C}(\mathcal{R})$ be defined by $f_n(R) = \sigma(R, n)$, and let $\tau(n) = \text{range}(f_n) \in \mathcal{A}$. Suppose that $B_n \in \tau(n) = \text{range}(f_n)$ for all $n < \omega$. Choose $R_n \in \mathcal{R}$ such that $B_n = f_n(R_n) = \sigma(R_n, n)$. Since σ is winning, $\{B_n : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. $\square$

Theorem 7. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*
Then $\mathrm{II} \underset{mark}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$ *if and only if* $\mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let σ witness $\mathrm{II} \underset{\text{mark}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. Let $n < \omega$. Suppose that for each $R \in \mathcal{R}$, there was $g(R) \in R$ such that for all $A \in \mathcal{A}$, $\sigma(A, n) \neq g(R)$. Then $g \in \mathbf{C}(\mathcal{R})$ and $\text{range}(g) \in \mathcal{A}$, thus $\sigma(\text{range}(g), n) \neq g(R)$ for all $R \in \mathcal{R}$, a contradiction.

So choose $\tau(n) \in \mathcal{R}$ such that for all $r \in \tau(n)$ there exists $A_{r,n} \in \mathcal{A}$ such that $\sigma(A_{r,n}, n) = r$. It follows that when $r_n \in \tau(n)$ for $n < \omega$, $\{r_n : n < \omega\} = \{\sigma(A_{r_n,n}) : n < \omega\} \in B$, so τ witnesses $\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Now let σ witness $\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$. Then $\sigma(n) \in \mathcal{R}$, so for $A \in \mathcal{A}$, let $f_A \in \mathbf{C}(\mathcal{R})$ satisfy $\text{range}(f_A) \subseteq A$, and let $\tau(A, n) = f_A(\sigma(n)) \in A \cap \sigma(n)$. Then if $A_n \in \mathcal{A}$ for $n < \omega$, $\tau(A_n, n) \in \sigma(n)$, so $\{\tau(A_n, n) : n < \omega\} \in \mathcal{B}$. Thus τ witnesses $\mathrm{II} \underset{\text{mark}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. $\square$

Definition 8. A pair of games $G(X), H(X)$ are *perfect information dual* if both of the following hold.

- $I \uparrow G(X)$ if and only if $II \uparrow H(X)$.
- $II \uparrow G(X)$ if and only if $I \uparrow H(X)$.

Theorem 9. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*
Then $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$ *if and only if* $\mathrm{II} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let σ witness $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$. Let $c(\emptyset) = \emptyset$. Suppose $c(s) \in (\bigcup A)^{<\omega}$ is defined for $s \in \mathcal{R}^{<\omega}$. Since $\sigma(c(s)) \in \mathcal{A}$, let $f_s \in \mathbf{C}(\mathcal{R})$ satisfy $\text{range}(f_s) \subseteq \sigma(c(s))$, and let $c(s^\frown\langle R \rangle) = c(s)^\frown\langle f_s(R) \rangle$. Then let $c(\alpha) = \bigcup\{c(\alpha \restriction n) : n < \omega\}$ for $\alpha \in \mathcal{R}^\omega$, so

$$c(\alpha)(n) = f_{\alpha \restriction n}(\alpha(n)) \in \text{range}(f_{\alpha \restriction n}) \subseteq \sigma(c(\alpha \restriction n))$$

demonstrating that $c(\alpha)$ is a legal attack against σ.

Let $\tau(s^\frown\langle R\rangle) = f_s(R)$. Consider the attack $\alpha \in \mathcal{R}^\omega$ against τ. Then since σ is winning and $\tau(\alpha \restriction n+1) = f_{\alpha\restriction n}(\alpha(n)) \in \operatorname{range}(f_{\alpha\restriction n}) \subseteq \sigma(c(\alpha \restriction n))$, it follows that $\{\tau(\alpha \restriction n+1) : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\mathrm{II} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$.

Now let σ witness $\mathrm{II} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$. For $s \in \mathcal{R}^{<\omega}$, define $f_s \in \mathbf{C}(\mathcal{R})$ by $f_s(R) = \sigma(s^\frown\langle R\rangle)$. Let $\tau(\emptyset) = \operatorname{range}(f_\emptyset) \in \mathcal{A}$, and for $x \in \tau(\emptyset)$, choose $R_{\langle x\rangle} \in \mathcal{R}$ such that $x = f_\emptyset(R_{\langle x\rangle})$ (for other $x \in \bigcup A$, choose $R_{\langle x\rangle}$ arbitrarily as it won't be used). Now let $s \in (\bigcup A)^{<\omega}$, and suppose $R_{s\restriction n^\frown\langle x\rangle} \in \mathcal{R}$ has been defined for $n \leq |s|$ and $x \in \bigcup A$. Then let $\tau(s^\frown\langle x\rangle) = \operatorname{range}(f_{\langle R_{s\restriction 0},\dots,R_s,R_{s^\frown\langle x\rangle}\rangle})$ and for $y \in \tau(s)$ choose $R_{s^\frown\langle x,y\rangle}$ such that $x = f_{\langle R_{s\restriction 0},\dots,R_s,R_{s^\frown\langle x\rangle}\rangle}(R_{s^\frown\langle x,y\rangle})$ (and again, choose $R_{s^\frown\langle x,y\rangle}$ arbitrarily for other $y \in \bigcup\mathcal{A}$ as it won't be used).

Then let α attack τ, so $\alpha(n) \in \tau(\alpha \restriction n)$ and thus $\alpha(n) = f_{\langle R_{\alpha\restriction 0},\dots,R_{\alpha\restriction n}\rangle}(R_{\alpha\restriction n+1}) = \sigma(\langle R_{\alpha\restriction 0}, \dots, R_{\alpha\restriction n+1}\rangle)$. Since σ is winning, $\{\sigma(\langle R_{\alpha\restriction 0}, \dots, R_{\alpha\restriction n+1}\rangle) : n < \omega\} = \{\alpha(n) : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$. $\square$

Theorem 10. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*
Then $\mathrm{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$ if and only if $\mathrm{I} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let σ witness $\mathrm{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$. Let $s \in (\bigcup A)^{<\omega}$ and assume $a(s) \in \mathcal{A}^{|s|}$ is defined (of course, $a(\emptyset) = \emptyset$). Suppose for all $R \in \mathcal{R}$ there existed $f(R) \in R$ such that for all $A \in \mathcal{A}$, $\sigma(a(s)^\frown\langle A\rangle) \neq f(R)$. Then $f \in \mathbf{C}(\mathcal{R})$ and $\operatorname{range}(f) \in \mathcal{A}$, and thus $\sigma(a(s)^\frown\langle \operatorname{range}(f)\rangle) \neq f(R)$ for all $R \in \mathcal{R}$, a contradiction. So let $\tau(s) \in \mathcal{R}$ satisfy for all $x \in \tau(s)$ there exists $a(s^\frown\langle x\rangle) \in \mathcal{A}^{|s|+1}$ extending $a(s)$ such that $x = \sigma(a(s^\frown\langle x\rangle))$.

If τ is attacked by $\alpha \in (\bigcup R)^\omega$, then $\alpha(n) \in \tau(\alpha \restriction n)$. So $\alpha(n) = \sigma(a(\alpha \restriction n+1))$, and since σ is winning, $\{\sigma(a(\alpha \restriction n+1)) : n < \omega\} = \{\alpha(n) : n < \omega\} \in \mathcal{B}$. Therefore τ witnesses $\mathrm{I} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$.

Now let σ witness $\mathrm{I} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$. Let $s \in \mathcal{A}^{<\omega}$, and suppose $r(s) \in (\bigcup\mathcal{R})^{|s|}$ is defined (again, $r(\emptyset) = \emptyset$). For $A \in \mathcal{A}$ choose $f_A \in \mathbf{C}(\mathcal{R})$ where $\operatorname{range}(f_A) \subseteq A$, and let $\tau(s^\frown\langle A\rangle) = f_A(\sigma(r(s)))$, and let $r(s^\frown\langle A\rangle)$ extend $r(s)$ by letting $r(s^\frown\langle A\rangle)(|s|) = \tau(s^\frown\langle A\rangle)$.

If τ is attacked by $\alpha \in \mathcal{A}^\omega$, then since $\tau(\alpha \restriction n+1) = f_{\alpha(n)}(\sigma(r(\alpha \restriction n)) \in \alpha(n) \cap \sigma(r(\alpha \restriction n))$ and σ is winning, we conclude that τ is a legal strategy and $\{\tau(\alpha \restriction n+1) : n < \omega\} \in \mathcal{B}$. Therefore τ witnesses $\mathrm{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$. $\square$

Corollary 11. *If $\mathcal{R}$ is a reflection of $\mathcal{A}$, then $G_1(\mathcal{A}, \mathcal{B})$ and $G_1(\mathcal{R}, \neg\mathcal{B})$ are both perfect information dual and Markov information dual.*

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