2-Markov Strategies in Selection Games

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Abstract

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1 Introduction

Let $S(\mathcal{A}, \mathcal{B})$ be the statement that whenever $A_n \in \mathcal{A}$ for $n < \omega$, there exist $B_n \in [A_n]^{<\omega}$ such that $\bigcup\{B_n : n < \omega\} \in \mathcal{B}$. This *selection principle* characterizes a property of a topological space X when $\mathcal{A}, \mathcal{B}$ are defined in terms of X. For example, if $\mathcal{O}_X$ is the collection of open covers of X, then $S(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known Menger covering property.

This property may be made stronger by considering the following two-player game of length ω: $G(\mathcal{A}, \mathcal{B})$. During each round $n < \omega$, the first player $\mathscr{A}$ chooses $A_n \in \mathcal{A}$, followed by $\mathscr{B}$ choosing $B_n \in [A_n]^{<\omega}$. $\mathscr{B}$ wins the game if $\bigcup\{B_n : n < \omega\} \in \mathcal{B}$; otherwise $\mathscr{A}$ wins. If $\mathscr{B}$ has a *winning strategy* for the game (a function which defines a move for each finite sequence of previous moves by $\mathscr{A}$, and beats every possible response by $\mathscr{A}$), then we write $\mathscr{B} \uparrow G(\mathcal{A}, \mathcal{B})$.

These concepts were first introduced by Scheepers in [MR1378387]. Of course, $\mathscr{B} \uparrow G(\mathcal{A}, \mathcal{B}) \Rightarrow S(\mathcal{A}, \mathcal{B})$, but the converse need not hold, since each B_n may be defined in $S(\mathcal{A}, \mathcal{B})$ using knowledge of all A_n, not just those "previously played". Thus for each topological property P characterized by $S(\mathcal{A}, \mathcal{B})$, we denote the (possibly) stronger property $\mathscr{B} \uparrow G(\mathcal{A}, \mathcal{B})$ as *strategic* P.

Such notions may be made even stronger using *limited information strategies*. A *k-Markov strategy* for $\mathscr{B}$ uses only the last k moves of $\mathscr{A}$ and the round number. When $\mathscr{B}$ has a winning k-Markov strategy for $G(\mathcal{A}, \mathcal{B})$, we write $\mathscr{B} \underset{k\text{-mark}}{\uparrow} G(\mathcal{A}, \mathcal{B})$. Similarly, for each topological property P characterized by $S(\mathcal{A}, \mathcal{B})$, we denote property $\mathscr{B} \underset{k\text{-mark}}{\uparrow} G(\mathcal{A}, \mathcal{B})$ as *k-Markov* P. When $k = 1$, we may omit the k.

2 k-Markov implies 2-Markov

In the case of the selection game $G(\mathcal{A}, \mathcal{B})$, we may see that a $(k+2)$-Markov strategy may always be improved to a 2-Markov strategy, as shown by the author in [clontzMengerPreprint] with regards to $G(\mathcal{O}_X, \mathcal{O}_X)$.

Theorem 1. *For each $k < \omega$, $\mathscr{B} \underset{(k+2)\text{-}mark}{\uparrow} G(\mathcal{A}, \mathcal{B})$ if and only if $\mathscr{B} \underset{2\text{-}mark}{\uparrow} G(\mathcal{A}, \mathcal{B})$.*

Proof. Let σ be a winning $(k+2)$-mark. We define the 2-mark τ as follows:

$$\tau(\langle A \rangle, 0) = \bigcup_{m<k+1} \sigma(\langle \underbrace{A, \dots, A}_{m+1} \rangle, m)$$

$$\tau(\langle A, A' \rangle, n+1) = \bigcup_{m<k+1} \sigma(\langle \underbrace{A, \dots, A}_{k+1-m}, \underbrace{A', \dots, A'}_{m+1} \rangle, (n+1)(k+1)+m)$$

Let $\langle A_0, A_1, \dots \rangle$ be an attack by $\mathscr{A}$ against τ. Then consider the attack

$$\langle \underbrace{A_0, \dots, A_0}_{k+1}, \underbrace{A_1, \dots, A_1}_{k+1}, \dots \rangle$$

by $\mathscr{A}$ against σ. Since σ is a winning $(k+2)$-mark,

$$\bigcup_{m<k+1} \sigma(\langle \underbrace{A_0, \dots, A_0}_{m+1} \rangle, m) \cup \bigcup_{n<\omega, m<k+1} \sigma(\langle \underbrace{A_n, \dots, A_n}_{k+1-m}, \underbrace{A_{n+1}, \dots, A_{n+1}}_{m+1} \rangle, (n+1)(k+1)+m)$$

$$= \tau(\langle A_0 \rangle, 0) \cup \bigcup_{n<\omega} \tau(\langle A_n, A_{n+1} \rangle, n+1) \in \mathcal{B}$$

Thus τ is a winning 2-mark. $\square$

The following natrual question is open:

Question 2. *Do there exist (interesting/topological) $\mathcal{A}, \mathcal{B}$ such that $\mathscr{B} \uparrow G(\mathcal{A}, \mathcal{B})$ but $\mathscr{B} \underset{2\text{-}mark}{\not\uparrow} G(\mathcal{A}, \mathcal{B})$?*

3 Menger game results

Consider the case that $\mathcal{A} = \mathcal{B} = \mathcal{O}_X$, i.e. the Menger game. The following summarize results from [MR1129143] [clontzMengerPreprint] and [clontzDowAlcompPreprint].

Definition 3. For any cardinal κ, let $\kappa^\dagger = \kappa \cup \{\infty\}$ denote the *one-point Lindelöf-ication* of discrete κ, where points in κ are isolated, and the neighborhoods of ∞ are co-countable.

Theorem 4. $\mathscr{B} \uparrow G(\mathcal{O}_{\kappa^\dagger}, \mathcal{O}_{\kappa^\dagger})$.

Definition 5. For two functions f, g we say f is *almost compatible* with g if $|\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(x) \neq g(x)\}| < \omega$.

Definition 6. $\mathcal{A}'(\kappa)$ states that there exists a collection of pairwise almost compactible finite-to-one functions $\{f_A \in \omega^A : A \in [\kappa]^{\leq \omega}\}$.

Theorem 7. *$\mathcal{A}'(\omega_n)$ holds for all $n < \omega$.*

Theorem 8. *$\mathcal{A}'(\kappa)$ implies $\mathscr{B} \underset{2\text{-}mark}{\uparrow} G(\mathcal{O}_{\kappa^\dagger}, \mathcal{O}_{\kappa^\dagger})$.*

Theorem 9. *For any cardinal κ, κ Cohen reals may be added to a model of $ZFC + CH$ while preserving $\mathcal{A}'(\mathfrak{c})$.*

Theorem 10. *There exists a model of ZFC where $\mathcal{A}'(\omega_\omega)$ fails.*

Theorem 11. $\mathscr{B} \underset{2\text{-}mark}{\uparrow} G(\mathcal{O}_{\omega_\omega^\dagger}, \mathcal{O}_{\omega_\omega^\dagger})$.

It remains open whether $\mathscr{B} \underset{\text{2-mark}}{\uparrow} G(\mathcal{O}_{\omega_{\omega+1}^\dagger}, \mathcal{O}_{\omega_{\omega+1}^\dagger})$ might fail when $\mathcal{A}'(\omega_\omega)$ fails. Due to the above, any attempt to show $\mathscr{B} \underset{\text{2-mark}}{\not\uparrow} G(\mathcal{O}_{\kappa^\dagger}, \mathcal{O}_{\kappa^\dagger})$ cannot happen solely within ZFC.