

**Definition 1.** A **uniform space**  $\langle X, \mathcal{D} \rangle$  is a set  $X$  paired with a filter  $\mathcal{D}$  (called its **uniformity**) of relations (called **entourages**) on  $X$  such that for each entourage  $D \in \mathcal{D}$ :

- $D$  is reflexive, i.e., the diagonal  $\Delta \subseteq D$ .
- Its inverse  $D^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in D\} \in \mathcal{D}$ .
- There exists  $\frac{1}{2}D \in \mathcal{D}$  such that

$$2(\frac{1}{2}D) = \frac{1}{2}D \circ \frac{1}{2}D = \{\langle x, z \rangle : \exists y(\langle x, y \rangle, \langle y, z \rangle \in \frac{1}{2}D)\} \subseteq D$$

Note that since  $\mathcal{D}$  is a filter, for each  $D \in \mathcal{D}$ , the symmetric relation  $D \cap D^{-1} \in \mathcal{D}$ .

**Proposition 2.** For each  $D \in \mathcal{D}$  and  $n < \omega$  there exists  $\frac{1}{2^{n+1}}D \in \mathcal{D}$  such that

$$2(\frac{1}{2^{n+1}}D) = \frac{1}{2^{n+1}}D \circ \frac{1}{2^{n+1}}D \subseteq \frac{1}{2^n}D$$

and if  $2E \subseteq \frac{1}{2^n}D$ , then  $E \subseteq \frac{1}{2^{n+1}}D$ .

**Definition 3.** For an entourage  $D \in \mathcal{D}$ , let  $D[x] = \{y : \langle x, y \rangle \in D\}$  be the  $D$ -**neighborhood** of  $x$ . The uniform topology for a uniform space  $\langle X, \mathcal{D} \rangle$  is generated by the base  $\{D[x] : x \in X, D \in \mathcal{D}\}$ .

**Theorem 4.** A space  $X$  is uniformizable (its topology is the uniform topology for some uniformity) if and only if  $X$  is completely regular ( $T_{3\frac{1}{2}}$ ).

**Proposition 5.** If  $X$  is a uniform space, then for all  $x \in X$  and symmetric entourages  $D$ :

$$x \in \frac{1}{2}D[y] \text{ and } y \in \frac{1}{2}D[z] \Rightarrow x \in D[z]$$

and

$$\frac{1}{2}D[x] \subseteq \overline{\frac{1}{2}D[x]} \subseteq D[x]$$

*Proof.* The first is by definition of  $\frac{1}{2}D$ .

If  $z \in \overline{\frac{1}{2}D[x]}$ , it follows that there is  $y \in \frac{1}{2}D[x] \cap \frac{1}{2}D[z]$  since  $\frac{1}{2}D[z]$  is an open neighborhood of  $z$ . Thus  $(x, z) \in D \Rightarrow z \in D[x] \Rightarrow \overline{\frac{1}{2}D[x]} \subseteq D[x]$ .  $\square$

**Definition 6.** For a uniform space  $X$ , Bell's proximity game proceeds as follows.

In round 0,  $\mathcal{D}$  chooses an entourage  $D_0$ , followed by  $\mathcal{P}$  choosing a point  $p_0 \in X$ .

In round  $n + 1$ ,  $\mathcal{D}$  chooses an entourage  $D_{n+1} \subseteq D_n$ , followed by  $\mathcal{P}$  choosing a point  $p_{n+1} \in 4D_n[p_n]$ .

Player  $\mathcal{D}$  wins if either  $\bigcap_{n < \omega} 4D_n[p_n] = \emptyset$  or  $\langle p_0, p_1, \dots \rangle$  converges.

**Definition 7.** For a uniform space  $X$ , the simplified proximal game  $Prox_{D,P}(X)$  can be defined as follows:

In round 0,  $\mathcal{D}$  chooses a symmetric entourage  $D_0$ , followed by  $\mathcal{P}$  choosing a point  $p_0 \in X$ .

In round  $n+1$ ,  $\mathcal{D}$  chooses a symmetric entourage  $D_{n+1}$ , followed by  $\mathcal{P}$  choosing a point  $p_{n+1} \in \left(\bigcap_{m \leq n} D_m\right)[p_n]$ .

Player  $\mathcal{D}$  wins if either  $\bigcap_{n < \omega} \left(\bigcap_{m \leq n} D_m\right)[p_n] = \emptyset$  or  $\langle p_0, p_1, \dots \rangle$  converges.

**Theorem 8.**  $\mathcal{D}$  has a winning perfect-information strategy in Bell's game if and only if  $\mathcal{D} \uparrow Prox_{D,P}(X)$ .

*Proof.* Let  $\sigma$  be a winning perfect information strategy for  $\mathcal{D}$  in Bell's game. We define a perfect information strategy  $\tau$  in the simplified game to yield symmetric entourages  $\tau(p \upharpoonright n) = \sigma(p \upharpoonright n) \cap (\sigma(p \upharpoonright n))^{-1}$  for all partial attacks  $p \upharpoonright n$ . Note that  $\tau(p \upharpoonright n) = \bigcap_{m \leq n} \tau(p \upharpoonright m)$ .

If  $p$  attacks  $\tau$  in the simplified game,  $p(n+1) \in \left(\bigcap_{m \leq n} \tau(p \upharpoonright m)\right)[p(n)] = \tau(p \upharpoonright n)[p(n)] \subseteq \sigma(p \upharpoonright n)[p(n)] \subseteq 4\sigma(p \upharpoonright n)[p(n)]$ , so  $p$  attacks  $\sigma$  in Bell's game. Thus either  $p$  converges, or

$$\emptyset = \bigcap_{n < \omega} 4\sigma(p \upharpoonright n)[p(n)] \supseteq \bigcap_{n < \omega} \tau(p \upharpoonright n)[p(n)] = \bigcap_{n < \omega} \left( \bigcap_{m \leq n} \tau(p \upharpoonright m) \right)[p(n)]$$

For the other direction, let  $\sigma$  be a winning perfect information strategy for  $\mathcal{D}$  in the simplified game such that  $\sigma(p \upharpoonright n) = \bigcap_{m \leq n} \sigma(p \upharpoonright m)$ . Define the perfect information strategy  $\tau$  in Bell's Game such that  $4\tau(p \upharpoonright n) \subseteq \sigma(p \upharpoonright n)$  and  $\tau(p \upharpoonright n) = \bigcap_{m \leq n} \tau(p \upharpoonright m)$  for all partial attacks  $p \upharpoonright n$ .

If  $p$  attacks  $\tau$  in Bell's game,  $p(n) \in 4\tau(p \upharpoonright n) \subseteq \sigma(p \upharpoonright n) = \bigcap_{m \leq n} \sigma(p \upharpoonright m)$ , so  $p$  attacks  $\sigma$  in the simplified game. Thus either  $p$  converges, or

$$\emptyset = \bigcap_{n < \omega} \left( \bigcap_{m \leq n} \sigma(p \upharpoonright m) \right)[p(n)] = \bigcap_{n < \omega} \sigma(p \upharpoonright n)[p(n)] \supseteq \bigcap_{n < \omega} 4\tau(p \upharpoonright n)[p(n)] \supseteq \bigcap_{n < \omega} \tau(p \upharpoonright n)[p(n)]$$

□

**Proposition 9.**  $\mathcal{P}$  has a winning perfect-information strategy in Bell's game if and only if  $\mathcal{P} \uparrow Prox_{D,P}(X)$ .

*Proof.* Similar to the previous. □

**Definition 10.** A uniform space is **proximal** if  $\mathcal{D} \uparrow \text{Prox}_{D,P}(X)$ .

**Definition 11.** For a space  $X$  and a point  $x \in X$ , the  **$W$ -convergence-game**  $\text{Con}_{O,P}(X, x)$  proceeds as follows.

In round 0,  $\mathcal{O}$  chooses a neighborhood  $U_n$  of  $x$ , followed by  $\mathcal{P}$  choosing a point  $p_n \in \bigcap_{m \leq n} U_m$ .

Player  $\mathcal{O}$  wins if  $\langle p_0, p_1, \dots \rangle$  converges.

**Definition 12.** A space is  **$W$**  if  $\mathcal{O} \uparrow \text{Con}_{O,P}(X, x)$  for all  $x \in X$ .

**Definition 13.** For each finite tuple  $(m_0, \dots, m_{n-1})$ , we define the  **$k$ -tactical fog-of-war**

$$T_k(\langle m_0, \dots, m_{n-1} \rangle) = \langle m_{n-k}, \dots, m_{n-1} \rangle$$

and the  **$k$ -Marköv fog-of-war**

$$M_k(\langle m_0, \dots, m_{n-1} \rangle) = \langle \langle m_{n-k}, \dots, m_{n-1} \rangle, n \rangle$$

So  $P \uparrow G$  if and only if there exists a winning strategy for  $P$  of the form  $\sigma \circ T_k$ , and  $P \uparrow_{k\text{-tact}} G$  if and only if there exists a winning strategy of the form  $\sigma \circ M_k$ .

**Theorem 14.** For all  $x \in X$ :

- $\mathcal{D} \uparrow \text{Prox}_{D,P}(X) \Rightarrow \mathcal{O} \uparrow \text{Con}_{O,P}(X, x)$
- $\mathcal{D} \uparrow_{2k\text{-tact}} \text{Prox}_{D,P}(X) \Rightarrow \mathcal{O} \uparrow_{k\text{-tact}} \text{Con}_{O,P}(X, x)$
- $\mathcal{D} \uparrow_{2k\text{-mark}} \text{Prox}_{D,P}(X) \Rightarrow \mathcal{O} \uparrow_{k\text{-mark}} \text{Con}_{O,P}(X, x)$

*Proof.* Let  $\sigma$  witness  $\mathcal{D} \uparrow_{2k\text{-tact}} \text{Prox}_{D,P}(X)$  (resp.  $\mathcal{D} \uparrow_{2k\text{-mark}} \text{Prox}_{D,P}(X)$ ,  $\mathcal{D} \uparrow \text{Prox}_{D,P}(X)$ ).

We define the  $k$ -tactical (resp.  $k$ -Marköv, perfect info) strategy  $\tau$  such that

$$\tau \circ L_k(p) = \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(|p| - 1) \rangle)[x] \cap \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(|p| - 1), x \rangle)[x]$$

where  $L_{2k}$  is the  $2k$ -tactical fog-of-war (resp.  $2k$ -Marköv fog-of-war, identity) and  $L_k$  is the  $k$ -tactical fog-of-war (resp.  $k$ -Marköv fog-of-war, identity).

Let  $p$  attack  $\tau$ . Consider the attack  $q$  against the winning strategy  $\sigma$  such that  $q(2n) = x$  and  $q(2n + 1) = p(n)$ , and let  $D_n = \sigma \circ L_{2k}(q)$  and  $E_n = \bigcap_{m \leq n} D_m$ .

Certainly,  $x \in E_{2n}[x] = E_{2n}[q(2n)]$  for any  $n < \omega$ . Note also for any  $n < \omega$  that

$$p(n) \in \bigcap_{m \leq n} \tau \circ L_k(p \upharpoonright n)$$

$$\begin{aligned}
&= \bigcap_{m \leq n} (\sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(m-1) \rangle)[x] \cap \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(m-1), x \rangle)[x]) \\
&= \bigcap_{m \leq n} (D_{2m}[x] \cap D_{2m+1}[x]) = \bigcap_{m \leq 2n+1} D_m[x] = E_{2n+1}[x]
\end{aligned}$$

so by the symmetry of  $E_{2n+1}$ ,  $x \in E_{2n+1}[p(n)] = E_{2n+1}[q(2n+1)]$ . Thus  $x \in \bigcap_{n < \omega} E_n[q(n)] \neq \emptyset$ , and since  $\sigma$  is a winning strategy, the attack  $q$  converges. Since  $q(2n) = x$ ,  $q$  must converge to  $x$ . Thus its subsequence  $p$  converges to  $x$ , and  $\tau$  is a winning strategy in  $Con_{O,P}(X, x)$ .  $\square$

**Corollary 15.** *For all  $x \in X$ :*

- $\mathcal{D} \uparrow_{k\text{-tact}} Prox_{D,P}(X) \Rightarrow \mathcal{O} \uparrow_{k\text{-tact}} Con_{O,P}(X, x)$
- $\mathcal{D} \uparrow_{k\text{-mark}} Prox_{D,P}(X) \Rightarrow \mathcal{O} \uparrow_{k\text{-mark}} Con_{O,P}(X, x)$

**Corollary 16.** *All proximal spaces are  $W$ -spaces.*

**Theorem 17.** *Let  $X \cup \{\infty\}$  be a uniformizable space such that  $X$  is discrete. Then*

- $\mathcal{O} \uparrow Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathcal{D} \uparrow Prox_{D,P}(X \cup \{\infty\})$
- $\mathcal{O} \uparrow_{k\text{-tact}} Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathcal{D} \uparrow_{k\text{-tact}} Prox_{D,P}(X \cup \{\infty\})$
- $\mathcal{O} \uparrow_{k\text{-mark}} Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathcal{D} \uparrow_{k\text{-mark}} Prox_{D,P}(X \cup \{\infty\})$

*Proof.* Note that the topology on  $X \cup \{\infty\}$  is induced by the uniformity with equivalence relation entourages  $D(U) = \Delta \cup U^2$  for each open neighborhood  $U$  of  $\infty$ .

Let  $\sigma$  witness  $\mathcal{D} \uparrow_{k\text{-tact}} Con_{O,P}(X \cap \{\infty\}, \infty)$  (resp.  $\mathcal{D} \uparrow_{k\text{-mark}} Con_{O,P}(X \cap \{\infty\}, \infty)$ ,  $\mathcal{D} \uparrow Con_{O,P}(X \cap \{\infty\}, \infty)$ ). We define the  $k$ -tactical (resp.  $k$ -Marköv, perfect info) strategy  $\tau$  such that

$$\tau \circ L(p) = D(\sigma \circ L(p))$$

where  $L$  is the  $k$ -tactical fog-of-war (resp.  $k$ -Marköv fog-of-war, identity).

Let  $p \in (X \cup \{\infty\})^\omega$  attack  $\tau$  such that  $\bigcap_{n < \omega} \tau(p \upharpoonright n)[p(n)] \neq \emptyset$ .

If  $\infty \in \bigcap_{n < \omega} \tau(p \upharpoonright n)[p(n)]$ , it follows that  $p$  is an attack on  $\sigma$ . Since  $\sigma$  is a winning strategy, it follows that  $q$  and its subsequence  $p$  must converge to  $\infty$ .

Otherwise,  $\infty \notin \tau(p \upharpoonright N)[p(N)]$  for some  $N < \omega$ , and then  $\tau(p \upharpoonright N)[p(N)] = \{p(N)\}$  implies  $p \rightarrow p(N)$ .

Thus  $\tau \circ L$  is a winning strategy.  $\square$

**Corollary 18.** *Let  $X \cup \{\infty\}$  be a uniformizable space such that  $X$  is discrete. Then*

- $\mathcal{O} \uparrow_{k\text{-tact}} \text{Con}_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathcal{D} \uparrow_{k\text{-tact}} \text{Prox}_{D,P}(X \cup \{\infty\})$
- $\mathcal{O} \uparrow_{k\text{-tact}} \text{Con}_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathcal{D} \uparrow_{k\text{-tact}} \text{Prox}_{D,P}(X \cup \{\infty\})$
- $\mathcal{O} \uparrow_{k\text{-mark}} \text{Con}_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathcal{D} \uparrow_{k\text{-mark}} \text{Prox}_{D,P}(X \cup \{\infty\})$

**Proposition 19.** *For any  $x \in X$  and  $k \geq 1$ ,*

- $\mathcal{O} \uparrow_{k\text{-tact}} \text{Con}_{O,P}(X, x) \Leftrightarrow \mathcal{O} \uparrow_{\text{tact}} \text{Con}_{O,P}(X, x)$
- $\mathcal{O} \uparrow_{k\text{-mark}} \text{Con}_{O,P}(X, x) \Leftrightarrow \mathcal{O} \uparrow_{\text{mark}} \text{Con}_{O,P}(X, x)$

*Proof.* If  $\sigma$  witnesses  $\mathcal{O} \uparrow_{k\text{-tact}} \text{Con}_{O,P}(X, x)$ , let  $\tau(\emptyset) = \sigma(\emptyset)$  and

$$\tau(\langle q \rangle) = \bigcap_{i < k} \sigma(\langle \underbrace{x, \dots, x}_{k-i-1}, \underbrace{x, \dots, x}_i \rangle)$$

This is easily verified to be a winning strategy. The proof for  $\mathcal{O} \uparrow_{k\text{-mark}} \text{Con}_{O,P}(X, x)$  is analogous.  $\square$

**Corollary 20.** *Let  $X \cup \{\infty\}$  be a uniformizable space such that  $X$  is discrete, and  $k \geq 1$ . Then*

- $\mathcal{D} \uparrow_{k\text{-tact}} \text{Prox}_{D,P}(X \cup \{\infty\}) \Leftrightarrow \mathcal{O} \uparrow_{\text{tact}} \text{Prox}_{D,P}(X \cup \{\infty\})$
- $\mathcal{D} \uparrow_{k\text{-mark}} \text{Prox}_{D,P}(X \cup \{\infty\}) \Leftrightarrow \mathcal{O} \uparrow_{\text{mark}} \text{Prox}_{D,P}(X \cup \{\infty\})$

**Proposition 21.** *For any uniform space  $X$ ,*

- $\mathcal{O} \uparrow_{k\text{-tact}} \text{Prox}_{D,P}(X) \Leftrightarrow \mathcal{O} \uparrow_{2\text{-tact}} \text{Prox}_{D,P}(X)$
- $\mathcal{O} \uparrow_{k\text{-mark}} \text{Prox}_{D,P}(X) \Leftrightarrow \mathcal{O} \uparrow_{2\text{-mark}} \text{Prox}_{D,P}(X)$

*Proof.* If  $\sigma$  witnesses  $\mathcal{O} \uparrow_{k\text{-tact}} \text{Con}_{O,P}(X, x)$ , let  $\tau(\emptyset) = \sigma(\emptyset)$  and

$$\tau(\langle q \rangle) = \bigcap_{i < k} \sigma(\langle \underbrace{q, \dots, q}_i \rangle)$$

$$\tau(\langle q, q' \rangle) = \bigcap_{i < k} \sigma(\underbrace{\langle q, \dots, q \rangle}_{k-i}, \underbrace{\langle q', \dots, q' \rangle}_i)$$

This is easily verified to be a winning strategy. The proof for  $\mathcal{O} \uparrow_{k\text{-mark}} \text{Con}_{O,P}(X, x)$  is analogous. □

**Definition 22.** The strong proximal game  $sProx_{D,P}(X)$  is analogous to  $Prox_{D,P}(X)$ , except  $\mathcal{D}$  may only win if  $p$  converges.

**Definition 23.** A **uniformly locally compact** space is a uniformizable space with a **uniformly compact entourage**  $M$  where  $\overline{M[x]}$  is compact for all  $x$ .

**Theorem 24.** For any uniformly locally compact space  $X$ ,  $\mathcal{D} \uparrow Prox_{D,P}(X) \Leftrightarrow \mathcal{D} \uparrow sProx_{D,P}(X)$

*Proof.* Let  $M$  be a uniformly locally compact entourage. Let  $\sigma$  witness  $\mathcal{D} \uparrow Prox_{D,P}(X)$  such that  $\sigma(a) \subseteq M$  always (so  $\overline{\sigma(a)[x]} \subseteq \overline{M[x]}$  is compact), and  $a \supseteq b$  implies  $\sigma(a) \subseteq \frac{1}{4}\sigma(b)$ .

Let  $\tau(p \upharpoonright n) = \frac{1}{2}\sigma(p \upharpoonright n)$ . If  $p$  attacks  $\tau$  in  $sProx_{D,P}(X)$ , then

$$p(n+1) \in \tau(p \upharpoonright n)[p(n)] = \frac{1}{2}\sigma(p \upharpoonright n)[p(n)]$$

and for

$$x \in \overline{\sigma(p \upharpoonright (n+1))[p(n+1)]} \subseteq \overline{\frac{1}{4}\sigma(p \upharpoonright n)[p(n+1)]} \subseteq \frac{1}{2}\sigma(p \upharpoonright n)[p(n+1)]$$

we can conclude  $x \in \sigma(p \upharpoonright n)[p(n)]$ . Thus

$$\sigma(p \upharpoonright (n+1))[p(n+1)] \subseteq \overline{\sigma(p \upharpoonright (n+1))[p(n+1)]} \subseteq \sigma(p \upharpoonright n)[p(n)]$$

Finally, note that  $p$  attacks the winning strategy  $\sigma$  in  $Prox_{D,P}(X)$ , but since the intersection of a chain of nonempty compact sets is nonempty:

$$\bigcap_{n < \omega} \sigma(p \upharpoonright n)[p(n)] = \bigcap_{n < \omega} \overline{\sigma(p \upharpoonright n)[p(n)]} \neq \emptyset$$

We conclude that  $p$  converges. □

**Corollary 25.** A uniformly locally compact space  $X$  is proximal if and only if  $\mathcal{D} \uparrow sProx_{D,P}(X)$ .

**Theorem 26.** For any uniformly locally compact proximal space  $X$ ,  $\mathcal{O} \uparrow Clus_{O,P}(X, H)$  for all compact  $H \subseteq X$ .

*Proof.* Let  $\sigma$  witness  $\mathcal{D} \uparrow sProx_{D,P}(X)$  such that  $p \supseteq q$  implies  $\sigma(p) \subseteq \frac{1}{4}\sigma(q)$ .

Let  $o(t)$  be the subsequence of  $t$  consisting of its odd-indexed terms.

We define  $T(\emptyset)$ , etc. as follows:

- Let  $\emptyset \in T(\emptyset)$ .
- Choose  $m_\emptyset < \omega$ ,  $h_{\emptyset,i} \in H$  for  $i < m_\emptyset$ , and  $h_{\emptyset,i,j} \in H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$  for  $i, j < m_\emptyset$  such that

$$\{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}] : i < m_\emptyset\}$$

is a cover for  $H$  and such that for each  $i < m_\emptyset$

$$\{\frac{1}{4}\sigma(\langle h_{\emptyset,i} \rangle)[h_{\emptyset,i,j}] : j < m_\emptyset\}$$

is a cover for  $H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$ .

- Let  $\langle i \rangle \in T(\emptyset)$ ,  $\langle i, h_{\emptyset,i} \rangle \in T(\emptyset)$ , and  $\langle i, h_{\emptyset,i}, j \rangle \in T(\emptyset)$  for  $i, j < m_\emptyset$ .

Suppose  $T(a)$ , etc. are defined. We then define  $T(a \smallfrown \langle x \rangle)$ , etc. for

$$x \in \bigcup_{s \smallfrown \langle i, h_{s,i}, j \rangle \in \max(T(a))} \frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]$$

as follows:

- Let  $T(a) \subseteq T(a \smallfrown \langle x \rangle)$ .
- Choose  $t = s \smallfrown \langle i, h_{s,i}, j, x \rangle$  such that  $s \smallfrown \langle i, h_{s,i}, j \rangle \in \max(T(a))$  and  $x \in \frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]$ .
- Note that, assuming  $o(s) \smallfrown \langle h_{s,i} \rangle$  is a legal partial attack against  $\sigma$ , then

$$x \in \frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}] \subseteq \frac{1}{4}\sigma(o(s))[h_{s,i,j}]$$

and

$$h_{s,i,j} \in \overline{\frac{1}{4}\sigma(o(s))[h_{s,i}]} \subseteq \frac{1}{2}\sigma(o(s))[h_{s,i}]$$

implies

$$x \in \sigma(o(s))[h_{s,i}]$$

and thus  $o(s) \smallfrown \langle h_{s,i}, x \rangle = o(t)$  is a legal partial attack against  $\sigma$ .

- Choose  $m_t < \omega$ ,  $h_{t,k} \in H \cap \overline{\frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]}$  for  $k < m_t$ , and  $h_{t,k,l} \in H \cap \overline{\frac{1}{4}\sigma(t)[h_{t,k}]}$  for  $k, l < m_t$  such that

$$\{\frac{1}{4}\sigma(o(t))[h_{t,k}] : k < m_t\}$$

is a cover for  $H \cap \overline{\frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]}$  and such that for each  $k < m_t$

$$\{\frac{1}{4}\sigma(o(t) \smallfrown \langle h_{t,k} \rangle)[h_{t,i,j}] : l < m_t\}$$

is a cover for  $H \cap \overline{\frac{1}{4}\sigma(o(t))[h_{t,k}]}$ .



- Note that, assuming  $o(t)$  is a legal partial attack against  $\sigma$ , then

$$h_{t,k} \in \overline{\frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]} \subseteq \frac{1}{2}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]$$

and

$$x \in \frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]$$

implies

$$h_{t,k} \in \sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[x]$$

and thus  $o(t) \smallfrown \langle h_{t,k} \rangle$  is a legal partial attack against  $\sigma$ .

- Let  $t \in T(a \smallfrown \langle x \rangle)$ ,  $t \smallfrown \langle k \rangle \in T(a \smallfrown \langle x \rangle)$ ,  $t \smallfrown \langle k, h_{t,k} \rangle \in T(a \smallfrown \langle x \rangle)$ , and  $t \smallfrown \langle k, h_{t,k}, l \rangle \in T(a \smallfrown \langle x \rangle)$  for  $k, l < m_t$ .
- Note that assuming

$$\{\frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}] : s \smallfrown \langle i, h_{s,i}, j \rangle \in \max(T(a))\}$$

covers  $H$ , then since

$$\{\frac{1}{4}\sigma(o(t) \smallfrown \langle h_{t,k} \rangle)[h_{t,k,l}] : s \smallfrown \langle i, h_{s,i}, j, x, k, h_{t,k}, l \rangle \in \max(T(a \smallfrown \langle x \rangle)) \setminus \max(T(a))\}$$

covers  $H \cap \frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]$ , we have that

$$\{\frac{1}{4}\sigma(o(t) \smallfrown \langle h_{t,k} \rangle)[h_{t,k,l}] : t \smallfrown \langle k, h_{t,k}, l \rangle \in \max(T(a \smallfrown \langle x \rangle))\}$$

covers  $H$ .

With this we may define the perfect information strategy  $\tau$  for  $\mathcal{O}$  in  $Con_{O,P}(X, H)$  such that:

$$\tau(p \upharpoonright n) = \bigcup_{s \smallfrown \langle i, h_{s,i}, j \rangle \in \max(T(p \upharpoonright n))} \frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]$$

If  $p$  attacks  $\tau$ , then it follows that  $T(p \upharpoonright n)$  is defined for all  $n < \omega$ , so let  $T(p) = \bigcup_{n < \omega} T(p \upharpoonright n)$ . We note  $T(p)$  is an infinite tree with finite levels:

- $\emptyset$  has exactly  $m_\emptyset$  successors  $\langle i \rangle$ .
- $s \smallfrown \langle i \rangle$  has exactly one successor  $t \smallfrown \langle i, h_{s,i} \rangle$
- $s \smallfrown \langle i, h_{s,i} \rangle$  has exactly  $m_s$  successors  $t \smallfrown \langle i, h_{s,i}, j \rangle$
- $s \smallfrown \langle i, h_{s,i}, j \rangle$  has either no successors or exactly one successor  $t \smallfrown \langle i, h_{s,i}, j, x \rangle$

- $t = s^\frown \langle i, h_{s,i}, j, x \rangle$  has exactly  $m_t$  successors  $t^\frown \langle k \rangle$

Let  $q' = \langle i_0, h_0, j_0, x_0, i_1, h_1, j_1, x_1, \dots \rangle$  correspond to this infinite branch in  $T(p)$ , and let  $q = o(q') = \langle h_0, x_0, h_1, x_1, \dots \rangle$ . Note that by the construction of  $T(p)$ ,  $q$  is an attack on the winning strategy  $\sigma$  in  $sProx_{D,P}(X)$ , so it must converge. Since every other term of  $q$  is in  $H$ , it must converge to  $H$ . Then since  $q$  is a subsequence of  $p$ ,  $p$  must cluster at  $H$ .  $\square$

**Corollary 27.** *For any uniformly locally compact proximal space,  $\mathcal{O} \uparrow Con_{O,P}(X, H)$  for all compact  $H \subseteq X$ .*

*Proof.*  $\mathcal{O} \uparrow Con_{O,P}(X, H)$  if and only if  $\mathcal{O} \uparrow Clus_{O,P}(X, H)$ .  $\square$

**Corollary 28.** *A compact uniform space  $X$  is Corson compact if and only if it is proximal.*

*Proof.* A characterization of Corson compact is having a  $W$ -set diagonal. If  $X$  is proximal compact, then  $X^2$  is proximal compact, and its compact diagonal is a  $W$ -set.  $\square$

**Theorem 29.**  $\mathcal{O} \uparrow_{pre} Con_{O,P}(X, H)$  if and only if there exists a countable base around  $H$ .

*Proof.* Let  $\{U_n : n < \omega\}$  be a countable base around  $H$ . We define the predetermined strategy  $\sigma(n) = \bigcap_{m \leq n} U_m$ . Let  $p$  attack  $\sigma(n)$  - then if  $U$  is any neighborhood of  $H$ , we may choose  $H \subseteq U_m \subseteq U$ , and note that  $\sigma(n) \subseteq U_m$  for  $n \geq m$ , and thus  $p(n) \in U_m \subseteq U$  for all  $n \geq m$ . Thus  $\sigma$  is a winning strategy.

For the other direction, suppose there does not exist a countable base around  $H$ , and let  $\sigma(n)$  be an arbitrary predetermined strategy. Since  $\{\bigcap_{m \leq n} \sigma(m) : n < \omega\}$  is not a countable base around  $H$ , we may choose an open set  $U$  around  $H$  such that  $\bigcap_{m \leq n} \sigma(m) \not\subseteq U$  for all  $n < \omega$ . We may easily verify that if  $p(n) \in \bigcap_{m \leq n} \sigma(m) \setminus U$  for all  $n < \omega$ , then  $p$  is a successful counterattack to  $\sigma$ .  $\square$

**Corollary 30.**  $X$  is first countable if and only if  $\mathcal{O} \uparrow_{pre} Con_{O,P}(X, x)$  for all  $x \in X$

**Corollary 31.**  $\mathcal{D} \uparrow_{pre} Prox_{D,P}(X)$  implies  $X$  is first countable.

**Definition 32.** Scattered Eberlein compact spaces are known as **strong Eberlein compact** spaces.

**Theorem 33** (folklore). *Scattered compact first-countable spaces are metrizable.*

**Corollary 34.** If  $X$  is scattered compact and  $\mathcal{O} \uparrow_{pre} Con_{O,P}(X, x)$  for all  $x \in X$  (or  $\mathcal{D} \uparrow_{pre} Prox_{D,P}(X)$ ), then  $X$  is metrizable.

**Example 35.**  $\mathcal{D} \nuparrow_{pre} Prox_{D,P}(\omega_1^*)$

*Proof.* There does not exist a countable base around  $\infty$ , so  $\mathcal{O} \nuparrow_{pre} Con_{O,P}(X, \omega_1)$ .  $\square$

**Example 36.**  $\mathcal{O} \uparrow_{tact} Con_{O,P}(\kappa^*, \infty)$  and  $\mathcal{D} \uparrow_{tact} Prox_{D,P}(\kappa^*)$  for all cardinals  $\kappa$

*Proof.* For  $Con_{O,P}(\kappa^*, \infty)$ , let  $\sigma() = \sigma(\infty) = \kappa^*$  and  $\sigma(x) = \kappa^* \setminus \{x\}$  otherwise.  $\square$

**Theorem 37.** If  $H$  is a closed subset of  $X$ , then  $\mathcal{D} \uparrow_{\text{limit}} \text{Prox}_{D,P}(X) \Rightarrow \mathcal{D} \uparrow_{\text{limit}} \text{Prox}_{D,P}(H)$  where  $\uparrow_{\text{limit}}$  is any of  $\uparrow$ ,  $\uparrow_{k\text{-tact}}$ , or  $\uparrow_{k\text{-mark}}$ .

*Proof.* Let  $\sigma \circ L$  witness  $\mathcal{D} \uparrow_{\text{limit}} \text{Prox}_{D,P}(X)$ . We define  $\tau \circ L$  for  $\mathcal{D}$  in  $\text{Prox}_{D,P}(H)$  as follows:

$$\tau \circ L(p \upharpoonright n) = \sigma \circ L(p \upharpoonright n) \cap H^2$$

Let  $p$  attack  $\tau \circ L$ .  $p$  also attacks the winning strategy  $\sigma \circ L$ , so either

$$\bigcap_{n < \omega} \left( \bigcap_{m \leq n} \tau \circ L(p \upharpoonright m) \right) [p(n)] \subseteq \bigcap_{n < \omega} \left( \bigcap_{m \leq n} \sigma \circ L(p \upharpoonright m) \right) [p(n)] = \emptyset$$

or  $p$  converges in  $X$ , and thus converges in  $H$ .  $\square$

**Theorem 38.** If  $\mathcal{D} \uparrow_{\text{limit}} \text{Prox}_{D,P}(X_i)$  for  $i < \omega$ , then  $\mathcal{D} \uparrow_{\text{limit}} \text{Prox}_{D,P}(\prod_{i < \omega} X_i)$ , where  $\uparrow_{\text{limit}}$  is either  $\uparrow$  or  $\uparrow_{k\text{-mark}}$ .

*Proof.* A subbase for  $\prod_{i < \omega} X_i$  is

$$\{\pi_i^{-1}(D) : i < \omega, D \in \mathcal{D}_i\}$$

where  $\pi_i$  is the natural projection from  $(\prod_{i < \omega} X_i)^2$  onto  $X_i^2$ . (See Bell.)

For  $p \in (\prod_{i < \omega} X_i)^\omega$ , let  $p_i \in X_i^\omega$  such that  $p_i(n) = p(n)(i)$ .

Let  $\sigma_i \circ L$  witness  $\mathcal{D} \uparrow_{\text{limit}} \text{Prox}_{D,P}(X_i)$  for  $i < \omega$ , and assume without loss of generality that  $\sigma_i \circ L$  always yields  $X_i^2$  before round  $i$ .

Then we define the strategy  $\tau \circ L$  for  $\mathcal{D}$  in  $\text{Prox}_{D,P}(\prod_{i < \omega} X_i)$  as follows:

$$\tau \circ L(p \upharpoonright n) = \bigcap_{i \leq n} \pi_i^{-1}(\sigma_i \circ L(p_i \upharpoonright n))$$

Let  $p$  attack  $\tau \circ L$ . If  $\bigcap_{n < \omega} \left( \bigcap_{m \leq n} \sigma_i(p_i \upharpoonright m) \right) [p_i(n)] = \emptyset$  for any  $i < \omega$ , it easily follows that  $\bigcap_{n < \omega} \left( \bigcap_{m \leq n} \tau(p \upharpoonright m) \right) [p(n)] = \emptyset$ .

Otherwise, we assume that for each  $i < \omega$ ,  $p_i$  converges to some  $x_i \in X_i$ . Thus  $p$  converges to  $x = \langle x_0, x_1, \dots \rangle$ .  $\square$

Note: I expect I should be able to do some clever things assuming  $S(\kappa, \omega, \omega)$  to get a similar result for sigma products of dimension  $\kappa$ .

**Example 39.**  $\mathcal{D} \uparrow_{\text{mark}} \text{Prox}_{D,P}((\kappa^*)^\omega)$

*Proof.*  $\mathcal{D} \uparrow_{\text{tact}} \text{Prox}_{D,P}(\kappa^*) + \text{previous result}$

□