RELATING GAMES OF MENGER, COUNTABLE FAN TIGHTNESS, AND SELECTIVE SEPARABILITY

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ABSTRACT. By adapting techniques of Arhangel'skii, Barman, and Dow, we may equate the existence of perfect-information, Markov, and tactical strategies between two interesting selection games. These results shed some light on Gruenhage's question asking whether all strategically selectively separable spaces are Markov selectively separable.

1. INTRODUCTION

Definition 1. The *selection principle* $S_{fin}(\mathcal{A}, \mathcal{B})$ states that given $A_n \in \mathcal{A}$ for $n < \omega$, there exist $B_n \in [A_n]^{<\omega}$ such that $\bigcup_{n<\omega} B_n \in \mathcal{B}$.

Definition 2. The *selection game* $G_{fin}(\mathcal{A}, \mathcal{B})$ is the analogous game to $S_{fin}(\mathcal{A}, \mathcal{B})$, where during each round $n < \omega$, Player I first chooses $A_n \in \mathcal{A}$, and then Player II chooses $B_n \in [A_n]^{<\omega}$. Player II wins in the case that $\bigcup_{n<\omega} B_n \in \mathcal{B}$, and Player I wins otherwise.

This game and property were first formally investigated by Scheepers in "Combinatorics of open covers" [6], which inspired a series of ten sequels with several co-authors. We may quickly observe that if II has a winning strategy for the game $G_{fin}(\mathcal{A}, \mathcal{B})$, then $S_{fin}(\mathcal{A}, \mathcal{B})$ will hold, but the converse need not follow.

The power of this selection principle and game comes from their ability to characterize several properties and games from the literature. Of interest to us are the following.

Definition 3. Let $\mathcal{O}_X$ be the collection of open covers for a topological space X. Then $S_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger property* for X (M for short), and $G_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger game*.

Definition 4. An *ω-cover* $\mathcal{U}$ for a topological space X is an open cover such that for every $F \in [X]^{<\omega}$, there exists some $U \in \mathcal{U}$ such that $F \subseteq U$.

Definition 5. Let Ω_X be the collection of ω-covers for a topological space X. Then $S_{fin}(\Omega_X, \Omega_X)$ is the *Ω-Menger property* for X (ΩM for short), and $G_{fin}(\Omega_X, \Omega_X)$ is the *Ω-Menger game*.

In [4, Theorem 3.9] it was shown that X is Ω-Menger if and only if X^n is Menger for all $n < \omega$.

Definition 6. Let $\mathcal{B}_{X,x}$ be the collection of subsets $A \subset X$ where $x \in \operatorname{cl} A$. (Call A a *blade* of x.) Then $S_{fin}(\mathcal{B}_{X,x}, \mathcal{B}_{X,x})$ is the *countable fan tightness property* for X at x (CFT_x for short), and $G_{fin}(\mathcal{B}_{X,x}, \mathcal{B}_{X,x})$ is the *countable fan tightness game* for X at x.

Definition 7. A space X has *countable fan tightness* (CFT for short) if it has countable fan tightness at each point $x \in X$.

Definition 8. Let $\mathcal{D}_X$ be the collection of dense subsets of a topological space X. (So, $\mathcal{D}_X \subseteq \mathcal{B}_{X,x}$ for all $x \in X$.) Then $S_{fin}(\mathcal{D}_X, \mathcal{B}_{X,x})$ is the *countable dense fan tightness property* for X at x ($CDFT_x$ for short), and $G_{fin}(\mathcal{D}_X, \mathcal{B}_{X,x})$ is the *countable dense fan tightness game* for X at x.

Definition 9. A space X has *countable dense fan tightness* ($CDFT$ for short) if it has countable dense fan tightness at each point $x \in X$.

Note that $CFT \Rightarrow CDFT$ for any space X as $\mathcal{D}_X \subseteq \mathcal{B}_{X,x}$.

The notion of countable fan tightness was first studied by by Arhangel'skii in [1]. A result of that paper showed that for $T_{3\frac{1}{2}}$ spaces X, the countable fan tightness of the space of real-vaued continuous functions with pointwise convergence $C_p(X)$ is characterized by the Ω-Menger property of X.

Definition 10. $S_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability property* for X (SS for short), and $G_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability game* for X.

Of course, one may easily observe that a selective separable space is separable. In [2] Barman and Dow demonstrated that all separable Frechet spaces are selectively separable. They were also able to produce a space which is selectively separable, but does not allow II a winning strategy in the selective separability game.

The object of this paper is to investigate the game-theoretic properties characterized by the presence of winning *limited information* strategies in these selection games.

Definition 11. A *strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function σ satisfying $\sigma(\langle A_0, \dots, A_n \rangle) \in [A_n]^{<\omega}$ for $\langle A_0 \dots, A_n \rangle \in \mathcal{A}^{n+1}$. We say this strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(\langle A_0, \dots, A_n \rangle)$ during each round $n < \omega$. If a winning strategy exists, then we write II $\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$.

Definition 12. A *Markov strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function σ satisfying $\sigma(A, n) \in [A_n]^{<\omega}$ for $A \in \mathcal{A}$ and $n < \omega$. We say this Markov strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(A_n, n)$ during each round $n < \omega$. If a winning Markov strategy exists, then we write II $\underset{\text{mark}}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$.

Notation 13. If $S_{fin}(\mathcal{A}, \mathcal{B})$ characterizes the property P, then we say II $\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes P^+ ("strategically P"), and II $\underset{\text{mark}}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes P^{+mark} ("Markov P"). Of course, $P^{+mark} \Rightarrow P^+ \Rightarrow P$.

In this notation, Barman and Dow showed that SS does not imply SS^+. We aim to make progress on the following question attributed to Gary Gruenhage:

Question 14. *Does SS^+ imply SS^{+mark}?*

The solution is known to be "yes" in the context of countable spaces [2]. However in general, winning strategies in selection games cannot be improved to be winning Markov strategies. In [3] the author showed that while M^+ implies M^{+mark} for second-countable spaces, there exists a simple example of a regular non-second-countable space which is M^+ but not M^{+mark}.

2. CFT, $CDFT$ AND SS

We begin by generalizing the following result:

Theorem 15 (Lem 2.7 of [2])**.** *The following are equivalent for any topological space X.*

- *X is SS.*
- *X is separable and $CDFT$.*
- *X has a countable dense subset D where $CDFT_x$ holds for all $x \in D$.*

Theorem 16. *The following are equivalent for any topological space X.*

- *X is SS (resp. SS^+, SS^{+mark}).*
- *X is separable and $CDFT$ (resp. $CDFT^+$, $CDFT^{+mark}$).*
- *X has a countable dense subset D where $CDFT_x$ (resp. $CDFT_x^+$, $CDFT_x^{+mark}$) holds for all $x \in D$.*

Proof. We need only show that the final condition implies the first. Let $D = \{d_i : i < \omega\}$.

Let σ_i be a winning strategy witnessing $CDFT^+_{d_i}$ for each $i < \omega$. We define the strategy τ for the SS game by

$$\tau(\langle D_0, \dots, D_n\rangle) = \bigcup_{i \le n} \sigma_i(\langle D_i, \dots, D_n\rangle).$$

By $CDFT^+_{d_i}$, we have

$$d_i \in \overline{\bigcup_{i \le n < \omega} \sigma_i(\langle D_i, \dots, D_n\rangle)} \subseteq \overline{\bigcup_{i \le n < \omega} \tau(\langle D_0, \dots, D_n\rangle)} \subseteq \overline{\bigcup_{n < \omega} \tau(\langle D_0, \dots, D_n\rangle)}$$

and as $D \subseteq \overline{\bigcup_{n<\omega} \tau(\langle D_0, \dots, D_n\rangle)}$ it follows that

$$X \subseteq \overline{D} \subseteq \overline{\overline{\bigcup_{n<\omega} \tau(\langle D_0, \dots, D_n\rangle)}} = \overline{\bigcup_{n<\omega} \tau(\langle D_0, \dots, D_n\rangle)}.$$

Therefore τ witnesses SS^+.

The above proof may be easily modified for the Markov case by replacing $\sigma_i(\langle D_i, \dots, D_n\rangle)$ with $\sigma_i(D_n, n)$ and $\tau(\langle D_0, \dots, D_n\rangle)$ with $\tau(D_n, n)$. □

So amongst separable spaces, we see that SS (resp. SS^+, SS^{+mark}) and $CDFT$ (resp. $CDFT^+$, $CDFT^{+mark}$) are equivalent. We now further bridge the gap between $CDFT$ and CFT in the context of function spaces. Consider the following result of Arhangel'skii.

Theorem 17 ([1])**.** *The following are equivalent for any $T_{3\frac{1}{2}}$ topological space X.*

- *X is ΩM.*
- *$C_p(X)$ is CFT.*

This result may similarly be generalized in a game theoretic sense. In addition, this proof will demonstrate the equivalence of CFT and $CDFT$ in $C_p(X)$. It is unknown to the author whether Arhangel'skii used a strategy similar to the following proof in [1], but Sakai employed a similar technique in [5] to relate the Ω-Rothberger and countable strong fan tightness properties (and essentially, the countable strong dense fan tightness property). Due to the difficulty in obtaining an English translation of [1], we reprove Arhangel'skii's theorem above in our more general context below.

Definition 18. Let X be a $T_{3\frac{1}{2}}$ topological space. For $\vec{x} \in C_p(X)$, $F \in [X]^{<\omega}$, and $\epsilon > 0$, let

$$[\vec{x}, F, \epsilon] = \{\vec{y} \in C_p(x) : |\vec{y}(t) - \vec{x}(t)| < \epsilon \text{ for all } t \in F\}$$

give a basic open neighborhood of $\vec{x}$.

Lemma 19. *Let X be a $T_{3\frac{1}{2}}$ topological space. If X is ΩM (resp. ΩM^+, ΩM^{+mark}). then $C_p(X)$ is $CFT_{\vec{0}}$ (resp. $CFT_{\vec{0}}^+$, $CFT_{\vec{0}}^{+mark}$).*

Proof. □

Lemma 20. *Let X be a $T_{3\frac{1}{2}}$ topological space. If $C_p(X)$ is $CDFT_{\vec{0}}$ (resp. $CDFT_{\vec{0}}^+$, $CDFT_{\vec{0}}^{+mark}$), then X is ΩM (resp. ΩM^+, ΩM^{+mark}).*

Proof. For each $\mathcal{U} \in \Omega_X$ define

$$D(\mathcal{U}) = \{\vec{y} \in C_p(X) : \vec{y}[X \setminus U_{\vec{y},\mathcal{U}}] = \{1\} \text{ for some } U_{\vec{y},\mathcal{U}} \in \mathcal{U}\}.$$

Consider the point $\vec{x} \in C_p(X)$ and its basic open neighborhood $[\vec{x}, G, \epsilon]$. If $\mathcal{U}$ is an ω-cover of X, $G \subseteq U$ for some $U_{\vec{y},\mathcal{U}} \in \mathcal{U}$. Since X is $T_{3\frac{1}{2}}$, $X \setminus U_{\vec{y},\mathcal{U}}$ is closed, and G is finite and disjoint from $X \setminus U_{\vec{y},\mathcal{U}}$, we may choose some function $\vec{y} \in C_p(X)$ where $\vec{y}[X \setminus U_{\vec{y},\mathcal{U}}] = \{1\}$ and $\vec{x}(t) = \vec{y}(t)$ for each $t \in G$. It follows $\vec{y} \in [\vec{x}, G, \epsilon] \cap D$, so $D(\mathcal{U})$ is dense in $C_p(X)$.

Consider the sequence of ω-covers $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle \in \Omega_X^\omega$, and the corresponding sequence of dense subsets $\langle D(\mathcal{U}_0), D(\mathcal{U}_1), \dots \rangle \in \mathcal{D}_{C_p(X)}^\omega$.

Assuming $C_p(X)$ is $CDFT_{\vec{0}}$, choose a witness $\langle F_0, F_1, \dots \rangle$ such that

$$\vec{0} \in \overline{\bigcup_{n<\omega} F_n}.$$

Now let

$$\mathcal{F}_n = \{U_{\vec{y},\mathcal{U}_n} : \vec{y} \in F_n\} \in [\mathcal{U}_n]^{<\omega}.$$

We claim that $\bigcup_{n<\omega} \mathcal{F}_n$ is an ω-cover. Let $G \in [X]^{<\omega}$. The neighborhood $[\vec{0}, G, \frac{1}{2}]$ contains some point $\vec{y} \in F_n$ for some $n < \omega$. It follows that $U_{\vec{y},\mathcal{U}_n} \in \mathcal{U}_n$ and $\vec{y}[X \setminus U_{\vec{y},\mathcal{U}_n}] = \{1\}$. It follows that $G \cap (X \setminus U_{\vec{y},\mathcal{U}_n}) = \emptyset$, and therefore $G \subseteq U_{\vec{y},\mathcal{U}_n} \in \mathcal{F}_n$.

Assuming $C_p(X)$ is $CDFT_{\vec{0}}^+$, choose a witness σ such that

$$\vec{0} \in \overline{\bigcup_{n<\omega} \sigma(\langle D(\mathcal{U}_0), \dots, D(\mathcal{U}_n)\rangle)}.$$

Now let

$$\tau(\langle \mathcal{U}_0, \dots, \mathcal{U}_n\rangle) = \{U_{\vec{y},\mathcal{U}_n} : \vec{y} \in \sigma(\langle D(\mathcal{U}_0), \dots, D(\mathcal{U}_n)\rangle)\} \in [\mathcal{U}_n]^{<\omega}.$$

We claim that $\bigcup_{n<\omega} \tau(\langle \mathcal{U}_0, \dots, \mathcal{U}_n\rangle)$ is an ω-cover. Let $G \in [X]^{<\omega}$. The neighborhood $[\vec{0}, G, \frac{1}{2}]$ contains some point $\vec{y} \in \sigma(\langle D(\mathcal{U}_0), \dots, D(\mathcal{U}_n)\rangle)$ for some $n < \omega$. It follows that $U_{\vec{y},\mathcal{U}_n} \in \mathcal{U}_n$ and $\vec{y}[X \setminus U_{\vec{y},\mathcal{U}_n}] = \{1\}$. As a result $G \cap (X \setminus U_{\vec{y},\mathcal{U}_n}) = \emptyset$, and therefore $G \subseteq U_{\vec{y},\mathcal{U}_n} \in \tau(\langle \mathcal{U}_0, \dots, \mathcal{U}_n\rangle)$.

Assuming $C_p(X)$ is $CDFT_{\vec{0}}^{+mark}$, choose a witness σ such that

$$\vec{0} \in \overline{\bigcup_{n<\omega} \sigma(D(\mathcal{U}_n), n)}.$$

Now let

$$\tau(\mathcal{U}_n, n) = \{U_{\vec{y},\mathcal{U}_n} : \vec{y} \in \sigma(D(\mathcal{U}_n), n)\} \in [\mathcal{U}_n]^{<\omega}.$$

We claim that $\bigcup_{n<\omega} \tau(\mathcal{U}_n, n)$ is an ω-cover. Let $G \in [X]^{<\omega}$. The neighborhood $[\vec{0}, G, \frac{1}{2}]$ contains some point $\vec{y} \in \sigma(D(\mathcal{U}_n), n)$ for some $n < \omega$. It follows that $U_{\vec{y},\mathcal{U}_n} \in \mathcal{U}_n$ and $\vec{y}[X \setminus U_{\vec{y},\mathcal{U}_n}] = \{1\}$. As a result $G \cap (X \setminus U_{\vec{y},\mathcal{U}_n}) = \emptyset$, and therefore $G \subseteq U_{\vec{y},\mathcal{U}_n} \in \tau(\mathcal{U}_n, n)$. □

Theorem 21. *The following are equivalent for any $T_{3\frac{1}{2}}$ topological space X.*

- *X is ΩM (resp. ΩM^+, ΩM^{+mark}).*
- *$C_p(X)$ is CFT (resp. CFT^+, CFT^{+mark}).*
- *$C_p(X)$ is $CDFT$ (resp. $CDFT^+$, $CDFT^{+mark}$).*

Proof. Since $\mathcal{D}_X \subseteq \mathcal{B}_{X,x}$, the second condition trivially implies the first. As $C_p(X)$ is homogeneous, the $C(D)FT$ properties are characterized by $C(D)FT_{\vec{0}}$. So the result follows from the previous lemmas. □

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