Definition 1. A **uniform space** $\langle X, \mathcal{D}\rangle$ is a set X paired with a filter $\mathcal{D}$ (called its **uniformity**) of relations (called **entourages**) on X such that for each entourage $D \in \mathcal{D}$:

- D is reflexive, i.e., the diagonal $\Delta \subseteq D$.
- Its inverse $D^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in D\} \in \mathcal{D}$.
- There exists $\frac{1}{2}D \in \mathcal{D}$ such that

$$2(\frac{1}{2}D) = \frac{1}{2}D \circ \frac{1}{2}D = \{\langle x, z\rangle : \exists y(\langle x, y\rangle, \langle y, z\rangle \in \frac{1}{2}D)\} \subseteq D$$

Note that since $\mathcal{D}$ is a filter, for each $D \in \mathcal{D}$, the symmetric relation $D \cap D^{-1} \in \mathcal{D}$.

Proposition 2. *For each $D \in \mathcal{D}$ and $n < \omega$ there exists $\frac{1}{2^{n+1}}D \in \mathcal{D}$ such that*

$$2(\frac{1}{2^{n+1}}D) = \frac{1}{2^{n+1}}D \circ \frac{1}{2^{n+1}}D \subseteq \frac{1}{2^n}D$$

and if $2E \subseteq \frac{1}{2^n}D$, then $E \subseteq \frac{1}{2^{n+1}}D$.

Definition 3. For an entourage $D \in \mathcal{D}$, let $D[x] = \{y : (x, y) \in D\}$ be the **D-neighborhood** of x. The uniform topology for a uniform space $\langle X, \mathcal{D}\rangle$ is generated by the base $\{D[x] : x \in X, D \in \mathcal{D}\}$.

Theorem 4. *A space X is uniformizable (its topology is the uniform topology for some uniformity) if and only if X is completely regular ($T_{3\frac{1}{2}}$).*

Proposition 5. *If X is a uniform space, then for all $x \in X$ and symmetric entourages D:*

$$x \in \frac{1}{2}D[y] \text{ and } y \in \frac{1}{2}D[z] \Rightarrow x \in D[z]$$

and

$$\frac{1}{2}D[x] \subseteq \overline{\frac{1}{2}D[x]} \subseteq D[x]$$

Proof. The first is by definition of $\frac{1}{2}D$.

If $z \in \overline{\frac{1}{2}D[x]}$, it follows that there is $y \in \frac{1}{2}D[x] \cap \frac{1}{2}D[z]$ since $\frac{1}{2}D[z]$ is an open neighborhood of z. Thus $(x, z) \in D \Rightarrow z \in D[x] \Rightarrow \overline{\frac{1}{2}D[x]} \subseteq D[x]$. □

Definition 6. For a uniform space X, Bell's proximity game proceeds as follows.

In round 0, $\mathscr{D}$ chooses an entourage D_0, followed by $\mathscr{P}$ choosing a point $p_0 \in X$.

In round $n + 1$, $\mathscr{D}$ chooses an entourage $D_{n+1} \subseteq D_n$, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in 4D_n[p_n]$.

Player $\mathscr{D}$ wins if either $\bigcap_{n<\omega} 4D_n[p_n] = \emptyset$ or $\langle p_0, p_1, \ldots\rangle$ converges.

Definition 7. For a uniform space X, the simplified proximal game $Prox_{D,P}(X)$ can be defined as follows:

In round 0, $\mathscr{D}$ chooses a symmetric entourage D_0, followed by $\mathscr{P}$ choosing a point $p_0 \in X$.

In round $n+1$, $\mathscr{D}$ chooses a symmetric entourage D_{n+1}, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in \left(\bigcap_{m \leq n} D_m\right)[p_n]$.

Player $\mathscr{D}$ wins if either $\bigcap_{n<\omega}\left(\bigcap_{m \leq n} D_m\right)[p_n] = \emptyset$ or $\langle p_0, p_1, \ldots \rangle$ converges.

Theorem 8. *$\mathscr{D}$ has a winning perfect-information strategy in Bell's game if and only if $\mathscr{D} \uparrow Prox_{D,P}(X)$.*

Proof. Let σ be a winning perfect information strategy for $\mathscr{D}$ in Bell's game. We define a perfect information strategy τ in the simplified game to yield symmetric entourages $\tau(p \upharpoonright n) = \sigma(p \upharpoonright n) \cap (\sigma(p \upharpoonright n))^{-1}$ for all partial attacks $p \upharpoonright n$. Note that $\tau(p \upharpoonright n) = \bigcap_{m \leq n} \tau(p \upharpoonright m)$.

If p attacks τ in the simplified game, $p(n+1) \in \left(\bigcap_{m \leq n} \tau(p \upharpoonright m)\right)[p(n)] = \tau(p \upharpoonright n)[p(n)] \subseteq \sigma(p \upharpoonright n)[p(n)] \subseteq 4\sigma(p \upharpoonright n)[p(n)]$, so p attacks σ in Bell's game. Thus either p converges, or

$$\emptyset = \bigcap_{n<\omega} 4\sigma(p \upharpoonright n)[p(n)] \supseteq \bigcap_{n<\omega} \tau(p \upharpoonright n)[p(n)] = \bigcap_{n<\omega}\left(\bigcap_{m \leq n} \tau(p \upharpoonright n)\right)[p(n)]$$

For the other direction, let σ be a winning perfect information strategy for $\mathscr{D}$ in the simplified game such that $\sigma(p \upharpoonright n) = \bigcap_{m \leq n} \sigma(p \upharpoonright m)$. Define the perfect information strategy τ in Bell's Game such that $4\tau(p \upharpoonright n) \subseteq \sigma(p \upharpoonright n)$ and $\tau(p \upharpoonright n) = \bigcap_{m \leq n} \tau(p \upharpoonright m)$ for all partial attacks $p \upharpoonright n$.

If p attacks τ in Bell's game, $p(n) \in 4\tau(p \upharpoonright n) \subseteq \sigma(p \upharpoonright n) = \bigcap_{m \leq n} \sigma(p \upharpoonright m)$, so p attacks σ in the simplified game. Thus either p converges, or

$$\emptyset = \bigcap_{n<\omega}\left(\bigcap_{m \leq n} \sigma(p \upharpoonright n)\right)[p(n)] = \bigcap_{n<\omega} \sigma(p \upharpoonright n)[p(n)] \supseteq \bigcap_{n<\omega} 4\tau(p \upharpoonright n)[p(n)] \supseteq \bigcap_{n<\omega} \tau(p \upharpoonright n)[p(n)]$$

□

Proposition 9. *$\mathscr{P}$ has a winning perfect-information strategy in Bell's game if and only if $\mathscr{P} \uparrow Prox_{D,P}(X)$.*

Proof. Similar to the previous. □

Definition 10. A uniform space is **proximal** if $\mathscr{D} \uparrow Prox_{D,P}(X)$.

Definition 11. For a space X and a point $x \in X$, the W**-convergence-game** $Con_{O,P}(X,x)$ proceeds as follows.

In round 0, $\mathscr{O}$ chooses a neighborhood U_n of x, followed by $\mathscr{P}$ choosing a point $p_n \in \bigcap_{m \leq n} U_m$.

Player $\mathscr{O}$ wins if $\langle p_0, p_1, \dots \rangle$ converges.

Definition 12. A space is W if $\mathscr{O} \uparrow Con_{O,P}(X,x)$ for all $x \in X$.

Definition 13. For each finite tuple $(m_0, \dots, m_{n-1})$, we define the k**-tactical fog-of-war**

$$T_k(\langle m_0, \dots, m_{n-1} \rangle) = \langle m_{n-k}, \dots, m_{n-1} \rangle$$

and the k**-Marköv fog-of-war**

$$M_k(\langle m_0, \dots, m_{n-1} \rangle) = \langle \langle m_{n-k}, \dots, m_{n-1} \rangle, n \rangle$$

So $P \uparrow_{k\text{-tact}} G$ if and only if there exists a winning strategy for P of the form $\sigma \circ T_k$, and $P \uparrow_{k\text{-mark}} G$ if and only if there exists a winning strategy of the form $\sigma \circ M_k$.

Theorem 14. *For all $x \in X$:*

- $\mathscr{D} \uparrow Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow Con_{O,P}(X,x)$
- $\mathscr{D} \uparrow_{2k\text{-}tact} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(X,x)$
- $\mathscr{D} \uparrow_{2k\text{-}mark} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(X,x)$

Proof. Let σ witness $\mathscr{D} \uparrow_{2k\text{-tact}} Prox_{D,P}(X)$ (resp. $\mathscr{D} \uparrow_{2k\text{-mark}} Prox_{D,P}(X)$, $\mathscr{D} \uparrow Prox_{D,P}(X)$). We define the k-tactical (resp. k-Marköv, perfect info) strategy τ such that

$$\tau \circ L_k(p) = \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(|p|-1) \rangle)[x] \cap \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(|p|-1), x \rangle)[x]$$

where L_{2k} is the $2k$-tactical fog-of-war (resp. $2k$-Marköv fog-of-war, identity) and L_k is the k-tactical fog-of-war (resp. k-Marköv fog-of-war, identity).

Let p attack τ. Consider the attack q against the winning strategy σ such that $q(2n) = x$ and $q(2n+1) = p(n)$, and let $D_n = \sigma \circ L_{2k}(q)$ and $E_n = \bigcap_{m \leq n} D_n$.

Certainly, $x \in E_{2n}[x] = E_{2n}[q(2n)]$ for any $n < \omega$. Note also for any $n < \omega$ that

$$p(n) \in \bigcap_{m \leq n} \tau \circ L_k(p \restriction n)$$

$$= \bigcap_{m \leq n} (\sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(m-1) \rangle)[x] \cap \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(m-1), x \rangle)[x])$$

$$= \bigcap_{m\leq n} (D_{2m}[x] \cap D_{2m+1}[x]) = \bigcap_{m\leq 2n+1} D_m[x] = E_{2n+1}[x]$$

so by the symmetry of E_{2n+1}, $x \in E_{2n+1}[p(n)] = E_{2n+1}[q(2n+1)]$. Thus $x \in \bigcap_{n<\omega} E_n[q(n)] \neq \emptyset$, and since σ is a winning strategy, the attack q converges. Since $q(2n) = x$, q must converge to x. Thus its subsequence p converges to x, and τ is a winning strategy in $Con_{O,P}(X, x)$. □

Corollary 15. *For all $x \in X$:*

- $\mathscr{D} \uparrow_{k\text{-tact}} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-tact}} Con_{O,P}(X, x)$
- $\mathscr{D} \uparrow_{k\text{-mark}} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-mark}} Con_{O,P}(X, x)$

Corollary 16. *All proximal spaces are W-spaces.*

Theorem 17. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete. Then*

- $\mathscr{O} \uparrow Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathscr{D} \uparrow Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \uparrow_{k\text{-tact}} Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathscr{D} \uparrow_{k\text{-tact}} Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \uparrow_{k\text{-mark}} Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathscr{D} \uparrow_{k\text{-mark}} Prox_{D,P}(X \cup \{\infty\})$

Proof. Note that the topology on $X \cup \{\infty\}$ is induced by the uniformity with equivalence relation entourages $D(U) = \Delta \cup U^2$ for each open neighborhood U of ∞.

Let σ witness $\mathscr{D} \uparrow_{k\text{-tact}} Con_{O,P}(X \cap \{\infty\}, \infty)$ (resp. $\mathscr{D} \uparrow_{k\text{-mark}} Con_{O,P}(X \cap \{\infty\}, \infty)$, $\mathscr{D} \uparrow Con_{O,P}(X \cap \{\infty\}, \infty)$). We define the k-tactical (resp. k-Marköv, perfect info) strategy τ such that

$$\tau \circ L(p) = D(\sigma \circ L(p))$$

where L is the k-tactical fog-of-war (resp. k-Marköv fog-of-war, identity).

Let $p \in (X \cup \{\infty\})^\omega$ attack τ such that $\bigcap_{n<\omega} \tau(p \upharpoonright n)[p(n)] \neq \emptyset$.

If $\infty \in \bigcap_{n<\omega} \tau(p \upharpoonright n)[p(n)]$, it follows that p is an attack on σ. Since σ is a winning strategy, it follows that q and its subsequence p must coverge to ∞.

Otherwise, $\infty \notin \tau(p \upharpoonright N)[p(N)]$ for some $N < \omega$, and then $\tau(p \upharpoonright N)[p(N)] = \{p(N)\}$ implies $p \to p(N)$.

Thus $\tau \circ L$ is a winning strategy. □

Corollary 18. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete. Then*

- $\mathscr{O} \uparrow Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \uparrow Prox_{D,P}(X \cup \{\infty\})$

- $\mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \uparrow_{k\text{-}tact} Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \uparrow_{k\text{-}mark} Prox_{D,P}(X \cup \{\infty\})$

Proposition 19. *For any $x \in X$ and $k \geq 1$,*

- $\mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(X, x) \Leftrightarrow \mathscr{O} \uparrow_{tact} Con_{O,P}(X, x)$
- $\mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(X, x) \Leftrightarrow \mathscr{O} \uparrow_{mark} Con_{O,P}(X, x)$

Proof. If σ witnesses $\mathscr{O} \uparrow_{k\text{-tact}} Con_{O,P}(X, x)$, let $\tau(\emptyset) = \sigma(\emptyset)$ and

$$\tau(\langle q \rangle) = \bigcap_{i<k} \sigma(\langle \underbrace{x, \dots, x}_{k-i-1}, q, \underbrace{x, \dots, x}_{i} \rangle)$$

This is easily verified to be a winning strategy. The proof for $\mathscr{O} \uparrow_{k\text{-mark}} Con_{O,P}(X, x)$ is analogous. $\square$

Corollary 20. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete, and $k \geq 1$. Then*

- $\mathscr{D} \uparrow_{k\text{-}tact} Prox_{D,P}(X \cup \{\infty\}) \Leftrightarrow O \uparrow_{tact} Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{D} \uparrow_{k\text{-}mark} Prox_{D,P}(X \cup \{\infty\}) \Leftrightarrow O \uparrow_{mark} Prox_{D,P}(X \cup \{\infty\})$

Proposition 21. *For any uniform space X,*

- $\mathscr{O} \uparrow_{k\text{-}tact} Prox_{D,P}(X) \Leftrightarrow \mathscr{O} \uparrow_{2\text{-}tact} Prox_{D,P}(X)$
- $\mathscr{O} \uparrow_{k\text{-}mark} Prox_{D,P}(X) \Leftrightarrow \mathscr{O} \uparrow_{2\text{-}mark} Prox_{D,P}(X)$

Proof. If σ witnesses $\mathscr{O} \uparrow_{k\text{-tact}} Con_{O,P}(X, x)$, let $\tau(\emptyset) = \sigma(\emptyset)$ and

$$\tau(\langle q \rangle) = \bigcap_{i<k} \sigma(\langle \underbrace{q, \dots, q}_{i} \rangle)$$

$$\tau(\langle q, q' \rangle) = \bigcap_{i<k} \sigma(\langle \underbrace{q, \dots, q}_{k-i}, \underbrace{q', \dots, q'}_{i} \rangle)$$

This is easily verified to be a winning strategy. The proof for $\mathscr{O} \uparrow_{k\text{-mark}} Con_{O,P}(X, x)$ is analogous. $\square$

Definition 22. The strong proximal game $sProx_{D,P}(X)$ is analogous to $Prox_{D,P}(X)$, except $\mathscr{D}$ may only win if p converges.

Definition 23. A **uniformly locally compact** space is a uniformizable space with a **uniformly compact entourage** M where $\overline{M[x]}$ is compact for all x.

Theorem 24. *For any uniformly locally compact space X, $\mathscr{D} \uparrow Prox_{D,P}(X) \Leftrightarrow \mathscr{D} \uparrow sProx_{D,P}(X)$*

Proof. Let M be a uniformly locally compact entourage. Let σ witness $\mathscr{D} \uparrow Prox_{D,P}(X)$ such that $\sigma(a) \subseteq M$ always (so $\overline{\sigma(a)[x]} \subseteq \overline{M[x]}$ is compact), and $a \supseteq b$ implies $\sigma(a) \subseteq \frac{1}{4}\sigma(b)$.

Let $\tau(p \upharpoonright n) = \frac{1}{2}\sigma(p \upharpoonright n)$. If p attacks τ in $sProx_{D,P}(X)$, then

$$p(n+1) \in \tau(p \upharpoonright n)[p(n)] = \frac{1}{2}\sigma(p \upharpoonright n)[p(n)]$$

and for

$$x \in \overline{\sigma(p \upharpoonright (n+1))[p(n+1)]} \subseteq \overline{\frac{1}{4}\sigma(p \upharpoonright n)[p(n+1)]} \subseteq \frac{1}{2}\sigma(p \upharpoonright n)[p(n+1)]$$

we can conclude $x \in \sigma(p \upharpoonright n)[p(n)]$. Thus

$$\sigma(p \upharpoonright (n+1))[p(n+1)] \subseteq \overline{\sigma(p \upharpoonright (n+1))[p(n+1)]} \subseteq \sigma(p \upharpoonright n)[p(n)]$$

Finally, note that p attacks the winning strategy σ in $Prox_{D,P}(X)$, but since the intersection of a chain of nonempty compact sets is nonempty:

$$\bigcap_{n<\omega} \sigma(p \upharpoonright n)[p(n)] = \bigcap_{n<\omega} \overline{\sigma(p \upharpoonright n)[p(n)]} \neq \emptyset$$

We conclude that p converges. $\square$

Corollary 25. *A uniformaly locally compact space X is proximal if and only if $\mathscr{D} \uparrow sProx_{D,P}(X)$.*

Theorem 26. *For any uniformly locally compact proximal space X, $\mathscr{O} \uparrow Clus_{O,P}(X, H)$ for all compact $H \subseteq X$.*

Proof. Let σ witness $\mathscr{D} \uparrow sProx_{D,P}(X)$ such that $p \supseteq q$ implies $\sigma(p) \subseteq \frac{1}{4}\sigma(q)$.

Let $o(t)$ be the subsequence of t consisting of its odd-indexed terms.

We define $T(\emptyset)$, etc. as follows:

- Let $\emptyset \in T(\emptyset)$.
- Choose $m_\emptyset < \omega$, $h_{\emptyset,i} \in H$ for $i < m_\emptyset$, and $h_{\emptyset,i,j} \in H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$ for $i, j < m_\emptyset$ such that
$$\{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}] : i < m_\emptyset\}$$
is a cover for H and such that for each $i < m_\emptyset$
$$\{\frac{1}{4}\sigma(\langle h_{\emptyset,i}\rangle)[h_{\emptyset,i,j}] : j < m_\emptyset\}$$
is a cover for $H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$.
- Let $\langle i\rangle \in T(\emptyset)$, $\langle i, h_{\emptyset,i}\rangle \in T(\emptyset)$, and $\langle i, h_{\emptyset,i}, j\rangle \in T(\emptyset)$ for $i, j < m_\emptyset$.

Suppose $T(a)$, etc. are defined. We then define $T(a^\frown\langle x\rangle)$, etc. for
$$x \in \bigcup_{s^\frown\langle i,h_{s,i},j\rangle \in \max(T(a))} \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$
as follows:

- Let $T(a) \subseteq T(a^\frown\langle x\rangle)$.
- Choose $t = s^\frown\langle i, h_{s,i}, j, x\rangle$ such that $s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(a))$ and $x \in \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$.
- Note that, assuming $o(s)^\frown\langle h_{s,i}\rangle$ is a legal partial attack against σ, then
$$x \in \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}] \subseteq \frac{1}{4}\sigma(o(s))[h_{s,i,j}]$$
and
$$h_{s,i,j} \in \overline{\frac{1}{4}\sigma(o(s))[h_{s,i}]} \subseteq \frac{1}{2}\sigma(o(s))[h_{s,i}]$$
implies
$$x \in \sigma(o(s))[h_{s,i}]$$
and thus $o(s)^\frown\langle h_{s,i}, x\rangle = o(t)$ is a legal partial attack against σ.
- Choose $m_t < \omega$, $h_{t,k} \in H \cap \overline{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]}$ for $k < m_t$, and $h_{t,k,l} \in H \cap \overline{\frac{1}{4}\sigma(t)[h_{t,k}]}$ for $k, l < m_t$ such that
$$\{\frac{1}{4}\sigma(o(t))[h_{t,k}] : k < m_t\}$$
is a cover for $H \cap \overline{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]}$ and such that for each $k < m_t$
$$\{\frac{1}{4}\sigma(o(t)^\frown\langle h_{t,k}\rangle)[h_{t,i,j}] : l < m_t\}$$
is a cover for $H \cap \overline{\frac{1}{4}\sigma(o(t))[h_{t,k}]}$.

- Note that, assuming $o(t)$ is a legal partial attack against σ, then

$$h_{t,k} \in \overline{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]} \subseteq \frac{1}{2}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

  and

$$x \in \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

  implies

$$h_{t,k} \in \sigma(o(s)^\frown\langle h_{s,i}\rangle)[x]$$

  and thus $o(t)^\frown\langle h_{t,k}\rangle$ is a legal partial attack against σ.

- Let $t \in T(a^\frown\langle x\rangle)$, $t^\frown\langle k\rangle \in T(a^\frown\langle x\rangle)$, $t^\frown\langle k, h_{t,k}\rangle \in T(a^\frown\langle x\rangle)$, and $t^\frown\langle k, h_{t,k}, l\rangle \in T(a^\frown\langle x\rangle)$ for $k, l < m_t$.

- Note that assuming

$$\{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}] : s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(a))\}$$

  covers H, then since

$$\{\frac{1}{4}\sigma(o(t)^\frown\langle h_{t,k}\rangle)[h_{t,k,l}] : s^\frown\langle i, h_{s,i}, j, x, k, h_{t,k}, l\rangle \in \max(T(a^\frown\langle x\rangle)) \setminus \max(T(a))\}$$

  covers $H \cap \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$, we have that

$$\{\frac{1}{4}\sigma(o(t)^\frown\langle h_{t,k}\rangle)[h_{t,k,l}] : t^\frown\langle k, h_{t,k}, l\rangle \in \max(T(a^\frown\langle x\rangle))\}$$

  covers H.

With this we may define the perfect information strategy τ for $\mathscr{O}$ in $Con_{O,P}(X, H)$ such that:

$$\tau(p \restriction n) = \bigcup_{s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(p\restriction n))} \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

If p attacks τ, then it follows that $T(p \restriction n)$ is defined for all $n < \omega$, so let $T(p) = \bigcup_{n<\omega} T(p \restriction n)$. We note $T(p)$ is an infinite tree with finite levels:

- $\emptyset$ has exactly $m_\emptyset$ successors $\langle i\rangle$.
- $s^\frown\langle i\rangle$ has exactly one successor $t^\frown\langle i, h_{s,i}\rangle$
- $s^\frown\langle i, h_{s,i}\rangle$ has exactly m_s successors $t^\frown\langle i, h_{s,i}, j\rangle$
- $s^\frown\langle i, h_{s,i}, j\rangle$ has either no successors or exactly one successor $t^\frown\langle i, h_{s,i}, j, x\rangle$

- $t = s^\frown\langle i, h_{s,i}, j, x\rangle$ has exactly m_t successors $t^\frown\langle k\rangle$

Let $q' = \langle i_0, h_0, j_0, x_0, i_1, h_1, j_1, x_1, \ldots\rangle$ correspond to this infinite branch in $T(p)$, and let $q = o(q') = \langle h_0, x_0, h_1, x_1, \ldots\rangle$. Note that by the construction of $T(p)$, q is an attack on the winning strategy σ in $sProx_{D,P}(X)$, so it must converge. Since every other term of q is in H, it must converge to H. Then since q is a subsequence of p, p must cluster at H. $\square$

Corollary 27. *For any uniformly locally compact proximal space, $\mathscr{O} \uparrow Con_{O,P}(X, H)$ for all compact $H \subseteq X$.*

Proof. $\mathscr{O} \uparrow Con_{O,P}(X, H)$ if and only if $\mathscr{O} \uparrow Clus_{O,P}(X, H)$. $\square$

Corollary 28. *A compact uniform space X is Corson compact if and only if it is proximal.*

Proof. A characterization of Corson compact is having a W-set diagonal. If X is proximal compact, then X^2 is proximal compact, and its compact diagonal is a W-set. $\square$

Theorem 29. *$\mathscr{D} \uparrow_{pre} Prox_{D,P}(X) \Leftrightarrow$ the uniformity on X is induced by a psuedometric*

Proof. Let σ witness $\mathscr{D} \uparrow_{\text{pre}} Prox_{D,P}(X)$, and assume without loss of generality that $\sigma(n+1) \subseteq \frac{1}{4}\sigma(n)$. Note that $\sigma(n)$ satisfies the hypotheses of Engleking 8.1.10, so there exists a psuedometric ρ such that for $n < \omega$,

$$\{\langle x, y\rangle : \rho(x, y) < 2^{-n}\} \subseteq \sigma(n) \subseteq \{\langle x, y\rangle : \rho(x, y) \leq 2^{-n}\}$$

(Note that if $\rho(x, y) = 0$, then $\langle x, y\rangle \in \bigcap_{n<\omega} \sigma(n)$ implies $\{x, y\} \subseteq \bigcap_{n<\omega} \sigma(n)[x] \cap \bigcap_{n<\omega} \sigma(n)[y]$, and since σ is a winning strategy, the attack $\langle x, y, x, y, \ldots\rangle$ must converge. If X is T_1, then $x = y$, and ρ is a metric.)

Now assume that the uniformity on X is induced by the metric d. Let $\sigma(n) = \{\langle x, y\rangle : d(x, y) < 2^{-n}\}$. Then if p attacks σ such that $x \in \bigcap_{m<\omega} \sigma(n)[p(n)]$, it follows that $p(n) \in \sigma(n)[x]$ for all $n < \omega$. Then for any entourage D, there is some $m < \omega$ such that $\sigma(m) \subseteq E$, and thus for $n \geq m$, $p(n) \in \sigma(n)[x] \subseteq \sigma(m)[x] \subseteq D[x]$. Thus p converges to x. $\square$