Limited information strategies for a topological proximal game

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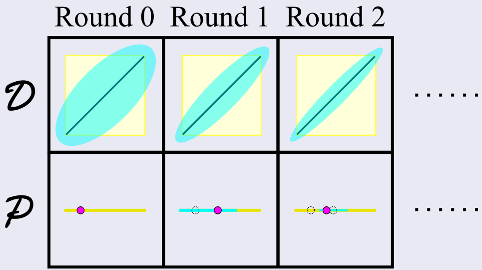
Game

Bell's absolutely proximal game $Bell^{\rightarrow}_{D,P}(X)$ [1] (2014)



$\mathscr{D}$ wins the game if the points chosen by $\mathscr{P}$ converge. Otherwise, $\mathscr{P}$ wins.

If $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$, then X is called an *absolutely proximal space*. "Absolutely proximal" is a strengthening of "proximal" characterized by an easier game (for $\mathscr{D}$), but these games are equivalent for compact spaces.

This game connects to a game of Gary Gruenhage: [1]

Theorem

Every proximal space is a W-space. So $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X) \Rightarrow \mathscr{O} \uparrow Gru^{\rightarrow}_{O,P}(X, x)$ *for all* $x \in X$.

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Proximal spaces have strong preservation properties, as any closed subset or Σ-product of proximal spaces is proximal.

Since any metrizable space is proximal, and any proximal space is collectionwise normal, Bell's game gives an elegant proof of the classic result of Rudin and Gulko:

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Nyikos [4] observed that

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Corson compact spaces are proximal.

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With Gruenhage, I showed that the answer is yes: [2]

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A compact space is Corson compact if and only if it is proximal.

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Player $\mathscr{D}$ chooses *entourages* of the diagonal: elements of a *uniformity* inducing the topology of the space.

A uniformity $\mathbb{D}$ on X is a filter of subsets of X^2 satisfying:

- $\bigcap \mathbb{D} = \Delta = \{\langle x, x\rangle : x \in X\}$
- $D \in \mathbb{D}$ implies $D^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in D\} \in \mathbb{D}$
- for each $D \in \mathbb{D}$ there is $\frac{1}{2}D \in \mathbb{D}$ such that $\frac{1}{2}D \circ \frac{1}{2}D \subseteq D$

The topology induced by a uniformity is the smallest topology such that $D[x] = \{y : \langle x, y\rangle \in D\}$ is a neighborhood of x for each $D \in \mathbb{D}$.

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Our goal is to obtain a purely topological characterization of the proximal property.

As it turns out, the union of all uniformities inducing a topology is itself a uniformity inducing that topology, called the *fine* or *universal uniformity*. Furthermore, the proximal property is agnostic to which uniformity is chosen for the space's topology.

If there's a winning strategy for $\mathscr{D}$ given any uniformity for the topology on X, then that strategy also works with the universal uniformity for X containing it. This reduces our goal to characterizing the entourages of the universal uniformity.

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If you look in the right textbook [5], you'll find the answer:

Theorem

A neighborhood U of the diagonal is a universal entourage if and only if there exist neighborhoods U_n for $n < \omega$ where $U = U_0$ and $U_{n+1} \circ U_{n+1} \subseteq U_n$.

As a bonus, for paracompact spaces, *all* neighborhoods of the diagonal have this property (entourages may be converted to open covers and then star-refined).

So we topologize Bell's game by simply saying an "entourage" is any open symmetric neighborhood of the diagonal with the above property, and discard the need to consider a specific uniform structure.

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A *perfect information strategy* uses full information of the previous moves of the opponent. ($\mathscr{A} \uparrow G$)

A *k-tactical strategy* only uses the last k previous moves of the opponent. ($\mathscr{A} \underset{k\text{-tact}}{\uparrow} G$)

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Bell observed that a winning perfect information strategy may always be passed down to win in a closed subspace.

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If $\mathscr{D} \uparrow Bell_{\vec{D},P}(X)$, then $\mathscr{D} \uparrow Bell_{\vec{D},P}(H)$ for every closed subspace H of X.

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Bell's also showed that winning strategies are preserved for Σ-products.

Theorem

If $\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(X_\alpha)$ *for* $\alpha < \kappa$, *then* $\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}\left(\sum_{\alpha<\kappa} X_\alpha\right)$.

Idea of proof: during round n, consider the first n non-zero coordinates of the previous n moves by $\mathscr{P}$ and use the winning strategies for those finite coordinates. Note that this uses the round number and perfect information of all previous moves.

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If we allow ourselves the round number, we may at least handle countable products:

Theorem

If $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell_{\vec{D},P}(X_i)$ for $i < \omega$, then $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell_{\vec{D},P}\left(\prod_{i<\omega} X_i\right)$.

It seems likely that $\mathscr{D} \underset{\text{mark}}{\not\uparrow} Bell_{\vec{D},P}\left(\sum_{\alpha<\omega_1} 2\right)$, but I don't have a proof. Whether $\mathscr{D} \underset{2\text{-mark}}{\uparrow} Bell_{\vec{D},P}\left(\sum_{\alpha<\omega_1} 2\right)$ holds is less clear.

Existence of a winning limited information strategy characterizes a stronger topological property than the existence of a winning perfect information strategy.

As it turns out:

Theorem

A compact space X is strongly Eberlein compact if and only if $\mathscr{D} \underset{tact}{\uparrow} Bell^{\rightarrow}_{D,P}(X)$.

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Sketch of Proof

Easy direction:

Definition

Strong Eberlein compacts embed in $\sigma 2^\kappa$ for some κ.

Lemma

$\mathscr{D} \underset{tact}{\uparrow} Bell_{\vec{D},P}(\sigma 2^\kappa)$.

Sketch of Proof (cont.)

Lemmas which give the other direction:

Lemma (Gruenhage [3])

Scattered proximal compacts are strong Eberlein compact.

Lemma

Non-scattered proximal compacts contain copies of the Cantor space 2^ω.

Lemma

$\mathscr{D} \underset{tact}{\not\uparrow} Bell_{\overrightarrow{D},P}(2^\omega)$.

A neat corollary:

- For compact spaces, $\mathscr{O} \uparrow Gru_{O,P}^{\rightarrow}(X^2, \Delta)$ if and only if $\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$.

but:

- Any metric space satisfies $\mathscr{O} \underset{\text{tact}}{\uparrow} Gru_{O,P}^{\rightarrow}(X^2, \Delta)$, but for compact spaces, $\mathscr{D} \underset{\text{tact}}{\uparrow} Bell_{D,P}^{\rightarrow}(X)$ implies X is scattered.

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Any questions?

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