Example 1. *Let X be a zero-dimensional, compact L-space (hereditarally Lindeloff and non-separable). It is a fact that there exists a point-countable collection $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ of clopen sets in X, and it is also true that any point-finite subcollection of $\mathcal{U}$ is countable.*

Let $C = \{c_\alpha : \alpha < \omega_1\}$ be any uncountable subset of the Cantor space 2^ω. Let $X_s = X \times \{s\}$ for each $s \in 2^{<\omega}$, and $U_{\alpha,s} = U_\alpha \times \{s\}$.

Finally, let

$$\mathbb{X} = C \cup \bigcup_{s \in 2^{<\omega}} X_s$$

be a tree of $2^{<\omega}$ copies of X, and where

$$c_\alpha \cup \bigcup_{n < \omega} U_{\alpha, x_\alpha \restriction n}$$

is an open set about each c_α.

Definition 2. Let $S \in [\omega_1]^{<\omega}$ and $m < \omega$. Define

$$K_S = \bigcup_{\alpha \in S} \left(c_\alpha \cup \left(\bigcup_{s < c_\alpha} U_{\alpha,s} \right) \right)$$

$$A = \{z ^\frown \langle 1 \rangle : z \in 1^{<\omega}\}$$

$$K_S^* = K_S \setminus \bigcup_{s \in A} X_s$$

and

$$L_m = \bigcup_{s \in 2^{<m}} X_s$$

and observe that every compact set is dominated by $K_S^* \cup L_m$ for some S, m. Intuitively, K_S^* collects the branches of U_α converging up to c_α for each $\alpha \in S$ while avoiding copies X_s of X for each s in an antichain A, and L_m collects the copies X_s of X with $|s| < m$ at the base of the tree.

Proposition 3. *Without loss of generality, P always plays points in $\bigcup_{s \in 2^{<\omega}} X_s$.*

Proposition 4. $K \uparrow LF_{K,P}(\mathbb{X})$.

Proof. In response to a point $\langle x, s \rangle$, K observes that there are only countably many α such that $U_\alpha \times \{s\}$ contains $\langle x, s \rangle$ (by point-countability of X). Enumerate these as α_n. K makes a promise that during round m, K will forbid some superset of $K_{\{\alpha_n : n \leq m\}}$. Finally, K also always forbids a superset of $L_{|s|+1}$.

Suppose P's moves clustered at some point. Since K forbade $L_{|s|+1}$ during each round, that point must be c_α for some α. P's play then must have included a subsequence of points $\langle x_0, s_0 \rangle, \langle x_1, s_1 \rangle, \langle x_2, s_2 \rangle \ldots$ such that $x_n \in U_\alpha$ and $s_n \leq s_{n+1} \leq c_\alpha$. However, in response to $\langle x_0, s_0 \rangle$, K made a promise to eventually forbid a superset of $K_{\{\alpha\}}$, making every $\langle x_n, t_n \rangle$ illegal after that round. $\square$

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Theorem 5. $K \not\Uparrow_{tact} LF_{K,P}(\mathbb{X})$.

Proof. This is actually a corollary of G's theorem in [?]. The following is a direct game-theoretic proof.

Suppose that $\sigma(\langle x, s\rangle)$ was a winning strategy for K and assume

$$\sigma(\langle x, s\rangle) = \bigcup_{|t| \leq |s|} \sigma(\langle x, t\rangle) = \sigma'(x, |s|)$$

Thus there exists some $f : \omega_1 \to \omega$ such that $\sigma'(x, f(\alpha))$ covers every neighborhood of c_α for all $x \in U_\alpha$. (If not, P wins by taking the α for which f is not defined, and may always play $\langle x, s\rangle$ in a neighborhood of c_α for which $\sigma'(x, |s|)$ doesn't cover a neighborhood of c_α.) Fix n for which $f(\alpha) = n$ for α in an uncountable set A.

Since the collection $\{U_\alpha : \alpha \in A\}$ is uncountable, it is not point-finite. Fix x so that x belongs to U_α for all α in an infinite $B \subseteq A$. Finally, consider $\sigma'(x, n)$. For each $\alpha \in B$, $\sigma'(x, f(\alpha)) = \sigma'(x, n)$ covers c_α. Since $\{c_\alpha : \alpha \in B\}$ is a closed infinite discrete set, we have a contradiction to the compactness of $\sigma'(x, n)$. $\qquad\square$

Theorem 6. $K \not\Uparrow_{2\text{-}tact} LF_{K,P}(\mathbb{X})$.

Proof. Suppose $\sigma(\langle x, s\rangle, \langle y, t\rangle)$ was a winning 2-tactical strategy. We may define $S(x, y, n) \in [\omega_1]^{<\omega}$ (increasing on n) and $n < m(x, y, n) < \omega$ such that for each (x, y),

$$\bigcup_{s,t \in 2^{\leq n}} \sigma(\langle x, s\rangle, \langle y, t\rangle) \subseteq K^*_{S(x,y,n)} \cup L_{m(x,y,n)}$$

and so we assume

$$\sigma(\langle x, s\rangle, \langle y, t\rangle) = K^*_{S(x,y,\max(|s|,|t|))} \cup L_{m(x,y,\max(|s|,|t|))}$$

Select an arbitrary point $x' \in X$. We define a tactical strategy

$$\tau(x, s) = K^*_{S(x,x',m(x,x',|s|)+1)} \cup L_{m(x,x',m(x,x',|s|)+1)}$$

We complete the proof by showing τ is a winning tactical strategy (a contradiction).

Suppose

$$\langle x_0, s_0\rangle, \langle x_1, s_1\rangle, \langle x_2, s_2\rangle, \ldots$$

successfully countered τ by clustering at $c \in C$ (the strategy trivially prevents clustering elsewhere). Let $z_n = \langle 0, \ldots, 0\rangle$ with n zeros. We claim

$$\langle x_0, s_0\rangle, \langle x', z_{m(x_0,x',|s_0|)} \frown \langle 1\rangle\rangle, \langle x_1, s_1\rangle, \langle x', z_{m(x_1,x',|s_1|)} \frown \langle 1\rangle\rangle, \langle x_2, s_2\rangle, \langle x', z_{m(x_2,x',|s_2|)} \frown \langle 1\rangle\rangle, \ldots$$

is a successful counter to σ.

We will need the fact that, as $\langle x_{i+1}, s_{i+1} \rangle$ was legal against τ:

$$|s_i| < m(x_i, x', |s_i|) + 1 = |z_{m(x_i,x',|s_i|)} {}^\frown \langle 1 \rangle|$$

$$< m(x_i, x', m(x_i, x', |s_i|) + 1) = m(x_i, x', |z_{m(x_i,x',|s_i|)} {}^\frown \langle 1 \rangle|) \le |s_{i+1}|$$

Note that $m(x, y, \max(|s|, |t|))$ is increasing throughout this play of the game versus σ:

$$m(x_i, x', \max(|s_i|, |z_{m(x_i,x',|s_i|)} {}^\frown \langle 1 \rangle|))$$

$$= m(x_i, x', |z_{m(x_i,x',|s_i|)} {}^\frown \langle 1 \rangle|)$$

$$\le |s_{i+1}|$$

$$< m(x_{i+1}, x', |s_{i+1}|)$$

$$= m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_i,x',|s_i|)} {}^\frown \langle 1 \rangle|))$$

$$= |z_{m(x_{i+1},x',|s_{i+1}|)}|$$

$$< |z_{m(x_{i+1},x',|s_{i+1}|)} {}^\frown \langle 1 \rangle|$$

$$< m(x_{i+1}, x', |z_{m(x_{i+1},x',|s_{i+1}|)} {}^\frown \langle 1 \rangle|)$$

$$= m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_{i+1},x',|s_{i+1}|)} {}^\frown \langle 1 \rangle|))$$

We turn to showing that $\langle x', z_{m(x_{i+1},x',|s_{i+1}|)} {}^\frown \langle 1 \rangle \rangle$ is always a legal move. Since $z_{m(x_{i+1},x',|s_{i+1}|)} {}^\frown \langle 1 \rangle$ is on the antichain avoided by any K^*, the problem is reduced to showing that this move isn't forbidden by

$$L_{m(x_{i+1},x',\max(|s_{i+1}|,|z_{m(x_i,x',|s_i|)} {}^\frown \langle 1 \rangle|))}$$

which we can see here:

$$m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_i,x',|s_i|)} {}^\frown \langle 1 \rangle|)) = m(x_{i+1}, x', |s_{i+1}|) < |z_{m(x_{i+1},x',|s_{i+1}|)} {}^\frown \langle 1 \rangle|$$

We can conclude by showing that $\langle x_{i+1}, s_{i+1} \rangle$ is always a legal move. We can see it avoids

$$L_{m(x_i,x',\max(|s_i|,|z_{m(x_i,x',|s_i|)} {}^\frown \langle 1 \rangle|))}$$

since

$$m(x_i, x', \max(|s_i|, |z_{m(x_i,x',|s_i|)} {}^\frown \langle 1 \rangle|)) = m(x_i, x', |z_{m(x_i,x',|s_i|)} {}^\frown \langle 1 \rangle|) \le |s_{i+1}|$$

Since $\langle x_{i+1}, s_{i+1} \rangle$ was legal against τ, it avoided

$$K^*_{S(x_h,x',m(x_h,x',|s_h|)+1)} = K^*_{S(x_h,x',\max(|s_h|,|z_{m(x_h,x',|s_h|)} {}^\frown \langle 1 \rangle|))}$$

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for $h \leq i$. And when $h < i$, we see it avoids:

$$K^*_{S(x_{h+1},x',\max(|s_{h+1}|,|z_{m(x_h,x',|s_h|)}\frown\langle 1\rangle|))} = K^*_{S(x_{h+1},x',|s_{h+1}|)}$$

$$\subseteq K^*_{S(x_{h+1},x',m(x_{h+1},x',|s_{h+1}|)+1)}$$

This concludes the proof. □

Theorem 7. $K \not\vdash_{k\text{-tact}} LF_{K,P}(\mathbb{X})$.

Proof. The proof proceeds in parallel to the proof of $K \not\vdash_{2\text{-tact}} LF_{K,P}(\mathbb{X})$.

Suppose $\sigma(\langle x_0, s_0\rangle, \ldots, \langle x_k, s_k\rangle)$ was a winning $(k+1)$-tactical strategy. We may define $S(x_0, \ldots, x_k, n) \in [\omega_1]^{<\omega}$ (increasing on n) and $n < m(x_0, \ldots, x_k, n) < \omega$ such that for each $(x_0, \ldots, x_k)$,

$$\bigcup_{s_0,\ldots,s_k \in 2^{\leq n}} \sigma(\langle x_0, s_0\rangle, \ldots, \langle x_k, s_k\rangle) \subseteq K^*_{S(x_0,\ldots,x_k,n)} \cup L_{m(x_0,\ldots,x_k,n)}$$

and so we assume

$$\sigma(\langle x_0, s_0\rangle, \ldots, \langle x_k, s_k\rangle) = K^*_{S(x_0,\ldots,x_k,\max(|s_0|,\ldots,|s_k|))} \cup L_{m(x_0,\ldots,x_k,\max(|s_0|,\ldots,|s_k|))}$$

Select an arbitrary point $x' \in X$. Let $M^0(x, n) = m(x, x', \ldots, x', n)$ and $M^{i+1}(x, n) = M^0(x, M^i(x, n) + 1)$. We define a tactical strategy

$$\tau(x, s) = K^*_{S(x,x',\ldots,x',M^{k-1}(x,|s|)+1)} \cup L_{m(x,x',\ldots,x',M^{k-1}(x,|s|)+1)}$$

We complete the proof by showing τ is a winning tactical strategy (a contradiction).

Suppose

$$\langle x_0, s_0\rangle, \langle x_1, s_1\rangle, \langle x_2, s_2\rangle, \ldots$$

successfully countered τ by clustering at $c \in C$ (the strategy trivially prevents clustering elsewhere). Let $z_n = \langle 0, \ldots, 0\rangle$ with n zeros. We claim

$$\langle x_0, s_0\rangle, \langle x', z_{M^0(x_0,|s_0|)}\frown\langle 1\rangle\rangle, \langle x', z_{M^1(x_0,|s_0|)}\frown\langle 1\rangle\rangle, \ldots, \langle x', z_{M^{k-1}(x_0,|s_0|)}\frown\langle 1\rangle\rangle,$$

$$\langle x_1, s_1\rangle, \langle x', z_{M^0(x_1,|s_1|)}\frown\langle 1\rangle\rangle, \langle x', z_{M^1(x_1,|s_1|)}\frown\langle 1\rangle\rangle, \ldots, \langle x', z_{M^{k-1}(x_1,|s_1|)}\frown\langle 1\rangle\rangle, \ldots$$

is a successful counter to σ.

We will need the fact that, as $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ:

$$|s_i| < M^0(x_i, |s_i|) + 1 = |z_{M^0(x_i,|s_i|)}\frown\langle 1\rangle| < M^0(x_i, M^0(x_i, |s_i|) + 1) + 1$$

$$= M^1(x_i, |s_i|) + 1 = |z_{M^1(x_i,|s_i|)}\frown\langle 1\rangle| < \cdots < |z_{M^{k-1}(x_i,|s_i|)}\frown\langle 1\rangle|$$

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$$= M^{k-1}(x_i, |s_i|) + 1 < m(x_i, x', \ldots, x', M^{k-1}(x_i, |s_i|) + 1) \le |s_{i+1}|$$

Note that $m(x_0, \ldots, x_k, \max(|s_0|, \ldots, |s_k|))$ is increasing throughout this play of the game versus σ:

$$m(x_i, x', \ldots, x', \max(|s_i|, |z_{M^0(x_i, |s_i|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_i, |s_i|)}{}^\frown\langle 1\rangle|))$$

$$= m(x_i, x', \ldots, x', |z_{M^{k-1}(x_i, |s_i|)}{}^\frown\langle 1\rangle|)$$

$$= m(x_i, x', \ldots, x', M^{k-1}(x_i, |s_i|) + 1)$$

$$\le |s_{i+1}|$$

$$< M^0(x_{i+1}, |s_{i+1}|)$$

$$= m(x_{i+1}, x', \ldots, x', |s_{i+1}|)$$

$$= m(x_{i+1}, x', \ldots, x', \max(|s_{i+1}|, |z_{M^0(x_i, |s_i|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_i, |s_i|)}{}^\frown\langle 1\rangle|))$$

$$= |z_{m(x_{i+1}, x', \ldots, x', |s_{i+1}|)}|$$

$$= |z_{M^0(x_{i+1}, |s_{i+1}|)}|$$

$$< |z_{M^0(x_{i+1}, |s_{i+1}|)}{}^\frown\langle 1\rangle|$$

$$< m(x_{i+1}, x', \ldots, x', |z_{M^0(x_{i+1}, |s_{i+1}|)}{}^\frown\langle 1\rangle|)$$

$$= m(x_{i+1}, x', \ldots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1}, |s_{i+1}|)}{}^\frown\langle 1\rangle|, |z_{M^1(x_i, |s_i|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_i, |s_i|)}{}^\frown\langle 1\rangle|))$$

$$\vdots$$

$$< m(x_{i+1}, x', \ldots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1}, |s_{i+1}|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_{i+1}, |s_{i+1}|)}{}^\frown\langle 1\rangle|))$$

We turn to showing that $\langle x', z_{M^j(x_{i+1}, |s_{i+1}|)}{}^\frown\langle 1\rangle\rangle$ is always a legal move. Since $z_{M^j(x_{i+1}, |s_{i+1}|)}{}^\frown\langle 1\rangle$ is on the antichain avoided by any K^*, the problem is reduced to showing that this move isn't forbidden by

$$L_{m(x_{i+1}, x', \ldots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1}, |s_{i+1}|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{j-1}(x_{i+1}, |s_{i+1}|)}{}^\frown\langle 1\rangle|, |z_{M^j(x_i, |s_i|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^k(x_i, |s_i|)}{}^\frown\langle 1\rangle|))}$$

$$= L_{m(x_{i+1}, x', \ldots, x', |z_{M^{j-1}(x_{i+1}, |s_{i+1}|)}{}^\frown\langle 1\rangle|)}$$

which we can see here:

$$m(x_{i+1}, x', \ldots, x', |z_{M^{j-1}(x_{i+1}, |s_{i+1}|)}{}^\frown\langle 1\rangle|)$$

$$= m(x_{i+1}, x', \ldots, x', M^{j-1}(x_{i+1}, |s_{i+1}|) + 1)$$

$$= M^0(x_{i+1}, M^{j-1}(x_{i+1}, |s_{i+1}|) + 1)$$

$$= M^j(x_{i+1}, s_{i+1})$$

$$< |z_{M^j(x_{i+1},|s_{i+1}|)} \frown \langle 1 \rangle|$$

We can conclude by showing that $\langle x_{i+1}, s_{i+1} \rangle$ is always a legal move. We can see it avoids

$$L_{m(x_i,x',\ldots,x',\max(|s_i|,|z_{M^0(x_i,|s_i|)} \frown \langle 1 \rangle|,\ldots,|z_{M^{k-1}(x_i,|s_i|)} \frown \langle 1 \rangle|))}$$

since

$$m(x_i, x', \ldots, x', \max(|s_i|, |z_{M^0(x_i,|s_i|)} \frown \langle 1 \rangle|, \ldots, |z_{M^{k-1}(x_i,|s_i|)} \frown \langle 1 \rangle|))$$
$$= m(x_i, x', \ldots, x', |z_{M^{k-1}(x_i,|s_i|)} \frown \langle 1 \rangle|)$$
$$= m(x_i, x', \ldots, x', M^{k-1}(x_i, |s_i|) + 1)$$
$$\leq |s_{i+1}|$$

Since $\langle x_{i+1}, s_{i+1} \rangle$ was legal against τ, it avoided

$$K^*_{S(x_h,x',\ldots,x',M^{k-1}(x_h,|s_h|)+1)}$$

$$= K^*_{S(x_h,x',\ldots,x',\max(|s_h|,|z_{M^0(x_h,|s_h|)} \frown \langle 1 \rangle|,\ldots,|z_{M^{k-1}(x_h,|s_h|)} \frown \langle 1 \rangle|))}$$

for $h \leq i$. And when $h < i$, we see it avoids both:

$$K^*_{S(x_{h+1},x',\ldots,x',\max(|s_{h+1}|,|z_{M^0(x_{h+1},|s_{h+1}|)} \frown \langle 1 \rangle|,\ldots,|z_{M^{j-1}(x_{h+1},|s_{h+1}|)} \frown \langle 1 \rangle|,|z_{M^j(x_h,|s_h|)} \frown \langle 1 \rangle|,\ldots,|z_{M^k(x_h,|s_h|)} \frown \langle 1 \rangle|))}$$

$$= K^*_{S(x_{h+1},x',\ldots,x',|z_{M^{j-1}(x_{h+1},|s_{h+1}|)} \frown \langle 1 \rangle|)}$$
$$= K^*_{S(x_{h+1},x',\ldots,x',M^{j-1}(x_{h+1},|s_{h+1}|)+1)}$$
$$\subseteq K^*_{S(x_{h+1},x',\ldots,x',M^{k-1}(x_{h+1},|s_{h+1}|)+1)}$$

and:

$$K^*_{S(x_{h+1},x',\ldots,x',\max(|s_{h+1}|,|z_{M^0(x_h,|s_h|)} \frown \langle 1 \rangle|,\ldots,|z_{M^k(x_h,|s_h|)} \frown \langle 1 \rangle|))}$$

$$= K^*_{S(x_{h+1},x',\ldots,x',|s_{k+1}|)}$$
$$\subseteq K^*_{S(x_{h+1},x',\ldots,x',M^{k-1}(x_{h+1},|s_{h+1}|)+1)}$$

This concludes the proof. $\square$