EQUIVALENCY OF PROXIMAL COMPACT AND CORSON COMPACT IN UNIFORM SPACES

GARY GRUENHAGE AND STEVEN CLONTZ

Abstract. A common generalization of metric spaces is the idea of a uniform space, determined by a uniformity of entourages on the diagonal of the square. Proximal uniform spaces are those for which the first player has a winning strategy in this ω-length game due to Bell: the first player chooses an entourage of the space, and her opponent chooses a point close to the point chosen in the previous round, with respect to the entourage chosen in the previous round. As proximal spaces are preserved under Σ-products, Nyikos has observed that every Corson compact space, a compact space embeddable in the Σ-product of real lines, must then be proximal. He asked if that implication may also be reversed. We answer his question postively, by way of defining a stronger version of the proximal game which is perfect-information equivalent to Bell's original game for uniformly locally compact spaces.

Jocelyn Bell introduced the concept of proximal spaces in her doctoral dissertation while working on the so-called uniform box product problems, due to her advisor Scott Williams: is the uniform box product of compact spaces normal, collectionwise normal, or paracompact? Proximal spaces, such as the one-point compactification $\omega_1^* = \omega_1 \cup \{\infty\}$ of the discrete space of cardinality ω_1, are defined to be the spaces for which the first player in the proximal game played on that space has a winning strategy. Bell has shown that proximal spaces have strong preservation properties, particularly, the Σ-product of proximal spaces is itself proximal [1]. This mirrors the analogous theorem for metric spaces; in fact, every metric space is easily seen to be proximal.

In [4], Peter Nyikos observed that compact subspaces of the Σ-product of real lines, known as Corson compacts, must be proximal, since the proximal property is preserved under Σ-products. He left open the question as to whether any proximal compact must then be Corson compact. Using a characterization of Corson compact due to Gruenhage in [3], we can answer that question in the affirmative.

1. Definitions and Properites of Uniform Spaces

In this paper, all spaces are assumed to be topological spaces induced by a uniformity. We relate some definitions and properties of uniform spaces.

Definition 1.1. A **uniform space** is a pair $\langle X, \mathcal{D}\rangle$ where X is a set, and $\mathcal{D}$ is a **uniformity**. A uniformity is a filter on subsets of X^2, called **entourages**, such that for each entourage $D \in \mathcal{D}$:

- D is reflexive, i.e., the diagonal $\Delta = \{\langle x, x\rangle : x \in X\} \subseteq D$.
- Its inverse $D^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in D\} \in \mathcal{D}$.

- There exists $\frac{1}{2}D \in \mathcal{D}$ such that
$$2\left(\frac{1}{2}D\right) = \frac{1}{2}D \circ \frac{1}{2}D = \left\{\langle x, z\rangle : \exists y \left(\langle x, y\rangle, \langle y, z\rangle \in \frac{1}{2}D\right)\right\} \subseteq D$$
(Let $\frac{1}{2^{n+1}}D$ be shorthand for $\frac{1}{2}(\frac{1}{2^n}D)$. Without loss of generality, assume that $\frac{1}{2}D$ is always symmetric, that is, $\langle x, y\rangle \in \frac{1}{2}D \Leftrightarrow \langle y, x\rangle \in \frac{1}{2}D$.)

Definition 1.2. The **uniform topology** induced by a uniformity declares a set U to be open if for every $x \in U$, there is some $D \in \mathcal{D}$ with $x \in D[x] = \{y : \langle x, y\rangle \in D\} \subseteq U$.

Theorem 1.3. *Every completely regular topology may be induced by a uniformity, and every uniform topology is completely regular.*

Theorem 1.4. *For every entourage D, there is an open symmetric entourage $E \subseteq D$. That is, $\langle x, y\rangle \in E \Leftrightarrow \langle y, x\rangle \in E$, and E is open in X^2 with the usual product topology induced by the uniform topology on X.*

Proposition 1.5. *If D is an open entourage, then for all $x \in X$, $D[x]$ is an open neighborhood of x.*

Proof. For the proofs of these, see [2]. $\square$

Proposition 1.6. *If X is a uniform space, then for all $x \in X$ and open symmetric entourages D:*
$$\frac{1}{2}D[x] \subseteq \overline{\frac{1}{2}D[x]} \subseteq D[x]$$

References

[1] Bell, Jocelyn. *An Infinite Game with Topological Consequences*. etc.
[2] Engelking, Ryszard. *General Topology*. etc.
[3] Gruenhage, Gary. *Covering properties on $X^2 \setminus \Delta$, W-sets, and compact subsets of Σ-products.* Topology Appl. 17 (1984), no. 3, 287304.
[4] Nyikos, Peter. *Proximal and Semi-Proximal Spaces*. etc.

Department of Mathematics, Auburn University, Auburn, AL 36830
E-mail address: gruengf@auburn.edu
URL: www.auburn.edu/∼gruengf

Department of Mathematics, Auburn University, Auburn, AL 36830
E-mail address: steven.clontz@auburn.edu
URL: www.stevenclontz.com