Proximal compact spaces are Corson compact

2015 Joint Mathematics Meetings at San Antonio

Steven Clontz
http://stevenclontz.com

Department of Mathematics and Statistics
Auburn University

January 11, 2015

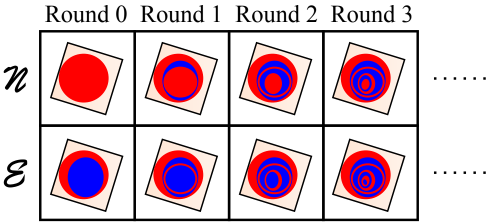
A *topological game* is a two-player game $G(X)$ of length $\omega = \{0, 1, 2, \dots\}$ defined for certain topological spaces X.

During each round n, the first and second player take turns choosing certain topological objects from X (e.g. point, open set, open cover, etc.).

At the "end" of the game, a winner is declared by inspecting the sequences of choices made throughout the game.

The study of such games involves finding when a player has a *winning strategy* which defeats every possible counterattack by the opponent.

Canonical example: *Banach-Mazur Game* (1935) [5]



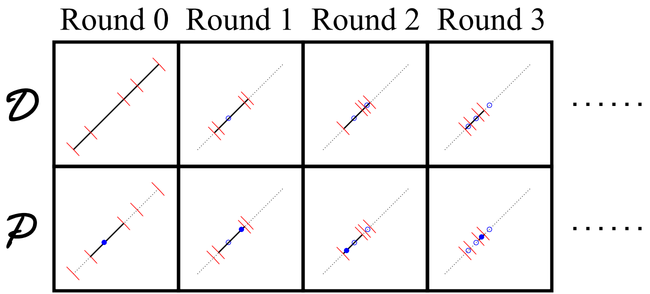
The first player $\mathscr{N}$ wins the game if the intersection of all the chosen open sets is nonempty.

Theorem

X is Baire if and only if $\mathscr{N}$ lacks a winning strategy in the Banach Mazur game.

See Telgarsky's excellent survey on topological games for more details: [8]

Proximal Game (2011) [1]



for compact T_1 0-dim spaces

The first player $\mathscr{D}$ wins the game if the points chosen by the second player $\mathscr{P}$ converge. If $\mathscr{D}$ has a winning strategy for this game, call X *proximal*.

Some results related to the Proximal Game due to Jocelyn Bell:

Proposition

If X is metrizable, then X is proximal.

Theorem

If X is proximal, then X is collectionwise normal.

Theorem

Σ-product and closed subspaces of proximal spaces are proximal.

Corollary

The Σ-product of metrizable spaces is collectionwise normal. [4] [7]

A *Corson compact* space is a space homeomorphic to a compact subset of the Σ-product of real lines.

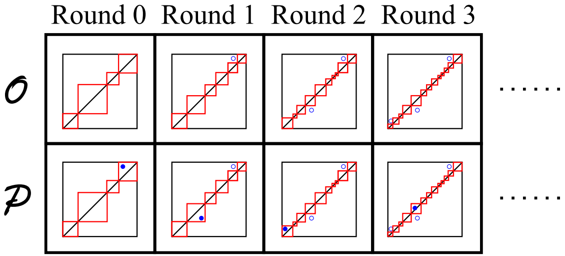
Peter Nyikos observed:

Proposition

Every Corson compact space is proximal compact. [6]

C. and Gruenhage showed in [2] that any compact proximal space must be Corson compact, using another game-theoretic characterization of Corson compact due to Gruenhage:

Diagonal Game (1984) [3]:



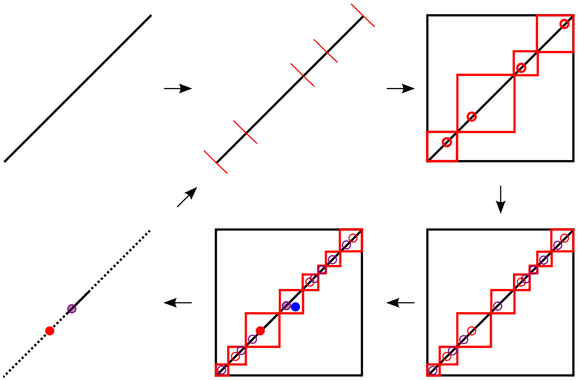
for compact T_1 0-dim spaces

The first player $\mathscr{O}$ wins the game if any open set containing the diagonal also contains infinitely many of $\mathscr{P}$'s chosen points.

Theorem

If $\mathscr{O}$ has a winning strategy for the diagonal game and X is compact, then X is Corson compact.

One may use a winning strategy σ for $\mathscr{D}$ in the proximal game to construct a strategy τ for $\mathscr{O}$ in the diagonal game.



In general:

$$\tau(a) = \bigcup_{s^\frown\langle i,h_{s,i},j\rangle \in \max(T(a))} \frac{1}{4}\sigma(o_s{}^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

Using the strategy τ defined for every proximal compact space, $\mathscr{O}$ cannot be defeated in the diagonal game, and therefore all proximal compacts are Corson compact. $\square$

Open questions:

- If compactness is dropped, does the proximal game characterize all copies of *closed* subspaces of a Σ-product of reals? (Nyikos)
- If the winning strategy for the proximal game is *Markov* (relies on only the latest move and round number) for a compact space, does that imply that the space is *Eberlein* compact? (This holds for the diagonal game.)

Jocelyn R. Bell.
An infinite game with topological consequences.
Topology Appl., 175:1–14, 2014.

Steven Clontz and Gary Gruenhage.
Proximal compact spaces are Corson compact.
Topology Appl., 173:1–8, 2014.

Gary Gruenhage.
Covering properties on $X^2 \setminus \Delta$, W-sets, and compact subsets of Σ-products.
Topology Appl., 17(3):287–304, 1984.

S. P. Gulko.
Properties of sets that lie in Σ-products.
Dokl. Akad. Nauk SSSR, 237(3):505–508, 1977.

R. Daniel Mauldin, editor.
The Scottish Book.
Birkhäuser, Boston, Mass., 1981.
Mathematics from the Scottish Café, Including selected papers presented at the Scottish Book Conference held at North Texas State University, Denton, Tex., May 1979.

Peter J. Nyikos.
Proximal and semi-proximal spaces (preprint).
2013.

Mary Ellen Rudin.
The shrinking property.
Canad. Math. Bull., 26(4):385–388, 1983.

Rastislav Telgársky.
Topological games: on the 50th anniversary of the Banach-Mazur game.
Rocky Mountain J. Math., 17(2):227–276, 1987.

Any questions?