By convention, $n = \{0, 1, \ldots, n-1\}$ for each natural number n.

Definition 1. A function $F : n \to S$ is called a **formula** for a set S of dice, where $F(k)$ returns which die in the set S has a face with the number $k < n$.

For example, the formula $F : 4 \to \{A, B\}$ defined by $ABBA$ represents an assignment of values to the faces of dice in the set $\{A, B\}$. In particular A has two sides $\{0, 3\}$ while die B has sides $\{1, 2\}$.

Definition 2. Given a formula $F : n \to S$ and $T \subseteq S$, let $F_T : m \to T$ be the **subformula** obtained by removing the dice in $S \setminus T$ from the sequence.

So for $F : 7 \to \{A, B, C\}$ defined by $ABCACBA$, $F_{\{A,C\}}$ is defined by $ACACA$.

Definition 3. Given a formula $F : n \to S$ and $m \leq n$, let $F \upharpoonright m : m \to S$ be the **restriction formula** where $(F \upharpoonright m)(k) = F(k)$ for $k < m$.

So for $F : 7 \to \{A, B, C\}$ defined by $ABCACBA$, $F \upharpoonright 4$ is defined by $ABCA$.

Definition 4. A **sample** for a formula $F : n \to S$ is a function $f : S \to n$ such that $F(f(A)) = A$ for all $A \in S$.

Definition 5. The **winner** of a given sample is $A \in S$ such that $f(A) = \max\{f(B) : B \in S\}$.

Definition 6. A formula $F : n \to S$ is called **go-first-fair** if each $A \in S$ is the winner of an equal number of samples of F.

Definition 7. etc.

Proposition 8. *Suppose $F : n \to S$ is a permutation-fair (resp. place-fair, go-first-fair) formula. Then F_T is permutation-fair (resp. place-fair, go-first-fair) for all $T \subseteq S$.*

Theorem 9. *Suppose $F : n \to S \cup \{X\}$ is a go-first-fair formula such that for each $m \leq n$ where $f(m) = X$, $(F \upharpoonright m)_S$ is permutation-fair. Then F is permutation-fair.*

Proof. Since go-first-fair implies permutation-fair in the base case $|S| = 0$, assume the theorem holds when $|S| \leq k$, and let $|S| = k + 1$.

For each $T \subsetneq S$, we note that $F_{T \cup \{X\}}$ is a go-first-fair formula such that for each $m \leq |F_{T\cup\{X\}}|$ where $F_{T\cup\{X\}}(m) = X$, $F_{T\cup\{X\}} \upharpoonright m = F \upharpoonright m'$ for some $m \leq m' < n$ and $F(m') = X$. Thus $(F_{T\cup\{X\}} \upharpoonright m)_T = (F \upharpoonright m')_T = ((F \upharpoonright m')_S)_T$ is permutation-fair, and since $|T| \leq k$, $F_{T\cup\{X\}}$ is permutation-fair.

□