Two mark in DS game

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Let $[f, F, \epsilon] = \{g \in C_p(X) : |g(x) - f(x)| < \epsilon \text{ for all } x \in F\}$.

Game 1. Let G be the following game. During round n, player I chooses $\beta_n < \omega_1$, and player II chooses $F_n \in [\omega_1]^{<\aleph_0}$. II wins if whenever $\gamma < \beta_n$ for co-finitely many $n < \omega$, $\gamma \in F_n$ for infinitely many $n < \omega$.

Proposition 2. II $\underset{2\text{-}mark}{\uparrow}$ G.

Theorem 3. I $\underset{2\text{-}mark}{\uparrow}$ $DS(C_p(\omega_1 + 1))$

Proof. Let σ be a winning 2-mark for II in G.

Given a point $f \in C_p(\omega_1 + 1)$, let $\alpha_f < \omega_1$ satisfy $f(\beta) = f(\gamma)$ for all $\alpha_f \leq \beta \leq \gamma \leq \omega_1$.

Let $\tau(\emptyset, 0) = [\mathbf{0}, \{\omega_1\}, 4]$, $\tau(\langle f \rangle, 1) = [\mathbf{0}; \sigma(\langle \alpha_f \rangle, 0) \cup \{\omega_1\}; 2]$, and

$$\tau(\langle f, g \rangle, n + 2) = [\mathbf{0}; \sigma(\langle \alpha_f, \alpha_g \rangle, n + 1) \cup \{\omega_1\}; 2^{-n}].$$

Let ν be a legal attack by II against σ. For $\beta \leq \omega_1$, if $\beta < \alpha_{\nu(n)}$ for co-finitely many $n < \omega$, then $\beta \in \sigma(\langle \alpha_{\nu(n)}, \alpha_{\nu(n+1)} \rangle)$ for infinitely-many $n < \omega$, and thus $0 \in \mathrm{cl}\{\nu(n)(\beta) : n < \omega\}$. Otherwise $\beta \geq \alpha_{\nu(n)}$ for infinitely many $n < \omega$, and thus $0 \in \mathrm{cl}\{\nu(n)(\beta) : n < \omega\}$ as well. Thus $\mathbf{0} \in \mathrm{cl}\{\nu(n) : n < \omega\}$. □