In this paper we investigate an open question posed to us by Gruenhage:

Question 1. *Let P be the subspace of the Sorgenfrey line containing only irrational numbers. Does there exist a base $\mathcal{B}$ for P such that for every $\mathcal{C} \subseteq \mathcal{B}$ which is also a base for X, there exists a locally finite subcover $\mathcal{C}' \subseteq \mathcal{C}$?*

We begin by tackling a simpler (solved) problem:

Proposition 2. *Let R be the Sorgenfrey line, the set of real numbers with the topology generated by the base $\mathcal{B} = \{[a, b) : a < b\}$ (where $[a, b) = \{x : a \leq x < b\}$).*

For every $\mathcal{C} \subseteq \mathcal{B}$ which is also a base for R, there exists a pairwise disjoint subcover $\mathcal{C}' \subseteq \mathcal{C}$ (and thus a locally finite subcover).

Proof. We begin by letting $b_{n,0} = n$ for each $n < \omega$, and if $b_{n,\alpha}$ is defined for some ordinal $\alpha < \omega_1$ and $b_{n,\alpha} < n + 1$, we define its successor $b_{n,\alpha+1}$ as follows:

- $b_{n,\alpha} < b_{n,\alpha+1} \leq n + 1$
- $[b_{n,\alpha}, b_{n,\alpha+1}) \in \mathcal{C}$

  (This is possible as $\mathcal{C}$ is a base, and there must be some element of $\mathcal{C}$ which contains $b_{n,\alpha}$ and is a subset of $[b_{n,\alpha}, n + 1)$.)

(If $b_{n,\alpha} = n + 1$, then let $b_{n,\alpha+1} = n + 1$ as well.) Finally, if $\alpha < \omega_1$ is a limit ordinal, let $b_{n,\alpha} = \lim\limits_{\beta \to \alpha} b_{n,\beta}$.

Let $C_{n,\alpha} = [b_{n,\alpha}, b_{n,\alpha+1})$. We claim $\mathcal{C}' = \{C_{n,\alpha} : n < \omega, \alpha < \omega_1\}$ is a pairwise disjoint cover of R. Pairwise disjoint is evident by definition. To see that it is a cover, suppose it wasn't and missed some $x \in [n, n + 1)$. Then we have an uncountable increasing sequence of numbers $\{b_{n,\alpha} : \alpha < \omega_1\}$, which contradicts the countable chain condition on the real line. $\square$

This idea can actually be generalized for any dense subspace, but it takes some machinery:

Theorem 3. *Let P be a dense subspace of the Sorgenfrey line with the induced topology, which has the base $\mathcal{B} = \{[a, b) : a < b, a \in P\}$.*

For every $\mathcal{C} \subseteq \mathcal{B}$ which is also a base for P, there exists a pairwise disjoint subcover $\mathcal{C}' \subseteq \mathcal{C}$ (and thus a locally finite subcover).

Proof. It suffices to show for $[0,1) \cap P$. We begin by constructing a collection of functions $S \subseteq \omega^{\omega_1}$ and numbers c_s, d_s defined by those functions as follows:

- Let $S_0 = \{\emptyset\}$. Let $d_\emptyset = 0$ and $c_{\langle -1 \rangle} = 1$.
- Suppose S_α has been defined, as well as $d_s \leq c_{s^\frown\langle -1 \rangle}$. For $s \in S$, consider the following:
  - If $d_s = c_{s^\frown\langle -1 \rangle}$, do nothing.
  - If $d_s < c_{s^\frown\langle -1 \rangle}$ and $d_s \in P$, let $S_{\alpha+1}$ contain $s^\frown\langle 0 \rangle$ and define $c_{s^\frown\langle 0 \rangle}$, $d_{s^\frown\langle 0 \rangle}$, $c_{s^\frown\langle 0,-1 \rangle}$ such that

    $$d_s = c_{s^\frown\langle 0 \rangle} < d_{s^\frown\langle 0 \rangle} \leq c_{s^\frown\langle 0,-1 \rangle} = c_{s^\frown\langle -1 \rangle}$$

    where $[c_{s^\frown\langle 0 \rangle}, d_{s^\frown\langle 0 \rangle}) \in \mathcal{C}$.
  - If $d_s < c_{s^\frown\langle -1 \rangle}$ and $d_s \in P$, let $S_{\alpha+1}$ contain $s^\frown\langle n \rangle$ for all $n < \omega$ and define $c_{s^\frown\langle n \rangle}$, $d_{s^\frown\langle n \rangle}$, $c_{s^\frown\langle n,-1 \rangle}$ for all $n < \omega$ such that

    $$d_s < \cdots \leq c_{s^\frown\langle 2,-1 \rangle} = c_{s^\frown\langle 1 \rangle} < d_{s^\frown\langle 1 \rangle} \leq c_{s^\frown\langle 1,-1 \rangle} = c_{s^\frown\langle 0 \rangle} < d_{s^\frown\langle 0 \rangle} \leq c_{s^\frown\langle 0,-1 \rangle} = c_{s^\frown\langle -1 \rangle}$$

    where $[c_{s^\frown\langle n \rangle}, d_{s^\frown\langle n \rangle}) \in \mathcal{C}$ for all $n < \omega$ and $c_{s^\frown\langle n \rangle} \to d_s$.
- Suppose α is a limit ordinal and S_β has been defined for all $\beta < \alpha$. If $s \in \omega^\alpha$ and $t \in \bigcup_{\beta<\alpha} S_\beta$ for all $t < s$, let S_α contain s and define $d_s = \lim_{t<s} d_t$ and $c_{s^\frown\langle -1 \rangle} = \lim_{t<s} c_{t^\frown\langle -1 \rangle}$.

Let $S = \bigcup_{\alpha<\omega_1} S_\alpha$. By construction, $\mathcal{C}' = \{[c_s, d_s) : s \in S\}$ is a disjoint subcollection of $\mathcal{C}$. We claim it also must cover $[0,1) \cap P$.

Suppose not: $x \in [0,1) \cap P$ is not contained in $[c_s, d_s)$ for any $s \in S$. Note that $d_\emptyset < x$ (or else $x \in [c_{\langle 0 \rangle}, d_{\langle 1 \rangle}))$. Assume $n_\beta < \omega$ is defined for all $\beta < \alpha$, and consider $s \in \omega^\alpha$ where $s(\beta) = n_\beta$. If $d_s < x$, we claim there is a minimal $n_\alpha < \omega$ where $d_{s^\frown\langle n_\alpha \rangle} < x$.

- This possible when $d_s \notin P$ since $d_{s^\frown\langle n \rangle} \to d_s$.
- This is also possible when $d_s \in P$ since $[c_{s^\frown\langle 0 \rangle}, d_{s^\frown\langle 0 \rangle})$ does not contain x, and thus $d_s = c_{s^\frown\langle 0 \rangle} < d_{s^\frown\langle 0 \rangle} \leq x$. If $d_{s^\frown\langle 0 \rangle} = x$ then $d_{s^\frown\langle 0 \rangle} \in P$ and $[d_{s^\frown\langle 0 \rangle}, d_{s^\frown\langle 0,0 \rangle}) = [c_{s^\frown\langle 0,0 \rangle}, d_{s^\frown\langle 0,0 \rangle})$ contains x, which is a contradiction.

Finally, we notice that by defining $f \in \omega_1^\omega$ such that $f(\alpha) = n_\alpha$, then $d_{f(\alpha)}$ is an increasing sequence defined for all $\alpha < \omega_1$, which is a contradiction. $\square$