

Definition 1. Let $Men_{C,F}(X)$ be the Menger game played on X . Let $BD_{B,F}(X, x)$ be the countable fan tightness game introduced by Barman and Dow in [2]: \mathcal{B} chooses a set B_n with $x \in \overline{B_n}$, followed by \mathcal{F} choosing $F_n \in [B_n]^{<\omega}$, where \mathcal{F} wins if $x \in \overline{\bigcup_{n<\omega} F_n}$.

Theorem 2. X^k is Menger for all $k < \omega$ if and only if $C_p(X)$ has countable fan tightness.

Proof. Arhangel'skiĭ in [1]. □

Theorem 3. If $\mathcal{F} \uparrow_{\text{mark}} Men_{C,F}(X)$, then $\mathcal{F} \uparrow_{\text{mark}} BD_{B,F}(C_p(X), \vec{0})$.

Proof. Essentially [3, 2.6], once one observes that $\mathcal{F} \uparrow_{\text{mark}} Men_{C,F}(X)$ characterizes σ -relative-compactness (equivalent to σ -compactness in regular spaces). Note further that this property is preserved for finite powers. □

Theorem 4. If $\mathcal{F} \uparrow Men_{C,F}(X^k)$ for all $k < \omega$, then $\mathcal{F} \uparrow BD_{B,F}(C_p(X), \vec{0})$.

Proof. For a sequence of blades $s^\frown \langle B \rangle \in (\mathcal{P}(X))^{\leq \omega}$ satisfying $\vec{0} \in \overline{s(i)}$ for all $i < |s|$, let $t_k^m(s^\frown \langle B \rangle)$ be the sequence defined by

$$t_k^m(s^\frown \langle B \rangle)(i) = \{(f^{-1}[-2^{-i-m}, 2^{-i-m}])^k : f \in s^\frown \langle B \rangle(i)\}$$

for each $i \leq |s|$. In the case $i = |s|$, note

$$t_k^m(s^\frown \langle B \rangle)(|s|) = \{(f^{-1}[-2^{-|s|-m}, 2^{-|s|-m}])^k : f \in B\}$$

We claim that $t_k(s^\frown \langle B \rangle)(i)$ is an open cover of X^k for each $i \leq |s|$. Indeed, for $\alpha \in X^k$, $U(\vec{0}, \text{ran } \alpha, 2^{-i-m})$ is an open neighborhood of $\vec{0}$, and thus must hit some $f \in s^\frown \langle B \rangle(i)$. Note then that $\alpha \in (f^{-1}[-2^{-i-m}, 2^{-i-m}])^k$.

For a sequence s of length $\leq \omega$, define its offset by n , $s \downarrow n$, as follows. Let $s \downarrow n = \emptyset$ when $n \geq |s|$ and otherwise satisfy $|s \downarrow n| + n = |s|$ and $s \downarrow n(i) = s(n+i)$.

If σ_k is the winning strategy for \mathcal{F} in $Men_{C,F}(X^k)$, then let τ^m be a strategy for \mathcal{F} in $BD_{B,F}(C_p(X), \vec{0})$ satisfying

$$\sigma_k(t_k^m(s^\frown \langle B \rangle)) \subseteq \{(f^{-1}[-2^{-|s|-m}, 2^{-|s|-m}])^k : f \in \tau^m(s^\frown \langle B \rangle)\}$$

and τ be a winning strategy satisfying

$$\tau(s^\frown \langle B \rangle) = \bigcup_{m \leq |s|} \tau^m((s \downarrow m)^\frown \langle B \rangle)$$

Attack τ with a . Let $\alpha \in X^k$ and $m < \omega$. Since $t_k^m(a \restriction m)$ is an attack on σ in $Men_{C,F}(X^k)$, there is some $n < \omega$ where $\alpha \in \bigcup \sigma_k(t_k^m((a \restriction m) \restriction (n+1)))$, and therefore $\alpha \in (f^{-1}[(-2^{-n-m}, 2^{-n-m})])^k$ for some $f \in \tau^m((a \restriction m) \restriction (n+1))$, so $f \in U(\vec{0}, \text{ran } \alpha, 2^{-n}) \cap \tau(a \restriction (n+m+1))$. \square

References

- [1] A. V. Arhangel'skiĭ. The structure and classification of topological spaces and cardinal invariants. *Uspekhi Mat. Nauk*, 33(6(204)):29–84, 272, 1978.
- [2] Doyel Barman and Alan Dow. Selective separability and SS^+ . *Topology Proc.*, 37:181–204, 2011.
- [3] Doyel Barman and Alan Dow. Proper forcing axiom and selective separability. *Topology Appl.*, 159(3):806–813, 2012.