Limited Information Strategies for Topological Games

by

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Abstract

A topological game $G(X)$ is a two-player game which characterizes properties of a topological space X based upon the existence of winning perfect-information strategies for players in the game. If the property P is characterized by a player having a winning perfect information strategy, then a property stronger than P is characterized by that player having a winning limited information strategy. This paper investigates the existence of limited information strategies for four types of topological games from the literature, and the topological properties characterized by those strategies.

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This dissertation is dedicated to my wife Jessica, for her love and encouragement.

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Chapter 1

Introduction

Topological games have been studied since the 1930s to characterize properties of topological spaces. Generally speaking, a topological game $G(X)$ is a two player game defined for each topological space X such that a topological property P is characterized by a player $\mathscr{A}$ having a so-called "winning" strategy for $G(X)$ which cannot be countered by the opponent, denoted $\mathscr{A} \uparrow G(X)$.

The study of limited information strategies in topological games involves the following observation. If the existence of a winning strategy for a player in $G(X)$ characterizes property P, then the existence of a winning strategy which doesn't require perfect information about the history of the game characterizes a (perhaps non-strictly) stronger property Q.

This document is organized into five chapters in addition to this introductory chapter. The second provides preliminary definitions and conventions used throughout the paper, and the remaining chapters each consider different topological games from the literature and extend results on these games by considering limited information strategies.

The first game considered is Gruenhage's convergence game $Gru_{O,P}^{\rightarrow}(X,x)$, originally called the W-game and introduced in [8] to answer a question of Phil Zenor, and later used to characterize the W-space property generalizing first-countability. With regards to limited information and the one-point compactification of an uncountable cardinal, denoted κ^*, Peter Nyikos noted that the first player $\mathscr{O}$ has a winning tactical strategy considering only the most recent move of the opponent, denoted $\mathscr{O} \underset{\text{tact}}{\uparrow} Gru_{O,P}^{\rightarrow}(\kappa^*,\infty)$. If the game is slightly altered to $Gru_{O,P}^{\rightarrow,\star}(X,x)$, perfect information strategies are preserved. However, Nyikos demonstrated that $\mathscr{O}$ lacks a winning Markov strategy considering only the most recent move of the opponent and the round number, denoted $\mathscr{O} \underset{\text{mark}}{\not\uparrow} Gru^{\rightarrow,\star}_{O,P}(\kappa^*, \infty)$. This result has been extended as follows:

Theorem. *$\mathscr{O} \underset{k\text{-mark}}{\not\uparrow} Gru^{\rightarrow,\star}_{O,P}(\kappa^*, \infty)$ for $\kappa > \omega$, $k < \omega$. That is, $\mathscr{O}$ cannot force a win in the game with a k-Markov strategy which only considers k previous moves of the opponent and the round number.*

The game is made easier for $\mathscr{O}$ by weakening the game to $Gru^{\rightsquigarrow,\star}_{O,P}(X, x)$, exchanging the convergence requirement with clustering. In this scenario, the author has shown the size of the uncountable cardinal κ matters.

Theorem. *$\mathscr{O} \underset{mark}{\uparrow} Gru^{\rightsquigarrow,\star}_{O,P}(\kappa^*, \infty)$ if and only if $\kappa \leq \omega_1$.*

Based upon this game is a relatively new game from the literature, Bell's "proximal" game $Bell^{\rightarrow}_{D,P}(X)$. Peter Nyikos noted that Corson compact spaces satisfy $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$, and asked if this in fact was a characterization of Corson compacts. The author answered this question with Gary Gruenhage in [3].

Theorem. *Among compact Hausdorff spaces, X is Corson compact if and only if $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$.*

Many perfect information results on $Bell^{\rightarrow}_{D,P}(X)$ may be easily extended to limited information analogs, but others are more elusive. Bell showed in [2] that $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$ extends to arbitrary sigma-products, but the proof makes non-trivial use of perfect information. Thus for k-Markov strategies, the result has only been shown to hold for countable products.

Theorem. *For $k < \omega$, if $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell^{\rightarrow}_{D,P}(X_i)$ for all $i < \omega$, then $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell^{\rightarrow}_{D,P}\left(\prod_{i<\omega} X_i\right)$.*

Another game due to Gruenhage is the compact-point game $Gru_{K,P}(X)$ which may be used to characterize metacompactness and σ-metacompactness among locally compact spaces using tactical and Markov strategies. Using predetermined strategies which only consider the round number:

Theorem. *For locally compact spaces,* $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(X)$ *if and only if* X *is hemicompact.*

The compact-compact variation $Gru_{K,L}(X)$ may be used to characterize paracompactness among locally compact spaces using perfect information strategies. However, the existence of winning predetermined strategies for $Gru_{K,L}(X)$ coincides with $Gru_{K,P}(X)$ when considering locally compact or even compactly generated spaces. By investigating the subspace $\omega \cup \{\mathcal{F}\}$ of the Stone-Cech compactification $\beta\omega$ consisting of a single free ultrafilter, a distinction between the two games is revealed.

Theorem. *There exists a free ultrafilter* $\mathcal{F}$ *such that* $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(\omega \cup \{\mathcal{F}\})$*, but* $\mathscr{K} \underset{pre}{\not\uparrow} Gru_{K,L}(\omega \cup \{\mathcal{F}\})$ *for any free ultrafilter* $\mathcal{F}$*.*

Theorem. *Assuming* CH*, there exists a free ultrafilter* $\mathcal{S}$ *such that* $\mathscr{K} \underset{pre}{\not\uparrow} Gru_{K,P}(\omega \cup \{\mathcal{S}\})$*.*

A winning $k+1$-Markov strategy for $Gru_{\vec{O},P}(X, x)$ may be improved to a winning Markov strategy; however, it remains open whether this same result holds true for $Gru_{K,P}(X)$. The chapter on $Gru_{K,P}(X)$ concludes with a class of spaces which may seem to provide a counterexample at first, however:

Theorem. *The space* $\boldsymbol{X}$ *satisfies* $\mathscr{K} \uparrow Gru_{K,P}(\boldsymbol{X})$*, but* $\mathscr{K} \underset{k\text{-}mark}{\not\uparrow} Gru_{K,P}(\boldsymbol{X})$ *for all* $k < \omega$*.*

The final chapter is based upon one of the oldest topological games: the Menger game $Men_{C,F}(X)$, characterizing Menger's covering property. In this game, k-Markov strategies need only consider at most two moves of the opponent.

Theorem. $\mathscr{F} \underset{k+2\text{-}mark}{\uparrow} Men_{C,F}(X)$ *if and only if* $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}(X)$*.*

For a one-point Lindelöf-ication $\kappa^\dagger$ of discrete κ, the topological game $Men_{C,F}\left(\kappa^\dagger\right)$ is equivalent to a set-theoretic game $Fill^{\cap}_{C,F}(\kappa)$ with respect to $\mathscr{F}$'s k-Markov strategies. Using this game, one may see the following.

This game has similarities to a game introduced by Scheepers [26], who also introduced a set-theoretic axiom $S(\kappa)$ to study it. This axiom may also be applied to study $Men_{C,F}\left(\kappa^\dagger\right)$.

Theorem. *Assume* $S(\kappa)$*. Then* $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}\left(\kappa^{\dagger}\right)$.

Scheepers observed that $S(\omega_1)$ is a theorem of ZFC.

Theorem. $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}\left(\omega_1^{\dagger}\right)$ *but* $\mathscr{F} \underset{mark}{\not\uparrow} Men_{C,F}\left(\omega_1^{\dagger}\right)$.

Telgarsky [29] and Scheepers [28] provided different proofs to show that among metrizable spaces, $\mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$ characterizes σ-compactness. These results may be extended by considering Markov strategies. Recall that for Lindelöf spaces, metrizability is characterized by regularity and second-countability.

Theorem. *For regular spaces,* $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$ *if and only if* X *is* σ*-compact.*

Theorem. *For second-countable spaces,* $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$ *if and only if* $\mathscr{F} \uparrow Men_{C,F}(X)$.

The topological property characterized by $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$ seems to be heretofore unstudied. The final chapter concludes by introducing the sufficient robustly Menger property, and uses it to study the space R_ω finer than the usual Euclidean real line.

Theorem. *Assume* $S(2^\omega)$*. Then* $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}(R_\omega)$.

Chapter 2

Definitions and Examples

This chapter outlines the set-theoretic and game-theoretic preliminaries required for this paper.

2.1 Set Theory

In general, all required set-theoretic concepts are taken from Kunen's text [15].

Definition 2.1.1. Define *ordinals* and *cardinals* as in Kunen's text. In particular, $0 = \emptyset$, $\omega = \{0, 1, 2, \ldots\}$, ω_1 is the smallest uncountable ordinal, and $\alpha + 1 = \alpha \cup \{\alpha\}$. The order on ordinals and cardinals is given by $\alpha < \beta$ if and only if $\alpha \in \beta$.

Definition 2.1.2. Let R^D denote the set of functions from D to R. For an ordinal α, $R^{<\alpha} = \bigcup_{\beta<\alpha} R^\beta$ and $R^{\leq\alpha} = \bigcup_{\beta\leq\alpha} R^\beta$.

Definition 2.1.3. A *sequence* is a function $f \in A^\omega$, denoted $f = \langle f(0), f(1), \ldots\rangle$. A *finite sequence* is a function $t \in A^{<\omega}$, denoted $t = \langle t(0), \ldots, t(|t| - 1)\rangle$.

Ordered pairs, triples, etc. are considered to be finite sequences.

Definition 2.1.4. For $s, t \in A^{<\omega}$, let $s^\frown t \in A^{<\omega}$ be the *concatenation* of s and t, so that $(s^\frown t)(i) = s(i)$ for $i < |s|$, and $(s^\frown t)(i + |s|) = t(i)$ for $i < |t|$.

Definition 2.1.5. For $f : D \to R$ and $A \subseteq D$, let $f \restriction A : A \to R$ be the *restriction* of f to A.

2.2 Games

Intuitively, the games studied in this paper are two-player games for which each player takes turns making a choice from a set of possible moves. At the conclusion of the game, the choices made by both players are examined, and one of the players is declared the winner of that playthrough.

Games may be modeled mathematically in various ways, but we will find it convenient to think of them in terms defined by Gale and Stewart [6].

Definition 2.2.1. A *game* is a tuple $\langle M, W\rangle$ such that $W \subseteq M^\omega$. M is the set of *moves* for the game, and M^ω is the set of all possible *playthroughs* of the game.

W is the set of *winning playthroughs* or *victories* for the first player, and $M^\omega \setminus W$ is the set of victories for the second player. (W is often called the *payoff set* for the first player.)

Within this model, we may imagine two players $\mathscr{A}$ and $\mathscr{B}$ playing a game which consists of *rounds* enumerated for each $n < \omega$. During round n, $\mathscr{A}$ chooses $a_n \in M$, followed by $\mathscr{B}$ choosing $b_n \in M$. The playthrough corresponding to those choices would be the sequence $p = \langle a_0, b_0, a_1, b_1, \ldots\rangle$. If $p \in W$, then $\mathscr{A}$ is the winner of that playthrough, and if $p \notin W$, then $\mathscr{B}$ is the winner. Note that no ties are allowed.

Rather than explicitly defining W, we typically define games by declaring the *rules* that each player must follow and the *winning condition* for the first player. Then a playthrough is in W if either the first player made only *legal moves* which observed the game's rules and the playthrough satisified the winning condition, or the second player made an *illegal move* which contradicted the game's rules. Often, we will consider *legal playthoughs* where both players only made legal moves, in which case only the winning condition need be considered.

As an illustration, we could model a game of chess (ignoring stalemates) by letting

$$M = \{\langle p, s\rangle : p \text{ is a chess piece and } s \text{ is a space on the board }\}$$

representing moving a piece p to the space s on the board. Then the rules of chess restict White from moving pieces which belong to Black, or moving a piece to an illegal space on the board.[1] The winning condition could then "inspect" the resulting positions of pieces on the board after each move to see if White attained a checkmate. This winning condition along with the rules implicitly define the set W of winning playthroughs for White.

2.2.1 Infinite and Topological Games

Games never technically end within this model, since playthroughs of the game are infinite sequences. However, for all practical purposes many games end after a finite number of turns.

Definition 2.2.2. A game is said to be an *finite game* if for every playthrough $p \in M^\omega$ there exists a round $n < \omega$ such that $[p \restriction n] = \{q \in M^\omega : q \supseteq p \restriction n\}$ is a subset of either W or $M^\omega \setminus W$.

Put another way, a finite game is decided after a finite number of rounds, after which the game's winner could not change even if further rounds were played. Games which are not finite are called *infinite games*.

As an illustration of an infinite game, we may consider a simple example due to Baker [1].

Game 2.2.3. Let $Bak_{A,B}(X)$ denote a game with players $\mathscr{A}$ and $\mathscr{B}$, defined for each subset $X \subset \mathbb{R}$. In round 0, $\mathscr{A}$ chooses a number a_0, followed by $\mathscr{B}$ choosing a number b_0 such that $a_0 < b_0$. In round $n+1$, $\mathscr{A}$ chooses a number a_{n+1} such that $a_n < a_{n+1} < b_n$, followed by $\mathscr{B}$ choosing a number b_{n+1} such that $a_{n+1} < b_{n+1} < b_n$.

$\mathscr{A}$ wins the game if the sequence $\langle a_n : n < \omega \rangle$ converges to a point in X, and $\mathscr{B}$ wins otherwise.

[1] In practice, M is often defined as the union of two sets, such as white pieces and black pieces in chess. For example, the first player may choose open sets in a topology, while the second player chooses points within the topological space.

Certainly, $\mathscr{A}$ and $\mathscr{B}$ will never be in a position without (infinitely many) legal moves available, and provided that A is non-trivial, there is a playthrough such that for all $n < \omega$, the segment (a_n, b_n) intersects both A and $\mathbb{R} \setminus A$. Such a playthrough could never be decided in a finite number of moves, so the winning condition considers the infinite sequence of moves made by the players and declares a victor at the "end" of the game.

Definition 2.2.4. A *topological game* is a game defined in terms of an arbitrary topological space.

Topological games are usually infinite games, ignoring trivial examples. One of the earliest examples of a topological game is the Banach-Mazur game, proposed by Stanislaw Mazur as Problem 43 in Stefan Banach's Scottish Book in 1935, and solved by Banach later that year [19]. A more comprehensive history of the Banacy-Mazur and other topological games may be found in Telgarsky's survey on the subject [30].

The original game was defined for subsets of the real line; however, we give a more general definition here.

Game 2.2.5. Let $BM_{E,N}(X)$ denote the *Banach-Mazur game* with players $\mathscr{E}$, $\mathscr{N}$ defined for each topological space X. In round 0, $\mathscr{E}$ chooses a nonempty open set $E_0 \subseteq X$, followed by $\mathscr{N}$ choosing a nonempty open subset $N_0 \subseteq E_0$. In round $n+1$, $\mathscr{E}$ chooses a nonempty open subset $E_{n+1} \subseteq N_n$, followed by $\mathscr{N}$ choosing a nonempty open subset $N_{n+1} \subseteq E_{n+1}$.

$\mathscr{E}$ wins the game if $\bigcap_{n<\omega} E_n = \emptyset$, and $\mathscr{N}$ wins otherwise.

For example, if X is a locally compact Hausdorff space, $\mathscr{N}$ can "force" a win by choosing N_0 such that $\overline{N_0}$ is compact, and choosing N_{n+1} such that $N_{n+1} \subseteq \overline{N_{n+1}} \subseteq E_{n+1} \subseteq N_n$. Since $\bigcap_{n<\omega} E_n = \bigcap_{n<\omega} N_n$ is the decreasing intersection of compact sets, it cannot be empty.

This concept of when (and how) a player can "force" a win in certain topological games is the focus of this manuscript.

2.3 Strategies

We shall make the notion of forcing a win in a game rigorous by introducing "strategies" and "attacks" for games.

Definition 2.3.1. A *strategy* for a game $G = \langle M, W \rangle$ is a function from $M^{<\omega}$ to M.

Definition 2.3.2. An *attack* for a game $G = \langle M, W \rangle$ is a function from ω to M.

Intuitively, a strategy is a rule for one of the players on how to play the game based upon the previous (finite) moves of the opponent, while an attack is a fixed strike by an opponent indexed by round number.

Definition 2.3.3. The *result* of a game given a strategy σ for the first player and an attack $\langle a_0, a_1, \ldots \rangle$ by the second player is the playthrough

$$\langle \sigma(\emptyset), a_0, \sigma(\langle a_0 \rangle), a_1, \sigma(\langle a_0, a_1 \rangle), \ldots \rangle$$

Likewise, if σ is a strategy for the second player, and $\langle a_0, a_1, \ldots \rangle$ is an attack by the first player, then the result is the playthrough

$$\langle a_0, \sigma(\langle a_0 \rangle), a_1, \sigma(\langle a_0, a_1 \rangle), \ldots \rangle$$

We now rigorously define the notion of "forcing" a win in a game.

Definition 2.3.4. A strategy σ is a *winning strategy* for a player if for every attack by the opponent, the result of the game is a victory for that player.

If a winning strategy exists for a player $\mathscr{A}$ in the game G, then we write $\mathscr{A} \uparrow G$. Otherwise, we write $\mathscr{A} \not\uparrow G$.

To show that a winning strategy exists for a player (i.e. $\mathscr{A} \uparrow G$), we typically begin by defining it and showing that it is *legal*: it only yields moves which are legal according to the rules of the game. Then, we consider an arbitrary legal attack, and prove that the result of the game is a victory for that player.

If we wish to show that a winning strategy does not exist for a player (i.e. $\mathscr{A} \not\uparrow G$), we often consider an arbitrary legal strategy, and use it to define a legal *counter-attack* for the opponent. If we can prove that the result of the game for that strategy and counter-attack is a victory for the opponent, then a winning strategy does not exist.

Unlike finite games, is not the case that a winning strategy must exist for one of the players in an infinite game.

Definition 2.3.5. A game G with players $\mathscr{A}$, $\mathscr{B}$ is said to be *determined* if either $\mathscr{A} \uparrow G$ or $\mathscr{B} \uparrow G$. Otherwise, the game is *undetermined*.

The Borel Determinacy Theorem states that $G = \langle M, W \rangle$ is determined whenever W is a Borel subset of M^ω [18]. It's an easy corollary that all finite games are determined; W must be clopen.

However, as stated earlier, most topological games are infinite, and many are undetermined for certain spaces constructed using the Axiom of Choice.[2]

Often, we will build new strategies based on others.

Definition 2.3.6. A strategy τ is a *strengthening* of another strategy σ for a player if whenever the result of the game for σ and an attack a by the opponent is a victory for the player, then the result of the game for τ and a is also a victory for the player.

Proposition 2.3.7. *If σ is a winning strategy, and τ strengthens σ, then τ is also a winning strategy.*

2.3.1 Applications of Strategies

The power of studying these infinite-length games can be illustrated by considering the following proposition.

[2]These spaces cannot be constructed just only the axioms of ZF. In fact, mathematicians have studied an Axiom of Determinacy which declares that that all Gale-Stewart games are determined (and implies that the Axiom of Choice is false). [21]

Proposition 2.3.8. *If X is countable, then $\mathscr{B} \uparrow Bak_{A,B}(X)$.*

Proof. Adapted from [1]. Let $X = \{x_i : i < \omega\}$. Let $i(a, b)$ be the least integer such that $a < x_{i(x,y)} < b$, if it exists. We define a strategy σ for $\mathscr{B}$ such that:

- If $i(a_0, \infty)$ exists, $\sigma(\langle a_0 \rangle) = x_{i(a_0,\infty)}$.
- Let $b_n = \sigma(\langle a_0, \ldots, a_n \rangle)$ If $i(a_{n+1}, b_n)$ exists, $\sigma(\langle a_0, ..., a_{n+1} \rangle) = x_{i(a_{n+1},b_n)}$.
- Otherwise, the choice of $\sigma(t)$ may be any legal move.

Observe that σ is a legal strategy according to the rules of the game since $a_0 < \sigma(\langle a_0 \rangle)$ and $a_{n+1} < \sigma(\langle a_0, \ldots, a_{n+1} \rangle) < b_n$. Note every x_i was either chosen by $\mathscr{B}$ during a round, or it was illegal to choose. In either case, there exist a_n, b_n with $x_i \notin (a_n, b_n)$, so $\lim_{n\to\infty} a_n \notin X$ and thus σ is a winning strategy. $\square$

This yields the classical result from undergraduate set theory.

Corollary 2.3.9. *$\mathbb{R}$ is uncountable.*

Proof. $\mathscr{A} \uparrow Bak_{A,B}(\mathbb{R})$, since a_n must converge to some real number. This implies $\mathscr{B} \not\uparrow Bak_{A,B}(\mathbb{R})$, and thus $\mathbb{R}$ is not countable. $\square$

Infinite games thus provide a rich framework for considering questions in set theory and topology. In general, the presence or absence of a winning strategy for a player in a topological game characterizes a property of the topological space in question.

Theorem 2.3.10. *$\mathscr{E} \not\uparrow BM_{E,N}(X)$ if and only if X is a Baire space. [13]*

2.3.2 Limited Information Strategies

So far we have assumed both players enjoy *perfect information*, and may develop strategies which use all of the previous moves of the opponent as input.

Definition 2.3.11. For a game $G = \langle M, W\rangle$, the *k-tactical fog-of-war* is the function $\nu_k : M^{<\omega} \to M^{\leq k}$ defined by

$$\nu_k(\langle m_0, \dots, m_{n-1}\rangle) = \langle m_{n-k}, \dots, m_{n-1}\rangle$$

and the *k-Markov fog-of-war* is the function $\mu_k : M^{<\omega} \to (M^{\leq k} \times \omega)$ defined by

$$\mu_k(\langle m_0, \dots, m_{n-1}\rangle) = \langle\langle m_{n-k}, \dots, m_{n-1}\rangle, n\rangle$$

Essentially, these fogs-of-war represent a limited memory: ν_k filters out all but the last k moves of the opponent, and μ_k filters out all but the last k moves of the opponent and the round number.

We call strategies which do not require full recollection of the opponent's moves *limited information strategies*.

Definition 2.3.12. A *k-tactical strategy* or *k-tactic* is a function $\sigma : M^{\leq k} \to M$ yielding a corresponding strategy $\sigma \circ \nu_k : M^{<\omega} \to M$.

A *k-Markov strategy* or *k-mark* is a function $\sigma : M^{\leq k} \times \omega \to M$ yielding a corresponding strategy $\sigma \circ \mu_k : M^{<\omega} \to M$.

k-tactics and k-marks may then only use the last k moves of the opponent, and in the latter case, also the round number.

The k is usually omitted when $k = 1$. A (1-)tactic is called a *stationary strategy* by some authors. 0-tactics are not usually interesting (such strategies would be constant functions); however, we will discuss 0-Markov strategies, called *predetermined strategies* since such a strategy only uses the round number and does not rely on knowing which moves the opponent will make. Of course, a limited information strategy σ is *winning* for a player if its corresponding strategy $\sigma \circ \nu_k$ or $\sigma \circ \mu_k$ is winning for that player.

Definition 2.3.13. If a winning k-tactical strategy exists for a player $\mathscr{A}$ in the game G, then we write $\mathscr{A} \underset{k\text{-tact}}{\uparrow} G$. If $k = 1$, then we may write $\mathscr{A} \underset{\text{tact}}{\uparrow} G$.

If a winning k-Markov strategy exists for a player $\mathscr{A}$ in the game G, then we write $\mathscr{A} \underset{k\text{-mark}}{\uparrow} G$. If $k = 1$, then we may write $\mathscr{A} \underset{\text{mark}}{\uparrow} G$; if $k = 0$, then we may write $\mathscr{A} \underset{\text{pre}}{\uparrow} G$.

The existence of a winning limited information strategy can characterize a stronger property than the property characterized by a perfect information strategy.

Definition 2.3.14. X is said to be an *α-favorable space* when $\mathscr{N} \underset{\text{tact}}{\uparrow} BM_{E,N}(X)$. X is said to be a *weakly α-favorable space* when $\mathscr{N} \uparrow BM_{E,N}(X)$.

Proposition 2.3.15. *X is α-favorable $\Rightarrow$ X is weakly α-favorable $\Rightarrow$ X is Baire*

Those arrows may not be reversed. A Bernstein subset of the real line is an example of a Baire space which is not weakly α-favorable, and Gabriel Debs constructed an example of a completely regular space for which $\mathscr{N}$ has a winning 2-tactic, but lacks a winning 1-tactic. [4]

Chapter 3

Gruenhage's Convergence and Clustering Games

We begin by investigating a game due to Gary Gruenhage, introduced in his doctoral dissertation to solve a problem due to Phil Zenor.

3.1 Definitions

Game 3.1.1. Let $Gru_{\vec{O,P}}(X, S)$ denote the *W-convergence game* with players $\mathscr{O}$, $\mathscr{P}$, for a topological space X and $S \subseteq X$.

In round n, $\mathscr{O}$ chooses an open neighborhood $O_n \supseteq S$, followed by $\mathscr{P}$ choosing a point $x_n \in \bigcap_{m \leq n} O_m$.

$\mathscr{O}$ wins the game if the points x_n converge to the set S; that is, for every open neighborhood $U \supseteq S$, $x_n \in U$ for all but finite $n < \omega$.

If $S = \{x\}$ then we write $Gru_{\vec{O,P}}(X, x)$ for short.

The "W" in the name merely refers to $\mathscr{O}$'s goal: to "win" the game. Gruenhage defined this game in his doctoral dissertation to define a class of spaces generalizing first-countability. [9]

Definition 3.1.2. The spaces X for which $\mathscr{O} \uparrow Gru_{\vec{O,P}}(X, x)$ for all $x \in X$ are called *W-spaces.*

In fact, using limited information strategies, one may characterize the first-countable spaces using this game.

Proposition 3.1.3. *X is first countable if and only if $\mathscr{O} \underset{pre}{\uparrow} Gru_{\vec{O,P}}(X, x)$ for all $x \in X$.*

Proof. The forward implication shows that all first-countable spaces are W spaces, and was proven in [9]: if $\{U_n : n < \omega\}$ is a countable base at x, let $\sigma(n) = \bigcap_{m \leq n} U_m$. σ is easily seen to be a winning predetermined strategy.

If X is not first countable at some x, let σ be a predetermined strategy for $\mathscr{O}$ in $Gru_{O,P}^{\rightarrow}(X, x)$. There exists an open neighborhood U of x which does not contain any $\bigcap_{m \leq n} \sigma(m)$ (otherwise $\{\bigcap_{m \leq n} \sigma(m) : n < \omega\}$ would be a countable base at x). Let x_n be an element of $\bigcap_{m \leq n} \sigma(m) \setminus U$ for all $n < \omega$. Then $\langle x_0, x_1, \ldots \rangle$ is a winning counter-attack to σ for $\mathscr{P}$, so $\mathscr{O}$ lacks a winning predetermined strategy. $\square$

At first glance, the difficulty of $Gru_{O,P}^{\rightarrow}(X, S)$ could be increased for $\mathscr{O}$ by only restricting the choices for $\mathscr{P}$ to be within the most recent open set played by $\mathscr{O}$, rather than all the previously played open sets.

Definition 3.1.4. Let $Gru_{O,P}^{\rightarrow,\star}(X, S)$ denote the *hard W-convergence game* which proceeds as $Gru_{O,P}^{\rightarrow}(X, S)$, except that $\mathscr{P}$ need only choose $x_n \in O_n$ rather than $x_n \in \bigcap_{m \leq n} O_m$ during each round.

This seemingly more difficult game for $\mathscr{O}$ is Gruenhage's original formulation. But with perfect information, there is no real difference for $\mathscr{O}$.

Proposition 3.1.5. *$\mathscr{O} \underset{limit}{\uparrow} Gru_{O,P}^{\rightarrow}(X, S)$ if and only if $\mathscr{O} \underset{limit}{\uparrow} Gru_{O,P}^{\rightarrow,\star}(X, S)$, where $\underset{limit}{\uparrow}$ is either $\uparrow$ or $\underset{pre}{\uparrow}$.*

Proof. The backwards implication is immediate.

For the forward implication, let σ be a winning predetermined (perfect information) strategy, and λ be the 0-Markov fog-of-war μ_0 (the identity).

We define a new predetermined (perfect information) strategy τ by

$$\tau \circ \lambda(\langle x_0, \ldots, x_{n-1} \rangle) = \bigcap_{m \leq n} \sigma \circ \lambda(\langle x_0, \ldots, x_{m-1} \rangle)$$

so that each move by $\mathscr{O}$ according to $\tau \circ \lambda$ is the intersection of $\mathscr{O}$'s previous moves. Then any attack against $\tau \circ \lambda$ is an attack against $\sigma \circ \lambda$, and since $\sigma \circ \lambda$ is a winning strategy, so is $\tau \circ \lambda$. $\square$

Put more simply, $\tau(n) = \bigcap_{m \leq n} \sigma(m)$ in the predetermined case, and $\tau(\langle x_0, \dots, x_{n-1} \rangle) = \bigcap_{m \leq n} \sigma(\langle x_0, \dots, x_{m-1} \rangle)$ in the perfect information case. The original proof would have been invalid if λ was required to be, say, the tactical fog-of-war ν_1, since the value of $\mathscr{O}$'s own round 1 move $\sigma \circ \nu_1(\langle x_0 \rangle) = \sigma(\langle x_0 \rangle)$ could not be determined from the information she has during round 2: $\nu_1(\langle x_0, x_1 \rangle) = \langle x_1 \rangle$.

Due to the equivalency of the "hard" and "normal" variations of the convergence game in the perfect information case, many authors use them interchangibly. However, it is possible to find spaces for which the games are not equivalent when considering $k+1$-tactics and $k+1$-marks, as we will soon see.

In addition to the W-convergence games, we will also investigate "clustering" analogs to both variations.

Game 3.1.6. Let $Gru_{O,P}^{\leadsto}(X,S)$ ($Gru_{O,P}^{\leadsto,\star}(X,S)$) be a variation of $Gru_{O,P}^{\rightarrow}(X,S)$ ($Gru_{O,P}^{\rightarrow,\star}(X,S)$) such that x_n need only cluster at S, that is, for every open neighborhood U of S, $x_n \in U$ for infinitely many $n < \omega$.

This variation seems to make $\mathscr{O}$'s job easier, but Gruenhage noted that the clustering game is perfect-information equivalent to the convergence game for $\mathscr{O}$. This can easily be extended for some limited information cases as well.

Proposition 3.1.7. *$\mathscr{O} \underset{limit}{\uparrow} Gru_{O,P}^{\rightarrow}(X,S)$ if and only if $\mathscr{O} \underset{limit}{\uparrow} Gru_{O,P}^{\leadsto}(X,S)$ where $\underset{limit}{\uparrow}$ is any of $\uparrow$, $\underset{pre}{\uparrow}$, $\underset{tact}{\uparrow}$, or $\underset{mark}{\uparrow}$.*

Proof. For the perfect information case we refer to [9].

In the predetermined (resp. tactical) case, suppose that σ is a winning predetermined (resp. tactical) strategy for $\mathscr{O}$ in $Gru_{O,P}^{\leadsto}(X,S)$. Let p be a legal attack against σ, and q be a subsequence of p. It's easily seen that q is also a legal attack against σ, so q clusters at S. Since every subsequence of p clusters at S, p converges to S, and σ is a winning predetermined (resp. tactical) strategy for $\mathscr{O}$ in $Gru_{\vec{O},P}(X, S)$ as well.

In the final case, note that any Markov strategy σ' for $\mathscr{O}$ may be strengthened to σ defined by $\sigma(x,n) = \bigcap_{m \leq n} \sigma'(x,m)$. So, suppose that σ is a winning Markov strategy for $\mathscr{O}$ in $Gru_{\overset{\leadsto}{O},P}(X, S)$ such that $\sigma(x,m) \supseteq \sigma(x,n)$ for all $m \leq n$.

Let p be a legal attack against σ, and q be a subsequence of p. For $m < \omega$, there exists $f(m) \geq m$ such that $q(m) = p(f(m))$. It follows that $q(0) = p(f(0)) \in \sigma(\emptyset, 0) \cap \bigcap_{m \leq f(0)} \sigma(\langle p(m) \rangle, m) \subseteq \sigma(\emptyset, 0)$ and

$$\begin{aligned} q(n+1) = p(f(n+1)) &\in \sigma(\emptyset,0) \cap \bigcap_{m<f(n+1)} \sigma(\langle p(m)\rangle, m+1) \\ &\subseteq \sigma(\emptyset,0) \cap \bigcap_{m<n+1} \sigma(\langle p(f(m))\rangle, f(m)+1) \\ &= \sigma(\emptyset,0) \cap \bigcap_{m<n+1} \sigma(\langle q(m)\rangle, f(m)+1) \\ &\subseteq \sigma(\emptyset,0) \cap \bigcap_{m<n+1} \sigma(\langle q(m)\rangle, m+1) \end{aligned}$$

so q is also a legal attack against σ. Since σ is a winning strategy, q clusters at S, and since every subsequence of p clusters at S, p must converge to S. Thus σ is also a winning Markov strategy for $\mathscr{O}$ in $Gru_{\vec{O},P}(X, S)$ as well. $\square$

Proposition 3.1.8. *For any $x \in X$ and $k < \omega$,*

- $\mathscr{O} \underset{k+1\text{-}tact}{\uparrow} Gru_{\vec{O},P}(X, x) \Leftrightarrow \mathscr{O} \underset{tact}{\uparrow} Gru_{\vec{O},P}(X, x)$
- $\mathscr{O} \underset{k+1\text{-}mark}{\uparrow} Gru_{\vec{O},P}(X, x) \Leftrightarrow \mathscr{O} \underset{mark}{\uparrow} Gru_{\vec{O},P}(X, x)$

Proof. If σ witnesses $\mathscr{O} \underset{k+1\text{-tact}}{\uparrow} Gru_{\vec{O},P}(X, x)$, let $\tau(\emptyset) = \sigma(\emptyset)$ and

$$\tau(\langle q \rangle) = \bigcap_{i<k} \sigma(\langle \underbrace{x, \ldots, x}_{k-i}, q, \underbrace{x, \ldots, x}_{i+1} \rangle)$$

Then τ is easily verified to be a winning tactic, and the proof for the second part is analogous. $\square$

Two types of questions emerge from these results.

Question 3.1.9. Does $\mathscr{O} \underset{\text{2-tact}}{\uparrow} Gru_{O,P}^{\rightsquigarrow}(X,S)$ imply $\mathscr{O} \underset{\text{2-tact}}{\uparrow} Gru_{O,P}^{\rightarrow}(X,S)$? What about for $\underset{\text{2-mark}}{\uparrow}$?

Question 3.1.10. Could $\mathscr{O} \underset{k+1\text{-tact}}{\uparrow} Gru_{O,P}^{\rightarrow}(X,S)$ actually imply $\mathscr{O} \underset{\text{tact}}{\uparrow} Gru_{O,P}^{\rightarrow}(X,S)$? What about for $Gru_{O,P}^{\rightsquigarrow}(X,S)$?

3.2 Fort spaces

In his original paper, Gruenhage suggested the one-point-compactification of a discrete space as an example of a W-space which is not first-countable.

Definition 3.2.1. A *Fort space* $\kappa^* = \kappa \cup \{\infty\}$ is defined for each cardinal κ. Its subspace κ is discrete, and the neighborhoods of ∞ are of the form $\kappa^* \setminus F$ for each $F \in [\kappa]^{<\omega}$.

Proposition 3.2.2. *$\mathscr{O} \underset{tact}{\uparrow} Gru_{O,P}^{\rightarrow}(\kappa^*, \infty)$ for all cardinals κ*

Proof. Let $\sigma(\emptyset) = \sigma(\langle\infty\rangle) = \kappa^*$ and $\sigma(\langle\alpha\rangle) = \kappa^* \setminus \{\alpha\}$. Any legal attack against the tactic σ could not repeat non-∞ points, so it must converge to ∞. □

Corollary 3.2.3. *$\mathscr{O} \uparrow Gru_{O,P}^{\rightarrow,\star}(\kappa^*, \infty)$ for all cardinals κ*

Proof. Propositions 3.1.5 and 3.2.2. □

Since it's trivial to show that $\mathscr{O} \underset{\text{pre}}{\uparrow} Gru_{O,P}^{\rightarrow}(\kappa^*, \infty)$ if and only if $\kappa \leq \omega$, this closes the question on limited information strategies for $Gru_{O,P}^{\rightarrow}(\kappa^*, \infty)$. However, limited information analysis of the harder $Gru_{O,P}^{\rightarrow,\star}(\kappa^*, \infty)$ is more interesting.

Peter Nyikos noted Proposition 3.2.2 and the following in [22].

Theorem 3.2.4. *$\mathscr{O} \underset{mark}{\not\uparrow} Gru_{O,P}^{\rightarrow,\star}(\omega_1^*, \infty)$.*

This actually can be generalized to any k-Markov strategy with just a little more bookkeeping.

Theorem 3.2.5. $\mathscr{O} \underset{k\text{-}mark}{\not\uparrow} Gru^{\rightarrow,\star}_{O,P}(\omega_1^*, \infty)$.

Proof. Let σ be a k-mark for $\mathscr{O}$. Since the set

$$D_\sigma = \bigcap_{n<\omega, s\in\omega^{\leq k}} \sigma(s,n)$$

is co-countable, we may choose $\alpha_\sigma \in D_\sigma \cap \omega_1$. Thus, we may choose $n_0 < n_1 < \cdots < \omega$ such that

$$\langle n_0, \ldots, n_{k-1}, \alpha_\sigma, n_k, \ldots, n_{2k-1}, \alpha_\sigma, \ldots \rangle$$

is a legal counterattack, which fails to converge to ∞ since α_σ is repeated infinitely often. $\square$

However, while the clustering and convergence variants are equivalent for Markov strategies in the "normal" version of the W game, they are *not* equivalent in the "hard" version.

Theorem 3.2.6. $\mathscr{O} \underset{mark}{\uparrow} Gru^{\rightsquigarrow,\star}_{O,P}(\omega_1^*, \infty)$.

Proof. For each $\alpha < \omega_1$ let $A_\alpha = \langle A_\alpha(0), A_\alpha(1), \ldots \rangle$ be a countable sequence of finite sets such that $A_\alpha(n) \subset A_\alpha(n+1)$ and $\bigcup_{n<\omega} A_\alpha(n) = \alpha + 1$.

We define the Markov strategy σ by setting

$$\sigma(\emptyset, 0) = \sigma(\langle \infty \rangle, n) = \omega_1^*$$

and for all $\alpha < \omega_1$ setting

$$\sigma(\langle \alpha \rangle, n) = \omega_1^* \setminus A_\alpha(n)$$

Note that for any $\alpha_0 < \cdots < \alpha_{k-1}$, there is some $n < \omega$ such that $\{\alpha_0, \ldots, \alpha_{k-1}\} \subseteq A_{\alpha_i}(n)$ for all $i < k$. Thus for any legal attack p against σ, the range of p cannot be finite. Since the range of p is infinite, every open neighborhood of ∞ contains infinitely many points of p, so p clusters at ∞. $\square$

However, knowledge of the round number is critical. First recall the following "closing-up lemma" (see [15]):

Lemma 3.2.7. *For any function $f : \omega_1^{<\omega} \to \omega_1$, the following set is closed and unbounded in ω_1:*

$$\{\alpha < \omega_1 : s \in \alpha^{<\omega} \Rightarrow f(s) \subseteq \alpha\}$$

Theorem 3.2.8. $\mathscr{O} \underset{k\text{-}tact}{\not\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$.

Proof. Let σ be a k-tactic for $\mathscr{O}$ in $Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$. By the closing-up lemma, the set

$$C_\sigma = \{\alpha < \omega_1 : s \in \alpha^{\leq k} \Rightarrow \omega_1^* \setminus \sigma(s) \subseteq \alpha\}$$

is closed and unbounded. Let $a_\sigma : \omega_1 \to C_\sigma$ be an order isomorphism.

Choose $n_0 < \cdots < n_{k-1} < \omega$ such that for each $i < k$:

$$a_\sigma(n_i) \in \sigma(\langle a_\sigma(n_0), \ldots, a_\sigma(n_{i-1}), a_\sigma(\omega + i), \ldots, a_\sigma(\omega + k - 1)\rangle)$$

Finally, observe that the legal counterattack

$$\langle a_\sigma(n_0), \ldots, a_\sigma(n_{k-1}), a_\sigma(\omega), \ldots, a_\sigma(\omega+k-1), a_\sigma(n_0), \ldots, a_\sigma(n_{k-1}), a_\sigma(\omega), \ldots, a_\sigma(\omega+k-1), \ldots\rangle$$

has a range outside the open neighborhood

$$\omega_1^* \setminus \{a_\sigma(n_0), \ldots, a_\sigma(n_{k-1}), a_\sigma(\omega), \ldots, a_\sigma(\omega + k - 1)\}$$

of ∞. Thus σ is not a winning k-tactic. $\square$

Once the discrete space is larger than ω_1, knowing the round number is not sufficient to construct a limited information strategy, due to a similar argument.

Theorem 3.2.9. $\mathscr{O} \underset{k\text{-}mark}{\not\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_2^*, \infty)$.

Proof. Let σ be a k-mark for $\mathscr{O}$ in $Gru_{O,P}^{\rightsquigarrow}(\omega_2^*, \infty)$. By the closing-up lemma, the set

$$C_\sigma = \{\alpha < \omega_2 : s \in \alpha^{<\omega} \Rightarrow \omega_2^* \setminus \sigma \circ \mu_k(s) \subseteq \alpha\}$$

is closed and unbounded. (Recall that μ_k is the k-Markov fog-of-war which turns perfect information into the last k moves and the round number.) Let $a_\sigma : \omega_2 \to C_\sigma$ be an order isomorphism.

Choose $\beta_0 < \cdots < \beta_{k-1} < \omega_1$ such that for each $i < k$:

$$a_\sigma(\beta_i) \in \bigcap_{n<\omega} \sigma(\langle a_\sigma(\beta_0), \ldots, a_\sigma(\beta_{i-1}), a_\sigma(\omega_1 + i), \ldots, a_\sigma(\omega_1 + k - 1)\rangle, n)$$

Finally, observe that the legal counterattack

$$\langle a_\sigma(\beta_0), \ldots, a_\sigma(\beta_{k-1}), a_\sigma(\omega_1), \ldots, a_\sigma(\omega_1+k-1), a_\sigma(\beta_0), \ldots, a_\sigma(\beta_{k-1}), a_\sigma(\omega_1), \ldots, a_\sigma(\omega_1+k-1), \ldots\rangle$$

has a range outside the open neighborhood

$$\omega_2^* \setminus \{a_\sigma(\beta_0), \ldots, a_\sigma(\beta_{k-1}), a_\sigma(\omega_1), \ldots, a_\sigma(\omega_1 + k - 1)\}$$

of ∞. Thus σ is not a winning k-mark. $\square$

3.3 Sigma-products

Knowing the status of W-games in simpler spaces yields insight to larger spaces.

Proposition 3.3.1. *Suppose $S \subseteq Y \subseteq X$, $\underset{limit}{\uparrow}$ is any of $\uparrow$, $\underset{k\text{-}tact}{\uparrow}$, or $\underset{k\text{-}mark}{\uparrow}$, and $G(X, S)$ is any of $Gru_{O,P}^{\rightarrow}(X, S)$, $Gru_{O,P}^{\rightarrow,\star}(X, S)$, $Gru_{O,P}^{\rightsquigarrow}(X, S)$, or $Gru_{O,P}^{\rightsquigarrow,\star}(X, S)$.*

Then $\mathscr{O} \underset{limit}{\uparrow} G(X, S)$ implies $\mathscr{O} \underset{limit}{\uparrow} G(Y, S)$.

Proof. Simply intersect the output of the winning strategy in $G(X, S)$ with Y. $\square$

A natural superspace of a Fort space is the sigma-product of a discrete cardinal.

Definition 3.3.2. Let

$$\sum_{\alpha<\kappa}^{y} X_\alpha = \left\{ x \in \prod_{\alpha<\kappa} X_\alpha : |\{x(\alpha) \neq y(\alpha) : \alpha < \kappa\}| \leq \omega \right\}$$

denote the *sigma-product* of $\{X_\alpha : \alpha < \kappa\}$ with base point $y \in \prod_{\alpha<\kappa} X_\alpha$. The topology on this space is given by the subspace topology.

Note also the following syntactic sugar: $\sum_{\alpha<\kappa} X_\alpha = \sum_{\alpha<\kappa}^{\vec{0}} X_\alpha$, $\sum^y X^\kappa = \sum_{\alpha<\kappa}^{y} X$, and $\sum X^\kappa = \sum_{\alpha<\kappa}^{\vec{0}} X$.

Proposition 3.3.3. *κ^* is homeomorphic to the space*

$$\left\{ x \in \sum 2^\kappa : |\{\alpha < \kappa : x(\alpha) = 1\}| \leq 1 \right\}$$

Proof. Map $\alpha < \kappa$ to x_α such that

$$x_\alpha(\beta) = \begin{cases} 0 : \beta \neq \alpha \\ 1 : \beta = \alpha \end{cases}$$

and map ∞ to the zero vector $\vec{0}$. $\square$

Corollary 3.3.4. *$\mathscr{O} \underset{k\text{-tact}}{\not\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}\left(\Sigma\mathbb{R}^{\omega_1}, \vec{0}\right)$, $\mathscr{O} \underset{k\text{-mark}}{\not\uparrow} Gru_{O,P}^{\rightarrow,\star}\left(\Sigma\mathbb{R}^{\omega_1}, \vec{0}\right)$, and $\mathscr{O} \underset{k\text{-mark}}{\not\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}\left(\Sigma\mathbb{R}^{\omega_2}, \vec{0}\right)$.*

While this closes the question on tactics and marks for high dimensional sigma- (and Tychonoff-) products of the real line, there is another type of limited information strategy to investigate.

Definition 3.3.5. For a game $G = \langle M, W \rangle$ and *coding strategy* or *code* $\sigma : M^{\leq 2} \to M$, the *$\sigma$-coding fog-of-war* $\gamma_\sigma : M^{<\omega} \to M^{\leq 2}$ is the function defined such that

$$\gamma_\sigma(\emptyset) = \emptyset$$

and

$$\gamma_\sigma(s^\frown\langle x\rangle) = \langle \sigma \circ \gamma_\sigma(s), x\rangle$$

For a coding strategy σ, its coresponding strategy is $\sigma \circ \gamma_\sigma$. For a game G, if $\sigma \circ \gamma_\sigma$ is a winning strategy for $\mathscr{A}$, then σ is a winning coding strategy and we write $\mathscr{A} \underset{\text{code}}{\uparrow} G$.

Intuitively, a σ-coding fog-of-war converts perfect information of the game into the last moves of both the player and her opponent, so a player has a winning coding strategy when she only needs to know the move of her opponent and her own last move. The term "coding" comes from the fact that a player may encode information about the history of the game into her own moves, and use this encoded information in later rounds.

As an example, the existence of a winning coding strategy is necessary for the second player to force a win the Banach-Mazur game.

Theorem 3.3.6. *$\mathscr{N} \uparrow BM_{E,N}(X)$ if and only if $\mathscr{N} \underset{code}{\uparrow} BM_{E,N}(X)$ [4] [7].*

We are interested in whether the same holds for W games.

The hard and normal versions of the W games are all equivalent with regards to coding strategies since $\mathscr{O}$ may always ensure her new move is a subset of her previous move. For Fort spaces, the question is immediately closed.

Proposition 3.3.7. $\mathscr{O} \underset{code}{\uparrow} Gru_{\vec{O},P}(\kappa^*, \infty)$.

Proof. Let $\sigma(\emptyset) = \kappa^*$, $\sigma(\langle U, \alpha\rangle) = U \setminus \{\alpha\}$ for $\alpha < \kappa$, and $\sigma(\langle U, \infty\rangle) = U$. $\mathscr{P}$ cannot legally repeat non-∞ points of the set, so her points converge to ∞. $\square$

This trick does not simply extend to the $\Sigma\mathbb{R}^\kappa$ case, however. An open set may only restrict finitely many coordinates of the product, and a point in $\Sigma\mathbb{R}^\kappa$ may have countably infinite non-zero coordinates. Thus, information about the previous non-zero coordinates cannot be directly encoded into the open set.

Circumventing this takes a bit of extra machinery. We proceed by defining a simpler infinite game for each cardinal κ.

Game 3.3.8. Let $PtFin_{F,C}(\kappa)$ denote the *point-finite game* with players $\mathscr{F}$, $\mathscr{C}$ for each cardinal κ.

In round n, $\mathscr{F}$ chooses $F_n \in [\kappa]^{<\omega}$, followed by $\mathscr{C}$ choosing $C_n \in [\kappa \setminus \bigcup_{m\leq n} F_m]^{\leq\omega}$.

$\mathscr{F}$ wins the game if the collection $\{C_n : n < \omega\}$ is a point-finite cover of its union $\bigcup_{n<\omega} C_n$, that is, each point in $\bigcup_{n<\omega} C_n$ is in C_n only for finitely many $n < \omega$.

This game has a strong resemblance to a game defined by Scheepers in [26] in relationship to the Banach-Mazur game and studied specifically with finite and countable sets in [27]. Scheeper's game and the results pertaining to it aren't of use here; however, they will be referenced in a later chapter in studying a different topological game.

This game of finite and countable sets is directly applicable to the W games played upon the sigma-product of real lines.

Lemma 3.3.9. *$\mathscr{F} \underset{code}{\uparrow} PtFin_{F,C}(\kappa)$ implies $\mathscr{O} \underset{code}{\uparrow} Gru_{O,P}^{\rightarrow}\left(\Sigma\mathbb{R}^\kappa, \vec{0}\right)$.*

Proof. Let σ be a winning coding strategy for $\mathscr{F}$ in $PtFin_{F,C}(\kappa)$ such that $\sigma(\emptyset) \supset \emptyset$ and $\sigma(F, C) \supset F$.

For $F \in [\kappa]^{<\omega}$ and $\epsilon > 0$ let $U(F, \epsilon)$ be the basic open set in $\mathbb{R}^\kappa$ such that each projection is of the form

$$\pi_\alpha(U(F,\epsilon)) = \begin{cases} (-\epsilon, \epsilon) & \alpha \in F \\ \mathbb{R} & \alpha \notin F \end{cases}$$

Note that $F \supset \emptyset$ and ϵ are uniquely identifiable given $U(F, \epsilon) \cap \Sigma\mathbb{R}^\kappa$.

For each point $x \in \Sigma\mathbb{R}^\kappa$ and $\epsilon > 0$, let $C_\epsilon(x) \in [\kappa]^{\leq\omega}$ such that $\alpha \in C_\epsilon(\alpha)$ if and only if $|x(\alpha)| \geq \epsilon$.

We define the coding strategy τ for $\mathscr{O}$ in $Gru_{O,P}^{\rightarrow}\left(\Sigma\mathbb{R}^\kappa, \vec{0}\right)$ as follows:

$$\tau(\emptyset) = U(\sigma(\emptyset), 1) \cap \Sigma\mathbb{R}^\kappa$$

$$\tau(\langle U(F,\epsilon) \cap \Sigma\mathbb{R}^\kappa, x\rangle) = U\left(\sigma(\langle F, C_\epsilon(x)\rangle), \frac{\epsilon}{2}\right) \cap \Sigma\mathbb{R}^\kappa$$

Let $\langle a_0, a_1, a_2, \ldots \rangle$ be a legal attack by $\mathscr{P}$ against τ. It then follows that

$$b = \langle C_1(a_0), C_{1/2}(a_1), C_{1/4}(a_2), \ldots \rangle$$

is a legal attack by $\mathscr{C}$ against σ. Since σ is a winning strategy, each ordinal in $\bigcup_{n<\omega} C_{2^{-n}}(a_n)$ is in $C_{2^{-n}}(a_n)$ only for finitely many $n < \omega$. Thus for every coordinate $\alpha < \kappa$ it follows that there exists some $n_\alpha < \omega$ such that $a_n(\alpha) \leq 2^{-n}$ for $n \geq n_\alpha$. We conclude $a_n \to \vec{0}$, showing that τ is a winning strategy. $\square$

This lemma simplifies our notation in proving the main result. Intuitively, we aim to show that when κ has cofinality ω, $\mathscr{F}$ can split up the game among ω-many smaller cardinals converging to κ, and when κ has a larger cofinality, $\mathscr{F}$ may exploit the fact that $\mathscr{C}$ may only play within some ordinal smaller than κ.

Theorem 3.3.10. *$\mathscr{F} \underset{code}{\uparrow} PtFin_{F,C}(\kappa)$ for all cardinals κ.*

Proof. For each cardinal κ and $\lambda < \kappa$, assume σ_λ is a winning strategy for $\mathscr{F}$ in $PtFin_{F,C}(\lambda)$ such that $\sigma_\lambda(\emptyset) \supset \emptyset$ and $\sigma_\lambda(\langle F, C \rangle) \supset F$.

In the case that $\mathrm{cf}(\kappa) = \omega$, let $\langle \kappa_0, \kappa_1, \ldots \rangle$ be an increasing sequence of cardinals limiting to κ. Then we define the coding strategy σ for $\mathscr{F}$ as follows:

$$\sigma(\emptyset) = \sigma_{\kappa_0}(\emptyset)$$

$$\sigma(\langle F, C \rangle) = \bigcup_{n \leq |F|} \sigma_{\kappa_n}(\langle F \cap \kappa_n, C \cap \kappa_n \rangle)$$

Then for each legal attack $a = \langle a(0), a(1), \ldots \rangle$ by $\mathscr{C}$ against σ and each $n < \omega$, the sequence $b_n = \langle a(n) \cap \kappa_n, a(n+1) \cap \kappa_n, \ldots \rangle$ is a legal attack by $\mathscr{C}$ against the winning coding strategy σ_{κ_n}. It follows then that $\{a(i+n) \cap \kappa_n : i < \omega\}$ is a point-finite cover of $\bigcup_{i<\omega} a(i+n) \cap \kappa_n$. We conclude that $\{a(i) : i < \omega\}$ is a point-finite cover of $\bigcup_{i<\omega} a(i)$ and σ is a winning strategy.

It remains to consider the case where $\mathrm{cf}(\kappa) > \omega$. Note that now, for each $C \in [\kappa]^{\leq\omega}$, C is bounded above in κ. So we define the coding strategy σ for $\mathscr{F}$ as follows:

$$\sigma(\emptyset) = \emptyset$$

$$\sigma(\langle F, C\rangle) = \{\sup(C)\} \cup \bigcup_{\alpha\in F} \sigma_{\alpha+1}(\langle F \cap (\alpha+1), C \cap (\alpha+1)\rangle)$$

Then for each legal attack $a = \langle a(0), a(1), \ldots\rangle$ by $\mathscr{C}$ against σ and each $n < \omega$, the sequence $b_n = \langle a(n) \cap (\sup(a(n)) + 1), a(n+1) \cap (\sup(a(n)) + 1), \ldots\rangle$ is a legal attack by $\mathscr{C}$ against the winning coding strategy $\sigma_{\sup(a(n))+1}$. It follows then that $\{a(i+n) \cap (\sup(a(n)) + 1) : i < \omega\}$ is a point-finite cover of $\bigcup_{i<\omega} a(i+n) \cap (\sup(a(n)) + 1)$. We conclude that $\{a(i) : i < \omega\}$ is a point-finite cover of $\bigcup_{i<\omega} a(i)$ and σ is a winning strategy. $\square$

Corollary 3.3.11. *$\mathscr{O} \underset{code}{\uparrow} Gru_{\vec{O},P}\left(\Sigma\mathbb{R}^\kappa, \vec{0}\right)$ for all cardinals κ.*

This leaves open the question analogous to Theorem 3.3.6.

Question 3.3.12. Does $\mathscr{O} \uparrow Gru_{\vec{O},P}(X, x)$ imply $\mathscr{O} \underset{\text{code}}{\uparrow} Gru_{\vec{O},P}(X, x)$?

Chapter 4

Bell's Convergence Games

A very recent development related to Gruenhage's convergence and clustering games comes from Jocelyn Bell.

4.1 A Game on Uniform Spaces

Definition 4.1.1. A *uniformity* on a set X is a filter $\mathbb{D}$ of subsets of X^2, known as $\mathbb{D}$-*entourages*, such that $\bigcap \mathbb{D} = \Delta = \{\langle x, x\rangle : x \in X\}$ and, for each entourage $D \in \mathbb{D}$:

- There exists $E \in \mathbb{D}$ such that

$$E \circ E = \{\langle x, z\rangle : \exists y \in X\,(\langle x, y\rangle, \langle y, z\rangle \in E)\} \subseteq D$$

- $D^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in D\} \in \mathbb{D}$

A set X with a uniformity is called a *uniform space.* As $\mathbb{D}$ is a filter, we also have that $D \cap E \in \mathbb{D}$ for all $E \in \mathbb{D}$, and $F \in \mathbb{D}$ for all $F \supseteq D$. Note that if $\mathbb{E}$ is a filter base satisfying the conditions for a uniformity, then we say $\mathbb{E}$ is a *uniformity base* which may be extended to a uniformity by closing it under the superset operation.

A uniformity is a generalization of a metric.

Definition 4.1.2. For a $\mathbb{D}$-entourage D and a point $x \in X$, the *D-ball around x* is the set $D[x] = \{y : \langle x, y\rangle \in D\}$.

Definition 4.1.3. If d is a metric for the space X, then the *metric uniformity* for X is generated by the uniformity base $\{D_\epsilon : \epsilon > 0\}$ where $D_\epsilon = \{\langle x, y\rangle : d(x, y) < \epsilon\}$.

Bell first introduced what she called the "proximal game" in [2]. This game was used to prove that the Σ-product of spaces for which $\mathscr{D}$ has a winning strategy is collectionwise normal, as well as to show the collectionwise normality of certain uniform box products.

Game 4.1.4. Let $Bell^{\text{uni}}_{D,P}(X, \mathbb{D})$ denote the *Bell uniform space game* with players $\mathscr{D}$, $\mathscr{P}$ which proceeds as follows for a uniform space X with uniformity $\mathbb{D}$. In round 0, $\mathscr{D}$ chooses a $\mathbb{D}$-entourage D_0, followed by $\mathscr{P}$ choosing a point $p_0 \in X$. In round $n+1$, $\mathscr{D}$ chooses a $\mathbb{D}$-entourage D_{n+1}, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in D_n[p_n]$.

$\mathscr{D}$ wins in the case that either $\langle p_0, p_1, \ldots \rangle$ converges with respect to the uniformity $\mathbb{D}$ (there exists $N < \omega$ such that $p(n) \in D[x]$ for $n \geq N$), or $\bigcap_{n<\omega} D_n[p_n] = \emptyset$.

4.2 Topologizing Bell's Game

Like metrics, uniformities induce natural topological structures.

Definition 4.2.1. The *uniform topology* induced by a uniformity $\mathbb{D}$ on X declares U open if for each $x \in U$, there exists $D \in \mathbb{D}$ such that $D[x] \subseteq U$.

Theorem 4.2.2. *The uniform topology induced by a uniformity $\mathbb{D}$ on X is the coarsest topology such that for each $x \in X$ and $D \in \mathbb{D}$, $D[x]$ is a neighborhood (not necessarily open) of x.*

Proof. Let $\mathcal{T}$ be a topology such that $D[x]$ is a neighborhood of x for each $x \in X$, and let U be open in the uniform topology. For each $x \in U$, there exists $D_x \in \mathbb{D}$ such that $D_x[x] \subseteq U$, and since $D_x[x]$ is a neighborhood of x, there is $U_x \in \mathcal{T}$ such that $x \in U_x \subseteq D_x[x]$. Thus $\mathcal{T}$ contains the uniform topology. □

The uniform topology for a metric uniformity is simply the usual metric topology, and $D_\epsilon[x]$ is the usual metric ϵ-ball around x.

Definition 4.2.3. A topological space X is *uniformizable* if there exists a uniformity which induces the given topology on X.

Bell's game may thus be used to characterize the structure of uniformizable spaces.

Definition 4.2.4. A uniformizable space X is *proximal* if there exists a compatible uniformity $\mathbb{D}$ such that $\mathscr{D} \uparrow Bell^{\text{uni}}_{D,P}(X, \mathbb{D})$.

However, since the focus of this manuscript is on topological spaces, it will be useful to recharacterize the proximal property in terms of a topological game. This may be attained by considering a few known results on uniform spaces. See e.g. [31] for proofs.

Theorem 4.2.5. *Every uniform topology is $T_{3\frac{1}{2}}$, and every $T_{3\frac{1}{2}}$ topology is uniformizable.*

Theorem 4.2.6. *The union of all uniformities which induce a particular topology is itself a uniformity and induces the same topology.*

Definition 4.2.7. The *universal uniformity* for a uniformizable topology is the uniformity finer than all uniformities which induce the given topology.

Definition 4.2.8. For a uniformizable space X, a *universal entourage D* is a $\mathbb{D}$-entourage of the universal uniformity $\mathbb{D}$.

Theorem 4.2.9. *For every uniformizable space, if D is a neighborhood of the diagonal Δ such that there exist neighborhoods D_n of Δ with $D \supseteq D_0$ and $D_n \supseteq D_{n+1} \circ D_{n+1}$, then D is a universal entourage.*

Theorem 4.2.10. *Every neighborhood of the diagonal is a universal entourage for paracompact uniformizable spaces.*

Definition 4.2.11. An *open symmetric $\mathbb{D}$-entourage D* is a $\mathbb{D}$-entourage which is open in the product topology induced by $\mathbb{D}$ and where $D = D^{-1}$.

Theorem 4.2.12. *For every $\mathbb{D}$-entourage D, there exists an open symmetric $\mathbb{D}$-entourage $U \subseteq D$.*

We will simply use the word *entourage* to refer to open symmetric universal entourages. Note that if D is an entourage, then $D[x]$ is an open neighborhood of x.

Definition 4.2.13. For every entourage D, let $\frac{1}{2^n}D$ denote entourages for $n < \omega$ such that $\frac{1}{1}D = D$ and $\frac{1}{2^{n+1}}D \circ \frac{1}{2^{n+1}}D \subseteq \frac{1}{2^n}D$.

The proof of the following is routine.

Proposition 4.2.14. *If X is a uniformizable space, then for all $x \in X$ and entourages D:*

$$x \in \frac{1}{2}D[y] \text{ and } y \in \frac{1}{2}D[z] \Rightarrow x \in D[z]$$

and

$$\frac{1}{2}D[x] \subseteq \overline{\frac{1}{2}D[x]} \subseteq D[x]$$

The natural adaptation of Bell's game simply replaces the $\mathbb{D}$-entourages of the uniform space with the (open symmetric universal) entourages of a uniformizable space.

Game 4.2.15. Let $Bell_{D,P}^{\rightarrow,\star}(X)$ denote the *hard Bell convergence game* with players $\mathscr{D}$, $\mathscr{P}$ which proceeds as follows for a uniformizable space X. In round 0, $\mathscr{D}$ chooses an entourage D_0, followed by $\mathscr{P}$ choosing a point $p_0 \in X$. In round $n+1$, $\mathscr{D}$ chooses an entourage D_{n+1}, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in D_n[p_n]$.

$\mathscr{D}$ wins in the case that either $\langle p_0, p_1, \dots \rangle$ converges in X, or $\bigcap_{n<\omega} D_n[p_n] = \emptyset$. $\mathscr{P}$ wins otherwise.

Like $Gru_{O,P}^{\rightarrow,\star}(X, x)$, $\mathscr{D}$ may choose to intersect her plays with her previous plays given perfect information. Since this cannot be guaranteed with limited information, a simpler variation for $\mathscr{D}$ may also be considered, which will be the focus of this chapter.

Game 4.2.16. Let $Bell_{D,P}^{\rightarrow}(X)$ denote the *Bell convergence game* with players $\mathscr{D}$, $\mathscr{P}$ which proceeds analogously to $Bell_{D,P}^{\rightarrow,\star}(X)$, except for the following. Let $E_n = \bigcap_{m \leq n} D_n$, where D_n is the entourage played by $\mathscr{D}$ in round n. Then $\mathscr{P}$ must ensure that $p_{n+1} \in E_n[p_n]$, and $\mathscr{D}$ wins when either $\langle p_0, p_1, \dots \rangle$ converges in X or $\bigcap_{n<\omega} E_n[p_n] = \emptyset$.

These games are all essentially equivalent with respect to perfect information for $\mathscr{D}$.

Theorem 4.2.17. *$\mathscr{D} \uparrow Bell^{\rightarrow,\star}_{D,P}(X)$ if and only if $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$ if and only if X is proximal.*

Proof. If $\mathscr{D} \uparrow Bell^{\rightarrow,\star}_{D,P}(X)$, then we immediately see that $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$. If σ is a winning strategy for $\mathscr{D}$ in $Bell^{\rightarrow}_{D,P}(X)$, then τ defined by $\tau(s) = \bigcap_{t \leq s} \sigma(t)$ is easily seen to be a winning strategy for $\mathscr{D}$ in $Bell^{\rightarrow,\star}_{D,P}(X)$.

If $\mathscr{D} \uparrow Bell^{\rightarrow,\star}_{D,P}(X)$, then $\mathscr{D} \uparrow Bell^{\mathrm{uni}}_{D,P}(X, \mathbb{D})$ where $\mathbb{D}$ is the universal uniformity, showing X is proximal. Finally, if X is proximal, then there exists a winning strategy σ for $Bell^{\mathrm{uni}}_{D,P}(X, \mathbb{D})$ where $\mathbb{D}$ is a uniformity inducing the topology on X. Then a winning strategy for $\mathscr{D}$ in $Bell^{\rightarrow,\star}_{D,P}(X)$ may be constructed by converting every $\mathbb{D}$-entourage into a smaller open symmetric universal entourage. $\square$

Bell showed the following results in [2].

Theorem 4.2.18. *If X is metrizable, then $\mathscr{D} \underset{pre}{\uparrow} Bell^{\rightarrow}_{D,P}(X)$.*

Theorem 4.2.19. *If $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$, then $\mathscr{O} \uparrow Gru^{\rightarrow}_{O,P}(X, x)$ for all $x \in X$. Thus proximal spaces are W spaces.*

Theorem 4.2.20. *Proximal spaces are collectionwise normal.*

Theorem 4.2.21. *Every closed subspace of a proximal space is proximal.*

Theorem 4.2.22. *Every Σ-product of proximal spaces is proximal.*

Since the empty intersection winning condition of her game can obfuscate things, Bell suggested that $\mathscr{D}$ "absolutely wins" her uniform space game if the points played by $\mathscr{P}$ always converge, inspiring the following game.

Definition 4.2.23. Let $Bell^{\rightarrow}_{\vec{D},P}(X)$ denote the *absolute Bell convergence game* which proceeds analogously to $Bell^{\rightarrow}_{D,P}(X)$, except that $\mathscr{D}$ must always ensure that $\langle p_0, p_1, \ldots \rangle$ converges in X in order to win.

Definition 4.2.24. A uniformizable space X is *absolutely proximal* if $\mathscr{D} \uparrow Bell_{\vec{D},P}(X)$.

4.3 Characterizing Corson compactness

Gruenhage and the author showed in [3] that Bell's uniform space game yields an internal characterization of Corson compact spaces, answering a question of Nyikos in [23]. We include a proof of this result using the topological version of the game instead.

Definition 4.3.1. A compact space is *Corson compact* if it is homeomorphic to a compact subset of a Σ-product of real lines.

Nyikos observed the following in [23].

Proposition 4.3.2. *Corson compact spaces are proximal.*

Proof. The real line is metrizable and thus proximal, and closed subsets of Σ-products of proximal spaces are proximal. $\square$

A result of Gruenhage [10] gives a useful game characterization of Corson compactness.

Theorem 4.3.3. *A compact space is Corson compact if and only if* $\mathscr{O} \uparrow Gru_{\vec{O},P}(X^2, \Delta)$.

Thus our desired result will follow if we may show that a winning strategy in $Bell_{\vec{D},P}(X)$ may be used to construct a winning strategy in $Gru_{\vec{O},P}(X^2, \Delta)$. However, due to the secondary winning condition for $\mathscr{D}$ in $Bell_{\vec{D},P}(X)$, it will be more convenient if we may use a winning strategy for $\mathscr{D}$ in $Bell_{\vec{D},P}(X)$ instead.

Definition 4.3.4. A uniformizable space X is *uniformly locally compact* if there exists an entourage D such that $\overline{D[x]}$ is compact for all x.

Of course, any compact uniformizable space is uniformly locally compact. However, note that a space may be locally compact without being uniformly locally compact.

Theorem 4.3.5. *A uniformizable space is uniformly locally compact if and only if it is locally compact and paracompact. [16]*

As an example, ω_1 with the linear order topology is locally compact, but not paracompact or uniformly locally compact.

Theorem 4.3.6. *If X is a uniformly locally compact space, then $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$ if and only if $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$.*

Proof. Let L be an entourage such that $\overline{L[x]}$ is compact for all x. Let σ be a strategy witnessing $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$. Without loss of generality, we may assume $\sigma(t) \subseteq L$ and that $t \supseteq s$ implies $\sigma(t) \subseteq \frac{1}{4}\sigma(s)$. Note then that $\overline{\sigma(t)[x]} \subseteq \overline{L[x]}$ is compact.

Let $\tau(t) = \frac{1}{2}\sigma(t)$. If p attacks τ in $Bell^{\rightarrow}_{D,P}(X)$, then

$$p(n+1) \in \tau(p \restriction n)[p(n)] = \frac{1}{2}\sigma(p \restriction n)[p(n)]$$

and for

$$x \in \overline{\sigma(p \restriction (n+1))[p(n+1)]} \subseteq \overline{\frac{1}{4}\sigma(p \restriction n)[p(n+1)]} \subseteq \frac{1}{2}\sigma(p \restriction n)[p(n+1)]$$

we can conclude $x \in \sigma(p \restriction n)[p(n)]$. Thus

$$\sigma(p \restriction (n+1))[p(n+1)] \subseteq \overline{\sigma(p \restriction (n+1))[p(n+1)]} \subseteq \sigma(p \restriction n)[p(n)]$$

Finally, note that since τ yields subsets of σ, p also attacks the winning strategy σ in $Bell^{\rightarrow}_{D,P}(X)$, but since the intersection of a descending chain of nonempty compact sets is nonempty, we have

$$\bigcap_{n<\omega} \sigma(p \restriction n)[p(n)] = \bigcap_{n<\omega} \overline{\sigma(p \restriction n)[p(n)]} \neq \emptyset$$

We conclude that p converges. $\square$

Lemma 4.3.7. *If $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$, then $\mathscr{O} \uparrow Gru^{\rightarrow}_{O,P}(X,H)$ for all compact H in X.*

Proof. Let σ be a winning strategy for $\mathscr{D}$ in $Bell^{\rightarrow}_{D,P}(X)$ game such that $p \supsetneq q$ implies $\sigma(p) \subseteq \frac{1}{4}\sigma(q)$. For any sequence t, let $o_t = \{\langle n, t(2n+1)\rangle : 2n+1 \in \mathrm{dom}(t)\}$ be the subsequence of t consisting of its odd-indexed terms. We proceed by constructing a winning strategy for $\mathscr{O}$ in $Gru^{\rightsquigarrow}_{O,P}(X,H)$. Since $\mathscr{O} \uparrow Gru^{\rightsquigarrow}_{O,P}(X,H)$ if and only if $\mathscr{O} \uparrow Gru^{\rightarrow}_{O,P}(X,H)$, the result will follow.

We begin by defining a tree $T(\emptyset)$, during which we will define a number $m_\emptyset < \omega$ and points $h_{\emptyset,i}, h_{\emptyset,i,j}$ for $i,j < m_\emptyset$ which yield an open set

$$\bigcup_{i,j<m_\emptyset} \frac{1}{4}\sigma(\langle h_{\emptyset,i}\rangle)[h_{\emptyset,i,j}] = \bigcup_{\emptyset^\frown\langle i,h_{\emptyset,i},j\rangle\in\max(T(\emptyset))} \frac{1}{4}\sigma(o_\emptyset{}^\frown\langle h_{\emptyset,i}\rangle)[h_{\emptyset,i,j}]$$

containing H. $\mathscr{O}$ will use this as the inital move in her winning strategy for $Gru^{\rightsquigarrow}_{O,P}(X,H)$.

- Choose $m_\emptyset < \omega$, $h_{\emptyset,i} \in H$ for $i < m_\emptyset$, and $h_{\emptyset,i,j} \in H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$ for $i,j < m_\emptyset$ such that

  $$\left\{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}] : i < m_\emptyset\right\}$$

  is a cover for H and such that for each $i < m_\emptyset$

  $$\left\{\frac{1}{4}\sigma(\langle h_{\emptyset,i}\rangle)[h_{\emptyset,i,j}] : j < m_\emptyset\right\}$$

  is a cover for $H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$.

- Let $\langle i, h_{\emptyset,i}, j\rangle$ and its initial segments be in $T(\emptyset)$ for $i,j < m_\emptyset$.

It follows that

$$\left\{\frac{1}{4}\sigma(o_\emptyset{}^\frown\langle h_{\emptyset,i}\rangle)[h_{\emptyset,i,j}] : \emptyset^\frown\langle i, h_{\emptyset,i}, j\rangle \in \max(T(a))\right\}$$

covers H.

Now suppose that a is a partial attack by $\mathscr{P}$ in $Gru^{\rightsquigarrow}_{O,P}(X,H)$ for which we have defined a tree $T(a)$. We will define $T(a^\frown\langle x\rangle) \supseteq T(a)$ for each $x \in X$, and whenever

$x \in \frac{1}{4}\sigma(o_s{}^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$ for some $s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(a))$, we will set $t = s^\frown\langle i, h_{s,i}, j, x\rangle$ and define a number $m_t < \omega$ and points $h_{t,k}, h_{t,k,l}$ for $k, l < m_t$ to yield an open set

$$\bigcup_{t^\frown\langle k, h_{t,k}, l\rangle \in \max(T(a^\frown\langle x\rangle))} \frac{1}{4}\sigma(o_t{}^\frown\langle h_{t,k}\rangle)[h_{t,k,l}]$$

which will be the next open neighborhood of H in $\mathscr{O}$'s winning strategy for $Gru_{O,P}^{\rightsquigarrow}(X, H)$.

- We will extend the nodes $s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(a))$ such that $x \in \frac{1}{4}\sigma(o_s{}^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$. For each such $s^\frown\langle i, h_{s,i}, j\rangle$, set $t = s^\frown\langle i, h_{s,i}, j, x\rangle$.

- Note that whenever $o_s{}^\frown\langle h_{s,i}\rangle$ is a legal partial attack against σ, then

$$x \in \frac{1}{4}\sigma(o_s{}^\frown\langle h_{s,i}\rangle)[h_{s,i,j}] \subseteq \frac{1}{4}\sigma(o_s)[h_{s,i,j}]$$

  and

$$h_{s,i,j} \in \overline{\frac{1}{4}\sigma(o_s)[h_{s,i}]} \subseteq \frac{1}{2}\sigma(o_s)[h_{s,i}]$$

  implies

$$x \in \sigma(o_s)[h_{s,i}]$$

  and thus $o_s{}^\frown\langle h_{s,i}, x\rangle = o_t$ is also a legal partial attack against σ.

- Choose $m_t < \omega$, $h_{t,k} \in H \cap \overline{\frac{1}{4}\sigma(o_s{}^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]}$ for $k < m_t$, and $h_{t,k,l} \in H \cap \overline{\frac{1}{4}\sigma(o_t)[h_{t,k}]}$ for $k, l < m_t$ such that

$$\left\{\frac{1}{4}\sigma(o_t)[h_{t,k}] : k < m_t\right\}$$

  is a cover for $H \cap \overline{\frac{1}{4}\sigma(o_s{}^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]}$ and such that for each $k < m_t$

$$\left\{\frac{1}{4}\sigma(o_t{}^\frown\langle h_{t,k}\rangle)[h_{t,k,l}] : l < m_t\right\}$$

  is a cover for $H \cap \overline{\frac{1}{4}\sigma(o_t)[h_{t,k}]}$.

- Note that whenever o_t is a legal partial attack against σ, then

$$h_{t,k} \in \overline{\frac{1}{4}\sigma(o_s{}^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]} \subseteq \frac{1}{2}\sigma(o_s{}^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

  and

$$x \in \frac{1}{4}\sigma(o_s{}^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

  implies

$$h_{t,k} \in \sigma(o_s{}^\frown\langle h_{s,i}\rangle)[x]$$

  and thus $o_t{}^\frown\langle h_{t,k}\rangle$ is a legal partial attack against σ.

- Include all initial segments of $t^\frown\langle k, h_{t,k}, l\rangle$ in $T(a^\frown\langle x\rangle)$ for $k, l < m_t$.

This completes the construction of $T(a^\frown\langle x\rangle)$. Note that since

$$\left\{\frac{1}{4}\sigma(o_s{}^\frown\langle h_{s,i}\rangle)[h_{s,i,j}] : s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(a))\right\}$$

covers H, then since

$$\left\{\frac{1}{4}\sigma(o_t{}^\frown\langle h_{t,k}\rangle)[h_{t,k,l}] : s^\frown\langle i, h_{s,i}, j, x, k, h_{t,k}, l\rangle \in \max(T(a^\frown\langle x\rangle)) \setminus \max(T(a))\right\}$$

covers $H \cap \frac{1}{4}\sigma(o_s{}^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$, we have that

$$\left\{\frac{1}{4}\sigma(o_t{}^\frown\langle h_{t,k}\rangle)[h_{t,k,l}] : t^\frown\langle k, h_{t,k}, l\rangle \in \max(T(a^\frown\langle x\rangle))\right\}$$

covers H.

We define a strategy τ for $\mathscr{O}$ in $Gru^{\leadsto}_{O,P}(X, H)$ such that:

$$\tau(a) = \bigcup_{s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(a))} \frac{1}{4}\sigma(o_s{}^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

If p is a legal attack by $\mathscr{P}$ against τ, then let $T(p) = \bigcup_{n<\omega} T(p \restriction n)$. We note $T(p)$ is an infinite tree (for each $n < \omega$, a node of $T(p \restriction n)$ was extended) with finite levels:

- $\emptyset$ has exactly $m_\emptyset$ successors $\langle i \rangle$.
- $s^\frown\langle i \rangle$ has exactly one successor $s^\frown\langle i, h_{s,i} \rangle$
- $s^\frown\langle i, h_{s,i} \rangle$ has exactly m_s successors $s^\frown\langle i, h_{s,i}, j \rangle$
- $s^\frown\langle i, h_{s,i}, j \rangle$ has either no successors or exactly one successor $s^\frown\langle i, h_{s,i}, j, x \rangle$
- $t = s^\frown\langle i, h_{s,i}, j, x \rangle$ has exactly m_t successors $t^\frown\langle k \rangle$

Hence $T(p)$ has an infinite branch

$$q' = \langle i_0, h_0, j_0, x_0, i_1, h_1, j_1, x_1, \ldots \rangle$$

Let $q = o_{q'} = \langle h_0, x_0, h_1, x_1, \ldots \rangle$. Note that by the construction of $T(p)$, q is a legal attack on the winning strategy σ in $Bell_{\vec{D},P}(X)$, so it must converge. Since every other term of q is in H, it must converge to H. Then since o_q is a subsequence of p, p must cluster at H. $\square$

Theorem 4.3.8. *A compact space is Corson compact if and only if $\mathscr{D} \uparrow Bell_{\vec{D},P}(X)$.*

Proof. If $\mathscr{D} \uparrow Bell_{\vec{D},P}(X)$, then $\mathscr{D} \uparrow Bell_{\vec{D},P}(X^2)$. As X^2 is (uniformly locally) compact, $\mathscr{D} \uparrow Bell_{\vec{D},P}(X^2)$. Thus $\mathscr{O} \uparrow Gru_{\vec{O},P}(X^2, \Delta)$, showing that X is Corson compact. $\square$

4.4 Limited information results

One may generalize many of the results originally shown by Bell [2] and Nyikos [23] by considering limited information strategies.

Theorem 4.4.1. *Let $k < \omega$. For all $x \in X$:*

- $\mathscr{D} \underset{2k\text{-}tact}{\uparrow} Bell_{\vec{D},P}(X) \Rightarrow \mathscr{O} \underset{k\text{-}tact}{\uparrow} Gru_{\vec{O},P}(X, x)$

- $\mathscr{D} \underset{2k\text{-}mark}{\uparrow} Bell^{\rightarrow}_{D,P}(X) \Rightarrow \mathscr{O} \underset{k\text{-}mark}{\uparrow} Gru^{\rightarrow}_{O,P}(X, x)$
- $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X) \Rightarrow \mathscr{O} \uparrow Gru^{\rightarrow}_{O,P}(X, x)$

Proof. The perfect-information result was originally shown by Bell. Let L_k represent either the k-tactical fog-of-war ν_k, the k-Markov fog-of-war μ_k, or the identity id.

Let $\sigma \circ L_{2k}$ be a winning strategy for $\mathscr{D}$ in $Bell^{\rightarrow}_{D,P}(X)$. We define the strategy $\tau \circ L_k$ for $\mathscr{O}$ in $Gru^{\rightarrow}_{O,P}(X, x)$ such that

$$\tau \circ L_k(t) = \sigma \circ L_{2k}\Big(\langle x, t(0), \ldots, x, t(|t|-1)\rangle\Big)[x] \cap \sigma \circ L_{2k}\Big(\langle x, t(0), \ldots, x, t(|t|-1), x\rangle\Big)[x]$$

Let p attack τ such that $p(n) \in \bigcap_{m \leq n} \tau \circ L_k(t)$. Consider the attack q against the winning strategy $\sigma \circ L_{2k}$ such that $q(2n) = x$ and $q(2n+1) = p(n)$. Let $D_n = \sigma \circ L_{2k}(q \restriction n)$ and $E_n = \bigcap_{m \leq n} D_m$.

Certainly, $x \in E_{2n}[x] = E_{2n}[q(2n)]$ for any $n < \omega$. Note also for any $n < \omega$ that

$$p(n) \in \bigcap_{m \leq n} \tau \circ L_k(p \restriction m)$$

$$= \bigcap_{m \leq n} (\sigma \circ L_{2k}(\langle x, p(0), \ldots, x, p(m-1)\rangle)[x] \cap \sigma \circ L_{2k}(\langle x, p(0), \ldots, x, p(m-1), x\rangle)[x])$$

$$= \bigcap_{m \leq n} (D_{2m}[x] \cap D_{2m+1}[x]) = \bigcap_{m \leq 2n+1} D_m[x] = E_{2n+1}[x]$$

so by the symmetry of E_{2n+1}, $x \in E_{2n+1}[p(n)] = E_{2n+1}[q(2n+1)]$. Also, $q(2n+1) = p(n) \in E_{2n}[x] = E_{2n}[q(2n)]$ and $p(n) \in E_{2n+1}[x] \Rightarrow q(2n+2) = x \in E_{2n+1}[p(n)] = E_{2n+1}[q(2n+1)]$, making q a legal attack.

Then as $x \in \bigcap_{n<\omega} E_n[q(n)] \neq \emptyset$, and σ is a winning strategy, the attack q converges. Since $q(2n) = x$, q must converge to x. Thus its subsequence p converges to x, and $\tau \circ L_k$ is a winning strategy for $\mathscr{O}$ in $Gru^{\rightarrow}_{O,P}(X, x)$. $\square$

Theorem 4.4.2. *Let $X \cup \{\infty\}$ be a T_1 space with points in X isolated (and therefore a uniformizable space), and $k < \omega$. Then*

- $\mathscr{O} \underset{k\text{-}tact}{\uparrow} Gru_{\vec{O},P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \underset{k\text{-}tact}{\uparrow} Bell_{\vec{D},P}(X \cup \{\infty\})$
- $\mathscr{O} \underset{k\text{-}mark}{\uparrow} Gru_{\vec{O},P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \underset{k\text{-}mark}{\uparrow} Bell_{\vec{D},P}(X \cup \{\infty\})$
- $\mathscr{O} \uparrow Gru_{\vec{O},P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \uparrow Bell_{\vec{D},P}(X \cup \{\infty\})$

Proof. The perfect-information result is due to Nyikos. Let L_k represent either the k-tactical fog-of-war ν_k, the k-Markov fog-of-war μ_k, or the identity id.

The right-to-left implications have already been shown. For any open neighborhood U of ∞, $D(U) = \Delta \cup U^2$ is an entourage of X.

Let $\sigma \circ L_k$ be a winning strategy for $Gru_{\vec{O},P}(X \cap \{\infty\}, \infty)$. We then define the strategy $\tau \circ L_k$ such that

$$\tau \circ L_k(t) = D(\sigma \circ L_k(t))$$

Let p attack τ such that $\bigcap_{n<\omega} \tau \circ L_k(p \upharpoonright n)[p(n)] \neq \emptyset$.

If $\infty \in \bigcap_{n<\omega} \tau \circ L_k(p \upharpoonright n)[p(n)]$, it follows that p is a legal attack on $\sigma \circ L_k$. Since $\sigma \circ L_k$ is a winning strategy, it follows that $p \to \infty$.

Otherwise, $\infty \notin \tau \circ L_k(p \upharpoonright N)[p(N)]$ for some $N < \omega$, and then $\tau \circ L_k(p \upharpoonright N)[p(N)] = \{p(N)\}$ implies $p \to p(N)$.

Thus $\tau \circ L_k$ is a winning strategy for $Gru_{\vec{O},P}(X \cup \{\infty\}, \infty)$. □

Corollary 4.4.3. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete, and $k < \omega$. Then*

- $\mathscr{D} \underset{k+1\text{-}tact}{\uparrow} Bell_{\vec{D},P}(X \cup \{\infty\}) \Leftrightarrow \mathscr{D} \underset{tact}{\uparrow} Bell_{\vec{D},P}(X \cup \{\infty\})$
- $\mathscr{D} \underset{k+1\text{-}mark}{\uparrow} Bell_{\vec{D},P}(X \cup \{\infty\}) \Leftrightarrow \mathscr{D} \underset{mark}{\uparrow} Bell_{\vec{D},P}(X \cup \{\infty\})$

Proof. The equivalencies hold for $Gru_{\vec{O},P}(X, x)$. □

A close result may be obtained for arbitrary uniformizable spaces.

Proposition 4.4.4. *For any uniformizable space X and $k < \omega$,*

- $\mathscr{D} \underset{k+2\text{-}tact}{\uparrow} Bell_{D,P}^{\rightarrow}(X) \Leftrightarrow \mathscr{D} \underset{2\text{-}tact}{\uparrow} Bell_{D,P}^{\rightarrow}(X)$
- $\mathscr{D} \underset{k+2\text{-}mark}{\uparrow} Bell_{D,P}^{\rightarrow}(X) \Leftrightarrow \mathscr{D} \underset{2\text{-}mark}{\uparrow} Bell_{D,P}^{\rightarrow}(X)$

Proof. If σ is a winning $k+2$-tactic, then define τ by

$$\tau(s) = \bigcap_{t \in \mathrm{range}(s)^{<k+2}} \sigma(t)$$

If σ is a winning $k+2$-mark, then define τ by

$$\tau(s,n) = \bigcap_{\substack{t \in \mathrm{range}(s)^{<k+2} \\ m \leq (k+2)n}} \sigma(t,m)$$

In either case, the proof that τ is a winning limited information strategy is routine. $\square$

Theorem 4.4.5. *Let X be a uniformizable space and H be a closed subset of X. If $k < \omega$, then*

- $\mathscr{D} \underset{k\text{-}tact}{\uparrow} Bell_{D,P}^{\rightarrow}(X) \Rightarrow \mathscr{D} \underset{k\text{-}tact}{\uparrow} Bell_{D,P}^{\rightarrow}(H)$
- $\mathscr{D} \underset{k\text{-}mark}{\uparrow} Bell_{D,P}^{\rightarrow}(X) \Rightarrow \mathscr{D} \underset{k\text{-}mark}{\uparrow} Bell_{D,P}^{\rightarrow}(H)$
- $\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(X) \Rightarrow \mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(H)$

Proof. The perfect-information result was originally shown by Bell. Let L_k represent either the k-tactical fog-of-war ν_k, the k-Markov fog-of-war μ_k, or the identity *id*.

Let $\sigma \circ L_k$ be a winning strategy for $\mathscr{D}$ in $Bell_{D,P}^{\rightarrow}(X)$. We define the strategy $\tau \circ L_k$ for $\mathscr{D}$ in $Bell_{D,P}^{\rightarrow}(H)$ as follows:

$$\tau \circ L(p \restriction n) = \sigma \circ L(p \restriction n) \cap H^2$$

Let p attack $\tau \circ L_k$. p also attacks the winning strategy $\sigma \circ L_k$, so either

$$\bigcap_{n<\omega} \left(\bigcap_{m \leq n} \tau \circ L(p \restriction n) \right) [p(n)] \subseteq \bigcap_{n<\omega} \left(\bigcap_{m \leq n} \sigma \circ L(p \restriction n) \right) [p(n)] = \emptyset$$

or $p : \omega \to H$ converges in X, and thus converges in H as H is closed. $\square$

Bell showed the following to obtain a result of Mary Ellen Rudin [24] and S. P. Gulko [12] as a corollary.

Theorem 4.4.6. *A Σ-product of proximal spaces is proximal.*

Corollary 4.4.7. *A Σ-product of metrizable spaces is collectionwise normal.*

A sketch of the proof: $\mathscr{D}$ may use the winning strategies for the proximal spaces coordinate-wise to ensure that either every coordinate converges, or one coordinate intersects to the empty set. But in order to do this throughout the entire game, perfect information is used in a non-trival way to remember all coordinates for which $\mathscr{P}$ played a point with non-zero value in that coordinate. But for countable products, this memory may be replaced with knowledge of the round number.

Lemma 4.4.8. *Let D_α be an entourage of X_α for $\alpha < \kappa$. Then $P_\alpha(D_\alpha) = \{\langle x, y\rangle : \langle x(\alpha), y(\alpha)\rangle \in D_\alpha\}$ is an entourage of $\prod_{\alpha<\kappa} X_\alpha$.*

Proof. Let $D_\alpha \supseteq U_0$ and $U_{n+1} \circ U_{n+1} \subseteq U_n$ for neighborhoods U_n of the diagonal of X_α. Then note that $P_\alpha(U_n)$ is a neighborhood of the diagonal of $\prod_{\alpha<\kappa} X_\alpha$. Thus as $P_\alpha(D_\alpha) \supseteq P_\alpha(U_0)$ and $P_\alpha(U_{n+1}) \circ P_\alpha(U_{n+1}) \subseteq P_\alpha(U_n)$, we conclude $P_\alpha(D_\alpha)$ is an entourage. $\square$

Theorem 4.4.9. *If $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell^{\rightarrow}_{D,P}(X_i)$ for $i < \omega$, then $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell^{\rightarrow}_{D,P}\left(\prod_{i<\omega} X_i\right)$.*

Proof. Let σ_i be a winning k-mark for $\mathscr{D}$ in $Bell^{\rightarrow}_{D,P}(X_i)$. For $s \in \left(\prod_{i<\omega} X_i\right)^{\leq\omega}$, let $s_i \in X_i^{\leq\omega}$ such that $s(j)(i) = s_i(j)$ for $j \in \mathrm{dom}(s)$. Recall that ν_k removes all but the last k elements of a finite sequence, and for an countably infinite sequence p, let $p \restriction^k n = \nu_k(p \restriction n)$.

Define the k-mark τ for $\mathscr{D}$ in $Bell^{\rightarrow}_{D,P}\left(\prod_{i<\omega} X_i\right)$ by

$$\tau(s, n) = \bigcap_{i\leq n} P_i(\sigma_i(s_i, n))$$

and let $p \in \left(\prod_{i<\omega} X_i\right)^\omega$ be a legal attack against τ, so the following must hold for $m, n < \omega$:

$$p(n+1) \in \left(\bigcap_{j\leq n} \tau(p \restriction^k j, j)\right)[p(n)] = \left(\bigcap_{i\leq j\leq n} P_i(\sigma_i(p_i \restriction^k j, j))\right)[p(n)]$$

$$p_m(m+n+1) \in \left(\bigcap_{j\leq n} \sigma_m(p_m \restriction^k (m+j), m+j)\right)[p_m(m+n)]$$

For $m < \omega$, attack σ_m with $q_m \in X_m^\omega$ defined by $q_m(n) = p_m(m+n)$. Note that

$$q_m(n+1) = p_m(m+n+1) \in \left(\bigcap_{j\leq n} \sigma_m(p_m \restriction^k (m+j), m+j)\right)[p_m(m+n)]$$

$$\subseteq \left(\bigcap_{j\leq n} \sigma_m(q_m \restriction^k (j), j)\right)[q_m(n)]$$

so q_m is a legal attack.

If for some $m < \omega$,

$$\emptyset = \left(\bigcap_{n<\omega} \sigma_m(q_m \restriction^k n, n)\right)[q_m(n)] \supseteq \left(\bigcap_{n<\omega} \sigma_m(p_m \restriction^k (m+n), n)\right)[p_m(m+n)]$$

$$\supseteq \left(\bigcap_{n<\omega} \sigma_m(p_m \restriction^k (m+n), m+n)\right)[p_m(m+n)]$$

$$= \left(\bigcap_{m\leq n<\omega} \sigma_m(p_m \restriction^k n, n)\right)[p_m(n)]$$

then

$$\bigcap_{n<\omega}\left(\bigcap_{j\leq n} \tau(p \restriction^k j, j)\right)[p(n)] = \bigcap_{n<\omega}\left(\bigcap_{i\leq j\leq n} P_i(\sigma_i(p_m \restriction^k j, j))\right)[p(n)]$$

$$\subseteq \bigcap_{m\leq n<\omega}\left(\bigcap_{m\leq j\leq n} P_m(\sigma_m(p_m \restriction^k j, j))\right)[p(n)] \subseteq \bigcap_{m\leq n<\omega}\left(\bigcap_{m\leq n} P_m(\sigma_m(p_m \restriction^k n, n))\right)[p(n)]$$

$$= \bigcap_{m\leq n<\omega}\left(\bigcap_{m\leq n} P_m(\sigma_m(p_m \restriction^k n, n)[p_m(n)])\right) = \bigcap_{m\leq n<\omega} P_m(\sigma_m(p_m \restriction^k n, n)[p_m(n)])$$

$$= P_m\left(\bigcap_{m\leq n<\omega}\sigma_m(p_m \restriction^k n, n)[p_m(n)]\right) = P_m(\emptyset) = \emptyset$$

Otherwise for all $m < \omega$, q_m converges to some $x_m \in X_m$, then p_m converges to $x_m \in X_m$ and thus p converges. This shows that τ is a winning k-mark. $\square$

Chapter 5

Gruenhage's Locally Finite Games

A variation of $Gru_{\vec{O},P}(X,x)$, also due to Gruenhage, may be used to characterize various covering properties, particularly for locally compact spaces. All spaces are assumed to be T_1 in this chapter.

5.1 Characterizations using $Gru_{K,P}(X)$, $Gru_{K,L}(X)$

Game 5.1.1. Let $Gru_{K,P}(X)$ denote the *Gruenhage compact/point game* with players $\mathscr{K}$, $\mathscr{P}$. During round n, $\mathscr{K}$ chooses a compact subset K_n of X, followed by $\mathscr{P}$ choosing a point $p_n \in X$ such that $p_n \notin \bigcup_{m \leq n} K_m$.

$\mathscr{K}$ wins the game if the collection $\{\{p_n\} : n < \omega\}$ is locally finite in the space, and $\mathscr{P}$ wins otherwise.

This game is often formulated by requiring that the collection $\{\{p_n\} : n < \omega\}$ be discrete. With the knowledge of at least the latest move of the opponent, $\mathscr{K}$ may guarantee that if $\{\{p_n\} : n < \omega\}$ is locally finite then it is also discrete, since she may require $p_n \in K_{n+1}$. Thus this formulation is essentially equivalent to the usual formulation for all existing applications.

We may relate this game to $Gru_{\vec{O},P}(X,x)$ as follows:

Theorem 5.1.2. *If X is locally compact, then for $k < 2$:*

- $\mathscr{K} \uparrow Gru_{K,P}(X)$ *if and only if* $\mathscr{O} \uparrow Gru_{\vec{O},P}(X^*, \infty)$.
- $\mathscr{K} \underset{k\text{-}mark}{\uparrow} Gru_{K,P}(X)$ *if and only if* $\mathscr{O} \underset{k\text{-}mark}{\uparrow} Gru_{\vec{O},P}(X^*, \infty)$.
- $\mathscr{K} \underset{k\text{-}tact}{\uparrow} Gru_{K,P}(X)$ *if and only if* $\mathscr{O} \underset{k\text{-}tact}{\uparrow} Gru_{\vec{O},P}(X^*, \infty)$.

Proof. Let L be any of ν_0, ν_1, μ_0, μ_1, or the identity. For any sequence s of points in X^*, let s' be the subsequence of non-∞ points in s.

If $\sigma \circ L$ is a winning strategy for $\mathscr{K}$ in $Gru_{K,P}(X)$, let $\tau \circ L$ be a strategy for $\mathscr{O}$ in $Gru_{\vec{O},P}(X^*, \infty)$ such that $\tau(L(s)) = X^* \setminus \sigma(L(s)')$. Then for any legal attack p against τ, p' is a legal attack against σ. (The proof of this claim uses the fact that $k < 2$.) If p' is a finite sequence, then p converges to ∞. Otherwise, the set $\{\{p'(n)\} : n < \omega\}$ is locally finite in X, so $\{p'(n) : n < \omega\}$ is a closed discrete subset of X. Then for every neighborhood U of ∞, $X \setminus U$ is contained in a compact set, so it cannot contain a closed discrete subset. Thus p' and p converge to ∞.

If $\sigma \circ L$ is a winning strategy for $\mathscr{O}$ in $Gru_{\vec{O},P}(X^*, \infty)$, let $\tau \circ L$ be a strategy for $\mathscr{K}$ in $Gru_{K,P}(X)$ such that $\tau(L(s)) = X \setminus \sigma(L(s))$. Then for any legal attack p against τ, p is a legal attack against σ, so the sequence p converges to ∞. For any point $x \in X$ distinct from the $p(n)$, we may choose a neighborhood U_x of x in X missing every point in $p(n)$. For every $n < \omega$, we may choose a neighborhood $U_{p(n)}$ of $p(n)$ in X which misses every distinct $p(m)$. Thus $\{\{p(n)\} : n < \omega\}$ is a locally finite collection. $\square$

The reason why $Gru_{\vec{O},P}(X^*, \infty)$ and $Gru_{K,P}(X)$ are not completely equivalent is due to the fact that $\mathscr{P}$ may hide information from $\mathscr{O}$ by playing ∞. These moves cannot trivially be ignored using a k-limited information strategy for $k \geq 2$, since $\mathscr{P}$ may ensure that only one "useful" move may be seen by $\mathscr{K}$ at a time by playing k ∞'s between each non-∞.

Applications of limited information strategies for $\mathscr{K}$ are already known; see [11].

Definition 5.1.3. A space is *metacompact* if for every open cover $\mathcal{U}$ there exists a point-finite open refinement $\mathcal{V}$ of $\mathcal{U}$ also covering the space.

Theorem 5.1.4. *The following are equivalent for a locally compact space X:*

- *X is metacompact*
- $\mathscr{K} \underset{tact}{\uparrow} Gru_{K,P}(X)$.

Definition 5.1.5. A space is *σ-metacompact* if for every open cover $\mathcal{U}$ there exist point-finite open refinements $\mathcal{V}_n$ of $\mathcal{U}$ such that $\bigcup_{n<\omega} \mathcal{V}_n$ also covers the space.

Theorem 5.1.6. *The following are equivalent for a locally compact space X:*

- *X is σ-metacompact*
- $\mathscr{K} \underset{mark}{\uparrow} Gru_{K,P}(X)$.

In addition, it's trivial to show the following.

Proposition 5.1.7. *The following are equivalent for any space X:*

- *X is compact*
- $\mathscr{K} \underset{0\text{-}tact}{\uparrow} Gru_{K,P}(X)$.

Proof. A 0-tactic is seeded with zero information about the moves of the opponent or the round number, so it must be a constant function valued at X. □

A similar game may be considered by allowing the second player to choose compact sets rather than points, which provided in [11] a game-theoretic characterization of paracompactness for locally compact spaces.

Game 5.1.8. Let $Gru_{K,L}(X)$ denote the *Gruenhage compact/compact game* with players $\mathscr{K}$, $\mathscr{L}$. This game proceeds analogously to $Gru_{K,P}(X)$, except the second player $\mathscr{L}$ chooses compact sets L_n missing $\bigcup_{m\leq n} K_n$, and $\mathscr{K}$ wins if the collection $\{L_n : n < \omega\}$ is locally finite.

As above, this formulation is equivalent to requiring $\{L_n : n < \omega\}$ be discrete when considering strategies for $\mathscr{K}$ which use at least the latest move of $\mathscr{L}$.

Definition 5.1.9. A space is *paracompact* if for every open cover $\mathcal{U}$ there exists a locally-finite open refinement $\mathcal{V}$ of $\mathcal{U}$ also covering the space.

Theorem 5.1.10. *The following are equivalent for a locally compact space X:*

- *X is paracompact*
- *$\mathscr{K} \uparrow Gru_{K,L}(X)$.*

5.2 Locally compact spaces and predetermined strategies

As mentioned above, adding knowledge of the round number to a tactic changes the characterization from metacompact to σ-metacompact. In fact, the analogous result holds for 0-tactics to 0-marks, known as predetermined strategies since they rely only on the round number and not the moves of the opponent.

Theorem 5.2.1. *If X is a locally compact Lindelöf space, then $\mathscr{K} \underset{pre}{\uparrow} G_{K,L}(X)$.*

Proof. For each $x \in X$, let U_x be an open neighborhood of x with $\overline{U_x}$ compact. Then as X is Lindelöf, choose $x_n \in X$ for $n < \omega$ such that $\{U_{x_n} : n < \omega\}$ covers X. Define the predetermined strategy σ for $\mathscr{K}$ by $\sigma(n) = \overline{U_{x_n}}$.

Let $L : \omega \to \mathcal{K}(X)$ legally attack σ, so $L(n) \cap \bigcup_{m \leq n} \sigma(m) = \emptyset$. For each $x \in X$, choose $n < \omega$ with $x \in U_{x_n}$. Then U_{x_n} is a neighborhood of x which intersects finitely many $L(n)$, so $\{L(n) : n < \omega\}$ is locally finite. □

Definition 5.2.2. A space X is *hemicompact* if there exist compact sets K_n for $n < \omega$ such that every compact set in X is a subset of some K_n.

Theorem 5.2.3. *If $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(X)$, then X is hemicompact.*

Proof. Let σ be a winning predetermined strategy for $\mathscr{K}$ in $Gru_{K,P}(X)$. If $C \in \mathcal{K}(X)$ is compact, then for each $x \in C$ let U_x be an open neighborhood of x which intersects finitely many $\sigma(n)$. Choose $x_i \in C$ for $i < n < \omega$ such that $\{U_{x_i} : i < n\}$ covers C. Then $\bigcup_{i<n} U_{x_i}$ contains C and intersects finitely many $\sigma(n)$, and thus $\{\bigcup_{m \leq n} \sigma(m) : n < \omega\}$ witnesses hemicompactness. □

Corollary 5.2.4. *The following are equivalent for any locally compact space X:*

- *X is Lindelöf.*
- *X is σ-compact.*
- *X is hemicompact.*
- *$\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(X)$.*
- *$\mathscr{K} \underset{pre}{\uparrow} Gru_{K,L}(X)$.*

5.3 Compactly generated spaces and predetermined strategies

Definition 5.3.1. A space X is *compactly generated* if a set is closed if and only if its intersection with every compact set is closed. Such spaces are also known as *k-spaces.*

All locally compact spaces are k-spaces. As will be shown, the games $Gru_{K,P}(X)$, $Gru_{K,L}(X)$ are equivalent for $\mathscr{K}$'s predetermined strategies in Hausdorff k-spaces.

Definition 5.3.2. A space X is a k_ω-space if there exist compact sets K_n for $n < \omega$ such that a set is closed if and only if its intersection with every K_n is closed.

Theorem 5.3.3. *If X is a k_ω-space, then $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,L}(X)$.*

Proof. Let K_n witness that X is a k_ω-space. Define the predetermined strategy σ for $\mathscr{K}$ by $\sigma(n) = K_n$.

Let $L : \omega \to \mathcal{K}(X)$ be a legal attack against σ, and let $L_{\omega\setminus n} = \bigcup_{n\leq m<\omega} L(m)$. Then as

$$L_{\omega\setminus n} \cap K_p = \bigcup_{n\leq m<p} L(m) \cap \sigma(p)$$

is compact for each $p < \omega$, $L_{\omega\setminus n}$ is closed.

For each $x \in X$, $x \in \sigma(p)$ for some p, so $x \in X \setminus L_{\omega\setminus p}$ which misses all but finitely many $L(n)$, showing that $\{L(n) : n < \omega\}$ is locally finite and σ is a winning predetermined strategy. $\square$

The following result was observed in [5]; a proof is provided for convenience.

Proposition 5.3.4. *Hemicompact k-spaces are k_ω-spaces.*

Proof. Let K_n for $n < \omega$ witness hemicompactness. If $C \cap K_n$ is closed for each $n < \omega$, then let K be any compact set. Since $K \subseteq K_n$ for some $n < \omega$, $C \cap K$ is closed, and therefore C is closed. $\square$

As we've already seen that $\mathscr{K} \underset{\text{pre}}{\uparrow} Gru_{K,P}(X)$ implies hemicompactness:

Corollary 5.3.5. *The following are equivalent for any k-space X:*

- *X is k_ω.*
- *X is hemicompact.*
- *$\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(X)$.*
- *$\mathscr{K} \underset{pre}{\uparrow} Gru_{K,L}(X)$.*

5.4 Non-equivalence of $Gru_{K,P}(X)$, $Gru_{K,L}(X)$

For k-spaces, it has been shown that $Gru_{K,P}(X)$ and $Gru_{K,L}(X)$ are equivalent with respect to $\mathscr{K}$'s winning predetermined strategies. Looking at a subspace of the Stone-Cech compactification $\beta\omega$ of ω reveals an example for which the predetermined strategies are not equivalent.

Definition 5.4.1. An *ultrafilter* on a cardinal κ is a maximal filter of non-empty subsets of κ. For each $\alpha \in \kappa$, the ultrafilter $\mathcal{F}_\alpha$ containing all supersets of $\{\alpha\}$ is called a *principal ultrafilter*. All ultrafilters not of this form are called *free ultrafilters*.

Definition 5.4.2. The *Stone-Cech compactification* of a cardinal κ is the space $\beta\kappa$ consisting of all ultrafilters on κ, with open sets of the form $U_S = \{\mathcal{F} \in \beta\kappa : S \in \mathcal{F}\}$ for $S \subseteq \kappa$.

From these definitions it is easily verified that principal ultrafilters are isolated, so κ with the discrete topology may be viewed as a dense open subspace of $\beta\kappa$. $\beta\kappa$ is also compact, so $\mathscr{K} \underset{\text{0-tact}}{\uparrow} Gru_{K,L}(\beta\kappa)$; of greater interest is the subspace of $\beta\omega$ consisting of all principal ultrafilters and a single free ultrafilter $\mathcal{F}$, denoted $\omega \cup \{\mathcal{F}\}$.

Lemma 5.4.3. *All compact subsets of $\omega \cup \{\mathcal{F}\} \subset \beta\omega$ are finite. In particular, the difference of compact sets in $\omega \cup \{\mathcal{F}\}$ is compact.*

Proof. Let $I = \{n_i : i < \omega\} \cup \{\mathcal{F}\}$ be infinite. Then $\{U_{\omega \setminus \{n_i : i \geq j\}} : j < \omega\}$ is an open cover of $I \cup \{\mathcal{F}\}$ with no finite subcover. $\square$

Theorem 5.4.4. $\mathscr{K} \underset{pre}{\not\uparrow} Gru_{K,L}(\omega \cup \{\mathcal{F}\})$ *for any free ultrafilter $\mathcal{F}$.*

Proof. Let σ be a predetermined strategy for $\mathscr{K}$, and define the legal counter-attack $H : \omega \to \mathcal{K}(X)$ by $H(n) = (n \cup \sigma(n+1)) \setminus \sigma(n)$. Then for any neighborhood U_S of $\mathcal{F}$, S is infinite, and since $\bigcup_{n<\omega} H(n) \supseteq \omega \setminus \sigma(0)$, U_S meets infinitely many of the finite $H(n)$. Thus σ is not a winning predetermined strategy. $\square$

Theorem 5.4.5. *There exists a free ultrafilter $\mathcal{F}$ such that* $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(\omega \cup \{\mathcal{F}\})$.

Proof. Let $\mathcal{F}$ be any free ultrafilter, and define the predetermined strategy σ by $\sigma(n) = n^2 \cup \{\mathcal{F}\}$.

Consider the set of all legal attacks $A \subseteq \omega^\omega$ by $\mathscr{P}$ against σ. For $\{f_i : i \leq m\} \in [A]^{<\omega}$ and $m < n < \omega$, each f_i maps only n points into n^2, so $\bigcup_{i \leq m} \operatorname{range}(f_i)$ is coinfinite. Then $\mathcal{G}' = \{\omega \setminus \operatorname{range}(f) : f \in A\}$ is contained in a free ultrafilter $\mathcal{G}$, and if $\mathcal{F} = \mathcal{G}$, then σ is a winning predetermined strategy. $\square$

It is not possible to prove in ZFC that $\mathscr{K} \underset{\text{pre}}{\uparrow} Gru_{K,P}(\omega \cup \{\mathcal{F}\})$ for arbitrary free ultrafilters.

Definition 5.4.6. A *selective ultrafilter* $\mathcal{S}$ is a free ultrafilter with the property that for every partition $\{B_n : n < \omega\}$ of nonempty subsets of ω such that $B_n \notin \mathcal{S}$ for all n, there exists $A \in \mathcal{S}$ such that $|A \cap B_n| = 1$ for all n.

Theorem 5.4.7. *CH implies the existence of a selective ultrafilter. [25]*

Theorem 5.4.8. *If $\mathcal{S}$ is a selective ultrafilter, then $\mathscr{K} \underset{pre}{\not\uparrow} Gru_{K,P}(\omega \cup \{\mathcal{S}\})$.*

Proof. Let σ be a predetermined strategy for $\mathscr{K}$ such that $\sigma(n) \supset \bigcup_{m<n} \sigma(m)$. Then define $B_n = \omega \cap (\sigma(n+1) \setminus \sigma(n))$. Since B_n is always nonempty finite, $B_n \notin \mathcal{F}$ and there exists $A \in \mathcal{S}$ such that $|A \cap B_n| = 1$.

Define the legal counter-attack $p : \omega \to \omega \cup \{\mathcal{S}\}$ by $p(n) \in A \cap B_n = A \cap (\sigma(n+1) \setminus \sigma(n))$. Since $A = (A \cap \sigma(0)) \cup \{p(n) : n < \omega\}$, $\{p(n) : n < \omega\} \in \mathcal{S}$. Therefore, every neighborhood of $\mathcal{F}$ intersects infinitely many of the $p(n)$, and p defeats the predetermined strategy σ. $\square$

Of particular note is that the author knows of no examples of a non-k-space such that $K \underset{\text{pre}}{\uparrow} Gru_{K,P}(X)$.

Question 5.4.9. Does $K \underset{\text{pre}}{\uparrow} Gru_{K,P}(X)$ imply X is a k-space?

5.5 Tactics and marks for $Gru_{K,P}(X)$

While $\mathscr{O} \underset{k+1\text{-tact}}{\uparrow} Gru_{\vec{O},P}(X^*, \infty)$ implies $\mathscr{O} \underset{\text{tact}}{\uparrow} Gru_{\vec{O},P}(X^*, \infty)$, and likewise for Markov strategies, this result cannot be immediately extended to $Gru_{K,P}(X)$. However, this section will demonstrate a non-trivial example of a locally-compact space $\boldsymbol{X}$ for which $\mathscr{K} \uparrow Gru_{K,P}(\boldsymbol{X})$ but $\mathscr{K} \underset{k\text{-mark}}{\not\uparrow} Gru_{K,P}(\boldsymbol{X})$ for any $k < \omega$.

Definition 5.5.1. Let $\boldsymbol{X} = (X \times 2^{<\omega}) \cup C$ denote a Cantor tree of copies of a zero-dimensional, compact space X with a point-countable cover $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ of distinct clopen sets, along with an uncountable subset of the Cantor set $C = \{c_\alpha : \alpha < \omega_1\} \in [2^\omega]^{\omega_1}$. The topology on $\boldsymbol{X}$ is given by decaring $U \times \{s\}$ to be a open neighborhood of $\langle x, s\rangle \in X \times 2^{<\omega}$ for each open neighborhood U of x in X, and declaring $B_{\alpha,m} = (U_\alpha \times \{c_\alpha \restriction n : m \leq n < \omega\}) \cup \{c_\alpha\}$ to be a clopen neighborhood of $c_\alpha \in C$ for each $\alpha < \omega_1$, $m < \omega$.

Definition 5.5.2. Let $F \in \omega_1^{<\omega}$ and $m, n < \omega$.

$$K_F = \bigcup_{\alpha \in F} B_{\alpha,0}$$

$$a_n = \{\langle i, 0\rangle : i < n\} \cup \{\langle n, 1\rangle\}$$

$$A = \{a_n : n < \omega\}$$

$$K'_F = K_F \setminus (X \times A)$$

$$L_m = X \times 2^{<m}$$

Lemma 5.5.3. *K_F, K'_F, and L_m are compact in $\boldsymbol{X}$. Furthermore, every compact set is contained in a union of K'_F, L_m for some $F \in C^{<\omega}$ and $m < \omega$.*

Proof. K_F contains $C_F = \{c_\alpha : \alpha \in F\} \subseteq C$, so any cover of basic open sets must include B_{α,n_α} for each $\alpha \in F$, and the remaining uncovered portion of K_F is a closed subset of a finite union of copies of compact X. Then K'_F is also compact as it is a closed subset of K_F, and L_m is compact as it is a finite union of copies of compact X.

Let D be compact. Consider the open cover

$$\{B_{\alpha,0} : \alpha < \omega_1\} \cup \{X \times \{s\} : s \in 2^{<\omega}\}$$

and note that the finite subcover for D contains subsets of some $K'_F \cup L_m$. □

Theorem 5.5.4. *$\mathscr{K} \uparrow Gru_{K,P}(\boldsymbol{X})$.*

Proof. Since $\{U_\alpha : \alpha < \omega_1\}$ is a point-countable cover, for each $x \in X$ let $\alpha_{x,n} < \omega_1$ yield ordinals such that $x \in U_{\alpha_{x,n}}$ for $n < \omega$.

Let $M : \boldsymbol{X} \times \omega \to \mathcal{K}(\boldsymbol{X})$ as follows:

$$M(\boldsymbol{x}, n) = \begin{cases} K_{\{\alpha_{x,m} : m \leq n\}} & : \boldsymbol{x} = \langle x, s\rangle \in X \times 2^{<\omega} \\ K_{\{\alpha\}} & : \boldsymbol{x} = c_\alpha \in C \end{cases}$$

and use M to define the strategy σ for each $\boldsymbol{a} \in \boldsymbol{X}^{<\omega}$:

$$\sigma(\boldsymbol{a}) = L_{|\boldsymbol{a}|} \cup \bigcup_{i < |\boldsymbol{a}|} M(\boldsymbol{a}(i), |\boldsymbol{a}|)$$

Let $\boldsymbol{p} : \omega \to \boldsymbol{X}$ be a legal attack against σ. Then as $\boldsymbol{p}(n) \notin L_n$, for each $\boldsymbol{x} = \langle x, s\rangle \in X \times 2^{<\omega}$, $X \times \{s\}$ is an open neighborhood of $\boldsymbol{x}$ which contains finitely many $\boldsymbol{p}(n)$.

Now consider $\boldsymbol{x} = c_\alpha$ for some $\alpha < \omega_1$, and let $n < \omega$. Then if $\boldsymbol{p}(n) = \langle x, s\rangle$ with $\alpha = \alpha_{x,N}$ for some $N < \omega$, then $\boldsymbol{p}(m) \notin B_{\alpha,0}$ for $\max(n, N) < m < \omega$. Or, if $\boldsymbol{p}(n) = c_\alpha$, then $\boldsymbol{p}(m) \notin B_{\alpha,0}$ for $n < m < \omega$. Otherwise, $\boldsymbol{p}(m) \notin B_{\alpha,0}$ for any $m < \omega$. In any case, $B_{\alpha,0}$ is a neighborhood of $\boldsymbol{x}$ which contains finitely many $\boldsymbol{p}(n)$. Therefore, σ is a winning strategy. $\square$

We will show that there does not exist any winning k-Markov strategy for $\mathscr{K}$ in this game. Knowledge of round number does not assist $\mathscr{K}$, since she may force $\mathscr{P}$ to either stay within C, or to seed a growing integer by forcing her to play outside $L_{|s|+1}$ in response to $\langle x, s\rangle \in X \times 2^{<\omega}$.

Lemma 5.5.5. *If* $\mathscr{K} \underset{k+1\text{-}mark}{\uparrow} Gru_{K,P}(\boldsymbol{X})$, *then* $\mathscr{K} \underset{k+1\text{-}tact}{\uparrow} Gru_{K,P}(\boldsymbol{X})$.

Proof. Let σ be a winning $k+1$-mark for $\mathscr{K}$ such that $m \leq n$ and $\mathrm{range}(r) \subseteq \mathrm{range}(s)$ implies $\sigma(r, m) \subseteq \sigma(s, n)$. For a sequence p, let $p \restriction^k n = \nu_k(p \restriction n)$ give the last k terms of $p \restriction n$.

Define $r : \boldsymbol{X} \to \omega$ by

$$r(\boldsymbol{x}) = \begin{cases} |s| & : \ \boldsymbol{x} = \langle x, s\rangle \in X \times 2^{<\omega} \\ 0 & : \ \boldsymbol{x} \in C \end{cases}$$

and use r to define the $k+1$-tactic τ by

$$\tau(\emptyset) = \sigma(\emptyset, 0)$$

$$\tau(\boldsymbol{t}^\frown\langle \boldsymbol{x}\rangle) = L_{r(\boldsymbol{x})+1} \cup \{\boldsymbol{x}\} \cup \sigma(\boldsymbol{t}^\frown\langle \boldsymbol{x}\rangle, r(\boldsymbol{x}) + 1)$$

Let $\boldsymbol{p} : \omega \to \boldsymbol{X}$ be a legal attack by $\mathscr{P}$ against τ. If $\boldsymbol{p}(n) \in C$ for $N < n < \omega$, then since no $\boldsymbol{p}(n)$ may be legally repeated, $\{\{\boldsymbol{p}(n)\} : N < n < \omega\}$ is a discrete collection, making $\{\{\boldsymbol{p}(n)\} : n < \omega\}$ locally finite.

Otherwise, let $f \in \omega^\omega$ be increasing and define $\boldsymbol{q} : \omega \to X \times 2^{<\omega}$ such that $\boldsymbol{q}(i) = \boldsymbol{p}(f(i))$, and $\boldsymbol{p}(j) \in X \times 2^{<\omega}$ implies there is some i with $j = f(i)$. It follows that

$$\boldsymbol{q}(0) = \boldsymbol{p}(f(0)) \notin \bigcup_{m \leq f(0)} \tau(\boldsymbol{p} \restriction^{k+1} m) \supseteq \tau(\emptyset) = \sigma(\emptyset, 0)$$

Denoting $\boldsymbol{q}(n) = \langle x_n, s_n \rangle$, it's trivial to note that $|s_0| \geq 0$. Assuming that $|s_m| \geq m$ for $m \leq n$, it then follows that

$$\boldsymbol{q}(n+1) = \boldsymbol{p}(f(n+1)) \notin \bigcup_{m \leq f(n+1)} \tau(\boldsymbol{p} \restriction^{k+1} m)$$

$$\supseteq \bigcup_{m \leq n} \tau(\boldsymbol{q} \restriction^{k+1} m) \supseteq \sigma(\emptyset, 0) \cup \bigcup_{m<n} \sigma(\boldsymbol{q} \restriction^{k+1} (m+1), |s_m| + 1)$$

$$\supseteq \sigma(\emptyset, 0) \cup \bigcup_{m<n} \sigma(\boldsymbol{q} \restriction^{k+1} (m+1), m+1)$$

and

$$\boldsymbol{q}(n+1) \notin \tau(\boldsymbol{q} \restriction^{k+1} (n+1)) \supseteq L_{r(\boldsymbol{q}(n))} = L_{|s_n|+1}$$

gives $|s_{n+1}| \geq |s_n| + 1 \geq n+1$. Thus $\boldsymbol{q}$ is a legal attack on the winning $k+1$-Markov strategy σ, so the collection $\{\{\boldsymbol{q}(n)\} : n < \omega\}$ is locally finite. and it follows that $\{\{\boldsymbol{p}(n)\} : n < \omega\}$ is also locally finite. $\square$

Corollary 5.5.6. *$\boldsymbol{X}$ is σ-metacompact if and only if $\boldsymbol{X}$ is metacompact.*

Lemma 5.5.7. *If $\mathscr{K} \underset{k+1\text{-}tact}{\uparrow} Gru_{K,P}(\boldsymbol{X})$, then $\mathscr{K} \underset{tact}{\uparrow} Gru_{K,P}(\boldsymbol{X})$.*

Proof. Let σ be a winning $(k+1)$-tactical strategy, and without loss of generality assume it ignores order.

Define $F(x_0, \dots, x_k, n) \in [C]^{<\omega}$ and $m(x_0, \dots, x_k, n) \in \omega \setminus (n+1)$, both increasing on n, such that for each $\langle x_0, \dots, x_k \rangle \in X^{k+1}$,

$$\bigcup_{s_0, \dots, s_k \in 2^{\leq n}} \sigma(\langle x_0, s_0 \rangle, \dots, \langle x_k, s_k \rangle) \subseteq K'_{F(x_0, \dots, x_k, n)} \cup L_{m(x_0, \dots, x_k, n)}$$

Select an arbitrary point $y \in X$. Let

$$M^0(x, n) = n$$

$$M^{i+1}(x, n) = m(x, y, \ldots, y, M^i(x, n) + 1)$$

and define the tactical strategy τ as follows:

$$\tau(\emptyset) = \sigma(\emptyset)$$

$$\tau(\langle c_\alpha \rangle) = \{c_\alpha\}$$

$$\tau(\langle\langle x, s \rangle\rangle) = K'_{F(x,y,\ldots,y,M^k(x,|s|)+1)} \cup L_{m(x,y,\ldots,y,M^k(x,|s|)+1)}$$

Let $\boldsymbol{p} : \omega \to \boldsymbol{X}$ be a legal attack against τ, and assume $\boldsymbol{p}(n) = \langle x_n, s_n \rangle \in X \times 2^{<\omega}$. Then consider the attack $\boldsymbol{q} : \omega \to X \times 2^{<\omega}$ against σ defined by, for $n < \omega$ and $i < k$,

$$\boldsymbol{q}((k+1)n) = \boldsymbol{p}(n) = \langle x_n, s_n \rangle$$

$$\boldsymbol{q}((k+1)n + (i+1)) = \langle y, a_{M^{i+1}(x_n, |s_n|)} \rangle$$

Since

$$\langle x_{n+1}, s_{n+1} \rangle = \boldsymbol{p}(n+1) \notin \tau(\langle \boldsymbol{p}(n) \rangle) \supseteq L_{M^{k+1}(x_n, |s_n|)+1}$$

it follows that $|s_{n+1}| \geq M^{i+1}(x_n, |s_n|) + 1$ for $i < k$; furthermore,

$$|s_n| \leq M^i(x_n, |s_n|) < M^i(x_n, |s_n|) + 1 \leq M^{i+1}(x_n, |s_n|) < M^{i+1}(x_n, |s_n|) + 1 \leq |s_{n+1}|$$

so the second coordinate of $\boldsymbol{q}(n)$ is always strictly increasing.

By the definition of τ,

$$\boldsymbol{q}((k+1)n) = \boldsymbol{p}(n) \notin \bigcup_{m \leq n} \tau(\boldsymbol{p} \restriction^1 m) \supseteq \bigcup_{m \leq (k+1)n} \sigma(\boldsymbol{q} \restriction^{k+1} m)$$

Since

$$\boldsymbol{q}((k+1)n + (i+1)) = \langle y, a_{M^{i+1}(x_n, |s_n|)} \rangle \in X \times A$$

it follows that $\boldsymbol{q}((k+1)n + (i+1)) \notin K'_F$ for any $F \in [\omega_1]^{<\omega}$.

Then it's sufficient to note that

$$|a_{M^{i+1}(x_n,|s_n|)}| = M^{i+1}(x_n, |s_n|) + 1 > m(x_n, y, \ldots, y, M^i(x_n, |s_n|) + 1)$$

to show that

$$\boldsymbol{q}((k+1)n + (i+1)) = \langle y, a_{M^{i+1}(x_n,|s_n|)} \rangle \notin L_{m(x_n, y, \ldots, y, M^i(x_n, |s_n|)+1)}$$

and therefore $\boldsymbol{q}((k+1)n + (i+1))$ is a legal move.

As a result, $\boldsymbol{q}$ is a legal attack against σ, and $\{\{\boldsymbol{q}(n)\} : n < \omega\} \supseteq \{\{\boldsymbol{p}(n)\} : n < \omega\}$ are both locally finite.

Finally, if the range of $\boldsymbol{p}$ intersects C, those moves may be safely ignored as they cannot be repeated and lay in a closed discrete set, so the proof is complete. □

Therefore if $\boldsymbol{X}$ can be constructed which is not metacompact, then $\mathscr{K}$ lacks a k-Markov strategy for any $k < \omega$.

Theorem 5.5.8. *$\boldsymbol{X}$ is metacompact if and only if $\{U_\alpha : \alpha < \omega_1\}$ is σ-point-finite.*

Proof. Let $\omega_1 = \bigcup_{n<\omega} A_n$ such that $\{U_\alpha : \alpha \in A_n\}$ is point-finite for each $n < \omega$.

Let $\mathcal{U}$ be a cover of $\boldsymbol{X}$, and for each $s \in 2^\omega$ let $\mathcal{V}_s$ be a finite open refinement of $\mathcal{C}$ covering the compact set $X \times \{s\}$. Then let $\mathcal{W}_n = \{B_{\alpha,n_\alpha} : \alpha \in A_n\}$ be an open refinement of $\mathcal{C}$ for each $n < \omega$, and note that it is point-finite. It follows that $\mathcal{U}' = \bigcup_{s\in 2^{<\omega}} \mathcal{V}_s \cup \bigcup_{n<\omega} \mathcal{W}_n$ is an open σ-point-finite refinement of $\mathcal{U}$, so $\boldsymbol{X}$ is σ-metacompact, and therefore it is metacompact.

For the other direction, consider the open cover $\mathcal{U} = \{B_{\alpha,0} : \alpha < \omega_1\}$ of the closed subset C of metacompact $\boldsymbol{X}$, and let $\{B_{\alpha,n_\alpha} : \alpha < \omega_1\}$ be a point-finite refinement. Then $\mathcal{U}_s = \{U_\alpha : c_\alpha \restriction n_\alpha = s\}$ is point-finite for $s \in 2^{<\omega}$ and $U_\alpha \in \mathcal{U}_{c_\alpha \restriction n_\alpha}$ for each $\alpha < \omega_1$. Therefore $\mathcal{U} = \bigcup_{s\in 2^{<\omega}} \mathcal{U}_s$ is σ-point-finite. □

The following example was suggested by Gary Gruenhage.

Theorem 5.5.9. *There exists a compact, zero-dimensional topological space X with a clopen cover $\{U_\alpha : \alpha < \omega_1\}$ of distinct sets which is not σ-point-finite.*

Proof. Let Y be a zero-dimensional Corson compact space which is not Eberlein compact; one such space was constructed in [17]. ¿et $X = Y^2$. Then by characterizations of Corson and Eberlien compacts found in [10], $Y^2 \setminus \Delta$ is meta-Lindelöf but not σ-metacompact, so there exists a point-countable clopen cover $\mathcal{U}$ of $Y^2 \setminus \Delta$ which is not σ-point-finite. Then $\mathcal{U} \cup \{X\}$ is a point-countable clopen cover of X which is not σ-point-finite. □

Corollary 5.5.10. *There exists a locally compact space X such that $\mathscr{K} \uparrow Gru_{K,P}(X)$ but $\mathscr{K} \underset{k\text{-mark}}{\not\uparrow} Gru_{K,P}(X)$ for all $k < \omega$.*

So the following questions remain open:

Question 5.5.11. Does $\mathscr{K} \underset{k+1\text{-tact}}{\uparrow} Gru_{K,P}(X)$ imply $\mathscr{K} \underset{\text{tact}}{\uparrow} Gru_{K,P}(X)$ for all locally compact spaces? Equivalently, is metacompactness characterized by $\mathscr{K} \underset{k+1\text{-tact}}{\uparrow} Gru_{K,P}(X)$ for all locally compact spaces? What about for Markov strategies and σ-metacompactness?

Chapter 6

Menger's Game

In 1924 Karl Menger introduced a covering property generalizing σ-compactness [20].

6.1 The Menger property and game

Definition 6.1.1. A space X is Menger if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ of open covers of X there exists a sequence $\langle \mathcal{F}_0, \mathcal{F}_1, \dots \rangle$ such that $\mathcal{F}_n \subseteq \mathcal{U}_n$, $|\mathcal{F}_n| < \omega$, and $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Proposition 6.1.2. *X is σ-compact $\Rightarrow$ X is Menger $\Rightarrow$ X is Lindelöf.*

None of these implications may be reversed; the irrationals are a simple example of a Lindelöf space which is not Menger, and we'll see several examples of Menger spaces which are not σ-compact.

It can be shown via a non-trivial proof that the following game can be used to characterize the Menger property.

Definition 6.1.3. For each cover $\mathcal{U}$ of X, $S \subseteq X$ is *$\mathcal{U}$-finite* if there exists a finite subcollection of $\mathcal{U}$ which covers C.

Of course, a compact space is $\mathcal{U}$-finite for all open covers $\mathcal{U}$.

Game 6.1.4. Let $Men_{C,F}(X)$ denote the *Menger game* with players $\mathscr{C}$, $\mathscr{F}$. In round n, $\mathscr{C}$ chooses an open cover $\mathcal{U}_n$, followed by $\mathscr{F}$ choosing a $\mathcal{U}_n$-finite subset F_n of X.

$\mathscr{F}$ wins the game if $X = \bigcup_{n<\omega} F_n$, and $\mathscr{C}$ wins otherwise.

Theorem 6.1.5. *A space X is Menger if and only if $\mathscr{C} \not\uparrow Men_{C,F}(X)$ [14].*

The typical characterization of the Menger game involves $\mathscr{F}$ choosing a finite subcollection $\mathcal{F}_n$ of $\mathcal{U}_n$, but it is easy to see that the characterization given above is equivalent, and will be convenient for use in our proofs.

6.2 Markov strategies

To the author's knowledge, no other direct work has been done on limited information strategies pertaining to the Menger game, although as we'll see there are results which can be sharpened when considering them. However, we immediately see that tactics are not of any real interest.

Proposition 6.2.1. *X is compact if and only if $\mathscr{F} \underset{tact}{\uparrow} Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{k\text{-}tact}{\uparrow} Men_{C,F}(X)$.*

Proof. If σ is a winning k-tactic, then for each open cover $\mathcal{U}$, σ defeats the attack $\langle \mathcal{U}, \mathcal{U}, \ldots \rangle$. Then

$$\bigcup_{i \leq k} \sigma(\langle \underbrace{\mathcal{U}, \ldots, \mathcal{U}}_{i} \rangle) = X$$

and X is $\mathcal{U}$-finite. □

Essentially, because $\mathscr{C}$ may repeat the same finite sequence of open covers, $\mathscr{F}$ needs to be seeded with knowledge of the round number to prevent being trapped in a loop.

If $\mathscr{F}$'s memory of $\mathscr{C}$'s past moves is bounded, then there is no need to consider more than the two most recent moves. The intuitive reason is that $\mathscr{C}$ could simply play the same cover repeatedly until $\mathscr{F}$'s memory is exhausted, in which case $\mathscr{F}$ would only ever see the change from one cover to another.

Theorem 6.2.2. *For each $k < \omega$, $F \underset{(k+2)\text{-}mark}{\uparrow} Men_{C,F}(X)$ if and only if $F \underset{2\text{-}mark}{\uparrow} Men_{C,F}(X)$.*

Proof. Let σ be a winning $(k+2)$-mark. We define the 2-mark τ as follows:

$$\tau(\langle \mathcal{U} \rangle, 0) = \bigcup_{m < k+2} \sigma(\langle \underbrace{\mathcal{U}, \ldots, \mathcal{U}}_{m+1} \rangle, m)$$

$$\tau(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) = \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}, \ldots, \mathcal{U}}_{k+1-m}, \underbrace{\mathcal{V}, \ldots, \mathcal{V}}_{m+1} \rangle, (n+1)(k+2)+m)$$

Let $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ be an attack by $\mathscr{C}$ against τ. Then consider the attack

$$\langle \underbrace{\mathcal{U}_0, \ldots, \mathcal{U}_0}_{k+2}, \underbrace{\mathcal{U}_1, \ldots, \mathcal{U}_1}_{k+2}, \ldots \rangle$$

by $\mathscr{C}$ against σ. Since σ is a winning $(k+2)$-mark,

$$X = \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}_0, \ldots, \mathcal{U}_0}_{m+1} \rangle, m) \cup \bigcup_{n<\omega} \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}_n, \ldots, \mathcal{U}_n}_{k+1-m}, \underbrace{\mathcal{U}_{n+1}, \ldots, \mathcal{U}_{n+1}}_{m+1} \rangle, (n+1)(k+2)+m)$$

$$= \tau(\langle \mathcal{U}_0 \rangle, 0) \cup \bigcup_{n<\omega} \tau(\langle \mathcal{U}_n, \mathcal{U}_{n+1} \rangle, n+1)$$

Thus τ is a winning 2-mark. $\square$

A natural question arises: is there an example of a space X for which $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$ but $\mathscr{F} \underset{\text{mark}}{\not\uparrow} Men_{C,F}(X)$? We quickly see that perhaps the simplest example of a Lindelöf non-σ-compact space has this property.

Definition 6.2.3. For any cardinal κ, let $\kappa^\dagger = \kappa \cup \{\infty\}$ denote the *one-point Lindelöfication* of discrete κ, where points in κ are isolated, and the neighborhoods of ∞ are the co-countable sets containing it.

Theorem 6.2.4. $\mathscr{F} \underset{mark}{\not\uparrow} Men_{C,F}\left(\omega_1^\dagger\right)$.

Proof. This result will later follow from the fact that $\omega_1^\dagger$ is not a σ-compact space (all its compact subsets are finite).

For now, let σ be a Markov strategy for $\mathscr{F}$. For each $\alpha < \omega_1$, let $\mathcal{U}_\alpha$ be the open cover $\{\{\beta\} : \beta < \alpha\} \cup \{\omega_1^\dagger \setminus \alpha\}$ of $\omega_1^\dagger$, and set $F(\alpha, n)$ to be the finite set $\alpha \cap \sigma(\langle \mathcal{U}_\alpha \rangle, n)$.

If $P_n = \{\beta : \beta < \alpha < \omega_1 \Rightarrow \beta \in F(\alpha, n)\}$, then $P_n \subseteq F(\sup(P_n)+1, n)$. Thus P_n is finite for $n < \omega$. Choose $\beta \in \omega_1 \setminus \bigcup_{n<\omega} P_n$ and $\alpha_n > \beta$ such that $\beta \notin F(\alpha_n, n)$. Then $\mathscr{C}$ may attack σ with $\langle \mathcal{U}_{\alpha_0}, \mathcal{U}_{\alpha_1}, \ldots \rangle$, and it follows that $\beta \notin \bigcup_{n<\omega} F(\alpha_n, n)$ and $\beta \notin \bigcup_{n<\omega} \sigma(\langle \mathcal{U}_{\alpha_n} \rangle, n)$. $\square$

The greatest advantage of a strategy which has knowledge of two or more previous moves of the opponent, versus only knowledge of the most recent move, is the ability to react to changes from one round to the next. It's this ability to react that will give $\mathscr{F}$ her winning 2-Markov strategy in the Menger game on $\omega_1^\dagger$.

For inspiration, we turn to a game whose n-tactics were studied by Marion Scheepers in [26] which has similar goals to the Menger game when played upon $\kappa^\dagger$.

Game 6.2.5. Let $Fill_{C,F}^{\cup,\subset}(\kappa)$ denote the *strict union filling game* with two players $\mathscr{C}$, $\mathscr{F}$. In round 0, $\mathscr{C}$ chooses $C_0 \in [\kappa]^{\leq\omega}$, followed by $\mathscr{F}$ choosing $F_0 \in [\kappa]^{<\omega}$. In round $n+1$, $\mathscr{C}$ chooses $C_{n+1} \in [\kappa]^{\leq\omega}$ such that $C_{n+1} \supset C_n$, followed by $\mathscr{F}$ choosing $F_{n+1} \in [\kappa]^{<\omega}$.

$\mathscr{F}$ wins the game if $\bigcup_{n<\omega} F_n \supseteq \bigcup_{n<\omega} C_n$; otherwise, $\mathscr{C}$ wins.

In $Men_{C,F}\left(\kappa^\dagger\right)$, $\mathscr{C}$ essentially chooses a countable set to not include in her neighborhood of ∞, followed by $\mathscr{F}$ choosing a finite subset of this complement to cover during each round. Thus, $\mathscr{F}$ need only be concerned with the *intersection* of the countable sets chosen by $\mathscr{C}$ in $Men_{C,F}\left(\kappa^\dagger\right)$, rather than the union as in $Fill_{C,F}^{\cup,\subset}(\kappa)$.

Another difference: Scheepers required that $\mathscr{C}$ always choose strictly growing countable sets. The reasoning is clear: if the goal is to study tactics, then $\mathscr{C}$ cannot be allowed to trap $\mathscr{F}$ in a loop by repeating the same moves. But by eliminating this requirement, the study can then turn to Markov srategies, bringing the game further in line with the Menger game played upon $\kappa^\dagger$.

We introduce a few games to make the relationship between Scheeper's $Fill_{C,F}^{\cup,\subset}(\kappa)$ and $Men_{C,F}\left(\kappa^\dagger\right)$ more precise.

Game 6.2.6. Let $Fill_{C,F}^{\cup,\subseteq}(\kappa)$ denote the *union filling game* which proceeds analogously to $Fill_{C,F}^{\cup,\subset}(\kappa)$, except that $\mathscr{C}$'s restriction in round $n+1$ is reduced to $C_{n+1} \supseteq C_n$.

Game 6.2.7. Let $Fill_{C,F}^{1,\subseteq}(\kappa)$ denote the *initial filling game* which proceeds analogously to $Fill_{C,F}^{\cup,\subseteq}(\kappa)$, except that $\mathscr{F}$ wins whenever $\bigcup_{n<\omega} F_n \supseteq C_0$.

Game 6.2.8. Let $Fill^{\cap}_{C,F}(\kappa)$ denote the *intersection filling game* which proceeds analogously to $Fill^{1,\subseteq}_{C,F}(\kappa)$, except that $\mathscr{C}$ may choose any $C_n \in [\kappa]^{\leq\omega}$ each round, and $\mathscr{F}$ wins whenever $\bigcup_{n<\omega} F_n \supseteq \bigcap_{n<\omega} C_n$.

$$\begin{tikzcd}
\mathscr{F} \underset{k\text{-mark}}{\uparrow} Men_{C,F}\left(\kappa^{\dagger}\right) \arrow[r, leftrightarrow] \arrow[d] & \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa) \arrow[d] \\
\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{1,\subseteq}_{C,F}(\kappa) \arrow[r, leftrightarrow] & \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cup,\subseteq}_{C,F}(\kappa) \\
& \mathscr{F} \underset{k\text{-tact}}{\uparrow} Fill^{\cup,\subset}_{C,F}(\kappa) \arrow[u]
\end{tikzcd}$$

Figure 6.1: Diagram of Filling/Menger game implications

Theorem 6.2.9. *For any cardinal $\kappa > \omega$ and integer $k < \omega$, Figure 6.1 holds.*

Proof. $\mathscr{F} \underset{k\text{-mark}}{\uparrow} Men_{C,F}\left(\kappa^{\dagger}\right) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Men_{C,F}\left(\kappa^{\dagger}\right)$. Let $\mathcal{U}(C)$ (resp. $\mathcal{U}(s)$) convert each countable subset C of κ (resp. finite sequence s of such subsets) into the open cover $[C]^1 \cup \{\kappa^{\dagger} \setminus C\}$ (resp. finite sequence of such open covers). Then τ defined by

$$\tau(s^\frown\langle C\rangle, n) = C \cap \sigma(\mathcal{U}(s^\frown\langle C\rangle), n)$$

is a winning k-mark for $\mathscr{F}$ in $Fill^{\cap}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Men_{C,F}\left(\kappa^{\dagger}\right)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Fill^{\cap}_{C,F}(\kappa)$. Let $C(\mathcal{U})$ (resp. $C(s)$) convert each open cover $\mathcal{U}$ of $\kappa^{\dagger}$ (resp. finite sequence s of such covers) into a countable set C which is the complement of some neighborhood of ∞ in $\mathcal{U}$ (resp. finite sequence of such countable sets). Then τ defined by

$$\tau(s^\frown\langle\mathcal{U}\rangle, n) = (\kappa^{\dagger} \setminus C(\mathcal{U})) \cup \sigma(C(s^\frown\langle\mathcal{U}\rangle), n)$$

is a winning k-mark for $\mathscr{F}$ in $Men_{C,F}\left(\kappa^{\dagger}\right)$.

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{1,\subsetneq}_{C,F}(\kappa)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Fill^{\cap}_{C,F}(\kappa)$. σ is also a winning k-mark for $\mathscr{F}$ in $Fill^{1,\subsetneq}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{1,\subsetneq}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cup,\subsetneq}_{C,F}(\kappa)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Fill^{1,\subsetneq}_{C,F}(\kappa)$. For each finite sequence s, let $t \preceq s$ mean t is a final subsequence of s. Then τ defined by

$$\tau(s^\frown\langle C\rangle, n) = \bigcup_{t\preceq s, m\leq n} \sigma(t^\frown\langle C\rangle, m)$$

is a winning k-mark for $\mathscr{F}$ in $Fill^{\cup,\subsetneq}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cup,\subsetneq}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{1,\subsetneq}_{C,F}(\kappa)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Fill^{\cup,\subsetneq}_{C,F}(\kappa)$. σ is also a winning k-mark for $\mathscr{F}$ in $Fill^{1,\subsetneq}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k\text{-tact}}{\uparrow} Fill^{\cup,\subset}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cup,\subsetneq}_{C,F}(\kappa)$: Let σ be a winning k-tactic for $\mathscr{F}$ in $Fill^{\cup,\subset}_{C,F}(\kappa)$. For each countable subset C of κ, let $C+n$ be the union of C with the n least ordinals in $\kappa \setminus C$. Then τ defined by

$$\tau(\langle C_0, \ldots, C_i\rangle, n) = \sigma(\langle C_0 + (n-i), \ldots, C_i + n\rangle)$$

is a winning k-mark for $\mathscr{F}$ in $Fill^{\cup,\subsetneq}_{C,F}(\kappa)$. □

While we have not proven a direct implication between the Menger game and Scheeper's original filling game, Scheepers introduced the statement $S(\kappa)$ relating to the almost-compatability of functions from countable subsets of κ into ω which may be applied to both.

Definition 6.2.10. For two functions f, g we say f is **μ-almost compatible** with g ($f\|^*_\mu g$) if $|\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(x) \neq g(x)\}| < \mu$. If $\mu = \omega$ then we say f, g are **almost compatible** ($f\|^* g$).

Definition 6.2.11. $S(\kappa)$ states that there exist functions $f_A : A \to \omega$ for each $A \in [\kappa]^{\leq\omega}$ such that $|\{\alpha \in A : f_A(\alpha) \leq n\}| < \omega$ for all $n < \omega$ and $f_A\|^* f_B$ for all $A, B \in [\kappa]^\omega$. [1]

[1]This is equivalent to the original characterization given in [26]: there exist injections $g_A : A \to \omega$ such that $g_A\|^* g_B$ for all $A, B \in [\kappa]^\omega$ and $A \subset B$.

Scheepers went on to show that $S(\kappa)$ implies $\mathscr{F} \underset{\text{2-tact}}{\uparrow} Fill^{\cup,\subset}_{C,F}(\kappa)$. This proof, along with the following facts, give us inspiration for finding a winning 2-Markov strategy in the Menger game played on $\kappa^{\dagger}$.

Theorem 6.2.12. *$S(\omega_1)$ and $\kappa > 2^{\omega} \Rightarrow \neg S(\kappa)$ are theorems of ZFC. $S(2^{\omega})$ is a theorem of $ZFC + CH$ and consistent with $ZFC + \neg CH$.*

Proof. For $S(\omega_1)$, look at pg. 70 of [15]; this of course implies $S(2^{\omega})$ under CH. $\neg S((2^{\omega})^{+})$ is shown by a cardinality argument in [26]. The consistency result under $ZFC + \neg CH$ is a lemma for the main theorem in [26]. $\square$

$$\begin{tikzcd}
 & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}\left(\kappa^{\dagger}\right) \arrow[r, leftrightarrow] \arrow[d] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa) \arrow[d] \\
S(\kappa) \arrow[ur] \arrow[r] \arrow[drr] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill^{1,\subseteq}_{C,F}(\kappa) \arrow[r, leftrightarrow] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill^{\cup,\subseteq}_{C,F}(\kappa) \\
 & & \mathscr{F} \underset{\text{2-tact}}{\uparrow} Fill^{\cup,\subset}_{C,F}(\kappa) \arrow[u]
\end{tikzcd}$$

Figure 6.2: Diagram of Filling/Menger game implications with $S(\kappa)$

Theorem 6.2.13. *$S(\kappa)$ implies the game-theoretic results in Figure 6.2.*

Proof. Since $S(\kappa) \Rightarrow \mathscr{F} \underset{\text{2-tact}}{\uparrow} Fill^{\cup,\subset}_{C,F}(\kappa)$ was a main result of [26], it is sufficient to show that $S(\kappa) \Rightarrow \mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa)$.

Let f_A for $A \in [\kappa]^{\leq\omega}$ witness $S(\kappa)$. We define the 2-mark σ as follows:

$$\sigma(\langle A\rangle, 0) = \{\alpha \in A : f_A(\alpha) \leq 0\}$$

$$\sigma(\langle A, B\rangle, n+1) = \{\alpha \in A \cap B : f_B(\alpha) \leq n+1 \text{ or } f_A(\alpha) \neq f_B(\alpha)\}$$

For any attack $\langle A_0, A_1, \ldots \rangle$ by $\mathscr{C}$ and $\alpha \in \bigcap_{n<\omega} A_n$, either $f_{A_n}(\alpha)$ is constant for all n, or $f_{A_n}(\alpha) \neq f_{A_{n+1}}(\alpha)$ for some n; either way, α is covered. $\square$

Corollary 6.2.14. $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}\left(\omega_1^\dagger\right)$.

6.3 Menger game derived covering properties

$$\begin{tikzcd}
\mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X) \arrow[r] \arrow[d, leftrightarrow] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X) \arrow[r] \arrow[d, leftrightarrow] \arrow[l, bend right, dotted, "/"'] & \mathscr{F} \uparrow Men_{C,F}(X) \arrow[r] \arrow[d, leftrightarrow] \arrow[l, bend right, dotted, "?"'] & \mathscr{C} \not\uparrow Men_{C,F}(X) \arrow[d, leftrightarrow] \arrow[l, bend right, dotted, "/"'] \\
\text{Markov Menger} \arrow[r] & \text{2-Markov Menger} \arrow[r] \arrow[l, bend right, dotted, "/"'] & \text{Perfect Menger} \arrow[r] \arrow[l, bend right, dotted, "?"'] & \text{Menger} \arrow[l, bend right, dotted, "/"']
\end{tikzcd}$$

Figure 6.3: Diagram of covering properties related to the Menger game

Limited information strategies for the Menger game naturally define a spectrum of covering properties, see Figure 6.3. However, we do not know if the middle two properties are actually distinct.

Question 6.3.1. Does there exist a space X such that $\mathscr{F} \uparrow Men_{C,F}(X)$ but $\mathscr{F} \underset{\text{2-mark}}{\not\uparrow} Men_{C,F}(X)$?

Note that while it's consistent that $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}\left((2^\omega)^\dagger\right)$, $\kappa^\dagger$ for $\kappa > 2^\omega$ is a candidate to answer the above question.

We are also interested in non-game-theoretic characterizations of these covering properties. It has been known for some time that for metrizable spaces, winning Menger spaces are exactly the σ-compact spaces, shown first by Telgarksy in [29] and later directly by Scheepers in [28].

In the interest of generality, we will first characterize the Markov Menger spaces without any separation axioms.

Definition 6.3.2. A subset Y of X is *relatively compact* to X if for every open cover of X, there exists a finite subcollection which covers Y.

For example, any bounded subset of Euclidean space is relatively compact whether it is closed or not. Actually, relative compactness can be thought of as an analogue of boundedness for regular spaces.

Proposition 6.3.3. *For regular spaces, Y is relatively compact to X if and only if $\overline{Y}$ is compact in X.* [2]

Proof. For any space, any subset of a compact set is relatively compact.

Assume Y is relatively compact, let $\mathcal{U}$ be an open cover of $\overline{Y}$, and define $x \in V_x \subseteq \overline{V_x} \subseteq U_x \in \mathcal{U}$ for each $x \in X$. Then if we take a subcollection $\mathcal{F} = \{V_{x_i} : i < n\}$ covering Y by relative compactness, then $\{U_{x_i} : i < n\}$ is a finite subcollection of $\mathcal{U}$ covering $\overline{Y}$, showing compactness. □

We now begin the process of factoring out Scheeper's proof to reveal the limited information implications at work.

Lemma 6.3.4. *Let $\sigma(\mathcal{U}, n)$ be a Markov strategy for F in $Men_{C,F}(X)$, and $\mathfrak{C}$ collect all open covers of X. Then the set*

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \sigma(\mathcal{U}, n)$$

is relatively compact to X. If σ is a winning Markov startegy, then $\bigcup_{n<\omega} R_n = X$.

Proof. First, for every open cover $\mathcal{U} \in \mathfrak{C}$, $R_n \subseteq \sigma(\mathcal{U}, n)$ is covered by a finite subcollection of $\mathcal{U}$.

Suppose that $x \notin R_n$ for any $n < \omega$. Then for each n, pick $\mathcal{U}_n \in \mathfrak{C}$ such that $x \notin \sigma(\mathcal{U}_n, n)$. Then $\mathscr{C}$ may counter σ with the attack $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$. □

[2] It should be noted that some authors define relative compactness in this way, but such a definition creates pathological implications for non-regular spaces. For example, the singleton containing the particular point of an infinite space with the particular point topology would not be relatively compact since its closure is not compact, even though it is finite.

Definition 6.3.5. A *σ-relatively-compact* space is the countable union of relatively compact subsets.

Corollary 6.3.6. *The following are equivalent:*

- X *is* σ*-relatively-compact*
- $\mathscr{F} \underset{pre}{\uparrow} Men_{C,F}(X)$
- $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$

Proof. If $X = \bigcup_{n<\omega} R_n$ for R_n relatively compact, then $\sigma(n) = R_n$ is a winning predetermined strategy, which yields a winning Markov strategy. The previous lemma finishes the proof. □

Corollary 6.3.7. *Let* X *be a regular space. The following are equivalent:*

- X *is* σ*-compact*
- X *is* σ*-relatively-compact*
- $\mathscr{F} \underset{pre}{\uparrow} Men_{C,F}(X)$
- $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$

For Lindelöf spaces, metrizability is characterized by regularity and second-countability, the latter of which was essentially used by Scheepers in this way:

Lemma 6.3.8. *Let* X *be a second-countable space.* $\mathscr{F} \uparrow Men_{C,F}(X)$ *if and only if* $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$.

Proof. Let σ be a strategy for $\mathscr{F}$, and note that it's sufficient to consider playthroughs with only basic open covers.

So if $\mathcal{U}_t$ is a basic open cover for $t < s \in \omega^{<\omega}$, and $\mathcal{V}$ is any basic open cover, we may choose a finite subcollection $\mathcal{F}(s, \mathcal{V})$ of $\mathcal{V}$ such that

$$\sigma(\langle \mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{V}\rangle) \subseteq \bigcup \mathcal{F}(s, \mathcal{V})$$

Note that there are only countably-many finite collections of basic open sets. Thus we may choose basic open covers $\mathcal{U}_{s^\frown\langle n\rangle}$ for $n < \omega$ such that for any basic open cover $\mathcal{V}$, there exists $n < \omega$ where $\mathcal{F}(s, \mathcal{V}) = \mathcal{F}(s, \mathcal{U}_{s^\frown\langle n\rangle})$.

Let $t : \omega \to \omega^{<\omega}$ be a bijection. We define the Markōv strategy τ as follows:

$$\tau(\langle\mathcal{V}\rangle, n) = \bigcup \mathcal{F}(t(n), \mathcal{V})$$

Suppose there exists a counter-attack $\langle\mathcal{V}_0, \mathcal{V}_1, \ldots\rangle$ of basic open covers which defeats τ. Then there exists $f : \omega \to \omega$ such that, letting $t(m_n) = f \restriction n$:

$$\begin{aligned} x &\notin \tau(\langle\mathcal{V}_{m_n}\rangle, m_n) \\ &= \bigcup \mathcal{F}(f \restriction n, \mathcal{V}_{m_n}) \\ &= \bigcup \mathcal{F}(f \restriction n, \mathcal{U}_{f\restriction(n+1)}) \\ &\supseteq \sigma(\langle\mathcal{U}_{f\restriction 1}, \ldots, \mathcal{U}_{f\restriction(n+1)}\rangle) \end{aligned}$$

Thus $\langle\mathcal{U}_{f\restriction 1}, \mathcal{U}_{f\restriction 2}, \ldots\rangle$ is a successful counter-attack by $\mathscr{C}$ against the perfect information strategy σ. $\square$

Corollary 6.3.9. *Let X be a second-countable space. The following are equivalent:*

- *X is σ-relatively-compact*
- $F \underset{pre}{\uparrow} Men_{C,F}(X)$
- $F \underset{mark}{\uparrow} Men_{C,F}(X)$
- $F \uparrow Men_{C,F}(X)$

Corollary 6.3.10. *Let X be a metrizable space. The following are equivalent:*

- *X is σ-compact*
- *X is σ-relatively-compact*

- $F \underset{pre}{\uparrow} Men_{C,F}(X)$

- $F \underset{mark}{\uparrow} Men_{C,F}(X)$

- $F \uparrow Men_{C,F}(X)$

Proof. Each property implies Lindelöf, so X may be assumed to be regular and second-countable. $\square$

6.4 Robustly Lindelöf

To help describe $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$ topologically, we introduce a subset variant of the Menger game and a related covering property.

Game 6.4.1. Let $Men_{C,F}(X,Y)$ denote the *Menger subspace game* which proceeds analogously to the Menger game, except that $\mathscr{F}$ wins whenever $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover for $Y \subseteq X$.

Note of course that $Men_{C,F}(X,X) = Men_{C,F}(X)$.

Definition 6.4.2. A subset Y of X is *relatively robustly Menger* if there exist functions $r_{\mathcal{V}} : Y \to \omega$ for each open cover $\mathcal{V}$ of X such that for all open covers $\mathcal{U}, \mathcal{V}$ and numbers $n < \omega$, the following sets are $\mathcal{V}$-finite:

$$c(\mathcal{V}, n) = \{x \in Y : r_{\mathcal{V}}(x) \leq n\}$$

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{x \in Y : n < r_{\mathcal{U}}(x) < r_{\mathcal{V}}(x)\}$$

Definition 6.4.3. A space X is *robustly Menger* if it is relatively robustly Menger to itself.

Proposition 6.4.4. *All σ-relatively-compact spaces are robustly Menger.*

Proof. If $X = \bigcup_{n<\omega} R_n$, then for all $\mathcal{U}$, let $r_{\mathcal{U}}(x)$ be the least n such that $x \in R_n$. Then $c(\mathcal{V}, n) = \bigcup_{m \leq n} R_m$ and $p(\mathcal{U}, \mathcal{V}) = \emptyset$. $\square$

Theorem 6.4.5. *If $Y \subseteq X$ is relatively robustly Menger, then $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X,Y)$.*

Proof. We define the Markov strategy σ as follows. Let $\sigma(\langle\mathcal{U}\rangle, 0) = c(\mathcal{U}, 0)$, and let $\sigma(\langle\mathcal{U},\mathcal{V}\rangle, n+1) = c(\mathcal{V}, n+1) \cup p(\mathcal{U},\mathcal{V}, n+1)$.

For any attack $\langle\mathcal{U}_0, \mathcal{U}_1, \ldots\rangle$ by $\mathscr{C}$ and $x \in Y$, one of the following must occur:

- $r_{\mathcal{U}_0}(x) = 0$ and thus $x \in c(\mathcal{U}_0, 0) \subseteq \sigma(\langle\mathcal{U}_0\rangle, 0)$.
- $r_{\mathcal{U}_0}(x) = N+1$ for some $N \geq 0$ and:
  - For all $n \leq N$,

    $$r_{\mathcal{U}_{n+1}}(x) \leq N+1$$

    and thus $x \in c(\mathcal{U}_{N+1}, N+1) \subseteq \sigma(\langle U_N, U_{N+1}\rangle, N+1)$.
  - For some $n \leq N$,

    $$r_{\mathcal{U}_n}(x) \leq n$$

    and thus $x \in c(\mathcal{U}_{n+1}, n+1) \subseteq \sigma(\langle U_n, U_{n+1}\rangle, n+1)$.
  - For some $n \leq N$,

    $$n < r_{\mathcal{U}_n}(x) \leq N+1 < r_{\mathcal{U}_{n+1}}(x)$$

    and thus $x \in p(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1) \subseteq \sigma(\langle U_n, U_{n+1}\rangle, n+1)$

□

Theorem 6.4.6. *$S(\kappa)$ implies $\kappa^\dagger$ is robustly Menger, and thus $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(\kappa^\dagger)$.*

Proof. Let f_A for $A \in [\kappa]^{\leq\omega}$ witness $S(\kappa)$ and fix $A(\mathcal{U}) \in [\kappa]^{\leq\omega}$ for each open cover $\mathcal{U}$ such that $\kappa^\dagger \setminus A(\mathcal{U})$ is contained in some element of $\mathcal{U}$. Then let $r_{\mathcal{U}}(x) = 0$ for $x \in \kappa^\dagger \setminus A(\mathcal{U})$, and $r_{\mathcal{U}}(\alpha) = f_{A(\mathcal{U})}(\alpha)$ for $\alpha \in A(\mathcal{U})$.

It follows that

$$c(\mathcal{U}, n) = \left(\kappa^\dagger \setminus A(\mathcal{U})\right) \cup \{\alpha \in A(\mathcal{U}) : f_{A(\mathcal{U})}(\alpha) \leq n\}$$

is $\mathcal{U}$-finite, $\bigcup_{n<\omega} c(\mathcal{U}, n) = X$, and

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{\alpha \in A(\mathcal{U}) \cap A(\mathcal{V}) : n < f_{A(\mathcal{U})}(\alpha) < f_{A(\mathcal{V})}(\alpha)\}$$

is finite. $\square$

We may also consider common (non-regular) counterexamples which are finer than the usual Euclidean line.

Definition 6.4.7. Let $R_{\mathbb{Q}}$ be the real line with the topology generated by open intervals with or without the rationals removed.

Theorem 6.4.8. *$R_{\mathbb{Q}}$ is non-regular and non-σ-compact, but is second-countable and σ-relatively-compact.*

Proof. Compact sets in $R_{\mathbb{Q}}$ can be shown to not contain open intervals, and thus are nowhere dense in nonmeager $\mathbb{R}$, so $R_{\mathbb{Q}}$ is not σ-compact. The usual base of intervals with rational endpoints (with or without rationals removed) witnesses second-countability.

To see that $R_{\mathbb{Q}}$ is σ-relatively compact, consider $[a, b] \setminus \mathbb{Q}$. Let $\mathcal{U}$ be a cover of $R_{\mathbb{Q}}$, and let $\mathcal{U}'$ fill in the missing rationals for any open set in $\mathcal{U}$. There is a finite subcover $\mathcal{V}' \subseteq \mathcal{U}'$ for $[a, b]$ since $\mathcal{U}'$ contains open sets from the Euclidean topology. Let $\mathcal{V} = \{V \setminus \mathbb{Q} : V \in V'\}$: this is a finite refinment of $\mathcal{U}$ covering $[a, b] \setminus \mathbb{Q}$, so $[a, b] \setminus \mathbb{Q}$ is relatively compact. It follows then that $R_{\mathbb{Q}} \setminus \mathbb{Q}$ is σ-relatively-compact, and since $\mathbb{Q}$ is countable, $R_{\mathbb{Q}}$ is σ-relatively-compact. Non-regularity follows since regular and σ-relatively-compact implies σ-compact. $\square$

Definition 6.4.9. Let R_ω be the real line with the topology generated by open intervals with countably many points removed.

Theorem 6.4.10. *R_ω is non-regular, non-second-countable, and non-σ-relatively-compact, but $\mathscr{F} \uparrow Men_{C,F}(R_\omega)$.*

Proof. The closure of any open set is its closure in the usual Euclidean topology, so R_ω is not regular. If $S \supseteq \{s_n : n < \omega\}$ for s_n discrete, then $U_m = R_\omega \setminus \{s_n : m < n < \omega\}$ yields an infinite cover $\{U_m : m < \omega\}$ with no finite subcollection covering S, showing that all relatively compact sets are finite, and R_ω is not σ-relatively-compact.

Define the winning strategy σ for $\mathscr{F}$ in $Men_{C,F}(R_\omega)$ as follows: let $\sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{2n}) = [-n, n] \setminus C_n$ for some countable $C_n = \{c_{n,m} : m < \omega\}$, and let $\sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{2n+1}) = \{c_{i,j} : i, j < n\}$. Non-second-countable follows since second-countable and $\mathscr{F} \uparrow Men_{C,F}(X)$ implies σ-relatively-compact. $\square$

We will soon see that, assuming $S(2^\omega)$, $\mathscr{F}$ has a winning 2-Marköv strategy for $Men_{C,F}(R_\omega)$ as well.

Proposition 6.4.11. *Let* $\underset{limit}{\uparrow}$ *be either* $\underset{k\text{-}mark}{\uparrow}$ *or* $\uparrow$*. If* $X = \bigcup_{i<\omega} X_i$ *and* $\mathscr{F} \underset{limit}{\uparrow} Men_{C,F}(X, X_i)$ *for* $i < \omega$*, then* $\mathscr{F} \underset{limit}{\uparrow} Men_{C,F}(X)$

Proof. Let L be the k-Markov fog-of-war μ_k (resp. the identity), and let σ_i be a k-Markov strategy (resp. perfect information strategy) for $\mathscr{F}$ in $Men_{C,F}(X, X_i)$.

We define the k-Markov strategy (resp. perfect information strategy) σ for $Men_{C,F}(X)$ as follows:

$$\sigma \circ L(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle) = \bigcup_{i \le n} \sigma_i \circ L(\langle \mathcal{U}_i, \ldots, \mathcal{U}_n \rangle)$$

Let $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ be a successful counter-attack by $\mathscr{C}$ against σ. Then there exists $x \in X_i$ for some $i < \omega$ such that x is not covered by $\bigcup_{n<\omega} \sigma \circ L(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$. It follows that x is not covered by $\bigcup_{n<\omega} \sigma_i \circ L(\langle \mathcal{U}_i, \ldots, \mathcal{U}_{i+n} \rangle)$, and $\langle \mathcal{U}_i, \mathcal{U}_{i+1}, \ldots \rangle$ is a successful counter-attack by $\mathscr{C}$ against σ_i. $\square$

Theorem 6.4.12. *If* $S(2^\omega)$*, then* $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}(R_\omega)$.

Proof. It's sufficient to show that $[0, 1] \subseteq R_\omega$ is relatively robustly Menger. Let f_A witness $S(2^\omega)$ for $A \in [[a, b]]^{\le\omega}$. For each open cover $\mathcal{U}$, let $A_\mathcal{U}$ be such that $[0, 1] \setminus A_\mathcal{U}$ is $\mathcal{U}$-finite. Let $r_\mathcal{U}(x) = 0$ if $x \in [0, 1] \setminus A_\mathcal{U}$ and $r_\mathcal{U}(x) = f_{\mathcal{A}_\mathcal{U}}(x)$ otherwise.

It follows then that

$$c(\mathcal{U}, n) = [0, 1] \setminus \{x \in A_\mathcal{U} : f_{\mathcal{A}_\mathcal{U}}(x) > n\}$$

is $\mathcal{U}$-finite and

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{x \in A_{\mathcal{U}} \cap A_{\mathcal{V}} : n < f_{\mathcal{A}_{\mathcal{U}}}(x) < f_{\mathcal{A}_{\mathcal{V}}}(x)\}$$

is finite. $\square$

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