APPLICATIONS OF LIMITED INFORMATION STRATEGIES IN MENGER'S GAME

STEVEN CLONTZ

ABSTRACT. I need an abstract

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1. The Menger property and game

Recall the following definition.

Definition 1.1. A space X is Menger if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ of open covers of X there exists a sequence $\langle \mathcal{F}_0, \mathcal{F}_1, \dots \rangle$ such that $\mathcal{F}_n \subseteq \mathcal{U}_n$, $|\mathcal{F}_n| < \omega$, and $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Note that many authors refer to this property as $S_{fin}(\mathcal{O}, \mathcal{O})$, where $\mathcal{O}$ is the collection of open covers of X, and $S_{fin}(A, B)$ denotes the selection property such that for each sequence in A^ω, there are finite subsets of each entry for which the union of these subsets belongs in B.

Proposition 1.2. *X is σ-compact $\Rightarrow$ X is Menger $\Rightarrow$ X is Lindelöf.*

None of these implications may be reversed; the irrationals are a simple example of a Lindelöf space which is not Menger, and we'll see several examples of Menger spaces which are not σ-compact.

It can be shown via a non-trivial proof that the following game can be used to characterize the Menger property.

Definition 1.3. For each cover $\mathcal{U}$ of X, $S \subseteq X$ is *$\mathcal{U}$-finite* if there exists a finite subcollection of $\mathcal{U}$ which covers C.

Of course, a compact space is $\mathcal{U}$-finite for all open covers $\mathcal{U}$.

Game 1.4. Let $Men_{C,F}(X)$ denote the *Menger game* with players $\mathscr{C}$, $\mathscr{F}$. In round n, $\mathscr{C}$ chooses an open cover $\mathcal{U}_n$, followed by $\mathscr{F}$ choosing a $\mathcal{U}_n$-finite subset F_n of X.

$\mathscr{F}$ wins the game if $X = \bigcup_{n<\omega} F_n$, and $\mathscr{C}$ wins otherwise.

Theorem 1.5. *A space X is Menger if and only if $\mathscr{C} \not\uparrow Men_{C,F}(X)$* [1].

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The typical characterization of the Menger game involves $\mathscr{F}$ choosing a finite subcollection $\mathcal{F}_n$ of $\mathcal{U}_n$, but it is easy to see that the characterization given above is equivalent, and will be convenient for use in our proofs.

2. Markov strategies

To the author's knowledge, no other direct work has been done on limited information strategies pertaining to the Menger game, although as we'll see there are results which can be sharpened when considering them. However, we immediately see that tactics are not of any real interest.

Proposition 2.1. *X is compact if and only if $\mathscr{F} \underset{tact}{\uparrow} Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{k\text{-}tact}{\uparrow} Men_{C,F}(X)$.*

Proof. If σ is a winning k-tactic, then for each open cover $\mathcal{U}$, σ defeats the attack $\langle \mathcal{U}, \mathcal{U}, \dots \rangle$. Then

$$\bigcup_{i \leq k} \sigma(\langle \underbrace{\mathcal{U}, \dots, \mathcal{U}}_{i} \rangle) = X$$

and X is $\mathcal{U}$-finite. □

Essentially, because $\mathscr{C}$ may repeat the same finite sequence of open covers, $\mathscr{F}$ needs to be seeded with knowledge of the round number to prevent being trapped in a loop.

If $\mathscr{F}$'s memory of $\mathscr{C}$'s past moves is bounded, then there is no need to consider more than the two most recent moves. The intuitive reason is that $\mathscr{C}$ could simply play the same cover repeatedly until $\mathscr{F}$'s memory is exhausted, in which case $\mathscr{F}$ would only ever see the change from one cover to another.

Theorem 2.2. *For each $k < \omega$, $F \underset{(k+2)\text{-}mark}{\uparrow} Men_{C,F}(X)$ if and only if $F \underset{2\text{-}mark}{\uparrow} Men_{C,F}(X)$.*

Proof. Let σ be a winning $(k+2)$-mark. We define the 2-mark τ as follows:

$$\tau(\langle \mathcal{U} \rangle, 0) = \bigcup_{m < k+2} \sigma(\langle \underbrace{\mathcal{U}, \dots, \mathcal{U}}_{m+1} \rangle, m)$$

$$\tau(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) = \bigcup_{m < k+2} \sigma(\langle \underbrace{\mathcal{U}, \dots, \mathcal{U}}_{k+1-m}, \underbrace{\mathcal{V}, \dots, \mathcal{V}}_{m+1} \rangle, (n+1)(k+2) + m)$$

Let $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ be an attack by $\mathscr{C}$ against τ. Then consider the attack

$$\langle \underbrace{\mathcal{U}_0, \dots, \mathcal{U}_0}_{k+2}, \underbrace{\mathcal{U}_1, \dots, \mathcal{U}_1}_{k+2}, \dots \rangle$$

by $\mathscr{C}$ against σ. Since σ is a winning $(k+2)$-mark,

$$X = \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}_0, \dots, \mathcal{U}_0}_{m+1} \rangle, m) \cup \bigcup_{n<\omega} \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}_n, \dots, \mathcal{U}_n}_{k+1-m}, \underbrace{\mathcal{U}_{n+1}, \dots, \mathcal{U}_{n+1}}_{m+1} \rangle, (n+1)(k+2)+m)$$

$$= \tau(\langle \mathcal{U}_0 \rangle, 0) \cup \bigcup_{n<\omega} \tau(\langle \mathcal{U}_n, \mathcal{U}_{n+1} \rangle, n+1)$$

Thus τ is a winning 2-mark. ☐

A natural question arises: is there an example of a space X for which $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$ but $\mathscr{F} \underset{\text{mark}}{\not\uparrow} Men_{C,F}(X)$? We quickly see that perhaps the simplest example of a Lindelöf non-σ-compact space has this property.

Definition 2.3. For any cardinal κ, let $\kappa^\dagger = \kappa \cup \{\infty\}$ denote the *one-point Lindelöfication* of discrete κ, where points in κ are isolated, and the neighborhoods of ∞ are the co-countable sets containing it.

Theorem 2.4. $\mathscr{F} \underset{mark}{\not\uparrow} Men_{C,F}\left(\omega_1^\dagger\right)$.

Proof. This result will later follow from the fact that $\omega_1^\dagger$ is not a σ-compact space (all its compact subsets are finite).

For now, let σ be a Markov strategy for $\mathscr{F}$. For each $\alpha < \omega_1$, let $\mathcal{U}_\alpha$ be the open cover $\{\{\beta\} : \beta < \alpha\} \cup \{\omega_1^\dagger \setminus \alpha\}$ of $\omega_1^\dagger$, and set $F(\alpha, n)$ to be the finite set $\alpha \cap \sigma(\langle \mathcal{U}_\alpha \rangle, n)$.

If $P_n = \{\beta : \beta < \alpha < \omega_1 \Rightarrow \beta \in F(\alpha, n)\}$, then $P_n \subseteq F(\sup(P_n)+1, n)$. Thus P_n is finite for $n < \omega$. Choose $\beta \in \omega_1 \setminus \bigcup_{n<\omega} P_n$ and $\alpha_n > \beta$ such that $\beta \notin F(\alpha_n, n)$. Then $\mathscr{C}$ may attack σ with $\langle \mathcal{U}_{\alpha_0}, \mathcal{U}_{\alpha_1}, \dots \rangle$, and it follows that $\beta \notin \bigcup_{n<\omega} F(\alpha_n, n)$ and $\beta \notin \bigcup_{n<\omega} \sigma(\langle \mathcal{U}_{\alpha_n} \rangle, n)$. ☐

The greatest advantage of a strategy which has knowledge of two or more previous moves of the opponent, versus only knowledge of the most recent move, is the ability to react to changes from one round to the next. It's this ability to react that will give $\mathscr{F}$ her winning 2-Markov strategy in the Menger game on $\omega_1^\dagger$.

For inspiration, we turn to a game whose n-tactics were studied by Marion Scheepers in [3] which has similar goals to the Menger game when played upon $\kappa^\dagger$.

Game 2.5. Let $Fill_{C,F}^{\cup,\subset}(\kappa)$ denote the *strict union filling game* with two players $\mathscr{C}$, $\mathscr{F}$. In round 0, $\mathscr{C}$ chooses $C_0 \in [\kappa]^{\leq\omega}$, followed by $\mathscr{F}$ choosing $F_0 \in [\kappa]^{<\omega}$. In round $n+1$, $\mathscr{C}$ chooses $C_{n+1} \in [\kappa]^{\leq\omega}$ such that $C_{n+1} \supset C_n$, followed by $\mathscr{F}$ choosing $F_{n+1} \in [\kappa]^{<\omega}$.

$\mathscr{F}$ wins the game if $\bigcup_{n<\omega} F_n \supseteq \bigcup_{n<\omega} C_n$; otherwise, $\mathscr{C}$ wins.

In $Men_{C,F}\left(\kappa^\dagger\right)$, $\mathscr{C}$ essentially chooses a countable set to not include in her neighborhood of ∞, followed by $\mathscr{F}$ choosing a finite subset of this complement to cover during each round. Thus, $\mathscr{F}$ need only be concerned with the *intersection* of the countable sets chosen by $\mathscr{C}$ in $Men_{C,F}\left(\kappa^\dagger\right)$, rather than the union as in $Fill_{C,F}^{\cup,\subset}(\kappa)$.

Another difference: Scheepers required that $\mathscr{C}$ always choose strictly growing countable sets. The reasoning is clear: if the goal is to study tactics, then $\mathscr{C}$ cannot be allowed to trap $\mathscr{F}$ in a loop by repeating the same moves. But by eliminating this requirement, the study can then turn to Markov strategies, bringing the game further in line with the Menger game played upon $\kappa^\dagger$.

We introduce a few games to make the relationship between Scheeper's $Fill^{\cup,\subset}_{C,F}(\kappa)$ and $Men_{C,F}\left(\kappa^\dagger\right)$ more precise.

Game 2.6. Let $Fill^{\cup,\subseteq}_{C,F}(\kappa)$ denote the *union filling game* which proceeds analogously to $Fill^{\cup,\subset}_{C,F}(\kappa)$, except that $\mathscr{C}$'s restriction in round $n+1$ is reduced to $C_{n+1} \supseteq C_n$.

Game 2.7. Let $Fill^{1,\subseteq}_{C,F}(\kappa)$ denote the *initial filling game* which proceeds analogously to $Fill^{\cup,\subseteq}_{C,F}(\kappa)$, except that $\mathscr{F}$ wins whenever $\bigcup_{n<\omega} F_n \supseteq C_0$.

Game 2.8. Let $Fill^{\cap}_{C,F}(\kappa)$ denote the *intersection filling game* which proceeds analogously to $Fill^{1,\subseteq}_{C,F}(\kappa)$, except that $\mathscr{C}$ may choose any $C_n \in [\kappa]^{\leq\omega}$ each round, and $\mathscr{F}$ wins whenever $\bigcup_{n<\omega} F_n \supseteq \bigcap_{n<\omega} C_n$.

Figure 1. Diagram of Filling/Menger game implications

Theorem 2.9. *For any cardinal $\kappa > \omega$ and integer $k < \omega$, Figure 1 holds.*

Proof. $\mathscr{F} \underset{k\text{-mark}}{\uparrow} Men_{C,F}\left(\kappa^\dagger\right) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Men_{C,F}\left(\kappa^\dagger\right)$. Let $\mathcal{U}(C)$ (resp. $\mathcal{U}(s)$) convert each countable subset C of κ (resp. finite sequence s of such subsets) into the open cover $[C]^1 \cup \{\kappa^\dagger \setminus C\}$ (resp. finite sequence of such open covers). Then τ defined by

$$\tau(s^\frown\langle C\rangle, n) = C \cap \sigma(\mathcal{U}(s^\frown\langle C\rangle), n)$$

is a winning k-mark for $\mathscr{F}$ in $Fill^{\cap}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Men_{C,F}\left(\kappa^\dagger\right)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Fill^{\cap}_{C,F}(\kappa)$. Let $C(\mathcal{U})$ (resp. $C(s)$) convert each open cover $\mathcal{U}$ of $\kappa^\dagger$ (resp. finite sequence s of such covers) into a countable set C which is the complement of some neighborhood of ∞ in $\mathcal{U}$ (resp. finite sequence of such countable sets). Then τ defined by

$$\tau(s^\frown\langle\mathcal{U}\rangle, n) = (\kappa^\dagger \setminus C(\mathcal{U})) \cup \sigma(C(s^\frown\langle\mathcal{U}\rangle)), n)$$

is a winning k-mark for $\mathscr{F}$ in $Men_{C,F}\left(\kappa^\dagger\right)$.

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{1,\subseteq}_{C,F}(\kappa)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Fill^{\cap}_{C,F}(\kappa)$. σ is also a winning k-mark for $\mathscr{F}$ in $Fill^{1,\subseteq}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{1,\subseteq}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cup,\subseteq}_{C,F}(\kappa)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Fill^{1,\subseteq}_{C,F}(\kappa)$. For each finite sequence s, let $t \preceq s$ mean t is a final subsequence of s. Then τ defined by

$$\tau(s^\frown\langle C\rangle, n) = \bigcup_{t \preceq s, m \leq n} \sigma(t^\frown\langle C\rangle, m)$$

is a winning k-mark for $\mathscr{F}$ in $Fill^{\cup,\subsetneq}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cup,\subsetneq}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{1,\subsetneq}_{C,F}(\kappa)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Fill^{\cup,\subsetneq}_{C,F}(\kappa)$. σ is also a winning k-mark for $\mathscr{F}$ in $Fill^{1,\subsetneq}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k\text{-tact}}{\uparrow} Fill^{\cup,\subset}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cup,\subsetneq}_{C,F}(\kappa)$: Let σ be a winning k-tactic for $\mathscr{F}$ in $Fill^{\cup,\subset}_{C,F}(\kappa)$. For each countable subset C of κ, let $C+n$ be the union of C with the n least ordinals in $\kappa \setminus C$. Then τ defined by

$$\tau(\langle C_0, \dots, C_i \rangle, n) = \sigma(\langle C_0 + (n-i), \dots, C_i + n \rangle)$$

is a winning k-mark for $\mathscr{F}$ in $Fill^{\cup,\subsetneq}_{C,F}(\kappa)$. □

While we have not proven a direct implication between the Menger game and Scheeper's original filling game, Scheepers introduced the statement $S(\kappa)$ relating to the almost-compatability of functions from countable subsets of κ into ω which may be applied to both.

Definition 2.10. For two functions f, g we say f is **μ-almost compatible** with g ($f \|^*_\mu g$) if $|\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(x) \neq g(x)\}| < \mu$. If $\mu = \omega$ then we say f, g are **almost compatible** ($f \|^* g$).

Definition 2.11. $S(\kappa)$ states that there exist functions $f_A : A \to \omega$ for each $A \in [\kappa]^{\leq\omega}$ such that $|\{\alpha \in A : f_A(\alpha) \leq n\}| < \omega$ for all $n < \omega$ and $f_A \|^* f_B$ for all $A, B \in [\kappa]^\omega$. [1]

Scheepers went on to show that $S(\kappa)$ implies $\mathscr{F} \underset{2\text{-tact}}{\uparrow} Fill^{\cup,\subset}_{C,F}(\kappa)$. This proof, along with the following facts, give us inspiration for finding a winning 2-Markov strategy in the Menger game played on $\kappa^\dagger$.

Theorem 2.12. *$S(\omega_1)$ and $\kappa > 2^\omega \Rightarrow \neg S(\kappa)$ are theorems of ZFC. $S(2^\omega)$ is a theorem of $ZFC + CH$ and consistent with $ZFC + \neg CH$.*

Proof. For $S(\omega_1)$, look at pg. 70 of [2]; this of course implies $S(2^\omega)$ under CH. $\neg S((2^\omega)^+)$ is shown by a cardinality argument in [3]. The consistency result under $ZFC + \neg CH$ is a lemma for the main theorem in [3]. □

FIGURE 2. Diagram of Filling/Menger game implications with $S(\kappa)$

Theorem 2.13. *$S(\kappa)$ implies the game-theoretic results in Figure 2.*

Proof. Since $S(\kappa) \Rightarrow \mathscr{F} \underset{2\text{-tact}}{\uparrow} Fill^{\cup,\subset}_{C,F}(\kappa)$ was a main result of [3], it is sufficient to show that $S(\kappa) \Rightarrow \mathscr{F} \underset{2\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa)$.

[1] This is equivalent to the original characterization given in [3]: there exist injections $g_A : A \to \omega$ such that $g_A \|^* g_B$ for all $A, B \in [\kappa]^\omega$ and $A \subset B$.

Let f_A for $A \in [\kappa]^{\leq\omega}$ witness $S(\kappa)$. We define the 2-mark σ as follows:

$$\sigma(\langle A \rangle, 0) = \{\alpha \in A : f_A(\alpha) \leq 0\}$$

$$\sigma(\langle A, B \rangle, n+1) = \{\alpha \in A \cap B : f_B(\alpha) \leq n+1 \text{ or } f_A(\alpha) \neq f_B(\alpha)\}$$

For any attack $\langle A_0, A_1, \ldots \rangle$ by $\mathscr{C}$ and $\alpha \in \bigcap_{n<\omega} A_n$, either $f_{A_n}(\alpha)$ is constant for all n, or $f_{A_n}(\alpha) \neq f_{A_{n+1}}(\alpha)$ for some n; either way, α is covered. $\square$

Corollary 2.14. $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}\left(\omega_1^\dagger\right)$.

3. Menger game derived covering properties

Figure 3. Diagram of covering properties related to the Menger game

Limited information strategies for the Menger game naturally define a spectrum of covering properties, see Figure 3. However, we do not know if the middle two properties are actually distinct.

Question 3.1. Does there exist a space X such that $\mathscr{F} \uparrow Men_{C,F}(X)$ but $\mathscr{F} \underset{\text{2-mark}}{\not\uparrow} Men_{C,F}(X)$?

Note that while it's consistent that $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}\left((2^\omega)^\dagger\right)$, $\kappa^\dagger$ for $\kappa > 2^\omega$ is a candidate to answer the above question.

We are also interested in non-game-theoretic characterizations of these covering properties. It has been known for some time that for metrizable spaces, winning Menger spaces are exactly the σ-compact spaces, shown first by Telgarksy in [5] and later directly by Scheepers in [4].

In the interest of generality, we will first characterize the Markov Menger spaces without any separation axioms.

Definition 3.2. A subset Y of X is *relatively compact* to X if for every open cover of X, there exists a finite subcollection which covers Y.

For example, any bounded subset of Euclidean space is relatively compact whether it is closed or not. Actually, relative compactness can be thought of as an analogue of boundedness for regular spaces.

Proposition 3.3. *For regular spaces, Y is relatively compact to X if and only if $\overline{Y}$ is compact in X.* [2]

[2] It should be noted that some authors define relative compactness in this way, but such a definition creates pathological implications for non-regular spaces. For example, the singleton containing the particular point of an infinite space with the particular point topology would not be relatively compact since its closure is not compact, even though it is finite.

Proof. For any space, any subset of a compact set is relatively compact.

Assume Y is relatively compact, let $\mathcal{U}$ be an open cover of $\overline{Y}$, and define $x \in V_x \subseteq \overline{V_x} \subseteq U_x \in \mathcal{U}$ for each $x \in X$. Then if we take a subcollection $\mathcal{F} = \{V_{x_i} : i < n\}$ covering Y by relative compactness, then $\{U_{x_i} : i < n\}$ is a finite subcollection of $\mathcal{U}$ covering $\overline{Y}$, showing compactness. □

We now begin the process of factoring out Scheeper's proof to reveal the limited information implications at work.

Lemma 3.4. *Let $\sigma(\mathcal{U}, n)$ be a Markov strategy for F in $Men_{C,F}(X)$, and $\mathfrak{C}$ collect all open covers of X. Then the set*

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \sigma(\mathcal{U}, n)$$

is relatively compact to X. If σ is a winning Markov startegy, then $\bigcup_{n<\omega} R_n = X$.

Proof. First, for every open cover $\mathcal{U} \in \mathfrak{C}$, $R_n \subseteq \sigma(\mathcal{U}, n)$ is covered by a finite subcollection of $\mathcal{U}$.

Suppose that $x \notin R_n$ for any $n < \omega$. Then for each n, pick $\mathcal{U}_n \in \mathfrak{C}$ such that $x \notin \sigma(\mathcal{U}_n, n)$. Then $\mathscr{C}$ may counter σ with the attack $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$. □

Definition 3.5. A *σ-relatively-compact* space is the countable union of relatively compact subsets.

Corollary 3.6. *The following are equivalent:*

- *X is σ-relatively-compact*
- $\mathscr{F} \underset{pre}{\uparrow} Men_{C,F}(X)$
- $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$

Proof. If $X = \bigcup_{n<\omega} R_n$ for R_n relatively compact, then $\sigma(n) = R_n$ is a winning predetermined strategy, which yields a winning Markov strategy. The previous lemma finishes the proof. □

Corollary 3.7. *Let X be a regular space. The following are equivalent:*

- *X is σ-compact*
- *X is σ-relatively-compact*
- $\mathscr{F} \underset{pre}{\uparrow} Men_{C,F}(X)$
- $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$

For Lindelöf spaces, metrizability is characterized by regularity and second-countability, the latter of which was essentially used by Scheepers in this way:

Lemma 3.8. *Let X be a second-countable space. $\mathscr{F} \uparrow Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$.*

Proof. Let σ be a strategy for $\mathscr{F}$, and note that it's sufficient to consider playthroughs with only basic open covers.

So if $\mathcal{U}_t$ is a basic open cover for $t < s \in \omega^{<\omega}$, and $\mathcal{V}$ is any basic open cover, we may choose a finite subcollection $\mathcal{F}(s, \mathcal{V})$ of $\mathcal{V}$ such that

$$\sigma(\langle \mathcal{U}_{s\restriction 1}, \ldots, \mathcal{U}_s, \mathcal{V}\rangle) \subseteq \bigcup \mathcal{F}(s, \mathcal{V})$$

Note that there are only countably-many finite collections of basic open sets. Thus we may choose basic open covers $\mathcal{U}_{s^\frown\langle n\rangle}$ for $n < \omega$ such that for any basic open cover $\mathcal{V}$, there exists $n < \omega$ where $\mathcal{F}(s, \mathcal{V}) = \mathcal{F}(s, \mathcal{U}_{s^\frown\langle n\rangle})$.

Let $t : \omega \to \omega^{<\omega}$ be a bijection. We define the Marköv strategy τ as follows:

$$\tau(\langle \mathcal{V}\rangle, n) = \bigcup \mathcal{F}(t(n), \mathcal{V})$$

Suppose there exists a counter-attack $\langle \mathcal{V}_0, \mathcal{V}_1, \ldots\rangle$ of basic open covers which defeats τ. Then there exists $f : \omega \to \omega$ such that, letting $t(m_n) = f \restriction n$:

$$\begin{array}{rcl} x & \notin & \tau(\langle \mathcal{V}_{m_n}\rangle, m_n) \\ & = & \bigcup \mathcal{F}(f \restriction n, \mathcal{V}_{m_n}) \\ & = & \bigcup \mathcal{F}(f \restriction n, \mathcal{U}_{f\restriction(n+1)}) \\ & \supseteq & \sigma(\langle \mathcal{U}_{f\restriction 1}, \ldots, \mathcal{U}_{f\restriction(n+1)}\rangle) \end{array}$$

Thus $\langle \mathcal{U}_{f\restriction 1}, \mathcal{U}_{f\restriction 2}, \ldots\rangle$ is a successful counter-attack by $\mathscr{C}$ against the perfect information strategy σ. $\square$

Corollary 3.9. *Let X be a second-countable space. The following are equivalent:*

- *X is σ-relatively-compact*
- $F \underset{pre}{\uparrow} Men_{C,F}(X)$
- $F \underset{mark}{\uparrow} Men_{C,F}(X)$
- $F \uparrow Men_{C,F}(X)$

Corollary 3.10. *Let X be a metrizable space. The following are equivalent:*

- *X is σ-compact*
- *X is σ-relatively-compact*
- $F \underset{pre}{\uparrow} Men_{C,F}(X)$
- $F \underset{mark}{\uparrow} Men_{C,F}(X)$
- $F \uparrow Men_{C,F}(X)$

Proof. Each property implies Lindelöf, so X may be assumed to be regular and second-countable. $\square$

4. Robustly Lindelöf

To help describe $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$ topologically, we introduce a subset variant of the Menger game and a related covering property.

Game 4.1. Let $Men_{C,F}(X,Y)$ denote the *Menger subspace game* which proceeds analogously to the Menger game, except that $\mathscr{F}$ wins whenever $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover for $Y \subseteq X$.

Note of course that $Men_{C,F}(X,X) = Men_{C,F}(X)$.

Definition 4.2. A subset Y of X is *relatively robustly Menger* if there exist functions $r_\mathcal{V} : Y \to \omega$ for each open cover $\mathcal{V}$ of X such that for all open covers $\mathcal{U}, \mathcal{V}$ and numbers $n < \omega$, the following sets are $\mathcal{V}$-finite:

$$c(\mathcal{V}, n) = \{x \in Y : r_\mathcal{V}(x) \leq n\}$$

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{x \in Y : n < r_\mathcal{U}(x) < r_\mathcal{V}(x)\}$$

Definition 4.3. A space X is *robustly Menger* if it is relatively robustly Menger to itself.

Proposition 4.4. *All σ-relatively-compact spaces are robustly Menger.*

Proof. If $X = \bigcup_{n<\omega} R_n$, then for all $\mathcal{U}$, let $r_\mathcal{U}(x)$ be the least n such that $x \in R_n$. Then $c(\mathcal{V}, n) = \bigcup_{m \leq n} R_m$ and $p(\mathcal{U}, \mathcal{V}) = \emptyset$. □

Theorem 4.5. *If $Y \subseteq X$ is relatively robustly Menger, then $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}(X,Y)$.*

Proof. We define the Markov strategy σ as follows. Let $\sigma(\langle \mathcal{U} \rangle, 0) = c(\mathcal{U}, 0)$, and let $\sigma(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) = c(\mathcal{V}, n+1) \cup p(\mathcal{U}, \mathcal{V}, n+1)$.

For any attack $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ by $\mathscr{C}$ and $x \in Y$, one of the following must occur:

- $r_{\mathcal{U}_0}(x) = 0$ and thus $x \in c(\mathcal{U}_0, 0) \subseteq \sigma(\langle \mathcal{U}_0 \rangle, 0)$.
- $r_{\mathcal{U}_0}(x) = N+1$ for some $N \geq 0$ and:
  - For all $n \leq N$,
    $$r_{\mathcal{U}_{n+1}}(x) \leq N+1$$
    and thus $x \in c(\mathcal{U}_{N+1}, N+1) \subseteq \sigma(\langle U_N, U_{N+1} \rangle, N+1)$.
  - For some $n \leq N$,
    $$r_{\mathcal{U}_n}(x) \leq n$$
    and thus $x \in c(\mathcal{U}_{n+1}, n+1) \subseteq \sigma(\langle U_n, U_{n+1} \rangle, n+1)$.
  - For some $n \leq N$,
    $$n < r_{\mathcal{U}_n}(x) \leq N+1 < r_{\mathcal{U}_{n+1}}(x)$$
    and thus $x \in p(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1) \subseteq \sigma(\langle U_n, U_{n+1} \rangle, n+1)$

□

Theorem 4.6. *$S(\kappa)$ implies $\kappa^\dagger$ is robustly Menger, and thus $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}\left(\kappa^\dagger\right)$.*

Proof. Let f_A for $A \in [\kappa]^{\leq\omega}$ witness $S(\kappa)$ and fix $A(\mathcal{U}) \in [\kappa]^{\leq\omega}$ for each open cover $\mathcal{U}$ such that $\kappa^\dagger \setminus A(\mathcal{U})$ is contained in some element of $\mathcal{U}$. Then let $r_\mathcal{U}(x) = 0$ for $x \in \kappa^\dagger \setminus A(\mathcal{U})$, and $r_\mathcal{U}(\alpha) = f_{A(\mathcal{U})}(\alpha)$ for $\alpha \in A(\mathcal{U})$.

It follows that

$$c(\mathcal{U}, n) = \left(\kappa^\dagger \setminus A(\mathcal{U})\right) \cup \{\alpha \in A(\mathcal{U}) : f_{A(\mathcal{U})}(\alpha) \leq n\}$$

is $\mathcal{U}$-finite, $\bigcup_{n<\omega} c(\mathcal{U}, n) = X$, and

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{\alpha \in A(\mathcal{U}) \cap A(\mathcal{V}) : n < f_{A(\mathcal{U})}(\alpha) < f_{A(\mathcal{V})}(\alpha)\}$$

is finite. $\square$

We may also consider common (non-regular) counterexamples which are finer than the usual Euclidean line.

Definition 4.7. Let $R_{\mathbb{Q}}$ be the real line with the topology generated by open intervals with or without the rationals removed.

Theorem 4.8. *$R_{\mathbb{Q}}$ is non-regular and non-σ-compact, but is second-countable and σ-relatively-compact.*

Proof. Compact sets in $R_{\mathbb{Q}}$ can be shown to not contain open intervals, and thus are nowhere dense in nonmeager $\mathbb{R}$, so $R_{\mathbb{Q}}$ is not σ-compact. The usual base of intervals with rational endpoints (with or without rationals removed) witnesses second-countability.

To see that $R_{\mathbb{Q}}$ is σ-relatively compact, consider $[a, b] \setminus \mathbb{Q}$. Let $\mathcal{U}$ be a cover of $R_{\mathbb{Q}}$, and let $\mathcal{U}'$ fill in the missing rationals for any open set in $\mathcal{U}$. There is a finite subcover $\mathcal{V}' \subseteq \mathcal{U}'$ for $[a, b]$ since $\mathcal{U}'$ contains open sets from the Euclidean topology. Let $\mathcal{V} = \{V \setminus \mathbb{Q} : V \in V'\}$: this is a finite refinment of $\mathcal{U}$ covering $[a, b] \setminus \mathbb{Q}$, so $[a, b] \setminus \mathbb{Q}$ is relatively compact. It follows then that $R_{\mathbb{Q}} \setminus \mathbb{Q}$ is σ-relatively-compact, and since $\mathbb{Q}$ is countable, $R_{\mathbb{Q}}$ is σ-relatively-compact. Non-regularity follows since regular and σ-relatively-compact implies σ-compact. $\square$

Definition 4.9. Let R_ω be the real line with the topology generated by open intervals with countably many points removed.

Theorem 4.10. *R_ω is non-regular, non-second-countable, and non-σ-relatively-compact, but $\mathscr{F} \uparrow Men_{C,F}(R_\omega)$.*

Proof. The closure of any open set is its closure in the usual Euclidean topology, so R_ω is not regular. If $S \supseteq \{s_n : n < \omega\}$ for s_n discrete, then $U_m = R_\omega \setminus \{s_n : m < n < \omega\}$ yields an infinite cover $\{U_m : m < \omega\}$ with no finite subcollection covering S, showing that all relatively compact sets are finite, and R_ω is not σ-relatively-compact.

Define the winning strategy σ for $\mathscr{F}$ in $Men_{C,F}(R_\omega)$ as follows: let $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{2n}) = [-n, n] \setminus C_n$ for some countable $C_n = \{c_{n,m} : m < \omega\}$, and let $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{2n+1}) = \{c_{i,j} : i, j < n\}$. Non-second-countable follows since second-countable and $\mathscr{F} \uparrow Men_{C,F}(X)$ implies σ-relatively-compact. $\square$

We will soon see that, assuming $S(2^\omega)$, $\mathscr{F}$ has a winning 2-Markōv strategy for $Men_{C,F}(R_\omega)$ as well.

Proposition 4.11. *Let $\underset{limit}{\uparrow}$ be either $\underset{k\text{-}mark}{\uparrow}$ or $\uparrow$. If $X = \bigcup_{i<\omega} X_i$ and $\mathscr{F} \underset{limit}{\uparrow} Men_{C,F}(X, X_i)$ for $i < \omega$, then $\mathscr{F} \underset{limit}{\uparrow} Men_{C,F}(X)$*

Proof. Let L be the k-Markov fog-of-war μ_k (resp. the identity), and let σ_i be a k-Markov strategy (resp. perfect information strategy) for $\mathscr{F}$ in $Men_{C,F}(X, X_i)$.

We define the k-Markov strategy (resp. perfect information strategy) σ for $Men_{C,F}(X)$ as follows:

$$\sigma \circ L(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle) = \bigcup_{i \leq n} \sigma_i \circ L(\langle \mathcal{U}_i, \ldots, \mathcal{U}_n \rangle)$$

Let $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ be a successful counter-attack by $\mathscr{C}$ against σ. Then there exists $x \in X_i$ for some $i < \omega$ such that x is not covered by $\bigcup_{n<\omega} \sigma \circ L(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$. It follows that x is not covered by $\bigcup_{n<\omega} \sigma_i \circ L(\langle \mathcal{U}_i, \ldots, \mathcal{U}_{i+n} \rangle)$, and $\langle \mathcal{U}_i, \mathcal{U}_{i+1}, \ldots \rangle$ is a successful counter-attack by $\mathscr{C}$ against σ_i. $\square$

Theorem 4.12. *If $S(2^\omega)$, then $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}(R_\omega)$.*

Proof. It's sufficient to show that $[0,1] \subseteq R_\omega$ is relatively robustly Menger. Let f_A witness $S(2^\omega)$ for $A \in [[a,b]]^{\leq \omega}$. For each open cover $\mathcal{U}$, let $A_\mathcal{U}$ be such that $[0,1] \setminus A_\mathcal{U}$ is $\mathcal{U}$-finite. Let $r_\mathcal{U}(x) = 0$ if $x \in [0,1] \setminus A_\mathcal{U}$ and $r_\mathcal{U}(x) = f_{A_\mathcal{U}}(x)$ otherwise.

It follows then that

$$c(\mathcal{U}, n) = [0,1] \setminus \{x \in A_\mathcal{U} : f_{\mathcal{A}_\mathcal{U}}(x) > n\}$$

is $\mathcal{U}$-finite and

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{x \in A_\mathcal{U} \cap A_\mathcal{V} : n < f_{\mathcal{A}_\mathcal{U}}(x) < f_{\mathcal{A}_\mathcal{V}}(x)\}$$

is finite. $\square$

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Department of Mathematics and Statistics, UNC Charlotte, Charlotte, NC 28262

E-mail address: `steven.clontz@gmail.edu`

URL: `stevenclontz.com`