Limited Information Strategies for Topological Games

by

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Abstract

I talk a lot about topological games.

TODO: Write this.

Acknowledgments

TODO: Thank people.

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Chapter 1

W convergence and clustering games

We begin by investigating a game due to Gary Gruenhage.

Game 1.0.1. Let $Con_{O,P}(X, S)$ denote the *W-convergence game* with players $\mathscr{O}$, $\mathscr{P}$, for a topological space X and $S \subseteq X$.

In round n, $\mathscr{O}$ chooses an open neighborhood $O_n \supseteq S$, followed by $\mathscr{P}$ choosing a point $x_n \in \bigcap_{m \leq n} O_m$.

$\mathscr{O}$ wins the game if the points x_n converge to the set S; that is, for every open neighborhood $U \supseteq S$, $x_n \in U$ for all but finite $n < \omega$.

If $S = \{x\}$ then we write $Con^{\star}_{O,P}(X, x)$ for short. $\Diamond$

(TODO: Any reason for the "W"?)

Gruenhage defined this game in his doctoral dissertation to define a class of spaces generalizing first-countability. [1]

Definition 1.0.2. The spaces X for which $\mathscr{O} \uparrow Con^{\star}_{O,P}(X, x)$ for all $x \in X$ are called *W-spaces.* $\Diamond$

In fact, using limited information strategies, one may characterize the first-countable spaces using this game.

Proposition 1.0.3. *X is first countable if and only if $\mathscr{O} \underset{pre}{\uparrow} Con^{\star}_{O,P}(X, x)$ for all $x \in X$.* $\Diamond$

Proof. The forward implication shows that all W spaces are first-countable spaces, and was proven in [1]: if $\{U_n : n < \omega\}$ is a countable base at x, let $\sigma(n) = \bigcap_{m \leq n} U_m$. σ is easily seen to be a winning predetermined strategy.

If X is not first countable at some x, let σ be a predetermined strategy for $\mathscr{O}$ in $Con^{\star}_{O,P}(X,x)$. There exists an open neighborhood U such that U is not a subset of any $\sigma(n)$ (otherwise $\{\sigma(n) : n < \omega\}$ would be a countable base at x). Let x_n be an element of $\sigma(n) \setminus U$ for all $n < \omega$. Then $\langle x_0, x_1, \dots \rangle$ is a winning counter-attack to σ for $\mathscr{P}$, so $\mathscr{O}$ lacks a winning predetermined strategy. □

At first glance, the difficulty of $Con_{O,P}(X,S)$ could be increased for $\mathscr{O}$ by only restricting the choices for $\mathscr{P}$ to be within the most recent open set played by $\mathscr{O}$, rather than all the previously played open sets.

Definition 1.0.4. Let $Con_{O,P}(X,S)$ denote the *hard W-convergence game* which proceeds as $Con^{\star}_{O,P}(X,S)$, except that $\mathscr{P}$ need only play within O_n rather than $\bigcap_{m \leq n} O_n$. ◊

Of course, this is normally easily circumvented.

Proposition 1.0.5. *$\mathscr{O} \underset{limit}{\uparrow} Con_{O,P}(X,S)$ if and only if $\mathscr{O} \underset{limit}{\uparrow} Con^{\star}_{O,P}(X,S)$, where $\underset{limit}{\uparrow}$ is either $\uparrow$ or $\underset{pre}{\uparrow}$.* ◊

Proof. The backwards implication is immediate.

For the forward implication, let σ be a winning predetermined (perfect information) strategy, and λ be μ_0 (the identity).

We define a new predetermined (perfect information) strategy τ by

$$\tau \circ \lambda(\langle x_0, \dots, x_{n-1} \rangle) = \bigcap_{m \leq n} \sigma \circ \lambda(\langle x_0, \dots, x_{m-1} \rangle)$$

so that each move by $\mathscr{O}$ according to τ is the intersection of $\mathscr{O}$'s previous moves. Then any attack against τ is an attack against σ, and since σ is a winning strategy, so is τ. □

It's important to note that τ is only well-defined in the case $\lambda = \mu_0$ because the value of $\sigma \circ \lambda(\langle x_0, \dots, x_{m-1} \rangle)$ does not rely on knowledge of the points $\langle x_0, \dots, x_{m-1} \rangle$ for any $m \leq n$. The proof would be invalid if λ was required to be, say, ν_{k+1}, since the value of $\sigma \circ \nu_{k+1}(\langle x_0, \dots, x_k \rangle) = \sigma(\langle x_0, \dots, x_k \rangle)$ could not be uniquely determined from $\nu_{k+1}(\langle x_0, \dots, x_{k+1} \rangle) = \langle x_1, \dots, x_{k+1} \rangle$.

Due to the equivalency of the "hard" and "normal" variations of the convergence game in the perfect information case, most authors use them interchangibly. However, it is easy to find spaces for which the games are not equivalent when considering $k+1$-tactics and $k+1$-marks.

In addition to the W-convergence games, we will also investigate "clustering" analogs to both variations.

Game 1.0.6. Let $Clus_{O,P}(X,S)$ (resp. $Clus^{\star}_{O,P}(X,S)$) be a variation of $Con_{O,P}(X,S)$ (resp. $Con^{\star}_{O,P}(X,S)$) such that x_n need only cluster at S, that is, for every open neighborhood U of S, $x_n \in U$ for infinitely many $n < \omega$. $\diamondsuit$

Gruenhage noted that the clustering game is perfect-information equivalent to the convergence game for $\mathscr{O}$. This can easily be extended for some limited information cases as well.

Proposition 1.0.7. *$\mathscr{O} \underset{limit}{\uparrow} Con_{O,P}(X,S)$ if and only if $\mathscr{O} \underset{limit}{\uparrow} Clus_{O,P}(X,S)$ where $\underset{limit}{\uparrow}$ is any of $\uparrow$, $\underset{pre}{\uparrow}$, $\underset{tact}{\uparrow}$, or $\underset{mark}{\uparrow}$.* $\diamondsuit$

Proof. For the perfect information case we refer to [1].

In the predetermined (resp. tactical) case, suppose that σ is a winning predetermined (resp. tactical) strategy for $\mathscr{O}$ in $Clus_{O,P}(X,S)$. Let p be a legal attack against σ, and q be a subsequence of p. It's easily seen that q is also a legal attack against σ, so q clusters at S. Since every subsequence of p clusters at S, p converges to S, and σ is a winning predetermined (resp. tactical) strategy for $\mathscr{O}$ in $Con_{O,P}(X,S)$ as well.

In the final case, note that any Markōv strategy σ for $\mathscr{O}$ may be strengthened by setting $\sigma'(x,n) = \bigcap_{m \leq n} \sigma(x,m)$. So, suppose that σ is a winning Markōv strategy for $\mathscr{O}$ in $Clus_{O,P}(X,S)$ such that $\sigma(x,m) \supseteq \sigma(x,n)$ for all $m \leq n$.

Let p be a legal attack against σ, and q be a subsequence of p. For $m < \omega$, there exists $f(m) \leq m$ such that $q(m) = p(f(m))$. It follows that

$$q(n+1) \in \bigcap_{m \leq n} \sigma(q(m), m+1) = \bigcap_{m \leq n} \sigma(p(f(m)), m+1)$$

$$\subseteq \bigcap_{m \leq n} \sigma(p(f(m)), f(m)+1) \subseteq \bigcap_{m \leq f(n)} \sigma(p(m), m+1)$$

so q is also a legal attack against σ. Since σ is a winning strategy, q clusters at S, and since every subsequence of p clusters at S, p must converge to S. Thus σ is also a winning Marköv strategy for $\mathscr{O}$ in $Con_{O,P}(X, S)$ as well. $\square$

(TODO: Maybe $k+2$ tacts/marks as well, but not as obvious if so.)

1.1 Fort spaces

Gruenhage suggested the one-point-compactification of a discrete space as an example of a W-space which is not first-countable.

Definition 1.1.1. A *Fort space* $\kappa^* = \kappa \cup \{\infty\}$ is defined for each cardinal κ. Its subspace κ is discrete, and the neighborhoods of ∞ are of the form $\kappa^* \setminus F$ for each $F \in [\kappa]^{<\omega}$. $\Diamond$

Theorem 1.1.2. *$O \underset{k\text{-tact}}{\not\uparrow} Clus_{O,P}(\omega_1^*, \infty)$.* $\Diamond$

Theorem 1.1.3. *$O \underset{mark}{\uparrow} Clus_{O,P}(\omega_1^*, \infty)$.* $\Diamond$

Theorem 1.1.4. *(Nyikos) $O \underset{mark}{\not\uparrow} Con_{O,P}(\omega_1^*, \infty)$.* $\Diamond$

Theorem 1.1.5. *$O \underset{k\text{-mark}}{\not\uparrow} Clus_{O,P}(\kappa^*, \infty)$ for $\kappa > \omega_1$.* $\Diamond$

(TODO: It's feasible that k-limit $\Leftrightarrow$ 1-limit.)

1.2 Sigma-products

Theorem 1.2.1. *Let $cf([\kappa]^{\leq\omega}) = \kappa$. Then $F \underset{code}{\uparrow} PF_{F,C}(\kappa)$.* $\Diamond$

Theorem 1.2.2. *Let κ be the limit of cardinals κ_n such that $cf([\kappa_n]^{\leq\omega}, \subseteq) = \kappa_n$. Then $F \underset{code}{\uparrow} PF_{F,C}(\kappa)$.* $\Diamond$

Theorem 1.2.3. *$F \underset{code}{\uparrow} PF_{F,C}(\kappa)$ for all cardinals κ.* $\Diamond$

Corollary 1.2.4. *$O \underset{code}{\uparrow} Con_{O,P}(\Sigma\mathbb{R}^{\kappa}, \vec{0})$ for all cardinals κ.* $\Diamond$

Bibliography

[1] Gary Gruenhage. Infinite games and generalizations of first-countable spaces. *General Topology and Appl.*, 6(3):339–352, 1976.