# Scheeper's Meager-NWD Game and the Menger Game

## AU Topology Seminar

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# Abstract

Marion Scheepers designed the Meager-NWD game $Fill^{\subsetneq}_{\mathscr{M},\mathscr{N}}(J)$ in the 80s to study the existence of $k$-tactics in set-theoretic and topological games.

There are strong similarities between Dr. Scheeper's game and the special case of the Menger game $Cov_{\mathscr{C},\mathscr{F}}(\kappa^{\dagger})$ played upon the one-point "Lindelöfication" of a discrete cardinal $\kappa$.

We will explore the relationship between $k$-tactical stratgies in $Fill^{\subsetneq}_{\mathscr{M},\mathscr{N}}(J)$ and $k$-Marköv strategies in $Fill^{\subseteq}_{\mathscr{M},\mathscr{N}}(J)$ or $Cov_{\mathscr{C},\mathscr{F}}(\kappa^{\dagger})$, as well as a sentence $S(\kappa,\omega,\omega)$ which is consistent with ZFC.

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# Menger Game

## Game

The two-player Menger Game $Cov_{\mathscr{C},\mathscr{F}}(X)$ proceeds as follows:

- Round $n$: player $\mathscr{C}$ chooses an open cover $\mathcal{U}_n$ of $X$
- Round $n$: player $\mathscr{F}$ chooses finite $\mathcal{F}_n \subseteq \mathcal{U}_n$.

$\mathscr{F}$ wins if $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of $X$.

- Easy to see that $\mathscr{F}$ can win for any $\sigma$-compact space.
- The existence or non-existence of various limited info strategies in this game characterize covering properties of $X$.

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# $Cov_{\mathscr{C}\mathscr{F}}(X)$ characterizations

$$\begin{array}{ccccccc}
\mathscr{F} \uparrow_{\text{mark}} Cov_{\mathscr{C}\mathscr{F}}(X) & \Rightarrow & \mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(X) & \Rightarrow & \mathscr{F} \uparrow Cov_{\mathscr{C}\mathscr{F}}(X) & \Rightarrow & \mathscr{C} \not\uparrow Cov_{\mathscr{C}\mathscr{F}}(X) \\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \\
X \text{ is } \sigma\text{-(rel. compact)} & \Rightarrow & ??? & \Rightarrow & ??? & \Rightarrow & X \text{ is Menger}
\end{array}$$

- $\uparrow$ denotes a player with a **winning strategy**
- $\uparrow_{\text{mark}}$ denotes a player with a winning **Markö v** strategy (using only the round number and most recent move of opponent)
- $\uparrow_{k\text{-mark}}$ denotes a player with a winning $k$-**Marköv** strategy (using only the round number and $k$ most recent moves of opponent)

**Theorem**

*Assume $k \geq 2$. $\mathscr{F} \uparrow_{k\text{-mark}} Cov_{\mathscr{C}\mathscr{F}}(X) \Leftrightarrow \mathscr{F} \uparrow_{2\text{-mark}} Cov_{\mathscr{C}\mathscr{F}}(X)$*

**Theorem**

*For $X$ second-countable, $\mathscr{F} \uparrow_{\text{mark}} Cov_{\mathscr{C}\mathscr{F}}(X) \Leftrightarrow \mathscr{F} \uparrow Cov_{\mathscr{C}\mathscr{F}}(X)$*

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# $Cov_{\mathscr{C}\mathscr{F}}(X)$ characterizations

Here are a couple properties between $\sigma$-compact and Menger:

- Alster
- Hurewicz

An example of a Menger space which doesn't yield a Markov strategy for $\mathscr{F}$ in the Menger game is $\omega_1^\dagger$.

($\kappa^\dagger = \kappa \cup \{\infty\}$ is the one-point "Lindelöfication" of discrete $\kappa$.)

**Theorem**

$\mathscr{F} \not\uparrow_{mark} Cov_{\mathscr{C}\mathscr{F}}(\omega_1^\dagger)$ but $\mathscr{F} \uparrow_{2\text{-}mark} Cov_{\mathscr{C}\mathscr{F}}(\omega_1^\dagger)$

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# What about $Cov_{\mathscr{C}\mathscr{F}}(\kappa^\dagger)$?

- The direct proof of $\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(\omega_1^\dagger)$ uses injective functions $f_\alpha : \alpha \to \omega$ for each $\alpha < \omega_1$ such that for $\alpha < \beta$:

$$|\{\gamma < \alpha : f_\alpha(\gamma) \neq f_\beta(\gamma)\}| < \omega$$

  (Proof in Kunen's set theory text, used for construction of an Aronszajn tree)
- Would like to extend this idea for $\kappa > \omega_1$ to show $\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(\kappa^\dagger)$...

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# $Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J)$

## Game

The **strict filling game** $Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J)$ on an ideal $J$ proceeds as follows:

- Round 0: player $\mathscr{M}$ chooses $M_0 \in \langle J \rangle$, the $\sigma$-completion of $J$ (closure under countable unions).
- Round 0: player $\mathscr{N}$ chooses $N_0 \in J$.
- Round $n+1$: player $\mathscr{M}$ chooses $M_{n+1}$ where $M_n \subsetneq M_{n+1} \in \langle J \rangle$
- Round $n+1$: player $\mathscr{N}$ replies with $N_{n+1} \in J$.

Player $\mathscr{N}$ wins the game if $\bigcup_{n<\omega} N_n \supseteq \bigcup_{n<\omega} M_n$.

- The sets in $\langle J \rangle$ and $J$ are referred to as meager and nowhere-dense sets, respectively.
  - For any topological space, the set of nowhere dense sets $J$ forms an ideal.
  - For every ideal $J$, there is a topological space where $J$ is the set of nowhere dense sets.
- This game was defined and studied by Marion Scheepers. Here's some facts.

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$$\mathscr{N} \uparrow Fill^{\subsetneq}_{\mathscr{M},\mathscr{N}}(J)$$

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## Proposition

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## Theorem

$$\mathscr{N} \uparrow_{tact} Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J) \Leftrightarrow J = \langle J \rangle$$

- $\uparrow_{\text{tact}}$ denotes a player with a winning **tactical** strategy (using only the most recent move of opponent)
- $\uparrow_{k\text{-tact}}$ denotes a player with a winning $k$-**tactical** strategy (using only the $k$ most recent moves of opponent)

## Theorem

*Assume $cf(\langle J \rangle) = \omega_1$. Let $J_X = \{N \cap X : N \in J\}$. $\mathscr{N} \uparrow_{k\text{-tact}} Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J) \Leftrightarrow \mathscr{N} \uparrow_{k\text{-tact}} Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J_X)$ for each $X \in \langle J \rangle \setminus J$*

**Proof:** $\Rightarrow$ is straight-forward.

Sketch of $\Leftarrow$: Let $S_\alpha$ for $\alpha < \omega_1$ enumerate a cofinal set of $\langle J \rangle$, with $\beta \leq \alpha \Rightarrow S_\beta \subseteq S_\alpha$. Assume the latest move by $\mathscr{M}$ is contained by $S_\alpha$. There are two types of attacks that $\mathscr{N}$ must defeat.

1. $\mathscr{M}$'s attack may never go outside $S_\alpha$, so $\mathscr{N}$ can cover according to the strategy for $\mathscr{N} \uparrow_{k\text{-tact}} Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(S_\alpha)$.
2. $\mathscr{M}$'s attack may eventually exceed $S_\alpha$, but by using tree arrangments $<_n$ of $\omega_1$ of finite height approximating $<$, $\mathscr{N}$ can cover according to the *winning perfect information strategy* as though $\mathscr{M}$ had played sets $S_\beta$ for $\beta \leq_n \alpha$ instead.

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### Corollary

*If $|\bigcup J| \leq \omega_1$ and $|M| \leq \omega$ for $M \in \langle J \rangle$, then $\mathscr{N} \uparrow_{2\text{-tact}} Fill^{\subsetneq}_{\mathscr{M}\mathscr{N}}(J)$.*

**Proof:** Assume $\omega \in \langle J \rangle$ and assume the two latest moves of $\mathscr{M}$ are $M \subsetneq M' \subseteq \omega$. Let $n = \min(M' \setminus M)$, and have $\mathscr{N}$ cover $\{0, \dots, n\}$. It follows that the generated $n$ must be unbounded for any legal attack by $\mathscr{M}$, making it a winning 2-tactic for $Fill^{\subsetneq}_{\mathscr{M}\mathscr{N}}(J_\omega)$.

Apply the previous theorem to finish the result. □

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# Countable Finite Game

## Game

The special case of $Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J)$ where $J = [\kappa]^{<\omega}$ is the Countable-Finite game $Fill^{\subsetneq}_{\mathscr{C}\,\mathscr{F}}(\kappa)$.

## Corollary

$$\mathscr{F} \uparrow_{2\text{-}tact} Fill^{\subsetneq}_{\mathscr{C}\,\mathscr{F}}(\omega_1)$$

So $\mathscr{F} \uparrow_{\text{2-tact}} Fill^{\subsetneq}_{\mathscr{C}\,\mathscr{F}}(\omega_1)$ and $\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\,\mathscr{F}}(\omega_1^{\dagger})$. In addition, the basic goal of $\mathscr{F}$ in $Cov_{\mathscr{C}\,\mathscr{F}}(\omega_1^{\dagger})$ is similar to the goal of $\mathscr{F}$ in $Fill^{\subsetneq}_{\mathscr{C}\,\mathscr{F}}(\omega_1)$: $\mathscr{F}$ can cover a co-countable neighborhood of $\infty$ in the initial round, and is trying to cover the countable remainder in the following rounds (most likely using finitely many singletons from $\mathscr{C}$'s covers).

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- Question: why does $\mathscr{F}$ need the round number in $Cov_{\mathscr{C}\mathscr{F}}(\omega_1^\dagger)$ and not $Fill^{\subseteq}_{\mathscr{C}\mathscr{F}}(\omega_1)$?

**Proposition**

*$\mathscr{F} \uparrow_{k\text{-tact}} Cov_{\mathscr{C}\mathscr{F}}(X) \Leftrightarrow X$ is compact*

**Proof:** If $X$ isn't compact, and $\mathscr{C}$ constantly chooses an open cover $\mathcal{U}$ without a finite subcover for $X$ throughout the entire game, then $\mathscr{F}$ only chooses $k$ different finite subcollections of $\mathcal{U}$ by the game's end, which cannot cover $X$.

If $X$ is compact, $\mathscr{F} \uparrow_{\text{tact}} Cov_{\mathscr{C}\mathscr{F}}(X)$ trivially. □

- Answer: $\mathscr{C}$ cannot choose a constant strategy in $Fill^{\subseteq}_{\mathscr{C}\mathscr{F}}(\kappa)$, but $\mathscr{C}$ can in $Cov_{\mathscr{C}\mathscr{F}}(\kappa^\dagger)$.

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This provides the motivation to change the rules of Scheeper's game to bring it more in line with the Menger game.

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$Fill^{\subseteq}_{\mathscr{C}\,\mathscr{F}}(\kappa)$ is identical to $Fill^{\subseteq}_{\mathscr{M}\,\mathscr{N}}([\kappa]^{<\omega})$

It seems reasonable to ask if $k$-tactics in $Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J)$ correspond to $k$-Marköv strategies in $Fill^{\subseteq}_{\mathscr{M}\,\mathscr{N}}(J)$.

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## Theorem

$$\mathscr{N} \uparrow_{2\text{-}tact} Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(J) \Rightarrow \mathscr{N} \uparrow_{2\text{-}mark} Fill^{\subseteq}_{\mathscr{M}\,\mathscr{N}}(J)$$

**Proof:** Enumerate the sets in $J$ as $A_\alpha$ for $\alpha < |J|$. For $M \in \langle J \rangle$ and $n < \omega$, let $M + 0 = M$ and $M + n + 1$ be the union of $M + n$ and the least $A_\alpha$ not contained in $M + n$.

Let $\sigma$ be a winning 2-tactical strategy for $N$ in $Fill^{\subsetneq}_{\mathscr{M}\,\mathscr{N}}(\kappa)$, and assume $\sigma(M) \cup \sigma(M') \subseteq \sigma(M, M')$.

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$$\tau(M_0, 0) = \sigma(M_0)$$

$$\tau(M_n, M_{n+1}, n+1) = \begin{cases} \sigma(M_n, M_{n+1}) & \text{if } M_n \subsetneq M_{n+1} \\ \bigcup_{m \leq n} \sigma(M_n + m, M_{n+1} + m + 1) & \text{otherwise} \end{cases}$$

- (Essentially, if $\mathscr{M}$ tries to be tricky and not increase the size of her meager set, $\mathscr{N}$ can pretend she added a few extra nowhere dense sets based on the round number.)

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Let $M_0 \subseteq M_1 \subseteq \dots$ be an attack by $\mathscr{M}$ against $\tau$. There are two possible cases:

- Assume $M_n = M_N$ for all $n \geq N$.
  The collection produced by $\sigma$ versus the attack

$$M_N + 0 \subsetneq M_N + 1 \subsetneq \dots$$

must cover $M_N$ as $\sigma$ is a winning strategy.
Let $x \in M_N$. If $x \in \sigma(M_N + 0)$, then $x$ will be covered in round $N + 1$ by

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But the converse need not hold.

**Theorem**

*There is a free ideal $J$ such that $\mathscr{N} \not\uparrow_{2\text{-tact}} Fill^{\subseteq}_{\mathscr{M}\mathscr{N}}(J)$ but $\mathscr{N} \uparrow_{2\text{-mark}} Fill^{\subseteq}_{\mathscr{M}\mathscr{N}}(J)$.*

**Proof:** This counterexample was constructed by Scheepers for another purpose, but works for us as well. Assume $\mathbb{R}$ has the usual Euclidean topology.

Choose $A \subseteq \mathbb{R}$ such that $|A| = \omega$ and $A$ is meager but not nowhere dense. Then choose $V \subseteq \mathbb{R}$ such that $|V| = 2^\omega$, $V$ is meager, and $V$ is disjoint from $A$. Assume $A = \{a_n : n < \omega\}$.

Certainly, if $J$ is the collection of nowhere dense subsets of $A \cup V$, then $F \uparrow_{2\text{-mark}} Fill^{\subseteq}_{\mathscr{M}\mathscr{N}}(J)$. In fact, since $A \cup V$ is meager, $F \uparrow_{\text{pre}} Fill^{\subseteq}_{\mathscr{M}\mathscr{N}}(J)$ ($\mathscr{F}$ has a **predetermined strategy** using only the round number).

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By Cor 28 of Scheepers' "Partition relation for partially ordered sets", for every partition $\{K_n : n < \omega\}$ of the comparable pairs in $[\mathcal{P}(V)]^2$ there is some $n' < \omega$ and sequence $C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq V$ where $\{C_m, C_{m+1}\} \in K_{n'}$ for all $m < \omega$.

Define $K_n$ to be the collection of pairs of sets $\{B, C\}$ such that $B \subsetneq C$ and $n$ is the least integer where $a_n \in A \setminus \sigma(A \cup B, A \cup C)$.

Then $\sigma$ may be countered by the attack $A \cup C_0, A \cup C_1, \ldots,$ since $a_{n'} \in A \setminus \sigma(A \cup C_m, A \cup C_{m+1})$ for all $m < \omega$ and thus is never covered. □

**Question**

$\mathscr{N} \uparrow_{\text{2-mark}} Fill^{\subsetneq}_{\mathscr{C}\mathscr{F}}(\kappa) \Rightarrow \mathscr{N} \uparrow_{\text{2-tact}} Fill^{\subsetneq}_{\mathscr{C}\mathscr{F}}(\kappa)$?

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# $S(\kappa, \omega, \omega)$

Scheepers introduced the sentence $S(\kappa, \omega, \omega)$ (or rather, a sentence equivalent to the one I use below).

**Definition**

For two functions $f, g$ we say $f$ is **almost compatible** with $g$ ($f \|^* g$) if $|\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(x) \neq g(x)\}| < \omega$.

**Definition**

$S(\kappa, \omega, \omega)$ is shorthand for the sentence: there exist injective functions $f_A : A \to \omega$ for each $A \in [\kappa]^\omega$ such that $f_A \|^* f_B$ for all $A, B \in [\kappa]^\omega$.

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**Theorem**

$S(\omega_1, \omega, \omega)$

**Proof:** Use Kunen's $f_\alpha$ mentioned earlier. □

**Theorem**

$\neg S(\kappa, \omega, \omega)$ *for* $\kappa > 2^\omega$

**Proof:** Let $A_\alpha = \{\alpha \cdot \omega + n : n < \omega\} \in [\kappa]^\omega$ and $f_{A_\alpha} : A_\alpha \to \omega$ be injective for $\alpha < \kappa$. Since there are $\kappa > |[\omega]^\omega|$ different $A_\alpha$, there must be $\alpha, \beta$ where $\text{ran}(f_{A_\alpha}) = \text{ran}(f_{A_\beta})$. Then there is no way to define $f_{A_\alpha \cup A_\beta}$ so that it is almost compatible with both $f_{A_\alpha}$ and $f_{A_\beta}$. □

**Corollary**

$S(\omega_2, \omega, \omega) \Rightarrow \neg CH$

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**Proof:** Let $A_\alpha = \{\alpha\cdot\omega + n : n < \omega\} \in [\kappa]^\omega$ and $f_{A_\alpha} : A_\alpha \to \omega$ be injective for $\alpha < \kappa$. Since there are $\kappa > |[\omega]^\omega|$ different $A_\alpha$, there must be $\alpha, \beta$ where $\mathrm{ran}(f_{A_\alpha}) = \mathrm{ran}(f_{A_\beta})$. Then there is no way to define $f_{A_\alpha \cup A_\beta}$ so that it is almost compatible with both $f_{A_\alpha}$ and $f_{A_\beta}$. $\square$

**Corollary**

$S(\omega_2,\omega,\omega) \Rightarrow \neg CH$

So what about the consistency of $\neg CH + S(\omega_2,\omega,\omega)$? It turns out that's fine (to be shown later).

**Theorem**

$S(\kappa,\omega,\omega) \Rightarrow \mathscr{F} \uparrow_{2\text{-}tact} Fill^{\subsetneq}_{\mathscr{C},\mathscr{F}}(\kappa)$

**Proof:** Due to Todorcevic. Let $f_A : A \to \omega$ for $A \in [\kappa]^\omega$ witness $S(\kappa,\omega,\omega)$, and let $g_A(\alpha)$ be the number of ordinals "skipped" by $f_A$ below $f_A(\alpha)$, that is, $f_A(\alpha) - |\{\beta \in A : f_A(\beta) < f_A(\alpha)\}|$.

Note that for $A \subsetneq B$, $|\{\alpha \in A : g_A(\alpha) \leq g_B(\alpha)\}| < \omega$ since the difference in $f_A$ and $f_B \restriction A$ is finite, and $f_B$ has to map at least one more ordinal than $f_A$.

Let $\sigma(C, C') = \{\alpha \in C : g_C(\alpha) \leq g_{C'}(\alpha)\}$. If $C_0 \subsetneq C_1 \subsetneq \dots$ was an attack defeating $\sigma$, then let $\alpha \in C_N \setminus \bigcup_{n<\omega} \sigma(C_n, C_{n+1})$.

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## Theorem

$S(\kappa,\omega,\omega) \Rightarrow \mathscr{F} \uparrow_{2\text{-}mark} Fill^{\subseteq}_{\mathscr{C}\,\mathscr{F}}(\kappa)$

**Proof:** Corollary of the previous theorem. Alternatively, $\mathscr{F}$ can use the winning strategy

$$\sigma(C, C', n+1) = f_C^{-1}(\{0,\dots,n-1\}) \cup \{\alpha \in C : f_C(\alpha) \neq f_{C'}(\alpha)\}$$

# Back to $Cov_{\mathscr{C}\mathscr{F}}(\kappa^\dagger)$

While a proof $\mathscr{F} \uparrow_{\text{2-mark}} Fill^{\subseteq}_{\mathscr{C}\mathscr{F}}(\kappa) \Rightarrow \mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(\kappa^\dagger)$ has eluded me, the techniques used previously are very useful for dealing with $Cov_{\mathscr{C}\mathscr{F}}(\kappa^\dagger)$ directly.

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It will be useful to define a sufficient property for $\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(X)$, which I've called almost-$\sigma$-(relatively compact).

## Definition

Let $\mathcal{U}$ be a cover of $X$. We say $C \subseteq X$ is $\mathcal{U}$-compact if there exists a finite subcover of $\mathcal{U}$ which covers $C$.

We say $X$ is almost-$\sigma$-(relatively compact) if there exist functions $r_{\mathcal{V}} : X \to \omega$ for each open cover $\mathcal{V}$ of $X$ such that both of the following sets are $\mathcal{V}$-compact for all open covers $\mathcal{U}$, $\mathcal{V}$ and $n < \omega$:

$$c(\mathcal{V}, n) = \{x \in X : r_{\mathcal{V}}(x) \leq n\}$$

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**Theorem**

*If $X$ is almost-$\sigma$-(relatively compact), then $\mathscr{F} \uparrow_{2\text{-mark}} Cov_{\mathscr{C}\mathscr{F}}(X)$.*

**Proof:** Let $\sigma(\mathcal{U}_0, 0)$ cover $c(\mathcal{U}_0, 0)$, and let $\sigma(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1)$ cover both $c(\mathcal{U}_{n+1}, n+1)$ and $p(\mathcal{U}_n, \mathcal{U}_{n+1})$. If $\mathcal{U}_0, \mathcal{U}_1, \dots$ is any play by $C$, then for each $x \in X$, we note that one of the following must occur:

- $r_{\mathcal{U}_0}(x) = 0$ and thus $x \in c(\mathcal{U}_0, 0)$.
- $r_{\mathcal{U}_0}(x) = N + 1$ for some $N \geq 0$ and:
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*If $S(\kappa,\omega,\omega)$, then $\kappa^\dagger$ is almost-$\sigma$-(relatively compact).*

**Proof:** Take the injective funcions $f_A : A \to \omega$ witnessing $S(\kappa,\omega,\omega)$. For each cover $\mathcal{V}$ of $\kappa^\dagger$ let $A(\mathcal{V})$ define a set such that $\kappa^\dagger \setminus A(\mathcal{V})$ is in a refinement of $\mathcal{V}$.
Then $r_\mathcal{V}$ defined by

$$r_\mathcal{V}(x) = \begin{cases} 0 & x \in \kappa^\dagger \setminus A(\mathcal{V}) \\ f_{A(\mathcal{V})}(x) + 1 & x \in A(\mathcal{V}) \end{cases}$$

witnesses the property as $c(\mathcal{V}, 0)$ is contained in a single open set in $\mathcal{V}$, $c(\mathcal{V}, n+1)$ is a singleton or empty set, and

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This result becomes more interesting if we can show $S(\kappa, \omega, \omega)$ is consistent for $\kappa > \omega_1$.

## Definition

A finite partial function $p$ from $A$ to $B$ has a domain which is a finite subset of $A$ and a range which is a finite subset of $B$. Let the set of all finite partial functions from $A$ to $B$ be denoted by $Fn(A, B)$.

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## Definition

For $\kappa > \omega_1$, let $\mathbb{P}_\kappa \subset Fn^2([\kappa]^\omega, \omega)$ be such that each $p_A$ is injective, and give it the partial order $\leq$ defined by $q \leq p$ if and only if:

- $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$
- For each $A \in \mathrm{dom}(p)$, $q_A \supseteq p_A$
- For each $A, B \in \mathrm{dom}(p)$, if $p_A$ and $p_B$ are not defined for some $\alpha \in A \cap B$, but $q_A$ is, then $q_B$ is also defined for $\alpha$ and $q_A(\alpha) = q_B(x)$. That is, for $\alpha \in A \cap B$

$$\alpha \in \mathrm{dom}(q_A) \setminus (\mathrm{dom}(p_A) \cup \mathrm{dom}(p_B))$$

$$\Downarrow$$

$$\alpha \in \mathrm{dom}(q_B) \text{ and } q_A(x) = q_B(x)$$

**Lemma**

*$\mathbb{P}_\kappa$ has property K (and thus is c.c.c.). That is, let $P \subseteq \mathbb{P}_\kappa$ be uncountable: there is an uncountable $Q \subseteq P$ such that points in $Q$ are pairwise compatible.*

**Proof:** If $|\{\mathrm{dom}(p) : p \in P\}| > \omega$, we will use the $\Delta$-system lemma to find an uncountable $P' \subseteq P$ such that for $p, q \in P'$, $\mathrm{dom}(p) \cap \mathrm{dom}(q) = \mathcal{R}$. Otherwise, we may fix an uncountable $P' \subseteq P$ such that for $p, q \in P'$, $\mathrm{dom}(p) = \mathrm{dom}(q) = \mathcal{R}$.

Similarly, for each $A \in \mathcal{R}$ we may find that $|\{\mathrm{dom}(p_A) : p \in P'\}| > \omega$, and we can use the $\Delta$-system lemma to find an uncountable $P'' \subseteq P'$ where $\mathrm{dom}(p_A) \cap \mathrm{dom}(q_A) = A'$ for all $p, q \in P''$, or otherwise we may find $P'' \subseteq P'$ where $\mathrm{dom}(p_A) = \mathrm{dom}(q_A) = A'$ for all $p, q \in P''$.

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Finally, for each $A \in \mathcal{R}$ and $\alpha \in A'$, we may find $n_{A,\alpha}$ such that there are uncountable $p \in P''$ with $p_A(\alpha) = n_{A,\alpha}$, and thus we may choose $Q \subseteq P''$ to be an uncountable collection such that for $p, q \in Q$, $p_A = q_A$ for $A \in \mathcal{R}$.

Then it is easily verified that $p \cup q \in \mathbb{P}_\kappa$ and $p \cup q \leq p, q$ for all $p, q \in Q$. $\square$

Since $\mathbb{P}_\kappa$ is c.c.c.:

**Corollary**

*Any forcing using a $\mathbb{P}_\kappa$-generic filter preserves cardinals and cofinalities.*

**Corollary**

*If $cf(\kappa) > \omega$, any forcing using a $\mathbb{P}_\kappa$-generic filter results in $2^\omega \leq \kappa$.*

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## Proposition

*For $A \in [\kappa]^\omega$ and $\alpha \in A$, the sets*

$$D_A = \{p \in \mathbb{P}_\kappa : A \in dom(p)\}$$

$$D_{A,\alpha} = \{p \in \mathbb{P}_\kappa : A \in dom(p), \alpha \in dom(p_A)\}$$

*are dense in $\mathbb{P}_\kappa$.*

## Theorem

*If $cf(\kappa) > \omega$, $S(\kappa, \omega, \omega) + (\kappa = 2^\omega)$ is consistent with ZFC.*

**Proof:** We adapt a forcing argument due to Scheepers (which used a slightly different poset). Let $M$ be a countable transitive submodel of ZFC. Consider the c.c.c. poset $\mathbb{P}_\kappa$ realized in the model $M$. Let $G$ be a $\mathbb{P}_\kappa$-generic filter over $M$.

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We now work in the smallest model $M[G]$ extending $M$ and containing $G$.

For each $A \in [\kappa]^\omega$, note $[\kappa]^\omega \cap M$ is cofinal in $[\kappa]^\omega$, so let $A' \supseteq A$ be in $[\kappa]^\omega \cap M$ and let $f_A = \bigcup_{p \in G \cap D_{A'}} p_{A'} \restriction A$. Since $G$ is a $\mathbb{P}_\kappa$-generic filter over $M$, it is easily verified (considering the dense sets $D_{A,\alpha}$) that $f_A$ is an injective function from $A$ into $\omega$.

In addition, for $A, B \in [\kappa]^\omega \cap M$, let $p \in G \cap D_A \cap D_B$. For all $q \leq p$ it follows that

$$\{\alpha \in \mathrm{dom}(q_A) \cap \mathrm{dom}(q_B) : q_A(\alpha) \neq q_B(\alpha)\} \subseteq \mathrm{dom}(p_A) \cup \mathrm{dom}(p_B)$$

Thus $|\{\alpha \in A \cap B : f_A(\alpha) \neq f_B(\alpha)\}| < \omega$ and $f_A \|^* f_B$ for $A, B \in [\kappa]^\omega \cap M$, and it's immediate that $f_A \|^* f_B$ for $A, B \in [\kappa]^\omega$ as well.

The $f_A$ witness $S(\kappa, \omega, \omega)$. Since $\kappa \geq 2^\omega$ and $S(\kappa, \omega, \omega)$ is a contradiction for $\kappa > 2^\omega$, we know $\kappa = 2^\omega$.

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## Corollary

*For all $\kappa$, $\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(\kappa^\dagger)$ is consistent with ZFC.*

## Question

Is $\mathscr{F} \uparrow_{\text{2-mark}} Cov_{\mathscr{C}\mathscr{F}}(\omega_2^\dagger)$ a theorem of ZFC?