Remark 1. Scheeper's $S(\kappa)$ requiring injections is stronger than my $S'(\kappa)$ requiring finite-to-one maps. Dow suggests that $S'(\omega_\omega)$ holds in ZFC by the following.

Definition 2. A topological space is said to be ω-bounded if each countable subset of the space has compact closure.

Theorem 3. *There exists a topology on ω_2 which is ω-bounded and locally countable.*

Proof. Note that the natural linear order on ω_1 induces such a topology for it.

Let $\alpha < \omega_2$, and suppose we've defined compatible topologies on $\omega_1 \cdot (\beta + 1)$ for all $0 \leq \beta < \alpha$. If $\alpha = \beta + 1$, then let $\omega_1 \cdot (\alpha + 1) = \omega_1 \cdot (\beta + 2)$ be the topological sum of the previously defined $\omega_1 \cdot (\beta + 1)$ and the linear order on $\omega_1 \cdot (\beta + 2) \setminus \omega_1 \cdot (\beta + 1) \cong \omega_1$. Similarly, if $cf(\alpha) > \omega$, then let $\omega_1 \cdot (\alpha + 1)$ be the topological sum of $\bigcup_{\beta<\alpha} \omega_1 \cdot (\beta + 1)$ and the linear order on $\omega_1 \cdot (\alpha + 1) \setminus \omega_1 \cdot \alpha \cong \omega_1$.

The remaining case is where α is the limit of increasing α_n for $n < \omega$. Fix a bijection $f_\alpha : \omega_1 \cdot (\alpha + 1) \setminus \omega_1 \cdot \alpha \to \omega_1 \cdot \alpha$. Points in $\omega_1 \cdot (\alpha_n + 1)$ for some $n < \omega$ have their usual base induced by that previously defined topology. So let $\gamma \in \omega_1 \cdot (\alpha + 1) \setminus \omega_1 \cdot \alpha$. Basic open neighborhoods of γ are of the form $[\gamma', \gamma] \cup f_\alpha[[\omega_1 \cdot \alpha, \gamma]] \setminus \omega_1 \cdot (\alpha_n + 1)$, where $\omega_1 \cdot \alpha \leq \gamma' < \gamma$ and $n < \omega$.

We wish to show that ω_2 with the topology induced by $\bigcup_{\alpha<\omega_2} \omega_1 \cdot (\alpha + 1)$ is ω-bounded and locally countable. If $\gamma \in \omega_1 \cdot (\alpha + 1) \setminus \omega_1 \cdot \alpha$ where $cf(\alpha) \neq \omega$, then we immediately see that it is in a clopen copy of ω_1 giving us local countability immediately. Otherwise, γ has a basic open neighborhood of the form $[\gamma', \gamma] \cup f_\alpha[[\omega_1 \cdot \alpha, \gamma]] \setminus \omega_1 \cdot (\alpha_n + 1)$, which is obviously countable.

Let C be a countable subset of $\omega_1 \cdot (\alpha + 1)$. In the case that $\alpha = \beta + 1$, we may use the ω-boundedness of each part in the clopen partition $\omega_1 \cdot (\beta + 1)$ and $\omega_1 \cdot (\beta + 2) \setminus \omega_1 \cdot (\beta + 1) \cong \omega_1$ to conclude that the closure of C is compact. Similarly, if $cf(\alpha) > \omega$, then we may use the ω-boundedness of each part in the clopen partition $\bigcup_{\beta<\alpha} \omega_1 \cdot (\beta + 1)$ and $\omega_1 \cdot (\alpha + 1) \setminus \omega_1 \cdot \alpha \cong \omega_1$ to conclude that the closure of C is compact.

The remaining case is again where α is the limit of increasing α_n for $n < \omega$. Then $C \subseteq [\gamma', \gamma] \cup f_\alpha[[\omega_1 \cdot \alpha, \gamma]]$ for some $\omega_1 \cdot \alpha \leq \gamma' < \gamma < \omega_1 \cdot (\alpha + 1)$. Its closure is compact: the closure operation does not add any ordinals greater than γ, and any open cover contains another basic open neighborhood of γ such as $[\gamma'', \gamma] \cup f_\alpha[[\omega_1 \cdot \alpha, \gamma]] \setminus \omega_1 \cdot (\alpha_m + 1)$ which misses only the compact set $[\gamma', \gamma'']$ and the closure of the countable set $f_\alpha[[\omega_1 \cdot \alpha, \gamma]] \cap \omega_1 \cdot (\alpha_{\min(m,n)} + 1)$, which is compact by the ω-boundedness of $\omega_1 \cdot (\alpha_{\min(m,n)} + 1)$.

Finally, since every countable subset of ω_2 is contained in some $\omega_1 \cdot (\alpha + 1)$, we conclude ω_2 is ω-bounded. $\square$

Theorem. $S'(\omega_2)$.