

ALMOST COMPATIBLE FUNCTIONS AND INFINITE LENGTH GAMES

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ABSTRACT. TODO

1. INTRODUCTION

Definition 1. Two functions f, g are almost compatible, that is, $f \sim g$ when $\{a \in \text{dom } f \cap \text{dom } g : f(a) \neq g(a)\}$ is finite.

Marion Scheepers used almost compatible functions in [4] in order to study the existence of limited information strategies on a variation of the meager-nowhere dense game he introduced in [5].

Game 2. Let $Sch_{C,F}^{\cup, \subset}(\kappa)$ denote *Scheepers' strict countable-finite union game* with two players \mathcal{C} , \mathcal{F} . In round 0, \mathcal{C} chooses $C_0 \in [\kappa]^{\leq \omega}$, followed by \mathcal{F} choosing $F_0 \in [\kappa]^{< \omega}$. In round $n + 1$, \mathcal{C} chooses $C_{n+1} \in [\kappa]^{\leq \omega}$ such that $C_{n+1} \supset C_n$, followed by \mathcal{F} choosing $F_{n+1} \in [\kappa]^{< \omega}$.

\mathcal{F} wins the game if $\bigcup_{n < \omega} F_n \supseteq \bigcup_{n < \omega} C_n$; otherwise, \mathcal{C} wins.

Of course, with perfect information this game is trivial: during round n player \mathcal{F} simply chooses n ordinals from each of the n countable sets played by \mathcal{C} . However, if \mathcal{F} is limited to using information from the last k moves by \mathcal{C} during each round, the task becomes more difficult. Call such a strategy a *k-tactical strategy* or *k-tactic*; if using the round number is allowed, then the strategy is called a *k-Markov strategy* or a *k-mark*.

Definition 3. The statement $S(\kappa)$ (given as $S(\kappa, \aleph_0, \omega)$ in [4]) claims that there exist one-to-one functions $f_A : A \rightarrow \omega$ for each $A \in [\kappa]^{\leq \aleph_0}$ such that the collection $\{f_A : A \in [\kappa]^{\leq \aleph_0}\}$ is pairwise almost compatible.

In the same paper, Scheepers noted that $S(\omega_1)$ holds in *ZFC*, and that it's possible to force \mathfrak{c} to be arbitrarily large while preserving $S(\mathfrak{c})$. However, $S(\mathfrak{c}^+)$ always fails. This axiom may be applied to obtain a winning 2-tactic for \mathcal{F} in the countable-finite game.

In [1], Clontz related this game to a game which may be used to characterize the Menger covering property of a topological space.

Key words and phrases. TODO.

Game 4. Let $Men_{C,F}(X)$ denote the *Menger game* with players \mathcal{C} , \mathcal{F} . In round n , \mathcal{C} chooses an open cover \mathcal{U}_n , followed by \mathcal{F} choosing subset F_n of X which may be finitely covered by \mathcal{U}_n .

\mathcal{F} wins the game if $X = \bigcup_{n < \omega} F_n$, and \mathcal{C} wins otherwise.

This characterization is slightly different than the typical characterization in which the second player first chooses a specific finite subcollection \mathcal{F}_n of the cover itself and lets $F_n = \bigcup \mathcal{F}_n$, denoted as $G_{fin}(\mathcal{O}, \mathcal{O})$ in [6]. However, it's easily seen that these games are equivalent for perfect information strategies (so both characterize the Menger property), and this characterization is more convenient for our concerns.

Definition 5. Let $\kappa^\dagger = \kappa \cup \{\infty\}$ where κ is discrete and ∞ 's neighborhoods are the co-countable sets containing it.

The relationship between $Sch_{C,F}^{\cup, \subset}(\kappa)$ and $Men_{C,F}(\kappa^\dagger)$ is strong; in both games \mathcal{C} essentially chooses a countable subset of κ followed by \mathcal{F} choosing a finite subset of that choice, and it's easy to see the winning perfect information strategy for \mathcal{F} in both games. In addition, it was shown in [1] that when $S(\kappa)$ holds, \mathcal{F} has a winning 2-Markov strategy in $Men_{C,F}(\kappa^\dagger)$.

One source of motivation is to make progress on the following open question:

Question 6. *Does there exist a topological space X for which $\mathcal{F} \uparrow Men_{C,F}(X)$ but $\mathcal{F} \not\uparrow_{2\text{-mark}} Men_{C,F}(X)$? (That is, the second player can win the Menger game on X with perfect information but not with 2-Markov information.)*

One might hope that $(\mathfrak{c}^+)^\dagger$ might answer this question in the affirmative as $S(\mathfrak{c}^+)$ fails, but we will show that assuming $V = L$, $\mathcal{F} \uparrow_{2\text{-mark}} Men_{C,F}(\kappa^\dagger)$ for all cardinals κ .

2. ONE-TO-ONE AND FINITE-TO-ONE ALMOST COMPATIBLE FUNCTIONS

We may weaken Scheeper's $S(\kappa)$ as follows:

Definition 7. The statement $S'(\kappa)$ weakens $S(\kappa)$ by only requiring the witness functions $f_A : A \rightarrow \omega$ to be finite-to-one.

The following observation will be convenient.

Proposition 8. $S(\kappa)$ and $S'(\kappa)$ need only be witnessed by functions $\{f_A : A \in \mathcal{S}\}$ for some family \mathcal{S} cofinal in $[\kappa]^{\leq \aleph_0}$.

Proof. For each $A \in [\kappa]^{\leq \aleph_0}$ choose $A' \supseteq A$ from \mathcal{S} and let $g_A = f_{A'} \upharpoonright A$. □

In the next section we will show that $S'(\kappa)$ is sufficient for the applications to the Scheepers and Menger games. In the meantime, we will demonstrate that $S'(\kappa)$ is strictly weaker than $S(\kappa)$.

Recall the following.

Definition 9. A Kurepa family $\mathcal{K} \subseteq [\kappa]^{\aleph_0}$ on κ satisfies that $\mathcal{K} \restriction A = \{K \cap A : K \in \mathcal{K}\}$ is countable for each $A \in [\kappa]^{\aleph_0}$.

Theorem 10. $S'(\kappa)$ holds whenever there exists a cofinal Kurepa family on κ .

Proof. Let $\mathcal{K} = \{K_\alpha : \alpha < \theta\}$ be a cofinal Kurepa family on κ . We first define $f_\alpha : K_\alpha \rightarrow \omega$ for each $\alpha < \theta$.

Suppose we've already defined pairwise almost compatible finite-to-one functions $\{f_\beta : \beta < \alpha\}$. To define f_α , we first recall that $\mathcal{K} \restriction K_\alpha$ is countable, so we may choose $\beta_n < \alpha$ for $n < \omega$ such that $\{K_\beta : \beta < \alpha\} \restriction K_\alpha \setminus \{\emptyset\} = \{K_\alpha \cap K_{\beta_n} : n < \omega\}$. Let $K_\alpha = \{\delta_{i,j} : i \leq \omega, j < w_i\}$ where $w_i \leq \omega$ for each $i \leq \omega$, $K_\alpha \cap (K_{\beta_n} \setminus \bigcup_{m < n} K_{\beta_m}) = \{\delta_{n,j} : j < w_n\}$, and $K_\alpha \setminus \bigcup_{n < \omega} K_{\beta_n} = \{\delta_{\omega,j} : j < w_\omega\}$. Then let $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ for $n < \omega$ and $f_\alpha(\delta_{\omega,j}) = j$ otherwise.

We should show that f_α is finite-to-one. Let $n < \omega$. Since $f_\alpha(\delta_{m,j}) \geq m$, we only consider the finite cases where $m \leq n$. Since each f_{β_m} is finite-to-one, $f_{\beta_m}(\delta_{m,j}) \leq n$ for only finitely many j . Thus $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$ maps to n for only finitely many j .

We now want to demonstrate that $f_\alpha \sim f_{\beta_n}$ for all $n < \omega$. Note $\delta_{m,j} \in K_{\beta_n}$ implies $m \leq n$. For $m = n$, we have $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ which differs from $f_{\beta_n}(\delta_{n,j})$ for only the finitely many j which are mapped below n by f_{β_n} . For $m < n$ and $\delta_{m,j} \in K_{\beta_n}$, we have $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$ which can only differ from $f_{\beta_n}(\delta_{m,j})$ for only the finitely many j which are mapped below m by f_{β_m} or the finitely many j for which the almost compatible $f_{\beta_n} \sim f_{\beta_m}$ differ.

Finally for any $\beta < \alpha$, we may conclude $f_\alpha \sim f_\beta$ since there is some β_n with $K_\alpha \cap K_\beta = K_\alpha \cap K_{\beta_n}$, $f_\alpha \sim f_{\beta_n}$, and $f_{\beta_n} \sim f_\beta$. \square

We now construct a topology on ω_n for each $n < \omega$ which will witness a Kurepa family on \aleph_n .

Proposition 11. Let X be a T_2 space with a base of countable and compact neighborhoods. Then X is locally metrizable with a base of compact open countable sets.

Proof. For each point x let K be a countable and compact neighborhood of x , and it follows that it is contained in a countable, open, and locally compact neighborhood W of x , which in turn is zero-dimensional and metrizable. So choose V clopen in W such that $x \in V \subseteq K$; V is a compact open neighborhood of x in X . \square

Definition 12. A topological space is said to be ω -bounded if each countable subset of the space has compact closure.

Proposition 13. Let X be a T_2 space with cardinality less than \aleph_ω which is locally countable and ω -bounded. Then the closure operation preserves cardinality and weight.

Proof. Note that the closure of any countable neighborhood is compact, and any Lindelöf set is countable. This space is locally metrizable and thus first-countable, so cardinality and weight coincide for any subspace. The result is obvious if A is countable; otherwise let $A = \{a_\alpha : \alpha < \omega_{n+1}\}$ and since basic neighborhoods are

countable note any limit point is a limit point of $A_\beta = \{a_\alpha : \alpha < \beta\}$ for some $\beta < \omega_{n+1}$. Thus $\overline{A} = \bigcup_{\beta < \omega_{n+1}} \overline{A_\beta}$ and by induction $|\overline{A}| = |A|$. \square

Lemma 14. *Let X be a T_2 space with cardinality less than \aleph_ω which is locally countable and locally compact, and such that its closure operation preserves cardinalities. Then X has an ω -bounded extension \tilde{X} with the same properties where $\tilde{X} \setminus X$ has the same cardinality as X .*

Proof. We prove this by induction on n . If $n = 0$, then we can just use the one-point compactification of two copies of X . So suppose $n > 0$ and that $X = \omega_n$ has an appropriate topology. Note that X has a base of countable and compact neighborhoods since the closure operation preserves cardinalities.

For each $\alpha < \omega_n$, γ_α may be chosen such that both the closure of the set α in X and a countable neighborhood of the point α are subsets of γ_α . Note that the set $\{\lambda < \omega_n : \alpha < \lambda \Rightarrow \gamma_\alpha < \lambda\}$ is a cub subset of ω_n containing a cub subset C of limit ordinals. Now for each $\lambda \in C$, the set λ is open as $\alpha < \lambda$ belongs to the neighborhood $\gamma_\alpha \subseteq \lambda$. Also, if λ has uncountable cofinality, then for $\beta \geq \lambda$ and any countable neighborhood U of β , $U \cap \lambda = U \cap \alpha$ for some $\alpha < \lambda$; thus $U \setminus \overline{\alpha} = U \setminus \lambda$ is a neighborhood of β , showing that λ is clopen.

Let $\tilde{X} = \omega_n \times 2$. By induction on $\lambda \in C$ we will define compatible topologies for $\tilde{X}_\lambda = \omega_n \times \{0\} \cup \lambda \times \{1\}$ such that

- $\omega_n \times \{0\}$ is an open copy of X ,
- $\lambda \times 2$ is open, and when $\text{cf}(\lambda) > \omega$ also closed,
- the space has a base of countable and compact neighborhoods, and
- when λ is a successor, for each $\alpha < \lambda$ the closure of $\alpha \times 2$ is an ω -bounded subset of $\lambda \times 2$.

We first consider the case $n = 1$. If λ is a limit in C , then $\tilde{X}_\lambda = \bigcup_{\mu \in C \cap \lambda} \tilde{X}_\mu$ satisfies the induction requirements. Otherwise we choose an increasing sequence of ordinals $\{\alpha_k : k \in \omega\}$ with limit λ such that α_0 is the predecessor of λ in C , or $\alpha_0 = 0$ if λ is the least element of C .

The subspace $\overline{\lambda} \times \{0\} \cup \alpha_0 \times 2$ of X is countable and locally compact; therefore it is metrizable and zero-dimensional. So we may choose increasing sets U_k for $k < \omega$ which are clopen in this topology and satisfy

$$\overline{\alpha_k \times \{0\} \cup \alpha_0 \times 2} = \overline{\alpha_k} \times \{0\} \cup \alpha_0 \times 2 \subseteq U_k \subseteq \lambda \times \{0\} \cup \alpha_0 \times 2$$

Note that U_k is also clopen in \tilde{X}_{α_0} since it is closed in $\overline{\lambda} \times \{0\} \cup \alpha_0 \times 2$ and open in $\lambda \times \{0\} \cup \alpha_0 \times 2$.

We need only describe a base for the points $\langle \alpha, 1 \rangle \in (\lambda \setminus \alpha_0) \times \{1\}$. We do so by letting $\langle \alpha, 1 \rangle$ be isolated when $\alpha \notin \{\alpha_k : k < \omega\}$, and giving $\langle \alpha_k, 1 \rangle$ the open neighborhoods $(U_k \cup ((\alpha_k + 1) \times \{1\})) \setminus K$ for each compact subset K of $U_k \cup (\alpha_k \times \{1\})$; that is, $\langle \alpha_k, 1 \rangle$ is the one point compactifying $U_k \cup (\alpha_k \times \{1\})$.

The first two requirements of our inductive hypothesis are obviously satisfied. Note points in $\lambda \times 2$ are covered by the compact countable neighborhood $U_k \cup ((\alpha_k + 1) \times \{1\})$ for some $k < \omega$, and for points in $(\omega_n \setminus \lambda) \times \{0\}$ we may use a compact countable neighborhood from X . For the final requirement, note that for $\alpha < \lambda$,

we may choose $\alpha < \alpha_k < \lambda$ and note that $\alpha \times 2$ is contained in the compact subset $U_k \cup ((\alpha_k + 1) \times \{1\})$ of $\lambda \times 2$.

For the case $n > 1$, we may assume that the successors in C have uncountable cofinality. We again proceed by induction on $\lambda \in C$. Again when λ is a limit in C , $\tilde{X}_\lambda = \bigcup_{\mu \in C \cap \lambda} \tilde{X}_\mu$ satisfies the given requirements; in particular if $\alpha < \lambda$, then $\alpha < \mu < \lambda$ for some successor $\mu \in C$ with uncountable cofinality. As such, the closure of $\alpha \times 2$ is an ω -bounded subset of the clopen $\mu \times 2$ and therefore also of $\lambda \times 2$. In case λ is not a limit of C , then λ has uncountable cofinality and a predecessor $\mu \in C$. We therefore have that $\lambda \times \{0\}$ is clopen in $\omega_n \times \{0\}$. Since the cardinality of $\lambda \times \{0\} \cup \mu \times 2$ is less than \aleph_n , we may simply apply the induction hypothesis to choose an appropriate topology for $\lambda \times 2$.

As a result, $\tilde{X} = \bigcup_{\lambda \in C} \tilde{X}_\lambda$ is ω -bounded as any countable set is contained in some $\alpha \times 2$ for $\alpha < \lambda \in C$. \square

Theorem 15. *For each $n \in \omega$, there is a T_2 , locally countable, ω -bounded topology on ω_n .*

Proof. Apply the previous lemma to ω_n with the discrete topology. \square

Corollary 16. *There exists a Kurepa family cofinal in $[\omega_k]^\omega$ for each $k < \omega$.*

Proof. We use the family \mathcal{K} of all compact open sets in the constructed topology on ω_n as our witness. Of course, every Lindelöf set in a locally countable space is countable, and the closure of every countable set is a compact countable set; thus \mathcal{K} is cofinal in $[\omega_n]^\omega$. It is Kurepa since for every countable set A , there are only countably many distinct compact open sets in \bar{A} : TODO finish \square

This is alternatively a corollary of an observation of Todorcevic communicated by Dow in [2]: if every Kurepa family of size at most θ extends to a cofinal Kurepa family, then the same is true of θ^+ . So the result follows as every Kurepa family \mathcal{K} of size ω extends to the cofinal Kurepa family $[\bigcup \mathcal{K}]^\omega$.

So we have our desired result.

Corollary 17. *$S'(\omega_n)$ holds for all $n < \omega$. Under CH , we have both $S'(\omega_2)$ and $\neg S(\omega_2)$.*

3. TODO ALL THIS STUFF NEEDS EDITING STILL

As noted in [TODO cite Dow], Jensen's one-gap two-cardinal theorem under $V = L$ [TODO cite] can be used to show that there exist cofinal Kurepa families on every cardinal.

Corollary 18 ($V = L$). *$S'(\theta)$ holds for all cardinals.*

In particular, $S(\omega_2)$ fails under CH , showing the two are distinct. Actually, CH is not required to have $S(\omega_2)$ fail.

We are going to need a technical lemma (available in Kunen).

Lemma 19. *Assume that $G \subset \text{Fn}(\omega_2, 2)$ is a generic filter, and let $\mu \in \omega_2$. Then the final model $V[G]$ can be regarded as forcing with $\text{Fn}(\omega_2 \setminus \mu, 2)$ over the model $V[G_\mu]$. In addition, for each $\text{Fn}(\omega_2, 2)$ -name \dot{A} of a subset of ω (treat as a subset of $\omega \times \text{Fn}(\omega_2, 2)$), there is a canonical name $\dot{A}(G_\mu)$ where,*

$$\dot{A}(G_\mu) = \{(n, p \restriction [\mu, \omega_2)) : (n, p) \in \dot{A} \text{ and } p \restriction \mu \in G_\mu\}$$

and we get that the valuation of $\dot{A}(G_\mu)$ by the tail of the generic, $G_{\omega_2 \setminus \mu}$, is the same as the valuation of \dot{A} by the full generic.

Theorem 20. *If we add ω_2 Cohen reals to a model of CH we get that Scheepers' $S(\omega_2)$ (still) fails.*

Proof. The forcing poset is $\text{Fn}(\omega_2, 2)$. Let $\{\dot{f}_A : A \in [\omega_2]^\omega\}$ be a family of names such that \dot{f}_A is a one-to-one function from A into ω . It suffices to only consider sets A from the ground model.

Put all the lemma stuff in an elementary submodel M of the universe (technically of $H(\kappa)$, or of V_κ , for some large κ). Standard methods says that we can assume that $|M| = \omega_1 = \mathfrak{c}$ and that $M^\omega \subset M$ (which means that every countable subset of M is a member of M).

Let $\lambda = M \cap \omega_2$ (same as the supremum of $M \cap \omega_2$). Consider the name $\dot{f}_{[\lambda, \lambda + \omega)}$. What is such a name? We can assume that it is a set of pairs of the form $((\lambda + k, m), p)$ where $p \in \text{Fn}(\omega_2, 2)$ and, of course, $k, m \in \omega$. This is (almost) equivalent to saying that p forces that $\dot{f}_{[\lambda, \lambda + \omega)}(\lambda + k) = m$. We don't take all such p , in fact for each k, m it is enough to take a countable set of such p to get an equivalent name (Kunen calls it a nice name if we take, for each k, m an antichain that is maximal among such conditions). Given any such name \dot{f} , let $\text{supp}(\dot{f})$ denote the union of the domains of conditions p appearing in the name.

Also let Y equal $\text{supp}(\dot{f}_{[\lambda, \lambda + \omega)}) \setminus \lambda$. Let δ denote the order type of Y and let the 2-parameter notation $\varphi_{\mu, \lambda}$ be the order-preserving function from $\mu \cup Y$ onto the ordinal $\mu + \delta$. This lifts canonically to an order-preserving bijection $\varphi_{\mu, \lambda} : \text{Fn}(\mu \cup Y, 2) \rightarrow \text{Fn}(\mu + \delta, 2)$. Similarly, we make sense of the name $\varphi_{\mu, \lambda}(\dot{f}_{[\lambda, \lambda + \omega)})$, call it F_M . Here simply, for each tuple $((k, m), p) \in \dot{f}_{[\lambda, \lambda + \omega)}$, we have that $((k, m), \varphi_{\mu, \lambda}(p))$ is in F_M . Again, let $\varphi_{\mu, \lambda}(\dot{f}_{[\lambda, \lambda + \omega)})$ be interpreted in the above sense as giving F_M (which is an element of M). Note that we do not regard δ

as fixed here, but rather simply depending on the $\text{supp}(\dot{f}_{[\lambda, \lambda + \omega)})$ described above. Other values replacing $\lambda > \mu$ will result in their own set Y and canonical map $\varphi_{\mu, \lambda}$; but one thing we do have to assume (or arrange) for other values α replacing λ is that $\text{supp}(\dot{f}_{[\alpha, \alpha + \omega)})$ intersected with α is contained in μ .

Now the object F_M is an element of M , and M believes this statement is true:

$$(\forall \beta \in \omega_2) (\exists \beta < \lambda \in \omega_2) \text{supp}(\dot{f}_{[\lambda, \lambda + \omega)}) \cap \lambda \subset \mu \text{ and } F_M = \varphi_{\mu, \lambda}(\dot{f}_{[\lambda, \lambda + \omega)})$$

But now, this means that, not only is there an $\alpha \in M$, $F_M = \varphi_{\mu, \alpha}(\dot{f}_{[\alpha, \alpha + \omega)})$ but also that there is an increasing sequence $\{\alpha_\xi : \xi \in \omega_1\} \subset \lambda$ of such α 's satisfying that, for each ξ we have that $\text{supp}(\dot{f}_{[\alpha_\xi, \alpha_\xi + \omega)})$ is contained in $\alpha_{\xi+1}$.

Choose such a sequence. This means that if we let $A = \bigcup_{n > 0} [\alpha_n, \alpha_n + \omega)$ we have the name \dot{f}_A in M . This then means that all the $((\beta, m), p)$ appearing in \dot{f}_A have the property that $\text{dom}(p)$ is contained in M . There is, within M , a name \dot{g} satisfying that $\dot{f}_A(\alpha_n + k) = \dot{f}_{[\alpha_n, \alpha_n + \omega)}(\alpha_n + k)$ for all $k > \dot{g}(n)$.

We now apply the above Lemma using $\mu = \mu_0$ and we are now working in the extension $V[G_\mu]$. We will abuse the notation and use $\dot{f}_{[\alpha_n, \alpha_n + \omega)}$ instead of $\dot{f}_{[\alpha_n, \alpha_n + \omega)}(G_\mu)$ as defined in the Lemma. We work for a contradiction. Something special has now happened, namely, the supports of the names $\{\dot{f}_{[\alpha_n, \alpha_n + \omega)} : 0 < n < \omega\}$ are pairwise disjoint and also disjoint from the support of the name $\dot{f}_{[\lambda, \lambda + \omega)}$ (under the same convention about G_μ . And not only that, these names are pairwise isomorphic (in the way that they all map to F_M).

Since A is disjoint from $[\lambda, \lambda + \omega)$, there must be an integer ℓ together with a condition $q \in \text{Fn}(\omega_2 \setminus \mu, 2)$ satisfying that for all $n > \ell$, q forces that

$$\text{“if } k > \dot{g}(n) \text{ (since } \alpha_n + k \in A) \text{ then } \dot{f}_{[\alpha_n, \alpha_n + \omega)}(\alpha_n + k) \neq \dot{f}_{[\lambda, \lambda + \omega)}(\lambda + k)\text{”}.$$

Choose n large enough so that $\text{dom}(q) \cap [\alpha_n, \mu_{n+1})$ is empty. Choose $q_1 < q \restriction \lambda$ (in M) so that

$$\varphi_{\mu, \alpha_n}(q_1 \restriction \text{supp}(\dot{f}_{[\alpha_n, \alpha_n + \omega)})) = \varphi_{\mu, \lambda}(q \restriction \text{supp}(\dot{f}_{[\lambda, \lambda + \omega)}))$$

and then (again in M) choose $q_2 < q_1$ so that it both forces a value L on $\ell + \dot{g}(n)$ and subsequently forces a value m on $\dot{f}_{[\alpha_n, \alpha_n + \omega)}(\alpha_n + L + 1)$. But now, again calculate

$$q_3 = \varphi_{\mu, \lambda}^{-1} \circ \varphi_{\mu, \alpha_n}(q_2 \restriction \text{supp}(\dot{f}_{[\alpha_n, \alpha_n + \omega)}))$$

and, by the isomorphisms, we have that q_3 forces that $\dot{f}_{[\lambda, \lambda + \omega)}(\lambda + L + 1) = m$.

Technically (or with more care) all of this is taking place in the poset $\text{Fn}(\omega_2 \setminus \mu, 2)$ and this means that q_3 and q are all compatible with each other.

Follow the bouncing ball: it suffices to consider $q(\beta) = e$ and to assume that $q_3(\beta)$ is defined. Since $q_3(\beta)$ is defined, we have that there is a $\beta' \in \text{dom}(q_2)$ such that $\varphi_{\mu, \lambda}(\beta) = \varphi_{\mu, \alpha_n}(\beta')$, and that $q_3(\beta) = q_2(\beta')$. But, by definition of q_1 , $\beta' \in \text{dom}(q_1)$ and even that $q_1(\beta') = q(\beta)$. Then, since $q_2 < q_1$, we have that $q_2(\beta') = q_1(\beta') = q(\beta)$. This completes the circle that $q_3(\beta) = q(\beta)$.

Finally, our contradiction is that $q_3 \cup q_2 \cup q$ forces that $k = L + 1$ violates the quoted statement above. \square

On the other hand, it's also consistent that $S'(\theta)$ can fail.

Theorem 21. *There's a model where $S'(\omega_\omega)$ fails.*

Proof. We will need the model constructed in [3] in which an instance of Chang's conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is shown to fail.

We can take as a given (as shown in [3, Theorem 5]) that we may assume that we have a model V of GCH in which there are regular limit cardinals $\kappa < \lambda$ satisfying that $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$.

What this says is that if L is a countable language with at least one unary relation symbol R and M is a model of L with base set $\lambda^{+\omega+1}$ in which the interpretation of R has cardinality $\lambda^{+\omega}$, then M has an elementary submodel N of cardinality $\kappa^{+\omega+1}$ in which $R \cap N$ has cardinality $\kappa^{+\omega}$ (of course $R \cap N$ is the interpretation of R in N because $N \prec M$).

The interested reader will want to know that it is shown in [3] that if κ is a 2-huge cardinal and j is the 2-huge embedding with critical point κ , then with $\lambda = j(\kappa)$ one has that $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$ holds.

Let $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$ be a scale in $\Pi\{\lambda^{+n+1} : n \in \omega\}$ ordered by the usual mod finite coordinatewise ordering. For convenience we may assume that $h_\xi(n) \geq \lambda^{+n}$ for all ξ and all n . If P is any poset of cardinality less than $\lambda^{+\omega}$, then in the forcing extension by P , the sequence $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$ remains cofinal in $\Pi\{\lambda^{+n+1} : n \in \omega\}$.

The forcing notion \mathbb{P}_0 is simply the finite condition collapse of $\kappa^{+\omega}$, i.e. $\mathbb{P}_0 = (\kappa^{+\omega})^{<\omega}$. In the forcing extension by \mathbb{P}_0 , one now has that the ordinal $\kappa^{+\omega+1}$ from V is the first uncountable cardinal \aleph_1 . Then in this forcing extension we let \mathbb{P}_1 be the countable condition Levy collapse, $Lv(\lambda, \omega_2)$, which collapses all cardinals less than λ to have cardinality at most \aleph_1 . The poset \mathbb{P}_1 has cardinality λ . We treat \mathbb{P}_1 as containing \mathbb{P}_0 as a subposet by identifying each $(p_0, 1)$ with p_0 . After forcing with $\mathbb{P}_0 * \mathbb{P}_1$ we will have that ω_1 is the ordinal $(\kappa^{+\omega+1})^V$, ω_2 is the ordinal λ , and ω_ω is the ordinal $(\lambda^{+\omega})^V$.

Now we assume that we have an assignment $\dot{f}_{\dot{A}}$ of a $\mathbb{P}_0 * \mathbb{P}_1$ -name of a finite-to-one function from \dot{A} into ω for each $\mathbb{P}_0 * \mathbb{P}_1$ -name of a countable subset of $\lambda^{+\omega+1}$. We will obtain a contradiction.

Let $\{\dot{A}_\xi : \xi \in \lambda^{+\omega+1}\}$ be an enumeration of all the nice \mathbb{P}_0 -names of countable subsets of $\lambda^{+\omega}$. For each $\xi \in \lambda^{+\omega+1}$, let \dot{f}_ξ be another notation for $\dot{f}_{\dot{A}_\xi}$. Since \mathbb{P}_0 forces that \mathbb{P}_1 is countably closed, the collection of all nice \mathbb{P}_0 -names will produce all the countable sets in the extension by $\mathbb{P}_0 * \mathbb{P}_1$, but $\mathbb{P}_0 * \mathbb{P}_1$ can introduce new enumerations of these names. For each $\xi \in \lambda^{+\omega+1}$, there is a minimal ζ_ξ so that \dot{A}_{ζ_ξ} is the canonical name for the range of h_ξ . This means that $\dot{f}_{\zeta_\xi} \circ h_\xi$ is simply the $\mathbb{P}_0 * \mathbb{P}_1$ -name of a finite-to-one function from ω to ω . For each $\xi \in \lambda^{+\omega+1}$, choose any $p_\xi \in \mathbb{P}_0 * \mathbb{P}_1$ so that there is a nice \mathbb{P}_0 -name, \dot{H}_ξ , that is forced by p_ξ to equal

$\dot{f}_{\zeta_\xi} \circ h_\xi$. Choose $\Lambda \subset \lambda^{+\omega+1}$ of cardinality $\lambda^{+\omega+1}$ and so that there is a pair p, \dot{H} satisfying that $p_\xi = p$ and $\dot{H}_\xi = \dot{H}$ for all $\xi \in \Lambda$. We may assume that p is in a generic filter G .

Let $\{x_\xi : \xi \in \lambda^{+\omega+1}\}$ be any enumeration of $H(\lambda^{+\omega+1})$ such that $\{x_\xi : \xi \in \lambda^{+\omega}\}$ is also equal to $H(\lambda^{+\omega})$. We choose this enumeration in such a way that $x_\xi \in x_\eta$ implies $\xi < \eta$. We use relation symbol R_0 to code (and well order) $(H(\lambda^{+\omega+1}), \in)$ as follows: $(\xi, \eta) \in R_0$ if and only if $x_\xi \in x_\eta$. Let R_1 be a binary relation on $\kappa^{+\omega}$ so that $(\kappa^{+\omega}, R_1)$ is isomorphic to \mathbb{P}_0 . Let R_2 be a binary relation on λ so that $R_2 \cap (\kappa^{+\omega} \times \kappa^{+\omega}) = R_1$ and (λ, R_2) is isomorphic to $\mathbb{P}_0 * \mathbb{P}_1$. Let ψ be the poset isomorphism from λ to $\mathbb{P}_0 * \mathbb{P}_1$.

We continue coding. We can code the sequence $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$ as another binary relation R_3 on $\lambda^{+\omega+1}$ where $R_3 \cap (\{\xi\} \times \lambda^{+\omega+1}) = \{(\xi, h_\xi(n)) : n \in \omega\}$ for each $\xi \in \lambda^{+\omega+1}$. The relation symbol R_4 can code the sequence $\{\dot{A}_\xi : \xi \in \lambda^{+\omega+1}\}$ where $(\xi, \alpha, \zeta) \in R_4$ if and only if $(\check{\alpha}, \psi(\zeta))$ is in the name \dot{A}_ξ . Let R_5 code this collection, i.e. $(\gamma, n, m, \eta) \in R_5$ if and only if $((n, m), \psi(\eta)) \in \dot{H}_\gamma$. Also let R_6 code (equal) the set Λ . Finally we use the relation symbol R_7 to similarly code the sequence $\{\dot{f}_\xi : \xi \in \lambda^{+\omega+1}\}$: $(\xi, \alpha, n, \zeta) \in R_7$ if and only if $((\check{\alpha}, n), \psi(\zeta))$ is in the name \dot{f}_ξ .

Needless to say, the unary relation symbol R is interpreted as the set $\lambda^{+\omega}$ for the application of $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$. Now we have defined our model M of the language $L = \{\in, R, R_0, \dots, R_7\}$, and we choose an elementary submodel N witnessing $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$. Of course N is really just a $\kappa^{+\omega+1}$ sized subset of $\lambda^{+\omega+1}$ with the additional property that $N \cap \lambda^{+\omega}$ has cardinality $\kappa^{+\omega}$. In the forcing extension N has cardinality ω_1 and $A = N \cap \lambda^{+\omega}$ is countable.

We will need the following claim from [3]

Claim. *We may assume that N satisfies that $N \cap \kappa^{+\omega+1}$ is transitive (i.e. an initial segment).*

Proof of Claim. Suppose our originally supplied N fails the conclusion of the claim. We know that $\kappa^{+\omega} \in N$, (via R_1) in which case so is $\kappa^{+\omega+1}$.

Then set $\beta_0 = \sup(N \cap \kappa^{+\omega+1})$ and consider the Skolem closure $Hull(N \cup \beta_0, M)$. A little informally (in that we have to formalize the enumeration of formulas) let $\{\varphi_n : n \in \omega\}$ is the enumeration of all formulas in the language L , and let ℓ_n be the minimal integer such that the free variables of φ_n are among $\{v_0, \dots, v_{\ell_n}\}$. Then, for each tuple $\langle \xi_1, \dots, \xi_{\ell_n} \rangle$ of elements of $\lambda^{+\omega+1}$, we define $f_n(\xi_1, \dots, \xi_{\ell_n})$ to be the minimal $\xi_0 \in \lambda^{+\omega+1}$ such that $M \models \varphi_n(\xi_0, \dots, \xi_{\ell_n})$. If there is no such ξ_0 , in other words if $M \models \neg \exists x \varphi_n(x, \xi_1, \dots, \xi_{\ell_n})$, then set $f_n(\xi_1, \dots, \xi_{\ell_n})$ to be 0. Now $Hull(N \cup \beta_0, M)$ is just the minimal superset X of $N \cup \beta_0$ that satisfies that $f_n[X^{\{1, \dots, \ell_n\}}] \subset X$ for all n . Since this is simply a large algebra, we can generate all the terms t of the algebraic operations $\{f_n : n \in \omega\}$. It is easily seen that for each $\zeta \in X$, there is a term $t(v_1, \dots, v_m)$ such that $\zeta = t(\delta_1, \dots, \delta_m)$ for some sequence $\langle \delta_1, \dots, \delta_m \rangle$ with each $\delta_i \in N \cup \beta_0$. Assume that $\zeta \in \kappa^{+\omega+1}$. By re-indexing the variables in the term we can assume that there is an $n \leq m$ so that $\delta_i < \beta_0$ for $1 \leq i \leq n$ and $\kappa^{+\omega+1} \leq \delta_i$ for $n < i \leq m$. Let \vec{a} denote the tuple $\langle \delta_{n+1}, \dots, \delta_m \rangle$. Choose $\eta \in N \cap \kappa^{+\omega+1}$ large enough so that $\{\delta_1, \dots, \delta_n\}$ is contained in η . Since

set-membership in M is coded by R_0 rather than \in we have to argue a little less naturally. Consider the set $s_0(\eta, \vec{a}) = \{t(\gamma_1, \dots, \gamma_n, \vec{a}) : \{\gamma_1, \dots, \gamma_n\} \in [\eta]^{\leq n}\}$. Clearly $s_0(\eta, \vec{a})$ is a member of $H(\lambda^{+\omega+1})$. Now define $s_1(\eta, \vec{a})$ to be $\{x_\alpha : \alpha \in s_0(\eta, \vec{a})\}$, and choose the unique $\zeta_1 \in \lambda^{+\omega+1}$ such that $x_{\zeta_1} = s_1(\eta, \vec{a})$. We claim that $\zeta_1 \in N$. Note that $\alpha R_0 \zeta_1$ holds if and only if $\alpha \in s_0(\eta, \vec{a})$, and therefore

$$M \models (\forall \alpha) [\alpha R_0 \zeta_1 \text{ iff } (\exists \gamma_1 \in \eta) \cdots (\exists \gamma_n \in \eta) (\alpha = t(\gamma_1, \dots, \gamma_n, \vec{a}))] .$$

By elementarity then we have that $\zeta_1 \in N$, and by similar reasoning the supremum, ζ_0 , of $\zeta_1 \cap \kappa^{+\omega+1}$ is also in N . This of course means that $\zeta < \xi_0$. \square

We use the elementarity of N to deduce properties of the families $\{\dot{A}_\xi : \xi \in N\}$ and $\{\dot{f}_\xi : \xi \in N\}$. Actually the collection we are most interested in is the family $\{h_\xi : \xi \in \Lambda \cap N\}$.

Since $\mathfrak{c} < \kappa^{+\omega+1}$ there is a function $\langle \varrho_n : n \in \omega \rangle$ in $\Pi_n \lambda^{+\omega}$ such that the sequence $\{h_\xi : \xi \in N\}$ is unbounded mod finite in $\Pi_n \varrho_n$ (by Shelah's pcf theory). This is in Jech somewhere. For each n , $\rho_n \leq \sup(N \cap \lambda^{+n+2})$.

Since \mathbb{P}_0 has cardinality less than $|N| = \kappa^{+\omega+1}$, the sequence $\{h_\xi : \xi \in \Lambda \cap N\}$ remains unbounded mod finite in $\Pi_n \varrho_n$ (and in $\Pi_n(\varrho_n \cap N)$). Now pass to the extension by $G \cap \mathbb{P}_0$ and let H be the function $\text{val}_G(H)$, and we recall that $f_{\zeta_\xi}(h_\xi(n)) = H(n)$ for all $n \in \omega$. Now pass to the full extension $V[G]$ and again, since \mathbb{P}_1 was forced to be countably closed, the family $\{h_\xi : \xi \in \Lambda \cap N\}$ is still unbounded in $\Pi_n(\varrho_n \cap N)$. We let A be the countable set $N \cap \lambda^{+\omega}$, and for each $\xi \in \Lambda \cap N$, there is an n_ξ such that $f_\xi(h_\xi(m)) = f_A(h_\xi(m))$ for all $m > n_\xi$. There is a single n so that $\Lambda_n = \{\xi \in \Lambda \cap N : n_\xi = n\}$ has cardinality ω_1 , and thus $\{h_\xi : \xi \in \Lambda_n \cap N\}$ is also unbounded in $\Pi_n(\varrho_n \cap N)$. This certainly implies that there is an $m > n$ such that $\{h_\xi(m) : \xi \in \Lambda_n \cap N\}$ is infinite. This completes the proof since $f_A(h_\xi(m)) = H(m)$ for all $\xi \in \Lambda_n \cap N$. \square

Question 22. *Is $S'(\theta)$ equivalent to having a Kurepa family on θ ?*

APPLICATIONS!

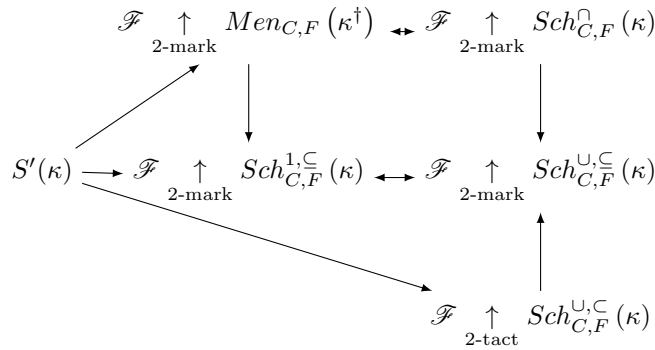


FIGURE 1. Diagram of Scheeper/Menger game implications with $S'(\kappa)$

Theorem 23. *Figure 1 holds. (Proven in [TODO cite]) (Actually, TODO double-check that it works with just S' , particularly the strict game)*

It was left open if these implications can be reversed. The answer is consistently no.

Theorem 24. *Let α be the limit of increasing ordinals β_n for $n < \omega$. If $\mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^\cap(\omega_{\beta_n})$ for all $n < \omega$, then $\mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^\cap(\omega_\alpha)$.*

Proof. Let σ_n be a winning 2-mark for \mathcal{F} in $Sch_{C,F}^\cap(\omega_{\beta_n})$. Define the 2-mark σ for \mathcal{F} in $Sch_{C,F}^\cap(\omega_\alpha)$ as follows:

$$\sigma(\langle C \rangle, 0) = \sigma_0(\langle C \cap \omega_{\beta_0} \rangle, 0)$$

$$\sigma(\langle C, D \rangle, n+1) = \sigma_{n+1}(\langle D \cap \omega_{\beta_{n+1}} \rangle, 0) \cup \bigcup_{m \leq n} \sigma_m(\langle C \cap \omega_{\beta_m}, D \cap \omega_{\beta_m} \rangle, n-m+1)$$

Let $\langle C_0, C_1, \dots \rangle$ be an attack by \mathcal{C} in $Sch_{C,F}^\cap(\omega_\alpha)$, and $\alpha \in \bigcap_{n < \omega} C_n$. Choose $N < \omega$ with $\alpha < \omega_{\beta_{N+1}}$. Consider the attack $\langle C_{N+1} \cap \omega_{\beta_{N+1}}, C_{N+2} \cap \omega_{\beta_{N+1}}, \dots \rangle$ by \mathcal{C} in $Sch_{C,F}^\cap(\omega_{\beta_{N+1}})$. Since σ_{N+1} is a winning strategy and $\alpha \in \bigcap_{n < \omega} C_{N+n+1} \cap \omega_{\beta_{N+1}}$, either $\alpha \in \sigma_{N+1}(\langle C_{N+1} \cap \omega_{\beta_{N+1}} \rangle, 0)$ and thus $\alpha \in \sigma(\langle C_N, C_{N+1} \rangle, N+1)$, or $\alpha \in \sigma_{N+1}(\langle C_{N+M+1} \cap \omega_{\beta_{N+1}}, C_{N+M+2} \cap \omega_{\beta_{N+1}} \rangle, M+1)$ for some $M < \omega$ and thus $\alpha \in \sigma(\langle C_{N+M+1}, C_{N+M+2} \rangle, N+M+2)$. Thus σ is a winning strategy. \square

Theorem 25. *Let α be the limit of increasing ordinals β_n for $n < \omega$. If $\mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^{1,\subseteq}(\omega_{\beta_n})$ for all $n < \omega$, then $\mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^{1,\subseteq}(\omega_\alpha)$.*

Proof. Let σ_n be a winning 2-mark for \mathcal{F} in $Sch_{C,F}^{1,\subseteq}(\omega_{\beta_n})$. Define the 2-mark σ for \mathcal{F} in $Sch_{C,F}^{1,\subseteq}(\omega_\alpha)$ as follows:

$$\sigma(\langle C \rangle, 0) = \sigma_0(\langle C \cap \omega_{\beta_0} \rangle, 0)$$

$$\sigma(\langle C, D \rangle, n+1) = \sigma_{n+1}(\langle D \cap \omega_{\beta_{n+1}} \rangle, 0) \cup \bigcup_{m \leq n} \sigma_m(\langle C \cap \omega_{\beta_m}, D \cap \omega_{\beta_m} \rangle, n-m+1)$$

Let $\langle C_0, C_1, \dots \rangle$ be an attack by \mathcal{C} in $Sch_{C,F}^{1,\subseteq}(\omega_\alpha)$, and $\alpha \in C_0$. Choose $N < \omega$ with $\alpha < \omega_{\beta_{N+1}}$. Consider the attack $\langle C_{N+1} \cap \omega_{\beta_{N+1}}, C_{N+2} \cap \omega_{\beta_{N+1}}, \dots \rangle$ by \mathcal{C} in $Sch_{C,F}^{1,\subseteq}(\omega_{\beta_{N+1}})$. Since σ_{N+1} is a winning strategy and $\alpha \in C_{N+1} \cap \omega_{\beta_{N+1}}$, either $\alpha \in \sigma_{N+1}(\langle C_{N+1} \cap \omega_{\beta_{N+1}} \rangle, 0)$ and thus $\alpha \in \sigma(\langle C_N, C_{N+1} \rangle, N+1)$, or $\alpha \in \sigma_{N+1}(\langle C_{N+M+1} \cap \omega_{\beta_{N+1}}, C_{N+M+2} \cap \omega_{\beta_{N+1}} \rangle, M+1)$ for some $M < \omega$ and thus $\alpha \in \sigma(\langle C_{N+M+1}, C_{N+M+2} \rangle, N+M+2)$. Thus σ is a winning strategy. \square

Corollary 26. *It is consistent that $S'(\omega_\omega)$ fails, but as $S'(\omega_k)$ holds for all $k < \omega$, we have $\mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^\cap(\omega_\omega)$ and $\mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^{1,\subseteq}(\omega_\omega)$.*

A tricky topological question: does $\mathcal{F} \uparrow_{2\text{-mark}} Men_{C,F}(X)$ imply $\mathcal{F} \uparrow_{2\text{-mark}} Men_{C,F}(X)$? (C showed that) Under $V = L$, we cannot hope to find a counterexample using $X = \kappa^\dagger$ since $S'(\kappa)$ and thus $\mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^\cap(\kappa)$ always holds.

Definition 27. Let R_ω be the real numbers with the topology of the usual open intervals with countably many elements removed.

Theorem 28. $\mathcal{F} \uparrow Men_{C,F}(R_\omega)$. If there exists a Kurepa family on the reals, then $\mathcal{F} \uparrow_{2\text{-mark}} Men_{C,F}(R_\omega)$.

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