Tactics and Marks in Banach Mazur Games

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My notes on Galvin/Telgarsky's Theorem 5 from [1].

Definition 1. Let $\mathbb{P}$ be partially ordered by $\leq$. Let $\mathbb{P}^{\downarrow} = \{f \in \mathbb{P}^\omega : f(n) \geq f(n+1)\}$. Then for $f, g \in \mathbb{P}^{\downarrow}$, we say that f, g zip into each other if for all $m < \omega$ there exists $n < \omega$ such that $f(m) \geq g(n)$ and $g(m) \geq f(n)$.

Definition 2. $BM_{po}(\mathbb{P}, W)$ is a game defined for all non-empty partial orders $\mathbb{P}$ and all subsets $W \subseteq \mathbb{P}^{\downarrow}$. During round 0, I chooses $a_0 \in \mathbb{P}$, and then II chooses $b_0 \leq a_0$; during around $n+1$, I chooses $a_{n+1} \leq b_n$, and then II chooses $b_{n+1} \leq a_{n+1}$. II wins this game if $\langle a_0, a_1, \dots \rangle \in W$.

Theorem 3. *Let $W \subseteq \mathbb{P}^{\downarrow}$ be closed under zipping.* II $\underset{mark}{\uparrow}$ $BM_{po}(\mathbb{P}, W)$ *if and only if* II $\underset{tact}{\uparrow}$ $BM_{po}(\mathbb{P}, W)$.

Proof. Let $\tau(p, n+1)$ be a winning mark for II, where p is the most recent move by I and $n+1$ is the number of moves made by I. Define $\tau^0(p) = p$ and $\tau^{n+1}(p) = \tau(\tau^n(p), n+1)$. Let $\preceq$ well-order $\mathbb{P}$.

For $p, q \in \mathbb{P}$, say $p \geq_n q$ if there exist $s_m(p) \in \mathbb{P}$ for $m \leq n$ such that

$$p \geq s_m(p) \geq \tau(s_m(p), n+1) \geq q.$$

Note that $p' \geq p \geq_n q \geq q'$ implies $p' \geq_n q'$, and $p \geq_n \tau^n(p)$.

Say $p \geq_\omega q$ whenever $p \geq_n q$ for all $n < \omega$. If $p \geq_\omega l(p)$ for some $l(p)$, then say p is long; otherwise call p short.

For p short, let

$$\mu(p) = \min_{\preceq}\{r \text{ short} : r \geq p\}$$

and since $\mu(p) \not\geq_n p$ for some n, let

$$N(p) = \min\{n < \omega : \mu(p) \not\geq_n p\}.$$

Note that whenever $\mu(p) = \mu(q)$ for $p \geq_n q$, it follows that $\mu(p) \geq_n q$ and therefore $N(p) < N(q)$.

We define

$$\sigma(p) = \begin{cases} l(p) & p \text{ is long} \\ \tau^{N(p)+1}(p) & p \text{ is short} \end{cases}.$$

Suppose σ is legally attacked by $a \in \mathbb{P}^\omega$. For $n \leq \omega$, if $a(n)$ is long, then $a(n) \geq_n l(a(n))$. Therefore,

$$a(n) \geq s_n(a(n)) \geq \tau(s_n(a(n)), n+1) \geq l(a(n)) = \sigma(a(n)) \geq a(n+1).$$

Thus if $a(n)$ is long for $n < \omega$, it follows that $c \in \mathbb{P}^{\downarrow}$ defined by $c(n) = s_n(a(n))$ is a legal attack against τ. Since τ is winning, $c \in W$, and since c zips into a, $a \in W$ as well.

Otherwise, we may choose a final subsequence b of a such that

- $b(n)$ is short for all $n < \omega$, since $a(m)$ short implies $a(n+m)$ short for all $n < \omega$.
- $\mu(b(n)) = \mu'$ is fixed for all $n < \omega$, since there cannot be an infinite $\preceq$-decreasing sequence.

As a result,

$$b(n) \geq_{N(b(n))} \tau^{N(b(n))+1}(b(n)) = \sigma(b(n)) \geq b(n+1)$$

and therefore $N(b(n)) < N(b(n+1))$. In particular, $N(b(n)) \geq n$.

Thus for $n < \omega$,

$$b(n) \geq \tau^n(b(n)) \geq \tau(\tau^n(b(n)), n+1) \geq \tau^{N(b(n))+1}(b(n)) = \sigma(b(n)) \geq b(n+1).$$

As a result, $c \in \mathbb{P}^{\downarrow}$ defined by $c(n) = \tau^n(b(n))$ is a legal attack against the winning strategy τ. Therefore $c \in W$, and since c zips into b and a, we conclude $a \in W$. $\square$

Observation 4. When $\mathbb{P} = T(X)\setminus\{\emptyset\}$ is ordered by set-inclusion and $W = \{U \in \mathbb{P}^{\downarrow} : \bigcap_{n<\omega} U(n) \neq \emptyset\}$, then $BM_{po}(\mathbb{P}, W)$ is exactly the topological Banach Mazur game $BM_{E,N}(X)$. Note W is closed under zipping.

Corollary 5. II $\underset{mark}{\uparrow}$ $BM_{E,N}(X)$ *if and only if* II $\underset{tact}{\uparrow}$ $BM_{E,N}(X)$.

Observation 6. When $\mathbb{P} = \{(U, x) : U \in T(X)\setminus\{\emptyset\}, x \in U\}$ is ordered by $(U, x) \geq (V, y)$ whenever $x \in V \subseteq U$ and $W = \{\langle U(n), x(n)\rangle_{n<\omega} \in \mathbb{P}^{\downarrow} : \bigcap_{n<\omega} U(n) \neq \emptyset\}$, then $BM_{po}(\mathbb{P}, W)$ is almost the Choquet game, except the first player also gets to choose a point. Note W is closed under zipping.

References

[1] Fred Galvin and Ratislav Telgársky. Stationary strategies in topological games. *Topology Appl.*, 22(1):51–69, 1986.