Remark 1. Scheeper's $S(\kappa)$ requiring injections is stronger than my $S'(\kappa)$ requiring finite-to-one maps. Dow suggests that $S'(\omega_\omega)$ holds in ZFC by the following.

Definition 2. A topological space is said to be ω-bounded if each countable subset of the space has compact closure.

Theorem 3. *For each $k < \omega$ there exists a topology on ω_k which is ω-bounded and locally countable.*

Proof. Assume we've defined such a topology for ω_k such that $\gamma + 1 = [0, \gamma]$ is clopen for all $\gamma < \omega_k$. Note that the usual linear order on ω_1 satisfies these requirements.

Let $\alpha < \omega_{k+1}$, and suppose we've defined compatible topologies on $\omega_k \cdot (\beta + 1)$ for all $0 \leq \beta < \alpha$. If $\alpha = \beta + 1$, then let $\omega_k \cdot (\alpha + 1) = \omega_k \cdot (\beta + 2)$ be the topological sum of the previously defined $\omega_k \cdot (\beta + 1)$ and the previously defined $\omega_k \cdot (\beta + 2) \setminus \omega_k \cdot (\beta + 1) \cong \omega_k$. Similarly, if $cf(\alpha) > \omega$, then let $\omega_k \cdot (\alpha + 1)$ be the topological sum of $\bigcup_{\beta<\alpha} \omega_k \cdot (\beta + 1)$ and the previously defined $\omega_k \cdot (\alpha + 1) \setminus \omega_k \cdot \alpha \cong \omega_k$.

The remaining case is where α is the limit of increasing α_n for $n < \omega$. Fix a bijection $f_\alpha : \omega_k \cdot (\alpha + 1) \setminus \omega_k \cdot \alpha \to \omega_k \cdot \alpha$. Points in $\omega_k \cdot (\alpha_n + 1)$ for some $n < \omega$ have their usual base induced by that previously defined topology. So let $\gamma \in \omega_k \cdot (\alpha + 1) \setminus \omega_k \cdot \alpha$. Basic open neighborhoods of γ are of the form $W \cup f_\alpha[W] \setminus \omega_k \cdot (\alpha_n + 1)$, where $n < \omega$ and $W \subseteq \gamma + 1$ is any countable neighborhood of γ.

We wish to show that ω_{k+1} with the topology induced by $\bigcup_{\alpha<\omega_{k+1}} \omega_k \cdot (\alpha + 1)$ is ω-bounded and locally countable. If $\gamma \in \omega_k \cdot (\alpha + 1) \setminus \omega_k \cdot \alpha$ where $cf(\alpha) \neq \omega$, then we immediately see that it is in a clopen copy of ω_k giving us local countability immediately. Otherwise, γ has a basic open neighborhood of the form $W \cup f_\alpha[W] \setminus \omega_k \cdot (\alpha_n + 1)$, which is obviously countable.

Let C be a countable subset of $\omega_k \cdot (\alpha+1)$. In the case that $\alpha = \beta+1$, we may use the ω-boundedness of each part in the clopen partition $\omega_k \cdot (\beta+1)$ and $\omega_k \cdot (\beta+2) \setminus \omega_k \cdot (\beta+1) \cong \omega_k$ to conclude that the closure of C is compact. Similarly, if $cf(\alpha) > \omega$, then we may use the ω-boundedness of each part in the clopen partition $\bigcup_{\beta<\alpha} \omega_k \cdot (\beta+1)$ and $\omega_k \cdot (\alpha+1) \setminus \omega_k \cdot \alpha \cong \omega_k$ to conclude that the closure of C is compact.

The remaining case is again where α is the limit of increasing α_n for $n < \omega$. Then TODO: generalize for $k + 1$

Finally, since every countable subset of ω_{k+1} is contained in some $\omega_k \cdot (\alpha+1)$, we conclude ω_{k+1} is ω-bounded. □

Theorem. $S'(\omega_\omega)$.