ZERO-MARKOV INFORMATION IN TOPOLOGICAL GAMES

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Abstract. A 0-Markov strategy in a topological game considers only the round number and ignores all moves by the opponent. The existence of a winning 0-Markov strategy in either of two games due to Gruenhage characterizes hemicompactness in either locally compact or compactly generated spaces. However, there exists a non-compactly generated space for which there exists a winning 0-Markov strategy in one game but not the other.

1. Introduction

The following two topological games were introduced by Gary Gruenhage in [3].

Game 1.1. Let $Gru_{K,P}(X)$ denote the *Gruenhage compact/point game* with players $\mathscr{K}$, $\mathscr{P}$ played on a topological space X. During round n, $\mathscr{K}$ chooses a compact subset K_n of X, followed by $\mathscr{P}$ choosing a point $p_n \in X$ such that $p_n \notin \bigcup_{m\leq n} K_m$.

$\mathscr{K}$ wins the game if the collection $\{\{p_n\} : n < \omega\}$ is locally finite in the space, and $\mathscr{P}$ wins otherwise.

Game 1.2. Let $Gru_{K,L}(X)$ denote the *Gruenhage compact/compact game* with players $\mathscr{K}$, $\mathscr{L}$ played on a topological space X. This game proceeds analogously to $Gru_{K,P}(X)$, except the second player $\mathscr{L}$ chooses compact sets L_n missing $\bigcup_{m\leq n} K_n$, and $\mathscr{K}$ wins if the collection $\{L_n : n < \omega\}$ is locally finite.

A *strategy* for a game defines the move a player makes each round as a function of the history of the game (previous moves, the round number, etc.). A *winning strategy* defeats every possible counterattack by the opponent. Note that a winning strategy in $Gru_{K,L}(X)$ is also a winning strategy in $Gru_{K,P}(X)$ since singletons are compact. In his paper, Gruenhage used these games to characterize several covering properties using the existence of various kinds of winning strategies for $\mathscr{K}$ in the games. These results hold in the context of *locally compact* spaces for which every point has a compact neighborhood.

Definition 1.3. A space is *paracompact* if for every open cover $\mathcal{U}$ there exists a locally-finite open refinement $\mathcal{V}$ of $\mathcal{U}$ also covering the space.

Theorem 1.4. *The following are equivalent for a locally compact space X:*

- *X is paracompact*
- *$\mathscr{K} \uparrow Gru_{K,L}(X)$. ($\mathscr{K}$ has a winning strategy for the game.)*

Definition 1.5. A space is *metacompact* if for every open cover $\mathcal{U}$ there exists a point-finite open refinement $\mathcal{V}$ of $\mathcal{U}$ also covering the space.

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Theorem 1.6. *The following are equivalent for a locally compact space X:*

- *X is metacompact*
- $\mathscr{K} \underset{tact}{\uparrow} Gru_{K,P}(X)$ *($\mathscr{K}$ has a tactical winning strategy which only considers the most recent move of the opponent each round)*

Definition 1.7. A space is *σ-metacompact* if for every open cover $\mathcal{U}$ there exist point-finite open refinements $\mathcal{V}_n$ of $\mathcal{U}$ such that $\bigcup_{n<\omega} \mathcal{V}_n$ also covers the space.

Theorem 1.8. *The following are equivalent for a locally compact space X:*

- *X is σ-metacompact*
- $\mathscr{K} \underset{mark}{\uparrow} Gru_{K,P}(X)$ *($\mathscr{K}$ has a Markov winning strategy which only considers the most recent move of the opponent and the round number each round)*

Tactical and Markov strategies are examples of *limited information* strategies. These may be generalized to *k-tactical* and *k-Markov* strategies by allowing the player to use the k most recent moves of the opponent; so 1-tactical strategies are simply tactical strategies, and similar for Markov. Of course, if $k < l$ then a winning k-tactical (resp. Markov) strategy is itself a winning l-tactical (resp. Markov) strategy. In [1] the author investigated $(k+1)$-tactical/Markov strategies in $Gru_{K,P}(X)$ and showed that even for a complexly constructed space they can often be improved to simply a tactical strategy; it remains open if this is always the case.

In this paper we investigate the applications of 0-Markov strategies in both $Gru_{K,P}(X)$ and $Gru_{K,L}(X)$, which we will call *predetermined* strategies as each move is determined completely by the round number of the game and ignores all moves of the opponent. It will be shown that for compactly generated spaces that a predetermined winning strategy in $Gru_{K,P}(X)$ can be used to get a predetermined winning strategy in $Gru_{K,L}(X)$. However, there exists a non-compactly generated space for which this does not work out.

2. Locally compact spaces and predetermined strategies

It's a known fact [5] that amongst locally compact spaces the following properties are equivalent.

Definition 2.1. A space X is *Lindelöf* if for every open cover of X there exists a countable subcover.

Definition 2.2. A space X is *σ-compact* if $X = \bigcup_{n<\omega} K_n$ for K_n compact.

Definition 2.3. A space X is *hemicompact* if $X = \bigcup_{n<\omega} K_n$ for K_n compact and every compact subset of X is contained in some K_n.

In general, hemicompact spaces are σ-compact, and σ-compact spaces are Lindelöf. By considering the games $Gru_{K,P}(X)$ and $Gru_{K,L}(X)$ we will obtain an alternate proof that locally compact Lindelöf spaces are hemicompact.

Theorem 2.4. *If X is a locally compact Lindelöf space, then* $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,L}(X)$ *($\mathscr{K}$ has a winning 0-Markov a.k.a. predetermined strategy for the game.)*

Proof. For each $x \in X$, let U_x be an open neighborhood of x with $\overline{U_x}$ compact. Then as X is Lindelöf, choose $x_n \in X$ for $n < \omega$ such that $\{U_{x_n} : n < \omega\}$ covers X. Define the predetermined strategy σ for $\mathscr{K}$ by $\sigma(n) = \overline{U_{x_n}}$.

Let $L : \omega \to \mathcal{K}(X)$ legally attack σ, so $L(n) \cap \bigcup_{m \leq n} \sigma(m) = \emptyset$. For each $x \in X$, choose $n < \omega$ with $x \in U_{x_n}$. Then U_{x_n} is a neighborhood of x which intersects finitely many $L(n)$, so $\{L(n) : n < \omega\}$ is locally finite. $\square$

Theorem 2.5. *If* $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(X)$*, then* X *is hemicompact.*

Proof. Let σ be a winning predetermined strategy for $\mathscr{K}$ in $Gru_{K,P}(X)$. If $C \in \mathcal{K}(X)$ is compact, then for each $x \in C$ let U_x be an open neighborhood of x which intersects finitely many $\sigma(n)$. Choose $x_i \in C$ for $i < n < \omega$ such that $\{U_{x_i} : i < n\}$ covers C. Then $\bigcup_{i<n} U_{x_i}$ contains C and intersects finitely many $\sigma(n)$, and thus $\{\bigcup_{m \leq n} \sigma(m) : n < \omega\}$ witnesses hemicompactness. $\square$

Corollary 2.6. *The following are equivalent for any locally compact space* X*:*

- X *is Lindelöf.*
- X *is* σ*-compact.*
- X *is hemicompact.*
- $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(X)$.
- $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,L}(X)$.

3. Compactly generated spaces and predetermined strategies

Definition 3.1. A space X is *compactly generated* if a set is closed if and only if its intersection with every compact set is closed. Such spaces are also known as *k-spaces.*

All locally compact spaces are k-spaces. As will be shown, the games $Gru_{K,P}(X)$, $Gru_{K,L}(X)$ are equivalent for $\mathscr{K}$'s predetermined strategies in Hausdorff k-spaces.

Definition 3.2. A space X is a k_ω-space if there exist compact sets K_n for $n < \omega$ such that a set is closed if and only if its intersection with every K_n is closed.

Theorem 3.3. *If* X *is a* k_ω*-space, then* $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,L}(X)$.

Proof. Let K_n witness that X is a k_ω-space. Define the predetermined strategy σ for $\mathscr{K}$ by $\sigma(n) = K_n$.

Let $L : \omega \to \mathcal{K}(X)$ be a legal attack against σ, and let $L_{\omega \setminus n} = \bigcup_{n \leq m < \omega} L(m)$. Then as

$$L_{\omega \setminus n} \cap K_p = \bigcup_{n \leq m < p} L(m) \cap \sigma(p)$$

is compact for each $p < \omega$, $L_{\omega \setminus n}$ is closed.

For each $x \in X$, $x \in \sigma(p)$ for some p, so $x \in X \setminus L_{\omega \setminus p}$ which misses all but finitely many $L(n)$, showing that $\{L(n) : n < \omega\}$ is locally finite and σ is a winning predetermined strategy. $\square$

The following result was observed in [2]; a proof is provided for convenience.

Proposition 3.4. *Hemicompact k-spaces are k_ω-spaces.*

Proof. Let K_n for $n < \omega$ witness hemicompactness. If $C \cap K_n$ is closed for each $n < \omega$, then let K be any compact set. Since $K \subseteq K_n$ for some $n < \omega$, $C \cap K$ is closed, and therefore C is closed. $\square$

As we've already seen that $\mathscr{K} \underset{\text{pre}}{\uparrow} Gru_{K,P}(X)$ implies hemicompactness:

Corollary 3.5. *The following are equivalent for any k-space X:*

- *X is k_ω.*
- *X is hemicompact.*
- *$\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(X)$.*
- *$\mathscr{K} \underset{pre}{\uparrow} Gru_{K,L}(X)$.*

4. Non-equivalence of $\mathscr{K} \underset{\text{PRE}}{\uparrow} Gru_{K,P}(X)$, $\mathscr{K} \underset{\text{PRE}}{\uparrow} Gru_{K,L}(X)$

For k-spaces, it has been shown that $Gru_{K,P}(X)$ and $Gru_{K,L}(X)$ are equivalent with respect to $\mathscr{K}$'s winning predetermined strategies. Looking at a subspace of the Stone-Cech compactification $\beta\omega$ of ω reveals an example for which the predetermined strategies are not equivalent.

Definition 4.1. An *ultrafilter* on a cardinal κ is a maximal filter of non-empty subsets of κ. For each $\alpha \in \kappa$, the ultrafilter $\mathcal{F}_\alpha$ containing all supersets of $\{\alpha\}$ is called a *principal ultrafilter*. All ultrafilters not of this form are called *free ultrafilters*.

Definition 4.2. The *Stone-Cech compactification* of a cardinal κ is the space $\beta\kappa$ consisting of all ultrafilters on κ, with open sets of the form $U_S = \{\mathcal{F} \in \beta\kappa : S \in \mathcal{F}\}$ for $S \subseteq \kappa$.

From these definitions it is easily verified that principal ultrafilters are isolated, so κ with the discrete topology may be viewed as a dense open subspace of $\beta\kappa$. We wish to consider the subspace of $\beta\omega$ consisting of all principal ultrafilters and a single free ultrafilter $\mathcal{F}$, denoted by $\omega \cup \{\mathcal{F}\}$.

Lemma 4.3. *All compact subsets of $\omega \cup \{\mathcal{F}\} \subset \beta\omega$ are finite. In particular, the difference of compact sets in $\omega \cup \{\mathcal{F}\}$ is compact.*

Proof. Let $I = \{n_i : i < \omega\} \cup \{\mathcal{F}\}$ be infinite. Then $\{U_{\omega \setminus \{n_i : i \geq j\}} : j < \omega\}$ is an open cover of $I \cup \{\mathcal{F}\}$ with no finite subcover. $\square$

Theorem 4.4. $\mathscr{K} \underset{pre}{\not\uparrow} Gru_{K,L}(\omega \cup \{\mathcal{F}\})$ *for any free ultrafilter $\mathcal{F}$.*

Proof. Let σ be a predetermined strategy for $\mathscr{K}$, and define the legal counter-attack $H : \omega \to \mathcal{K}(X)$ by $H(n) = (n \cup \sigma(n+1)) \setminus \sigma(n)$. Then for any neighborhood U_S of $\mathcal{F}$, S is infinite, and since $\bigcup_{n<\omega} H(n) \supseteq \omega \setminus \sigma(0)$, U_S meets infinitely many of the finite $H(n)$. Thus σ is not a winning predetermined strategy. $\square$

Theorem 4.5. *There exists a free ultrafilter $\mathcal{F}$ such that $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(\omega \cup \{\mathcal{F}\})$.*

Proof. Let $\mathcal{F}$ be any free ultrafilter, and define the predetermined strategy σ by $\sigma(n) = n^2 \cup \{\mathcal{F}\}$.

Consider the set of all legal attacks $A \subseteq \omega^\omega$ by $\mathscr{P}$ against σ. For $\{f_i : i \leq m\} \in [A]^{<\omega}$ and $m < n < \omega$, each f_i maps only n points into n^2, so $\bigcup_{i \leq m} \mathrm{range}(f_i)$ is coinfinite. Then $\mathcal{G}' = \{\omega \setminus \mathrm{range}(f) : f \in A\}$ is contained in a free ultrafilter $\mathcal{G}$, and if $\mathcal{F} = \mathcal{G}$, then σ is a winning predetermined strategy. $\square$

It is not possible to prove in ZFC that $\mathscr{K} \underset{\text{pre}}{\uparrow} Gru_{K,P}\,(\omega \cup \{\mathcal{F}\})$ for arbitrary free ultrafilters.

Definition 4.6. A *selective ultrafilter* $\mathcal{S}$ is a free ultrafilter with the property that for every partition $\{B_n : n < \omega\}$ of nonempty subsets of ω such that $B_n \notin \mathcal{S}$ for all n, there exists $A \in \mathcal{S}$ such that $|A \cap B_n| = 1$ for all n.

Theorem 4.7. *CH implies the existence of a selective ultrafilter.* [4]

Theorem 4.8. *If $\mathcal{S}$ is a selective ultrafilter, then $\mathscr{K} \underset{pre}{\not\uparrow} Gru_{K,P}\,(\omega \cup \{\mathcal{S}\})$.*

Proof. Let σ be a predetermined strategy for $\mathscr{K}$ such that $\sigma(n) \supset \bigcup_{m<n} \sigma(m)$. Then define $B_n = \omega \cap (\sigma(n+1) \setminus \sigma(n))$. Since B_n is always nonempty finite, $B_n \notin \mathcal{F}$ and there exists $A \in \mathcal{S}$ such that $|A \cap B_n| = 1$.

Define the legal counter-attack $p : \omega \to \omega \cup \{\mathcal{S}\}$ by $p(n) \in A \cap B_n = A \cap (\sigma(n+1) \setminus \sigma(n))$. Since $A = (A \cap \sigma(0)) \cup \{p(n) : n < \omega\}$, $\{p(n) : n < \omega\} \in \mathcal{S}$. Therefore, every neighborhood of $\mathcal{F}$ intersects infinitely many of the $p(n)$, and p defeats the predetermined strategy σ. □

Of particular note is that the author knows of no examples of a non-k-space such that $K \underset{\text{pre}}{\uparrow} Gru_{K,P}\,(X)$.

Question 4.9. Does $K \underset{\text{pre}}{\uparrow} Gru_{K,P}\,(X)$ imply X is a k-space?

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References

[1] Steven Clontz. On k-tactics in Gruenhage's compact-point game. *Questions Answers Gen. Topology*, to appear.

[2] Stanley P. Franklin and Barbara V. Smith Thomas. A survey of k_ω-spaces. In *Proceedings of the 1977 Topology Conference (Louisiana State Univ., Baton Rouge, La., 1977), I*, volume 2, pages 111–124 (1978), 1977.

[3] Gary Gruenhage. Games, covering properties and Eberlein compacts. *Topology Appl.*, 23(3):291–297, 1986.

[4] Walter Rudin. Homogeneity problems in the theory of Čech compactifications. *Duke Math. J.*, 23:409–419, 1956.

[5] Stephen Willard. *General topology.* Dover Publications, Inc., Mineola, NY, 2004. Reprint of the 1970 original [Addison-Wesley, Reading, MA; MR0264581].

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