Definition 1. Let a V-map be a u.s.c. idempotent surjection.

Definition 2. For any LOS $\langle L, \leq \rangle$, let $\check{L}$ be the collection of leftward subsets of L (subsets for which $b \in L, a \leq b \Rightarrow a \in L$) linearly ordered by $\subseteq$, and let $\hat{L}$ be the collection of left-open subsets of L (leftward subsets which are open) linearly ordered by $\subseteq$.

Proposition 3. *$\check{L}$, $\hat{L}$ are compact.*

Proof. Each subset S has a supremum $\cup S$ and infimum $\cap S$ (or $\text{int}(\cap S)$). $\square$

Note that $\check{L}$ this is not a "compactification" as L does not necessarily embed as a dense subspace of $\check{L}$: if $L = I$, we might attempt to embed $t \mapsto [0, t)$, but then note that the subspace topology induces the reverse Sorgenfrey interval as $([0, s), [0, t]) = ([0, s), [0, t)]$ is open. However $\hat{L}$ is basically the typical way of compactifying a linearly ordered space L (actually, left-closed is more typical, but this works similarly and fits our applications later), provided L lacks a maximum element (otherwise the whole space is an [easily removable] isolated point in $\hat{L}$).

Definition 4. For any compact LOTS K with minimum 0 and maximum 1, let γ be the V-map on K where $\gamma(0) = K$ and $\gamma(t) = \{1\}$ for $t > 0$.

Definition 5. For any LOTS M with minimum element 0, let ν be the V-map on M where $\nu(0) = K$ and $\nu(t) = \{t\}$ for $t > 0$.

Note for $K = M = 2$ that $\gamma = \nu$.

Theorem 6. $X = \varprojlim\{2, \nu, L\} \cong \check{L}$

Proof. We start by placing an order on X. Let $\vec{x} < \vec{y}$ if there exists $a \in L$ with $\vec{x}(a) = 0, \vec{y}(a) = 1$. We claim this is a total order inducing the topology on X.

We first observe that if $\vec{x}(b) = 1$, then for all $a \leq b$, $\vec{x}(a) \in \nu(1) = \{1\}$. If $\vec{x} \neq \vec{y}$, then assume without loss of generality that $\vec{x}(a) = 0, \vec{y}(a) = 1$, so $\vec{x} < \vec{y}$. Also, whenever $\vec{x}(b) = 1$, we have that $b < a$, so $\vec{y}(b) = 1$, preventing $\vec{y} < \vec{x}$. Finally if $\vec{x} < \vec{y}$ and $\vec{y} < \vec{z}$, take a, b with $\vec{x}(a) = 0$, $\vec{y}(a) = 1$,$\vec{y}(b) = 0$,$\vec{z}(b) = 1$. It follows that $a < b$ so $\vec{z}(a) = 1$ and $\vec{x} < \vec{z}$.

Consider the basic open set $B(\vec{x}, F)$ for a finite set $F \in [L]^{<\omega}$ about the sequence $\vec{x} \in X$ which contains all sequences $\vec{y}$ agreeing with $\vec{x}$ on F. If $\vec{x}(a) = 1$ for all $a \in F$, then let $\vec{w} \in X$ be 0 on the maximum of F, and 1 for anything less. It follows that $B(\vec{x}, F) = (\vec{w}, \rightarrow)$. If $\vec{x}(a) = 0$ for all $a \in F$, then let $\vec{y} \in X$ be 1 on the minimum of F, and 0 for anything greater. It follows that $B(\vec{x}, F) = (\leftarrow, \vec{y})$. Finally if $\vec{x}(a) = 1$ and $\vec{x}(b) = 0$ for $a < b$ in F and nothing between a, b is in F, then let $\vec{w} \in X$ be 0 on a and 1 for anything less, and let $\vec{y} \in X$ be 1 on b and 0 for anything greater. It follows that $B(\vec{x}, F) = (\vec{w}, \vec{y})$.

Let ϕ evaluate each $\vec{x} \in X \subseteq 2^L$ as the characteristic function for a subset of L. It's easy to see that ϕ is an order isomorphism between $\langle X, \leq\rangle$ and $\langle \check{L}, \subseteq\rangle$. $\square$

We introduce an alternate definition of an arbitrarily indexed inverse limit.

Definition 7. Let $\varprojlim^\star\{X, f, L\} \subseteq \varprojlim\{X, f, L\}$ satisfy that $\vec{x}(a) = \lim_{t\to a} \vec{x}(t)$ for all $a \in L$ (for any open neighborhood $U$of $\vec{x}(a)$ there is $b < a$ where $\vec{x}(t) \in U$ for all $t \in (b, a]$).

Theorem 8. *$Y = \varprojlim^\star\{2, \nu, L\} \cong \hat{L}$.*

Proof. Consider Y as a subspace of $X = \varprojlim\{2, \nu, L\}$ with the linear order described above. We claim that if ϕ is the characteristic function for a subset of L, then $\mathrm{int} \circ \phi$ is an order isomorphsim between $\langle Y, \leq\rangle$ and $\langle \hat{L}, \subseteq\rangle$.

Let A be a left-open subset of L. Then let $\vec{x}(a) = 1$ when $a \in \mathrm{cl}A$ and 0 otherwise. Then $\vec{x} \in Y$, $\phi(\vec{x}) = \mathrm{cl}A$, and $\mathrm{int} \circ \phi(\vec{x}) = \mathrm{intcl}A = A$.

Let $\vec{x}, \vec{y} \in Y$. If $\mathrm{int}\circ\phi(\vec{x}) = \mathrm{int}\circ\phi(\vec{y}) = A$, then A is a left-open set where $\vec{x}(a) = \vec{y}(a) = 1$ for $a \in A$. It follows that if $b \in \mathrm{cl}A$, then as $\vec{x}, \vec{y} \in Y$, $\vec{x}(b) = \vec{y}(b) = 1$ as well. And if $c \notin \mathrm{cl}A$, then as $c \notin \phi(x)$, $\vec{x}(c) = \vec{y}(c) = 0$, so $\vec{x} = \vec{y}$.

Finally let $\vec{x} < \vec{y}$, so there exists $a \in L$ with $\vec{x}(a) = 0, \vec{y}(a) = 1$. Then $\mathrm{int} \circ \phi(\vec{x}) \subseteq \phi(\vec{x}) \subseteq (\leftarrow, a) \subseteq \mathrm{int} \circ \phi(\vec{y})$. Thus $\mathrm{int} \circ \phi$ preserves order. $\square$

Corollary 9. *$\varprojlim\{2, \nu, \kappa\} \cong \varprojlim^\star\{2, \nu, \kappa\} \cong \kappa + 1$*

Proof. Since $\hat{\kappa} = \kappa + 1$ (actually equals, not just homeomorphic!), we get $\varprojlim^\star\{2, \nu, \kappa\} \cong \hat{\kappa} = \kappa + 1$ for free. Then observe that all leftward sets in κ are open: $A = [0, \min(\kappa + 1 \setminus A))$. Thus we have $\varprojlim\{2, \nu, \kappa\} \cong \check{\kappa} = \hat{\kappa} = \kappa + 1$. $\square$

References