Remark 1. Scheeper's $S(\kappa)$ requiring injections is stronger than my $S'(\kappa)$ requiring finite-to-one maps. Dow suggests that $S'(\omega_\omega)$ holds in ZFC by the following.

Definition 2. A topological space is said to be ω-bounded if each countable subset of the space has compact closure.

Theorem 3. *For each $k < \omega$ there exists a topology on ω_k which is ω-bounded and locally countable.*

Proof. TODO: write up Alan's proof $\square$

Definition 4. A Kurepa family $\mathcal{K}$ is a family of countable sets such that for each $A \in \mathcal{K}$, $\mathcal{K} \restriction A = \{K \cap A : K \in \mathcal{K}\}$ is countable.

Corollary 5. *There exists a Kurepa family cofinal in $[\omega_k]^\omega$ for each $k < \omega$.*

Proof. This is actually a corollary of an observation of Todorcevic communicated by Dow in [TODO cite Gen Prog in Top I]: if every Kurepa family of size at most θ extends to a cofinal Kurepa family, then the same is true of θ^+. So the result follows as every Kurepa family of size ω extends to the cofinal Kurepa family $[\omega]^\omega$. (TODO am I understanding this right?)

We may alternatively obtain the result from the previous topological argument by using the family $\mathcal{K}$ of compact sets in the constructed topology on ω_k as our witness. Of course, every Lindelöf set in a locally countable space is countable. Thus $\mathcal{K}$ is cofinal since for every countable set A, $\overline{A}$ is compact and countable. It is Kurepa since for every compact (countable) set A, (TODO). $\square$

Theorem 6. *$S'(\omega_k)$ for each $k < \omega$.*

Proof. Let $k < \omega$, and $\mathcal{K}$ be a cofinal Kurepa family on ω_k. (TODO finish) $\square$