

Definition 1. A **uniform space** $\langle X, \mathcal{D} \rangle$ is a set X paired with a filter \mathcal{D} (called its **uniformity**) of relations (called **entourages**) on X such that for each entourage $D \in \mathcal{D}$:

- D is reflexive, i.e., the diagonal $\Delta \subseteq D$.
- Its inverse $D^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in D\} \in \mathcal{D}$.
- There exists $\frac{1}{2}D \in \mathcal{D}$ such that

$$2(\frac{1}{2}D) = \frac{1}{2}D \circ \frac{1}{2}D = \{\langle x, z \rangle : \exists y(\langle x, y \rangle, \langle y, z \rangle \in \frac{1}{2}D)\} \subseteq D$$

Note that since \mathcal{D} is a filter, for each $D \in \mathcal{D}$, the symmetric relation $D \cap D^{-1} \in \mathcal{D}$.

Proposition 2. For each $D \in \mathcal{D}$ and $n < \omega$ there exists $\frac{1}{2^{n+1}}D \in \mathcal{D}$ such that

$$2(\frac{1}{2^{n+1}}D) = \frac{1}{2^{n+1}}D \circ \frac{1}{2^{n+1}}D \subseteq \frac{1}{2^n}D$$

and if $2E \subseteq \frac{1}{2^n}D$, then $E \subseteq \frac{1}{2^{n+1}}D$.

Definition 3. For an entourage $D \in \mathcal{D}$, let $D[x] = \{y : \langle x, y \rangle \in D\}$ be the D -**neighborhood** of x . The uniform topology for a uniform space $\langle X, \mathcal{D} \rangle$ is generated by the base $\{D[x] : x \in X, D \in \mathcal{D}\}$.

Theorem 4. A space X is uniformizable (its topology is the uniform topology for some uniformity) if and only if X is completely regular ($T_{3\frac{1}{2}}$).

Proposition 5. If X is a uniform space, then for all $x \in X$ and symmetric entourages D :

$$x \in \frac{1}{2}D[y] \text{ and } y \in \frac{1}{2}D[z] \Rightarrow x \in D[z]$$

and

$$\frac{1}{2}D[x] \subseteq \overline{\frac{1}{2}D[x]} \subseteq D[x]$$

Proof. The first is by definition of $\frac{1}{2}D$.

If $z \in \overline{\frac{1}{2}D[x]}$, it follows that there is $y \in \frac{1}{2}D[x] \cap \frac{1}{2}D[z]$ since $\frac{1}{2}D[z]$ is an open neighborhood of z . Thus $(x, z) \in D \Rightarrow z \in D[x] \Rightarrow \overline{\frac{1}{2}D[x]} \subseteq D[x]$. \square

Definition 6. For a uniform space X , Bell's proximity game proceeds as follows.

In round 0, \mathcal{D} chooses an entourage D_0 , followed by \mathcal{P} choosing a point $p_0 \in X$.

In round $n + 1$, \mathcal{D} chooses an entourage $D_{n+1} \subseteq D_n$, followed by \mathcal{P} choosing a point $p_{n+1} \in 4D_n[p_n]$.

Player \mathcal{D} wins if either $\bigcap_{n < \omega} 4D_n[p_n] = \emptyset$ or $\langle p_0, p_1, \dots \rangle$ converges.

Definition 7. For a uniform space X , the simplified proximal game $Prox_{D,P}(X)$ can be defined as follows:

In round 0, \mathcal{D} chooses a symmetric entourage D_0 , followed by \mathcal{P} choosing a point $p_0 \in X$.

In round $n+1$, \mathcal{D} chooses a symmetric entourage D_{n+1} , followed by \mathcal{P} choosing a point $p_{n+1} \in \left(\bigcap_{m \leq n} D_m\right)[p_n]$.

Player \mathcal{D} wins if either $\bigcap_{n < \omega} \left(\bigcap_{m \leq n} D_m\right)[p_n] = \emptyset$ or $\langle p_0, p_1, \dots \rangle$ converges.

Theorem 8. \mathcal{D} has a winning perfect-information strategy in Bell's game if and only if $\mathcal{D} \uparrow Prox_{D,P}(X)$.

Proof. Let σ be a winning perfect information strategy for \mathcal{D} in Bell's game. We define a perfect information strategy τ in the simplified game to yield symmetric entourages $\tau(p \upharpoonright n) = \sigma(p \upharpoonright n) \cap (\sigma(p \upharpoonright n))^{-1}$ for all partial attacks $p \upharpoonright n$. Note that $\tau(p \upharpoonright n) = \bigcap_{m \leq n} \tau(p \upharpoonright m)$.

If p attacks τ in the simplified game, $p(n+1) \in \left(\bigcap_{m \leq n} \tau(p \upharpoonright m)\right)[p(n)] = \tau(p \upharpoonright n)[p(n)] \subseteq \sigma(p \upharpoonright n)[p(n)] \subseteq 4\sigma(p \upharpoonright n)[p(n)]$, so p attacks σ in Bell's game. Thus either p converges, or

$$\emptyset = \bigcap_{n < \omega} 4\sigma(p \upharpoonright n)[p(n)] \supseteq \bigcap_{n < \omega} \tau(p \upharpoonright n)[p(n)] = \bigcap_{n < \omega} \left(\bigcap_{m \leq n} \tau(p \upharpoonright m) \right)[p(n)]$$

For the other direction, let σ be a winning perfect information strategy for \mathcal{D} in the simplified game such that $\sigma(p \upharpoonright n) = \bigcap_{m \leq n} \sigma(p \upharpoonright m)$. Define the perfect information strategy τ in Bell's Game such that $4\tau(p \upharpoonright n) \subseteq \sigma(p \upharpoonright n)$ and $\tau(p \upharpoonright n) = \bigcap_{m \leq n} \tau(p \upharpoonright m)$ for all partial attacks $p \upharpoonright n$.

If p attacks τ in Bell's game, $p(n) \in 4\tau(p \upharpoonright n) \subseteq \sigma(p \upharpoonright n) = \bigcap_{m \leq n} \sigma(p \upharpoonright m)$, so p attacks σ in the simplified game. Thus either p converges, or

$$\emptyset = \bigcap_{n < \omega} \left(\bigcap_{m \leq n} \sigma(p \upharpoonright m) \right)[p(n)] = \bigcap_{n < \omega} \sigma(p \upharpoonright n)[p(n)] \supseteq \bigcap_{n < \omega} 4\tau(p \upharpoonright n)[p(n)] \supseteq \bigcap_{n < \omega} \tau(p \upharpoonright n)[p(n)]$$

□

Proposition 9. \mathcal{P} has a winning perfect-information strategy in Bell's game if and only if $\mathcal{P} \uparrow Prox_{D,P}(X)$.

Proof. Similar to the previous. □

Definition 10. A uniform space is **proximal** if $\mathcal{D} \uparrow \text{Prox}_{D,P}(X)$.

Definition 11. For a space X and a point $x \in X$, the **W -convergence-game** $\text{Con}_{O,P}(X, x)$ proceeds as follows.

In round 0, \mathcal{O} chooses a neighborhood U_n of x , followed by \mathcal{P} choosing a point $p_n \in \bigcap_{m \leq n} U_m$.

Player \mathcal{O} wins if $\langle p_0, p_1, \dots \rangle$ converges.

Definition 12. A space is **W** if $\mathcal{O} \uparrow \text{Con}_{O,P}(X, x)$ for all $x \in X$.

Definition 13. For each finite tuple (m_0, \dots, m_{n-1}) , we define the **k -tactical fog-of-war**

$$T_k(\langle m_0, \dots, m_{n-1} \rangle) = \langle m_{n-k}, \dots, m_{n-1} \rangle$$

and the **k -Marköv fog-of-war**

$$M_k(\langle m_0, \dots, m_{n-1} \rangle) = \langle \langle m_{n-k}, \dots, m_{n-1} \rangle, n \rangle$$

So $P \uparrow G$ if and only if there exists a winning strategy for P of the form $\sigma \circ T_k$, and $P \uparrow_{k\text{-tact}} G$ if and only if there exists a winning strategy of the form $\sigma \circ M_k$.

Theorem 14. For all $x \in X$:

- $\mathcal{D} \uparrow \text{Prox}_{D,P}(X) \Rightarrow \mathcal{O} \uparrow \text{Con}_{O,P}(X, x)$
- $\mathcal{D} \uparrow_{2k\text{-tact}} \text{Prox}_{D,P}(X) \Rightarrow \mathcal{O} \uparrow_{k\text{-tact}} \text{Con}_{O,P}(X, x)$
- $\mathcal{D} \uparrow_{2k\text{-mark}} \text{Prox}_{D,P}(X) \Rightarrow \mathcal{O} \uparrow_{k\text{-mark}} \text{Con}_{O,P}(X, x)$

Proof. Let σ witness $\mathcal{D} \uparrow_{2k\text{-tact}} \text{Prox}_{D,P}(X)$ (resp. $\mathcal{D} \uparrow_{2k\text{-mark}} \text{Prox}_{D,P}(X)$, $\mathcal{D} \uparrow \text{Prox}_{D,P}(X)$).

We define the k -tactical (resp. k -Marköv, perfect info) strategy τ such that

$$\tau \circ L_k(p) = \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(|p| - 1) \rangle)[x] \cap \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(|p| - 1), x \rangle)[x]$$

where L_{2k} is the $2k$ -tactical fog-of-war (resp. $2k$ -Marköv fog-of-war, identity) and L_k is the k -tactical fog-of-war (resp. k -Marköv fog-of-war, identity).

Let p attack τ . Consider the attack q against the winning strategy σ such that $q(2n) = x$ and $q(2n + 1) = p(n)$, and let $D_n = \sigma \circ L_{2k}(q)$ and $E_n = \bigcap_{m \leq n} D_m$.

Certainly, $x \in E_{2n}[x] = E_{2n}[q(2n)]$ for any $n < \omega$. Note also for any $n < \omega$ that

$$p(n) \in \bigcap_{m \leq n} \tau \circ L_k(p \upharpoonright m)$$

$$\begin{aligned}
 &= \bigcap_{m \leq n} (\sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(m-1) \rangle)[x] \cap \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(m-1), x \rangle)[x]) \\
 &= \bigcap_{m \leq n} (D_{2m}[x] \cap D_{2m+1}[x]) = \bigcap_{m \leq 2n+1} D_m[x] = E_{2n+1}[x]
 \end{aligned}$$

so by the symmetry of E_{2n+1} , $x \in E_{2n+1}[p(n)] = E_{2n+1}[q(2n+1)]$. Thus $x \in \bigcap_{n < \omega} E_n[q(n)] \neq \emptyset$, and since σ is a winning strategy, the attack q converges. Since $q(2n) = x$, q must converge to x . Thus its subsequence p converges to x , and τ is a winning strategy in $Con_{O,P}(X, x)$. \square

Corollary 15. *For all $x \in X$:*

- $\mathcal{D} \uparrow_{k\text{-tact}} Prox_{D,P}(X) \Rightarrow \mathcal{O} \uparrow_{k\text{-tact}} Con_{O,P}(X, x)$
- $\mathcal{D} \uparrow_{k\text{-mark}} Prox_{D,P}(X) \Rightarrow \mathcal{O} \uparrow_{k\text{-mark}} Con_{O,P}(X, x)$

Corollary 16. *All proximal spaces are W -spaces.*

Theorem 17. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete. Then*

- $\mathcal{O} \uparrow Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathcal{D} \uparrow Prox_{D,P}(X \cup \{\infty\})$
- $\mathcal{O} \uparrow_{k\text{-tact}} Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathcal{D} \uparrow_{k\text{-tact}} Prox_{D,P}(X \cup \{\infty\})$
- $\mathcal{O} \uparrow_{k\text{-mark}} Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathcal{D} \uparrow_{k\text{-mark}} Prox_{D,P}(X \cup \{\infty\})$

Proof. Note that the topology on $X \cup \{\infty\}$ is induced by the uniformity with equivalence relation entourages $D(U) = \Delta \cup U^2$ for each open neighborhood U of ∞ .

Let σ witness $\mathcal{D} \uparrow_{k\text{-tact}} Con_{O,P}(X \cap \{\infty\}, \infty)$ (resp. $\mathcal{D} \uparrow_{k\text{-mark}} Con_{O,P}(X \cap \{\infty\}, \infty)$, $\mathcal{D} \uparrow Con_{O,P}(X \cap \{\infty\}, \infty)$). We define the k -tactical (resp. k -Marköv, perfect info) strategy τ such that

$$\tau \circ L(p) = D(\sigma \circ L(p))$$

where L is the k -tactical fog-of-war (resp. k -Marköv fog-of-war, identity).

Let $p \in (X \cup \{\infty\})^\omega$ attack τ such that $\bigcap_{n < \omega} \tau(p \upharpoonright n)[p(n)] \neq \emptyset$.

If $\infty \in \bigcap_{n < \omega} \tau(p \upharpoonright n)[p(n)]$, it follows that p is an attack on σ . Since σ is a winning strategy, it follows that q and its subsequence p must converge to ∞ .

Otherwise, $\infty \notin \tau(p \upharpoonright N)[p(N)]$ for some $N < \omega$, and then $\tau(p \upharpoonright N)[p(N)] = \{p(N)\}$ implies $p \rightarrow p(N)$.

Thus $\tau \circ L$ is a winning strategy. \square

Corollary 18. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete. Then*

- $\mathcal{O} \uparrow_{k\text{-tact}} \text{Con}_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathcal{D} \uparrow_{k\text{-tact}} \text{Prox}_{D,P}(X \cup \{\infty\})$
- $\mathcal{O} \uparrow_{k\text{-tact}} \text{Con}_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathcal{D} \uparrow_{k\text{-tact}} \text{Prox}_{D,P}(X \cup \{\infty\})$
- $\mathcal{O} \uparrow_{k\text{-mark}} \text{Con}_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathcal{D} \uparrow_{k\text{-mark}} \text{Prox}_{D,P}(X \cup \{\infty\})$

Proposition 19. *For any $x \in X$ and $k \geq 1$,*

- $\mathcal{O} \uparrow_{k\text{-tact}} \text{Con}_{O,P}(X, x) \Leftrightarrow \mathcal{O} \uparrow_{\text{tact}} \text{Con}_{O,P}(X, x)$
- $\mathcal{O} \uparrow_{k\text{-mark}} \text{Con}_{O,P}(X, x) \Leftrightarrow \mathcal{O} \uparrow_{\text{mark}} \text{Con}_{O,P}(X, x)$

Proof. If σ witnesses $\mathcal{O} \uparrow_{k\text{-tact}} \text{Con}_{O,P}(X, x)$, let $\tau(\emptyset) = \sigma(\emptyset)$ and

$$\tau(\langle q \rangle) = \bigcap_{i < k} \sigma(\langle \underbrace{x, \dots, x}_{k-i-1}, \underbrace{x, \dots, x}_i \rangle)$$

This is easily verified to be a winning strategy. The proof for $\mathcal{O} \uparrow_{k\text{-mark}} \text{Con}_{O,P}(X, x)$ is analogous. \square

Corollary 20. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete, and $k \geq 1$. Then*

- $\mathcal{D} \uparrow_{k\text{-tact}} \text{Prox}_{D,P}(X \cup \{\infty\}) \Leftrightarrow \mathcal{O} \uparrow_{\text{tact}} \text{Prox}_{D,P}(X \cup \{\infty\})$
- $\mathcal{D} \uparrow_{k\text{-mark}} \text{Prox}_{D,P}(X \cup \{\infty\}) \Leftrightarrow \mathcal{O} \uparrow_{\text{mark}} \text{Prox}_{D,P}(X \cup \{\infty\})$

Proposition 21. *For any uniform space X ,*

- $\mathcal{O} \uparrow_{k\text{-tact}} \text{Prox}_{D,P}(X) \Leftrightarrow \mathcal{O} \uparrow_{2\text{-tact}} \text{Prox}_{D,P}(X)$
- $\mathcal{O} \uparrow_{k\text{-mark}} \text{Prox}_{D,P}(X) \Leftrightarrow \mathcal{O} \uparrow_{2\text{-mark}} \text{Prox}_{D,P}(X)$

Proof. If σ witnesses $\mathcal{O} \uparrow_{k\text{-tact}} \text{Con}_{O,P}(X, x)$, let $\tau(\emptyset) = \sigma(\emptyset)$ and

$$\tau(\langle q \rangle) = \bigcap_{i < k} \sigma(\langle \underbrace{q, \dots, q}_i \rangle)$$

$$\tau(\langle q, q' \rangle) = \bigcap_{i < k} \sigma(\underbrace{\langle q, \dots, q \rangle}_{k-i}, \underbrace{\langle q', \dots, q' \rangle}_i)$$

This is easily verified to be a winning strategy. The proof for $\mathcal{O} \uparrow_{k\text{-mark}} \text{Con}_{O,P}(X, x)$ is analogous. □

Definition 22. The absolute proximal game $aProx_{D,P}(X)$ is analogous to $Prox_{D,P}(X)$, except \mathcal{D} may only win if p converges.

Definition 23. A **uniformly locally compact** space is a uniformizable space with a **uniformly compact entourage** M where $\overline{M[x]}$ is compact for all x .

Theorem 24. For any uniformly locally compact space X , $\mathcal{D} \uparrow Prox_{D,P}(X) \Leftrightarrow \mathcal{D} \uparrow aProx_{D,P}(X)$

Proof. Let M be a uniformly locally compact entourage. Let σ witness $\mathcal{D} \uparrow Prox_{D,P}(X)$ such that $\sigma(a) \subseteq M$ always (so $\overline{\sigma(a)[x]} \subseteq \overline{M[x]}$ is compact), and $a \supseteq b$ implies $\sigma(a) \subseteq \frac{1}{4}\sigma(b)$.

Let $\tau(p \upharpoonright n) = \frac{1}{2}\sigma(p \upharpoonright n)$. If p attacks τ in $aProx_{D,P}(X)$, then

$$p(n+1) \in \tau(p \upharpoonright n)[p(n)] = \frac{1}{2}\sigma(p \upharpoonright n)[p(n)]$$

and for

$$x \in \overline{\sigma(p \upharpoonright (n+1))[p(n+1)]} \subseteq \overline{\frac{1}{4}\sigma(p \upharpoonright n)[p(n+1)]} \subseteq \frac{1}{2}\sigma(p \upharpoonright n)[p(n+1)]$$

we can conclude $x \in \sigma(p \upharpoonright n)[p(n)]$. Thus

$$\sigma(p \upharpoonright (n+1))[p(n+1)] \subseteq \overline{\sigma(p \upharpoonright (n+1))[p(n+1)]} \subseteq \sigma(p \upharpoonright n)[p(n)]$$

Finally, note that p attacks the winning strategy σ in $Prox_{D,P}(X)$, but since the intersection of a chain of nonempty compact sets is nonempty:

$$\bigcap_{n < \omega} \sigma(p \upharpoonright n)[p(n)] = \bigcap_{n < \omega} \overline{\sigma(p \upharpoonright n)[p(n)]} \neq \emptyset$$

We conclude that p converges. □

Corollary 25. A uniformly locally compact space X is proximal if and only if $\mathcal{D} \uparrow aProx_{D,P}(X)$.

Theorem 26. For any uniformly locally compact proximal space X , $\mathcal{O} \uparrow Clus_{O,P}(X, H)$ for all compact $H \subseteq X$.

Proof. Let σ witness $\mathcal{D} \uparrow aProx_{D,P}(X)$ such that $p \supseteq q$ implies $\sigma(p) \subseteq \frac{1}{4}\sigma(q)$.

Let $o(t)$ be the subsequence of t consisting of its odd-indexed terms.

We define $T(\emptyset)$, etc. as follows:

- Let $\emptyset \in T(\emptyset)$.
- Choose $m_\emptyset < \omega$, $h_{\emptyset,i} \in H$ for $i < m_\emptyset$, and $h_{\emptyset,i,j} \in H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$ for $i, j < m_\emptyset$ such that

$$\{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}] : i < m_\emptyset\}$$

is a cover for H and such that for each $i < m_\emptyset$

$$\{\frac{1}{4}\sigma(\langle h_{\emptyset,i} \rangle)[h_{\emptyset,i,j}] : j < m_\emptyset\}$$

is a cover for $H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$.

- Let $\langle i \rangle \in T(\emptyset)$, $\langle i, h_{\emptyset,i} \rangle \in T(\emptyset)$, and $\langle i, h_{\emptyset,i}, j \rangle \in T(\emptyset)$ for $i, j < m_\emptyset$.

Suppose $T(a)$, etc. are defined. We then define $T(a \smallfrown \langle x \rangle)$, etc. for

$$x \in \bigcup_{s \smallfrown \langle i, h_{s,i}, j \rangle \in \max(T(a))} \frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]$$

as follows:

- Let $T(a) \subseteq T(a \smallfrown \langle x \rangle)$.
- Choose $t = s \smallfrown \langle i, h_{s,i}, j, x \rangle$ such that $s \smallfrown \langle i, h_{s,i}, j \rangle \in \max(T(a))$ and $x \in \frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]$.
- Note that, assuming $o(s) \smallfrown \langle h_{s,i} \rangle$ is a legal partial attack against σ , then

$$x \in \frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}] \subseteq \frac{1}{4}\sigma(o(s))[h_{s,i,j}]$$

and

$$h_{s,i,j} \in \overline{\frac{1}{4}\sigma(o(s))[h_{s,i}]} \subseteq \frac{1}{2}\sigma(o(s))[h_{s,i}]$$

implies

$$x \in \sigma(o(s))[h_{s,i}]$$

and thus $o(s) \smallfrown \langle h_{s,i}, x \rangle = o(t)$ is a legal partial attack against σ .

- Choose $m_t < \omega$, $h_{t,k} \in H \cap \overline{\frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]}$ for $k < m_t$, and $h_{t,k,l} \in H \cap \overline{\frac{1}{4}\sigma(t)[h_{t,k}]}$ for $k, l < m_t$ such that

$$\{\frac{1}{4}\sigma(o(t))[h_{t,k}] : k < m_t\}$$

is a cover for $H \cap \overline{\frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]}$ and such that for each $k < m_t$

$$\{\frac{1}{4}\sigma(o(t) \smallfrown \langle h_{t,k} \rangle)[h_{t,i,j}] : l < m_t\}$$

is a cover for $H \cap \overline{\frac{1}{4}\sigma(o(t))[h_{t,k}]}$.

- Note that, assuming $o(t)$ is a legal partial attack against σ , then

$$h_{t,k} \in \overline{\frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]} \subseteq \frac{1}{2}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]$$

and

$$x \in \frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]$$

implies

$$h_{t,k} \in \sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[x]$$

and thus $o(t) \smallfrown \langle h_{t,k} \rangle$ is a legal partial attack against σ .

- Let $t \in T(a \smallfrown \langle x \rangle)$, $t \smallfrown \langle k \rangle \in T(a \smallfrown \langle x \rangle)$, $t \smallfrown \langle k, h_{t,k} \rangle \in T(a \smallfrown \langle x \rangle)$, and $t \smallfrown \langle k, h_{t,k}, l \rangle \in T(a \smallfrown \langle x \rangle)$ for $k, l < m_t$.
- Note that assuming

$$\{\frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}] : s \smallfrown \langle i, h_{s,i}, j \rangle \in \max(T(a))\}$$

covers H , then since

$$\{\frac{1}{4}\sigma(o(t) \smallfrown \langle h_{t,k} \rangle)[h_{t,k,l}] : s \smallfrown \langle i, h_{s,i}, j, x, k, h_{t,k}, l \rangle \in \max(T(a \smallfrown \langle x \rangle)) \setminus \max(T(a))\}$$

covers $H \cap \frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]$, we have that

$$\{\frac{1}{4}\sigma(o(t) \smallfrown \langle h_{t,k} \rangle)[h_{t,k,l}] : t \smallfrown \langle k, h_{t,k}, l \rangle \in \max(T(a \smallfrown \langle x \rangle))\}$$

covers H .

With this we may define the perfect information strategy τ for \mathcal{O} in $Con_{O,P}(X, H)$ such that:

$$\tau(p \upharpoonright n) = \bigcup_{s \smallfrown \langle i, h_{s,i}, j \rangle \in \max(T(p \upharpoonright n))} \frac{1}{4}\sigma(o(s) \smallfrown \langle h_{s,i} \rangle)[h_{s,i,j}]$$

If p attacks τ , then it follows that $T(p \upharpoonright n)$ is defined for all $n < \omega$, so let $T(p) = \bigcup_{n < \omega} T(p \upharpoonright n)$. We note $T(p)$ is an infinite tree with finite levels:

- \emptyset has exactly m_\emptyset successors $\langle i \rangle$.
- $s \smallfrown \langle i \rangle$ has exactly one successor $t \smallfrown \langle i, h_{s,i} \rangle$
- $s \smallfrown \langle i, h_{s,i} \rangle$ has exactly m_s successors $t \smallfrown \langle i, h_{s,i}, j \rangle$
- $s \smallfrown \langle i, h_{s,i}, j \rangle$ has either no successors or exactly one successor $t \smallfrown \langle i, h_{s,i}, j, x \rangle$

- $t = s^\frown \langle i, h_{s,i}, j, x \rangle$ has exactly m_t successors $t^\frown \langle k \rangle$

Let $q' = \langle i_0, h_0, j_0, x_0, i_1, h_1, j_1, x_1, \dots \rangle$ correspond to this infinite branch in $T(p)$, and let $q = o(q') = \langle h_0, x_0, h_1, x_1, \dots \rangle$. Note that by the construction of $T(p)$, q is an attack on the winning strategy σ in $aProx_{D,P}(X)$, so it must converge. Since every other term of q is in H , it must converge to H . Then since q is a subsequence of p , p must cluster at H . \square

Corollary 27. *For any uniformly locally compact proximal space, $\mathcal{O} \uparrow Con_{O,P}(X, H)$ for all compact $H \subseteq X$.*

Proof. $\mathcal{O} \uparrow Con_{O,P}(X, H)$ if and only if $\mathcal{O} \uparrow Clus_{O,P}(X, H)$. \square

Corollary 28. *A compact uniform space X is Corson compact if and only if it is proximal.*

Proof. A characterization of Corson compact is having a W -set diagonal. If X is proximal compact, then X^2 is proximal compact, and its compact diagonal is a W -set. \square

Theorem 29. $\mathcal{O} \uparrow_{pre} Con_{O,P}(X, H)$ if and only if there exists a countable base around H .

Proof. Let $\{U_n : n < \omega\}$ be a countable base around H . We define the predetermined strategy $\sigma(n) = \bigcap_{m \leq n} U_m$. Let p attack $\sigma(n)$ - then if U is any neighborhood of H , we may choose $H \subseteq U_m \subseteq U$, and note that $\sigma(n) \subseteq U_m$ for $n \geq m$, and thus $p(n) \in U_m \subseteq U$ for all $n \geq m$. Thus σ is a winning strategy.

For the other direction, suppose there does not exist a countable base around H , and let $\sigma(n)$ be an arbitrary predetermined strategy. Since $\{\bigcap_{m \leq n} \sigma(m) : n < \omega\}$ is not a countable base around H , we may choose an open set U around H such that $\bigcap_{m \leq n} \sigma(m) \not\subseteq U$ for all $n < \omega$. We may easily verify that if $p(n) \in \bigcap_{m \leq n} \sigma(m) \setminus U$ for all $n < \omega$, then p is a successful counterattack to σ . \square

Corollary 30. X is first countable if and only if $\mathcal{O} \uparrow_{pre} Con_{O,P}(X, x)$ for all $x \in X$

Corollary 31. $\mathcal{D} \uparrow_{pre} Prox_{D,P}(X)$ implies X is first countable.

Definition 32. Scattered Eberlein compact spaces are known as **strong Eberlein compact** spaces.

Theorem 33 (folklore). *Scattered compact first-countable spaces are metrizable.*

Corollary 34. If X is scattered compact and $\mathcal{O} \uparrow_{pre} Con_{O,P}(X, x)$ for all $x \in X$ (or $\mathcal{D} \uparrow_{pre} Prox_{D,P}(X)$), then X is metrizable.

Example 35. $\mathcal{D} \nuparrow_{pre} Prox_{D,P}(\omega_1^*)$

Proof. There does not exist a countable base around ∞ , so $\mathcal{O} \nuparrow_{pre} Con_{O,P}(X, \omega_1)$. \square

Example 36. $\mathcal{O} \uparrow_{tact} Con_{O,P}(\kappa^*, \infty)$ and $\mathcal{D} \uparrow_{tact} Prox_{D,P}(\kappa^*)$ for all cardinals κ

Proof. For $Con_{O,P}(\kappa^*, \infty)$, let $\sigma() = \sigma(\infty) = \kappa^*$ and $\sigma(x) = \kappa^* \setminus \{x\}$ otherwise. \square

Theorem 37. If H is a closed subset of X , then $\mathcal{D} \uparrow_{\text{limit}} \text{Prox}_{D,P}(X) \Rightarrow \mathcal{D} \uparrow_{\text{limit}} \text{Prox}_{D,P}(H)$ where \uparrow_{limit} is any of \uparrow , $\uparrow_{k\text{-tact}}$, or $\uparrow_{k\text{-mark}}$.

Proof. Let $\sigma \circ L$ witness $\mathcal{D} \uparrow_{\text{limit}} \text{Prox}_{D,P}(X)$. We define $\tau \circ L$ for \mathcal{D} in $\text{Prox}_{D,P}(H)$ as follows:

$$\tau \circ L(p \upharpoonright n) = \sigma \circ L(p \upharpoonright n) \cap H^2$$

Let p attack $\tau \circ L$. p also attacks the winning strategy $\sigma \circ L$, so either

$$\bigcap_{n < \omega} \left(\bigcap_{m \leq n} \tau \circ L(p \upharpoonright n) \right) [p(n)] \subseteq \bigcap_{n < \omega} \left(\bigcap_{m \leq n} \sigma \circ L(p \upharpoonright n) \right) [p(n)] = \emptyset$$

or p converges in X , and thus converges in H . \square

Theorem 38. If $\mathcal{D} \uparrow_{\text{limit}} \text{Prox}_{D,P}(X_i)$ for $i < \omega$, then $\mathcal{D} \uparrow_{\text{limit}} \text{Prox}_{D,P}(\prod_{i < \omega} X_i)$, where \uparrow_{limit} is either \uparrow or $\uparrow_{k\text{-mark}}$.

Proof. A subbase for $\prod_{i < \omega} X_i$ is

$$\{\pi_i^{-1}(D) : i < \omega, D \in \mathcal{D}_i\}$$

where π_i is the natural projection from $(\prod_{i < \omega} X_i)^2$ onto X_i^2 . (See Bell.)

For $p \in (\prod_{i < \omega} X_i)^\omega$, let $p_i \in X_i^\omega$ such that $p_i(n) = p(n)(i)$.

Let $\sigma_i \circ L$ witness $\mathcal{D} \uparrow_{\text{limit}} \text{Prox}_{D,P}(X_i)$ for $i < \omega$, and assume without loss of generality that $\sigma_i \circ L$ always yields X_i^2 before round i .

Then we define the strategy $\tau \circ L$ for \mathcal{D} in $\text{Prox}_{D,P}(\prod_{i < \omega} X_i)$ as follows:

$$\tau \circ L(p \upharpoonright n) = \bigcap_{i \leq n} \pi_i^{-1}(\sigma_i \circ L(p_i \upharpoonright n))$$

Let p attack $\tau \circ L$. If $\bigcap_{n < \omega} \left(\bigcap_{m \leq n} \sigma_i(p_i \upharpoonright n) \right) [p_i(n)] = \emptyset$ for any $i < \omega$, it easily follows that $\bigcap_{n < \omega} \left(\bigcap_{m \leq n} \tau(p \upharpoonright n) \right) [p(n)] = \emptyset$.

Otherwise, we assume that for each $i < \omega$, p_i converges to some $x_i \in X_i$. Thus p converges to $x = \langle x_0, x_1, \dots \rangle$. \square

Note: I expect I should be able to do some clever things assuming $S(\kappa, \omega, \omega)$ to get a similar result for sigma products of dimension κ .

Example 39. $\mathcal{D} \uparrow_{\text{mark}} \text{Prox}_{D,P}((\kappa^*)^\omega)$

Proof. $\mathcal{D} \uparrow_{\text{tact}} \text{Prox}_{D,P}(\kappa^*) + \text{previous result}$

□

Theorem 40. For any predetermined absolutely proximal space X , $\mathcal{O} \uparrow_{pre} Con_{O,P}(X, H)$ for all compact $H \subseteq X$.

Proof. Let $\sigma(n)$ be a winning predetermined strategy for \mathcal{D} in the absolutely proximal game such that $\sigma(n+1) \subseteq \frac{1}{4}\sigma(n)$. For a given tree T , let $\max(T)$ denote its maximal nodes.

First we define $T(0) \subseteq \omega^{\leq 2}$.

- Let $\emptyset \in T(0)$.
- Choose $m_\emptyset < \omega$, $h_{\langle i \rangle} \in H$ for $i < m_\emptyset$, and $h_{\langle i, j \rangle} \in H \cap \overline{\frac{1}{4}\sigma(0)[h_{\langle i \rangle}]}$ for $i, j < m_\emptyset$ such that

$$\left\{ \frac{1}{4}\sigma(0)[h_{\langle i \rangle}] : i < m_\emptyset \right\}$$

is a cover for H and such that for each $i < m_\emptyset$

$$\left\{ \frac{1}{4}\sigma(1)[h_{\langle i, j \rangle}] : j < m_\emptyset \right\}$$

is a cover for $H \cap \overline{\frac{1}{4}\sigma(0)[h_{\langle i \rangle}]}$.

- Let $\langle i \rangle$ and $\langle i, j \rangle$ be in $T(0)$ for $i, j < m_\emptyset$.

Now suppose $T(n) \subseteq \omega^{\leq 2n+2}$ is defined. We then define $T(n+1) \subseteq \omega^{\leq 2n+4}$ as follows:

- Let $T(n) \subseteq T(n+1)$.
- For each $t \in \max(T(n))$, choose $m_t < \omega$, $h_{t \smallfrown \langle i \rangle} \in H \cap \overline{\frac{1}{4}\sigma(2n+2)[h_t]}$ for $i < m_t$, and $h_{t \smallfrown \langle i, j \rangle} \in H \cap \overline{\frac{1}{4}\sigma(2n+3)[h_{t \smallfrown \langle i \rangle}]}$ for $i, j < m_t$ such that

$$\left\{ \frac{1}{4}\sigma(2n+2)[h_{t \smallfrown \langle i \rangle}] : i < m_t \right\}$$

is a cover for $H \cap \overline{\frac{1}{4}\sigma(2n+1)[h_t]}$ and such that for each $i < m_t$

$$\left\{ \frac{1}{4}\sigma(2n+3)[h_{t \smallfrown \langle i, j \rangle}] : j < m_t \right\}$$

is a cover for $H \cap \overline{\frac{1}{4}\sigma(2n+2)[h_{t \smallfrown \langle i \rangle}]}$.

- For each $t \in \max(T(n))$ and each $i, j < m_t$, put $t \smallfrown \langle i \rangle$ and $t \smallfrown \langle i, j \rangle$ in $T(n+1)$.

We now define the predetermined strategy τ for \mathcal{O} in $Clus_{O,P}(X, H)$ such that:

$$\tau(n) = \bigcup_{t \in \max(T(n))} \frac{1}{4} \sigma(2n+1)[h_t]$$

In order for τ to be a legal strategy, we must show that $\tau(n)$ contains H . For $n = 0$,

$$\tau(0) = \bigcup_{i,j < m_\emptyset} \frac{1}{4} \sigma(1)[h_{\langle i,j \rangle}]$$

Since $\{\frac{1}{4} \sigma(1)[h_{\langle i,j \rangle}] : j < m_\emptyset\}$ is a cover for $H \cap \overline{\frac{1}{4} \sigma(0)[h_{\langle i \rangle}]}$, and since $\{\frac{1}{4} \sigma(0)[h_{\langle i \rangle}] : i < m_\emptyset\}$ is a cover for H , $\tau(0)$ must contain H . A similar argument follows for $\tau(n)$.

Let p be an attack against τ such that $p(n) \in \bigcap_{m \leq n} \tau(m)$. If we can construct an attack q against σ which shares a subsequence with p , then p must cluster. To find such a q , we construct a new tree T' .

We begin by setting

$$T'(0) = \{s : s \leq \langle i, h_{\langle i \rangle}, j \rangle \text{ for } i, j < m_\emptyset\}$$

Since

$$p(0) \in \frac{1}{4} \sigma(1)[h_{\langle i,j \rangle}]$$

for some $i, j < m_\emptyset$, we may find $t \in \max(T'(0))$ such that $p(0) \in \frac{1}{4} \sigma(|o(t)|)[h_{e(t)}]$.

Assume that $T'(n)$ is defined such that there is some $t \in \max(T'(n))$ where $p(n) \in \frac{1}{4} \sigma(|o(t)|)[h_{e(t)}]$. Then let

$$T'(n+1) = T'(n) \cup \{s : s \leq t \frown \langle p(n), i, h_{e(t) \frown \langle i \rangle}, j \rangle \text{ for } i, j < m_{e(t)}\}$$

Note that

$$p(n+1) \in \bigcap_{m \leq n+1} \tau(m) = \bigcap_{m \leq n+1} \left(\bigcup_{t \in \max(T(m))} \frac{1}{4} \sigma(2m+1)[h_t] \right)$$

(Need details on why there is some t in $\max(T'(n+1))$ such that $p(n+1) \in \frac{1}{4} \sigma(|o(t)|)[h_{e(t)}]$.)

(Follow this up with choosing an infinite branch $q' \in T'$, and showing that $q = o(q')$ is an attack on σ similar to perfect info result.)

□

Example 41. Let $X = I \times 2$ be the Alexandrov double interval. Then $\mathcal{D} \uparrow_{\text{pre}} \text{Prox}_{D,P}(X)$,
but $\mathcal{D} \uparrow_{\text{mark}} \text{Prox}_{D,P}(X)$.

Proof. We assume that the uniformity on X is given by entourages

$$D(\epsilon, F) = \{\langle x, 0 \rangle, \langle y, 0 \rangle : |x - y| < \epsilon\} \cup \{\langle x, 1 \rangle, \langle y, 0 \rangle : |x - y| < \epsilon \vee x \notin F\} \\ \cup \{\langle x, 0 \rangle, \langle y, 1 \rangle : |x - y| < \epsilon \vee y \notin F\} \cup \{\langle x, 1 \rangle, \langle y, 1 \rangle : x = y\}$$

That is, points are $D(\epsilon, F)$ -close if they are the same point, or the first coordinates are within ϵ of each other while neither second coordinate is in F .

Suppose \mathcal{D} had a predetermined winning strategy $\sigma(n) = D(\epsilon_n, F_n)$. Then \mathcal{P} can choose $x \notin \bigcup_{n < \omega} F_n$, and play $\langle x, 1 \rangle$ during even rounds, and $\langle x_{2n+1}, 0 \rangle$ where $|x - x_{2n+1}| < \epsilon_{2n}$ during odd rounds, preventing convergence.

However, assume \mathcal{D} uses the Marköv strategy $\sigma(x, n) = D(2^{-n}, \{x\})$. If \mathcal{P} repeats a point of the form $\langle x, 1 \rangle$, then since $D(2^{-n}, \{x\})[\langle x, 1 \rangle] = \{\langle x, 1 \rangle\}$, \mathcal{P} must repeat $\langle x, 1 \rangle$ for the rest of the game, and \mathcal{D} wins. Otherwise, \mathcal{P} cannot repeat points played in $I \times \{1\}$, and as the first coordinates form a Cauchy sequence and converge to some z , any open set about $\langle z, 0 \rangle$ contains all but finitely many points of \mathcal{P} 's sequence, and \mathcal{D} wins. \square

Theorem 42. For any uniformly locally compact space X , $\mathcal{D} \uparrow_{\text{pre}} \text{Prox}_{D,P}(X) \Leftrightarrow \mathcal{D} \uparrow_{\text{pre}} a\text{Prox}_{D,P}(X)$

Proof. Let M be a uniformly locally compact entourage. Let σ witness $\mathcal{D} \uparrow_{\text{pre}} \text{Prox}_{D,P}(X)$ such that $\sigma(n) \subseteq M$ always (so $\overline{\sigma(a)[x]} \subseteq \overline{M[x]}$ is compact), $\sigma(n+1) \subseteq \frac{1}{4}\sigma(n)$.

Let $\tau(n) = \frac{1}{2}\sigma(n)$. If p attacks τ in $a\text{Prox}_{D,P}(X)$, then

$$p(n+1) \in \tau(n)[p(n)] = \frac{1}{2}\sigma(n)[p(n)]$$

and for

$$x \in \overline{\sigma(n+1)[p(n+1)]} \subseteq \overline{\frac{1}{4}\sigma(n)[p(n+1)]} \subseteq \frac{1}{2}\sigma(n)[p(n+1)]$$

we can conclude $x \in \sigma(n)[p(n)]$. Thus

$$\sigma(n+1)[p(n+1)] \subseteq \overline{\sigma(n+1)[p(n+1)]} \subseteq \sigma(n)[p(n)]$$

Finally, note that p attacks the winning strategy σ in $Prox_{D,P}(X)$, but since the intersection of a chain of nonempty compact sets is nonempty:

$$\bigcap_{n < \omega} \sigma(n)[p(n)] = \bigcap_{n < \omega} \overline{\sigma(n)[p(n)]} \neq \emptyset$$

We conclude that p converges. □

Proposition 43. *If $\mathcal{D} \uparrow_{pre} Prox_{D,P}(X)$, then X has a G_δ diagonal.*

Proof. If $\mathcal{D} \uparrow_{pre} Prox_{D,P}(X)$ with strategy σ , then consider $\langle x, y \rangle \in \bigcap_{n < \omega} \sigma(n)$. It follows that $\langle x, y, x, y, \dots \rangle$ attacks σ , and $\{x, y\} \subseteq \bigcap_{n < \omega} \sigma(n)[x] \cap \bigcap_{n < \omega} \sigma(n)[y] \neq \emptyset$ so it must converge, and $x = y$. Thus $\bigcap_{n < \omega} \sigma(n) = \Delta$ is G_δ . □

Example 44. The Sorgenfrey line S has a G_δ diagonal but $\mathcal{D} \uparrow Prox_{D,P}(S)$.

Corollary 45. *For X with uniformity \mathbb{D} inducing the compact Hausdorff topology τ , the following are equivalent:*

- (a) $\mathcal{D} \uparrow_{pre} Prox_{D,P}(X)$
- (b) $\mathcal{D} \uparrow_{pre} aProx_{D,P}(X)$
- (c) X has a G_δ diagonal
- (d) \mathbb{D} is metrizable
- (e) τ is metrizable

Proof. For compact Hausdorff spaces, it is well known that there is exactly one uniformity inducing the topology. Thus (d) \Leftrightarrow (e). Since X is uniformly locally compact, (a) \Leftrightarrow (b). Also, compact spaces with a G_δ diagonal are metrizable, so (c) \Rightarrow (e). Bell noted (d) \Rightarrow (a) for arbitrary uniform spaces, and the previous proposition shows (a) \Rightarrow (c). □

Theorem 46. *A uniformly locally compact space with a G_δ diagonal is metrizable.*

Proof. Based on several folklore results.

Uniformly locally compact implies the topological sum of σ -compact spaces implies paracompact. Locally compact plus G_δ diagonal implies locally metrizable. Locally metrizable plus paracompact characterizes metrizable. □

Corollary 47. *If X is uniformly locally compact, then $\mathcal{D} \uparrow_{pre} Prox_{D,P}(X)$ implies X 's topology is metrizable.*