Definition 1. X is **Menger** if for all open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ there exist finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Proposition 2. *σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelof*

Definition 3. In the two-player game $Cov_{C,F}(X)$ player C chooses open covers $\mathcal{U}_n$ of X, followed by player F choosing a finite subcollection $\mathcal{F}_n \subseteq \mathcal{U}_n$. F wins if $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Theorem 4. *X is Menger if and only if $C \not\uparrow Cov_{C,F}(X)$.*

Proof. Result due to (???)

First, suppose X wasn't Menger. Then there would exist open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ of X such that for any choice of finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$, $\bigcup_{n<\omega} \mathcal{F}_n$ isn't a cover of X. Thus $C \uparrow_{\text{pre}} Cov_{C,F}(X) \Rightarrow S \not\uparrow Cov_{C,F}(X)$.

The other direction is based upon Gruenhage's topological game presentation. Assume X is Menger, and consider a strategy for C in $Cov_{C,F}(X)$.

Since X is Lindelof, we can assume C plays only countable covers of X. Then, since F is choosing finite subsets, we may assume F chooses some initial segement of the countable cover. In turn, we can assume C plays an increasing open cover $\{U_0, U_1, \dots\}$ where $U_n \subseteq U_{n+1}$. And in that case, it's suffient to assume F simply chooses a singleton subset of each cover. And finally, since choices made by F are already covered, we can assume that every open set in a cover played by C covers the sets chosen by F previously.

As a result, we have the following figure of a tree of plays which I need to draw:

(Insert figure here.)

Note that for $a, b \in \omega^{<\omega}$ and $m \leq n$, we know:

(a) $U_{a^\frown m} \subseteq U_{a^\frown n}$
(for example, $U_{1627} \subseteq U_{1629}$ - increasing the final digit yields supersets)

(b) $U_a \subseteq U_{a^\frown b}$
(for example, $U_{1627} \subseteq U_{162789}$ - appending any sequence to the end yields supersets)

(c) $U_{a^\frown m} \subseteq U_{a^\frown n} \subseteq U_{a^\frown n^\frown b} \subseteq U_{a^\frown n^\frown b^\frown m}$
(for example: $U_{1627} \subseteq U_{1629283287}$ - injecting a subseqence with initial number larger than the original's final number, prior to the final number, yields supersets)

We may observe that if F can find an $f : \omega \to \omega$ such that $\bigcup_{n<\omega} U_{f\restriction(n+1)} = X$, she can use $\{U_{f\restriction 0}\}, \{U_{f\restriction 1}\}, \dots$ to counter C's strategy.

Let $V_k^n = \bigcap_{a \in \omega^{\leq n}} U_{a^\frown k}$. We claim that (1) V_k^n is open, (2) $\mathcal{V}^n = \{V_0^n, V_1^n, \dots\}$ is increasing, and (3) $\mathcal{V}^n$ is a cover. Proofs:

1. Since due to (c) for each $b \in \omega^{\leq n} \setminus k^{\leq n}$, there is an $a \in k^{\leq n}$ with $U_{a^\frown k} \subseteq U_{b^\frown k}$:

$$V_k^n = \bigcap_{a \in \omega^{\leq n}} U_{a^\frown k} = \bigcap_{a \in k^{\leq n}} U_{a^\frown k} \cap \bigcap_{b \in \omega^{\leq n} \setminus k^{\leq n}} U_{b^\frown k} = \bigcap_{a \in k^{\leq n}} U_{a^\frown k}$$

   making V_k^n a finite intersection of open sets.

2. We show $V_k^0 \subseteq V_{k+1}^0$:

$$V_k^0 = U_k \subseteq U_{k+1} = V_{k+1}^0$$

   and then assume $V_k^n \subseteq V_{k+1}^n$:

$$V_k^{n+1} = \bigcap_{a \in \omega^{\leq n+1}} U_{a^\frown k} = V_k^n \cap \bigcap_{a \in \omega^{n+1}} U_{a^\frown k} \subseteq V_{k+1}^n \cap \bigcap_{a \in \omega^{n+1}} U_{a^\frown (k+1)} = V_{k+1}^{n+1}$$

3. We easily see that $\mathcal{V}^0 = \{U_0, U_1, \dots\}$ is a cover, and then assume $\mathcal{V}^n$ is a cover.

   Let $x \in X$ and pick $l < \omega$ such that $x \in V_l^n$. For $a \in l^{n+1}$ choose l_a such that $x \in U_{a^\frown l_a}$, giving

$$x \in \bigcap_{a \in l^{n+1}} U_{a^\frown l_a}$$

   We will assume $k > l, l_a$ for all $a \in l^{\leq n+1}$.

   For any $a \in k^{n+1} \setminus l^{n+1}$ note that $a = b^\frown c$ where $b \in l^{\leq n}$ and c begins with a number l or greater:

$$V_l^n \subseteq U_{b^\frown l} \subseteq U_{b^\frown c} \subseteq U_{b^\frown c^\frown l_a} = U_{a^\frown l_a}$$

   Thus:

$$x \in V_l^n \cap \left(\bigcap_{a \in l^{n+1}} U_{a^\frown l_a} \right)$$

$$= V_l^n \cap \left(\bigcap_{a \in k^{n+1} \setminus l^{n+1}} U_{a^\frown l_a} \right) \cap \left(\bigcap_{a \in l^{n+1}} U_{a^\frown l_a} \right)$$

$$= V_l^n \cap \left(\bigcap_{a \in k^{n+1}} U_{a^\frown l_a} \right)$$

$$\subseteq V_k^n \cap \left(\bigcap_{a \in k^{n+1}} U_{a^\frown k} \right)$$

$$= V_k^{n+1}$$

Finally, apply Menger to $\mathcal{V}^n$, resulting in the cover $\{V^0_{f(0)}, V^1_{f(1)}, \dots\}$, noting

$$X = \bigcup_{n<\omega} V^n_{f(n)} \subseteq \bigcup_{n<\omega} U_{(f\restriction n)^\frown f(n)} = \bigcup_{n<\omega} U_{f\restriction(n+1)}$$

□

Proposition 5. *X is compact if and only if $F \uparrow_{tact} Cov_{C,F}(X)$ if and only if $F \uparrow_{k\text{-}tact} Cov_{C,F}(X)$*

Proof. Assume X is compact. For each open cover played by C, pick a finite subcover, and this yields a winning tactical strategy.

Assume F has a winning k-tactical strategy. For any open cover, have C play only it during the entire game. F's only choice must be a finite subcover. □

Proposition 6. *If X is σ-compact then $F \uparrow_{mark} Cov_{C,F}(X)$*

Proof. Let $X = \bigcup_{n<\omega} X_n$ for compact X_n. On round n, F picks the finite subcover of C's open cover of X_n. □

For Menger's game, there is no useful distinction between a k-Markov strategy for F, and a 2-Markov strategy.

Theorem 7. *For any topological space X and all $k \geq 2$, $F \uparrow_{k\text{-}mark} Cov_{C,F}(X)$ if and only if $F \uparrow_{2\text{-}mark} Cov_{C,F}(X)$.*

Proof. Assume $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{k-1}, n)$ is a winning k-Markov strategy. Define the 2-Markov strategy $\tau(\mathcal{U}, \mathcal{V}, n)$ so that it contains $\sigma(\mathcal{W}_0, \dots, \mathcal{W}_{k-1}, m)$ for the following conditions on $(\mathcal{W}_0, \dots, \mathcal{W}_{k-1}, m)$:

- Each $\mathcal{W}_i \in \{\mathcal{U}, \mathcal{V}\}$
- $m \leq (n+1)k$; in particular, for $i < k$,

$$\sigma(\mathcal{W}_0, \dots, \mathcal{W}_{k-1}, (n+1)k + i) \subseteq \tau(\mathcal{U}, \mathcal{V}, n+1)$$

Considering an arbitrary play $\mathcal{U}_0, \mathcal{U}_1, \dots$ by C versus τ, we note that σ defeats the play

$$\underbrace{\mathcal{U}_0, \mathcal{U}_0, \dots, \mathcal{U}_0}_{k}, \underbrace{\mathcal{U}_1, \mathcal{U}_1, \dots, \mathcal{U}_1}_{k} \dots$$

So we have that

$$\bigcup_{i<k, n<\omega} \sigma(\underbrace{\mathcal{U}_n, \dots, \mathcal{U}_n}_{k-i-1}, \underbrace{\mathcal{U}_{n+1}, \dots, \mathcal{U}_{n+1}}_{i+1}, (n+1)k + i)$$

is a cover for X, and as

$$\sigma(\underbrace{\mathcal{U}_n,\dots,\mathcal{U}_n}_{k-i-1},\underbrace{\mathcal{U}_{n+1},\dots,\mathcal{U}_{n+1}}_{i+1},(n+1)k+i)\subseteq\tau(\mathcal{U}_n,\mathcal{U}_{n+1},n+1)$$

τ defeats the play $\mathcal{U}_0,\mathcal{U}_1,\dots$. □

But there are spaces for which there is no Markov strategy, but there is a 2-Markov strategy.

In a question I posed to G, he answered:

Lemma 8. *For all functions $\tau:\omega_1\times\omega\to[\omega_1]^{<\omega}$, there exists a sequence $\alpha_0,\alpha_1,\dots<\omega_1$ such that $\{\tau(\alpha_n,n):n<\omega\}$ is not a cover for $\{\beta:\forall n<\omega(\beta<\alpha_n)\}$.*

Proof. Let $P_n=\{\beta:\beta<\alpha\Rightarrow\beta\in\tau(\alpha,n)\}$. Observe that each P_n is finite; else there is some α larger than every member of some countably infinite $P_n^*\subseteq P_n$ such that $P_n^*\subseteq\tau(\alpha,n)$.

Choose $\beta\notin\bigcup_{n<\omega}P_n$. Then for each $n<\omega$, pick $\alpha_n>\beta$ such that $\beta\notin\tau(\alpha_n,n)$. □

Note that the one-point Lindelöfication of discrete ω_1, $\omega_1^\dagger$, is not σ-compact. With the above lemma, we may see that:

Example 9. *$F\uparrow Cov_{C,F}(\omega_1^\dagger)$ but $F\not\uparrow_{mark} Cov_{C,F}(\omega_1^\dagger)$.*

Proof. First, we see F has a simple perfect information strategy: in response to the initial cover of $\omega_1^\dagger$, F chooses a co-countable neighborhood of ∞. On successive turns she may pick a single set from C's covers to cover the countable remainder.

Now, suppose that $\sigma(\mathcal{U},n)$ was a winning Markov strategy and aim for a contradiction. Consider the covers

$$\mathcal{U}(\alpha)=\{[\alpha,\omega_1)\cup\{\infty\}\}\cup\{\{\beta\}:\beta<\alpha\}$$

and define $\tau(\alpha,n)$ to be the union of singletons chosen by $\sigma(\mathcal{U}(\alpha),n)$.

Using the sequence $\alpha_0,\alpha_1,\dots<\omega_1$ from the previous lemma, we consider the play $\mathcal{U}(\alpha_0),\mathcal{U}(\alpha_1),\dots$.

As σ was a winning strategy, $\{\sigma(\mathcal{U}(\alpha_n),n):n<\omega\}$ must cover $\omega_1^\dagger$, and thus $\{\tau(\alpha_n,n):n<\omega\}$ must cover $\{\beta:\forall n<\omega(\beta<\alpha_n)\}$, contradiction. □

Due to Telgarski in "On Games of Topsoe" (along with σ-compact implying $F\uparrow_{\mathrm{mark}} Cov_{C,F}(X)$).

Theorem 10. *For metrizable X, X is σ-compact if and only if $F \uparrow Cov_{C,F}(X)$ if and only if $F \uparrow_{mark} Cov_{C,F}(X)$.*

We adapt this result for regular spaces.

Lemma 11. *Let $\sigma(\mathcal{U}, n)$ be a winning Markov strategy for F in $Cov_{C,F}(X)$, and $\mathfrak{C}$ collect all open covers of X. Then for*

$$C_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \overline{\bigcup \sigma(\mathcal{U}, n)}$$

and

$$D_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

it follows that $\bigcup_{n<\omega} C_n = \bigcup_{n<\omega} D_n = X$.

Proof. Observe $D_n \subseteq C_n$. Suppose that $x \notin D_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$ for any $n < \omega$. Then for each n, pick $\mathcal{U}_n \in \mathfrak{C}$ such that $x \notin \bigcup \sigma(\mathcal{U}_n, n)$. Then σ does not defeat the play $\mathcal{U}_0, \mathcal{U}_1, \ldots$ since the $\sigma(\mathcal{U}_n, n)$ do not cover x, contradiction. □

Theorem 12. *For regular spaces X, X is σ-compact if and only if $F \uparrow_{mark} Cov_{C,F}(X)$.*

Proof. The reverse implication has already been shown. To complete the proof, we look to Scheepers for inspiration.

Let $\sigma(\mathcal{U}, n)$ be a winning Markov strategy for F in $Cov_{C,F}(X)$. Let $\mathfrak{C}$ collect all open covers of X. Define

$$C_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \overline{\bigcup \sigma(\mathcal{U}, n)}$$

as in the previous lemma. Note that $\bigcup_{n<\omega} C_n = X$, and we will show each C_n is compact as it is H-closed.

Let $\mathcal{U}$ be an open cover of C_n, and $\mathcal{V}$ be a cover of $X \setminus C_n$ by open sets whose closures are disjoint from C_n (possible by regularity).

Since $\mathcal{U} \cup \mathcal{V}$ covers X, $\overline{\bigcup \sigma(\mathcal{U} \cup \mathcal{V}, n)} \supseteq C_n$. Furthermore, if $\mathcal{F} = \sigma(\mathcal{U} \cup \mathcal{V}, n) \setminus \mathcal{V}$, then $\overline{\bigcup \mathcal{F}} \supseteq C_n$ (the closures of sets in $\mathcal{V}$ missed C_n). Thus $\mathcal{F}$ witnesses that C_n is H-closed. □

Example 13. *Let R be given the topology from example 63 from Counterexamples in Topology, the topology generated by open intervals with countable sets removed. This space is non-regular, non-σ-compact, and Lindelöf. It is also Menger as $F \uparrow Cov_{C,F}(R)$, but $F \not\uparrow_{mark} Cov_{C,F}(R)$.*

Proof. From Counterexamples: The irrationals are open, but contain no closed neighborhood, showing non-regular. Compact subsets are exactly finite subsets, showing non-σ-compact.

Take open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$. Define $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{2n})$ to be a finite subcover of $[-n, n] \setminus C_n$ for some countable $C_n = \{c_{n,0}, c_{n,1}, \dots\}$. For $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{2n+1})$, use any subcover of $\{c_{i,j} : i, j < n\}$. It is easily seen that σ is a winning perfect information strategy.

There cannot be a winning Markov strategy $\sigma(\mathcal{U}, n)$, however. Define

$$D_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

where $\mathfrak{C}$ is the collection of open covers of R. For any $x_0, x_1, \dots \in R$, we may define the open cover $\mathcal{U} = \{R \setminus \{x_i : i \neq n\} : n < \omega\}$, and observe that $\bigcup \sigma(\mathcal{U}, n) \supseteq D_n$ cannot contain every x_i. Thus D_n is finite, but since the previous lemma requires $\bigcup_{n<\omega} D_n = R$ if σ is a winning strategy, there exists a counter to σ. $\square$

Example 14. *Let R be given the topology from example 67 from Counterexamples in Topology, the topology generated by open intervals with or without the rationals removed. This space is non-regular, non-σ-compact, and Lindelöf.*

This space is an example of non-σ-compact but $F \uparrow_{mark} Cov_{C,F}(R)$ (and is thus also Menger).

Proof. From Counterexamples: The rationals are closed, but the closure of any open neighborhood is the whole real line, so they cannot be separated from any irrational point. Compact sets in this topology are nowhere dense in the Euclidean topology, so there cannot be countably many which union to the whole space. $\{(a, b) \setminus D : a, b \in \mathbb{Q}, D \in \{\emptyset, \mathbb{Q}\}\}$ is a countable base for the space, and second-countability implies Lindelöf.

To see $F \uparrow_{\text{mark}} Cov_{C,F}(R)$, we define $\sigma(\mathcal{U}_{2n}, 2n)$ to be a finite cover of $[-n, n] \setminus \mathbb{Q}$, and $\sigma(\mathcal{U}_{2n+1}, 2n + 1)$ to be a finite cover of $\{q_n\}$ for each $q_n \in \mathbb{Q}$. $\square$

We define a new property "σ-compactish" to describe a sufficient condition for $F \uparrow_{\text{2-mark}} Cov_{C,F}(X)$.

Definition 15. Let $\mathcal{U}$ be a cover of X. We say $C \subseteq X$ is $\mathcal{U}$-compact if there exists a finite subcover of $\mathcal{U}$ which covers C.

Let $\mathfrak{C}$ collect all the open covers of X. We say X is σ-compactish if there exists a fuction $f : \mathfrak{C} \times \omega \to \mathcal{P}(X)$ such that:

- $f(\mathcal{V}, n)$ is $\mathcal{V}$-compact
- $f(\mathcal{V}, n) \subseteq f(\mathcal{V}, n + 1)$

- $\bigcup_{n<\omega} f(\mathcal{V}, n) = X$
- The set
$$g(\mathcal{U}, \mathcal{V}, n) = \bigcup_{m \geq n} f(\mathcal{U}, m) \setminus (f(\mathcal{U}, m-1) \cup f(\mathcal{V}, m))$$
is $\mathcal{V}$-compact

Obviously σ-compact implies σ-compactish implies Lindelöf. We shall see that the non-σ-compact space $\omega_1^\dagger$ is σ-compactish .

Lemma 16. *There exist injective functions $f_\alpha : \alpha \to \omega$ such that if $\alpha < \beta$, then*
$$f_\beta \upharpoonright \alpha =^* f_\alpha$$
that is, $f_\beta \upharpoonright \alpha$ and f_α agree on all but finitely many ordinals. (In addition, the range of each f_α is co-infinite.)

Proof. Taken from Kunen (used for the construction of an ω_1-Aronszajn tree).

We begin with the empty function $f_0 : 0 \to \omega_1$ which satisfies the hypothesis, and assume f_α is defined by induction. Let $f_{\alpha+1} = f_\alpha \cup \{\langle \alpha, n \rangle\}$ where n is not defined for f_α, and this satisfies the hypothesis.

Finally, suppose γ is the limit of $\alpha_0, \alpha_1, \ldots$, and f_α is defined for $\alpha < \gamma$. Let $g_0 = f_{\alpha_0}$, and define $g_n : \alpha_n \to \omega$ to be injective, $g_n =^* f_{\alpha_n}$, and $g_{n+1} \upharpoonright \alpha_n = g_n$. $g = \bigcup_{n<\omega} g_n$ is an injective function from $\gamma \to \omega$ and $g =^* f_\alpha$ for $\alpha < \gamma$, but the range need not be coinfinte. So let
$$f_\gamma(\beta) = \begin{cases} g(\alpha_{2n}) & \beta = \alpha_n \\ g(\beta) & \text{otherwise} \end{cases}$$
which frees up $\{g(\alpha_{2n+1}) : n < \omega\}$ from the range of f_γ, but still agrees with all but finitely many points compared to previous f's. $\square$

Theorem 17. *The one-point Lindelöfication of the uncountable discrete space, $\omega_1^\dagger$, is σ-compactish .*

Proof. Take the injective funcions f_α from Kunen's lemma such that $f_\alpha \upharpoonright \beta =^* f_\beta$. Let $\gamma(\mathcal{U})$ identify the least ordinal such that $[\gamma(\mathcal{U}), \omega_1) \cup \{\infty\}$ is in a refinement of $\mathcal{U}$. Then $f(\mathcal{U}, n) = f^{-1}_{\gamma(\mathcal{U})}([0, n]) \cup [\gamma(\mathcal{U}), \omega_1) \cup \{\infty\}$ is easily seen to witness the property. $\square$

Theorem 18. *If X is σ-compactish , then $F \uparrow_{2\text{-}mark} Cov_{C,F}(X)$.*

Proof. Let $\sigma(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1)$ cover $f(\mathcal{U}_{n+1}, n+1)$ and $g(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1)$. If $\mathcal{U}_0, \mathcal{U}_1, \ldots$ is any play by C, then for each $x \in X$, we note that $x \in f(\mathcal{U}_0, N)$ for some N. So either $x \in \bigcap_{m \leq N} f(\mathcal{U}_m, N)$ and is covered by the strategy during round N, or for some $m < N$, $x \in f(\mathcal{U}_m, N) \setminus (f(\mathcal{U}_m, N-1) \cup f(\mathcal{U}_{m+1}, N))$ and is covered by the strategy during round $m+1$. $\square$

Corollary 19. $F \uparrow_{2\text{-}mark} Cov_{C,F}(\omega_1^\dagger)$