Limited Information Strategies for Topological Games

by

Steven Clontz

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Approved by

Gary Gruenhage, Chair, Professor of Mathematics
Stewart Baldwin, Professor of Mathematics
Chris Rodger, Professor of Mathematics
Michel Smith, Professor of Mathematics
George Flowers, Dean of the Graduate School

Abstract

I talk a lot about topological games.

TODO: Write this.

Acknowledgments

TODO: Thank people.

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Chapter 1

W convergence and clustering games

We begin by investigating a game due to Gary Gruenhage.

Game 1.0.1. Let $Con_{O,P}(X,S)$ denote the *W-convergence game* with players $\mathscr{O}$, $\mathscr{P}$, for a topological space X and $S \subseteq X$.

In round n, $\mathscr{O}$ chooses an open neighborhood $O_n \supseteq S$, followed by $\mathscr{P}$ choosing a point $x_n \in \bigcap_{m \leq n} O_m$.

$\mathscr{O}$ wins the game if the points x_n converge to the set S; that is, for every open neighborhood $U \supseteq S$, $x_n \in U$ for all but finite $n < \omega$.

If $S = \{x\}$ then we write $Con_{O,P}(X,x)$ for short. $\Diamond$

The "W" in the name merely refers to $\mathscr{O}$'s goal: to "win" the game. Gruenhage defined this game in his doctoral dissertation to define a class of spaces generalizing first-countability. [1]

Definition 1.0.2. The spaces X for which $\mathscr{O} \uparrow Con_{O,P}(X,x)$ for all $x \in X$ are called *W-spaces*. $\Diamond$

In fact, using limited information strategies, one may characterize the first-countable spaces using this game.

Proposition 1.0.3. *X is first countable if and only if $\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X,x)$ for all $x \in X$.* $\Diamond$

Proof. The forward implication shows that all W spaces are first-countable spaces, and was proven in [1]: if $\{U_n : n < \omega\}$ is a countable base at x, let $\sigma(n) = \bigcap_{m \leq n} U_m$. σ is easily seen to be a winning predetermined strategy.

If X is not first countable at some x, let σ be a predetermined strategy for $\mathscr{O}$ in $Con^{\star}_{O,P}(X,x)$. There exists an open neighborhood U of x which does not contain any $\bigcap_{m\leq n}\sigma(m)$ (otherwise $\{\bigcap_{m\leq n}\sigma(m) : n<\omega\}$ would be a countable base at x). Let x_n be an element of $\bigcap_{m\leq n}\sigma(m)\setminus U$ for all $n<\omega$. Then $\langle x_0, x_1, \ldots\rangle$ is a winning counter-attack to σ for $\mathscr{P}$, so $\mathscr{O}$ lacks a winning predetermined strategy. $\square$

At first glance, the difficulty of $Con_{O,P}(X,S)$ could be increased for $\mathscr{O}$ by only restricting the choices for $\mathscr{P}$ to be within the most recent open set played by $\mathscr{O}$, rather than all the previously played open sets.

Definition 1.0.4. Let $Con^{\star}_{O,P}(X,S)$ denote the *hard W-convergence game* which proceeds as $Con_{O,P}(X,S)$, except that $\mathscr{P}$ need only choose $x_n\in O_n$ rather than $x_n\in\bigcap_{m\leq n}O_m$ during each round. $\Diamond$

This seemingly more difficult game for $\mathscr{O}$ is Gruenhage's original formulation. But with perfect information, there is no real difference for $\mathscr{O}$.

Proposition 1.0.5. *$\mathscr{O} \underset{limit}{\uparrow} Con_{O,P}(X,S)$ if and only if $\mathscr{O} \underset{limit}{\uparrow} Con^{\star}_{O,P}(X,S)$, where $\underset{limit}{\uparrow}$ is either $\uparrow$ or $\underset{pre}{\uparrow}$.* $\Diamond$

Proof. The backwards implication is immediate.

For the forward implication, let σ be a winning predetermined (perfect information) strategy, and λ be the 0-Marköv fog-of-war μ_0 (the identity).

We define a new predetermined (perfect information) strategy τ by

$$\tau\circ\lambda(\langle x_0,\ldots,x_{n-1}\rangle)=\bigcap_{m\leq n}\sigma\circ\lambda(\langle x_0,\ldots,x_{m-1}\rangle)$$

so that each move by $\mathscr{O}$ according to $\tau\circ\lambda$ is the intersection of $\mathscr{O}$'s previous moves. Then any attack against $\tau\circ\lambda$ is an attack against $\sigma\circ\lambda$, and since $\sigma\circ\lambda$ is a winning strategy, so is $\tau\circ\lambda$. $\square$

Put more simply, $\tau(n) = \bigcap_{m \leq n} \sigma(m)$ in the predetermined case, and $\tau(\langle x_0, \dots, x_{n-1} \rangle) = \bigcap_{m \leq n} \sigma(\langle x_0, \dots, x_{m-1} \rangle)$ in the perfect information case. The original proof would have been invalid if λ was required to be, say, the tactical fog-of-war ν_1, since the value of $\mathscr{O}$'s own round 1 move $\sigma \circ \nu_1(\langle x_0 \rangle) = \sigma(\langle x_0 \rangle)$ could not be determined from the information she has during round 2: $\nu_1(\langle x_0, x_1 \rangle) = \langle x_1 \rangle$.

Due to the equivalency of the "hard" and "normal" variations of the convergence game in the perfect information case, many authors use them interchangibly. However, it is possible to find spaces for which the games are not equivalent when considering $k+1$-tactics and $k+1$-marks, as we will soon see.

In addition to the W-convergence games, we will also investigate "clustering" analogs to both variations.

Game 1.0.6. Let $Clus_{O,P}(X, S)$ ($Clus^{\star}_{O,P}(X, S)$) be a variation of $Con_{O,P}(X, S)$ ($Con^{\star}_{O,P}(X, S)$) such that x_n need only cluster at S, that is, for every open neighborhood U of S, $x_n \in U$ for infinitely many $n < \omega$. $\Diamond$

This variation seems to make $\mathscr{O}$'s job easier, but Gruenhage noted that the clustering game is perfect-information equivalent to the convergence game for $\mathscr{O}$. This can easily be extended for some limited information cases as well.

Proposition 1.0.7. *$\mathscr{O} \underset{limit}{\uparrow} Con_{O,P}(X, S)$ if and only if $\mathscr{O} \underset{limit}{\uparrow} Clus_{O,P}(X, S)$ where $\underset{limit}{\uparrow}$ is any of $\uparrow$, $\underset{pre}{\uparrow}$, $\underset{tact}{\uparrow}$, or $\underset{mark}{\uparrow}$.* $\Diamond$

Proof. For the perfect information case we refer to [1].

In the predetermined (resp. tactical) case, suppose that σ is a winning predetermined (resp. tactical) strategy for $\mathscr{O}$ in $Clus_{O,P}(X, S)$. Let p be a legal attack against σ, and q be a subsequence of p. It's easily seen that q is also a legal attack against σ, so q clusters at S. Since every subsequence of p clusters at S, p converges to S, and σ is a winning predetermined (resp. tactical) strategy for $\mathscr{O}$ in $Con_{O,P}(X, S)$ as well.

In the final case, note that any Marköv strategy σ' for $\mathscr{O}$ may be strengthened to σ defined by $\sigma(x,n) = \bigcap_{m \leq n} \sigma'(x,m)$. So, suppose that σ is a winning Marköv strategy for $\mathscr{O}$ in $Clus_{O,P}(X,S)$ such that $\sigma(x,m) \supseteq \sigma(x,n)$ for all $m \leq n$.

Let p be a legal attack against σ, and q be a subsequence of p. For $m < \omega$, there exists $f(m) \geq m$ such that $q(m) = p(f(m))$. It follows that $q(0) = p(f(0)) \in \sigma(\emptyset, 0) \cap \bigcap_{m \leq f(0)} \sigma(\langle p(m) \rangle, m) \subseteq \sigma(\emptyset, 0)$ and

$$\begin{aligned} q(n+1) = p(f(n+1)) &\in \sigma(\emptyset,0) \cap \bigcap_{m<f(n+1)} \sigma(\langle p(m)\rangle, m+1) \\ &\subseteq \sigma(\emptyset,0) \cap \bigcap_{m<n+1} \sigma(\langle p(f(m))\rangle, f(m)+1) \\ &= \sigma(\emptyset,0) \cap \bigcap_{m<n+1} \sigma(\langle q(m)\rangle, f(m)+1) \\ &\subseteq \sigma(\emptyset,0) \cap \bigcap_{m<n+1} \sigma(\langle q(m)\rangle, m+1) \end{aligned}$$

so q is also a legal attack against σ. Since σ is a winning strategy, q clusters at S, and since every subsequence of p clusters at S, p must converge to S. Thus σ is also a winning Marköv strategy for $\mathscr{O}$ in $Con_{O,P}(X,S)$ as well. $\square$

Two types of questions emerge from these results.

Question 1.0.8. Does $\mathscr{O} \underset{\text{2-tact}}{\uparrow} Clus_{O,P}(X,S)$ imply $\mathscr{O} \underset{\text{2-tact}}{\uparrow} Con_{O,P}(X,S)$? What about for $\underset{\text{2-mark}}{\uparrow}$? $\Diamond$

Question 1.0.9. Could $\mathscr{O} \underset{k+1\text{-tact}}{\uparrow} Con_{O,P}(X,S)$ actually imply $\mathscr{O} \underset{\text{tact}}{\uparrow} Con_{O,P}(X,S)$? What about for $Clus_{O,P}(X,S)$? $\Diamond$

1.1 Fort spaces

In his original paper, Gruenhage suggested the one-point-compactification of a discrete space as an example of a W-space which is not first-countable.

Definition 1.1.1. A *Fort space* $\kappa^* = \kappa \cup \{\infty\}$ is defined for each cardinal κ. Its subspace κ is discrete, and the neighborhoods of ∞ are of the form $\kappa^* \setminus F$ for each $F \in [\kappa]^{<\omega}$. $\Diamond$

Proposition 1.1.2. *$\mathscr{O} \underset{tact}{\uparrow} Con_{O,P}(\kappa^*, \infty)$ for all cardinals κ* $\Diamond$

Proof. Let $\sigma(\emptyset) = \sigma(\langle\infty\rangle) = \kappa^*$ and $\sigma(\langle\alpha\rangle) = \kappa^* \setminus \{\alpha\}$. Any legal attack against the tactic σ could not repeat non-∞ points, so it must converge to ∞. $\square$

Corollary 1.1.3. *$\mathscr{O} \uparrow Con^{\star}_{O,P}(\kappa^*, \infty)$ for all cardinals κ* $\Diamond$

Proof. Propositions 1.0.5 and 1.1.2. $\square$

Since it's trivial to show that $\mathscr{O} \underset{\text{pre}}{\uparrow} Con_{O,P}(\kappa^*,)$ if and only if $\kappa \leq \omega$, this closes the question on limited information strategies for $Con_{O,P}(\kappa^*, \infty)$. However, limited information analysis of the harder $Con^{\star}_{O,P}(\kappa^*, \infty)$ is more interesting.

Peter Nyikos noted Proposition 1.1.2 and the following in [2].

Theorem 1.1.4. *$\mathscr{O} \underset{mark}{\not\uparrow} Con^{\star}_{O,P}(\omega_1^*, \infty)$.* $\Diamond$

This actually can be generalized to any k-Markōv strategy with just a little more book-keeping.

Theorem 1.1.5. *$\mathscr{O} \underset{k\text{-}mark}{\not\uparrow} Con^{\star}_{O,P}(\omega_1^*, \infty)$.* $\Diamond$

Proof. Let σ be a k-mark for $\mathscr{O}$. Since the set

$$D_\sigma = \bigcap_{n<\omega, s\in\omega^{\leq k}} \sigma(s, n)$$

is co-countable, we may choose $\alpha_\sigma \in D_\sigma \cap \omega_1$. Thus, we may choose $n_0 < n_1 < \cdots < \omega$ such that

$$\langle n_0, \ldots, n_{k-1}, \alpha_\sigma, n_k, \ldots, n_{2k-1}, \alpha_\sigma, \ldots\rangle$$

is a legal counterattack, which fails to converge to ∞ since α_σ is repeated infinitely often. $\square$

However, while the clustering and convergence variants are equivalent for Marköv strategies in the “normal” version of the W game, they are *not* equivalent in the “hard” version.

Theorem 1.1.6. $\mathscr{O} \underset{mark}{\uparrow} Clus^{\star}_{O,P}(\omega_1^*, \infty)$. $\Diamond$

Proof. For each $\alpha < \omega_1$ let $A_\alpha = \langle A_\alpha(0), A_\alpha(1), \dots \rangle$ be a countable sequence of finite sets such that $A_\alpha(n) \subset A_\alpha(n+1)$ and $\bigcup_{n<\omega} A_\alpha(n) = \alpha + 1$.

We define the Marköv strategy σ by setting

$$\sigma(\emptyset, 0) = \sigma(\langle \infty \rangle, n) = \omega_1^*$$

and for all $\alpha < \omega_1$ setting

$$\sigma(\langle \alpha \rangle, n) = \omega_1^* \setminus A_\alpha(n)$$

Note that for any $\alpha_0 < \cdots < \alpha_{k-1}$, there is some $n < \omega$ such that $\{\alpha_0, \dots, \alpha_{k-1}\} \subseteq A_{\alpha_i}(n)$ for all $i < k$. Thus for any legal attack p against σ, the range of p cannot be finite. Since the range of p is infinite, every open neighborhood of ∞ contains infinitely many points of p, so p clusters at ∞. $\Box$

However, knowledge of the round number is critical.

Theorem 1.1.7. $O \underset{k\text{-}tact}{\not\uparrow} Clus^{\star}_{O,P}(\omega_1^*, \infty)$. $\Diamond$

Proof. Let σ be a k-tactic for $\mathscr{O}$ in $Clus_{O,P}(\omega_1^*, \infty)$. By the closing-up lemma, the set

$$C_\sigma = \{\alpha < \omega_1 : s \in \alpha^{\leq k} \Rightarrow \omega_1^* \setminus \sigma(s) \subset \alpha\}$$

is closed and unbounded. Let $a_\sigma : \omega_1 \to C_\sigma$ be an order isomorphism.

Choose $n_0 < \cdots < n_{k-1} < \omega$ such that for each $i < k$:

$$a_\sigma(n_i) \in \sigma(\langle a_\sigma(n_0), \dots, a_\sigma(n_{i-1}), a_\sigma(\omega + i), \dots, a_\sigma(\omega + k - 1) \rangle)$$

Finally, observe that the legal counterattack

$$\langle a_\sigma(n_0), \dots, a_\sigma(n_{k-1}), a_\sigma(\omega), \dots, a_\sigma(\omega+k-1), a_\sigma(n_0), \dots, a_\sigma(n_{k-1}), a_\sigma(\omega), \dots, a_\sigma(\omega+k-1), \dots\rangle$$

has a range outside the open neighborhood

$$\omega_1^* \setminus \{a_\sigma(n_0), \dots, a_\sigma(n_{k-1}), a_\sigma(\omega), \dots, a_\sigma(\omega+k-1)\}$$

of ∞. Thus σ is not a winning k-tactic. □

Once the discrete space is larger than ω_1, knowing the round number is not sufficient to construct a limited information strategy, due to a similar argument.

Theorem 1.1.8. $O \underset{k\text{-}mark}{\not\uparrow} Clus^\star_{O,P}(\omega_2^*, \infty)$. ◊

Proof. Let σ be a k-mark for $\mathscr{O}$ in $Clus_{O,P}(\omega_2^*, \infty)$. By the closing-up lemma, the set

$$C_\sigma = \{\alpha < \omega_2 : s \in \alpha^{<\omega} \Rightarrow \omega_2^* \setminus \sigma \circ \mu_k(s) \subset \alpha\}$$

is closed and unbounded. Let $a_\sigma : \omega_2 \to C_\sigma$ be an order isomorphism.

Choose $\beta_0 < \cdots < \beta_{k-1} < \omega_1$ such that for each $i < k$:

$$a_\sigma(\beta_i) \in \bigcup_{n<\omega} \sigma(\langle a_\sigma(\beta_0), \dots, a_\sigma(\beta_{i-1}), a_\sigma(\omega_1+i), \dots, a_\sigma(\omega_1+k-1)\rangle, n)$$

Finally, observe that the legal counterattack

$$\langle a_\sigma(\beta_0), \dots, a_\sigma(\beta_{k-1}), a_\sigma(\omega_1), \dots, a_\sigma(\omega_1+k-1), a_\sigma(\beta_0), \dots, a_\sigma(\beta_{k-1}), a_\sigma(\omega_1), \dots, a_\sigma(\omega_1+k-1), \dots\rangle$$

has a range outside the open neighborhood

$$\omega_2^* \setminus \{a_\sigma(\beta_0), \dots, a_\sigma(\beta_{k-1}), a_\sigma(\omega_1), \dots, a_\sigma(\omega_1+k-1)\}$$

of ∞. Thus σ is not a winning k-mark. $\square$

1.2 Sigma-products

Knowing the status of W-games in simpler spaces yields insight to larger spaces.

Proposition 1.2.1. *Suppose $S \subseteq Y \subseteq X$, $\underset{limit}{\uparrow}$ is any of $\uparrow$, $\underset{k\text{-}tact}{\uparrow}$, or $\underset{k\text{-}mark}{\uparrow}$, and $G(X,S)$ is any of $Con_{O,P}(X,S)$, $Con^{\star}_{O,P}(X,S)$, $Clus_{O,P}(X,S)$, or $Clus^{\star}_{O,P}(X,S)$.*

Then $\mathscr{O} \underset{limit}{\uparrow} G(X,S)$ implies $\mathscr{O} \underset{limit}{\uparrow} G(Y,S)$. $\lozenge$

Proof. Simply intersect the output of the winning strategy in $G(X,S)$ with Y. $\square$

A natural superspace of a Fort space is the sigma-product of a discrete cardinal.

Definition 1.2.2. Let $\Sigma_y X^\kappa$ be a *sigma product* of X with dimension κ for each $y \in X^\kappa$, the subset of the usual Tychonoff product space X^κ such that $x \in \Sigma_y X^\kappa$ if and only if $\{\alpha < \kappa : x(\alpha) \neq y(\alpha)\}$ is countable.

For homogeneous spaces X containing 0, y is usually assumed to be the zero vector $\vec{0}$ and the sigma product is written ΣX^κ. $\lozenge$

Proposition 1.2.3. *κ^* is homeomorphic to the space*

$$\{x \in \Sigma 2^\kappa : x(\alpha) = 0 \textit{ for all but one } \alpha < \kappa\}$$

$\lozenge$

Proof. Map $\alpha < \kappa$ to x_α such that

$$x_\alpha(\beta) = \begin{cases} 0 & \beta \neq \alpha \\ 1 & \beta = \alpha \end{cases}$$

and map ∞ to the zero vector $\vec{0}$. $\square$

Corollary 1.2.4. $\mathscr{O} \underset{k\text{-}tact}{\not\uparrow} Clus^{\star}_{O,P}(\Sigma\mathbb{R}^{\omega_1}, \vec{0})$, $\mathscr{O} \underset{k\text{-}mark}{\not\uparrow} Con^{\star}_{O,P}(\Sigma\mathbb{R}^{\omega_1}, \vec{0})$, *and* $\mathscr{O} \underset{k\text{-}mark}{\not\uparrow} Clus^{\star}_{O,P}(\Sigma\mathbb{R}^{\omega_2}, \vec{0})$. $\Diamond$

While this closes the question on tactics and marks for high dimensional sigma- (and Tychonoff-) products of the real line, there is another type of limited information strategy to investigate.

Definition 1.2.5. For a game $G = \langle M, W \rangle$ and *coding strategy* or *code* $\sigma : M^2 \to M$, the σ*-coding fog-of-war* $\gamma_\sigma : M^{<\omega} \to M^{\leq 2}$ is the function defined such that

$$\gamma_\sigma(\emptyset) = \emptyset$$

and

$$\gamma_\sigma(s^\frown\langle x \rangle) = \langle \sigma \circ \gamma_\sigma(s), x \rangle$$

For a coding strategy σ, its coresponding strategy is $\sigma \circ \gamma_\sigma$. For a game G, if $\sigma \circ \gamma_\sigma$ is a winning strategy for $\mathscr{A}$, then σ is a winning coding strategy and we write $\mathscr{A} \underset{\text{code}}{\uparrow} G$. $\Diamond$

Intuitively, a σ-coding fog-of-war converts perfect information of the game into the last moves of both the player and her opponent, so a player has a winning coding strategy when she only needs to know the move of her opponent and her own last move. The term "coding" comes from the fact that a player may encode information about the history of the game into her own moves, and use this encoded information in later rounds.

Coding strategies have been studied since the earliest days of the Banach-Mazur game.

Theorem 1.2.6. *(TODO: Cite the precise version of the BM game where a winning strategy implies a coding strategy.)* $\Diamond$

The hard and normal versions of the W games are all equivalent with regards to coding strategies since $\mathscr{O}$ may always ensure her new move is a subset of her previous move. For Fort spaces, the question is immediately closed.

Proposition 1.2.7. $\mathscr{O} \underset{code}{\uparrow} Con_{O,P}(\kappa^*, \infty)$. $\Diamond$

Proof. Let $\sigma(\emptyset) = \kappa^*$, $\sigma(\langle U, \alpha\rangle) = U \setminus \{\alpha\}$ for $\alpha < \kappa$, and $\sigma(\langle U, \infty\rangle) = U$. $\mathscr{P}$ cannot legally repeat non-∞ points of the set, so her points converge to ∞. $\square$

This trick does not simply extend to the $\Sigma\mathbb{R}^\kappa$ case, however. An open set may only restrict finitely many coordinates of the product, and a point in $\Sigma\mathbb{R}^\kappa$ may have countably infinite non-zero coordinates. Thus, information about the previous non-zero coordinates cannot be directly encoded into the open set.

Circumventing this takes a bit of extra machinery. We proceed by defining a simpler infinite game for each cardinal κ.

Game 1.2.8. Let $PF_{F,C}(\kappa)$ denote the *point-finite game* with players $\mathscr{F}$, $\mathscr{C}$ for each cardinal κ.

In round n, $\mathscr{F}$ chooses $F_n \in [\kappa]^{<\omega}$, followed by $\mathscr{C}$ choosing $C_n \in [\kappa \setminus \bigcup_{m\leq n} F_m]^{\leq\omega}$.

$\mathscr{F}$ wins the game if the collection $\{C_n : n < \omega\}$ is a point-finite cover of its union $\bigcup_{n<\omega} C_n$, that is, each point in $\bigcup_{n<\omega} C_n$ is in C_n only for finitely many $n < \omega$. $\Diamond$

Theorem 1.2.9. *$\mathscr{F} \underset{code}{\uparrow} PF_{F,C}(\kappa)$ implies $\mathscr{O} \underset{code}{\uparrow} Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$.* $\Diamond$

Proof. Let σ be a winning coding strategy for $\mathscr{F}$ in $PF_{F,C}(\kappa)$ such that $\sigma(\emptyset) \supset \emptyset$ and $\sigma(F, C) \supset F$.

For $F \in [\kappa]^{<\omega}$ and $\epsilon > 0$ let $U(F, \epsilon)$ be the basic open set in $\mathbb{R}^\kappa$ such that each projection is of the form

$$\pi_\alpha(U(F,\epsilon)) = \begin{cases} (-\epsilon, \epsilon) & \alpha \in F \\ \mathbb{R} & \alpha \notin F \end{cases}$$

Note that $F \supset \emptyset$ and ϵ are uniquely identifiable given $U(F, \epsilon) \cap \Sigma\mathbb{R}^\kappa$.

For each point $x \in \Sigma\mathbb{R}^\kappa$ and $\epsilon > 0$, let $C_\epsilon(x) \in [\kappa]^{\leq\omega}$ such that $\alpha \in C_\epsilon(\alpha)$ if and only if $|x(\alpha)| \geq \epsilon$.

We define the coding strategy τ for $\mathscr{O}$ in $Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$ as follows:

$$\tau(\emptyset) = U(\sigma(\emptyset), 1) \cap \Sigma\mathbb{R}^\kappa$$

$$\tau(\langle U(F, \epsilon) \cap \Sigma\mathbb{R}^\kappa, x\rangle) = U\left(\sigma(\langle F, C_\epsilon(x)\rangle), \frac{\epsilon}{2}\right) \cap \Sigma\mathbb{R}^\kappa$$

Let $\langle a_0, a_1, a_2, \dots\rangle$ be a legal attack by $\mathscr{P}$ against τ. It then follows that

$$b = \langle C_1(a_0), C_{1/2}(a_1), C_{1/4}(a_2), \dots\rangle$$

is a legal attack by $\mathscr{C}$ against σ. Since σ is a winning strategy, each ordinal in $\bigcup_{n<\omega} C_{2^{-n}}(a_n)$ is in $C_{2^{-n}}(a_n)$ only for finitely many $n < \omega$. Thus for every coordinate $\alpha < \kappa$ it follows that there exists some $n_\alpha < \omega$ such that $a_n(\alpha) \leq 2^{-n}$ for $n \geq n_\alpha$. We conclude $a_n \to \vec{0}$, showing that τ is a winning strategy. □

Theorem 1.2.10. *$\mathscr{F} \underset{code}{\uparrow} PF_{F,C}(\kappa)$ for all cardinals κ.* ◊

Proof. TODO: Fill in sketch.

SKETCH: For each cardinal κ, assume $F \underset{\text{code}}{\uparrow} PF_{F,C}(\lambda)$ for $\lambda < \kappa$. Always cover the upper bound of $\mathscr{O}$'s moves if possible, or use a trick assuming $\mathrm{cf}(\kappa) = \omega$. □

Corollary 1.2.11. *$O \underset{code}{\uparrow} Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$ for all cardinals κ.* ◊

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