Limited Information Strategies for Topological Games

by

Steven Clontz

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
May 4, 2015

Keywords: topology, uniform spaces, infinite games, limited information strategies

Approved by

Gary Gruenhage, Chair, Professor of Mathematics
Stewart Baldwin, Professor of Mathematics
Chris Rodger, Professor of Mathematics
Michel Smith, Professor of Mathematics
George Flowers, Dean of the Graduate School

Abstract

I talk a lot about topological games.

TODO: Write this.

Acknowledgments

TODO: Thank people.

Table of Contents

List of Figures

List of Tables

Chapter 1

Introduction

Basic overview of combinatorial games, topological games, limited info strategies, and applications in topology.

Chapter 2

Toplogical Games and Strategies of Perfect and Limited Information

The goal of this paper is to explore the applications of limited information strategies in existing topological games. There are a variety of frameworks for modeling such games, so we establish one within this chapter which we will use for this manuscript.

2.1 Games

Intuitively, the games studied in this paper are two-player games for which each player takes turns making a choice from a set of possible moves. At the conclusion of the game, the choices made by both players are examined, and one of the players is declared the winner of that playthrough.

Games may be modeled mathematically in various ways, but we will find it convenient to think of them in terms defined by Gale and Stewart. [2]

Definition 1. A *game* is a tuple $\langle M, W\rangle$ such that $W \subseteq M^\omega$. M is set of *moves* for the game, and M^ω is the set of all possible *playthroughs* of the game.

W is the set of *winning playthroughs* or *victories* for the first player, and $M^\omega \setminus W$ is the set of victories for the second player. (W is often called the *payoff set* for the first player.)

Within this model, we may imagine two players $\mathscr{A}$ and $\mathscr{B}$ playing a game which consists of *rounds* enumerated for each $n < \omega$. During round n, $\mathscr{A}$ chooses $a_n \in M$, followed by $\mathscr{B}$ choosing $b_n \in M$. The playthrough corresponding to those choices would be the sequence $p = \langle a_0, b_0, a_1, b_1, \ldots\rangle$. If $p \in W$, then $\mathscr{A}$ is the winner of that playthrough, and if $p \notin W$, then $\mathscr{B}$ is the winner. (No ties are allowed.)

Rather than explicitly defining W, we typically define games by declaring the *rules* that each player must follow and the *winning condition* for the first player. Then a playthrough is in W if either the first player made only *legal moves* which observed the game's rules and the playthrough satisified the winning condition, or the second player made an *illegal move* which contradicted the game's rules.

As an illustration, we could model a game of chess (ignoring stalemates) by letting

$$M = \{\langle p, s\rangle : p \text{ is a chess piece and } s \text{ is a space on the board }\}$$

representing moving a piece p to the space s on the board. Then the rules of chess restict White from moving pieces which belong to Black, or moving a piece to an illegal space on the board. [1] The winning condition could then "inspect" the resulting positions of pieces on the board after each move to see if White attained a checkmate. This winning condition along with the rules implicitly define the set W of winning playthroughs for White.

2.1.1 Infinite and Topological Games

Games never technically end within this model, since playthroughs of the game are infinite sequences. However, for all practical purposes many games end after a finite number of turns.

Definition 2. A game is said to be an *finite game* if for every playthrough $p \in M^\omega$ there exists a round $n < \omega$ such that $[p \restriction n] = \{q \in M^\omega : q \supseteq p \restriction n\}$ is a subset of either W or $M^\omega \setminus W$.

Put another way, a finite game is decided after a finite number of rounds, after which the game's winner could not change even if further rounds were played. Games which are not finite are called *infinite games*.

[1] In practice, M is often defined as the union of two sets, such as white pieces and black pieces in chess. For example, the first player may choose open sets in a topology, while the second player chooses points within the topological space.

As an illustration of an infinite game, we may consider a simple example due to Baker [1].

Game 3. Let $Con_{A,B}(A)$ denote a game with players $\mathscr{A}$ and $\mathscr{B}$, defined for each subset $A \subset \mathbb{R}$. In round 0, $\mathscr{A}$ chooses a number a_0, followed by $\mathscr{B}$ choosing a number b_0 such that $a_0 < b_0$. In round $n+1$, $\mathscr{A}$ chooses a number a_{n+1} such that $a_n < a_{n+1} < b_n$, followed by $\mathscr{B}$ choosing a number b_{n+1} such that $a_{n+1} < b_{n+1} < b_n$.

$\mathscr{A}$ wins the game if the sequence $\langle a_n : n < \omega \rangle$ converges to a point in A, and $\mathscr{B}$ wins otherwise.

Certainly, $\mathscr{A}$ and $\mathscr{B}$ will never be in a position without (infinitely many) legal moves available, and provided that A is non-trivial, there is a playthrough such that for all $n < \omega$, the segment (a_n, b_n) intersects both A and $\mathbb{R} \setminus A$. Such a playthrough could never be decided in a finite number of moves, so the winning condition considers the infinite sequence of moves made by the players and declares a victor at the "end" of the game.

Definition 4. A *topological game* is a game defined in terms of an arbitrary topological space.

Topological games are usually infinite games for non-trivial spaces. (The meaning of trivial here depends on the game played.) One of the earliest examples of a topological game is the Banach-Mazur game, proposed by Stanislaw Mazur as Problem 43 in Stefan Banach's Scottish Book (1935). A more comprehensive history of the Banacy-Mazur and other topological games may be found in Telgarsky's survey on the subject [6].

The original game was defined for subsets of the real line; however, we give a more general definition here.

Game 5. Let $Empty_{E,N}(X)$ denote the *Banach-Mazur game* with players $\mathscr{E}$, $\mathscr{N}$ defined for each topological space X. In round 0, $\mathscr{E}$ chooses a nonempty open set $E_0 \subseteq X$, followed by $\mathscr{N}$ choosing a nonempty open subset $N_0 \subseteq E_0$. In round $n+1$, $\mathscr{E}$ chooses a nonempty open subset $E_{n+1} \subseteq N_n$, followed by $\mathscr{N}$ choosing a nonempty open subset $N_{n+1} \subseteq E_{n+1}$.

$\mathscr{E}$ wins the game if $\bigcap_{n<\omega} E_n = \emptyset$, and $\mathscr{N}$ wins otherwise.

For example, if X is a locally compact Hausdorff space, $\mathscr{N}$ can "force" a win by choosing N_0 such that $\overline{N_0}$ is compact, and choosing N_{n+1} such that $N_{n+1} \subseteq \overline{N_{n+1}} \subseteq O_{n+1} \subseteq N_n$ (possible since N_n is a compact Hausdorff $\Rightarrow$ normal space). Since $\bigcap_{n<\omega} E_n = \bigcap_{n<\omega} N_n$ is the decreasing intersection of compact sets, it cannot be empty.

This concept of when (and how) a player can "force" a win in certain topological games is the focus of this manuscript.

2.2 Strategies

We shall make the notion of forcing a win in a game rigorous by introducing "strategies" and "attacks" for games.

Definition 6. A *strategy* for a game $G = \langle M, W \rangle$ is a function from $M^{<\omega}$ to M.

Definition 7. An *attack* for a game $G = \langle M, W \rangle$ is a function from ω to M.

Intuitively, a strategy is a rule for one of the players on how to play the game based upon the previous (finite) moves of her opponent, while an attack is a fixed strike by an opponent indexed by round number.

Definition 8. The *result* of a game given a strategy σ for the first player and an attack $\langle a_0, a_1, \ldots \rangle$ by the second player is the playthrough

$$\langle \sigma(\emptyset), a_0, \sigma(\langle a_0 \rangle), a_1, \sigma(\langle a_0, a_1 \rangle), \ldots \rangle$$

Likewise, if σ is a strategy for the second player, and $\langle a_0, a_1, \ldots \rangle$ is an attack by the first player, then the result is the playthrough

$$\langle a_0, \sigma(\langle a_0 \rangle), a_1, \sigma(\langle a_0, a_1 \rangle), \ldots \rangle$$

We now may rigorously define the notion of "forcing" a win in a game.

Definition 9. A strategy σ is a *winning strategy* for a player if for every attack by the opponent, the result of the game is a victory for that player.

If a winning strategy exists for player $\mathscr{A}$ in the game G, then we write $\mathscr{A} \uparrow G$. Otherwise, we write $\mathscr{A} \not\uparrow G$.

Of course, a strategy σ is not a winning strategy for a player if there exists some *counter-attack* by the opponent for which the result is a victory for the opponent. Typically this counter-attack is defined in terms of the strategy σ; else, the counter-attack is itself a winning strategy depending on only the round number (which we will investigate further in a later section).

Definition 10. A game G with players $\mathscr{A}$, $\mathscr{B}$ is said to be *determined* if either $\mathscr{A} \uparrow G$ or $\mathscr{B} \uparrow G$. Otherwise, the game is *indetermined.*

Theorem 11. *[2] If the move set M for a game $G = \langle M, W \rangle$ is given the discrete topology and W is either an open or closed subset of M^ω with the usual product topology, then G is determined.*

This is actually a special case of the powerful Borel Determinacy theorem which states that $G = \langle M, W \rangle$ is determined whenever W is a Borel subset of M^ω. [4]

It's an easy corollary that all finite games are determined (W must be clopen). As stated earlier, most topological games are infinite, and many are indetermined for certain spaces constructed using the Axiom of Choice. [2]

Example 12. Let B be a Bernstein subset of the real line. Then $Empty_{E,N}(B)$ is undetermined.

Proof. It can be shown that B is a Baire space, and we will soon see that $\mathscr{E} \not\uparrow Empty_{E,N}(X)$ characterizes Baire spaces. Also, if σ is a winning strategy for $\mathscr{N}$ in $Empty_{E,N}(Y)$ for $Y \subseteq X$, it can be shown that Y contains a closed uncountable set in X. Thus $\mathscr{N} \not\uparrow Empty_{E,N}(B)$. $\square$

[2] The Axiom of Choice is required, as the Axiom of Determinacy stating that all Gale-Stewart games are determined is another axiom independent of ZF. [5]

2.2.1 Applications of Strategies

The presence or absence of a winning strategie for a player in a topological game characterizes a property of the topological space in question.

A classical result follows.

Theorem 13. *$\mathscr{E} \not\uparrow Empty_{E,N}(X)$ if and only if X is a Baire space. [3]*

2.2.2 Limited Information Strategies

2.3 Examples of Topological Games

(TODO:)

Bibliography

[1] Matthew H. Baker. Uncountable sets and an infinite real number game. *Mathematics Magazine*, 80(5):377–380, December 2007.

[2] D. Gale and F. M. Stewart. Infinite games with perfect information. *Contributions to the Theory of Games, Volume II*, page 245266, 1953.

[3] R.C. Haworth and R.A. McCoy. Baire spaces. *Dissertationes Math*, 1977.

[4] Donald A. Martin. Borel determinacy. *Annals of Mathematics*, 102(2):pp. 363–371, 1975.

[5] Jan Mycielski and H. Steinhaus. A mathematical axiom contradicting the axiom of choice. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 10:1–3, 1962.

[6] Rastislav Telgarsky. Topological games: On the 50th anniversary of the banach-mazur game. *Rocky Mountain Journal of Mathematics*, 7(2), 1987.