

# 1 ARHANGELSKII'S $\alpha$ -PRINCIPLES AND SELECTION GAMES

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ABSTRACT. Arhangel'skii's properties  $\alpha_2$  and  $\alpha_4$  defined for convergent sequences may be characterized in terms of Scheeper's selection principles. We generalize these results to hold for more general collections and consider these results in terms of selection games.

3 The following characterizations were given as Definition 1 by Kocinac in [cite  
4 Kocinac selection principles related].

5 **Definition 1.** *Arhangel'skii's  $\alpha$ -principles  $\alpha_i(\mathcal{A}, \mathcal{B})$  are defined as follows for  $i \in$   
6  $\{1, 2, 3, 4\}$ . Let  $A_n \in \mathcal{A}$  for all  $n < \omega$ ; then there exists  $B \in \mathcal{B}$  such that:*

- 7  $\alpha_1$ :  $A_n \cap B$  is cofinite in  $A_n$  for all  $n < \omega$ .
- 8  $\alpha_2$ :  $A_n \cap B$  is infinite for all  $n < \omega$ .
- 9  $\alpha_3$ :  $A_n \cap B$  is infinite for infinitely-many  $n < \omega$ .
- 10  $\alpha_4$ :  $A_n \cap B$  is non-empty for infinitely-many  $n < \omega$ .

11 When  $(\mathcal{A}, \mathcal{B})$  is omitted, it is assumed that  $\mathcal{A} = \mathcal{B}$  is the collection  $\Gamma_{X,x}$  of  
12 sequences converging to some point  $x \in X$ , as introduced by Arhangel'skii in [cite  
13 Arhangel'skii frequency spectrum]. Provided  $\mathcal{A}$  only contains infinite sets, it's easy  
14 to see that  $\alpha_n(\mathcal{A}, \mathcal{B})$  implies  $\alpha_{n+1}(\mathcal{A}, \mathcal{B})$ .

15 We aim to relate these to the following games.

16 **Definition 2.** The *selection game*  $G_1(\mathcal{A}, \mathcal{B})$  (resp.  $G_{fin}(\mathcal{A}, \mathcal{B})$ ) is an  $\omega$ -length  
17 game involving Players I and II. During round  $n$ , I chooses  $A_n \in \mathcal{A}$ , followed  
18 by II choosing  $a_n \in A_n$  (resp.  $F_n \in [A_n]^{<\aleph_0}$ ). Player II wins in the case that  
19  $\{a_n : n < \omega\} \in \mathcal{B}$  (resp.  $\bigcup\{F_n : n < \omega\} \in \mathcal{B}$ ), and Player I wins otherwise.

20 Such games are well-represented in the literature; see [cite Scheepers combi-  
21 natorics ramsey] for example. We will also consider the similarly-defined games  
22  $G_{<2}(\mathcal{A}, \mathcal{B})$  (II chooses 0 or 1 points from each choice by I) and  $G_{cf}(\mathcal{A}, \mathcal{B})$  (II  
23 chooses cofinitely-many points).

24 **Definition 3.** Let  $P$  be a player in a game  $G$ .  $P$  has a *winning strategy* for  $G$ ,  
25 denoted  $P \uparrow G$ , if  $P$  has a strategy that defeats every possible counterplay by  
26 their opponent. If a strategy only relies on the round number and ignores the  
27 moves of the opponent, the strategy is said to be *predetermined*; the existence of a  
28 predetermined winning strategy is denoted  $P \uparrow_{\text{pre}} G$ .

29 We briefly note that the statement  $I \not\uparrow_{\text{pre}} G_\star(\mathcal{A}, \mathcal{B})$  is often denoted as the *selection*  
30 *principle*  $S_\star(\mathcal{A}, \mathcal{B})$ .

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*Key words and phrases.* Selection principle, selection game,  $\alpha_i$  property, convergence.

**Definition 4.** Let  $\Gamma_{X,x}$  be the collection of non-trivial sequences  $S \subseteq X$  converging to  $x$ , that is, infinite subsets of  $X \setminus \{x\}$  such that for each neighborhood  $U$  of  $x$ ,  $S \cap U$  is cofinite in  $S$ .

**Definition 5.** Let  $\Gamma_X$  be the collection of open  $\gamma$ -covers  $\mathcal{U}$  of  $X$ , that is, infinite open covers of  $X$  such that  $X \notin \mathcal{U}$  and for each  $x \in X$ ,  $\{U \in \mathcal{U} : x \in U\}$  is cofinite in  $\mathcal{U}$ .

The similarity in nomenclature follows from the observation that every non-trivial sequence in  $C_p(X)$  converging to the zero function  $\mathbf{0}$  naturally defines a corresponding  $\gamma$ -cover in  $X$ , see e.g. Theorem 4 of [Scheepers a sequential property and covering property].

The equivalence of  $\alpha_2(\Gamma_{X,x}\Gamma_{X,x})$  and  $\text{I} \not\preceq_{\text{pre}} G_1(\Gamma_{X,x}, \Gamma_{X,x})$  was briefly asserted by Sakai in the introduction of [cite Sakai sequence selection properties]; the similar equivalence of  $\alpha_4(\Gamma_{X,x}\Gamma_{X,x})$  and  $\text{I} \not\preceq_{\text{pre}} G_{fin}(\Gamma_{X,x}, \Gamma_{X,x})$  seems to be folklore. In fact, these relationships hold in more generality.

Note that by these definitions, convergent sequences (resp.  $\gamma$ -covers) may be uncountable, but any infinite subset of either would remain a convergent sequence (resp.  $\gamma$ -cover), in particular, countably infinite subsets. We capture this idea as follows.

**Definition 6.** Say a collection  $\mathcal{A}$  is  $\Gamma$ -like if it satisfies the following for each  $A \in \mathcal{A}$ .

- $|A| \geq \aleph_0$ .
- If  $A' \subseteq A$  and  $|A'| \geq \aleph_0$ , then  $A' \in \mathcal{A}$ .

We also require the following.

**Definition 7.** Say a collection  $\mathcal{A}$  is *almost- $\Gamma$ -like* if for each  $A \in \mathcal{A}$ , there is  $A' \subseteq A$  such that:

- $|A'| = \aleph_0$ .
- If  $A''$  is a cofinite subset of  $A'$ , then  $A'' \in \mathcal{A}$ .

So all  $\Gamma$ -like sets are almost- $\Gamma$ -like.

We are now able to prove a few general equivalences between  $\alpha$ -principles and selection games.

#### 1. ON $\alpha_2(\mathcal{A}, \mathcal{B})$ AND $G_1(\mathcal{A}, \mathcal{B})$

**Theorem 8.** Let  $\mathcal{A}$  be almost- $\Gamma$ -like and  $\mathcal{B}$  be  $\Gamma$ -like. Then  $\alpha_2(\mathcal{A}, \mathcal{B})$  holds if and only if  $\text{I} \not\preceq_{\text{pre}} G_1(\mathcal{A}, \mathcal{B})$ .

*Proof.* We first assume  $\alpha_2(\mathcal{A}, \mathcal{B})$  and let  $A_n \in \mathcal{A}$  for  $n < \omega$  define a predetermined strategy for I. We may apply  $\alpha_2(\mathcal{A}, \mathcal{B})$  to choose  $B \in \mathcal{B}$  such that  $|A_n \cap B| \geq \aleph_0$ . We may then choose  $a_n \in (A_n \cap B) \setminus \{a_i : i < n\}$  for each  $n < \omega$ . It follows that  $B' = \{a_n : n < \omega\} \in \mathcal{B}$  since  $B'$  is an infinite subset of  $B \in \mathcal{B}$ ; therefore  $A_n$  does not define a winning predetermined strategy for I.

Now suppose  $\text{I} \not\preceq_{\text{pre}} G_1(\mathcal{A}, \mathcal{B})$ . Given  $A_n \in \mathcal{A}$  for  $n < \omega$ , first choose  $A'_n \in \mathcal{A}$  such that  $A'_n = \{a_{n,j} : j < \omega\} \subseteq A_n$ ,  $j < k$  implies  $a_{n,j} \neq a_{n,k}$ , and  $A_{n,m} = \{a_{n,j} : m \leq j < \omega\} \in \mathcal{A}$ . Finally choose some  $\theta : \omega \rightarrow \omega$  such that  $|\theta^{\leftarrow}(n)| = \aleph_0$  for each  $n < \omega$ .

Since playing  $A_{\theta(m),m}$  during round  $m$  does not define a winning strategy for I in  $G_1(\mathcal{A}, \mathcal{B})$ , II may choose  $x_m \in A_{\theta(m),m}$  such that  $B = \{x_m : m < \omega\} \in \mathcal{B}$ . Choose

73  $i_m < \omega$  for each  $m < \omega$  such that  $x_m = a_{\theta(m), i_m}$ , noting  $i_m \geq m$ . It follows that  
 74  $A_n \cap B \supseteq \{a_{\theta(m), i_m} : m \in \theta^{\leftarrow}(n)\}$ . Since for each  $m \in \theta^{\leftarrow}(n)$  there exists  $M \in$   
 75  $\theta^{\leftarrow}(n)$  such that  $m \leq i_m < M \leq i_M$ , and therefore  $a_{\theta(m), i_m} \neq a_{\theta(m), i_M} = a_{\theta(M), i_M}$ ,  
 76 we have shown that  $A_n \cap B$  is infinite. Thus  $B$  witnesses  $\alpha_2(\mathcal{A}, \mathcal{B})$ .  $\square$

77 While  $\alpha_2(\mathcal{A}, \mathcal{B})$  involves infinite intersection and  $G_1(\mathcal{A}, \mathcal{B})$  involves single selec-  
 78 tions, the previous result is made more intuitive given the following result, shown  
 79 for  $\mathcal{A} = \mathcal{B} = \Gamma_{X,x}$  by Nogura in [cite product of alpha spaces].

80 **Definition 9.**  $\alpha'_2(\mathcal{A}, \mathcal{B})$  is the following claim: if  $A_n \in \mathcal{A}$  for all  $n < \omega$ , then there  
 81 exists  $B \in \mathcal{B}$  such that  $A_n \cap B$  is nonempty for all  $n < \omega$ .

82 (Note that  $\alpha_5$  is sometimes used in the literature in place of  $\alpha'_2$ .)

83 **Proposition 10.** If  $\mathcal{A}$  is almost- $\Gamma$ -like, then  $\alpha_2(\mathcal{A}, \mathcal{B})$  is equivalent to  $\alpha'_2(\mathcal{A}, \mathcal{B})$ .

84 *Proof.* The forward implication is immediate, so we assume  $\alpha'_2(\mathcal{A}, \mathcal{B})$ . Given  $A_n \in$   
 85  $\mathcal{A}$ , we apply the almost- $\Gamma$ -like property to obtain  $A'_n = \{a_{n,m} : m < \omega\} \subseteq A_n$  such  
 86 that  $A_{n,m} = A_n \setminus \{a_{i,j} : i, j < m\} \in \mathcal{A}$  for all  $m < \omega$ .

87 By applying  $\alpha'_2(\mathcal{A}, \mathcal{B})$  to  $A_{n,m}$ , we obtain  $B \in \mathcal{B}$  such that  $A_{n,m} \cap B$  is nonempty  
 88 for all  $n, m < \omega$ . Since it follows that  $A_n \cap B$  is infinite for all  $n < \omega$ , we have  
 89 established  $\alpha_2(\mathcal{A}, \mathcal{B})$ .  $\square$

## 90 2. ON $\alpha_4(\mathcal{A}, \mathcal{B})$ AND $G_{fin}(\mathcal{A}, \mathcal{B})$

91 A similar correspondence exists between  $\alpha_4(\mathcal{A}, \mathcal{B})$  and  $G_{fin}(\mathcal{A}, \mathcal{B})$ .

92 **Theorem 11.** Let  $\mathcal{A}$  be almost- $\Gamma$ -like and  $\mathcal{B}$  be  $\Gamma$ -like. Then  $\alpha_4(\mathcal{A}, \mathcal{B})$  holds if and  
 93 only if  $\text{I} \not\Uparrow_{pre} G_{<2}(\mathcal{A}, \mathcal{B})$  if and only if  $\text{I} \not\Uparrow_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$ .

94 *Proof.* We first assume  $\alpha_4(\mathcal{A}, \mathcal{B})$  and let  $A_n \in \mathcal{A}$  for  $n < \omega$  define a predetermined  
 95 strategy for I in  $G_{<2}(\mathcal{A}, \mathcal{B})$ . We then may choose  $A'_n \in \mathcal{A}$  where  $A'_n = \{a_{n,j} : j <$   
 96  $\omega\} \subseteq A_n$ ,  $j < k$  implies  $a_{n,j} \neq a_{n,k}$ , and  $A''_n = A'_n \setminus \{a_{i,j} : i, j < n\} \in \mathcal{A}$ .

97 By applying  $\alpha_4(\mathcal{A}, \mathcal{B})$  to  $A''_n$ , we obtain  $B \in \mathcal{B}$  such that  $A''_n \cap B \neq \emptyset$  for infinitely-  
 98 many  $n < \omega$ . We then let  $F_n = \emptyset$  when  $A''_n \cap B = \emptyset$ , and  $F_n = \{x_n\}$  for some  
 99  $x_n \in A''_n \cap B$  otherwise. Then we will have that  $B' = \bigcup \{F_n : n < \omega\} \subseteq B$  belongs  
 100 to  $\mathcal{B}$  once we show that  $B'$  is infinite. To see this, for  $m \leq n < \omega$  note that either  
 101  $F_m$  is empty (and we let  $j_m = 0$ ) or  $F_m = \{a_{m,j_m}\}$  for some  $j_m \geq m$ ; choose  $N < \omega$   
 102 such that  $j_m < N$  for all  $m \leq n$  and  $F_N = \{x_N\}$ . Thus  $F_m \neq F_N$  for all  $m \leq n$   
 103 since  $x_N \notin \{a_{i,j} : i, j < N\}$ . Thus II may defeat the predetermined strategy  $A_n$  by  
 104 playing  $F_n$  each round.

105 Since  $\text{I} \not\Uparrow_{pre} G_{<2}(\mathcal{A}, \mathcal{B})$  immediately implies  $\text{I} \not\Uparrow_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$ , we assume the latter.

106 Given  $A_n \in \mathcal{A}$  for  $n < \omega$ , we note this defines a (non-winning) predetermined  
 107 strategy for I, so II may choose  $F_n \in [A_n]^{<\aleph_0}$  such that  $B = \bigcup \{F_n : n < \omega\} \in \mathcal{B}$ .  
 108 Since  $B$  is infinite, we note  $F_n \neq \emptyset$  for infinitely-many  $n < \omega$ . Thus  $B$  witnesses  
 109  $\alpha_4(\mathcal{A}, \mathcal{B})$  since  $A_n \cap B \supseteq F_n \neq \emptyset$  for infinitely-many  $n < \omega$ .  $\square$

110 This shows that II gains no advantage from picking more than one point per  
 111 round. This in fact only depends on  $\mathcal{B}$  being  $\Gamma$ -like, which we formalize in the  
 112 following results.

113 **Theorem 12.** Let  $\mathcal{B}$  be  $\Gamma$ -like. Then  $\text{I} \uparrow_{pre} G_{<2}(\mathcal{A}, \mathcal{B})$  if and only if  $\text{I} \uparrow_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$ .

114 *Proof.* Assume  $\bigcup \mathcal{A}$  is well-ordered. Given a winning predetermined strategy  $A_n$   
 115 for I in  $G_{<2}(\mathcal{A}, \mathcal{B})$ , consider  $F_n \in [A_n]^{<\aleph_0}$ . We set

$$F_n^* = \begin{cases} \emptyset & \text{if } F_n \setminus \bigcup \{F_m : m < n\} = \emptyset \\ \{\min(F_n \setminus \bigcup \{F_m : m < n\})\} & \text{otherwise} \end{cases}$$

116 Since  $|F_n^*| < 2$ , we have that  $\bigcup \{F_n^* : n < \omega\} \notin \mathcal{B}$ . In the case that  $\bigcup \{F_n^* : n < \omega\}$   
 117 is finite, we immediately see that  $\bigcup \{F_n : n < \omega\}$  is also finite and therefore not in  
 118  $\mathcal{B}$ . Otherwise  $\bigcup \{F_n^* : n < \omega\} \notin \mathcal{B}$  is an infinite subset of  $\bigcup \{F_n : n < \omega\}$ , and thus  
 119  $\bigcup \{F_n : n < \omega\} \notin \mathcal{B}$  too. Therefore  $A_n$  is a winning predetermined strategy for I in  
 120  $G_{fin}(\mathcal{A}, \mathcal{B})$  as well.  $\square$

121 **Theorem 13.** *Let  $\mathcal{B}$  be  $\Gamma$ -like. Then  $I \uparrow G_{<2}(\mathcal{A}, \mathcal{B})$  if and only if  $I \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ .*

122 *Proof.* Assume  $\bigcup \mathcal{A}$  is well-ordered. Suppose  $I \uparrow G_{<2}(\mathcal{A}, \mathcal{B})$  is witnessed by the  
 123 strategy  $\sigma$ . Let  $\langle \rangle^* = \langle \rangle$ , and for  $s \frown \langle F \rangle \in ([\bigcup \mathcal{A}]^{<\aleph_0})^{<\omega} \setminus \{\langle \rangle\}$  let

$$(s \frown \langle F \rangle)^* = \begin{cases} s^* \frown \langle \emptyset \rangle & \text{if } F \setminus \bigcup \text{range}(s) = \emptyset \\ s^* \frown \langle \{\min(F \setminus \bigcup \text{range}(s))\} \rangle & \text{otherwise} \end{cases}$$

124 We then define the strategy  $\tau$  for I in  $G_{fin}(\mathcal{A}, \mathcal{B})$  by  $\tau(s) = \sigma(s^*)$ . Then given  
 125 any counterattack  $\alpha \in ([\bigcup \mathcal{A}]^{<\aleph_0})^\omega$  by II played against  $\tau$ , we note that  $\alpha^* =$   
 126  $\bigcup \{(\alpha \upharpoonright n)^* : n < \omega\}$  is a counterattack to  $\sigma$ , and thus loses. This means  $B =$   
 127  $\bigcup \text{range}(\alpha^*) \notin \mathcal{B}$ .

128 We consider two cases. The first is the case that  $\bigcup \text{range}(\alpha^*)$  is finite. Noting  
 129 that  $\alpha^*(m) \cap \alpha^*(n) = \emptyset$  whenever  $m \neq n$ , there exists  $N < \omega$  such that  $\alpha^*(n) = \emptyset$   
 130 for all  $n > N$ . As a result,  $\bigcup \text{range}(\alpha) = \bigcup \text{range}(\alpha \upharpoonright n)$ , and thus  $\bigcup \text{range}(\alpha)$  is  
 131 finite, and therefore not in  $\mathcal{B}$ .

132 In the other case,  $\bigcup \text{range}(\alpha^*) \notin \mathcal{B}$  is an infinite subset of  $\bigcup \text{range}(\alpha)$ , and  
 133 therefore  $\bigcup \text{range}(\alpha) \notin \mathcal{B}$  as well. Thus we have shown that  $\tau$  is a winning strategy  
 134 for I in  $G_{fin}(\mathcal{A}, \mathcal{B})$ .  $\square$

135 We note that the above proof technique could be used to establish that perfect-  
 136 information and limited-information strategies for II in  $G_{fin}(\mathcal{A}, \mathcal{B})$  may be improved  
 137 to be valid in  $G_{<2}(\mathcal{A}, \mathcal{B})$ , provided  $\mathcal{B}$  is  $\Gamma$ -like. As such,  $G_{<2}(\mathcal{A}, \mathcal{B})$  and  $G_{fin}(\mathcal{A}, \mathcal{B})$   
 138 are effectively equivalent games under this hypothesis, so we will no longer consider  
 139  $G_{<2}(\mathcal{A}, \mathcal{B})$ .

### 140 3. PERFECT INFORMATION AND PREDETERMINED STRATEGIES

141 We now demonstrate the following, in the spirit of Pawlikowski's celebrated  
 142 result that a winning strategy for the first player in the Rothberger game may  
 143 always be improved to a winning predetermined strategy [cite pawlikowski].

144 **Theorem 14.** *Let  $\mathcal{A}$  be almost- $\Gamma$ -like and  $\mathcal{B}$  be  $\Gamma$ -like. Then*

- 145 •  $I \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$  if and only if  $I \uparrow \overset{pre}{G_{fin}}(\mathcal{A}, \mathcal{B})$ , and
- 146 •  $I \uparrow G_1(\mathcal{A}, \mathcal{B})$  if and only if  $I \uparrow \overset{pre}{G_1}(\mathcal{A}, \mathcal{B})$ .

147 *Proof.* We assume  $I \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$  and let the symbol  $\dagger$  mean  $< \aleph_0$  (respectively,  
 148  $I \uparrow G_1(\mathcal{A}, \mathcal{B})$  and  $\dagger = 1$ , and for convenience we assume II plays singleton subsets  
 149 of  $\mathcal{A}$  rather than elements). As  $\mathcal{A}$  is almost- $\Gamma$ -like, there is a winning strategy  $\sigma$

where  $|\sigma(s)| = \aleph_0$  and  $\sigma(s) \cap \bigcup \text{range}(s) = \emptyset$  (that is,  $\sigma$  never replays the choices of II) for all partial plays  $s$  by II.

For each  $s \in \omega^{<\omega}$ , suppose  $F_{s \upharpoonright m} \in [\bigcup \mathcal{A}]^\dagger$  is defined for each  $0 < m \leq |s|$ . Then let  $s^* : |s| \rightarrow [\bigcup \mathcal{A}]^\dagger$  be defined by  $s^*(m) = F_{s \upharpoonright m+1}$ , and define  $\tau' : \omega^{<\omega} \rightarrow \mathcal{A}$  by  $\tau'(s) = \sigma(s^*)$ . Finally, set  $[\sigma(s^*)]^\dagger = \{F_{s \cap \langle n \rangle} : n < \omega\}$ , and for some bijection  $b : \omega^{<\omega} \rightarrow \omega$  let  $\tau(n) = \tau'(b(n))$  be a predetermined strategy for I in  $G_{fin}(\mathcal{A}, \mathcal{B})$  (resp.  $G_1(\mathcal{A}, \mathcal{B})$ ).

Suppose  $\alpha$  is a counterattack by II against  $\tau$ , so

$$\alpha(n) \in [\tau(n)]^\dagger = [\tau'(b(n))]^\dagger = [\sigma(b(n)^*)]^\dagger$$

It follows that  $\alpha(n) = F_{b(n) \cap \langle m \rangle}$  for some  $m < \omega$ . In particular, there is some infinite subset  $W \subseteq \omega$  and  $f \in \omega^\omega$  such that  $\{\alpha(n) : n \in W\} = \{F_{f \upharpoonright n+1} : n < \omega\}$ . Note here that  $(f \upharpoonright n+1)^* = (f \upharpoonright n)^* \cap \langle F_{f \upharpoonright n+1} \rangle$ . This shows that  $F_{f \upharpoonright n+1} \in [\sigma((f \upharpoonright n)^*)]^\dagger$  is an attempt by II to defeat  $\sigma$ , which fails. Thus  $\bigcup \{F_{f \upharpoonright n+1} : n < \omega\} = \bigcup \{\alpha(n) : n \in W\} \notin \mathcal{B}$ , and since this set is infinite (as  $\sigma$  prevents II from repeating choices) we have  $\bigcup \{\alpha(n) : n < \omega\} \notin \mathcal{B}$  too. Therefore  $\tau$  is winning.  $\square$

Note that the assumption in Theorem 14 that  $\mathcal{A}$  be almost- $\Gamma$ -like cannot be omitted. In [todo cite Clontz k-tactics in Gruenhage game] an example of a space  $X^*$  and point  $\infty \in X^*$  where  $I \upharpoonright G_1(\mathcal{A}, \mathcal{B})$  but  $I \not\upharpoonright_{pre} G_1(\mathcal{A}, \mathcal{B})$  is given, where  $\mathcal{A}$  is the set of open neighborhoods of  $\infty$  (which are all uncountable), and  $\mathcal{B}$  is the set  $\Gamma_{X^*, \infty}$  of sequences converging to that point. (Note that  $G_1(\mathcal{A}, \mathcal{B})$  is called  $Gru_{O,P}(X^*, \infty)$  in that paper, and an equivalent game  $Gru_{K,P}(X)$  is what is directly studied. In fact, more is shown: I has a winning perfect-information strategy, but for any natural number  $k$ , any strategy that only uses the most recent  $k$  moves of II and the round number can be defeated.)

While  $\mathcal{A}$  is often not almost- $\Gamma$ -like in general, it may satisfy that property in combination with the selection principles being considered.

**Proposition 15.** *Let  $\mathcal{B}$  be  $\Gamma$ -like,  $\mathcal{B} \subseteq \mathcal{A}$ , and  $I \not\upharpoonright_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$ . Then  $\mathcal{A}$  is almost- $\Gamma$ -like.*

*Proof.* Let  $A \in \mathcal{A}$ , and for all  $n < \omega$  let  $A_n = A$ . Then  $A_n$  is not a winning predetermined strategy for I, so II may choose finite sets  $B_n \subseteq A_n = A$  such that  $A' = \bigcup \{B_n : n < \omega\} \in \mathcal{B} \subseteq \mathcal{A}$ .

It follows that  $A' \subseteq A$  and  $|A'| = \aleph_0$ , and for any infinite subset  $A'' \subseteq A'$  (in particular, any cofinite subset),  $A'' \in \mathcal{B} \subseteq \mathcal{A}$ . Thus  $\mathcal{A}$  is almost- $\Gamma$ -like.  $\square$

Note that in the previous result,  $I \not\upharpoonright_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$  could be weakened to the choice principle  $(\mathcal{A} \setminus \mathcal{B})$ : for every member of  $\mathcal{A}$ , there is some countable subset belonging to  $\mathcal{B}$ .

**Corollary 16.** *Let  $\mathcal{B}$  be  $\Gamma$ -like and  $\mathcal{B} \subseteq \mathcal{A}$ . Then*

- $I \upharpoonright G_{fin}(\mathcal{A}, \mathcal{B})$  if and only if  $I \upharpoonright_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$ , and
- $I \upharpoonright G_1(\mathcal{A}, \mathcal{B})$  if and only if  $I \upharpoonright_{pre} G_1(\mathcal{A}, \mathcal{B})$ .

*Proof.* Assuming  $I \not\upharpoonright_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$ , we have  $I \not\upharpoonright_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$  by Proposition 15 and Theorem 14.

Similarly, assuming  $I \not\vdash_{\text{pre}} G_1(\mathcal{A}, \mathcal{B}) \Rightarrow I \not\vdash_{\text{pre}} G_{fin}(\mathcal{A}, \mathcal{B})$ , we have  $I \not\vdash G_1(\mathcal{A}, \mathcal{B})$  by Proposition 15 and Theorem 14.  $\square$

This corollary generalizes e.g. Theorems 26 and 30 of [cite Scheepers 1996 Ramsey], Theorem 5 of [cite MR2119791], and Corollary 36 of [cite Clontz dual games].

In summary, using the selection principle notation  $S_*(\mathcal{A}, \mathcal{B})$ :

**Corollary 17.** *Let  $\mathcal{B}$  be  $\Gamma$ -like and  $\mathcal{B} \subseteq \mathcal{A}$ . Then*

- $I \not\vdash G_{fin}(\mathcal{A}, \mathcal{B})$  if and only if  $S_{fin}(\mathcal{A}, \mathcal{B})$  if and only if  $\alpha_2(\mathcal{A}, \mathcal{B})$ , and
- $I \not\vdash G_1(\mathcal{A}, \mathcal{B})$  if and only if  $S_1(\mathcal{A}, \mathcal{B})$  if and only if  $\alpha_4(\mathcal{A}, \mathcal{B})$ .

#### 4. DISJOINT SELECTIONS

In each  $\alpha_i(\mathcal{A}, \mathcal{B})$  principle, it is not required for the collection  $\{A_n : n < \omega\}$  to be pairwise disjoint. However, in many cases it may as well be.

**Definition 18.** For  $i \in \{1, 2, 3, 4\}$  let  $\alpha_{i,1}(\mathcal{A}, \mathcal{B})$  denote the claim that  $\alpha_i(\mathcal{A}, \mathcal{B})$  holds provided the collection  $\{A_n : n < \omega\}$  is pairwise disjoint.

Of course,  $\alpha_i(\mathcal{A}, \mathcal{B})$  implies  $\alpha_{i,1}(\mathcal{A}, \mathcal{B})$ . It's also immediate that  $\alpha_{i,1}(\mathcal{A}, \mathcal{B})$  implies  $\alpha_{i,1+1}(\mathcal{A}, \mathcal{B})$  for the same reason that  $\alpha_i(\mathcal{A}, \mathcal{B})$  implies  $\alpha_{i+1}(\mathcal{A}, \mathcal{B})$ .

We take advantage of the following lemma.

**Lemma 19** (Lemma 1.2 of (cite Nyikos 92)). *Given a family  $\{A_n : n < \omega\}$  of infinite sets, there exist infinite subsets  $A'_n \subseteq A_n$  such that  $\{A'_n : n < \omega\}$  is pairwise disjoint.*

**Proposition 20.** *Let  $\mathcal{A}$  be  $\Gamma$ -like. For  $i \in \{2, 3, 4\}$ ,  $\alpha_i(\mathcal{A}, \mathcal{B})$  is equivalent to  $\alpha_{i,1}(\mathcal{A}, \mathcal{B})$ .*

*Proof.* Assume  $\alpha_{i,1}(\mathcal{A}, \mathcal{B})$ . Let  $A_n \in \mathcal{A}$ . By applying the previous lemma, we have  $\{A'_n : n < \omega\}$  pairwise disjoint with each  $A'_n$  being an infinite subset of  $A_n$ . Since  $\mathcal{A}$  is  $\Gamma$ -like,  $A'_n \in \mathcal{A}$ , so we have a witness  $B \in \mathcal{B}$  such that  $A'_n \cap B$  satisfies  $\alpha_{i,1}(\mathcal{A}, \mathcal{B})$  for all  $n < \omega$ . Since  $A'_n \subseteq A_n$ , it follows that  $A_n \cap B$  satisfies  $\alpha_i(\mathcal{A}, \mathcal{B})$  for all  $n < \omega$ .  $\square$

It's also true that  $\alpha_1(\Gamma_{X,x}, \Gamma_{X,x})$  is equivalent to  $\alpha_{1,1}(\Gamma_{X,x}, \Gamma_{X,x})$ , which is captured by the following theorem.

**Proposition 21.** *Let  $\mathcal{A}$  be a  $\Gamma$ -like collection closed under finite unions. Then  $\alpha_1(\mathcal{A}, \mathcal{A})$  is equivalent to  $\alpha_{1,1}(\mathcal{A}, \mathcal{A})$ .*

*Proof.* Let  $A_n \in \mathcal{A}$  and assume  $\alpha_{1,1}(\mathcal{A}, \mathcal{A})$ . To apply the assumption, we will define a pairwise disjoint collection  $\{A'_n : n < \omega\}$ . First let  $0' = 0$  and  $A'_0 = A_0$ . Then suppose  $m' \geq m$  and  $A'_m \subseteq A_{m'} \subseteq \bigcup_{i \leq m} A'_i$  are defined for all  $m \leq n$ .

If  $A_k \setminus \bigcup_{m \leq n} A'_m$  is finite for  $k > n$ , let  $B = \bigcup_{m \leq n'} A'_m \in \mathcal{A}$ . This  $B$  then witnesses  $\alpha_1(\mathcal{A}, \mathcal{A})$  since  $A_k \setminus B$  is finite for all  $k < \omega$ .

Otherwise pick the minimal  $(n+1)' > n$  where  $A'_{n+1} = A_{(n+1)'} \setminus \bigcup_{m \leq n} A'_m$  is infinite. It follows that  $A'_{n+1} \subseteq A_{(n+1)'} \subseteq \bigcup_{m \leq n+1} A'_m$ . By construction,  $\{A'_n : n < \omega\}$  is a pairwise disjoint collection of members of  $\mathcal{A}$ , and we may apply  $\alpha_{1,1}(\mathcal{A}, \mathcal{A})$  to obtain  $B \in \mathcal{A}$  where  $A'_n \setminus B$  is finite for all  $n < \omega$ .

Finally let  $k < \omega$ . If  $k = n'$  for some  $n < \omega$ , then  $A_k \setminus B = A_{n'} \setminus B \subseteq (\bigcup_{m \leq n} A'_m) \setminus B$  is finite. Otherwise,  $n' < k < (n+1)'$  for some  $n < \omega$ . Then

231  $(A_k \setminus \bigcup_{m \leq n} A'_m) \setminus B \subseteq A_k \setminus \bigcup_{m \leq n} A'_m$  is finite, and  $(A_k \cap \bigcup_{m \leq n} A'_m) \setminus B \subseteq$   
 232  $(\bigcup_{m \leq n} A'_m) \setminus B$  is finite, showing  $A_k \setminus B$  is finite.  $\square$

233 TODO add discussion of  $\alpha_{1.5}$  here.

234 For this paragraph we adopt the conventional assumption that  $\Gamma_{X,x}$  is restricted  
 235 to countable sets. Nyikos showed a consistent example where  $\alpha_2(\Gamma_{X,x}, \Gamma_{X,x})$  fails  
 236 to imply  $\alpha_{1.5}(\Gamma_{X,x}, \Gamma_{X,x})$ , and a consistent example where  $\alpha_{1.5}(\Gamma_{X,x}, \Gamma_{X,x})$  fails  
 237 to imply  $\alpha_1(\Gamma_{X,x}, \Gamma_{X,x})$  [cite Nyikos 92]. On the other hand, Dow showed that  
 238  $\alpha_2(\Gamma_{X,x}, \Gamma_{X,x})$  implies  $\alpha_1(\Gamma_{X,x}, \Gamma_{X,x})$  in the Laver model for the Borel conjecture  
 239 [cite Dow 1990]; the author conjectures that this model (specifically, the fact that  
 240 every  $\omega$ -splitting family contains an  $\omega$ -splitting family of size less than  $\mathfrak{b}$  in this  
 241 model) witnesses an affirmative answer to the following question.

242 **Definition 22.** A  $\Gamma$ -like collection is *strongly- $\Gamma$ -like* if the collection is closed under  
 243 finite unions and each member is countable.

244 **Question 23.** Let  $\mathcal{A}, \mathcal{B}$  be strongly- $\Gamma$ -like and  $\mathcal{B} \subseteq \mathcal{A}$ . Is it consistent that  $\alpha_2(\mathcal{A}, \mathcal{B})$   
 245 implies  $\alpha_1(\mathcal{A}, \mathcal{B})$ ?

## 246 5. CONCLUSION

247 We conclude with the following easy result, and a couple questions.

248 **Proposition 24.** Let  $\mathcal{B}$  be  $\Gamma$ -like. Then  $\alpha_1(\mathcal{A}, \mathcal{B})$  holds if and only if  $\text{I} \not\Uparrow_{\text{pre}} G_{cf}(\mathcal{A}, \mathcal{B})$ .

249 *Proof.* We first assume  $\alpha_1(\mathcal{A}, \mathcal{B})$  and let  $A_n \in \mathcal{A}$  for  $n < \omega$  define a predetermined  
 250 strategy for I. By  $\alpha_1(\mathcal{A}, \mathcal{B})$ , we immediately obtain  $B \in \mathcal{B}$  such that  $|A_n \setminus B| < \aleph_0$ .  
 251 Thus  $B_n = A_n \cap B$  is a cofinite choice from  $A_n$ , and  $B' = \bigcup \{B_n : n < \omega\}$  is an  
 252 infinite subset of  $B$ , so  $B' \in \mathcal{B}$ . Thus II may defeat I by choosing  $B_n \subseteq A_n$  each  
 253 round, witnessing  $\text{I} \not\Uparrow_{\text{pre}} G_{cf}(\mathcal{A}, \mathcal{B})$ .

254 On the other hand, let  $\text{I} \not\Uparrow_{\text{pre}} G_{cf}(\mathcal{A}, \mathcal{B})$ . Given  $A_n \in \mathcal{A}$  for  $n < \omega$ , we note that  
 255 II may choose a cofinite subset  $B_n \subseteq A_n$  such that  $B = \bigcup \{B_n : n < \omega\} \in \mathcal{B}$ . Then  
 256  $B$  witnesses  $\alpha_1(\mathcal{A}, \mathcal{B})$  since  $|A_n \setminus B| \leq |A_n \setminus B_n| \leq \aleph_0$ .  $\square$

257 **Question 25.** Is there a game-theoretic characterization of  $\alpha_3(\mathcal{A}, \mathcal{B})$ ?

258 Noting that  $\text{I} \uparrow G_1(\Gamma_X, \Gamma_X)$  if and only if  $\text{I} \uparrow G_{fin}(\Gamma_X, \Gamma_X)$  [cite Kocinac], but  
 259 the same is not true of  $G_\star(\Gamma_{X,x}, \Gamma_{X,x})$  (i.e. there are  $\alpha_4$  spaces that are not  $\alpha_2$   
 260 [cite Shakhmatov convergence algebraic structure]), we also ask the following.

261 **Question 26.** Is there a natural condition on  $\mathcal{A}, \mathcal{B}$  guaranteeing  $\text{I} \uparrow G_1(\mathcal{A}, \mathcal{B}) \Rightarrow$   
 262  $\text{I} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ ?

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