PREDETERMINED PROXIMAL SPACES ARE METRIZABLE

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ABSTRACT. TODO

1. Predetermined Proximal

We take the following from Willard's text.

Definition 1.1. A *normal covering sequence* for a space X is a sequence $\{\mathcal{U}_n : n < \omega\}$ of open covers such that $\mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$. Such a sequence is *compatible* with X if $\{St(x,\mathcal{U}_n) : n < \omega\}$ is a local base at each $x \in X$.

Theorem 1.2. *A space X is psuedometrizable if and only if it has a compatible normal covering sequence.*

For convenience, we will recast these results in terms of entourages.

Definition 1.3. An *entourage sequence* for a space X $\{D_n : n < \omega\}$ is *compatible* with X if $\{D_n[x] : n < \omega\}$ is a local base at each $x \in X$.

Theorem 1.4. *A space X is psuedometrizable if and only if it has a compatible entourage sequence.*

Proof. Let d be a psuedometric generating X; then $\{D_n : n < \omega\}$ given by $D_n = \{\langle x,y\rangle : d(x,y) < 2^{-n}\}$ is a entourage sequence, which is compatible since $D_n[x] = B_{2^{-n}}(x)$.

On the other hand, given a compatible entourage sequence $\{D_n : n < \omega\}$, let $E_0 = D_0$, $E_{n+1} = \frac{1}{4}D_n \cap \frac{1}{4}E_n$, and $\mathcal{U}_n = \{E_{n+1}[x] : x \in X\}$. Fix $x \in X$ and consider $St(x,\mathcal{U}_n) \subseteq St(E_{n+1}[x],\mathcal{U}_n) = \bigcup\{E_{n+1}[y] \in \mathcal{U}_n : E_{n+1}[x] \cap E_{n+1}[y] \neq \emptyset\}$.

Let $z \in St(E_{n+1},\mathcal{U}_n)$, so $z \in E_{n+1}[y]$ for some $y \in X$ where $w \in E_{n+1}[x] \cap E_{n+1}[y]$ for some $w \in X$. It follows that $\langle z,y\rangle, \langle y,w\rangle, \langle w,x\rangle \in E_{n+1} = \frac{1}{4}D_n \cap \frac{1}{4}E_n$; therefore $\langle z,x\rangle \in D_n \cap E_n$ and $z \in D_n[x] \cap E_n[x]$. Therefore $St(x,\mathcal{U}_n) \subseteq St(E_{n+1}[x],\mathcal{U}_n) \subseteq D_n[x] \cap E_n[x]$.

We can then observe that $\mathcal{U}_{n+1}$ star refines $\mathcal{U}_n$ since for each $U \in \mathcal{U}_{n+1}$, $U = E_{n+2}[x]$ for some $x \in X$ and $St(E_{n+2}[x],\mathcal{U}_{n+1}) \subseteq E_{n+1}[x] \in \mathcal{U}_n$, making $\{\mathcal{U}_n : n < \omega\}$ a normal covering sequence. Finally, the sequence is compatible since for $x \in X$, $St(x,\mathcal{U}_n) \subseteq D_n[x]$ and $\{D_n[x] : n < \omega\}$ is a local base at x. $\square$

Theorem 1.5. *A space X is psuedometrizable if and only if* $\mathrm{I} \underset{pre}{\uparrow} Bell_{D,P}^{\rightarrow,\emptyset}(X)$.

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Proof. Suppose X is psuedometrizable by d; then let σ be the predetermined strategy for $Bell_{D,P}^{\rightarrow,\emptyset}(X)$ defined by $\sigma(n) = \{\langle x,y\rangle : d(x,y) < 2^{-n}\}$. For any legal attack α against σ, $\alpha(n+1) \in \sigma(n)[\alpha(n)]$. It follows that if $x \in \bigcap_{n<\omega} \sigma(n)[\alpha(n)]$ and $\epsilon > 0$, we may choose $N < \omega$ such that $2^{-N} < \epsilon$. Therefore $d(x, \alpha(n)) < 2^{-n} \leq 2^{-N} < \epsilon$ for all $n \geq N$, showing α converges to x. Thus σ is a winning strategy.

Now let σ be any predetermined winning strategy satisfying $\sigma(n) \subseteq \sigma(m)$ for all $n \geq m$, and suppose $\left\{\frac{1}{2^{n+1}}\sigma(n)[x] : n < \omega\right\}$ is not a local base at some $x \in X$. Then we may pick an entourage D such that $\frac{1}{2^{n+1}}\sigma(n)[x] \not\subseteq D[x]$ for all $n < \omega$. So choose $\alpha(n) \in \frac{1}{2^{n+1}}\sigma(n)[x] \setminus D[x]$.

Observe that $\langle \alpha(n), x\rangle \in \frac{1}{2^{n+1}}\sigma(n)$ and $\langle \alpha(n+1), x\rangle \in \frac{1}{2^{n+2}}\sigma(n+1) \subseteq \frac{1}{2^{n+1}}\sigma(n)$. It follows that $\langle \alpha(n), \alpha(n+1)\rangle \in \frac{1}{2^n}\sigma(n) \subseteq \sigma(n)$, witnessing that $\alpha(n+1) \in \sigma(n)[\alpha(n)]$, that is, α is a legal counterattack to σ. Since $x \in \frac{1}{2^{n+1}}\sigma(n)[\alpha(n)] \subseteq \sigma(n)[\alpha(n)]$ for all $n < \omega$, σ can only win for I if α converges. But $\alpha(n) \notin D[x]$ for all $n < \omega$, so α fails to converge as well. Thus σ is not a winning strategy.

As a result, if σ is a winning predetermined strategy, we have that $\left\{\frac{1}{2^{n+1}}\sigma(n)[x] : n < \omega\right\}$ is a local base at each $x \in X$. This shows that $\left\{\frac{1}{2^{n+1}}\sigma(n) : n < \omega\right\}$ is a compatible entourage sequence; therefore by the previous lemma, X is psuedometrizable. □

2. A Selectively Proximal Game and a Dual Proximal Game

Because the choices of $P2$ do not depend solely on the choices of $P1$ each round, the proximal game is not a selection game. However, it can be somewhat emulated as a selection game as follows.

Definition 2.1. Let $\mathcal{A}_X$ be the collection of basic neighborhood assignments $N : X \to \mathcal{T}_X$ such that $x \in N(x)$ (equivalently, $x \in U$ for each $\langle x, U\rangle \in N$), and let $\mathcal{PR}_X$ be the collection of countable sets of tuples $\{\langle x_n, U_n\rangle : n < \omega\}$ satisfying all of the following:

- For each $n < \omega$, $x_n \in U_m$ for cofinitely-many $m < \omega$
- $\{x_n : n < \omega\}$ fails to converge to any point of X
- $\bigcap \{U_n : n < \omega\} \neq \emptyset$

Then we call $G_1(\mathcal{A}_X, \mathcal{PR}_X)$ the *selectively proximal game.*

Theorem 2.2. *The selectively proximal game is equivalent to the proximal game when considering only paracompact spaces.*

Proof. TODO

Let σ be a winning predetermined strategy for $P1$ in the proximal game such that $\sigma(n) \subseteq \sigma(m)$ for all $n \geq m$, and define the neighborhood assignment $\tau(n)$ for $P1$ in the selectively proximal game by $\tau(n)(x) = \sigma(n)[x]$. Then consider when $P2$ responds to τ by $\langle x_n, \sigma(n)[x_n]\rangle$ during round $n < \omega$, and suppose that for each $n < \omega$, $x_n \in \sigma(m)[x_m]$ for cofinitely-many $m < \omega$. Pick some $S(n) > n$ such that $x_n \in U_{S(n)}$, and let $\langle y_n, V_n\rangle = \left\langle x_{S^n(0)}, U_{S^n(0)}\right\rangle$. □

Definition 2.3. Let $\mathcal{N}_X = \{N_x : x \in X\}$ where $N_x = \{\langle x, U\rangle : U \in \mathcal{T}_{X,x}\}$. Then we call $G_1(\mathcal{N}_X, \neg\mathcal{PR}_X)$ the *dual selectively proximal game.*

We should defend this nominclature.

Proposition 2.4. *$G_1(\mathcal{A}_X, \mathcal{PR}_X)$ is dual to $G_1(\mathcal{N}_X, \neg\mathcal{PR}_X)$.*

Proof. We proceed by showing that $\mathcal{N}_X$ is a reflection of $\mathcal{A}_X$; that is,

$$\mathcal{A}'_X = \{\text{range}(f) : f \in \mathbf{C}(\mathcal{N}_X)\}$$

is a selection basis for $\mathcal{A}_X$. To see this, first observe that for each $f \in \mathbf{C}(\mathcal{N}_X)$ and $x \in X$, $f(N_x) \in N_x$, so $f(N_x) = \langle x, U\rangle$ for some open neighborhood U of x. It follows that range$(f) \in \mathcal{A}_X$ and $\mathcal{A}'_X \subseteq \mathcal{A}_X$. Furthermore for each neighborhood assignment $N \in \mathcal{A}_X$, we may define $f_N \in \mathbf{C}(\mathcal{N}_X)$ by $f_N(N_x) = \langle x, N(x)\rangle$. It follows that range$(f_N) \subseteq N$ yielding our result, but in fact we have shown that $N = \text{range}(f_N) \in \mathcal{A}'_X$ and thus $\mathcal{A}'_X = \mathcal{A}_X$. □

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