DUAL SELECTION GAMES

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Abstract. Often, a given selection game studied in the literature has a known dual game. In dual games, a winning strategy for a player in either game may be used to create a winning strategy for the opponent in the dual. For example, the Rothberger selection game involving open covers is dual to the point-open game. This extends to a general theorem: if $\{\mathrm{range}(f) : f \in \mathbf{C}(\mathcal{R})\}$ is coinitial in $\mathcal{A}$ with respect to $\subseteq$, where $\mathbf{C}(\mathcal{R}) = \{f \in (\bigcup \mathcal{R})^{\mathcal{R}} : R \in \mathcal{R} \Rightarrow f(R) \in R\}$ collects the choice functions on the set $\mathcal{R}$, then $G_1(\mathcal{A}, \mathcal{B})$ and $G_1(\mathcal{R}, \neg\mathcal{B})$ are dual selection games.

1. Introduction

Definition 1. An *ω-length game* is a pair $G = \langle M, W\rangle$ such that $W \subseteq M^\omega$. The set M is the *moveset* of the game, and the set W is the *payoff set* for the second player.

In such a game G, players I and II alternate making choices $a_n \in M$ and $b_n \in M$ during each round $n < \omega$, and II wins the game if and only if $\langle a_0, b_0, a_1, b_1, \dots\rangle \in W$.

Often when defining games, I and II are restricted to choosing from different movesets A, B. Of course, this can be modeled with $\langle M, W\rangle$ by simply letting $M = A \cup B$ and adding/removing sequences from W whenever player I/II makes the first "illegal" move.

A class of such games heavily studied in the literature (see [7] and its many sequels) are selection games.

Definition 2. The *selection game* $G_1(\mathcal{A}, \mathcal{B})$ is an ω-length game involving Players I and II. During round n, I chooses $A_n \in \mathcal{A}$, followed by II choosing $B_n \in A_n$. Player II wins in the case that $\{B_n : n < \omega\} \in \mathcal{B}$, and Player I wins otherwise.

For brevity, let

$$G_1(\mathcal{A}, \neg\mathcal{B}) = G_1(\mathcal{A}, \mathcal{P}\left(\bigcup \mathcal{A}\right) \setminus \mathcal{B}).$$

That is, II wins in the case that $\{B_n : n < \omega\} \notin \mathcal{B}$, and I wins otherwise.

Definition 3. For a set X, let $\mathbf{C}(X) = \{f \in (\bigcup X)^X : x \in X \Rightarrow f(x) \in x\}$ be the collection of all choice functions on X.

Definition 4. Write $X \preceq Y$ if X is coinitial in Y with respect to $\subseteq$; that is, $X \subseteq Y$, and for all $y \in Y$, there exists $x \in X$ such that $x \subseteq y$.

In the context of selection games, we will say $\mathcal{A}'$ is a *selection basis* for $\mathcal{A}$ when $\mathcal{A}' \preceq \mathcal{A}$.

2010 *Mathematics Subject Classification.* 54C30, 54D20, 54D45, 91A44.

Key words and phrases. Selection principle, selection game, limited information strategies.

Definition 5. The set $\mathcal{R}$ is said to be a *reflection* of the set $\mathcal{A}$ if

$$\{\mathrm{range}(f) : f \in \mathbf{C}(\mathcal{R})\}$$

is a selection basis for $\mathcal{A}$.

Put another way, $\mathcal{R}$ is a reflection of $\mathcal{A}$ if for every $A \in \mathcal{A}$, there exists $f \in \mathbf{C}(\mathcal{R})$ such that $\mathrm{range}(f) \in \mathcal{A}$ and $\mathrm{range}(f) \subseteq A$.

As we will see, reflections of selection sets are used frequently (but implicitly) throughout the literature to define dual selection games.

We use the following conventions to describe strategies for playing games.

Definition 6. For $f \in B^A$ and $X \subseteq A$, let $f \restriction X$ be the restrction of f to X. In particular, for $f \in B^\omega$ and $n < \omega$, $f \restriction n$ describes the first n terms of the sequence f.

Definition 7. A *strategy* for the first player I (resp. second player II) in a game G with moveset M is a function $\sigma : M^{<\omega} \to M$. This strategy is said to be *winning* if for all possible *attacks* $\alpha \in M^\omega$ by their opponent, where $\alpha(n)$ is played by the opponent during round n, the player wins the game by playing $\sigma(\alpha \restriction n)$ (resp. $\sigma(\alpha \restriction n+1)$) during round n.

That is, a strategy is a rule that determines the moves of a player based upon all previous moves of the opponent. (It could also rely on all previous moves of the player using the strategy, since these can be reconstructed from the previous moves of the opponent and the strategy itself.)

Definition 8. A *predetermined strategy* for the first player I in a game G with moveset M is a function $\sigma : \omega \to M$. This strategy is said to be winning if for all possible attacks $\alpha \in M^\omega$ by their opponent, the first player wins the game by playing $\sigma(n)$ during round n.

So a predetermined strategy ignores all moves of the opponent during the game (all moves were decided before the game began).

Definition 9. A *Markov strategy* for the second player II in a game G with moveset M is a function $\sigma : M \times \omega \to M$. This strategy is said to be winning if for all possible attacks $\alpha \in M^\omega$ by their opponent, the first player wins the game by playing $\sigma(\alpha(n), n)$ during round n.

So a Markov strategy may only consider the most recent move of the opponent, and the current round number. Note that unlike perfect-information or predetermined strategies, a Markov strategy cannot use knowledge of moves used previously by the player (since they depend on previous moves of the opponent that have been "forgotten").

Definition 10. Write $\mathrm{I} \uparrow G$ (resp. $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} G$) if player I has a winning strategy (resp. winning predetermined strategy) for the game G. Similarly, write $\mathrm{II} \uparrow G$ (resp. $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} G$) if player II has a winning strategy (resp. winning Markov strategy) for the game G.

Of course, $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} G \Rightarrow \mathrm{II} \uparrow G \Rightarrow \mathrm{I} \not\uparrow G \Rightarrow \mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} G$. In general, none of these implications (not even the second [4]) can be reversed.

It's worth noting that $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_1(\mathcal{A},\mathcal{B})$ is equivalent to the selection principle often denoted $S_1(\mathcal{A},\mathcal{B})$ in the literature.

The goal of this paper is to characerize when two games are "dual" in the following senses.

Definition 11. A pair of games $G(X), H(X)$ defined for a topological space X are *Markov information dual* if both of the following hold.

- $I \underset{\text{pre}}{\uparrow} G(X)$ if and only if $II \underset{\text{mark}}{\uparrow} H(X)$.
- $II \underset{\text{mark}}{\uparrow} G(X)$ if and only if $I \underset{\text{pre}}{\uparrow} H(X)$.

Definition 12. A pair of games $G(X), H(X)$ defeind for a topological space X are *perfect information dual* if both of the following hold.

- $I \uparrow G(X)$ if and only if $II \uparrow H(X)$.
- $II \uparrow G(X)$ if and only if $I \uparrow H(X)$.

2. Main Results

The following four theorems demonstrate that reflections characterize dual selection games for both perfect information strategies and certain limited information strategies.

The duality of the Rothberger game $G_1(\mathcal{O}_X,\mathcal{O}_X)$ and the point-open game on X for perfect information strategies was first noted by Galvin in [5], and for Markov-information strategies by Clontz and Holshouser in [3]. These proofs may be generalized as follows.

Theorem 13. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*
Then $\mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{A},\mathcal{B})$ *if and only if* $\mathrm{II} \underset{mark}{\uparrow} G_1(\mathcal{R},\neg\mathcal{B})$.

Proof. Let σ witness $\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathcal{A},\mathcal{B})$. Since $\sigma(n) \in \mathcal{A}$, $\mathrm{range}(f_n) \subseteq \sigma(n)$ for some $f_n \in \mathbf{C}(\mathcal{R})$. So let $\tau(R,n) = f_n(R)$ for all $R \in \mathcal{R}$ and $n < \omega$. Suppose $R_n \in \mathcal{R}$ for all $n < \omega$. Note that since σ is winning and $\tau(R_n,n) = f_n(R_n) \in \mathrm{range}(f_n) \subseteq \sigma(n)$, $\{\tau(R_n,n) : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\mathrm{II} \underset{\text{mark}}{\uparrow} G_1(\mathcal{R},\neg\mathcal{B})$.

Now let σ witness $\mathrm{II} \underset{\text{mark}}{\uparrow} G_1(\mathcal{R},\neg\mathcal{B})$. Let $f_n \in \mathbf{C}(\mathcal{R})$ be defined by $f_n(R) = \sigma(R,n)$, and let $\tau(n) = \mathrm{range}(f_n) \in \mathcal{A}$. Suppose that $B_n \in \tau(n) = \mathrm{range}(f_n)$ for all $n < \omega$. Choose $R_n \in \mathcal{R}$ such that $B_n = f_n(R_n) = \sigma(R_n,n)$. Since σ is winning, $\{B_n : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathcal{A},\mathcal{B})$. □

Theorem 14. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*
Then $\mathrm{II} \underset{mark}{\uparrow} G_1(\mathcal{A},\mathcal{B})$ *if and only if* $\mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{R},\neg\mathcal{B})$.

Proof. Let σ witness $\mathrm{II} \underset{\text{mark}}{\uparrow} G_1(\mathcal{A},\mathcal{B})$. Let $n < \omega$. Suppose that for each $R \in \mathcal{R}$, there was $g(R) \in R$ such that for all $A \in \mathcal{A}$, $\sigma(A,n) \neq g(R)$. Then $g \in \mathbf{C}(\mathcal{R})$ and $\mathrm{range}(g) \in \mathcal{A}$, thus $\sigma(\mathrm{range}(g),n) \neq g(R)$ for all $R \in \mathcal{R}$, a contradiction.

So choose $\tau(n) \in \mathcal{R}$ such that for all $r \in \tau(n)$ there exists $A_{r,n} \in \mathcal{A}$ such that $\sigma(A_{r,n},n) = r$. It follows that when $r_n \in \tau(n)$ for $n < \omega$, $\{r_n : n < \omega\} = \{\sigma(A_{r_n,n}) : n < \omega\} \in B$, so τ witnesses $\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathcal{R},\neg\mathcal{B})$.

Now let σ witness $\mathrm{I} \underset{\text{pre}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$. Then $\sigma(n) \in \mathcal{R}$, so for $A \in \mathcal{A}$, let $f_A \in \mathbf{C}(\mathcal{R})$ satisfy $\mathrm{range}(f_A) \subseteq A$, and let $\tau(A, n) = f_A(\sigma(n)) \in A \cap \sigma(n)$. Then if $A_n \in \mathcal{A}$ for $n < \omega$, $\tau(A_n, n) \in \sigma(n)$, so $\{\tau(A_n, n) : n < \omega\} \in \mathcal{B}$. Thus τ witnesses $\mathrm{II} \underset{\text{mark}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. □

Theorem 15. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*
Then $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$ if and only if $\mathrm{II} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let σ witness $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$. Let $c(\emptyset) = \emptyset$. Suppose $c(s) \in (\bigcup A)^{<\omega}$ is defined for $s \in \mathcal{R}^{<\omega}$. Since $\sigma(c(s)) \in \mathcal{A}$, let $f_s \in \mathbf{C}(\mathcal{R})$ satisfy $\mathrm{range}(f_s) \subseteq \sigma(c(s))$, and let $c(s^\frown\langle R\rangle) = c(s)^\frown\langle f_s(R)\rangle$. Then let $c(\alpha) = \bigcup\{c(\alpha \restriction n) : n < \omega\}$ for $\alpha \in \mathcal{R}^\omega$, so

$$c(\alpha)(n) = f_{\alpha\restriction n}(\alpha(n)) \in \mathrm{range}(f_{\alpha\restriction n}) \subseteq \sigma(c(\alpha \restriction n))$$

demonstrating that $c(\alpha)$ is a legal attack against σ.

Let $\tau(s^\frown\langle R\rangle) = f_s(R)$. Consider the attack $\alpha \in \mathcal{R}^\omega$ against τ. Then since σ is winning and $\tau(\alpha \restriction n+1) = f_{\alpha\restriction n}(\alpha(n)) \in \mathrm{range}(f_{\alpha\restriction n}) \subseteq \sigma(c(\alpha \restriction n))$, it follows that $\{\tau(\alpha \restriction n+1) : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\mathrm{II} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$.

Now let σ witness $\mathrm{II} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$. For $s \in \mathcal{R}^{<\omega}$, define $f_s \in \mathbf{C}(\mathcal{R})$ by $f_s(R) = \sigma(s^\frown\langle R\rangle)$. Let $\tau(\emptyset) = \mathrm{range}(f_\emptyset) \in \mathcal{A}$, and for $x \in \tau(\emptyset)$, choose $R_{\langle x\rangle} \in \mathcal{R}$ such that $x = f_\emptyset(R_{\langle x\rangle})$ (for other $x \in \bigcup A$, choose $R_{\langle x\rangle}$ arbitrarily as it won't be used). Now let $s \in (\bigcup A)^{<\omega}$, and suppose $R_{s\restriction n^\frown\langle x\rangle} \in \mathcal{R}$ has been defined for $n \leq |s|$ and $x \in \bigcup A$. Then let $\tau(s^\frown\langle x\rangle) = \mathrm{range}(f_{\langle R_{s\restriction 0},\dots,R_s,R_{s^\frown\langle x\rangle}\rangle})$ and for $y \in \tau(s)$ choose $R_{s^\frown\langle x,y\rangle}$ such that $x = f_{\langle R_{s\restriction 0},\dots,R_s,R_{s^\frown\langle x\rangle}\rangle}(R_{s^\frown\langle x,y\rangle})$ (and again, choose $R_{s^\frown\langle x,y\rangle}$ arbitrarily for other $y \in \bigcup\mathcal{A}$ as it won't be used).

Then let α attack τ, so $\alpha(n) \in \tau(\alpha \restriction n)$ and thus $\alpha(n) = f_{\langle R_{\alpha\restriction 0},\dots,R_{\alpha\restriction n}\rangle}(R_{\alpha\restriction n+1}) = \sigma(\langle R_{\alpha\restriction 0}, \dots, R_{\alpha\restriction n+1}\rangle)$. Since σ is winning, $\{\sigma(\langle R_{\alpha\restriction 0}, \dots, R_{\alpha\restriction n+1}\rangle) : n < \omega\} = \{\alpha(n) : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$. □

Theorem 16. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*
Then $\mathrm{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$ if and only if $\mathrm{I} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let σ witness $\mathrm{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$. Let $s \in (\bigcup A)^{<\omega}$ and assume $a(s) \in \mathcal{A}^{|s|}$ is defined (of course, $a(\emptyset) = \emptyset$). Suppose for all $R \in \mathcal{R}$ there existed $f(R) \in R$ such that for all $A \in \mathcal{A}$, $\sigma(a(s)^\frown\langle A\rangle) \neq f(R)$. Then $f \in \mathbf{C}(\mathcal{R})$ and $\mathrm{range}(f) \in \mathcal{A}$, and thus $\sigma(a(s)^\frown\langle \mathrm{range}(f)\rangle) \neq f(R)$ for all $R \in \mathcal{R}$, a contradiction. So let $\tau(s) \in \mathcal{R}$ satisfy for all $x \in \tau(s)$ there exists $a(s^\frown\langle x\rangle) \in \mathcal{A}^{|s|+1}$ extending $a(s)$ such that $x = \sigma(a(s^\frown\langle x\rangle))$.

If τ is attacked by $\alpha \in (\bigcup R)^\omega$, then $\alpha(n) \in \tau(\alpha \restriction n)$. So $\alpha(n) = \sigma(a(\alpha \restriction n+1))$, and since σ is winning, $\{\sigma(a(\alpha \restriction n+1)) : n < \omega\} = \{\alpha(n) : n < \omega\} \in \mathcal{B}$. Therefore τ witnesses $\mathrm{I} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$.

Now let σ witness $\mathrm{I} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$. Let $s \in \mathcal{A}^{<\omega}$, and suppose $r(s) \in (\bigcup\mathcal{R})^{|s|}$ is defined (again, $r(\emptyset) = \emptyset$). For $A \in \mathcal{A}$ choose $f_A \in \mathbf{C}(\mathcal{R})$ where $\mathrm{range}(f_A) \subseteq A$, and let $\tau(s^\frown\langle A\rangle) = f_A(\sigma(r(s)))$, and let $r(s^\frown\langle A\rangle)$ extend $r(s)$ by letting $r(s^\frown\langle A\rangle)(|s|) = \tau(s^\frown\langle A\rangle)$.

If τ is attacked by $\alpha \in \mathcal{A}^\omega$, then since $\tau(\alpha \restriction n+1) = f_{\alpha(n)}(\sigma(r(\alpha \restriction n)) \in \alpha(n) \cap \sigma(r(\alpha \restriction n))$ and σ is winning, we conclude that τ is a legal strategy and $\{\tau(\alpha \restriction n+1) : n < \omega\} \in \mathcal{B}$. Therefore τ witnesses $\mathrm{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$. □

Corollary 17. *If $\mathcal{R}$ is a reflection of $\mathcal{A}$, then $G_1(\mathcal{A}, \mathcal{B})$ and $G_1(\mathcal{R}, \neg\mathcal{B})$ are both perfect information dual and Markov information dual.*

3. Applications of Reflections

Definition 18. Let X be a topological space and $\mathcal{T}_X$ be a chosen basis of nonempty sets for its topology.

- Let $\mathcal{T}_{X,x} = \{U \in \mathcal{T}_X : x \in U\}$ be the local point-base at $x \in X$.
- Let $\Omega_{X,x} = \{Y \subseteq X : \forall U \in \mathcal{T}_{X,x}(U \cap Y \neq \emptyset)\}$ be the fan at $x \in X$.
- Let $\mathcal{T}_{X,F} = \{U \in \mathcal{T}_X : F \subseteq U\}$ be the local finite-base at $F \in [X]^{<\aleph_0}$.
- Let $\mathcal{O}_X = \{\mathcal{U} \subseteq \mathcal{T}_X : \bigcup \mathcal{U} = X\}$ be the collection of basic open covers of X.
- Let $\mathcal{P}_X = \{\mathcal{T}_{X,x} : x \in X\}$ be the collection of local point-bases of X.
- Let $\Omega_X = \{\mathcal{U} \subseteq \mathcal{T}_X : \forall F \in [X]^{<\aleph_0} \exists U \in \mathcal{U}(F \subseteq U)\}$ be the collection of basic ω-covers of X.
- Let $\mathcal{F}_X = \{\mathcal{T}_{X,F} : F \in [X]^{<\aleph_0}\}$ be the collection of local finite-bases of X.
- Let $\mathcal{D}_X = \{Y \subseteq X : \forall U \in \mathcal{T}_X(U \cap Y \neq \emptyset)\}$ be the collection of dense subsets of X.
- Let $\Gamma_{X,x} = \{Y \subseteq X : \forall U \in \mathcal{T}_{X,x}(Y \setminus U \in [X]^{<\aleph_0})\}$ be the collection of converging fans at $x \in X$. (When intersected with $[X]^{\aleph_0}$, these are the non-trivial sequences of X converging to x.)

While these notions were defined in terms of a particular basis, the reader may verify the the following.

Proposition 19. *Let $\mathcal{A}'$ be a selection basis for $\mathcal{A}$.*

- $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathrm{I} \uparrow G_1(\mathcal{A}', \mathcal{B})$.
- $\mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{A}', \mathcal{B})$.
- $\mathrm{II} \uparrow G_1(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathrm{II} \uparrow G_1(\mathcal{A}', \mathcal{B})$.
- $\mathrm{II} \underset{mark}{\uparrow} G_1(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathrm{II} \underset{mark}{\uparrow} G_1(\mathcal{A}', \mathcal{B})$.

Proposition 20. *Each selection set in Definition 18 is a selection basis for the set defined by replacing $\mathcal{T}_X$ with the set of all nonempty open sets in X.*

As such, the choice of topological basis is irrelevant when playing selection games using these sets.

We may now establish (or re-establish) the following dual games.

Proposition 21. *$\mathcal{P}_X$ is a reflection of $\mathcal{O}_X$.*

Proof. For every open cover $\mathcal{U}$, the corresponding choice function $f \in \mathbf{C}(\mathcal{P}_X)$ is simply the witness that $x \in f(\mathcal{T}_{X,x}) \in \mathcal{U}$. □

Corollary 22. *$G_1(\mathcal{O}_X, \mathcal{B})$ and $G_1(\mathcal{P}_X, \neg\mathcal{B})$ are perfect-information and Markov-information dual.*

In the case that $\mathcal{B} = \mathcal{O}_X$, $G_1(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known Rothberger game, and $G_1(\mathcal{P}_X, \neg\mathcal{O}_X)$ is isomorphic to the point-open game $PO(X)$: I chooses points of X, II chooses an open neighborhood of each chosen point, and I wins if II's choices are a cover. So this was simply the classic result that the Rothberger game and point-open game are perfect-information dual [5], and the more recent result that these games are Markov-information dual [3].

Proposition 23. *$\mathcal{F}_X$ is a reflection of Ω_X.*

Proof. For every ω-cover $\mathcal{U}$, the corresponding choice function $f \in \mathbf{C}(\mathcal{F}_X)$ is simply the witness that $F \subseteq f(\mathcal{T}_{X,F}) \in \mathcal{U}$. $\square$

Corollary 24. *$G_1(\Omega_X, \mathcal{B})$ and $G_1(\mathcal{F}_X, \neg\mathcal{B})$ are perfect-information and Markov-information dual.*

Note that in the case that $\mathcal{B} = \Omega_X$, $G_1(\Omega_X, \Omega_X)$ is the Rothberger game played with ω-covers, and $G_1(\mathcal{F}_X, \neg\Omega_X)$ is isomorphic to the Ω-finite-open game $\Omega FO(X)$: I chooses finite subsets of X, II chooses an open neighborhood of each chosen finite set, and I wins if II's choices are an ω-cover. These games were shown to be dual in [3].

Proposition 25. *$\mathcal{T}_X$ is a reflection of $\mathcal{D}_X$.*

Proof. For every dense D, the corresponding choice function $f \in \mathbf{C}(\mathcal{T}_X)$ is simply the witness that $f(U) \in U \cap D$. $\square$

Corollary 26. *$G_1(\mathcal{D}_X, \mathcal{B})$ and $G_1(\mathcal{T}_X, \neg\mathcal{B})$ are perfect-information and Markov-information dual.*

In the case that $\mathcal{B} = \Omega_{X,x}$ for some $x \in X$, $G_1(\mathcal{D}_X, \Omega_{X,x})$ is the strong countable dense fan-tightness game at x, see e.g. [1]. $G_1(\mathcal{T}_X, \neg\Omega_{X,x})$ is the game $CL(X, x)$ first studied by Tkachuk in [10]. Tkachuk showed in that paper that these games are perfect-information dual; Clontz and Holshouser previously showed these were Markov-information dual in the case that $X = C_p(Y)$ [3].

In the case that $\mathcal{B} = D_X$, then $G_1(\mathcal{D}_X, \mathcal{D}_X)$ is the strong selective separability game introduced in [8], and $G_1(\mathcal{T}_X, \neg\mathcal{D}_X)$ is the point-picking game of Berner and Juhász defined in [2]. Scheepers showed that these were perfect-information dual in his paper.

Proposition 27. *$\mathcal{T}_{X,x}$ is a reflection of $\Omega_{X,x}$.*

Proof. For every set Y with limit point x, the corresponding choice function $f \in \mathbf{C}(\mathcal{T}_{X,x})$ is simply the witness that $f(U) \in U \cap Y$. $\square$

Corollary 28. *$G_1(\Omega_{X,x}, \mathcal{B})$ and $G_1(\mathcal{T}_{X,x}, \neg\mathcal{B})$ are perfect-information and Markov-information dual.*

In the case that $\mathcal{B} = \Gamma_{X,x}$ for some $x \in X$, $G_1(\mathcal{T}_{X,x}, \neg\Gamma_{X,x})$ is Gruenhage's W game [6]. Its dual $G_1(\Omega_{X,x}, \Gamma_{X,x})$ characterizes the strong Fréchet-Urysohn property $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_1(\Omega_{X,x}, \Gamma_{X,x})$ at x, which now seen to be equivalent to $\mathrm{II} \underset{\text{mark}}{\not\uparrow} G_1(\mathcal{T}_{X,x}, \neg\Gamma_{X,x})$. This allows us to obtain the following result.

Corollary 29. $\mathrm{I} \underset{pre}{\not\uparrow} G_1(\Omega_{X,x}, \Gamma_{X,x})$ *if and only if* $\mathrm{I} \not\uparrow G_1(\Omega_{X,x}, \Gamma_{X,x})$.

Proof. As shown in [9], a space is w at x, that is, $\mathrm{II} \not\uparrow G_1(\mathcal{T}_{X,x}, \neg\Gamma_{X,x})$ if and only if $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_1(\Omega_{X,x}, \Gamma_{X,x})$ for all $x \in X$. $\square$

For $\mathcal{B} = \Omega_{X,x}$, $G_1(\mathcal{T}_{X,x}, \neg\Omega_{X,x})$ is the variant of Gruenhage's W game for clustering. This game is now seen to be dual to the strong countable fan tightness game $G_1(\Omega_{X,x}, \Omega_{X,x})$ at x.

4. Open Questions

Question 30. *Does there exist a natural reflection for* $\Gamma_{X,x}$ *or* $\Gamma_X = \{\mathcal{U} \subseteq \mathcal{T}_X : \forall x \in X(\mathcal{U} \setminus \mathcal{T}_{X,x} \in [T_X]^{<\aleph_0})\}$*?*

Question 31. *Can these results be extended for* $G_{fin}(\mathcal{A}, \mathcal{B})$*?*

5. Acknowledgements

Thanks to Prof. Jared Holshouser for his input during the writing of these results.

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