RELATING GAMES OF MENGER, COUNTABLE FAN TIGHTNESS, AND SELECTIVE SEPARABILITY

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ABSTRACT. By adapting techniques of Arhangel'skii, Barman, and Dow, we may equate the existence of perfect-information, Markov, and tactical strategies between two interesting selection games. These results shed some light on Gruenhage's question asking whether all strategically selectively separable spaces are Markov selectively separable.

1. INTRODUCTION

Definition 1. The *selection principle* $S_{fin}(\mathcal{A}, \mathcal{B})$ states that given $A_n \in \mathcal{A}$ for $n < \omega$, there exist $B_n \in [A_n]^{<\omega}$ such that $\bigcup_{n<\omega} B_n \in \mathcal{B}$.

Definition 2. The *selection game* $G_{fin}(\mathcal{A}, \mathcal{B})$ is the analogous game to $S_{fin}(\mathcal{A}, \mathcal{B})$, where during each round $n < \omega$, Player I first chooses $A_n \in \mathcal{A}$, and then Player II chooses $B_n \in [A_n]^{<\omega}$. Player II wins in the case that $\bigcup_{n<\omega} B_n \in \mathcal{B}$, and Player I wins otherwise.

This game and property were first formally investigated by Scheepers in "Combinatorics of open covers" [8], which inspired a series of ten sequels with several co-authors. We may quickly observe that if II has a winning strategy for the game $G_{fin}(\mathcal{A}, \mathcal{B})$, then $S_{fin}(\mathcal{A}, \mathcal{B})$ will hold, but the converse need not follow.

The power of this selection principle and game comes from their ability to characterize several properties and games from the literature. Of interest to us are the following.

Definition 3. Let $\mathcal{O}_X$ be the collection of open covers for a topological space X. Then $S_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger property* for X (*M* for short), and $G_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger game*.

Definition 4. An ω-*cover* $\mathcal{U}$ for a topological space X is an open cover such that for every $F \in [X]^{<\omega}$, there exists some $U \in \mathcal{U}$ such that $F \subseteq U$.

Definition 5. Let Ω_X be the collection of ω-covers for a topological space X. Then $S_{fin}(\Omega_X, \Omega_X)$ is the Ω-*Menger property* for X (ΩM for short), and $G_{fin}(\Omega_X, \Omega_X)$ is the Ω-*Menger game*.

In [5, Theorem 3.9] it was shown that X is Ω-Menger if and only if X^n is Menger for all $n < \omega$.

Definition 6. Let $\mathcal{B}_{X,x}$ be the collection of subsets $A \subset X$ where $x \in \text{cl}\, A$. (Call A a *blade* of x.) Then $S_{fin}(\mathcal{B}_{X,x}, \mathcal{B}_{X,x})$ is the *countable fan tightness property* for X at x (CFT_x for short), and $G_{fin}(\mathcal{B}_{X,x}, \mathcal{B}_{X,x})$ is the *countable fan tightness game* for X at x.

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Definition 7. A space X has *countable fan tightness* (CFT for short) if it has countable fan tightness at each point $x \in X$.

Definition 8. Let $\mathcal{D}_X$ be the collection of dense subsets of a topological space X. (So, $\mathcal{D}_X \subseteq \mathcal{B}_{X,x}$ for all $x \in X$.) Then $S_{fin}(\mathcal{D}_X, \mathcal{B}_{X,x})$ is the *countable dense fan tightness property* for X at x ($CDFT_x$ for short), and $G_{fin}(\mathcal{D}_X, \mathcal{B}_{X,x})$ is the *countable dense fan tightness game* for X at x.

Definition 9. A space X has *countable dense fan tightness* ($CDFT$ for short) if it has countable dense fan tightness at each point $x \in X$.

Note that $CFT \Rightarrow CDFT$ for any space X as $\mathcal{D}_X \subseteq \mathcal{B}_{X,x}$.

The notion of countable fan tightness was first studied by by Arhangel'skii in [1]. A result of that paper showed that for $T_{3\frac{1}{2}}$ spaces X, the countable fan tightness of the space of real-vaued continuous functions with pointwise convergence $C_p(X)$ is characterized by the Ω-Menger property of X.

Definition 10. $S_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability property* for X (SS for short), and $G_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability game* for X.

Of course, one may easily observe that a selective separable space is separable. In [2] Barman and Dow demonstrated that all separable Frechet spaces are selectively separable. They were also able to produce a space which is selectively separable, but does not allow II a winning strategy in the selective separability game.

The object of this paper is to investigate the game-theoretic properties characterized by the presence of winning *limited information* strategies in these selection games.

Definition 11. A *strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function σ satisfying $\sigma(\langle A_0, \dots, A_n \rangle) \in [A_n]^{<\omega}$ for $\langle A_0 \dots, A_n \rangle \in \mathcal{A}^{n+1}$. We say this strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(\langle A_0, \dots, A_n \rangle)$ during each round $n < \omega$. If a winning strategy exists, then we write II $\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$.

Definition 12. A *Markov strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function σ satisfying $\sigma(A, n) \in [A_n]^{<\omega}$ for $A \in \mathcal{A}$ and $n < \omega$. We say this Markov strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(A_n, n)$ during each round $n < \omega$. If a winning Markov strategy exists, then we write II $\underset{\text{mark}}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$.

Notation 13. If $S_{fin}(\mathcal{A}, \mathcal{B})$ characterizes the property P, then we say II $\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes P^+ ("strategically P"), and II $\underset{\text{mark}}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes P^{+mark} ("Markov P"). Of course, $P^{+mark} \Rightarrow P^+ \Rightarrow P$.

In this notation, Barman and Dow showed that SS does not imply SS^+. Our goal is to make progress on the following question attributed to Gary Gruenhage:

Question 14. *Does SS^+ imply SS^{+mark}?*

The solution is already known to be "yes" in the context of countable spaces [2]. However in general, winning strategies in selection games cannot be improved to be winning Markov strategies. In [4] the author showed that while M^+ implies M^{+mark} for second-countable spaces, there exists a simple example of a regular non-second-countable space which is M^+ but not M^{+mark}.

2. CFT, $CDFT$ AND SS

We begin by generalizing the following result:

Theorem 15 (Lem 2.7 of [2])**.** *The following are equivalent for any topological space X.*

- *X is SS.*
- *X is separable and $CDFT$.*
- *X has a countable dense subset D where $CDFT_x$ holds for all $x \in D$.*

Theorem 16. *The following are equivalent for any topological space X.*

- *X is SS (resp. SS^+, SS^{+mark}).*
- *X is separable and $CDFT$ (resp. $CDFT^+$, $CDFT^{+mark}$).*
- *X has a countable dense subset D where $CDFT_x$ (resp. $CDFT_x^+$, $CDFT_x^{+mark}$) holds for all $x \in D$.*

Proof. We need only show that the final condition implies the first. Let $D = \{d_i : i < \omega\}$.

Let σ_i be a winning strategy witnessing $CDFT_{d_i}^+$ for each $i < \omega$. We define the strategy τ for the SS game by

$$\tau(\langle D_0, \ldots, D_n \rangle) = \bigcup_{i \leq n} \sigma_i(\langle D_i, \ldots, D_n \rangle).$$

By $CDFT_{d_i}^+$, we have

$$d_i \in \overline{\bigcup_{i \leq n < \omega} \sigma_i(\langle D_i, \ldots, D_n \rangle)} \subseteq \overline{\bigcup_{i \leq n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)} \subseteq \overline{\bigcup_{n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)}$$

and as $D \subseteq \overline{\bigcup_{n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)}$ it follows that

$$X \subseteq \overline{D} \subseteq \overline{\overline{\bigcup_{n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)}} = \overline{\bigcup_{n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)}.$$

Therefore τ witnesses SS^+.

The above proof may be easily modified for the Markov case by replacing $\sigma_i(\langle D_i, \ldots, D_n \rangle)$ with $\sigma_i(D_n, n)$ and $\tau(\langle D_0, \ldots, D_n \rangle)$ with $\tau(D_n, n)$. $\square$

So amongst separable spaces, we see that SS (resp. SS^+, SS^{+mark}) and $CDFT$ (resp. $CDFT^+$, $CDFT^{+mark}$) are equivalent. We now further bridge the gap between $CDFT$ and CFT in the context of function spaces. Consider the following result of Arhangel'skii.

Theorem 17 ([1])**.** *The following are equivalent for any $T_{3\frac{1}{2}}$ topological space X.*

- *X is ΩM.*
- *$C_p(X)$ is CFT.*

This result may similarly be generalized in a game theoretic sense. In addition, this proof will demonstrate the equivalence of CFT and $CDFT$ in $C_p(X)$. It is unknown to the author whether Arhangel'skii used a strategy similar to the following proof in [1], but Sakai employed a similar technique in [7] to relate the Ω-Rothberger and countable strong fan tightness properties (and essentially, the countable strong dense fan tightness property). Due to the difficulty in obtaining an English translation of [1], we reprove Arhangel'skii's theorem above in our more general context below.

Proposition 18. *Any witness for ΩM (resp. $\Omega M^+, \Omega M^{+mark}$) may be improved such that any final sequence of the chosen finite subcollections is an ω-cover.*

Proof. Consider any sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ of open covers. For ΩM, choose the witness $\langle \mathcal{F}_n^n, \mathcal{F}_{n+1}^n, \ldots \rangle$ for the final sequence $\langle \mathcal{U}_n, \mathcal{U}_{n+1}, \ldots \rangle$ for each $n < \omega$, and let $\mathcal{F}_n = \bigcup_{i \leq n} \mathcal{F}_n^i$. For ΩM^+, the winning strategy σ may be improved to τ where $\tau(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle) = \bigcup_{i \leq n} \sigma(\langle \mathcal{U}_i, \ldots, \mathcal{U}_n \rangle)$. For ΩM^{+mark}, the winning Markov strategy σ may be improved to τ where $\tau(\mathcal{U}, n) = \bigcup_{i \leq n} \sigma(\mathcal{U}, i)$. $\square$

Definition 19. Let X be a $T_{3\frac{1}{2}}$ topological space. For $\mathbf{x} \in C_p(X)$, $F \in [X]^{<\omega}$, and $\epsilon > 0$, let

$$[\mathbf{x}, F, \epsilon] = \{\mathbf{y} \in C_p(x) : |\mathbf{y}(t) - \mathbf{x}(t)| < \epsilon \text{ for all } t \in F\}$$

give a basic open neighborhood of $\mathbf{x}$.

Lemma 20. *Let X be a $T_{3\frac{1}{2}}$ topological space. If X is ΩM (resp. ΩM^+, ΩM^{+mark}). then $C_p(X)$ is $CFT_{\mathbf{0}}$ (resp. $CFT_{\mathbf{0}}^+$, $CFT_{\mathbf{0}}^{+mark}$).*

Proof. For each $B \in \mathcal{B}_{C_p(X),\mathbf{0}}$ define

$$\mathcal{U}_n(B) = \left\{ \mathbf{x}^{\leftarrow}\left[\left(-\frac{1}{2^n}, \frac{1}{2^n}\right)\right] : \mathbf{x} \in B \right\}.$$

Consider the finite set $F \in [X]^{<\omega}$. Since $\mathbf{0} \in \overline{B}$, choose $\mathbf{x} \in B \cap [\mathbf{0}, F, 1]$. It follows that $F \subseteq x^{\leftarrow}[(-1,1)] \in \mathcal{U}_n(B)$, so $\mathcal{U}_n(B)$ is an ω-cover of X.

Consider the sequence of blades $\langle B_0, B_1, \ldots \rangle \in \mathcal{B}_{C_p(X),\mathbf{0}}^\omega$, and the corresponding sequence of ω-covers $\langle \mathcal{U}_0(B_0), \mathcal{U}_1(B_1), \ldots \rangle \in \Omega_X^\omega$.

Assuming X is ΩM, choose a witness $\langle \mathcal{F}_0, \mathcal{F}_1, \ldots \rangle$ such that $\mathcal{F}_n \in [\mathcal{U}_n(B_n)]^{<\omega}$ and

$$\bigcup_{i \leq n < \omega} \mathcal{F}_n$$

is an ω-cover of X for all $i < \omega$. Now let

$$F_n = \left\{ \mathbf{x} \in B_n : \mathbf{x}^{\leftarrow}\left[\left(-\frac{1}{2^n}, \frac{1}{2^n}\right)\right] \in \mathcal{F}_n \right\}.$$

We claim that $\mathbf{0} \in \overline{\bigcup_{n<\omega} F_n}$. Let $G \in [X]^{<\omega}, \epsilon > 0$. Choose $i < \omega$ such that $\frac{1}{2^i} < \epsilon$ and then choose $i \leq n < \omega, \mathbf{x}^{\leftarrow}[(-\frac{1}{2^n}, \frac{1}{2^n})] \in \mathcal{F}_n$ such that $G \subseteq \mathbf{x}^{\leftarrow}[(-\frac{1}{2^n}, \frac{1}{2^n})]$ and therefore $G \subseteq \mathbf{x}^{\leftarrow}[(-\epsilon, \epsilon)]$. It follows that $\mathbf{x} \in F_n \cap [\mathbf{0}, G, \epsilon]$, and therefore every basic open neighborhood of $\mathbf{0}$ intersects $\bigcup_{n<\omega} F_n$.

Assuming X is ΩM^+, choose a witness σ such that

$$\bigcup_{i \leq n < \omega} \sigma(\langle \mathcal{U}_0(B_0), \ldots, \mathcal{U}_n(B_n) \rangle)$$

is an ω-cover of X for all $i < \omega$. Now let

$$\tau(\langle B_0, \ldots, B_n \rangle) = \left\{ \mathbf{x} \in B_n : \mathbf{x}^{\leftarrow}\left[\left(-\frac{1}{2^n}, \frac{1}{2^n}\right)\right] \in \sigma(\langle \mathcal{U}_0(B_0), \ldots, \mathcal{U}_n(B_n) \rangle) \right\}.$$

We claim that $\mathbf{0} \in \overline{\bigcup_{n<\omega} \tau(\langle B_0, \ldots, B_n \rangle)}$. Let $G \in [X]^{<\omega}, \epsilon > 0$. Choose $i < \omega$ such that $\frac{1}{2^i} < \epsilon$ and then choose $i \leq n < \omega, \mathbf{x}^{\leftarrow}[(-\frac{1}{2^n}, \frac{1}{2^n})] \in \sigma(\langle \mathcal{U}_0(B_0), \ldots, \mathcal{U}_n(B_n) \rangle)$ such that $G \subseteq \mathbf{x}^{\leftarrow}[(-\frac{1}{2^n}, \frac{1}{2^n})]$ and therefore $G \subseteq \mathbf{x}^{\leftarrow}[(-\epsilon, \epsilon)]$. It follows that $\mathbf{x} \in \tau(\langle B_0, \ldots, B_n \rangle) \cap [\mathbf{0}, G, \epsilon]$, and therefore every basic open neighborhood of $\mathbf{0}$ intersects $\bigcup_{n<\omega} \tau(\langle B_0, \ldots, B_n \rangle)$.

Assuming X is ΩM^{+mark}, choose a witness σ such that

$$\bigcup_{i \leq n < \omega} \sigma(\mathcal{U}_n(B_n), n)$$

is an ω-cover of X for all $i < \omega$. Now let

$$\tau(B_n, n) = \left\{ \mathbf{x} \in B_n : \mathbf{x}^\leftarrow \left[\left(-\frac{1}{2^n}, \frac{1}{2^n} \right) \right] \in \sigma(\mathcal{U}_n(B_n), n) \right\}.$$

We claim that $\mathbf{0} \in \overline{\bigcup_{n < \omega} \tau(B_n, n)}$. Let $G \in [X]^{<\omega}, \epsilon > 0$. Choose $i < \omega$ such that $\frac{1}{2^i} < \epsilon$ and then choose $i \leq n < \omega, \mathbf{x}^\leftarrow[(-\frac{1}{2^n}, \frac{1}{2^n})] \in \sigma(\mathcal{U}_n(B_n), n)$ such that $G \subseteq \mathbf{x}^\leftarrow[(-\frac{1}{2^n}, \frac{1}{2^n})]$ and therefore $G \subseteq \mathbf{x}^\leftarrow[(-\epsilon, \epsilon)]$. It follows that $\mathbf{x} \in \tau(B_n, n) \cap [\mathbf{0}, G, \epsilon]$, and therefore every basic open neighborhood of $\mathbf{0}$ intersects $\bigcup_{n < \omega} \tau(B_n, n)$. $\square$

Lemma 21. *Let X be a $T_{3\frac{1}{2}}$ topological space. If $C_p(X)$ is $CDFT_\mathbf{0}$ (resp. $CDFT_\mathbf{0}^+$, $CDFT_\mathbf{0}^{+mark}$), then X is ΩM (resp. ΩM^+, ΩM^{+mark}).*

Proof. For each $\mathcal{U} \in \Omega_X$ define

$$D(\mathcal{U}) = \{ \mathbf{y} \in C_p(X) : \mathbf{y}[X \setminus U_{\mathbf{y}, \mathcal{U}}] = \{1\} \text{ for some } U_{\mathbf{y}, \mathcal{U}} \in \mathcal{U} \}.$$

Consider the point $\mathbf{x} \in C_p(X)$ and its basic open neighborhood $[\mathbf{x}, G, \epsilon]$. If $\mathcal{U}$ is an ω-cover of X, $G \subseteq U$ for some $U_{\mathbf{y}, \mathcal{U}} \in \mathcal{U}$. Since X is $T_{3\frac{1}{2}}$, $X \setminus U_{\mathbf{y}, \mathcal{U}}$ is closed, and G is finite and disjoint from $X \setminus U_{\mathbf{y}, \mathcal{U}}$, we may choose some function $\mathbf{y} \in C_p(X)$ where $\mathbf{y}[X \setminus U_{\mathbf{y}, \mathcal{U}}] = \{1\}$ and $\mathbf{x}(t) = \mathbf{y}(t)$ for each $t \in G$. It follows $\mathbf{y} \in [\mathbf{x}, G, \epsilon] \cap D$, so $D(\mathcal{U})$ is dense in $C_p(X)$.

Consider the sequence of ω-covers $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle \in \Omega_X^\omega$, and the corresponding sequence of dense subsets $\langle D(\mathcal{U}_0), D(\mathcal{U}_1), \ldots \rangle \in \mathcal{D}_{C_p(X)}^\omega$.

Assuming $C_p(X)$ is $CDFT_\mathbf{0}$, choose a witness $\langle F_0, F_1, \ldots \rangle$ such that

$$\mathbf{0} \in \overline{\bigcup_{n < \omega} F_n}.$$

Now let

$$\mathcal{F}_n = \{ U_{\mathbf{y}, \mathcal{U}_n} : \mathbf{y} \in F_n \} \in [\mathcal{U}_n]^{<\omega}.$$

We claim that $\bigcup_{n < \omega} \mathcal{F}_n$ is an ω-cover. Let $G \in [X]^{<\omega}$. The neighborhood $[\mathbf{0}, G, \frac{1}{2}]$ contains some point $\mathbf{y} \in F_n$ for some $n < \omega$. It follows that $U_{\mathbf{y}, \mathcal{U}_n} \in \mathcal{U}_n$ and $\mathbf{y}[X \setminus U_{\mathbf{y}, \mathcal{U}_n}] = \{1\}$. It follows that $G \cap (X \setminus U_{\mathbf{y}, \mathcal{U}_n}) = \emptyset$, and therefore $G \subseteq U_{\mathbf{y}, \mathcal{U}_n} \in \mathcal{F}_n$.

Assuming $C_p(X)$ is $CDFT_\mathbf{0}^+$, choose a witness σ such that

$$\mathbf{0} \in \overline{\bigcup_{n < \omega} \sigma(\langle D(\mathcal{U}_0), \ldots, D(\mathcal{U}_n) \rangle)}.$$

Now let

$$\tau(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle) = \{ U_{\mathbf{y}, \mathcal{U}_n} : \mathbf{y} \in \sigma(\langle D(\mathcal{U}_0), \ldots, D(\mathcal{U}_n) \rangle) \} \in [\mathcal{U}_n]^{<\omega}.$$

We claim that $\bigcup_{n < \omega} \tau(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$ is an ω-cover. Let $G \in [X]^{<\omega}$. The neighborhood $[\mathbf{0}, G, \frac{1}{2}]$ contains some point $\mathbf{y} \in \sigma(\langle D(\mathcal{U}_0), \ldots, D(\mathcal{U}_n) \rangle)$ for some $n < \omega$. It follows that $U_{\mathbf{y}, \mathcal{U}_n} \in \mathcal{U}_n$ and $\mathbf{y}[X \setminus U_{\mathbf{y}, \mathcal{U}_n}] = \{1\}$. As a result $G \cap (X \setminus U_{\mathbf{y}, \mathcal{U}_n}) = \emptyset$, and therefore $G \subseteq U_{\mathbf{y}, \mathcal{U}_n} \in \tau(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$.

Assuming $C_p(X)$ is $CDFT_{\mathbf{0}}^{+mark}$, choose a witness σ such that

$$\mathbf{0} \in \overline{\bigcup_{n<\omega} \sigma(D(\mathcal{U}_n), n)}.$$

Now let

$$\tau(\mathcal{U}_n, n) = \{U_{\mathbf{y}, \mathcal{U}_n} : \mathbf{y} \in \sigma(D(\mathcal{U}_n), n)\} \in [\mathcal{U}_n]^{<\omega} S.$$

We claim that $\bigcup_{n<\omega} \tau(\mathcal{U}_n, n)$ is an ω-cover. Let $G \in [X]^{<\omega}$. The neighborhood $[\mathbf{0}, G, \frac{1}{2}]$ contains some point $\mathbf{y} \in \sigma(D(\mathcal{U}_n), n)$ for some $n < \omega$. It follows that $U_{\mathbf{y}, \mathcal{U}_n} \in \mathcal{U}_n$ and $\mathbf{y}[X \setminus U_{\mathbf{y}, \mathcal{U}_n}] = \{1\}$. As a result $G \cap (X \setminus U_{\mathbf{y}, \mathcal{U}_n}) = \emptyset$, and therefore $G \subseteq U_{\mathbf{y}, \mathcal{U}_n} \in \tau(\mathcal{U}_n, n)$. $\qquad\square$

Theorem 22. *The following are equivalent for any $T_{3\frac{1}{2}}$ topological space X.*

- *X is ΩM (resp. ΩM^+, ΩM^{+mark}).*
- *$C_p(X)$ is CFT (resp. CFT^+, CFT^{+mark}).*
- *$C_p(X)$ is $CDFT$ (resp. $CDFT^+$, $CDFT^{+mark}$).*

Proof. Since $\mathcal{D}_X \subseteq \mathcal{B}_{X,x}$, the second condition trivially implies the first. As $C_p(X)$ is homogeneous, the $C(D)FT$ properties follow from $C(D)FT_{\mathbf{0}}$. So the result follows from the previous lemmas. $\qquad\square$

3. A SPACE WHICH IS CFT^+ BUT NOT $CDFT^{+mark}$

Our goal is to now demonstrate a space which is CFT^+, but not even $CDFT^{+mark}$. To do so, we will find a space X which is ΩM^+ but not ΩM^{+mark}, yielding $C_p(X)$ as our example.

Proposition 23. *A space X is compact if and only if for each ω-cover $\mathcal{U}$ of X and $n < \omega$, there exists a finite subcollection $\mathcal{F} \in [\mathcal{U}]^{<\omega}$ such that for each $F \in [X]^{\leq n}$ there exists some $U \in \mathcal{F}$ where $F \subseteq U$.*

Proof. Let X and therefore X^n be compact. Let $\mathcal{F}_n$ be the finite subcover of the open cover $\mathcal{U}_n = \{U^n : U \in \mathcal{U}\}$. Then $\mathcal{F} = \{U : U^n \in \mathcal{F}_n\}$ witnesses our desired result. $\qquad\square$

Lemma 24. *The following are equivalent for a regular topological space X:*

- *X is σ-compact*
- *X is ΩM^{+mark}*
- *X is M^{+mark}*

Proof. The equivalence of σ-compact and M^{+mark} in regular spaces was shown in [4]. As ΩM^{+mark} trivially implies M^{+mark}, we need only demonstrate that if $X = \bigcup_{n<\omega} K_n$ for K_n compact and increasing, then X is ΩM^{+mark}.

We define $\sigma(\mathcal{U}, n)$ for each ω-cover $\mathcal{U}$ and $n < \omega$ to witness Proposition 23 for $\mathcal{U}$, K_n, and n. It follows that for every sequence of ω-covers $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ and $F \in [X]^{<\omega}$, $F \in [K_n]^{\leq n}$ for some $n \geq |F|$, and thus there exists some $U \in \sigma(\mathcal{U}_n, n)$ where $F \subseteq U$. Therefore $\bigcup_{n<\omega} \sigma(\mathcal{U}, n)$ is an ω-cover of X. $\qquad\square$

Definition 25. Let X be a topological space such that all countable sets are closed. Then $L(X) = X \cup \{\infty\}$ is the *one-point Lindelöf-ication* of X, with a basis given by the usual open sets of X and the co-countable sets containing ∞.

Theorem 26. *Let ω_1 have the discrete topology. Then $L(\omega_1)$ is ΩM^+ but not ΩM^{+mark}.*

Proof. The proof that $X = L(\omega_1)$ is not σ-compact (and therefore not ΩM^{+mark}) is simply the fact that its countably infinite subsets are closed and discrete. We define the strategy σ for II in $G_{fin}(\Omega_X, \Omega_X)$ as follows.

For $n < \omega$ let $\sigma(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle) = \{U(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)\}$, where

$$U(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle) = L(\omega_1) \setminus \{\alpha_{n,m} : m < \omega\}$$

is a co-countable set containing $\{\infty\} \cup \{\alpha_{i,j} : i, j < n\}$.

Consider the arbitrary sequence of moves $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ by I. For $F \in [L(\omega_1)]^{<\omega}$, choose $n < \omega$ such that

$$F \cap \{\alpha_{i,j} : i, j < \omega\} = F \cap \{\alpha_{i,j} : i, j < n\} \subseteq U(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$$

It follows that as

$$F \setminus \{\alpha_{i,j} : i, j < \omega\} \subseteq F \setminus \{\alpha_{n,m} : m < \omega\} \subseteq U(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$$

F is a subset of $U(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$, making $\bigcup_{n<\omega} \sigma(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$ an ω-cover. $\square$

Corollary 27. $C_p(L(\omega_1))$ *is* CFT^+ *but not* $CDFT^{+mark}$.

If $C_p(L(\omega_1))$ were separable, it would be a negative solution to Question 14. However, it is not.

Lemma 28 ([6]). *For a* $T_{3\frac{1}{2}}$ *topological space* X, $C_p(X)$ *is separable if and only if* X *has a coarser separable metrizable topology.*

Corollary 29. $C_p(L(\omega_1))$ *is not separable.*

Proof. Every metrizable topological space has points G_δ. However, if every neighborhood of ∞ is co-countable, $\{\infty\}$ cannot be the intersection of countably many open sets. $\square$

4. EQUIVALENCE OF CERTAIN STRATEGIC AND MARKOV SELECITON PROPERTIES

Barman and Dow previously demonstrated that SS^+ is equivalent to SS^{+mark} amongst countable spaces. A similar result by the author showed that M^+ is equivalent to M^{+mark} in second-countable spaces. The following result generalizes both.

Lemma 30. *Let* $\mathcal{A}$ *be a family of sets where* $|\bigcup \mathcal{A}| \leq \omega$ *and let* $\mathcal{B}$ *be a family of sets closed under supersets. Then* II $\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ *if and only if* II $\underset{mark}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$.

Proof. Let σ witness II $\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$.

For $t \in \omega^{<\omega}$, suppose $Z_s \in \mathcal{A}$ has been defined for all $s \leq t$. Note then that $\{\sigma(\langle Z_{t \upharpoonright 1}, Z_{t \upharpoonright 2}, \ldots, Z_t, A \rangle : A \in \mathcal{A}\} \subseteq [\bigcup \mathcal{A}]^{<\omega}$ and therefore is countable. So choose $Z_{t \frown \langle n \rangle} \in \mathcal{A}$ for $n < \omega$ such that $\{\sigma(\langle Z_{t \upharpoonright 1}, Z_{t \upharpoonright 2}, \ldots, Z_t, A \rangle : A \in \mathcal{A}\} = \{\sigma(\langle Z_{t \upharpoonright 1}, Z_{t \upharpoonright 2}, \ldots, Z_t, Z_{t \frown \langle n \rangle} \rangle)) : n < \omega\}$.

Let $b : \omega \to \omega^{<\omega}$ be a bijection, and define $\tau(A, n) = \sigma(\langle Z_{b(n) \upharpoonright 1}, Z_{b(n) \upharpoonright 2}, \ldots, Z_{b(n)}, A \rangle)$. Consider $\langle A_0, A_1, \ldots \rangle \in \mathcal{A}^\omega$.

Define $f \in \omega^\omega$ as follows. Let $n < \omega$ and suppose that $f \upharpoonright n$ has been already been defined. Then choose $f(n) < \omega$ such that $\sigma(\langle Z_{f \upharpoonright 1}, Z_{f \upharpoonright 2}, \ldots, Z_{f \upharpoonright n}, A_{b \leftarrow (f \upharpoonright n)} \rangle) = \sigma(\langle Z_{f \upharpoonright 1}, Z_{f \upharpoonright 2}, \ldots, Z_{f \upharpoonright n}, Z_{f \upharpoonright (n+1)} \rangle)$.

Since $\langle Z_{f\restriction 1}, Z_{f\restriction 2}, \ldots \rangle \in \mathcal{A}^\omega$, it follows that $\bigcup_{n<\omega} \sigma(\langle Z_{f\restriction 1}, Z_{f\restriction 2}, \ldots, Z_{f\restriction n}\rangle) \in \mathcal{B}$. The result then follows from

$$
\begin{aligned}
\bigcup_{n<\omega} \tau(A_n, n) &= \bigcup_{n<\omega} \sigma(\langle Z_{b(n)\restriction 1}, Z_{b(n)\restriction 2}, \ldots, Z_{b(n)}, A_n\rangle) \\
&\supseteq \bigcup_{n<\omega, b(n)<f} \sigma(\langle Z_{b(n)\restriction 1}, Z_{b(n)\restriction 2}, \ldots, Z_{b(n)}, A_n\rangle) \\
&= \bigcup_{n<\omega, b(n)<f} \sigma(\langle Z_{f\restriction 1}, Z_{f\restriction 2}, \ldots, Z_{f\restriction|b(n)|}, A_{b\leftarrow(f\restriction|b(n)|)}\rangle) \\
&= \bigcup_{n<\omega, b(n)<f} \sigma(\langle Z_{f\restriction 1}, Z_{f\restriction 2}, \ldots, Z_{f\restriction|b(n)|}, Z_{f\restriction(|b(n)|+1)}\rangle) \\
&= \bigcup_{n<\omega} \sigma(\langle Z_{f\restriction 1}, Z_{f\restriction 2}, \ldots, Z_{f\restriction n}\rangle) \\
&\in \mathcal{B}
\end{aligned}
$$

as $\mathcal{B}$ is closed under supersets. $\qquad\square$

Corollary 31 ([3]). *A countable space X is SS^+ if and only if it is SS^{+mark}.*

Proof. $\bigcup \mathcal{D}_X = X$ is countable, and any set containing a dense set is dense. $\qquad\square$

Corollary 32. *A second-countable space X is ΩM^+ if and only if it is ΩM^{+mark}.*

Proof. First note that any witness for ΩM^+ is a witness for $\text{II} \uparrow G_{fin}(\Omega_X^*, \Omega_X^*)$. Fix a countable base $\mathcal{B}$ for X closed under finite unions, and let Ω_X^* be the collection of ω-covers of X which only use basic open sets. Then $\bigcup \Omega_X^* = \mathcal{B}$ is countable, and any set containing an ω-cover is an ω-cover; therefore $\text{II} \uparrow G_{fin}(\Omega_X^*, \Omega_X^*)$ if and only if $\text{II} \underset{\text{mark}}{\uparrow} G_{fin}(\Omega_X^*, \Omega_X^*)$.

So let σ witness $\text{II} \underset{\text{mark}}{\uparrow} G_{fin}(\Omega_X^*, \Omega_X^*)$. For each ω-cover $\mathcal{U}$ and finite set $F \in [X]^{<\omega}$, choose $U_{\mathcal{U},F} \in \mathcal{U}$ such that $F \subseteq U$. Then choose a basic open set $U_{\mathcal{U},F}^*$ such that $F \subseteq U_{\mathcal{U},F}^* \subseteq U_{\mathcal{U},F}$. Finally, for each ω-cover $\mathcal{U}$, let

$$\mathcal{U}^* = \{U_{\mathcal{U},F}^* : F \in [X]^{<\omega}\} \in \Omega_X^*.$$

Define the strategy τ such that $\tau(\mathcal{U}, n) = \{U_{\mathcal{U},F} : U_{\mathcal{U},F}^* \in \sigma(\mathcal{U}^*, n)\}$. For any sequence of ω-covers $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$, it follows that $\bigcup_{n<\omega} \sigma(\mathcal{U}_n^*, n)$ is an ω-cover, and therefore $\bigcup_{n<\omega} \tau(\mathcal{U}_n, n)$ is an ω-cover also. $\qquad\square$

Corollary 33 ([4]). *A second-countable space X is M^+ if and only if it is M^{+mark}.*

Proof. An easy adapatation of the preceding proof, replacing ω-covers with open covers, and replacing $F \in [X]^{<\omega}$ with $x \in X$. $\qquad\square$

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