Limited Information Strategies for Topological Games

by

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Abstract

I talk a lot about topological games.

TODO: Write this.

Acknowledgments

TODO: Thank people.

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Chapter 1

Introduction

Basic overview of combinatorial games, topological games, limited info strategies, and applications in topology.

Chapter 2

Toplogical Games and Strategies

of Perfect and Limited Information

The goal of this paper is to explore the applications of limited information strategies in existing topological games. There are a variety of frameworks for modeling such games, so we establish one within this chapter which we will use for this manuscript.

2.1 Games

Intuitively, the games studied in this paper are two-player games for which each player takes turns making a choice from a set of possible moves. At the conclusion of the game, the choices made by both players are examined, and one of the players is declared the winner of that playthrough.

Games may be modeled mathematically in various ways, but we will find it convenient to think of them in terms defined by Gale and Stewart. [3]

Definition 1. A *game* is a tuple $\langle M, W\rangle$ such that $W \subseteq M^\omega$. M is set of *moves* for the game, and M^ω is the set of all possible *playthroughs* of the game.

W is the set of *winning playthroughs* or *victories* for the first player, and $M^\omega \setminus W$ is the set of victories for the second player. (W is often called the *payoff set* for the first player.)

Within this model, we may imagine two players $\mathscr{A}$ and $\mathscr{B}$ playing a game which consists of *rounds* enumerated for each $n < \omega$. During round n, $\mathscr{A}$ chooses $a_n \in M$, followed by $\mathscr{B}$ choosing $b_n \in M$. The playthrough corresponding to those choices would be the sequence $p = \langle a_0, b_0, a_1, b_1, \ldots\rangle$. If $p \in W$, then $\mathscr{A}$ is the winner of that playthrough, and if $p \notin W$, then $\mathscr{B}$ is the winner. Note that no ties are allowed.

Rather than explicitly defining W, we typically define games by declaring the *rules* that each player must follow and the *winning condition* for the first player. Then a playthrough is in W if either the first player made only *legal moves* which observed the game's rules and the playthrough satisified the winning condition, or the second player made an *illegal move* which contradicted the game's rules.

As an illustration, we could model a game of chess (ignoring stalemates) by letting

$$M = \{\langle p, s\rangle : p \text{ is a chess piece and } s \text{ is a space on the board } \}$$

representing moving a piece p to the space s on the board. Then the rules of chess restict White from moving pieces which belong to Black, or moving a piece to an illegal space on the board.[1] The winning condition could then "inspect" the resulting positions of pieces on the board after each move to see if White attained a checkmate. This winning condition along with the rules implicitly define the set W of winning playthroughs for White.

2.1.1 Infinite and Topological Games

Games never technically end within this model, since playthroughs of the game are infinite sequences. However, for all practical purposes many games end after a finite number of turns.

Definition 2. A game is said to be an *finite game* if for every playthrough $p \in M^\omega$ there exists a round $n < \omega$ such that $[p \upharpoonright n] = \{q \in M^\omega : q \supseteq p \upharpoonright n\}$ is a subset of either W or $M^\omega \setminus W$.

Put another way, a finite game is decided after a finite number of rounds, after which the game's winner could not change even if further rounds were played. Games which are not finite are called *infinite games*.

[1] In practice, M is often defined as the union of two sets, such as white pieces and black pieces in chess. For example, the first player may choose open sets in a topology, while the second player chooses points within the topological space.

As an illustration of an infinite game, we may consider a simple example due to Baker [1].

Game 3. Let $Lim_{A,B}(X)$ denote a game with players $\mathscr{A}$ and $\mathscr{B}$, defined for each subset $A \subset \mathbb{R}$. In round 0, $\mathscr{A}$ chooses a number a_0, followed by $\mathscr{B}$ choosing a number b_0 such that $a_0 < b_0$. In round $n+1$, $\mathscr{A}$ chooses a number a_{n+1} such that $a_n < a_{n+1} < b_n$, followed by $\mathscr{B}$ choosing a number b_{n+1} such that $a_{n+1} < b_{n+1} < b_n$.

$\mathscr{A}$ wins the game if the sequence $\langle a_n : n < \omega \rangle$ limits to a point in X, and $\mathscr{B}$ wins otherwise.

Certainly, $\mathscr{A}$ and $\mathscr{B}$ will never be in a position without (infinitely many) legal moves available, and provided that A is non-trivial, there is a playthrough such that for all $n < \omega$, the segment (a_n, b_n) intersects both A and $\mathbb{R} \setminus A$. Such a playthrough could never be decided in a finite number of moves, so the winning condition considers the infinite sequence of moves made by the players and declares a victor at the "end" of the game.

Definition 4. A *topological game* is a game defined in terms of an arbitrary topological space.

Topological games are usually infinite games, ignoring trivial examples. One of the earliest examples of a topological game is the Banach-Mazur game, proposed by Stanislaw Mazur as Problem 43 in Stefan Banach's Scottish Book (1935). A more comprehensive history of the Banacy-Mazur and other topological games may be found in Telgarsky's survey on the subject [7].

The original game was defined for subsets of the real line; however, we give a more general definition here.

Game 5. Let $Empty_{E,N}(X)$ denote the *Banach-Mazur game* with players $\mathscr{E}$, $\mathscr{N}$ defined for each topological space X. In round 0, $\mathscr{E}$ chooses a nonempty open set $E_0 \subseteq X$, followed by $\mathscr{N}$ choosing a nonempty open subset $N_0 \subseteq E_0$. In round $n+1$, $\mathscr{E}$ chooses a nonempty open subset $E_{n+1} \subseteq N_n$, followed by $\mathscr{N}$ choosing a nonempty open subset $N_{n+1} \subseteq E_{n+1}$.

$\mathscr{E}$ wins the game if $\bigcap_{n<\omega} E_n = \emptyset$, and $\mathscr{N}$ wins otherwise.

For example, if X is a locally compact Hausdorff space, $\mathscr{N}$ can "force" a win by choosing N_0 such that $\overline{N_0}$ is compact, and choosing N_{n+1} such that $N_{n+1} \subseteq \overline{N_{n+1}} \subseteq O_{n+1} \subseteq N_n$ (possible since N_n is a compact Hausdorff $\Rightarrow$ normal space). Since $\bigcap_{n<\omega} E_n = \bigcap_{n<\omega} N_n$ is the decreasing intersection of compact sets, it cannot be empty.

This concept of when (and how) a player can "force" a win in certain topological games is the focus of this manuscript.

2.2 Strategies

We shall make the notion of forcing a win in a game rigorous by introducing "strategies" and "attacks" for games.

Definition 6. A *strategy* for a game $G = \langle M, W \rangle$ is a function from $M^{<\omega}$ to M.

Definition 7. An *attack* for a game $G = \langle M, W \rangle$ is a function from ω to M.

Intuitively, a strategy is a rule for one of the players on how to play the game based upon the previous (finite) moves of her opponent, while an attack is a fixed strike by an opponent indexed by round number.

Definition 8. The *result* of a game given a strategy σ for the first player and an attack $\langle a_0, a_1, \dots \rangle$ by the second player is the playthrough

$$\langle \sigma(\emptyset), a_0, \sigma(\langle a_0 \rangle), a_1, \sigma(\langle a_0, a_1 \rangle), \dots \rangle$$

Likewise, if σ is a strategy for the second player, and $\langle a_0, a_1, \dots \rangle$ is an attack by the first player, then the result is the playthrough

$$\langle a_0, \sigma(\langle a_0 \rangle), a_1, \sigma(\langle a_0, a_1 \rangle), \dots \rangle$$

We now may rigorously define the notion of "forcing" a win in a game.

Definition 9. A strategy σ is a *winning strategy* for a player if for every attack by the opponent, the result of the game is a victory for that player.

If a winning strategy exists for a player $\mathscr{A}$ in the game G, then we write $\mathscr{A} \uparrow G$. Otherwise, we write $\mathscr{A} \not\uparrow G$.

To show that a winning strategy exists for a player (i.e. $\mathscr{A} \uparrow G$), we typically begin by defining it and showing that it is *legal*: it only yields moves which are legal according to the rules of the game. Then, we consider an arbitrary legal attack, and prove that the result of the game is a victory for that player.

If we wish to show that a winning strategy does not exist for a player (i.e. $\mathscr{A} \not\uparrow G$), we often consider an arbitrary legal strategy, and use it to define a legal *counter-attack* for the opponent. If we can prove that the result of the game for that strategy and counter-attack is a victory for the opponent, then a winning strategy does not exist.

Unlike finite games, is not the case that a winning strategy must exist for one of the players in an infinite game.

Definition 10. A game G with players $\mathscr{A}$, $\mathscr{B}$ is said to be *determined* if either $\mathscr{A} \uparrow G$ or $\mathscr{B} \uparrow G$. Otherwise, the game is *undetermined*.

The Borel Determinacy Theorem states that $G = \langle M, W \rangle$ is determined whenever W is a Borel subset of M^ω [5]. It's an easy corollary that all finite games are determined; W must be clopen.

However, as stated earlier, most topological games are infinite, and many are undetermined for certain spaces constructed using the Axiom of Choice.[2]

[2]These spaces cannot be constructed just only the axioms of ZF. In fact, mathematicians have studied an Axiom of Determinacy which declares that that all Gale-Stewart games are determined (and implies that the Axiom of Choice is false). [6]

2.2.1 Applications of Strategies

The power of studying these infinite-length games can be illustrated by considering the following proposition.

Proposition 11. *If X is countable, then $\mathscr{B} \uparrow Lim_{A,B}(X)$.*

Proof. Adapted from [1]. Let $X = \{x_0, x_1, \dots\}$. Let $i(a, b)$ be the least integer such that $a < x_{i(x,y)} < b$, if it exists. We define a strategy σ for $\mathscr{B}$ such that:

- $\sigma(\langle a_0 \rangle) = x_{i(a_0,\infty)}$. If $i(a_0, \infty)$ does not exist, then the choice of $\sigma(\langle a_0 \rangle)$ is arbitary, say, $a_0 + 1$.

- $\sigma(\langle a_0, ..., a_{n+1} \rangle) = x_{i(a_{n+1},b_n)}$, where $b_n = \sigma(\langle a_0, \dots, a_n \rangle)$. If $i(a_{n+1}, b_n)$ does not exist, then the choice of $\sigma(\langle a_0, ..., a_{n+1} \rangle)$ is arbitrary, say, $\sigma(\langle a_0, ..., a_{n+1} \rangle) = \frac{a_{n+1}+b_n}{2}$.

Observe that σ is a legal strategy according to the rules of the game since $a_0 < \sigma(\langle a_0 \rangle)$ and $a_{n+1} < \sigma(\langle a_0, \dots, a_{n+1} \rangle) < b_n$. We claim this is a winning strategy for $\mathscr{B}$. Let $a = \langle a_0, a_1, \dots \rangle$ be a legal attack by $\mathscr{A}$ against σ: we will show that the resulting playthrough is a victory for $\mathscr{B}$, that is, $\lim_{n\to\infty} a_n \notin X$. Let $b_n = \sigma(\langle a_0, \dots, a_n \rangle)$. Note that

$$a_0 < a_1 < \dots < \lim_{n\to\infty} a_n < \dots < b_1 < b_0$$

If $i(a_0, \infty)$ does not exist, then a_0 is greater than every element of X, and thus $\lim_{n\to\infty} a_n \notin X$. A similar argument follows if some $i(a_{n+1}, b_n)$ does not exist.

Otherwise,

$$i(a_0, \infty) < i(a_1, b_0) < i(a_2, b_1) < \dots$$

and for each $i < \omega$, one of the following must hold.

- $i < i(a_0, \infty)$. Then $x_i \leq a_0 < \lim_{n\to\infty} a_n$.

- $i = i(a_0, \infty)$. Then $x_i = b_0 > \lim_{n\to\infty} a_n$.

- $i(a_0, \infty) < i < i(a_1, b_0)$. Then $x_i \leq a_1 < \lim_{n\to\infty} a_n$ or $x_i \geq b_0 > \lim_{n\to\infty} a_n$.
- $i = i(a_{n+1}, b_n)$ for some $n < \omega$. Then $x_i = b_{n+1} > \lim_{n\to\infty} a_n$.
- $i(a_{n+1}, b_n) < i < i(a_{n+2}, b_{n+1})$ for some $n < \omega$. Then $x_i \leq a_{n+2} < \lim_{n\to\infty} a_n$ or $x_i \geq b_{n+1} > \lim_{n\to\infty} a_n$.

In any case, $x_i \neq \lim_{n\to\infty} a_n$, and thus $\lim_{n\to\infty} a_n \notin X$. □

More informally, $\mathscr{B}$ can force a win by enumerating the countable set X and playing every legal choice by the end of the game. This yields a classical result.

Corollary 12. $\mathbb{R}$ *is uncountable.*

Proof. $\mathscr{A} \uparrow Lim_{A,B}(\mathbb{R})$, since a_n must converge to some real number. This implies $\mathscr{B} \not\uparrow Lim_{A,B}(\mathbb{R})$, and thus $\mathbb{R}$ is not countable. □

Infinite games thus provide a rich framework for considering questions in set theory and topology. In general, the presence or absence of a winning strategy for a player in a topological game characterizes a property of the topological space in question.

Theorem 13. $\mathscr{E} \not\uparrow Empty_{E,N}(X)$ *if and only if* X *is a Baire space. [4]*

2.2.2 Limited Information Strategies

So far we have assumed both players enjoy *perfect information*, and may develop strategies which use all of the previous moves of the opponent as input.

Definition 14. For a game $G = \langle M, W\rangle$, the *k-tactical fog-of-war* is the function $\nu_k : M^{<\omega} \to M^{\leq k}$ defined by

$$\nu_k(\langle m_0, \ldots, m_{n-1}\rangle) = \langle m_{n-k}, \ldots, m_{n-1}\rangle$$

and the *k-Marköv fog-of-war* is the function $\mu_k : M^{<\omega} \to (M^{\leq k} \times \omega)$ defined by

$$\mu_k(\langle m_0, \ldots, m_{n-1} \rangle) = \langle \langle m_{n-k}, \ldots, m_{n-1} \rangle, n \rangle$$

Essentially, these fogs-of-war represent a limited memory: ν_k filters out all but the last k moves of the opponent, and μ_k filters out all but the last k moves of the opponent and the round number.

We call strategies which do not require full recollection of the opponent's moves *limited information strategies*.

Definition 15. A *k-tactical strategy* or *k-tactic* is a limited information strategy of the form $\sigma \circ \nu_k$.

A *k-Marköv strategy* or *k-mark* is a limited information strategy of the form $\sigma \circ \mu_k$.

k-tactics and k-marks may then only use the last k moves of the opponent, and in the latter case, also the round number.

The k is usually omitted when $k = 1$. A (1-)tactic is called a *stationary strategy* by some authors. 0-tactics are not usually interesting (such strategies would be constant functions); however, we will discuss 0-Marköv strategies, called *predetermined strategies* since such a strategy only uses the round number and does not rely on knowing which moves the opponent will make.

Definition 16. If a winning k-tactical strategy exists for a player $\mathscr{A}$ in the game G, then we write $\mathscr{A} \underset{k\text{-tact}}{\uparrow} G$. If $k = 1$, then $\mathscr{A} \underset{\text{tact}}{\uparrow} G$.

If a winning k-Marköv strategy exists for a player $\mathscr{A}$ in the game G, then we write $\mathscr{A} \underset{k\text{-mark}}{\uparrow} G$. If $k = 1$, then $\mathscr{A} \underset{\text{mark}}{\uparrow} G$, and if $k = 0$, then $\mathscr{A} \underset{\text{pre}}{\uparrow} G$.

The existence of a winning limited information strategy can characterize a stronger property than the property characterized by a perfect information strategy.

Definition 17. X is an *α-favorable space* when $\mathscr{N} \underset{\text{tact}}{\uparrow} \mathit{Empty}_{E,N}(X)$. X is a *weakly α-favorable space* when $\mathscr{N} \uparrow \mathit{Empty}_{E,N}(X)$.

Observation 18. *X is α-favorable $\Rightarrow$ X is weakly α-favorable $\Rightarrow$ X is Baire*

Those arrows may not be reversed. A Bernstein subset of the real line is an example of a Baire space which is not weakly α-favorable, and Gabriel Debs constructed an example of a completely regular space for which $\mathscr{N}$ has a winning 2-tactic, but lacks a winning 1-tactic. [2]

Chapter 3

W convergence and clustering games

Results related to Gruenhage's "W"-convergence game and variants.

(these are related to the "hard" version for $\mathscr{O}$ where $\mathscr{P}$ need only play within the most recent open set)

3.1 Fort spaces

Theorem 19. $O \underset{k\text{-}tact}{\not\uparrow} Clus_{O,P}(\omega_1^*, \infty)$.

Theorem 20. $O \underset{mark}{\uparrow} Clus_{O,P}(\omega_1^*, \infty)$.

Theorem 21. *(Nyikos)* $O \underset{mark}{\not\uparrow} Con_{O,P}(\omega_1^*, \infty)$.

Theorem 22. $O \underset{k\text{-}mark}{\not\uparrow} Clus_{O,P}(\kappa^*, \infty)$ *for* $\kappa > \omega_1$.

(TODO: It's feasible that k-limit $\Leftrightarrow$ 1-limit.)

3.2 Sigma-products

Theorem 23. *Let* $cf([\kappa]^{\leq\omega}) = \kappa$*. Then* $F \underset{code}{\uparrow} PF_{F,C}(\kappa)$.

Theorem 24. *Let* κ *be the limit of cardinals* κ_n *such that* $cf([\kappa_n]^{\leq\omega}, \subseteq) = \kappa_n$*. Then* $F \underset{code}{\uparrow} PF_{F,C}(\kappa)$.

Theorem 25. $F \underset{code}{\uparrow} PF_{F,C}(\kappa)$ *for all cardinals* κ.

Corollary 26. $O \underset{code}{\uparrow} Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$ *for all cardinals* κ.

Chapter 4

Proximal Game

Results pertaining to Bell's proximal game for uniform spaces.

Theorem 27. *For all $x \in X$:*

- $\mathscr{D} \uparrow Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow Con_{O,P}(X,x)$
- $\mathscr{D} \underset{2k\text{-}tact}{\uparrow} Prox_{D,P}(X) \Rightarrow \mathscr{O} \underset{k\text{-}tact}{\uparrow} Con_{O,P}(X,x)$
- $\mathscr{D} \underset{2k\text{-}mark}{\uparrow} Prox_{D,P}(X) \Rightarrow \mathscr{O} \underset{k\text{-}mark}{\uparrow} Con_{O,P}(X,x)$

Theorem 28. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete. Then*

- $\mathscr{O} \uparrow Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \uparrow Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \underset{k\text{-}tact}{\uparrow} Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \underset{k\text{-}tact}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \underset{k\text{-}mark}{\uparrow} Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \underset{k\text{-}mark}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$

Proposition 29. *For any $x \in X$ and $k \geq 1$,*

- $\mathscr{O} \underset{k\text{-}tact}{\uparrow} Con_{O,P}(X,x) \Leftrightarrow \mathscr{O} \underset{tact}{\uparrow} Con_{O,P}(X,x)$
- $\mathscr{O} \underset{k\text{-}mark}{\uparrow} Con_{O,P}(X,x) \Leftrightarrow \mathscr{O} \underset{mark}{\uparrow} Con_{O,P}(X,x)$

Corollary 30. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete, and $k \geq 1$. Then*

- $\mathscr{D} \underset{k\text{-}tact}{\uparrow} Prox_{D,P}(X \cup \{\infty\}) \Leftrightarrow O \underset{tact}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{D} \underset{k\text{-}mark}{\uparrow} Prox_{D,P}(X \cup \{\infty\}) \Leftrightarrow O \underset{mark}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$

Proposition 31. *For any uniform space X,*

- $\mathscr{D} \underset{k\text{-}tact}{\uparrow} Prox_{D,P}(X) \Leftrightarrow \mathscr{D} \underset{2\text{-}tact}{\uparrow} Prox_{D,P}(X)$
- $\mathscr{D} \underset{k\text{-}mark}{\uparrow} Prox_{D,P}(X) \Leftrightarrow \mathscr{D} \underset{2\text{-}mark}{\uparrow} Prox_{D,P}(X)$

Theorem 32. *For any uniformly locally compact space X, $\mathscr{D} \uparrow Prox_{D,P}(X) \Leftrightarrow \mathscr{D} \uparrow aProx_{D,P}(X)$*

Theorem 33. *For any uniformly locally compact proximal space X, $\mathscr{O} \uparrow Con_{O,P}(X,H)$ for all compact $H \subseteq X$.*

Corollary 34. *A compact uniform space X is Corson compact if and only if it is proximal.*

Theorem 35. *$\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X,H)$ if and only if there exists a countable base around H.*

Corollary 36. *X is first countable if and only if $\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X,x)$ for all $x \in X$*

Corollary 37. *$\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$ implies X is first countable.*

Corollary 38. *If X is scattered compact and $\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X,x)$ for all $x \in X$ (or $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$), then X is metrizable.*

Theorem 39. *If H is a closed subset of X, then $\mathscr{D} \underset{limit}{\uparrow} Prox_{D,P}(X) \Rightarrow \mathscr{D} \underset{limit}{\uparrow} Prox_{D,P}(H)$ where $\underset{limit}{\uparrow}$ is any of $\uparrow$, $\underset{k\text{-}tact}{\uparrow}$, or $\underset{k\text{-}mark}{\uparrow}$.*

Theorem 40. *If $\mathscr{D} \underset{limit}{\uparrow} Prox_{D,P}(X_i)$ for $i < \omega$, then $\mathscr{D} \underset{limit}{\uparrow} Prox_{D,P}(\prod_{i<\omega} X_i)$, where $\underset{limit}{\uparrow}$ is either $\uparrow$ or $\underset{k\text{-}mark}{\uparrow}$.*

(TODO: I expect I should be able to do some clever things assuming $S(\kappa, \omega, \omega)$ to get a similar result for sigma products of dimension κ.)

Lemma 41. *$\mathscr{O} \underset{pre}{\uparrow} Clus_{O,P}(X,S)$ if and only if $\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X,S)$.*

Theorem 42. *For any predetermined absolutely proximal space X, $\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X,H)$ for all compact $H \subseteq X$.*

Example 43. Let $X = I \times 2$ be the Alexandrov double interval. Then $\mathscr{D} \underset{\text{pre}}{\not\uparrow} Prox_{D,P}(X)$, but $\mathscr{D} \underset{\text{mark}}{\uparrow} Prox_{D,P}(X)$.

Theorem 44. *For any uniformly locally compact space X, $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X) \Leftrightarrow \mathscr{D} \underset{pre}{\uparrow} aProx_{D,P}(X)$*

Proposition 45. *If $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$, then X has a G_δ diagonal.*

Example 46. The Sorgenfrey line S has a G_δ diagonal but $\mathscr{P} \uparrow Prox_{D,P}(S)$.

Corollary 47. *For X with uniformity $\mathbb{D}$ inducing the compact Hausdorff topology τ, the following are equivalent:*

(a) $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$

(b) $\mathscr{D} \underset{pre}{\uparrow} aProx_{D,P}(X)$

(c) X has a G_δ diagonal

(d) $\mathbb{D}$ is metrizable

(e) τ is metrizable

Theorem 48. *A uniformly locally compact space with a G_δ diagonal is metrizable.*

Corollary 49. *If X is uniformly locally compact, then $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$ implies X's topology is metrizable.*

Example 50. Let R be the Michael Line. Then $\mathscr{P} \uparrow Prox_{D,P}(X)$.

Proof. During round 0, $\mathscr{P}$ may choose $m(0) = 0$ and $p(0) = 1$, and during round $n+1$, $\mathscr{P}$ may choose $m(n+1) > m(n)$ and $p(n+1) = p(n) + \frac{1}{10^{m(n+1)}}$ such that p is a legal attack.

It follows that p "converges" to $x = \sum_{n<\omega} \frac{1}{10^{m(n)}}$, except x is an irrational number composed of 1s separated by strings of 0s of strictly increasing size. $\square$

Example 51. Let ω_1 be given a ladder topology:

- All successor ordinals are isolated.
- Strictly increasing sequences (ladders) $L_\alpha : \omega \to \alpha$ are defined for each limit ordinal α such that L_α converges to α in the order topology, and each limit α is given neighborhoods of the form $L(\alpha, m) = \{\alpha\} \cup \{L_\alpha(n) : n \geq m\}$.
- $\omega_1 = \bigcup_{\alpha \in \omega_1^L} L(\alpha, 0)$

Let

$$A(\alpha, n) = [L(\alpha, 0) \setminus L(\alpha, n)]^1 \cup \{\omega_1^* \setminus (L(\alpha, 0) \setminus L(\alpha, n))\}$$

$$B(\alpha) = \{L(\alpha, 0), \omega_1^* \setminus L(\alpha, 0)\}$$

Finite refinements of $A(\alpha, n)$ and $B(\alpha)$ give partitions witnessing a uniformization of the ladder topology.

Then $Prox_{D,P}(\omega_1^*)$ is indetermined.

(TODO: finish proof)

Chapter 5

Locally Finite Games

Results pertaining to the Locally Finite games related to W games.

5.1 Characterizations using $LF_{K,P}(X)$, $LF_{K,L}(X)$

Theorem 52. *(G) The following are equivalent for a locally compact space X:*

- *X is paracompact*
- $\mathscr{K} \uparrow LF_{K,L}(X)$.

Theorem 53. *(G) The following are equivalent for a locally compact space X:*

- *X is metacompact*
- $\mathscr{K} \underset{tact}{\uparrow} LF_{K,P}(X)$.

Theorem 54. *(G) The following are equivalent for a locally compact space X:*

- *X is σ-metacompact*
- $\mathscr{K} \underset{mark}{\uparrow} LF_{K,P}(X)$.

Observation 55. *The following are equivalent for any space X:*

- *X is compact*
- $\mathscr{K} \underset{0\text{-}tact}{\uparrow} LF_{K,P}(X)$.

Theorem 56. *$P \underset{mark}{\uparrow} G_{K,P}(X)$ where X is a first-countable non-locally countably compact space.*

Theorem 57. *$P \underset{tact}{\not\uparrow} G_{K,P}(M)$ where M is the metric fan space.*

Theorem 58. *The following are equivalent for any locally compact space X:*

- *X is Lindelöf.*
- *X is σ-compact.*
- *X is hemicompact.*
- *$\mathscr{K} \underset{pre}{\uparrow} G_{K,L}(X)$.*
- *$\mathscr{K} \underset{pre}{\uparrow} G_{K,P}(X)$.*

Theorem 59. *The following are equivalent for any Hausdorff k-space X:*

- *X is hemicompact.*
- *X is k_ω.*
- *$\mathscr{K} \underset{pre}{\uparrow} LF_{K,L}(X)$.*
- *$\mathscr{K} \underset{pre}{\uparrow} LF_{K,P}(X)$.*

5.2 Non-locally compact spaces

Proposition 60. *If $X = \omega \cup \mathcal{F} \subset \beta\omega$ (a non-k space), then $K \underset{pre}{\not\uparrow} LF_{K,L}(X)$.*

Proposition 61. *If a selective ultrafilter $\mathcal{F}$ exists (independent of ZFC), then $K \underset{pre}{\not\uparrow} LF_{K,P}(X)$ for $X = \omega \cup \{\mathcal{F}\} \subset \beta\omega$.*

Theorem 62. *Then there is an ultrafilter $\mathcal{F}$ such that then $K \underset{pre}{\uparrow} LF_{K,P}(X)$ and $K \underset{tact}{\uparrow} LF_{K,P}(X)$ for $X = \omega \cup \{\mathcal{F}\} \subset \beta\omega$.*

Theorem 63. *Let M be the metric fan, a non-locally compact k space. Then $P \uparrow LF_{K,P}(M)$.*

Theorem 64. *Let S be the sequential fan, a non-locally compact k space. Then $K \underset{pre}{\uparrow} LF_{K,P}(S)$ and $K \underset{tact}{\uparrow} LF_{K,P}(S)$.*

(TODO: Maybe consider claim $K \underset{\text{pre}}{\uparrow} LF_{K,P}(X) \Rightarrow X$ is a k-space)

5.3 Cantor tree space example

Example 65. Let X be a zero-dimensional, compact L-space (hereditarally Lindeloff and non-separable). It is a fact that there exists a point-countable collection $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ of clopen sets in X, and it is also true that any point-finite subcollection of $\mathcal{U}$ is countable.

Let $C = \{c_\alpha : \alpha < \omega_1\}$ be any uncountable subset of the Cantor space 2^ω. Let $X_s = X \times \{s\}$ for each $s \in 2^{<\omega}$, and $U_{\alpha,s} = U_\alpha \times \{s\}$.

Finally, let

$$\mathbb{X} = C \cup \bigcup_{s \in 2^{<\omega}} X_s$$

be a tree of $2^{<\omega}$ copies of X, and where

$$c_\alpha \cup \bigcup_{n<\omega} U_{\alpha, x_\alpha \restriction n}$$

is an open set about each c_α.

Proposition 66. *$K \uparrow LF_{K,P}(\mathbb{X})$.*

Theorem 67. *(cor of G, game-theoretic proof by me) $K \underset{tact}{\not\uparrow} LF_{K,P}(\mathbb{X})$.*

Theorem 68. *$K \underset{k\text{-}tact}{/\!\!\uparrow} LF_{K,P}(\mathbb{X})$.*

Chapter 6

Menger Game

Results pertaining to the Menger game characterizing the Menger property.

Theorem 69. *(Hurewicz) X is Menger if and only if $C \not\uparrow Cov_{C,F}(X)$.*

Proposition 70. *X is compact if and only if $F \underset{tact}{\uparrow} Cov_{C,F}(X)$ if and only if $F \underset{k\text{-}tact}{\uparrow} Cov_{C,F}(X)$*

Proposition 71. *If X is σ-compact then $F \underset{mark}{\uparrow} Cov_{C,F}(X)$*

Theorem 72. *For any topological space X and all $k \geq 2$, $F \underset{k\text{-}mark}{\uparrow} Cov_{C,F}(X)$ if and only if $F \underset{2\text{-}mark}{\uparrow} Cov_{C,F}(X)$.*

Lemma 73. *(G) For all functions $\tau : \omega_1 \times \omega \to [\omega_1]^{<\omega}$, there exists a sequence $\alpha_0, \alpha_1, \dots < \omega_1$ such that $\{\tau(\alpha_n, n) : n < \omega\}$ is not a cover for $\{\beta : \forall n < \omega(\beta < \alpha_n)\}$.*

Example 74. $F \uparrow Cov_{C,F}(\omega_1^\dagger)$ but $F \underset{\text{mark}}{\not\uparrow} Cov_{C,F}(\omega_1^\dagger)$.

Theorem 75. *A space X is σ-(relatively compact) if and only if $F \underset{mark}{\uparrow} Cov_{C,F}(X)$.*

Corollary 76. *For regular spaces X, the following are equivalent:*

(a) X is σ-compact

(b) X is σ-(relatively compact)

(c) $F \underset{mark}{\uparrow} Cov_{C,F}(X)$

Theorem 77. *For second-countable X, the following are equivalent:*

(a) X is σ-(relatively compact)

(b) $F \uparrow Cov_{C,F}(X)$

(c) $F \underset{mark}{\uparrow} Cov_{C,F}(X)$

Corollary 78. *(Telgarsky) For metric spaces X, the following are equivalent:*

(a) X *is* σ*-compact*

(b) X *is* σ*-(relatively compact)*

(c) $F \uparrow Cov_{C,F}(X)$

(d) $F \underset{mark}{\uparrow} Cov_{C,F}(X)$

Example 79. Let R be given the topology from example 63 from Counterexamples in Topology, the topology generated by open intervals with countable sets removed. This space is a non-regular example where $F \uparrow Cov_{C,F}(R)$, but $F \underset{\text{mark}}{\not\uparrow} Cov_{C,F}(R)$, that is, R is not σ-(relatively compact).

Example 80. Let R be given the topology from example 67 from Counterexamples in Topology, the topology generated by open intervals with or without the rationals removed. This space is non-regular, and non-σ-compact, but is second-countable and σ-(relatively compact).

Definition 81. Let $\mathcal{U}$ be a cover of X. We say $C \subseteq X$ is $\mathcal{U}$-compact if there exists a finite subcover of $\mathcal{U}$ which covers C.

We say X is almost-σ-(relatively compact) if there exist functions $r_{\mathcal{V}} : X \to \omega$ for each open cover $\mathcal{V}$ of X such that both of the following sets are $\mathcal{V}$-compact for all open covers $\mathcal{U}$, $\mathcal{V}$ and $n < \omega$:

$$c(\mathcal{V}, n) = \{x \in X : r_{\mathcal{V}}(x) \leq n\}$$

$$p(\mathcal{U}, \mathcal{V}) = \{x \in X : 0 < r_{\mathcal{U}}(x) < r_{\mathcal{V}}(x)\}$$

Definition 82. For two functions f, g we say f is **μ-almost compatible** with g ($f \|^*_\mu g$) if $|\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(x) \neq g(x)\}| < \mu$. If $\mu = \omega$ then we say f, g are **almost compatible** ($f\|^* g$).

Example 83. The one-point Lindelöfication of the uncountable discrete space, $\omega_1^\dagger$, is almost-σ-(relatively compact).

Theorem 84. *If X is almost-σ-(relatively compact), then $F \underset{2\text{-}mark}{\uparrow} Cov_{C,F}(X)$.*

Corollary 85. $F \underset{2\text{-}mark}{\uparrow} Cov_{C,F}(\omega_1^\dagger)$

Proposition 86. $\neg S(\kappa, \omega, \omega)$ *for* $\kappa > 2^\omega$

Theorem 87. $S(\kappa, \omega, \omega)$ *implies* $\kappa^\dagger$ *is almost-σ-(relatively compact).*

Corollary 88. $S(\kappa, \omega, \omega)$ *implies* $F \underset{2\text{-}mark}{\uparrow} Cov_{C,F}(\kappa^\dagger)$.

Theorem 89. $S(\kappa, \omega, \omega) + (\kappa = 2^\omega)$ *is consistent with ZFC for any cardinal* κ *with* $cf(\kappa) > \omega$.

Corollary 90. *For each* κ, $F \underset{2\text{-}mark}{\uparrow} Cov_{C,F}(\kappa^\dagger)$ *is consistent with ZFC.*

6.1 Alster property

Besides various limited information characterizations of $Cov_{C,F}(X)$, there are other interesting covering properties between σ-(relatively compact) and Menger.

Proposition 91. *Every ample cover of a regular space X is really ample.*

Proposition 92. *Every regular relatively Alster space is Alster.*

Theorem 93. *(Aurichi, Tall) X σ-compact $\Rightarrow$ X Alster $\Rightarrow$ X Menger*

Proposition 94. *X σ-(relatively compact) $\Rightarrow$ X relatively Alster $\Rightarrow$ X Menger*

Example 95. Let the real numbers R be given the topology generated by open intervals with countable sets removed. R is not relatively Alster and $F \uparrow Cov_{C,F}(R)$. If $S(2^\omega, \omega, \omega)$ holds, then $F \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(R)$.

6.2 Filling Games

Definition 96. The **filling game** $Fill^{\subseteq}_{M,N}(J)$ on an ideal J proceeds as follows: player M chooses $M_0 \in \langle J \rangle$, the σ-completion of J, in the initial round, followed by N choosing $N_0 \in J$. In round $n+1$, player M chooses M_{n+1} where $M_n \subseteq M_{n+1} \in \langle J \rangle$, and player N replies with $N_{n+1} \in J$. Player N wins the game if $\bigcup_{n<\omega} N_n \supseteq \bigcup_{n<\omega} M_n$. (The sets in J and $\langle J \rangle$ are thought of as nowhere-dense and meager sets, respectively.)

The **strict filling game** $Fill^{\subsetneq}_{M,N}(J)$ proceeds analogously, with the added requirement that $M_n \subsetneq M_{n+1}$. This game has been studied by Scheepers.

Theorem 97. $N \underset{2\text{-}tact}{\uparrow} Fill^{\subsetneq}_{M,N}(J) \Rightarrow N \underset{2\text{-}mark}{\uparrow} Fill^{\subseteq}_{M,N}(J)$

Example 98. There is a free ideal J such that $N \underset{\text{2-tact}}{\not\uparrow} Fill^{\subsetneq}_{M,N}(J)$ but $N \underset{\text{2-mark}}{\uparrow} Fill^{\subsetneq}_{M,N}(J)$.

6.3 Rothberger property

Theorem 99. *(Pawlikowski) X is Rothberger if and only if $C \not\uparrow Cov_{C,S}(X)$.*

Theorem 100. *The following are equivalent for compact T_2 X:*

(a) X is Rothberger

(b) X is scattered

(c) $S \uparrow Cov_{C,S}(X)$

(d) $C \not\uparrow Cov_{C,S}(X)$

Theorem 101. *(Galvin) $Cov_{P,O}(X)$ is "perfect information equivalent" to $Cov_{C,S}(X)$. That is:*

- *$P \uparrow Cov_{P,O}(X)$ if and only if $S \uparrow Cov_{C,S}(X)$*
- *$O \uparrow Cov_{P,O}(X)$ if and only if $C \uparrow Cov_{C,S}(X)$.*

Theorem 102.
- *$P \underset{pre}{\uparrow} Cov_{P,O}(X)$ if and only if $S \underset{mark}{\uparrow} Cov_{C,S}(X)$*
- *$O \underset{mark}{\uparrow} Cov_{P,O}(X)$ if and only if $C \underset{pre}{\uparrow} Cov_{C,S}(X)$.*

Theorem 103. *For any space X, the following are equivalent:*

- *$S \underset{mark}{\uparrow} Cov_{C,S}(X)$*
- *$P \underset{pre}{\uparrow} Cov_{P,O}(X)$*
- *X is almost countable*

Theorem 104. *For any T_1 space X, the following are equivalent:*

- *$S \underset{mark}{\uparrow} Cov_{C,S}(X)$*
- *$P \underset{pre}{\uparrow} Cov_{P,O}(X)$*
- *X is almost countable*
- *$|X| \leq \omega$*

Example 105. Let $X = \omega_1 \cup \{\infty\}$ be a "weak Lindelöfication" of discrete ω_1 such that open neighborhoods of ∞ contain $\omega_1 \setminus \omega$. This space is T_0 but not T_1, and note that $S \underset{\text{mark}}{\uparrow} Cov_{C,S}(X)$ and $|X| > \omega$.

Theorem 106. *The following are equivalent for points-G_δ X:*

(a) $S \uparrow Cov_{C,S}(X)$

(b) $P \uparrow Cov_{P,O}(X)$

(c) $S \underset{k\text{-}mark}{\uparrow} Cov_{C,S}(X)$ *for some* $k \geq 1$

(d) $P \underset{k\text{-}mark}{\uparrow} Cov_{P,O}(X)$ *for some* $k \geq 1$

(e) $S \underset{mark}{\uparrow} Cov_{C,S}(X)$

(f) $P \underset{pre}{\uparrow} Cov_{P,O}(X)$

(g) X *is almost countable*

(h) $|X| \leq \omega$

Corollary 107. *The following are equivalent for compact points-G_δ X:*

(a) $S \uparrow Cov_{C,S}(X)$

(b) $P \uparrow Cov_{P,O}(X)$

(c) $S \underset{k\text{-}mark}{\uparrow} Cov_{C,S}(X)$ *for some* $k \geq 1$

(d) $P \underset{k\text{-}mark}{\uparrow} Cov_{P,O}(X)$ *for some* $k \geq 1$

(e) $S \underset{mark}{\uparrow} Cov_{C,S}(X)$

(f) $P \underset{pre}{\uparrow} Cov_{P,O}(X)$

(g) X *is almost countable*

(h) $|X| \leq \omega$

(i) $C \not\uparrow Cov_{C,S}(X)$

(j) $O \not\uparrow Cov_{P,O}(X)$

(k) X *is Rothberger*

(l) X *is scattered*

Definition 108. The game $Rec^m_{F,S}(\kappa)$ proceeds as follows: during round 0, player F chooses $F_0 \in [\kappa]^m$, followed by player S choosing $x_0 \in F_0 \cup \{\infty\}$. During round $n+1$, F chooses $F_{n+1} \in [\kappa]^{m^{n+2}}$ such that $F_{n+1} \supset F_n$, followed by S choosing $x_{n+1} \in F_{n+1} \cup \{\infty\}$.

S wins the game if $\{x_n : n < \omega\} \supseteq F_0 \cup \{\infty\}$, and F wins otherwise.

Proposition 109. $S \underset{limit}{\uparrow} Cov_{C,S}(\kappa^{\dagger}) \Rightarrow S \underset{limit}{\uparrow} Rec^{m}_{F,S}(\kappa)$

Proposition 110. $S \underset{k\text{-}mark}{\uparrow} Rec^{m}_{F,S}(\kappa) \Leftrightarrow S \underset{k\text{-}tact}{\uparrow} Rec^{m}_{F,S}(\kappa)$

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