Definition 1. X is **Menger** if for all open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ there exist finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Proposition 2. *σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelof*

Definition 3. In the two-player game $Cov_{C,F}(X)$ player C chooses open covers $\mathcal{U}_n$ of X, followed by player F choosing a finite subcollection $\mathcal{F}_n \subseteq \mathcal{U}_n$. F wins if $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Theorem 4. *X is Menger if and only if $C \not\uparrow Cov_{C,F}(X)$.*

Proof. Result due to (???)

First, suppose X wasn't Menger. Then there would exist open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ of X such that for any choice of finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$, $\bigcup_{n<\omega} \mathcal{F}_n$ isn't a cover of X. Thus $C \uparrow_{\text{pre}} Cov_{C,F}(X) \Rightarrow S \not\uparrow Cov_{C,F}(X)$.

The other direction is based upon Gruenhage's topological game presentation. Assume X is Menger, and consider a strategy for C in $Cov_{C,F}(X)$.

Since X is Lindelof, we can assume C plays only countable covers of X. Then, since F is choosing finite subsets, we may assume F chooses some initial segement of the countable cover. In turn, we can assume C plays an increasing open cover $\{U_0, U_1, \dots\}$ where $U_n \subseteq U_{n+1}$. And in that case, it's suffient to assume F simply chooses a singleton subset of each cover. And finally, since choices made by F are already covered, we can assume that every open set in a cover played by C covers the sets chosen by F previously.

As a result, we have the following figure of a tree of plays which I need to draw:

(Insert figure here.)

Note that for $a, b \in \omega^{<\omega}$ and $m \leq n$, we know:

(a) $U_{a^\frown m} \subseteq U_{a^\frown n}$
(for example, $U_{1627} \subseteq U_{1629}$ - increasing the final digit yields supersets)

(b) $U_a \subseteq U_{a^\frown b}$
(for example, $U_{1627} \subseteq U_{162789}$ - appending any sequence to the end yields supersets)

(c) $U_{a^\frown m} \subseteq U_{a^\frown n} \subseteq U_{a^\frown n^\frown b} \subseteq U_{a^\frown n^\frown b^\frown m}$
(for example: $U_{1627} \subseteq U_{1629283287}$ - injecting a subseqence with initial number larger than the original's final number, prior to the final number, yields supersets)

We may observe that if F can find an $f : \omega \to \omega$ such that $\bigcup_{n<\omega} U_{f \restriction (n+1)} = X$, she can use $\{U_{f\restriction 0}\}, \{U_{f\restriction 1}\}, \dots$ to counter C's strategy.

Let $V_k^n = \displaystyle\bigcap_{a \in \omega^{\leq n}} U_{a^\frown k}$. We claim that (1) V_k^n is open, (2) $\mathcal{V}^n = \{V_0^n, V_1^n, \dots\}$ is increasing, and (3) $\mathcal{V}^n$ is a cover. Proofs:

1. Since due to (c) for each $b \in \omega^{\leq n} \setminus k^{\leq n}$, there is an $a \in k^{\leq n}$ with $U_{a^\frown k} \subseteq U_{b^\frown k}$:

$$V_k^n = \bigcap_{a \in \omega^{\leq n}} U_{a^\frown k} = \bigcap_{a \in k^{\leq n}} U_{a^\frown k} \cap \bigcap_{b \in \omega^{\leq n} \setminus k^{\leq n}} U_{b^\frown k} = \bigcap_{a \in k^{\leq n}} U_{a^\frown k}$$

   making V_k^n a finite intersection of open sets.

2. We show $V_k^0 \subseteq V_{k+1}^0$:

$$V_k^0 = U_k \subseteq U_{k+1} = V_{k+1}^0$$

   and then assume $V_k^n \subseteq V_{k+1}^n$:

$$V_k^{n+1} = \bigcap_{a \in \omega^{\leq n+1}} U_{a^\frown k} = V_k^n \cap \bigcap_{a \in \omega^{n+1}} U_{a^\frown k} \subseteq V_{k+1}^n \cap \bigcap_{a \in \omega^{n+1}} U_{a^\frown (k+1)} = V_{k+1}^{n+1}$$

3. We easily see that $\mathcal{V}^0 = \{U_0, U_1, \dots\}$ is a cover, and then assume $\mathcal{V}^n$ is a cover.

   Let $x \in X$ and pick $l < \omega$ such that $x \in V_l^n$. For $a \in l^{n+1}$ choose l_a such that $x \in U_{a^\frown l_a}$, giving

$$x \in \bigcap_{a \in l^{n+1}} U_{a^\frown l_a}$$

   We will assume $k > l, l_a$ for all $a \in l^{\leq n+1}$.

   For any $a \in k^{n+1} \setminus l^{n+1}$ note that $a = b^\frown c$ where $b \in l^{\leq n}$ and c begins with a number l or greater:

$$V_l^n \subseteq U_{b^\frown l} \subseteq U_{b^\frown c} \subseteq U_{b^\frown c^\frown l_a} = U_{a^\frown l_a}$$

   Thus:

$$x \in V_l^n \cap \left(\bigcap_{a \in l^{n+1}} U_{a^\frown l_a} \right)$$

$$= V_l^n \cap \left(\bigcap_{a \in k^{n+1} \setminus l^{n+1}} U_{a^\frown l_a} \right) \cap \left(\bigcap_{a \in l^{n+1}} U_{a^\frown l_a} \right)$$

$$= V_l^n \cap \left(\bigcap_{a \in k^{n+1}} U_{a^\frown l_a} \right)$$

$$\subseteq V_k^n \cap \left(\bigcap_{a \in k^{n+1}} U_{a^\frown k} \right)$$

$$= V_k^{n+1}$$

Finally, apply Menger to $\mathcal{V}^n$, resulting in the cover $\{V^0_{f(0)}, V^1_{f(1)}, \dots\}$, noting

$$X = \bigcup_{n<\omega} V^n_{f(n)} \subseteq \bigcup_{n<\omega} U_{(f\restriction n)^\frown f(n)} = \bigcup_{n<\omega} U_{f\restriction(n+1)}$$

$\square$

Proposition 5. *X is compact if and only if $F \uparrow_{tact} Cov_{C,F}(X)$*

Proof. Assume X is compact. For each open cover played by C, pick the finite subcover.

Assume F has a winning tactical strategy. For any open cover, have C play only it during the entire game. F's only choice must be a finite subcover. $\square$

Proposition 6. *If X is σ-compact then $F \uparrow_{mark} Cov_{C,F}(X)$*

Proof. Let $X = \bigcup_{n<\omega} X_n$ for compact X_n. On round n, F picks the finite subcover of C's open cover of X_n. $\square$

Due to Telgarski in "On Games of Topsoe":

Theorem 7. *For metrizable X, X is σ-compact if and only if $F \uparrow Cov_{C,F}(X)$.*

In a question I posed to G, he answered:

Lemma 8. *For all functions $\tau : \omega_1 \times \omega \to [\omega_1]^{<\omega}$, there exists a sequence $\alpha_0, \alpha_1, \dots < \omega_1$ such that $\{\tau(\alpha_n, n) : n < \omega\}$ is not a cover for $\{\beta : \forall n < \omega(\beta < \alpha_n)\}$.*

Proof. Let $P_n = \{\beta : \beta < \alpha \Rightarrow \beta \in \tau(\alpha, n)\}$. Observe that each P_n is finite; else there is some α larger than every member of some countably infinite $P^*_n \subseteq P_n$ such that $P^*_n \subseteq \tau(\alpha, n)$.

Choose $\beta \notin \bigcup_{n<\omega} P_n$. Then for each $n < \omega$, pick $\alpha_n > \beta$ such that $\beta \notin \tau(\alpha_n, n)$. $\square$

Note that the one-point Lindelöfication of discrete ω_1, $\omega_1^\dagger$, is not σ-compact. With the above lemma, we may see that:

Example 9. *$F \uparrow Cov_{C,F}(\omega_1^\dagger)$ but $F \not\uparrow_{mark} Cov_{C,F}(\omega_1^\dagger)$.*

Proof. First, we see F has a simple perfect information strategy: in response to the initial cover of $\omega_1^\dagger$, F chooses a co-countable neighborhood of ∞. On successive turns she may pick a single set from C's covers to cover the countable remainder.

Now, suppose that $\sigma(\mathcal{U}, n)$ was a winning Markov strategy and aim for a contradiction. Consider the covers

$$\mathcal{U}(\alpha) = \{[\alpha, \omega_1) \cup \{\infty\}\} \cup \{\{\beta\} : \beta < \alpha\}$$

and define $\tau(\alpha, n)$ to be the union of singletons chosen by $\sigma(\mathcal{U}(\alpha), n)$.

Using the sequence $\alpha_0, \alpha_1, \dots < \omega_1$ from the previous lemma, we consider the play $\mathcal{U}(\alpha_0), \mathcal{U}(\alpha_1), \dots$.

As σ was a winning strategy, $\{\sigma(\mathcal{U}(\alpha_n), n) : n < \omega\}$ must cover $\omega_1^\dagger$, and thus $\{\tau(\alpha_n, n) : n < \omega\}$ must cover $\{\beta : \forall n < \omega(\beta < \alpha_n)\}$, contradiction. □

We require a lemma.

Lemma 10. *There exist injective functions $f_\alpha : \alpha \to \omega$ such that if $\alpha < \beta$, then*

$$f_\beta \restriction \alpha =^* f_\alpha$$

that is, $f_\beta \restriction \alpha$ and f_α agree on all but finitely many ordinals.

Example 11. $F \uparrow_{2\text{-}mark} Cov_{C,F}(\omega_1^\dagger)$

Proof. Using the functions f_α from the previous lemma, let

$$\tau(\alpha_n, \alpha_{n+1}, n+1) = f_{\alpha_n}^{-1}([0, n]) \cup \{\beta < \alpha_n, \alpha_{n+1} : f_{\alpha_n}(\beta) \neq f_{\alpha_{n+1}}(\beta)\}$$

For any sequence α_n, suppose that $\beta < \alpha_n$ for all n, and β is not covered by

$$\{\tau(\alpha_n, \alpha_{n+1}, n+1) : n < \omega\}$$

Then we see first that $f_{\alpha_n}(\beta) = f_{\alpha_{n+1}}(\beta)$ for all n. However, $f_n(\beta) > n$ for all n as well, which is a contradiction.

Finally, for each open cover $\mathcal{U}$, assign arbitrary $\alpha(\mathcal{U})$, $U(\mathcal{U})$ such that $[\alpha(\mathcal{U}), \omega_1) \cup \{\infty\}$ is a subset of $U(\mathcal{U}) \in \mathcal{U}$. Then a 2-Markov strategy $\sigma(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1)$ which chooses $U(\mathcal{U}_n)$ and covers the finite set $\tau(\alpha(\mathcal{U}_n), \alpha(\mathcal{U}_{n+1}), n+1)$ is a winning strategy for F. □

Lemma 12. *Let $\sigma(\mathcal{U}, n)$ be a winning Markov strategy for F in $Cov_{C,F}(X)$, and $\mathfrak{C}$ collect all open covers of X. Then for*

$$C_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \overline{\bigcup \sigma(\mathcal{U}, n)}$$

and

$$D_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

it follows that $\bigcup_{n<\omega} C_n = \bigcup_{n<\omega} D_n = X$.

Proof. Observe $D_n \subseteq C_n$. Suppose that $x \notin D_n = \bigcap_{\mathcal{U}\in\mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$ for any $n < \omega$. Then for each n, pick $\mathcal{U}_n \in \mathfrak{C}$ such that $x \notin \bigcup \sigma(\mathcal{U}_n, n)$. Then σ does not defeat the play $\mathcal{U}_0, \mathcal{U}_1, \ldots$ since the $\sigma(\mathcal{U}_n, n)$ do not cover x, contradiction. □

Theorem 13. *For regular spaces X, $F \uparrow_{mark} Cov_{C,F}(X)$ if and only if X is σ-compact.*

Proof. The reverse implication has already been shown. To complete the proof, we look to Scheepers for inspiration.

Let $\sigma(\mathcal{U}, n)$ be a winning Markov strategy for F in $Cov_{C,F}(X)$. Let $\mathfrak{C}$ collect all open covers of X. Define

$$C_n = \bigcap_{\mathcal{U}\in\mathfrak{C}} \overline{\bigcup \sigma(\mathcal{U}, n)}$$

as in the previous lemma. Note that $\bigcup_{n<\omega} C_n = X$, and we will show each C_n is compact as it is H-closed.

Let $\mathcal{U}$ be an open cover of C_n, and $\mathcal{V}$ be a cover of $X \setminus C_n$ by open sets whose closures are disjoint from C_n (possible by regularity).

Since $\mathcal{U} \cup \mathcal{V}$ covers X, $\overline{\bigcup \sigma(\mathcal{U} \cup \mathcal{V}, n)} \supseteq C_n$. Furthermore, if $\mathcal{F} = \sigma(\mathcal{U} \cup \mathcal{V}, n) \setminus \mathcal{V}$, then $\overline{\bigcup \mathcal{F}} \supseteq C_n$ (the closures of sets in $\mathcal{V}$ missed C_n). Thus $\mathcal{F}$ witnesses that C_n is H-closed. □

Example 14. *Let R be given the topology from example 63 from Counterexamples in Topology, the topology generated by open intervals with countable sets removed. This space is non-regular, non-σ-compact, and Lindelöf. It is also Menger as $F \uparrow Cov_{C,F}(R)$, but $F \not\uparrow_{mark} Cov_{C,F}(R)$.*

Proof. From Counterexamples: The irrationals are open, but contain no closed neighborhood, showing non-regular. Compact subsets are exactly finite subsets, showing non-σ-compact.

Take open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$. Define $\sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{2n})$ to be a finite subcover of $[-n, n] \setminus C_n$ for some countable $C_n = \{c_{n,0}, c_{n,1}, \ldots\}$. For $\sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{2n+1})$, use any subcover of $\{c_{i,j} : i, j < n\}$. It is easily seen that σ is a winning perfect information strategy.

There cannot be a winning Markov strategy $\sigma(\mathcal{U}, n)$, however. Define

$$D_n = \bigcap_{\mathcal{U}\in\mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

where $\mathfrak{C}$ is the collection of open covers of R. For any $x_0, x_1, \dots \in R$, we may define the open cover $\mathcal{U} = \{R \setminus \{x_i : i \neq n\} : n < \omega\}$, and observe that $\bigcup \sigma(\mathcal{U}, n) \supseteq D_n$ cannot contain every x_i. Thus D_n is finite, but since the previous lemma requires $\bigcup_{n<\omega} D_n = R$ if σ is a winning strategy, there exists a counter to σ. □

Theorem 15. *For any topological space X and all $k \geq 2$, $F \uparrow_{k\text{-}mark} Cov_{C,F}(X)$ if and only if $F \uparrow_{2\text{-}mark} Cov_{C,F}(X)$.*

Proof. Assume $\sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{k-1}, n)$ is a winning k-Markov strategy. Define the 2-Markov strategy $\tau(\mathcal{U}, \mathcal{V}, n)$ so that it contains $\sigma(\mathcal{W}_0, \ldots, \mathcal{W}_{k-1}, m)$ for the following conditions on $(\mathcal{W}_0, \ldots, \mathcal{W}_{k-1}, m)$:

- Each $\mathcal{W}_i \in \{\mathcal{U}, \mathcal{V}\}$
- $m \leq (n+1)k$; in particular, for $i < k$,

$$\sigma(\mathcal{W}_0, \ldots, \mathcal{W}_{k-1}, (n+1)k + i) \subseteq \tau(\mathcal{U}, \mathcal{V}, n+1)$$

Considering an arbitrary play $\mathcal{U}_0, \mathcal{U}_1, \ldots$ by C versus τ, we note that σ defeats the play

$$\underbrace{\mathcal{U}_0, \mathcal{U}_0, \ldots, \mathcal{U}_0}_{k}, \underbrace{\mathcal{U}_1, \mathcal{U}_1, \ldots, \mathcal{U}_1}_{k} \ldots$$

So we have that

$$\bigcup_{i<k, n<\omega} \sigma(\underbrace{\mathcal{U}_n, \ldots, \mathcal{U}_n}_{k-i-1}, \underbrace{\mathcal{U}_{n+1}, \ldots, \mathcal{U}_{n+1}}_{i+1}, (n+1)k + i)$$

is a cover for X, and as

$$\sigma(\underbrace{\mathcal{U}_n, \ldots, \mathcal{U}_n}_{k-i-1}, \underbrace{\mathcal{U}_{n+1}, \ldots, \mathcal{U}_{n+1}}_{i+1}, (n+1)k + i) \subseteq \tau(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1)$$

τ defeats the play $\mathcal{U}_0, \mathcal{U}_1, \ldots$. □

The question remains:

Question 16. *In general, does $F \uparrow_{mark} Cov_{C,F}(X)$ imply X is σ-compact?*