Definition 1. Let a V-map be a u.s.c. idempotent surjection.

Definition 2. For any LOS $\langle L, \leq \rangle$, let $\check{L}$ be the collection of leftward subsets of L (subsets for which $b \in L, a \leq b \Rightarrow a \in L$) linearly ordered by $\subseteq$, and let $\hat{L}$ be the collection of left-closed subsets of L (leftward subsets which are closed) linearly ordered by $\subseteq$.

Proposition 3. *$\check{L}$, $\hat{L}$ are compact.*

Proof. Each subset S has an infimum $\cap S$ and a supremum $\cup S$ (or $\mathrm{cl}(\cap S)$). □

Note that $\check{L}$ is not a "compactification" as L does not necessarily embed as a dense subspace of $\check{L}$: if $L = I$, we might attempt to embed $t \mapsto [0, t)$, but then note that the subspace topology induces the reverse Sorgenfrey interval as $([0, s), [0, t]) = ([0, s), [0, t)]$ is open. However $\hat{L}$ is the typical way of compactifying a linearly ordered space L, provided L lacks a least element (otherwise the empty set is an [easily removable] isolated point in $\hat{L}$).

Definition 4. For any compact LOTS K with minimum 0 and maximum 1, let γ be the V-map on K where $\gamma(0) = K$ and $\gamma(t) = \{1\}$ for $t > 0$.

Definition 5. For any LOTS M with minimum element 0, let ν be the V-map on M where $\nu(0) = K$ and $\nu(t) = \{t\}$ for $t > 0$.

Note for $K = M = 2$ that $\gamma = \nu$.

Theorem 6. $X = \varprojlim\{2, \nu, L\} \cong \check{L}$

Proof. We start by placing an order on X. Let $\vec{x} < \vec{y}$ if there exists $a \in L$ with $\vec{x}(a) = 0, \vec{y}(a) = 1$. We claim this is a total order inducing the topology on X.

We first observe that if $\vec{x}(b) = 1$, then for all $a \leq b$, $\vec{x}(a) \in \nu(1) = \{1\}$. If $\vec{x} \neq \vec{y}$, then assume without loss of generality that $\vec{x}(a) = 0, \vec{y}(a) = 1$, so $\vec{x} < \vec{y}$. Also, whenever $\vec{x}(b) = 1$, we have that $b < a$, so $\vec{y}(b) = 1$, preventing $\vec{y} < \vec{x}$. Finally if $\vec{x} < \vec{y}$ and $\vec{y} < \vec{z}$, take a, b with $\vec{x}(a) = 0$, $\vec{y}(a) = 1$,$\vec{y}(b) = 0$,$\vec{z}(b) = 1$. It follows that $a < b$ so $\vec{z}(a) = 1$ and $\vec{x} < \vec{z}$.

Consider the basic open set $B(\vec{x}, F)$ for a finite set $F \in [L]^{<\omega}$ about the sequence $\vec{x} \in X$ which contains all sequences $\vec{y}$ agreeing with $\vec{x}$ on F. If $\vec{x}(a) = 1$ for all $a \in F$, then let $\vec{w} \in X$ be 0 on the maximum of F, and 1 for anything less. It follows that $B(\vec{x}, F) = (\vec{w}, \rightarrow)$. If $\vec{x}(a) = 0$ for all $a \in F$, then let $\vec{y} \in X$ be 1 on the minimum of F, and 0 for anything greater. It follows that $B(\vec{x}, F) = (\leftarrow, \vec{y})$. Finally if $\vec{x}(a) = 1$ and $\vec{x}(b) = 0$ for $a < b$ in F and nothing between a, b is in F, then let $\vec{w} \in X$ be 0 on a and 1 for anything less, and let $\vec{y} \in X$ be 1 on b and 0 for anything greater. It follows that $B(\vec{x}, F) = (\vec{w}, \vec{y})$.

Let ϕ evaluate each $\vec{x} \in X \subseteq 2^L$ as the characteristic function for a subset of L. It's easy to see that ϕ is an order isomorphism between $\langle X, \leq \rangle$ and $\langle \check{L}, \subseteq \rangle$. □

Corollary 7. $\varprojlim\{2,\nu,\alpha\} \cong \alpha+1$ *for every ordinal* α.

Proof. Since $\breve{\alpha} = \alpha + 1$ (actually equals, not just homeomorphic!), we get $\varprojlim^\star\{2,\nu,\alpha\} \cong \breve{\alpha} = \alpha + 1$ for free. $\square$

We introduce an alternate definition of an arbitrarily indexed inverse limit.

Definition 8. Let $\varprojlim^\star\{X,f,L\} \subseteq \varprojlim\{X,f,L\}$ satisfy that $\vec{x}(a) = \lim_{t\to a}\vec{x}(t)$ for all $a \in L$ (for any open neighborhood $U$of $\vec{x}(a)$ there is $b < a$ where $\vec{x}(t) \in U$ for all $t \in (b,a]$).

Theorem 9. $Y = \varprojlim^\star\{2,\nu,L\} \cong \hat{L}$.

Proof. Consider Y as a subspace of $X = \varprojlim\{2,\nu,L\}$ with the linear order described above. We claim that if ϕ is the characteristic function for a subset of L, then ϕ is an order isomorphsim between $\langle Y,\leq\rangle$ and $\langle \hat{L},\subseteq\rangle$.

Let A be a left-closed subset of L. Let $\vec{x}(a) = 1$ when $a \in A$ and $\vec{x}(a) = 0$ otherwise. Then $\vec{x} \in Y$ and $\phi(\vec{x}) = A$.

Let $\vec{x},\vec{y} \in Y$. If $\phi(\vec{x}) = \phi(\vec{y}) = A$, then A is a left-closed set where $\vec{x}(a) = \vec{y}(a) = 1$ for $a \in A$ and $\vec{x}(a) = \vec{y}(a) = 0$ otherwise, so $\vec{x} = \vec{y}$.

Finally let $\vec{x} < \vec{y}$, so there exists $a \in L$ with $\vec{x}(a) = 0$, $\vec{y}(a) = 1$. Then $\phi(\vec{x}) \subseteq (\leftarrow, a) \subsetneq \phi(\vec{y})$. Thus ϕ preserves order. $\square$

Corollary 10. $\varprojlim^\star\{2,\nu,\alpha\} \cong \alpha+1$ *for every infinite limit or finite ordinal* α.

Proof. If α is finite, then of course all (leftward) sets are closed and we get $\hat{\alpha} = \breve{\alpha} = \alpha + 1$ for free. Otherwise, since α lacks a greatest point, $\hat{\alpha}$ is homeomorphic to its usual compactification $\alpha + 1$. $\square$

In fact, $\hat{\alpha} = \alpha + 1 \setminus L(\alpha)$ where $L(\alpha)$ is the colleciton of all limit ordinals less than α, which also shows $\hat{\alpha} \cong \alpha$ for infinite successor ordinals α.

Theorem 11. *Let* M *be a LOTS with minimum* 0. $Z = \varprojlim^\star\{M,\nu,L\} \cong \{-\infty\} \cup (\hat{L} \setminus \emptyset) \times_{lex} M'$ *where* $M' = M \setminus \{0\}$, $\times_{lex}$ *induces the lexicographic ordering on the product, and* $-\infty$ *is a least element.*

Proof. Let $\rho(\vec{x}) = \{a \in L : \vec{x}(a) > 0\}$; this is obviously leftwards, and it's closed for all $\vec{x} \in Z$. Then we give Z the linear order where $\vec{x} < \vec{y}$ if $\rho(\vec{x}) \subsetneq \rho(\vec{y})$ or both $\rho(\vec{x}) = \rho(\vec{y})$ and $\vec{x}(l) < \vec{y}(l)$ for some $l \in L$.

Define $r : Z \to \hat{L} \times_{\text{lex}} M$ by $r(\vec{0}) = -\infty$ and $r(\vec{x}) = \langle\rho(\vec{x}), m\rangle$ where $\vec{x}(l) = m > 0$ for some $l \in L$ otherwise.

We first show that r is a bijection. Let A be left-closed and $m \in M'$. Then let $\vec{x}_{A,m}$ satisfy $\vec{x}_{A,m}(a) = m$ for $a \in A$ and $\vec{x}_{A,m}(a) = 0$ otherwise and note $\vec{x}_{A,m} \in Z$, $r(\vec{x}_{A,m}) = \langle A, m \rangle$. Showing one-to-one is similarly trivial.

Finally let $\vec{0} < \vec{x} < \vec{y}$. If $\rho(\vec{x}) \subsetneq \rho(\vec{y})$, then $r(\vec{x}) < r(\vec{y})$. Otherwise $\rho(\vec{x}) = \rho(\vec{y})$, but $0 < \vec{x}(l) < \vec{y}(l)$ for some $l \in L$ and therefore $r(\vec{x}) < r(\vec{y})$. $\square$

This gives us the previous result as a corollary: $\varprojlim^\star\{2, \nu, L\} \cong \{-\infty\} \cup (\hat{L} \setminus \{\emptyset\}) \times_{\text{lex}} \{1\} \cong \hat{L}$. Unfortunately, the result is not always compact.

Example 12. $\varprojlim^\star\{I, \nu, I\} \cong \{-\infty\} \cup (0, 1] \times_{\text{lex}} (0, 1]$ is not compact.

Proof. The infinite set $\{\langle 1 - \frac{1}{2^n}, 1 - \frac{1}{2^n} \rangle : 0 < n < \omega\}$ is closed discrete. $\square$

References