TACTIC-PROXIMAL COMPACT SPACES ARE STRONG EBERLEIN COMPACT

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Abstract. The author and G. Gruenhage previously showed that J. Bell's proximal game may be used to characterize Corson compactness in compact Hausdorff spaces. Using limited information strategies, the proximal game may also be used to characterize the strong Eberlein compactness property. In doing so, a purely topological characterization of the proximal game is introduced, and several existing results on the proximal game are generalized to hold for limited information strategies.

Two papers published in 2014 introduced the *proximal uniform space game* $Bell^{\mathrm{uni}}_{D,P}(X)$ due to Jocelyn Bell. If X is a topological space, and there exists a uniform structure inducing its topology which gives the first player in this game has a winning stategy, then X is said to be a *proximal* space. Bell used this game as a tool in [1] for investigating uniform box products, and the author showed with Gary Gruenhage in [2] that this game characterizes Corson compactness amongst compact Hausdorff spaces, answering a question of Peter Nyikos in [3].

All spaces in this paper are assumed to be completely regular, so that they have a uniform structure inducing the topology on the space. Unlike many game-theoretic topological properties, the game $Bell^{\mathrm{uni}}_{D,P}(X)$ for which the proximal property was defined by is not itself a topological game. However, by considering entourages of the universal uniformity inducing the topology of a space, this original uniform space game may be easily modified to a purely topological game $Bell^{\rightarrow}_{D,P}(X)$.

The aim of this paper is to use this topological interpretation of the proximal game to give a new game-theoretic characterization of the strong Eberlein compactness property. Strong Eberlein compacts are Corson compacts, and therefore proximal compact spaces; in fact, it will be shown that strong Eberlein compacts are exactly the compact spaces for which the first player has a *tactical* winning strategy for the proximal game, a strategy which relies on only the most recent move of the opponent.

1. Topologizing $Bell^{\mathrm{UNI}}_{D,P}(X)$

We refer to [2] for definitions, notation, and basic theorems on uniform spaces and the proximal game $Bell^{\mathrm{uni}}_{D,P}(X)$ (denoted as $Prox_{D,P}(X)$ in that paper). In particular recall that:

Definition 1.1. $\mathscr{P} \uparrow G$ denotes that the player $\mathscr{P}$ has a winning strategy in the ω-length game G.

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Definition 1.2. X is a *proximal* space in the case that there exists a uniformity for X such that $\mathscr{D} \uparrow Bell^{\text{uni}}_{D,P}(X)$.

As it turns out, the search for such a uniformity is trivial. We refer to [4] for any non-trivial proofs.

Definition 1.3. The *universal uniformity* for a uniformizable topology is the union of all uniformities which induce the given topology.

Theorem 1.4. *The universal uniformity is itself a uniformity compatibile with the topology on the space.*

Definition 1.5. For a uniformizable space X, a *universal entourage* D is a entourage of the universal uniformity.

Theorem 1.6. *For every uniformizable space, if D is a neighborhood of the diagonal Δ such that there exist neighborhoods D_n of Δ with $D \supseteq D_0$ and $D_n \supseteq D_{n+1} \circ D_{n+1}$, then D is a universal entourage.*

Definition 1.7. An *open symmetric entourage* D is a entourage which is open in the product topology induced by the uniformity and where $D = D^{-1}$ ($\langle x, y\rangle \in D$ if and only if $\langle y, x\rangle \in D$).

Theorem 1.8. *For every entourage D, there exists an open symmetric entourage $U \subseteq D$.*

Due to this theorem, we will simply use the word *entourage* to refer to open symmetric universal entourages. Note that if D is an entourage, then $D[x]$ is an open neighborhood of x. One may consider $D[x]$ to be an entourage-"ball" about x, generalizing the notion of an ϵ-ball given by a metric structure.

In the case that the space is paracompact, entourages are even more easily found.

Theorem 1.9. *Every open neighborhood of the diagonal is a universal entourage for paracompact uniformizable spaces.*

Definition 1.10. For every entourage D, let $\frac{1}{2^n}D$ denote entourages for $n < \omega$ such that $\frac{1}{1}D = D$ and $\frac{1}{2^{n+1}}D \circ \frac{1}{2^{n+1}}D \subseteq \frac{1}{2^n}D$.

1.1. **Using (universal) entourages to characterize the proximal property.** The natural adaptation of the original uniform space game $Bell^{\text{uni}}_{D,P}(X)$ to a topological game requires the use of the universal uniformity on X.

Definition 1.11. Let $Bell^{\rightarrow,\star}_{D,P}(X)$ denote the *hard Bell convergence game* with players $\mathscr{D}$, $\mathscr{P}$ which proceeds as follows for a uniformizable space X. In round 0, $\mathscr{D}$ chooses an entourage D_0, followed by $\mathscr{P}$ choosing a point $p_0 \in X$. In round $n+1$, $\mathscr{D}$ chooses an entourage D_{n+1}, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in D_n[p_n]$.

$\mathscr{D}$ wins in the case that either $\langle p_0, p_1, \dots\rangle$ converges in X, or $\bigcap_{n<\omega} D_n[p_n] = \emptyset$. $\mathscr{P}$ wins otherwise.

This game is considered "hard" due to the requirement that $\mathscr{D}$ keep track of the history of the game to ensure that successive moves refine previous moves. This record-keeping may be eliminated by requiring that $\mathscr{P}$ respect all moves made by $\mathscr{D}$ rather than only the most recent move.

Definition 1.12. Let $Bell^{\rightarrow}_{D,P}(X)$ denote the *Bell convergence game* with players $\mathscr{D}$, $\mathscr{P}$ which proceeds analogously to $Bell^{\rightarrow,\star}_{D,P}(X)$, except for the following. Let

$E_n = \bigcap_{m\leq n} D_n$, where D_n is the entourage played by $\mathscr{D}$ in round n. Then $\mathscr{P}$ must ensure that $p_{n+1} \in E_n[p_n]$, and $\mathscr{D}$ wins when either $\langle p_0, p_1, \dots\rangle$ converges in X or $\bigcap_{n<\omega} E_n[p_n] = \emptyset$.

These games are all essentially equivalent with respect to perfect information for $\mathscr{D}$.

Theorem 1.13. *$\mathscr{D} \uparrow Bell^{\rightarrow,\star}_{D,P}(X)$ if and only if $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$ if and only if X is proximal.*

Proof. If $\mathscr{D} \uparrow Bell^{\rightarrow,\star}_{D,P}(X)$, then we immediately see that $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$. If σ is a winning strategy for $\mathscr{D}$ in $Bell^{\rightarrow}_{D,P}(X)$, then τ defined by $\tau(s) = \bigcap_{t\leq s} \sigma(t)$ is easily seen to be a winning strategy for $\mathscr{D}$ in $Bell^{\rightarrow,\star}_{D,P}(X)$.

If $\mathscr{D} \uparrow Bell^{\rightarrow,\star}_{D,P}(X)$, then $\mathscr{D} \uparrow Bell^{\mathrm{uni}}_{D,P}(X)$ for the universal uniformity, showing X is proximal. Finally, if X is proximal, then there exists a winning strategy σ for $Bell^{\mathrm{uni}}_{D,P}(X)$ for a uniformity inducing the topology on X. Then a winning strategy for $\mathscr{D}$ in $Bell^{\rightarrow,\star}_{D,P}(X)$ may be constructed by converting every entourage in this uniformity into a smaller open symmetric universal entourage. $\square$

The secondary winning condition in $Bell^{\rightarrow}_{D,P}(X)$ essentially handles the case where $\mathscr{P}$'s points converge to a "hole" in the space. Uniformly locally compact spaces (and in particular, compact spaces) lack such holes, so it will be convenient to eliminate this technicality when it is irrelevant.

Definition 1.14. Let $Bell^{\rightarrow}_{D,P}(X)$ denote the *absolute Bell convergence game* which proceeds analogously to $Bell^{\rightarrow}_{D,P}(X)$, except that $\mathscr{D}$ must always ensure that $\langle p_0, p_1, \dots\rangle$ converges in X in order to win.

Definition 1.15. A uniformizable space X is *absolutely proximal* if $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$.

As was shown in [2]:

Definition 1.16. A uniformizable space X is *uniformly locally compact* if there exists an entourage D such that $\overline{D[x]}$ is compact for all x.

Theorem 1.17. *If X is a uniformly locally compact space, then $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$ if and only if $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$.*

2. Limited information generalizations

References

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