Definition 1. For any partition $\mathcal{R}$ of a space X and $x \in X$, let $\mathcal{R}[x]$ be such that $x \in \mathcal{R}[x] \in \mathcal{R}$.

Proposition 2. $x \in \mathcal{R}[y] \Leftrightarrow y \in \mathcal{R}[x]$

Definition 3. For zero-dimensional X, the proximity game $Prox_{R,P}(X)$ proceeds as follows: in round n, $\mathscr{R}$ chooses a clopen partition $\mathcal{R}_n$ of X, followed by $\mathscr{P}$ choosing a point $p_n \in X$.

For clopen partitions $\mathcal{R}_0, \dots, \mathcal{R}_n$, let $\mathcal{H}_n$ be the coarsest partition which refines each $\mathcal{R}_m$. Player $\mathscr{R}$ wins if either $\bigcap_{n<\omega} \mathcal{H}_n[p_n] = \emptyset$ or p_n converges.

Proposition 4. *This game is perfect-information equivalent to the analogous game studied by Bell, requiring $\mathscr{P}$'s play p_{n+1} to be in $\mathcal{H}_n[p_n]$ in rounds $n+1$, and requiring $\mathscr{O}$ choose refinements.*

Proof. Allowing $\mathscr{P}$ to play $p_{n+1} \notin \mathcal{H}_n[p_n] \Rightarrow \mathcal{H}_n[p_{n+1}] \neq \mathcal{H}_n[p_n]$ does not introduce any new winning plays for $\mathscr{P}$ as for any such move, $\bigcap_{m<\omega} \mathcal{H}_n[p_n] \subseteq \mathcal{H}_{n+1}[p_{n+1}] \cap \mathcal{H}_n[p_n] \subseteq \mathcal{H}_n[p_{n+1}] \cap \mathcal{H}_n[p_n] = \emptyset$.

Allowing $\mathscr{R}$ to play non-refining clopen partitions does not introduce any new winning plays for $\mathscr{R}$ as the winning condition relies on the refinement of all $\mathcal{R}_n$ anyway. $\square$

Definition 5. A space X is **proximal** iff X is zero-dimensional and $\mathscr{R} \uparrow Prox_{R,P}(X)$.

Definition 6. For any space X and a point $x \in X$, the W**-convergence-game** $Con_{O,P}(X, x)$ proceeds as follows: in round n, $\mathscr{O}$ chooses a neighborhood U_n of x, followed by $\mathscr{P}$ choosing a point $p_n \in X$.

For open sets $U_0, \dots, U_n$, let $V_n = \bigcap_{m \leq n} U_m$. Player $\mathscr{O}$ wins if either $p_n \notin V_n$ for some $n < \omega$, or if p_n converges.

Definition 7. A space X is a W**-space** iff $\mathscr{O} \uparrow Con_{O,P}(X, x)$ for all $x \in X$.

Definition 8. For each finite tuple $(m_0, \dots, m_{n-1})$, we define the k**-tactical fog-of-war**

$$T_k(m_0, \dots, m_{n-1}) = (m_{n-k}, \dots, m_{n-1})$$

and the k**-Marköv fog-of-war**

$$M_k(m_0, \dots, m_{n-1}) = (m_{n-k}, \dots, m_{n-1}, n-1)$$

So $P \uparrow_{k\text{-tact}} G$ if and only if there exists a winning strategy for P of the form $\sigma \circ T_k$, and $P \uparrow_{k\text{-mark}} G$ if and only if there exists a winning strategy of the form $\sigma \circ M_k$.

Theorem 9. *For all $x \in X$:*

- $\mathscr{R} \uparrow Prox_{R,P}(X) \Rightarrow \mathscr{O} \uparrow Con_{O,P}(X,x)$
- $\mathscr{R} \uparrow_{pre} Prox_{R,P}(X) \Rightarrow \mathscr{O} \uparrow_{pre} Con_{O,P}(X,x)$
- $\mathscr{R} \uparrow_{2k\text{-}tact} Prox_{R,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(X,x)$
- $\mathscr{R} \uparrow_{2k\text{-}mark} Prox_{R,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(X,x)$

Proof. Let σ witness $\mathscr{R} \uparrow_{2k\text{-tact}} Prox_{R,P}(X)$ (resp. $\mathscr{R} \uparrow_{2k\text{-mark}} Prox_{R,P}(X)$, $\mathscr{R} \uparrow Prox_{R,P}(X)$). We define the k-tactical (resp. k-Markōv, perfect info) strategy τ such that

$$\tau \circ L_k(p_0, \ldots, p_{n-1}) = \sigma \circ L_{2k}(x, p_0, \ldots, x, p_{n-1})[x] \cap \sigma \circ L_{2k}(x, p_0, \ldots, x, p_{n-1}, x)[x]$$

where L_{2k} is the $2k$-tactical fog-of-war (resp. $2k$-Markōv fog-of-war, identity) and L_k is the k-tactical fog-of-war (resp. k-Markōv fog-of-war, identity).

Let $p_0, p_1, \ldots$ attack τ such that $p_n \in V_n = \bigcap_{m \leq n} \tau \circ L_k(p_0, \ldots, p_{m-1})$ for all $n < \omega$. Consider the attack $q_0, q_1, \ldots$ against the winning strategy σ such that $q_{2n} = x$ and $q_{2n+1} = p_n$.

Certainly, $x \in \mathcal{H}_{2n}[x] = \mathcal{H}_{2n}[q_{2n}]$ for any $n < \omega$. Note also for any $n < \omega$ that

$$p_n \in V_n = \bigcap_{m \leq n} \tau \circ L_k(p_0, \ldots, p_{m-1})$$

$$= \bigcap_{m \leq n} (\sigma \circ L_{2k}(x, p_0, \ldots, x, p_{m-1})[x] \cap \sigma \circ L_{2k}(x, p_0, \ldots, x, p_{m-1}, x)[x])$$

$$= \bigcap_{m \leq n} (\sigma \circ L_{2k}(q_0, q_1, \ldots, q_{2m-2}, q_{2m-1})[x] \cap \sigma \circ L_{2k}(q_0, q_1, \ldots, q_{2m-2}, q_{2m-1}, q_{2m})[x])$$

$$\bigcap_{m \leq n} \mathcal{R}_{2m}[x] \cap R_{2m+1}[x] = \mathcal{H}_{2n+1}[x]$$

so $x \in \mathcal{H}_{2n+1}[p_n] = \mathcal{H}_{2n+1}[q_{2n+1}]$. Thus $x \in \bigcap_{n<\omega} \mathcal{H}_n[q_n]$, and since σ is a winning strategy, the attack $q_0, q_1, \ldots$ converges, and must converge to x. Thus $p_0, p_1, \ldots$ converges to x, and τ is also a winning strategy. $\square$

Corollary 10. *For all $x \in X$:*

- $\mathscr{R} \uparrow_{k\text{-}tact} Prox_{R,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(X,x)$
- $\mathscr{R} \uparrow_{k\text{-}mark} Prox_{R,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(X,x)$

Corollary 11. *All proximal spaces are W-spaces.*

Definition 12. In the one-point compactification $\kappa^* = \kappa \cup \{\infty\}$ of discrete κ, define the clopen partition $\mathcal{C}(F) = [F]^1 \cup \{\kappa^* \setminus F\}$.

Theorem 13. $\mathscr{R} \uparrow_{code} Prox_{R,P}(\kappa^*)$

Proof. Use the coding strategy $\sigma() = \mathcal{C}(\emptyset) = \{\kappa^*\}$, $\sigma(\mathcal{C}(F), \alpha) = \mathcal{C}(F \cup \{\alpha\})$ for $\alpha < \kappa$ and $\sigma(\mathcal{C}(F), \infty) = \mathcal{C}(F)$. Note $\mathcal{R}_n = \mathcal{H}_n$. For any attack $p_0, p_1, \dots$ against σ such that $\bigcap_{n<\omega} \mathcal{H}_n[p_n] \neq \emptyset$, suppose

- $\infty \in \bigcap_{n<\omega} \mathcal{H}_n[p_n]$. Then $p_n \in \kappa^* \setminus \{p_m : m < n\}$ shows that the non-∞ p_n are all distinct. If co-finite $p_n = \infty$, we have $p_n \to \infty$. Otherwise, there are infinite distinct p_n, and since neighorhoods of ∞ are co-finite, we have $p_n \to \infty$.
- $\infty \notin \ \mathcal{H}_N[p_N]$ for some $N < \omega$, so $\alpha \in \bigcap_{n<\omega} \mathcal{H}_n[p_n]$ for some $\alpha < \kappa$. Then $\mathcal{H}_n[p_n] = \{\alpha\}$ for all $n \geq N$, and thus $p_n \to \alpha$.

Thus σ is a winning coding strategy. □

Theorem 14. $\mathscr{O} \uparrow Con_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow Prox_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{pre} Con_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{pre} Prox_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{k\text{-}tact} Prox_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{k\text{-}mark} Prox_{R,P}(\kappa^*)$

Proof. Let $\sigma \circ L$ be a winning strategy where L is the identify (resp. a k-tactical fog-of-war, a k-Marköv fog-of-war).

Define $\tau \circ L$ such that

$$\tau \circ L(p_0, \dots, p_{n-1}) = \mathcal{R}(\kappa^* \setminus (\sigma \circ L(p_0, \dots, p_{n-1})))$$

For any attack $p_0, p_1, \dots$ against τ such that $\bigcap_{n<\omega} \mathcal{H}_n[p_n] \neq \emptyset$, suppose

- $\mathcal{H}_n[p_n] = \mathcal{H}_n[\infty] = \bigcap_{m \leq n} \sigma \circ L(p_0, \dots, p_{m-1}) = \bigcap_{m \leq n} U_m = V_n$ for all $n < \omega$. Since σ is a winning strategy, the p_n converge at ∞.
- $\mathcal{H}_N[p_N] \neq \mathcal{H}_N[\infty]$ for some $N < \omega$. Then $\mathcal{H}_N[p_N] = \{p_N\}$, and since $\bigcap_{n<\omega} \mathcal{H}_n[p_n] \neq \emptyset$, we have $\mathcal{H}_n[p_n] = \mathcal{H}_N[p_N] = \{p_N\} \Rightarrow p_n = p_N$ for all $n \geq N$, and the p_n converge at p_N.

□

Corollary 15. $\mathscr{O} \uparrow Con_{O,P}(\kappa^*, \infty) \Leftrightarrow \mathscr{R} \uparrow Prox_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{pre} Con_{O,P}(\kappa^*, \infty) \Leftrightarrow \mathscr{R} \uparrow_{pre} Prox_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(\kappa^*, \infty) \Leftrightarrow \mathscr{R} \uparrow_{k\text{-}tact} Prox_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(\kappa^*, \infty) \Leftrightarrow \mathscr{R} \uparrow_{k\text{-}mark} Prox_{R,P}(\kappa^*)$

Corollary 16. *$O \uparrow_{pre} Prox_{R,P}(\omega^*)$.*

$O \uparrow_{tact} Prox_{R,P}(\omega^*)$.

$O \not\uparrow_{k\text{-}mark} Prox_{R,P}(\kappa^*)$ for $\kappa \geq \omega_1$.

Proof. Results hold for $\mathscr{O}$ and $Con_{O,P}(\kappa^*, \infty)$. $\square$

Definition 17. The **almost-proximal game** $aProx_{R,P}(X)$ is analogous to $Prox_{R,P}(X)$ except that the points p_n need only cluster for $\mathscr{R}$ to win the game.

Definition 18. The W**-clustering game** $Clus_{O,P}(X, x)$ is analogous to $Con_{O,P}(X, x)$ except that the points p_n need only cluster at x for $\mathscr{O}$ to win the game.

Proposition 19. $\mathscr{O} \uparrow Clus_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow aProx_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{pre} Clus_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{pre} aProx_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}tact} Clus_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{k\text{-}tact} aProx_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}mark} Clus_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{k\text{-}mark} aProx_{R,P}(\kappa^*)$

Proof. Same proof as before, replacing "converge" with "cluster". $\square$

Corollary 20. *$\mathscr{R} \uparrow_{mark} aProx_{R,P}(\omega_1^*)$.*

Proof. Holds for $\mathscr{O}$ and $Clus_{O,P}(\omega_1^*, \infty)$. $\square$