

ALMOST COMPATIBLE FUNCTIONS AND INFINITE LENGTH GAMES

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ABSTRACT. $\mathcal{A}'(\kappa)$ asserts the existence of pairwise almost compatible finite-to-one functions $A \rightarrow \omega$ for each countable subset A of κ . The existence of winning 2-Markov strategies in several infinite-length games, including the Menger game on the one-point Lindelöfication κ^\dagger of κ , are guaranteed by $\mathcal{A}'(\kappa)$. $\mathcal{A}'(\kappa)$ is implied by the existence of cofinal Kurepa families of size κ , and thus holds for all cardinals less than \aleph_ω . It's consistent that $\mathcal{A}'(\aleph_\omega)$ fails, but there must always be a winning 2-Markov strategy for the second player in the Menger game on ω_ω^\dagger .

1. INTRODUCTION

Definition 1. Two functions f, g are almost compatible, that is, $f \sim g$ when $\{a \in \text{dom } f \cap \text{dom } g : f(a) \neq g(a)\}$ is finite.

Marion Scheepers used almost compatible functions in [9] in order to study the existence of limited information strategies on a variation of the meager-nowhere dense game he introduced in [10].

Game 2. Let $Sch_{C,F}^{\cup, \subset}(\kappa)$ denote *Scheepers' strict countable-finite union game* with two players \mathcal{C}, \mathcal{F} . In round 0, \mathcal{C} chooses $C_0 \in [\kappa]^{\leq \omega}$, followed by \mathcal{F} choosing $F_0 \in [\kappa]^{< \omega}$. In round $n + 1$, \mathcal{C} chooses $C_{n+1} \in [\kappa]^{\leq \omega}$ such that $C_{n+1} \supset C_n$, followed by \mathcal{F} choosing $F_{n+1} \in [\kappa]^{< \omega}$.

\mathcal{F} wins the game if $\bigcup_{n < \omega} F_n \supseteq \bigcup_{n < \omega} C_n$; otherwise, \mathcal{C} wins.

Of course, with perfect information this game is trivial: during round n player \mathcal{F} simply chooses n ordinals from each of the n countable sets played by \mathcal{C} . However, if \mathcal{F} is limited to using information from the last k moves by \mathcal{C} during each round, the task becomes more difficult. Call such a strategy a *k-tactical strategy* or *k-tactic*; if using the round number is allowed, then the strategy is called a *k-Markov strategy* or a *k-mark*.

Definition 3. The statement $\mathcal{A}(\kappa)$ (given as $S(\kappa, \aleph_0, \omega)$ in [9] and $S(\kappa)$ in [1]) claims that there exist one-to-one functions $f_A : A \rightarrow \omega$ for each $A \in [\kappa]^{\leq \aleph_0}$ such that the collection $\{f_A : A \in [\kappa]^{\leq \aleph_0}\}$ is pairwise almost compatible.

In the same paper, Scheepers noted that $\mathcal{A}(\omega_1)$ holds in *ZFC*, and that it's possible to force \mathfrak{c} to be arbitrarily large while preserving $\mathcal{A}(\mathfrak{c})$. However, $\mathcal{A}(\mathfrak{c}^+)$ always fails. This axiom may be applied to obtain a winning 2-tactic for \mathcal{F} in the countable-finite game.

In [1], Clontz related this game to a game which may be used to characterize the Menger covering property of a topological space.

Game 4. Let $Men_{C,F}(X)$ denote the *Menger game* with players \mathcal{C} , \mathcal{F} . In round n , \mathcal{C} chooses an open cover \mathcal{U}_n , followed by \mathcal{F} choosing subset F_n of X which may be finitely covered by \mathcal{U}_n .

\mathcal{F} wins the game if $X = \bigcup_{n < \omega} F_n$, and \mathcal{C} wins otherwise.

This characterization is slightly different than the typical characterization in which the second player first chooses a specific finite subcollection \mathcal{F}_n of the cover itself and lets $F_n = \bigcup \mathcal{F}_n$, denoted as $G_{fin}(\mathcal{O}, \mathcal{O})$ in [11]. However, it's easily seen that these games are equivalent for perfect information strategies (so both characterize the Menger property), and this characterization is more convenient for our concerns.

Definition 5. Let $\kappa^\dagger = \kappa \cup \{\infty\}$ where κ is discrete and ∞ 's neighborhoods are the co-countable sets containing it.

The relationship between $Sch_{C,F}^{\cup, \subseteq}(\kappa)$ and $Men_{C,F}(\kappa^\dagger)$ is strong; in both games \mathcal{C} essentially chooses a countable subset of κ followed by \mathcal{F} choosing a finite subset of that choice, and it's easy to see the winning perfect information strategy for \mathcal{F} in both games. In addition, it was shown in [1] that when $\mathcal{A}(\kappa)$ holds, \mathcal{F} has a winning 2-Markov strategy in $Men_{C,F}(\kappa^\dagger)$.

One source of motivation is to make progress on the following open question:

Question 6. *Does there exist a topological space X for which $\mathcal{F} \uparrow Men_{C,F}(X)$ but $\mathcal{F} \not\uparrow_{2\text{-mark}} Men_{C,F}(X)$? (That is, the second player can win the Menger game on X with perfect information but not with 2-Markov information.)*

2. ONE-TO-ONE AND FINITE-TO-ONE ALMOST COMPATIBLE FUNCTIONS

We may weaken Scheeper's $\mathcal{A}(\kappa)$ as follows:

Definition 7. The statement $\mathcal{A}'(\kappa)$ weakens $\mathcal{A}(\kappa)$ by only requiring the witnessing almost-compatible functions $f_A : A \rightarrow \omega$ to be finite-to-one.

Proposition 8. $\mathcal{A}(\kappa)$ and $\mathcal{A}'(\kappa)$ need only be witnessed by functions $\{f_A : A \in \mathcal{S}\}$ for some family \mathcal{S} cofinal in $[\kappa]^{\leq \aleph_0}$.

Proof. For each $A \in [\kappa]^{\leq \aleph_0}$ choose $A' \supseteq A$ from \mathcal{S} and let $g_A = f_{A'} \upharpoonright A$. □

In the next section we will show that $\mathcal{A}'(\kappa)$ is sufficient for the applications to the Scheepers and Menger games. In the meantime, we will demonstrate that $\mathcal{A}'(\kappa)$ is strictly weaker than $\mathcal{A}(\kappa)$.

Recall the following.

Definition 9. A Kurepa family $\mathcal{K} \subseteq [\kappa]^{\aleph_0}$ on κ satisfies that $\mathcal{K} \upharpoonright A = \{K \cap A : K \in \mathcal{K}\}$ is countable for each $A \in [\kappa]^{\aleph_0}$. Let $\mathcal{K}(\kappa)$ be the statement claiming there exists a Kurepa family on κ cofinal in $[\kappa]^{\aleph_0}$.

Theorem 10. $\mathcal{K}(\kappa) \Rightarrow \mathcal{A}'(\kappa)$.

Proof. Let $\mathcal{K} = \{K_\alpha : \alpha < \theta\}$ be a cofinal Kurepa family on κ . We first define $f_\alpha : K_\alpha \rightarrow \omega$ for each $\alpha < \theta$.

Suppose we've already defined pairwise almost compatible finite-to-one functions $\{f_\beta : \beta < \alpha\}$. To define f_α , we first recall that $\mathcal{K} \restriction K_\alpha$ is countable, so we may choose $\beta_n < \alpha$ for $n < \omega$ such that $\{K_\beta : \beta < \alpha\} \restriction K_\alpha \setminus \{\emptyset\} = \{K_\alpha \cap K_{\beta_n} : n < \omega\}$. Let $K_\alpha = \{\delta_{i,j} : i \leq \omega, j < w_i\}$ where $w_i \leq \omega$ for each $i \leq \omega$, $K_\alpha \cap (K_{\beta_n} \setminus \bigcup_{m < n} K_{\beta_m}) = \{\delta_{n,j} : j < w_n\}$, and $K_\alpha \setminus \bigcup_{n < \omega} K_{\beta_n} = \{\delta_{\omega,j} : j < w_\omega\}$. Then let $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ for $n < \omega$ and $f_\alpha(\delta_{\omega,j}) = j$ otherwise.

We should show that f_α is finite-to-one. Let $n < \omega$. Since $f_\alpha(\delta_{m,j}) \geq m$, we only consider the finite cases where $m \leq n$. Since each f_{β_m} is finite-to-one, $f_{\beta_m}(\delta_{m,j}) \leq n$ for only finitely many j . Thus $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$ maps to n for only finitely many j .

We now want to demonstrate that $f_\alpha \sim f_{\beta_n}$ for all $n < \omega$. Note $\delta_{m,j} \in K_{\beta_n}$ implies $m \leq n$. For $m = n$, we have $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ which differs from $f_{\beta_n}(\delta_{n,j})$ for only the finitely many j which are mapped below n by f_{β_n} . For $m < n$ and $\delta_{m,j} \in K_{\beta_n}$, we have $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$ which can only differ from $f_{\beta_n}(\delta_{m,j})$ for only the finitely many j which are mapped below m by f_{β_m} or the finitely many j for which the almost compatible $f_{\beta_n} \sim f_{\beta_m}$ differ.

Finally for any $\beta < \alpha$, we may conclude $f_\alpha \sim f_\beta$ since there is some β_n with $K_\alpha \cap K_\beta = K_\alpha \cap K_{\beta_n}$, $f_\alpha \sim f_{\beta_n}$, and $f_{\beta_n} \sim f_\beta$. \square

We now construct a topology on ω_n for each $n < \omega$ which will witness a Kurepa family of size \aleph_n . A similar construction was previously shown by Juhász et. al. in [5], and the relationship of Kurepa families and such spaces has also been investigated in a preprint of Nyikos [8].

Proposition 11. *Let X be a T_2 space with a base of countable and compact neighborhoods. Then X is locally metrizable with a base of compact open countable sets.*

Proof. For each point x let K be a countable and compact neighborhood of x , and it follows that it is contained in a countable, open, and locally compact neighborhood W of x , which in turn is zero-dimensional and metrizable. So choose V clopen in W such that $x \in V \subseteq K$; V is a compact open neighborhood of x in X . \square

Definition 12. A topological space is said to be ω -bounded if each countable subset of the space has compact closure. As in [5] we call a T_2 , locally countable, ω -bounded space *splendid*, and let $\mathcal{S}(\kappa)$ represent the claim that there exists a splendid space of cardinality κ .

Proposition 13. *Let X be a T_2 space with cardinality less than \aleph_ω which is locally countable and ω -bounded. Then the closure operation preserves cardinality and weight.*

Proof. Note that the closure of any countable neighborhood is compact, and any Lindelöf set is countable. This space is locally metrizable and thus first-countable, so cardinality and weight coincide for any subspace. The result is obvious if A is countable; otherwise let $A = \{a_\alpha : \alpha < \omega_{n+1}\}$ and since basic neighborhoods are countable note any limit point of A is a limit point of $A_\beta = \{a_\alpha : \alpha < \beta\}$ for some $\beta < \omega_{n+1}$. Thus $\overline{A} = \bigcup_{\beta < \omega_{n+1}} \overline{A_\beta}$ and by induction $|\overline{A}| = |A|$. \square

Lemma 14. *Let X be a T_2 space with cardinality less than \aleph_ω which is locally countable and locally compact, and such that its closure operation preserves cardinalities. Then X has an ω -bounded extension \tilde{X} with the same properties where $\tilde{X} \setminus X$ has the same cardinality as X .*

Proof. We prove this by induction on n . If $n = 0$, then we can just use the one-point compactification of two copies of X . So suppose $n > 0$ and that $X = \omega_n$ has an appropriate topology. Note that X has a base of countable and compact neighborhoods since the closure operation preserves cardinalities.

For each $\alpha < \omega_n$, γ_α may be chosen such that both the closure of the set α in X and a countable neighborhood of the point α are subsets of γ_α . Note that the set $\{\lambda < \omega_n : \alpha < \lambda \Rightarrow \gamma_\alpha < \lambda\}$ is a cub subset of ω_n containing a cub subset C of limit ordinals. Now for each $\lambda \in C$, the set λ is open as $\alpha < \lambda$ belongs to the neighborhood $\gamma_\alpha \subsetneq \lambda$. Also, if λ has uncountable cofinality, then for $\beta \geq \lambda$ and any countable neighborhood U of β , $U \cap \lambda = U \cap \alpha$ for some $\alpha < \lambda$; thus $U \setminus \bar{\alpha} = U \setminus \lambda$ is a neighborhood of β , showing that λ is clopen.

Let $\tilde{X} = \omega_n \times 2$. By induction on $\lambda \in C$ we will define compatible topologies for $\tilde{X}_\lambda = \omega_n \times \{0\} \cup \lambda \times \{1\}$ such that

- $\omega_n \times \{0\}$ is an open copy of X ,
- $\lambda \times 2$ is open, and when $\text{cf}(\lambda) > \omega$ also closed,
- the space has a base of countable and compact neighborhoods, and
- when λ is a successor, for each $\alpha < \lambda$ the closure of $\alpha \times 2$ is an ω -bounded subset of $\lambda \times 2$.

We first consider the case $n = 1$. If λ is a limit in C , then $\tilde{X}_\lambda = \bigcup_{\mu \in C \cap \lambda} \tilde{X}_\mu$ satisfies the induction requirements. Otherwise we choose an increasing sequence of ordinals $\{\alpha_k : k \in \omega\}$ with limit λ such that α_0 is the predecessor of λ in C , or $\alpha_0 = 0$ if λ is the least element of C .

The subspace $\bar{\lambda} \times \{0\} \cup \alpha_0 \times 2$ of X is countable and locally compact; therefore it is metrizable and zero-dimensional. So we may choose increasing sets U_k for $k < \omega$ which are clopen in this topology and satisfy

$$\overline{\alpha_k \times \{0\} \cup \alpha_0 \times 2} = \bar{\alpha}_k \times \{0\} \cup \alpha_0 \times 2 \subseteq U_k \subseteq \lambda \times \{0\} \cup \alpha_0 \times 2$$

Note that U_k is also clopen in \tilde{X}_{α_0} since it is closed in $\bar{\lambda} \times \{0\} \cup \alpha_0 \times 2$ and open in $\lambda \times \{0\} \cup \alpha_0 \times 2$.

We need only describe a base for the points $\langle \alpha, 1 \rangle \in (\lambda \setminus \alpha_0) \times \{1\}$. We do so by letting $\langle \alpha, 1 \rangle$ be isolated when $\alpha \notin \{\alpha_k : k < \omega\}$, and giving $\langle \alpha_k, 1 \rangle$ the open neighborhoods $(U_k \cup ((\alpha_k + 1) \times \{1\})) \setminus K$ for each compact subset K of $U_k \cup (\alpha_k \times \{1\})$; that is, $\langle \alpha_k, 1 \rangle$ is the one point compactifying $U_k \cup (\alpha_k \times \{1\})$.

The first two requirements of our inductive hypothesis are obviously satisfied. Note points in $\lambda \times 2$ are covered by the compact countable neighborhood $U_k \cup ((\alpha_k + 1) \times \{1\})$ for some $k < \omega$, and for points in $(\omega_n \setminus \lambda) \times \{0\}$ we may use a compact countable neighborhood from X . For the final requirement, note that for $\alpha < \lambda$, we may choose $\alpha < \alpha_k < \lambda$ and note that $\alpha \times 2$ is contained in the compact subset $U_k \cup ((\alpha_k + 1) \times \{1\})$ of $\lambda \times 2$.

For the case $n > 1$, we may assume that the successors in C have uncountable cofinality. We again proceed by induction on $\lambda \in C$. Again when λ is a limit in C , $\tilde{X}_\lambda = \bigcup_{\mu \in C \cap \lambda} \tilde{X}_\mu$ satisfies the given requirements; in particular if $\alpha < \lambda$, then $\alpha < \mu < \lambda$ for some successor $\mu \in C$ with uncountable cofinality. As such, the closure of $\alpha \times 2$ is an ω -bounded subset of the clopen $\mu \times 2$ and therefore also of $\lambda \times 2$. In case λ is not a limit of C , then λ has uncountable cofinality and a predecessor $\mu \in C$. We therefore have that $\lambda \times \{0\}$ is clopen in $\omega_n \times \{0\}$. Since the cardinality of $\lambda \times \{0\} \cup \mu \times 2$ is less than \aleph_n , we may simply apply the induction hypothesis to choose an appropriate topology for $\lambda \times 2$.

As a result, $\tilde{X} = \bigcup_{\lambda \in C} \tilde{X}_\lambda$ is ω -bounded as any countable set is contained in some $\alpha \times 2$ for $\alpha < \lambda \in C$. \square

Theorem 15. *For each $k < \omega$, there is a T_2 , locally countable, ω -bounded topology on ω_k . That is, $\mathcal{S}(\aleph_k)$ for all $k < \omega$.*

Proof. Apply the previous lemma to ω_n with the discrete topology. \square

Lemma 16. *The family of compact open sets in a locally countable, ω -bounded topological space X is a Kurepa family cofinal in $[X]^\omega$. That is, $\mathcal{S}(\kappa) \Rightarrow \mathcal{K}(\kappa)$.*

Proof. Let \mathcal{K} collect all compact open subsets of X . Of course, every Lindelöf set in a locally countable space is countable, and the closure of every countable set is a compact countable set; thus \mathcal{K} is cofinal in $[X]^\omega$. It is Kurepa since every countable set is contained in a countable compact open subspace of X ; this subspace has a countable base of compact open sets, which closed under finite unions enumerates all compact open subsets of the subspace. \square

Corollary 17. $\mathcal{K}(\aleph_k)$ for all $k < \omega$.

Alternatively, the previous corollary may be obtained via an observation of Todorćević communicated by Dow in [2]: if every Kurepa family of size at most κ extends to a cofinal Kurepa family, then the same is true of κ^+ .

Nyikos points out in [8] that a cofinal Kurepa family may be used to construct a locally metrizable, ω -bounded, zero-dimensional space with appropriate cardinality, but whether this can be strengthened to locally countable and ω -bounded (as asked in [5]) remains an open question.

Also left open is this extension of the question asked in [8] and [5] on the possible equivalence of $\mathcal{S}(\kappa)$ and $\mathcal{K}(\kappa)$.

Question 18. *May any of the implications in the theorem $\mathcal{S}(\kappa) \Rightarrow \mathcal{K}(\kappa) \Rightarrow \mathcal{A}'(\kappa)$ be reversed?*

Regardless, we have obtained our desired result.

Corollary 19. $\mathcal{A}'(\aleph_k)$ for all $k < \omega$.

3. CONSISTENCY RESULTS

As noted in [2], Jensen's one-gap two-cardinal theorem under $V = L$ introduced in [4] implies that $\mathcal{K}(\kappa)$ holds for all cardinals κ .

Corollary 20 ($V = L$). $\mathcal{A}'(\kappa)$ for all cardinals κ .

Weakening to the continuum hypothesis, we see an obvious consequence.

Corollary 21 (CH). $\mathcal{A}'(\mathfrak{c}^+)$, but $\neg\mathcal{A}(\mathfrak{c}^+)$.

But CH is not required to have $\mathcal{A}(\aleph_2)$ fail.

Lemma 22 ([6]). Assume that $G \subset \text{Fn}(\omega_2, 2)$ is a generic filter, and let $\mu \in \omega_2$. Then the final model $V[G]$ can be regarded as forcing with $\text{Fn}(\omega_2 \setminus \mu, 2)$ over the model $V[G_\mu]$. In addition, for each $\text{Fn}(\omega_2, 2)$ -name \dot{A} of a subset of ω (treat as a subset of $\omega \times \text{Fn}(\omega_2, 2)$), there is a canonical name $\dot{A}(G_\mu)$ where,

$$\dot{A}(G_\mu) = \{(n, p \restriction [\mu, \omega_2)) : (n, p) \in \dot{A} \text{ and } p \restriction \mu \in G_\mu\}$$

and we get that the valuation of $\dot{A}(G_\mu)$ by the tail of the generic, $G_{\omega_2 \setminus \mu}$, is the same as the valuation of \dot{A} by the full generic.

Theorem 23. There exists a model of ZFC for which $\mathfrak{c} = \aleph_2$ and $\neg\mathcal{A}(\aleph_2)$.

Proof. The forcing poset is $\text{Fn}(\omega_2, 2)$. Let $\{\dot{f}_A : A \in [\omega_2]^\omega\}$ be a family of names such that \dot{f}_A is a one-to-one function from A into ω . It suffices to only consider sets A from the ground model.

Put all the lemma stuff in an elementary submodel M of the universe (technically of $H(\kappa)$, or of V_κ , for some large κ). Standard methods says that we can assume that $|M| = \omega_1 = \mathfrak{c}$ and that $M^\omega \subset M$ (which means that every countable subset of M is a member of M).

Let $\lambda = M \cap \omega_2$ (same as the supremum of $M \cap \omega_2$). Consider the name $\dot{f}_{[\lambda, \lambda + \omega)}$. What is such a name? We can assume that it is a set of pairs of the form $((\lambda + k, m), p)$ where $p \in \text{Fn}(\omega_2, 2)$ and, of course, $k, m \in \omega$. This is (almost) equivalent to saying that p forces that $\dot{f}_{[\lambda, \lambda + \omega)}(\lambda + k) = m$. We don't take all such p , in fact for each k, m it is enough to take a countable set of such p to get an equivalent name (Kunen calls it a nice name if we take, for each k, m an antichain that is maximal among such conditions). Given any such name \dot{f} , let $\text{supp}(\dot{f})$ denote the union of the domains of conditions p appearing in the name.

Also let Y equal $\text{supp}(\dot{f}_{[\lambda, \lambda + \omega)}) \setminus \lambda$. Let δ denote the order type of Y and let the 2-parameter notation $\varphi_{\mu, \lambda}$ be the order-preserving function from $\mu \cup Y$ onto the ordinal $\mu + \delta$. This lifts canonically to an order-preserving bijection $\varphi_{\mu, \lambda} : \text{Fn}(\mu \cup Y, 2) \rightarrow \text{Fn}(\mu + \delta, 2)$. Similarly, we make sense of the name $\varphi_{\mu, \lambda}(\dot{f}_{[\lambda, \lambda + \omega)})$, call it F_M . Here simply, for each tuple $((k, m), p) \in \dot{f}_{[\lambda, \lambda + \omega)}$, we have that $((k, m), \varphi_{\mu, \lambda}(p))$ is in F_M . Again, let $\varphi_{\mu, \lambda}(\dot{f}_{[\lambda, \lambda + \omega)})$ be interpreted in the above sense as giving F_M (which is an element of M). Note that we do not regard δ as fixed here, but rather simply depending on the $\text{supp}(\dot{f}_{[\lambda, \lambda + \omega)})$ described above. Other values replacing $\lambda > \mu$ will result in their own set Y and canonical map $\varphi_{\mu, \lambda}$; but one thing we do have to assume (or arrange) for other values α replacing λ is that $\text{supp}(\dot{f}_{[\alpha, \alpha + \omega)})$ intersected with α is contained in μ .

Now the object F_M is an element of M , and M believes this statement is true:

$$(\forall \beta \in \omega_2) (\exists \beta < \lambda \in \omega_2) \text{ supp}(\dot{f}_{[\lambda, \lambda + \omega)}) \cap \lambda \subset \mu \text{ and } F_M = \varphi_{\mu, \lambda}(\dot{f}_{[\lambda, \lambda + \omega)})$$

But now, this means that, not only is there an $\alpha \in M$, $F_M = \varphi_{\mu, \alpha}(\dot{f}_{[\alpha, \alpha + \omega)})$ but also that there is an increasing sequence $\{\alpha_\xi : \xi \in \omega_1\} \subset \lambda$ of such α 's satisfying that, for each ξ we have that $\text{supp}(\dot{f}_{[\alpha_\xi, \alpha_\xi + \omega)})$ is contained in $\alpha_{\xi+1}$.

Choose such a sequence. This means that if we let $A = \bigcup_{n > 0} [\alpha_n, \alpha_n + \omega)$ we have the name \dot{f}_A in M . This then means that all the $((\beta, m), p)$ appearing in \dot{f}_A have the property that $\text{dom}(p)$ is contained in M . There is, within M , a name \dot{g} satisfying that $\dot{f}_A(\alpha_n + k) = \dot{f}_{[\alpha_n, \alpha_n + \omega)}(\alpha_n + k)$ for all $k > \dot{g}(n)$.

We now apply the above Lemma using $\mu = \mu_0$ and we are now working in the extension $V[G_\mu]$. We will abuse the notation and use $\dot{f}_{[\alpha_n, \alpha_n + \omega)}$ instead of $\dot{f}_{[\alpha_n, \alpha_n + \omega)}(G_\mu)$ as defined in the Lemma. We work for a contradiction. Something special has now happened, namely, the supports of the names $\{\dot{f}_{[\alpha_n, \alpha_n + \omega)} : 0 < n < \omega\}$ are pairwise disjoint and also disjoint from the support of the name $\dot{f}_{[\lambda, \lambda + \omega)}$ (under the same convention about G_μ . And not only that, these names are pairwise isomorphic (in the way that they all map to F_M).

Since A is disjoint from $[\lambda, \lambda + \omega)$, there must be an integer ℓ together with a condition $q \in Fn(\omega_2 \setminus \mu, 2)$ satisfying that for all $n > \ell$, q forces that

$$\text{“if } k > \dot{g}(n) \text{ (since } \alpha_n + k \in A) \text{ then } \dot{f}_{[\alpha_n, \alpha_n + \omega)}(\alpha_n + k) \neq \dot{f}_{[\lambda, \lambda + \omega)}(\lambda + k)\text{”}.$$

Choose n large enough so that $\text{dom}(q) \cap [\alpha_n, \mu_{n+1})$ is empty. Choose $q_1 < q \restriction \lambda$ (in M) so that

$$\varphi_{\mu, \alpha_n}(q_1 \restriction \text{supp}(\dot{f}_{[\alpha_n, \alpha_n + \omega)})) = \varphi_{\mu, \lambda}(q \restriction \text{supp}(\dot{f}_{[\lambda, \lambda + \omega)}))$$

and then (again in M) choose $q_2 < q_1$ so that it both forces a value L on $\ell + \dot{g}(n)$ and subsequently forces a value m on $\dot{f}_{[\alpha_n, \alpha_n + \omega)}(\alpha_n + L + 1)$. But now, again calculate

$$q_3 = \varphi_{\mu, \lambda}^{-1} \circ \varphi_{\mu, \alpha_n}(q_2 \restriction \text{supp}(\dot{f}_{[\alpha_n, \alpha_n + \omega)}))$$

and, by the isomorphisms, we have that q_3 forces that $\dot{f}_{[\lambda, \lambda + \omega)}(\lambda + L + 1) = m$.

Technically (or with more care) all of this is taking place in the poset $\text{Fn}(\omega_2 \setminus \mu, 2)$ and this means that q_3 and q are all compatible with each other.

Follow the bouncing ball: it suffices to consider $q(\beta) = e$ and to assume that $q_3(\beta)$ is defined. Since $q_3(\beta)$ is defined, we have that there is a $\beta' \in \text{dom}(q_2)$ such that $\varphi_{\mu, \lambda}(\beta) = \varphi_{\mu, \alpha_n}(\beta')$, and that $q_3(\beta) = q_2(\beta')$. But, by definition of q_1 , $\beta' \in \text{dom}(q_1)$ and even that $q_1(\beta') = q(\beta)$. Then, since $q_2 < q_1$, we have that $q_2(\beta') = q_1(\beta') = q(\beta)$. This completes the circle that $q_3(\beta) = q(\beta)$.

Finally, our contradiction is that $q_3 \cup q_2 \cup q$ forces that $k = L + 1$ violates the quoted statement above. \square

We are also able to force $\mathcal{A}'(\kappa)$ to fail for every cardinal other than the first ω -many we've already guaranteed.

Theorem 24. *It follows from the existence of a 2-huge cardinal that there is a model of ZFC for which $\neg\mathcal{A}'(\aleph_\omega)$.*

Proof. We will need the model constructed in [7] in which an instance of Chang's conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is shown to fail.

We can take as a given (as shown in [7, Theorem 5]) that we may assume that we have a model V of GCH in which there are regular limit cardinals $\kappa < \lambda$ satisfying that $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$.

What this says is that if L is a countable language with at least one unary relation symbol R and M is a model of L with base set $\lambda^{+\omega+1}$ in which the interpretation of R has cardinality $\lambda^{+\omega}$, then M has an elementary submodel N of cardinality $\kappa^{+\omega+1}$ in which $R \cap N$ has cardinality $\kappa^{+\omega}$ (of course $R \cap N$ is the interpretation of R in N because $N \prec M$).

The interested reader will want to know that it is shown in [7] that if κ is a 2-huge cardinal and j is the 2-huge embedding with critical point κ , then with $\lambda = j(\kappa)$ one has that $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$ holds. There is no loss of generality to also assume that GCH holds in this model.

Let $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$ be a scale in $\Pi\{\lambda^{+n+1} : n \in \omega\}$ ordered by the usual mod finite coordinatewise ordering. For convenience we may assume that $h_\xi(n) \geq \lambda^{+n}$ for all ξ and all n . For each integer m the cofinality of the mod finite ordering on $\Pi\{\lambda^{+n+1} : m < n \in \omega\}$ is the same as it is for the entire product $\Pi\{\lambda^{+n+1} : n \in \omega\}$. If P is any poset of cardinality less than λ^{+m} then, in the forcing extension by P , every function in $\Pi\{\lambda^{+n+1} : m < n \in \omega\}$ is bounded above by a ground model function. It therefore follows easily that in the forcing extension by P , the sequence $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$ remains cofinal in $\Pi\{\lambda^{+n+1} : n \in \omega\}$.

The forcing notion \mathbb{P}_0 is simply the finite condition collapse of $\kappa^{+\omega}$, i.e. $\mathbb{P}_0 = (\kappa^{+\omega})^{<\omega}$. In the forcing extension by \mathbb{P}_0 , one now has that the ordinal $\kappa^{+\omega+1}$ from V is the first uncountable cardinal \aleph_1 . Then in this forcing extension we let \mathbb{P}_1 be the countable condition Levy collapse, $Lv(\lambda, \omega_2)$, which collapses all cardinals less than λ to have cardinality at most \aleph_1 . The poset \mathbb{P}_1 has cardinality λ . We treat $\mathbb{P} * \mathbb{P}_1$ as containing \mathbb{P}_0 as a subposet by identifying each $(p_0, 1)$ with p_0 . After forcing with $\mathbb{P}_0 * \mathbb{P}_1$ we will have that ω_1 is the ordinal $(\kappa^{+\omega+1})^V$, ω_2 is the ordinal λ , and ω_ω is the ordinal $(\lambda^{+\omega})^V$.

Now we assume that we have an assignment $\dot{f}_{\dot{A}}$ of a $\mathbb{P}_0 * \mathbb{P}_1$ -name of a finite-to-one function from \dot{A} into ω for each $\mathbb{P}_0 * \mathbb{P}_1$ -name of a countable subset of $\lambda^{+\omega+1}$. We will obtain a contradiction.

Let $\{\dot{A}_\xi : \xi \in \lambda^{+\omega+1}\}$ be an enumeration of all the nice \mathbb{P}_0 -names of countable subsets of $\lambda^{+\omega}$. For each $\xi \in \lambda^{+\omega+1}$, let \dot{f}_ξ be another notation for $\dot{f}_{\dot{A}_\xi}$. Since \mathbb{P}_0 forces that \mathbb{P}_1 is countably closed, the collection of all nice \mathbb{P}_0 -names will produce all the countable sets in the extension by $\mathbb{P}_0 * \mathbb{P}_1$, but $\mathbb{P}_0 * \mathbb{P}_1$ can introduce new enumerations of these names. For each $\xi \in \lambda^{+\omega+1}$, there is a minimal ζ_ξ so that \dot{A}_{ζ_ξ} is the canonical name for the range of h_ξ . This means that $\dot{f}_{\zeta_\xi} \circ h_\xi$ is simply the $\mathbb{P}_0 * \mathbb{P}_1$ -name of a finite-to-one function from ω to ω . For each $\xi \in \lambda^{+\omega+1}$, choose any $p_\xi \in \mathbb{P}_0 * \mathbb{P}_1$ so that there is a nice \mathbb{P}_0 -name, \dot{H}_ξ , that is forced by p_ξ to equal $\dot{f}_{\zeta_\xi} \circ h_\xi$. Choose $\Lambda \subset \lambda^{+\omega+1}$ of cardinality $\lambda^{+\omega+1}$ and so that there is a pair p, \dot{H} satisfying that $p_\xi = p$ and $\dot{H}_\xi = \dot{H}$ for all $\xi \in \Lambda$. We may assume that p is in a generic filter G .

Let $\{x_\xi : \xi \in \lambda^{+\omega+1}\}$ be any enumeration of $H(\lambda^{+\omega+1})$ such that $\{x_\xi : \xi \in \lambda^{+\omega}\}$ is also equal to $H(\lambda^{+\omega})$. We choose this enumeration in such a way that $x_\xi \in x_\eta$ implies $\xi < \eta$. We use relation symbol R_0 to code (and well order) $(H(\lambda^{+\omega+1}), \in)$ as follows: $(\xi, \eta) \in R_0$ if and only if $x_\xi \in x_\eta$. Let R_1 be a binary relation on $\kappa^{+\omega}$ so that $(\kappa^{+\omega}, R_1)$ is isomorphic to \mathbb{P}_0 . Let R_2 be a binary relation on λ so that $R_2 \cap (\kappa^{+\omega} \times \kappa^{+\omega}) = R_1$ and (λ, R_2) is isomorphic to $\mathbb{P}_0 * \mathbb{P}_1$. Let ψ be the poset isomorphism from (λ, R_2) to $\mathbb{P}_0 * \mathbb{P}_1$.

We continue coding. We can code the sequence $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$ as another binary relation R_3 on $\lambda^{+\omega+1}$ where $R_3 \cap (\{\xi\} \times \lambda^{+\omega+1}) = \{(\xi, h_\xi(n)) : n \in \omega\}$ for each $\xi \in \lambda^{+\omega+1}$. The relation symbol R_4 can code the sequence $\{\dot{A}_\xi : \xi \in \lambda^{+\omega+1}\}$ where $(\xi, \alpha, \zeta) \in R_4$ if and only if $(\dot{\alpha}, \psi(\zeta))$ is in the name \dot{A}_ξ . Let R_5 code this collection, i.e. $(\gamma, n, m, \eta) \in R_5$ if and only if $((n, m), \psi(\eta)) \in \dot{H}_\gamma$. Also let R_6 code (equal) the set Λ . Finally we use the relation symbol R_7 to similarly code the sequence $\{\dot{f}_\xi : \xi \in \lambda^{+\omega+1}\}$: $(\xi, \alpha, n, \zeta) \in R_7$ if and only if $((\alpha, n), \psi(\zeta))$ is in the name \dot{f}_ξ .

Needless to say, the unary relation symbol R is interpreted as the set $\lambda^{+\omega}$ for the application of $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$. Now we have defined our model M of the language $L = \{\in, R, R_0, \dots, R_7\}$, and we choose an elementary submodel N witnessing $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$. Of course N is really just a $\kappa^{+\omega+1}$ sized subset of $\lambda^{+\omega+1}$ with the additional property that $N \cap \lambda^{+\omega}$ has cardinality $\kappa^{+\omega}$. In the forcing extension N has cardinality ω_1 and $A = N \cap \lambda^{+\omega}$ is countable.

We will need the following claim from [7]:

Claim. *We may assume that N satisfies that $N \cap \kappa^{+\omega+1}$ is transitive (i.e. an initial segment).*

Proof of Claim: Suppose our originally supplied N fails the conclusion of the claim. We know that $\kappa^{+\omega} \in N$, (via R_1) in which case so is $\kappa^{+\omega+1}$.

Then set $\beta_0 = \sup(N \cap \kappa^{+\omega+1})$ and consider the Skolem closure $Hull(N \cup \beta_0, M)$. A little informally (in that we have to formalize the enumeration of formulas as per Gödel coding) let $\{\varphi_n : n \in \omega\}$ be an enumeration of all formulas in the language

L , and let ℓ_n be the minimal integer such that the free variables of φ_n are among $\{v_0, \dots, v_{\ell_n}\}$. Then, for each tuple $\langle \xi_1, \dots, \xi_{\ell_n} \rangle$ of elements of $\lambda^{+\omega+1}$, we define $f_n(\xi_1, \dots, \xi_{\ell_n})$ to be the minimal $\xi_0 \in \lambda^{+\omega+1}$ such that $M \models \varphi_n(\xi_0, \dots, \xi_{\ell_n})$. If there is no such ξ_0 , in other words if $M \models \neg \exists x \varphi_n(x, \xi_1, \dots, \xi_{\ell_n})$, then set $f_n(\xi_1, \dots, \xi_{\ell_n})$ to be 0. Now $Hull(N \cup \beta_0, M)$ is just the minimal superset X of $N \cup \beta_0$ that satisfies that $f_n[X^{\{1, \dots, \ell_n\}}] \subset X$ for all n . Since this is simply a large algebra, we can generate all the terms t of the algebraic operations $\{f_n : n \in \omega\}$. It is easily seen that for each $\zeta \in X$, there is a term $t(v_1, \dots, v_m)$ such that $\zeta = t(\delta_1, \dots, \delta_m)$ for some sequence $\langle \delta_1, \dots, \delta_m \rangle$ with each $\delta_i \in N \cup \beta_0$. Assume that $\zeta \in \kappa^{+\omega+1}$. By re-indexing the variables in the term we can assume that there is an $n \leq m$ so that $\delta_i < \beta_0$ for $1 \leq i \leq n$ and $\kappa^{+\omega+1} \leq \delta_i$ for $n < i \leq m$. Let \vec{a} denote the tuple $\langle \delta_{n+1}, \dots, \delta_m \rangle$. Choose $\eta \in N \cap \kappa^{+\omega+1}$ large enough so that $\{\delta_1, \dots, \delta_n\}$ is contained in η . Since set-membership in M is coded by R_0 rather than \in we have to argue a little less naturally. Consider the set $s_0(\eta, \vec{a}) = \{t(\gamma_1, \dots, \gamma_n, \vec{a}) : \{\gamma_1, \dots, \gamma_n\} \in [\eta]^{\leq n}\}$. Clearly $s_0(\eta, \vec{a})$ is a member of $H(\lambda^{+\omega+1})$. Now define $s_1(\eta, \vec{a})$ to be $\{x_\alpha : \alpha \in s_0(\eta, \vec{a})\}$, and choose the unique $\zeta_1 \in \lambda^{+\omega+1}$ such that $x_{\zeta_1} = s_1(\eta, \vec{a})$. We claim that $\zeta_1 \in N$. Note that $\alpha R_0 \zeta_1$ holds if and only if $\alpha \in s_0(\eta, \vec{a})$, and therefore

$$M \models (\forall \alpha) [\alpha R_0 \zeta_1 \text{ iff } (\exists \gamma_1 \in \eta) \cdots (\exists \gamma_n \in \eta) (\alpha = t(\gamma_1, \dots, \gamma_n, \vec{a}))] .$$

By elementarity then we have that $\zeta_1 \in N$, and by similar reasoning the supremum, ζ_0 , of $\zeta_1 \cap \kappa^{+\omega+1}$ is also in N . This of course means that $\zeta < \xi_0$. \square

We use the elementarity of N to deduce properties of the families $\{\dot{A}_\xi : \xi \in N\}$ and $\{\dot{f}_\xi : \xi \in N\}$. Actually the collection we are most interested in is the family $\{h_\xi : \xi \in \Lambda \cap N\}$.

Now we need a result from Shelah's pcf theory which is proven in Jech [3, 24.9]. Since $\aleph_1 = \mathfrak{c} < \kappa^{+\omega+1}$ there is a function $\langle \varrho_n : n \in \omega \rangle$ in $\Pi_n \lambda^{+\omega}$ such that the sequence $\{h_\xi : \xi \in N\}$ is unbounded mod finite in $\Pi_n \varrho_n$. For each n , $\varrho_n \leq \sup(N \cap \lambda^{+n+2})$. Since \mathbb{P}_0 has cardinality $\kappa^{+\omega}$, and so less than $|N| = \kappa^{+\omega+1}$, a standard argument (analogous to the fact that adding a Cohen real does not add a dominating real) shows that the sequence $\{h_\xi : \xi \in \Lambda \cap N\}$ remains unbounded mod finite in $\Pi_n \varrho_n$ (and in $\Pi_n(\varrho_n \cap N)$).

Now pass to the extension by $G \cap \mathbb{P}_0$ and let H be the function $\text{val}_G(\dot{H})$, and we recall that $f_{\xi_\xi}(h_\xi(n)) = H(n)$ for all $n \in \omega$ and $\xi \in \Lambda$. Now pass to the full extension $V[G]$ and again, since \mathbb{P}_1 was forced to be countably closed, the family $\{h_\xi : \xi \in \Lambda \cap N\}$ is still unbounded in $\Pi_n(\varrho_n \cap N)$ (no new elements were added). We let A be the countable set $N \cap \lambda^{+\omega}$, and for each $\xi \in \Lambda \cap N$, there is an n_ξ such that $f_\xi(h_\xi(m)) = f_A(h_\xi(m))$ for all $m > n_\xi$. There is a single n so that $\Lambda_n = \{\xi \in \Lambda \cap N : n_\xi = n\}$ has cardinality ω_1 , and thus $\{h_\xi : \xi \in \Lambda_n \cap N\}$ is also unbounded in $\Pi_n(\varrho_n \cap N)$. This certainly implies that there is an $m > n$ such that $\{h_\xi(m) : \xi \in \Lambda_n \cap N\}$ is infinite. This completes the proof since $f_A(h_\xi(m)) = H(m)$ for all $\xi \in \Lambda_n \cap N$. \square

4. APPLICATIONS TO INFINITE LENGTH GAMES

We introduce three variations of Scheeper's game which we defined in the introduction.

Game 25. Let $Sch_{C,F}^{\cup,\subseteq}(\kappa)$ denote the *Scheepers countable-finite union game* which proceeds analogously to $Sch_{C,F}^{\cup,\subseteq}(\kappa)$, except that \mathcal{C} 's restriction in round $n+1$ is weakened to $C_{n+1} \supseteq C_n$.

Game 26. Let $Sch_{C,F}^{1,\subseteq}(\kappa)$ denote the *Scheepers countable-finite initial game* which proceeds analogously to $Sch_{C,F}^{\cup,\subseteq}(\kappa)$, except that \mathcal{F} 's winning condition is weakened to $\bigcup_{n<\omega} F_n \supseteq C_0$.

Game 27. Let $Sch_{C,F}^{\cap}(\kappa)$ denote the *Scheepers countable-finite intersection game* which proceeds analogously to $Sch_{C,F}^{1,\subseteq}(\kappa)$, except that \mathcal{C} may choose any $C_n \in [\kappa]^{\leq \omega}$ each round, and \mathcal{F} 's winning condition is weakened to $\bigcup_{n<\omega} F_n \supseteq \bigcap_{n<\omega} C_n$.

In [1] Clontz extended Scheepers' application of almost-compatible injections to these game variants as well as $Men_{C,F}(\kappa^\dagger)$. However, when considering Markov strategies, finite-to-one functions suffice.

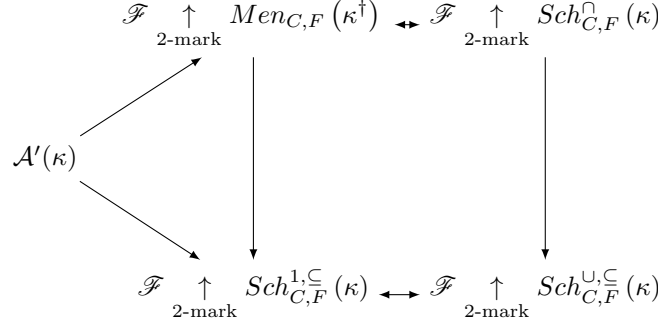


FIGURE 1. Diagram of Scheeper/Menger game implications with $\mathcal{A}'(\kappa)$

Theorem 28. $\mathcal{A}'(\kappa)$ implies the game-theoretic conditions shown in Figure 1.

Proof. The weaker claim $\mathcal{A}(\kappa) \Rightarrow \mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^{\cap}(\kappa)$ was proven in [1]; however, the strategy only required that the f_A be pairwise almost-compatible and that preimages of finite sets in each f_A are finite, which is also possible assuming $\mathcal{A}'(\kappa)$. Note that the relationships amongst the games were all shown in [1]. \square

We include the following proof from [9] for the convenience of the reader.

Theorem 29. $\mathcal{A}(\kappa) \Rightarrow \mathcal{F} \uparrow_{2\text{-tact}} Sch_{C,F}^{\cup,\subseteq}(\kappa)$

Proof. Let $\{f_A : A \in [\kappa]^{\leq \aleph_0}\}$ witness $\mathcal{A}(\kappa)$, and define $g_A : A \rightarrow \omega$ by $g_A(\alpha) = f_A(\alpha) - |\{\beta \in A : f_A(\beta) < f_A(\alpha)\}|$.

We claim that $\{\alpha \in A : g_A(\alpha) \leq g_B(\alpha)\}$ must be finite as it is bounded above by $\max\{M, f_A(\alpha), f_B(\alpha) : f_A(\alpha) \neq f_B(\alpha)\}$ where $M = f_B(\alpha)$ for some $\alpha \in B \setminus A$. To see this, let $f_A(\alpha) = f_B(\alpha) = N > M$ and assume $f_A(\beta) \neq f_B(\beta)$ implies $f_A(\beta), f_B(\beta) < N$. Then

$$g_A(\alpha) = N - |\{\beta \in A : f_A(\beta) < N\}| > N - |\{\beta \in B : f_B(\beta) < N\}| = g_B(\alpha)$$

with the strictness of the inequality witnessed by $f_B(\alpha) = M < N$ for some $\alpha \in B \setminus A$.

As a result,

$$\sigma(\langle A, B \rangle) = \{\alpha \in A : g_A(\alpha) \leq g_B(\alpha)\}$$

is a legal 2-tactic for \mathcal{F} . Let $C = \langle C_0, C_1, \dots \rangle$ be a strictly increasing sequence of countable sets and $\alpha \in C_n$. Noting that f_A is an injection and not just finite-to-one, $0 \leq g_{C_{n+m}}(\alpha)$ for all $m < \omega$, and it follows that $g_{C_{n+m}}(\alpha) \leq g_{C_{n+m+1}}(\alpha)$ for some $m < \omega$. Therefore $\alpha \in \sigma(\langle C_{n+m}, C_{n+m+1} \rangle)$. \square

So it would seem that $\mathcal{A}'(\kappa)$ is sufficient only when considering Markov strategies. (Of course, $\mathcal{A}'(\kappa) \Rightarrow \mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{\cup, \subseteq}(\kappa) \Rightarrow \mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{\cup, \subseteq}(\kappa)$.) We would like to demonstrate that $\mathcal{A}'(\kappa)$ is not necessary.

Theorem 30. *Let α be the limit of increasing ordinals β_n for $n < \omega$. If $\mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{\cap}(\aleph_{\beta_n})$ for all $n < \omega$, then $\mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{\cap}(\aleph_{\alpha})$.*

Proof. Let σ_n be a winning 2-mark for \mathcal{F} in $\text{Sch}_{C,F}^{\cap}(\aleph_{\beta_n})$. Define the 2-mark σ for \mathcal{F} in $\text{Sch}_{C,F}^{\cap}(\aleph_{\alpha})$ as follows:

$$\sigma(\langle C \rangle, 0) = \sigma_0(\langle C \cap \aleph_{\beta_0} \rangle, 0)$$

$$\sigma(\langle C, D \rangle, n+1) = \sigma_{n+1}(\langle D \cap \aleph_{\beta_{n+1}} \rangle, 0) \cup \bigcup_{m \leq n} \sigma_m(\langle C \cap \aleph_{\beta_m}, D \cap \aleph_{\beta_m} \rangle, n-m+1)$$

Let $\langle C_0, C_1, \dots \rangle$ be an attack by \mathcal{C} in $\text{Sch}_{C,F}^{\cap}(\aleph_{\alpha})$, and $\alpha \in \bigcap_{n < \omega} C_n$. Choose $N < \omega$ with $\alpha < \aleph_{\beta_{N+1}}$. Consider the attack $\langle C_{N+1} \cap \aleph_{\beta_{N+1}}, C_{N+2} \cap \aleph_{\beta_{N+1}}, \dots \rangle$ by \mathcal{C} in $\text{Sch}_{C,F}^{\cap}(\aleph_{\beta_{N+1}})$. Since σ_{N+1} is a winning strategy and $\alpha \in \bigcap_{n < \omega} C_{N+n+1} \cap \aleph_{\beta_{N+1}}$, either $\alpha \in \sigma_{N+1}(\langle C_{N+1} \cap \aleph_{\beta_{N+1}} \rangle, 0)$ and thus $\alpha \in \sigma(\langle C_N, C_{N+1} \rangle, N+1)$, or $\alpha \in \sigma_{N+1}(\langle C_{N+M+1} \cap \aleph_{\beta_{N+1}}, C_{N+M+2} \cap \aleph_{\beta_{N+1}} \rangle, M+1)$ for some $M < \omega$ and thus $\alpha \in \sigma(\langle C_{N+M+1}, C_{N+M+2} \rangle, N+M+2)$. Thus σ is a winning strategy. \square

Theorem 31. *Let α be the limit of increasing ordinals β_n for $n < \omega$. If $\mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{1, \subseteq}(\aleph_{\beta_n})$ for all $n < \omega$, then $\mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{1, \subseteq}(\aleph_{\alpha})$.*

Proof. The proof proceeds nearly identically to the previous proof. \square

Corollary 32. *It is consistent that $\mathcal{A}'(\aleph_{\omega})$ fails, but as $\mathcal{A}'(\aleph_k)$ holds in ZFC for all $k < \omega$, both $\mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{\cap}(\aleph_{\omega})$ and $\mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{1, \subseteq}(\aleph_{\omega})$ hold in ZFC.*

Note that Question 6 remains unsolved; however, our results have revealed that we cannot hope to find a ZFC counterexample where $X = \kappa^{\dagger}$. This is because if we also assume $V = L$, it follows that $\mathcal{A}'(\kappa)$ and therefore $\mathcal{F} \uparrow_{2\text{-mark}} \text{Men}_{C,F}(X)$ for every cardinal. Although, one may consider the following weaker question.

Question 33. *Is there a model of ZFC where both $\mathcal{F} \uparrow_{2\text{-mark}} \text{Men}_{C,F}(\aleph_{\omega+1}^{\dagger})$ and $\mathcal{A}'(\aleph_{\omega})$ fail?*

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