(joint work with Alan Dow)

Definition 1. Two functions f, g are almost compatible if $\{a \in dom\, f \cap dom\, g : f(a) \neq g(a)\}$ is finite.

Definition 2. $S'(\theta)$ states that there exists a cofinal family $\mathcal{S} \subseteq [\theta]^\omega$ and a collection of pairwise almost compatible finite-to-one functions $\{f_S \in \omega^S : S \in \mathcal{S}\}$

Definition 3. $S(\theta)$ strengthens $S'(\theta)$ by requiring the collection to contain one-to-one functions.

We wish to show that Scheeper's original $S(\theta)$ is strictly stronger than $S'(\theta)$.

Definition 4. A topological space is said to be ω-bounded if each countable subset of the space has compact closure.

Theorem 5. *For each $n \in \omega$, there is a locally countable, ω-bounded topology on ω_n. Note that this means that the closure of any set has the same cardinality and weight as the set.*

To prove the theorem, we must actually prove a stronger lemma.

Lemma 6. *Assume that X has cardinality at most ω_n (for any $n \in \omega$), and is locally countable, locally compact, and the closure of each set has the same cardinality as the set. Then X has an ω-bounded extension with the same properties.*

Proof. We prove this by induction on n. In fact, we make our inductive statement that if $\tilde{X}$ is the extension of X, then $\tilde{X} \setminus X$ also has cardinality ω_n. If $n = 0$, then we can just take the free union of two copies of X and then the one-point compactification. So suppose $n > 0$ and that X is such a topology on the ordinal ω_n. For each $\alpha < \omega_n$, the closure of the initial segment α is bounded by some γ_α. Also, because X is locally countable, γ_α can be chosen so that α is contained in the interior of γ_α. There is a cub $C \subset \omega_n$ with the property that for each $\delta \in C$ and $\alpha < \delta$, γ_α is also less than δ. This implies that for each $\delta \in C$, the initial segment δ is open, and if δ has uncountable cofinality, then δ is clopen.

The proof will be easier to visualize if we now identify the points of X with the point set $\omega_n \times \{0\}$ and we will add the points $\omega_n \times \{1\}$ to create the extension. By induction on $\lambda \in C$ we define a topology on $\omega_n \times \{0\} \cup \lambda \times \{1\}$ so that $\omega_n \times \{0\}$ is an open subset. We also ensure, by induction, for each $\alpha < \lambda$, the closure of $\alpha \times 2$ is an ω-bounded subset of $\lambda \times 2$.

In the case that $n = 1$, then choose any sequence $\lambda_n : n \in \omega$ increasing cofinal in λ. If λ is a limit in C, then we simply take the topology we have constructed so far on $\lambda \times 2$ and there's nothing more needs to be done. Otherwise we may assume that λ_0 is the predecessor of $\lambda \in C$ and we set Y_λ to equal the countable set $\overline{\lambda} \setminus \lambda$. For convenience, and with no loss, we assume that λ itself is a limit of limits. And we have a topology on

$$\lambda_0 \times 2 \cup (\lambda \cup Y_\lambda) \times \{0\} \ .$$

Recursively choose clopen sets U_n in this topology so that $\lambda_0 \times 2 \subset U_0$, $U_n \cup \lambda_{n+1} \times \{0\}$ is contained U_{n+1} while U_{n+1} is disjoint from Y_λ. It is easy to see that we can have all the points in $(\lambda \setminus \{\lambda_n : n \in \omega\}) \times \{1\}$ be isolated, and arrange that $(\lambda_n, 1)$ is the point at infinity in the one-point compactification $U_n \cup (\lambda_n \times \{1\})$.

Now we handle the case $n > 1$ and we can shrink C and now assume that C is the closure of $\{\lambda \in C : \mathrm{cf}(\lambda) > \omega\}$. We again proceed by induction on $\lambda \in C$. If λ is a limit in C, then there is nothing to do: we simply have defined an appropriate topology on $\omega_n \times \{0\} \cup \lambda \times \{1\}$ so that for each $\mu \in C \cap \lambda$ with $\mathrm{cf}(\mu) > \omega$, $\mu \times 2$ is a clopen ω-bounded subspace. In case λ is not a limit of C, then λ has uncountable cofinality and a predecessor $\mu \in C$. We therefore have that $\lambda \times \{0\}$ is clopen in $\omega_n \times \{0\}$. We apply the induction hypothesis to the space $\lambda \times \{0\} \cup \mu \times 2$ to choose the topology on $\lambda \times 2$.

□

Definition 7. A Kurepa family $\mathcal{K} \subseteq [\theta]^\omega$ on θ satisfies that $\mathcal{K} \restriction A = \{K \cap A : K \in \mathcal{K}\}$ is countable for each $A \in [\theta]^\omega$.

Corollary 8. *There exists a Kurepa family cofinal in $[\omega_k]^\omega$ for each $k < \omega$.*

Proof. This is actually a corollary of an observation of Todorcevic communicated by Dow in [TODO cite Gen Prog in Top I]: if every Kurepa family of size at most θ extends to a cofinal Kurepa family, then the same is true of θ^+. So the result follows as every Kurepa family $\mathcal{K}$ of size ω extends to the cofinal Kurepa family $[\bigcup \mathcal{K}]^\omega$.

We may alternatively obtain the result from the previous topological argument by using the family $\mathcal{K}$ of compact sets in the constructed topology on ω_k as our witness. Of course, every Lindelöf set in a locally countable space is countable. Thus $\mathcal{K}$ is cofinal in $[\omega_k]^\omega$ since for every countable set A, $\overline{A}$ is compact and countable. It is Kurepa since for every countable set A, let (TODO) □

Theorem 9. *$S'(\theta)$ holds whenever there exists a cofinal Kurepa family on θ.*

Proof. Let $k < \omega$, and $\mathcal{K} = \{K_\alpha : \alpha < \kappa\}$ be a cofinal Kurepa family on θ. We should define $f_\alpha : K_\alpha \to \omega$ for each $\alpha < \kappa$.

Suppose we've defined pairwise almost compatible $\{f_\beta : \beta < \alpha\}$. To define f_α, we first recall that $\mathcal{K} \restriction K_\alpha$ is countable, so we may choose $\beta_n < \alpha$ for $n < \omega$ such that $\{K_\beta : \beta < \alpha\} \restriction K_\alpha \setminus \{\emptyset\} = \{K_\alpha \cap K_{\beta_n} : n < \omega\}$. Let $K_\alpha = \{\delta_{i,j} : i \leq \omega, j < w_i\}$ where $w_i \leq \omega$ for each $i \leq \omega$, $K_\alpha \cap \left(K_{\beta_n} \setminus \bigcup_{m<n} K_{\beta_m}\right) = \{\delta_{n,j} : j < w_n\}$, and $K_\alpha \setminus \bigcup_{n<\omega} K_{\beta_n} = \{\delta_{\omega,j} : j < w_\omega\}$. Then let $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ for $n < \omega$ and $f_\alpha(\delta_{\omega,j}) = j$ otherwise.

We should show that f_α is finite-to-one. Let $n < \omega$. We need only worry about $\delta_{m,j}$ for $m \leq n$ since $f_\alpha(\delta_{m,j}) \geq m$. Since each f_{β_m} is finite-to-one, $f_{\beta_m}(\delta_{m,j}) \leq n$ for only finitely many j. Thus f_α maps to n only finitely often.

We now want to demonstrate that $f_\alpha \sim f_{\beta_n}$ for all $n < \omega$. We again need only concern ourselves with $\delta_{m,j}$ for $m \leq n$ since otherwise $\delta_{m,j} \notin K_{\beta_n}$. For $m = n$, we have $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ which differs from $f_{\beta_n}(\delta_{n,j})$ for only the finitely many j which are mapped below n by f_{β_n}. For $m < n$ and $\delta_{m,j} \in K_{\beta_n}$,we have $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$ which can only differ from $f_{\beta_n}(\delta_{m,j})$ for only the finitely many j which are mapped below m by f_{β_m} or the finitely many j for which the almost compatible $f_{\beta_n} \sim f_{\beta_m}$ differ. □

Corollary 10. *$S'(\omega_k)$ holds for all $k < \omega$.*

As noted in [TODO cite Dow], Jensen's one-gap two-cardinal theorem under $V = L$ [TODO cite] can be used to show that there exist cofinal Kurepa families on every cardinal.

Corollary 11 ($V = L$)**.** *$S'(\theta)$ holds for all cardinals.*

In particular, $S(\omega_2)$ fails under CH, showing the two are distinct. Actually, CH is not required to have $S(\omega_2)$ fail.

We are going to need a technical lemma (available in Kunen).

Lemma 12. *Assume that $G \subset \mathrm{Fn}(\omega_2, 2)$ is a generic filter, and let $\mu \in \omega_2$. Then the final model $V[G]$ can be regarded as forcing with $\mathrm{Fn}(\omega_2 \setminus \mu, 2)$ over the model $V[G_\mu]$. In addition, for each $\mathrm{Fn}(\omega_2, 2)$-name $\dot{A}$ of a subset of ω (treat as a subset of $\omega \times \mathrm{Fn}(\omega_2, 2)$), there is a canonical name $\dot{A}(G_\mu)$ where,*

$$\dot{A}(G_\mu) = \{(n, p \restriction [\mu, \omega_2)) : (n, p) \in \dot{A} \ \ \textit{and} \ \ p \restriction \mu \in G_\mu\}$$

and we get that the valuation of $\dot{A}(G_\mu)$ by the tail of the generic, $G_{\omega_2 \setminus \mu}$, is the same as the valuation of $\dot{A}$ by the full generic.

Theorem 13. *If we add ω_2 Cohen reals to a model of CH we get that Scheepers' $S(\omega_2)$ (still) fails.*

Proof. The forcing poset is $\mathrm{Fn}(\omega_2, 2)$. Let $\{\dot{f}_A : A \in [\omega_2]^\omega\}$ be a family of names such that $\dot{f}_A$ is a one-to-one function from A into ω. It suffices to only consider sets A from the ground model.

Put all the lemma stuff in an elementary submodel M of the universe (technically of $H(\kappa)$, or of V_κ, for some large κ). Standard methods says that we can assume that $|M| = \omega_1 = \mathfrak{c}$ and that $M^\omega \subset M$ (which means that every countable subset of M is a member of M).

Let $\lambda = M \cap \omega_2$ (same as the supremum of $M \cap \omega_2$). Consider the name $\dot{f}_{[\lambda, \lambda+\omega)}$. What is such a name? We can assume that it is a set of pairs of the form $((\lambda + k, m), p)$ where

$p \in Fn(\omega_2, 2)$ and, of course, $k, m \in \omega$. This is (almost) equivalent to saying that p forces that $\dot{f}_{[\lambda,\lambda+\omega)}(\lambda+k) = m$. We don't take all such p, in fact for each k, m it is enough to take a countable set of such p to get an equivalent name (Kunen calls it a nice name if we take, for each k, m an antichain that is maximal among such conditions). Given any such name $\dot{f}$, let $\mathrm{supp}(\dot{f})$ denote the union of the domains of conditions p appearing in the name.

Also let Y equal $\mathrm{supp}(\dot{f}_{[\lambda,\lambda+\omega)}) \setminus \lambda$. Let δ denote the order type of Y and let the 2-parameter notation $\varphi_{\mu,\lambda}$ be the order-preserving function from $\mu \cup Y$ onto the ordinal $\mu + \delta$. This lifts canonically to an order-preserving bijection $\varphi_{\mu,\lambda} : \mathrm{Fn}(\mu \cup Y, 2) \mapsto \mathrm{Fn}(\mu + \delta, 2)$. Similarly, we make sense of the name $\varphi_{\mu,\lambda}(\dot{f}_{[\lambda,\lambda+\omega)})$, call it F_M. Here simply, for each tuple $(\ (k, m)\ , p) \in \dot{f}_{[\lambda,\lambda+\omega)}$, we have that $(\ (k, m)\ , \varphi_{\mu,\lambda}(p))$ is in F_M. Again, let $\varphi_{\mu,\lambda}(\dot{f}_{[\lambda,\lambda+\omega)})$ be interpreted in the above sense as giving F_M (which is an element of M). Note that we do not regard δ as fixed here, but rather simply depending on the $\mathrm{supp}(\dot{f}_{[\lambda,\lambda+\omega)})$ described above. Other values replacing $\lambda > \mu$ will result in their own set Y and canonical map $\varphi_{\mu,\lambda}$; but one thing we do have to assume (or arrange) for other values α replacing λ is that $\mathrm{supp}(\dot{f}_{[\alpha,\alpha+\omega)})$ intersected with α is contained in μ.

Now the object F_M is an element of M, and M believes this statement is true:

$$(\forall \beta \in \omega_2)\ (\exists \beta < \lambda \in \omega_2) \quad \mathrm{supp}(\dot{f}_{[\lambda,\lambda+\omega)}) \cap \lambda \subset \mu \text{ and } F_M = \varphi_{\mu,\lambda}(\dot{f}_{[\lambda,\lambda+\omega)})$$

But now, this means that, not only is there an $\alpha \in M$, $F_M = \varphi_{\mu,\alpha}(\dot{f}_{[\alpha,\alpha+\omega)})$ but also that there is an increasing sequence $\{\alpha_\xi : \xi \in \omega_1\} \subset \lambda$ of such α's satisfying that, for each ξ we have that $\mathrm{supp}(\dot{f}_{[\alpha_\xi,\alpha_\xi+\omega)})$ is contained in $\alpha_{\xi+1}$.

Choose such a sequence. This means that if we let $A = \bigcup_{n>0}[\alpha_n, \alpha_n + \omega)$ we have the name $\dot{f}_A$ in M. This then means that all the $((\beta, m), p)$ appearing in $\dot{f}_A$ have the property that $dom(p)$ is contained in M. There is, within M, a name $\dot{g}$ satisfying that $\dot{f}_A(\alpha_n + k) = \dot{f}_{[\alpha_n,\alpha_n+\omega)}(\alpha_n + k)$ for all $k > \dot{g}(n)$.

We now apply the above Lemma using $\mu = \mu_0$ and we are now working in the extension $V[G_\mu]$. We will abuse the notation and use $\dot{f}_{[\alpha_n,\alpha_n+\omega)}$ instead of $\dot{f}_{[\alpha_n,\alpha_n+\omega)}(G_\mu)$ as defined in the Lemma. We work for a contradiction. Something special has now happened, namely, the supports of the names $\{\dot{f}_{[\alpha_n,\alpha_n+\omega)} : 0 < n < \omega\}$ are pairwise disjoint and also disjoint from the support of the name $\dot{f}_{[\lambda,\lambda+\omega)}$ (under the same convention about G_μ. And not only that, these names are pairwise isomorphic (in the way that they all map to F_M).

Since A is disjoint from $[\lambda, \lambda + \omega)$, there must be an integer ℓ together with a condition $q \in Fn(\omega_2 \setminus \mu, 2)$ satisfying that for all $n > \ell$, q forces that
"if $k > \dot{g}(n)$ (since $\alpha_n + k \in A$) then $\dot{f}_{[\alpha_n,\alpha_n+\omega)}(\alpha_n + k) \neq \dot{f}_{[\lambda,\lambda+\omega)}(\lambda + k)$".

Choose n large enough so that $dom(q) \cap [\alpha_n, \mu_{n+1})$ is empty. Choose $q_1 < q \restriction \lambda$ (in M) so that

$$\varphi_{\mu,\alpha_n}(q_1 \restriction \mathrm{supp}(\dot{f}_{[\alpha_n,\alpha_n+\omega)}) = \varphi_{\mu,\lambda}(q \restriction \mathrm{supp}(\dot{f}_{[\lambda,\lambda+\omega)})$$

and then (again in M) choose $q_2 < q_1$ so that it both forces a value L on $\ell + \dot{g}(n)$ and subsequently forces a value m on $\dot{f}_{[\alpha_n,\alpha_n+\omega)}(\alpha_n + L + 1)$. But now, again calculate

$$q_3 = \varphi^{-1}_{\mu,\lambda} \circ \varphi_{\mu,\alpha_n}(q_2 \restriction \mathrm{supp}(\dot{f}_{[\alpha_n,\alpha_n+\omega)}))$$

and, by the isomorphisms, we have that q_3 forces that $\dot{f}_{[\lambda,\lambda+\omega)}(\lambda + L + 1) = m$.

Technically (or with more care) all of this is taking place in the poset $\mathrm{Fn}(\omega_2 \setminus \mu, 2)$ and this means that q_3 and q are all compatible with each other.

Follow the bouncing ball: it suffices to consider $q(\beta) = e$ and to assume that $q_3(\beta)$ is defined. Since $q_3(\beta)$ is defined, we have that there is a $\beta' \in dom(q_2)$ such that $\varphi_{\mu,\lambda}(\beta) = \varphi_{\mu,\alpha_n}(\beta')$, and that $q_3(\beta) = q_2(\beta')$. But, by definition of q_1, $\beta' \in dom(q_1)$ and even that $q_1(\beta') = q(\beta)$. Then, since $q_2 < q_1$, we have that $q_2(\beta') = q_1(\beta') = q(\beta)$. This completes the circle that $q_3(\beta) = q(\beta)$.

Finally, our contradiction is that $q_3 \cup q_2 \cup q$ forces that $k = L + 1$ violates the quoted statement above. □

On the other hand, it's also consistent that $S'(\theta)$ can fail.

Theorem 14. *There's a model where $S'(\omega_\omega)$ fails.*

Proof. We will need the model constructed in [1] in which an instance of Chang's conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is shown to fail.

We can take as a given (as shown in [1, Theorem 5]) that we may assume that we have a model V of GCH in which there are regular limit cardinals $\kappa < \lambda$ satisfying that $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$.

What this says is that if L is a countable language with at least one unary relation symbol R and M is a model of L with base set $\lambda^{+\omega+1}$ in which the interpretation of R has cardinality $\lambda^{+\omega}$, then M has an elementary submodel N of cardinality $\kappa^{+\omega+1}$ in which $R \cap N$ has cardinality $\kappa^{+\omega}$ (of course $R \cap N$ is the interpretation of R in N because $N \prec M$).

The interested reader will want to know that it is shown in [1] that if κ is a 2-huge cardinal and j is the 2-huge embedding with critical point κ, then with $\lambda = j(\kappa)$ one has that $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$ holds.

Let $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$ be a scale in $\Pi\{\lambda^{+n+1} : n \in \omega\}$ ordered by the usual mod finite coordinatewise ordering. For convenience we may assume that $h_\xi(n) \geq \lambda^{+n}$ for all ξ and all n. If P is any poset of cardinality less than $\lambda^{+\omega}$, then in the forcing extension by P, the sequence $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$ remains cofinal in $\Pi\{\lambda^{+n+1} : n \in \omega\}$.

The forcing notion $\mathbb{P}_0$ is simply the finite condition collapse of $\kappa^{+\omega}$, i.e. $\mathbb{P}_0 = (\kappa^{+\omega})^{<\omega}$. In the forcing extension by $\mathbb{P}_0$, one now has that the ordinal $\kappa^{+\omega+1}$ from V is the first uncountable cardinal $\aleph_1$. Then in this forcing extension we let $\mathbb{P}_1$ be the countable condition Levy collapse, $Lv(\lambda, \omega_2)$, which collapses all cardinals less than λ to have cardinality at most $\aleph_1$. The poset $\mathbb{P}_1$ has cardinality λ. We treat $\mathbb{P}_1$ as containing $\mathbb{P}_0$ as a subposet by identifying each $(p_0, 1)$ with p_0. After forcing with $\mathbb{P}_0 * \mathbb{P}_1$ we will have that ω_1 is the ordinal $\left(\kappa^{+\omega+1}\right)^V$, ω_2 is the ordinal λ, and ω_ω is the ordinal $(\lambda^{+\omega})^V$.

Now we assume that we have an assignment $\dot{f}_{\dot{A}}$ of a $\mathbb{P}_0 * \mathbb{P}_1$-name of a finite-to-one function from $\dot{A}$ into ω for each $\mathbb{P}_0 * \mathbb{P}_1$-name of a countable subset of $\lambda^{+\omega+1}$. We will obtain a contradiction.

Let $\{\dot{A}_\xi : \xi \in \lambda^{+\omega+1}\}$ be an enumeration of all the nice $\mathbb{P}_0$-names of countable subsets of $\lambda^{+\omega}$. For each $\xi \in \lambda^{+\omega+1}$, let $\dot{f}_\xi$ be another notation for $\dot{f}_{\dot{A}_\xi}$. Since $\mathbb{P}_0$ forces that $\mathbb{P}_1$ is countably closed, the collection of all nice $\mathbb{P}_0$-names will produce all the countable sets in the extension by $\mathbb{P}_0 * \mathbb{P}_1$, but $\mathbb{P}_0 * \mathbb{P}_1$ can introduce new enumerations of these names. For each $\xi \in \lambda^{+\omega+1}$, there is a minimal ζ_ξ so that $\dot{A}_{\zeta_\xi}$ is the canonical name for the range of h_ξ. This means that $\dot{f}_{\zeta_\xi} \circ h_\xi$ is simply the $\mathbb{P}_0 * \mathbb{P}_1$-name of a finite-to-one function from ω to ω. For each $\xi \in \lambda^{+\omega+1}$, choose any $p_\xi \in \mathbb{P}_0 * \mathbb{P}_1$ so that there is a nice $\mathbb{P}_0$-name, $\dot{H}_\xi$, that is forced by p_ξ to equal $\dot{f}_{\zeta_\xi} \circ h_\xi$. Choose $\Lambda \subset \lambda^{+\omega+1}$ of cardinality $\lambda^{+\omega+1}$ and so that there is a pair $p, \dot{H}$ satisfying that $p_\xi = p$ and $\dot{H}_\xi = \dot{H}$ for all $\xi \in \Lambda$. We may assume that p is in a generic filter G.

Let $\{x_\xi : \xi \in \lambda^{+\omega+1}\}$ be any enumeration of $H(\lambda^{+\omega+1})$ such that $\{x_\xi : \xi \in \lambda^{+\omega}\}$ is also equal to $H(\lambda^{+\omega})$. We choose this enumeration in such a way that $x_\xi \in x_\eta$ implies $\xi < \eta$. We use relation symbol R_0 to code (and well order) $(H(\lambda^{+\omega+1}), \in)$ as follows: $(\xi, \eta) \in R_0$ if and only if $x_\xi \in x_\eta$. Let R_1 be a binary relation on $\kappa^{+\omega}$ so that $(\kappa^{+\omega}, R_1)$ is isomorphic to $\mathbb{P}_0$. Let R_2 be a binary relation on λ so that $R_2 \cap (\kappa^{+\omega} \times \kappa^{+\omega}) = R_1$ and (λ, R_2) is isomorphic to $\mathbb{P}_0 * \mathbb{P}_1$. Let ψ be the poset isomorphism from λ to $\mathbb{P}_0 * \mathbb{P}_1$.

We continue coding. We can code the sequence $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$ as another binary relation R_3 on $\lambda^{+\omega+1}$ where $R_3 \cap \left(\{\xi\} \times \lambda^{+\omega+1}\right) = \{(\xi, h_\xi(n)) : n \in \omega\}$ for each $\xi \in \lambda^{+\omega+1}$. The relation symbol R_4 can code the sequence $\{\dot{A}_\xi : \xi \in \lambda^{+\omega+1}\}$ where $(\xi, \alpha, \zeta) \in R_4$ if and only if $(\check{\alpha}, \psi(\zeta))$ is in the name $\dot{A}_\xi$. Let R_5 code this collection, i.e. $(\gamma, n, m, \eta) \in R_5$ if and only if $((\check{n,m}), \psi(\eta)) \in \dot{H}_\gamma$. Also let R_6 code (equal) the set Λ. Finally we use the relation symbol R_7 to similarly code the sequence $\{\dot{f}_\xi : \xi \in \lambda^{+\omega+1}\}$: $(\xi, \alpha, n, \zeta) \in R_7$ if and only if $((\check{\alpha,n}), \psi(\zeta))$ is in the name $\dot{f}_\xi$.

Needless to say, the unary relation symbol R is interpreted as the set $\lambda^{+\omega}$ for the application of $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$. Now we have defined our model M of the language $L = \{\in, R, R_0, \dots, R_7\}$, and we choose an elementary submodel N witnessing $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$. Of course N is really just a $\kappa^{+\omega+1}$ sized subset of $\lambda^{+\omega+1}$ with the additional property that $N \cap \lambda^{+\omega}$ has cardinality $\kappa^{+\omega}$. In the forcing extension N

has cardinality ω_1 and $A = N \cap \lambda^{+\omega}$ is countable.

We will need the following claim from [1]

Claim. *We may assume that N satisfies that $N \cap \kappa^{+\omega+1}$ is transitive (i.e. an initial segment).*

Proof of Claim. Suppose our originally supplied N fails the conclusion of the claim. We know that $\kappa^{+\omega} \in N$, (via R_1) in which case so is $\kappa^{+\omega+1}$.

Then set $\beta_0 = \sup(N \cap \kappa^{+\omega+1})$ and consider the Skolem closure $Hull(N \cup \beta_0, M)$. A little informally (in that we have to formalize the enumeration of formulas) let $\{\varphi_n : n \in \omega\}$ is the enumeration of all formulas in the language L, and let ℓ_n be the minimal integer such that the free variables of φ_n are among $\{v_0, \ldots, v_{\ell_n}\}$. Then, for each tuple $\langle \xi_1, \ldots, \xi_{\ell_n}\rangle$ of elements of $\lambda^{+\omega+1}$, we define $f_n(\xi_1, \ldots, \xi_{\ell_n})$ to be the minimal $\xi_0 \in \lambda^{+\omega+1}$ such that $M \models \varphi_n(\xi_0, \ldots, \xi_{\ell_n})$. If there is no such ξ_0, in other words if $M \models \neg\exists x \ \ \varphi_n(x, \xi_1, \ldots, \xi_{\ell_n})$, then set $f_n(\xi_1, \ldots, \xi_{\ell_n})$ to be 0. Now $Hull(N \cup \beta_0, M)$ is just the minimal superset X of $N \cup \beta_0$ that satisfies that $f_n[X^{\{1,\ldots,\ell_n\}}] \subset X$ for all n. Since this is simply a large algebra, we can generate all the terms t of the algebraic operations $\{f_n : n \in \omega\}$. It is easily seen that for each $\zeta \in X$, there is a term $t(v_1, \ldots, v_m)$ such that $\zeta = t(\delta_1, \ldots, \delta_m)$ for some sequence $\langle \delta_1, \ldots, \delta_m\rangle$ with each $\delta_i \in N \cup \beta_0$. Assume that $\zeta \in \kappa^{+\omega+1}$. By re-indexing the variables in the term we can assume that there is an $n \leq m$ so that $\delta_i < \beta_0$ for $1 \leq i \leq n$ and $\kappa^{+\omega+1} \leq \delta_i$ for $n < i \leq m$. Let $\vec{a}$ denote the tuple $\langle \delta_{n+1}, \ldots, \delta_m\rangle$. Choose $\eta \in N \cap \kappa^{+\omega+1}$ large enough so that $\{\delta_1, \ldots, \delta_n\}$ is contained in η. Since set-membership in M is coded by R_0 rather than $\in$ we have to argue a little less naturally. Consider the set $s_0(\eta, \vec{a}) = \{t(\gamma_1, \ldots, \gamma_n, \vec{a}) : \{\gamma_1, \cdots, \gamma_n\} \in [\eta]^{\leq n}\}$. Clearly $s_0(\eta, \vec{a})$ is a member of $H(\lambda^{+\omega+1})$. Now define $s_1(\eta, \vec{a})$ to be $\{x_\alpha : \alpha \in s_0(\eta, \vec{a})\}$, and choose the unique $\zeta_1 \in \lambda^{+\omega+1}$ such that $x_{\zeta_1} = s_1(\eta, \vec{a})$. We claim that $\zeta_1 \in N$. Note that $\alpha R_0 \zeta_1$ holds if and only if $\alpha \in s_0(\eta, \vec{a})$, and therefore

$$M \models (\forall\alpha)\,[\alpha R_0 \zeta_1 \ \text{ iff } \ (\exists\gamma_1 \in \eta)\cdots(\exists\gamma_n \in \eta)(\alpha = t(\gamma_1, \ldots, \gamma_n, \vec{a}))]\ .$$

By elementarity then we have that $\zeta_1 \in N$, and by similar reasoning the supremum, ζ_0, of $\zeta_1 \cap \kappa^{+\omega+1}$ is also in N. This of course means that $\zeta < \xi_0$. $\square$

We use the elementarity of N to deduce properties of the families $\{\dot{A}_\xi : \xi \in N\}$ and $\{\dot{f}_\xi : \xi \in N\}$. Actually the collection we are most interested in is the family $\{h_\xi : \xi \in \Lambda \cap N\}$.

Since $\mathfrak{c} < \kappa^{+\omega+1}$ there is a function $\langle \varrho_n : n \in \omega\rangle$ in $\Pi_n \lambda^{+\omega}$ such that the sequence $\{h_\xi : \xi \in N\}$ is unbounded mod finite in $\Pi_n \varrho_n$ (by Shelah's pcf theory). This is in Jech somewhere. For each n, $\rho_n \leq \sup(N \cap \lambda^{+n+2})$.

Since $\mathbb{P}_0$ has cardinality less than $|N| = \kappa^{+\omega+1}$, the sequence $\{h_\xi : \xi \in \Lambda \cap N\}$ remains unbounded mod finite in $\Pi_n \varrho_n$ (and in $\Pi_n(\varrho_n \cap N)$). Now pass to the extension by $G \cap \mathbb{P}_0$ and let H be the function $\mathrm{val}_G(\dot{H})$, and we recall that $f_{\zeta_\xi}(h_\xi(n)) = H(n)$ for all $n \in \omega$.

Now pass to the full extension $V[G]$ and again, since $\mathbb{P}_1$ was forced to be countably closed, the family $\{h_\xi : \xi \in \Lambda \cap N\}$ is still unbounded in $\Pi_n(\varrho_n \cap N)$. We let A be the countable set $N \cap \lambda^{+\omega}$, and for each $\xi \in \Lambda \cap N$, there is an n_ξ such that $f_\xi(h_\xi(m)) = f_A(h_\xi(m))$ for all $m > n_\xi$. There is a single n so that $\Lambda_n = \{\xi \in \Lambda \cap N : n_\xi = n\}$ has cardinality ω_1, and thus $\{h_\xi : \xi \in \Lambda_n \cap N\}$ is also unbounded in $\Pi_n(\rho_n \cap N)$. This certainly implies that there is an $m > n$ such that $\{h_\xi(m) : \xi \in \Lambda_n \cap N\}$ is infinite. This completes the proof since $f_A(h_\xi(m)) = H(m)$ for all $\xi \in \Lambda_n \cap N$.

□

Question 15. *Is $S'(\theta)$ equivalent to having a Kurepa family on θ?*

Applications!

$$\begin{tikzcd}
 & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}\left(\kappa^\dagger\right) \arrow[r, leftrightarrow] \arrow[d] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}\left(\kappa\right) \arrow[d] \\
S'(\kappa) \arrow[ur] \arrow[r] \arrow[drr] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{1,\subseteq}_{C,F}\left(\kappa\right) \arrow[r, leftrightarrow] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cup,\subseteq}_{C,F}\left(\kappa\right) \\
 & & \mathscr{F} \underset{\text{2-tact}}{\uparrow} Sch^{\cup,\subseteq}_{C,F}\left(\kappa\right) \arrow[u]
\end{tikzcd}$$

Figure 1: Diagram of Scheeper/Menger game implications with $S'(\kappa)$

Theorem 16. *Figure 1 holds. (Proven in [TODO cite]) (Actually, TODO double-check that it works with just S', particularly the strict game)*

It was left open if these implications can be reversed. The answer is consistently no.

Theorem 17. *Let α be the limit of increasing ordinals β_n for $n < \omega$. If $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}\left(\omega_{\beta_n}\right)$ for all $n < \omega$, then $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}\left(\omega_\alpha\right)$.*

Proof. Let σ_n be a winning 2-mark for $\mathscr{F}$ in $Sch^{\cap}_{C,F}\left(\omega_{\beta_n}\right)$. Define the 2-mark σ for $\mathscr{F}$ in $Sch^{\cap}_{C,F}\left(\omega_\alpha\right)$ as follows:

$$\sigma(\langle C\rangle, 0) = \sigma_0(\langle C \cap \omega_{\beta_0}\rangle, 0)$$

$$\sigma(\langle C, D\rangle, n+1) = \sigma_{n+1}(\langle D \cap \omega_{\beta_{n+1}}\rangle, 0) \cup \bigcup_{m \le n} \sigma_m(\langle C \cap \omega_{\beta_m}, D \cap \omega_{\beta_m}\rangle, n-m+1)$$

Let $\langle C_0, C_1, \dots\rangle$ be an attack by $\mathscr{C}$ in $Sch^{\cap}_{C,F}(\omega_\alpha)$, and $\alpha \in \bigcap_{n<\omega} C_n$. Choose $N < \omega$ with $\alpha < \omega_{\beta_{N+1}}$. Consider the attack $\langle C_{N+1} \cap \omega_{\beta_{N+1}}, C_{N+2} \cap \omega_{\beta_{N+1}}, \dots\rangle$ by $\mathscr{C}$ in $Sch^{\cap}_{C,F}(\omega_{\beta_{N+1}})$. Since σ_{N+1} is a winning strategy and $\alpha \in \bigcap_{n<\omega} C_{N+n+1} \cap \omega_{\beta_{N+1}}$, either $\alpha \in \sigma_{N+1}(\langle C_{N+1} \cap \omega_{\beta_{N+1}}\rangle, 0)$ and thus $\alpha \in \sigma(\langle C_N, C_{N+1}\rangle, N+1)$, or $\alpha \in \sigma_{N+1}(\langle C_{N+M+1} \cap \omega_{\beta_{N+1}}, C_{N+M+2} \cap \omega_{\beta_{N+1}}\rangle, M+1)$ for some $M < \omega$ and thus $\alpha \in \sigma(\langle C_{N+M+1}, C_{N+M+2}\rangle, N+M+2)$. Thus σ is a winning strategy. □

Theorem 18. *Let α be the limit of increasing ordinals β_n for $n < \omega$. If $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Sch^{1,\subseteq}_{C,F}(\omega_{\beta_n})$ for all $n < \omega$, then $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Sch^{1,\subseteq}_{C,F}(\omega_\alpha)$.*

Proof. Let σ_n be a winning 2-mark for $\mathscr{F}$ in $Sch^{1,\subseteq}_{C,F}(\omega_{\beta_n})$. Define the 2-mark σ for $\mathscr{F}$ in $Sch^{1,\subseteq}_{C,F}(\omega_\alpha)$ as follows:

$$\sigma(\langle C\rangle, 0) = \sigma_0(\langle C \cap \omega_{\beta_0}\rangle, 0)$$

$$\sigma(\langle C, D\rangle, n+1) = \sigma_{n+1}(\langle D \cap \omega_{\beta_{n+1}}\rangle, 0) \cup \bigcup_{m \le n} \sigma_m(\langle C \cap \omega_{\beta_m}, D \cap \omega_{\beta_m}\rangle, n-m+1)$$

Let $\langle C_0, C_1, \dots\rangle$ be an attack by $\mathscr{C}$ in $Sch^{1,\subseteq}_{C,F}(\omega_\alpha)$, and $\alpha \in C_0$. Choose $N < \omega$ with $\alpha < \omega_{\beta_{N+1}}$. Consider the attack $\langle C_{N+1} \cap \omega_{\beta_{N+1}}, C_{N+2} \cap \omega_{\beta_{N+1}}, \dots\rangle$ by $\mathscr{C}$ in $Sch^{1,\subseteq}_{C,F}(\omega_{\beta_{N+1}})$. Since σ_{N+1} is a winning strategy and $\alpha \in C_{N+1} \cap \omega_{\beta_{N+1}}$, either $\alpha \in \sigma_{N+1}(\langle C_{N+1} \cap \omega_{\beta_{N+1}}\rangle, 0)$ and thus $\alpha \in \sigma(\langle C_N, C_{N+1}\rangle, N+1)$, or $\alpha \in \sigma_{N+1}(\langle C_{N+M+1} \cap \omega_{\beta_{N+1}}, C_{N+M+2} \cap \omega_{\beta_{N+1}}\rangle, M+1)$ for some $M < \omega$ and thus $\alpha \in \sigma(\langle C_{N+M+1}, C_{N+M+2}\rangle, N+M+2)$. Thus σ is a winning strategy. □

Corollary 19. *It is consistent that $S'(\omega_\omega)$ fails, but as $S'(\omega_k)$ holds for all $k < \omega$, we have $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Sch^{\cap}_{C,F}(\omega_\omega)$ and $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Sch^{1,\subseteq}_{C,F}(\omega_\omega)$.*

A tricky topological question: does $\mathscr{F} \uparrow Men_{C,F}(X)$ imply $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$? (C showed that) Under $V = L$, we cannot hope to find a counterexample using $X = \kappa^\dagger$ since $S'(\kappa)$ and thus $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}(\kappa)$ always holds.

Definition 20. Let R_ω be the real numbers with the topology of the usual open intervals with countably many elements removed.

Theorem 21. *$\mathscr{F} \uparrow Men_{C,F}(R_\omega)$. If there exists a Kurepa family on the reals, then $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}(R_\omega)$.*

References

[1] Jean-Pierre Levinski, Menachem Magidor, and Saharon Shelah. Chang's conjecture for $\aleph_\omega$. *Israel J. Math.*, 69(2):161–172, 1990.