$Con_{O,P}(X,x)$

Definition 1. Gruenhage's open-point convergence game $Con_{O,P}(X,x)$ has O choosing nested open sets and P choosing a point within the last chosen open set by O. O wins if the points chosen by P converge to x.

Definition 2. The one-point compactification of a space X is $X \cup \{\infty\}$, where neighborhoods of points in X are the same as they were originally, and neighborhoods of ∞ are sets $X \cup \{\infty\} \setminus K$ for compact K. If X is discrete then neighborhoods of ∞ are cofinite sets containing ∞, and the game is equivalent to O choosing finite "forbidden" sets and P choosing points not forbidden by O.

Proposition 3. *$O \uparrow_{code} Con_{O,P}(\kappa \cup \{\infty\}, \infty)$ for all cardinals κ.*

Proof. Use $F(N,p) = N \cup \{p\}$. □

Proposition 4. *$O \uparrow_{pre} Con_{O,P}(\omega \cup \{\infty\}, \infty)$.*

Proof. Use $F(n) = n$. □

Proposition 5. *$O \not\uparrow_{pre} Clus_{O,P}(\kappa \cup \{\infty\}, \infty)$ for $\kappa \geq \omega_1$.*

Proof. Let $F(n)$ be O's predetermined forbidding strategy, let $\alpha \in \kappa \setminus \bigcup_{n<\omega} F(n)$, and have P counter with $\langle \alpha, \alpha, \dots \rangle$. □

Proposition 6. *$O \uparrow_{tact} Con_{O,P}(\omega \cup \{\infty\}, \infty)$.*

Proof. Use $F(n) = n + 1$. □

Theorem 7. *If κ is a regular uncountable cardinal, for every function $f : [\kappa]^{<\omega} \to [\kappa]^{<\omega}$ the set $C_f = \{\alpha < \kappa : S \in [\alpha]^{<\omega} \Rightarrow f(S) \in [\alpha]^{<\omega}\}$ is club.*

Proof. First assume $\alpha_0 < \alpha_1 < \dots \in C_f$. It is easily seen that $\sup(\alpha_n) \in C_f$, showing C_f is closed.

Now assume $\gamma_0 \in C_f$. Let $\gamma_{n+1} > \gamma_n$ be the least ordinal such that if $S \in [\gamma_n + 1]^{<\omega}$ then $f(S) \in [\gamma_{n+1}]^{<\omega}$. We claim $\gamma_\omega = \sup(\gamma_n) \in C_f$. Let $S \in [\gamma_\omega]^{<\omega}$. Then $S \in [\gamma_n + 1]^{<\omega}$ for some n, and thus $f(S) \in [\gamma_{n+1}]^{<\omega} \subset [\gamma_\omega]^{<\omega}$. Therefore C_f is unbounded. □

We may thus assume, for the purposes of countering a tactical or Markov strategy, that the strategy is **downward** on some regular uncountable cardinal.

Theorem 8. *$O \not\uparrow_{k\text{-}tact} Clus_{O,P}(\kappa \cup \{\infty\}, \infty)$ for $\kappa \geq \omega_1$.*

Proof. Let $F : [\kappa]^{\leq k} \to [\kappa]^{<\omega}$ be a forbidding strategy by O against P which is downward on ω_1. We define n_i for $0 \leq i < k$ to be a natural number such that

$$n_i \in \omega \setminus (F(n_0, \ldots, n_{i-1}) \cup F(n_0, \ldots, n_{i-1}, \omega + i, \ldots, \omega + k - 1))$$

and note that

$$\langle n_0, n_1, \ldots, n_{k-1}, \omega, \omega + 1, \ldots, \omega + k - 1, n_0, n_1, \ldots, n_{k-1}, \omega, \omega + 1, \ldots, \omega + k - 1, \ldots \rangle$$

counters F. $\square$

Theorem 9. *$O \uparrow_{mark} Clus_{O,P}(\omega_1 \cup \{\infty\}, \infty)$.*

Proof. For $\alpha < \omega_1$ let $A_{\alpha,n}$ be a sequence of finite sets such that $A_{\alpha,n} \subset A_{\alpha,n+1}$ and $\bigcup_{n<\omega} A_{\alpha,n} = \alpha + 1$.

Give O the Markov forbidding strategy $F(n, \alpha) = A_{\alpha,n}$. To observe that any legal play by P against the strategy F has infinite range, we observe that for any $\alpha_0 < \cdots < \alpha_{k-1}$, there is some round n such that $\{\alpha_0, \ldots, \alpha_{k-1}\} \subseteq \bigcup_{0 \leq i < k} F(n, \alpha_i)$, and thus P cannot legally play any of these ordinals until another ordinal is played. $\square$

Peter J. Nyikos has shown the following:

Theorem 10. *$O \not\uparrow_{mark} Con_{O,P}(\omega_1 \cup \{\infty\}, \infty)$.*

which improves to:

Theorem 11. *$O \not\uparrow_{k\text{-}mark} Clus_{O,P}(\kappa \cup \{\infty\}, \infty)$ for $\kappa > \omega_1$.*

Proof. Let $F : \omega \times [\kappa]^{\leq k} \to [\kappa]^{<\omega}$ be a forbidding strategy by O against P which is downward on ω_2. We define α_i for $0 \leq i < k$ to be a countable ordinal such that

$$\alpha_i \in \omega_1 \setminus \bigcup_{n<\omega} (F(n, \{\alpha_0, \ldots, \alpha_{i-1}\}) \cup F(n, \{\alpha_0, \ldots, \alpha_{i-1}, \omega_1 + i, \ldots, \omega_1 + k - 1\}))$$

and note that

$$\langle \alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \omega_1, \omega_1 + 1, \ldots, \omega_1 + k - 1, \alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \omega_1, \omega_1 + 1, \ldots, \omega_1 + k - 1, \ldots \rangle$$

counters F. $\square$

$Con_{O,P}(X, x)$ for Sigma Product

Proposition 12. *$O \uparrow Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$.*

Proof. For $s \in \Sigma\mathbb{R}^\kappa$ let $C(s) = \{\alpha < \kappa : s(\alpha) \neq 0\}$ denote the countable nonzero coordinates of s. Let $\Phi : [\kappa]^{\leq\omega} \times \omega \to [\kappa]^{<\omega}$ be such that $\Phi(C, n) \subseteq \Phi(C, n+1)$ and $\bigcup_{n<\omega} \Phi(C, n) = C$.

If τ is the usual topology on $\mathbb{R}$, let $\sigma_\alpha : (\Sigma\mathbb{R}^\kappa)^{<\omega} \to \tau$ be such that

$$U_\alpha(s_0, \ldots, s_{n-1}) = \begin{cases} (-\frac{1}{n}, \frac{1}{n}) \text{ if } \alpha \in \bigcup_{i<n} \Phi(C(s_i), n) \\ \mathbb{R} \text{ otherwise} \end{cases}$$

Finally, give O the winning strategy $\sigma(s_0, \ldots, s_{n-1}) = \Sigma\mathbb{R}^\kappa \cap \prod_{\alpha<\kappa} U_\alpha(s_0, \ldots, s_{n-1})$. $\square$

Theorem 13. *For all cardinals $\kappa \leq 2^\omega$, $O \uparrow_{code} Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$.*

Proof. Note that $|\Sigma\mathbb{R}^\kappa| \leq 2^\omega = |\mathbb{R}|$. Define the following:

- Encode every $S \in (\Sigma\mathbb{R}^\kappa)^{<\omega}$ as a real number $0 < r(S) < 1$.
- Let $\gamma(U)$ be the function which, for basic open sets $U = \Sigma\mathbb{R}^\kappa \cap \prod_{\alpha<\kappa} U_\alpha$ where for all $\alpha < \kappa$ either $U_\alpha = \mathbb{R}$ or $(-\frac{1}{r}, \frac{1}{r})$, returns $\lfloor r \rfloor$.
- Let $n(U)$ be the number of non-$\mathbb{R}$ components of a basic open set U.
- Let $\psi(U, s) = r^{-1}(\gamma(U))^\frown \langle y \rangle$.
- For $s \in \Sigma\mathbb{R}^\kappa$ let $C(s) = \{\alpha < \kappa : s(\alpha) \neq 0\}$ denote the countable nonzero coordinates of s.
- Let $\Phi : [\kappa]^{\leq\omega} \times \omega \to [\kappa]^{<\omega}$ be such that $\Phi(C, n) \subseteq \Phi(C, n+1)$ and $\bigcup_{n<\omega} \Phi(C, n) = C$.
- For each $\alpha < \kappa$, define the interval $\sigma_\alpha(U, s)$ about 0 as follows:
  - If $\alpha \leq n(U)$ or $\alpha \in \bigcup_{s\in\psi(U,s)} \Phi(C(s), n(U))$ then $\sigma_\alpha(U, s) = (-\frac{1}{n(U)+r(\psi(U,s))}, -\frac{1}{n(U)+r(\psi(U,s))})$.
  - Otherwise, $\sigma_\alpha(U, s) = \mathbb{R}$.

It follows that $\sigma(U, s) = \Sigma\mathbb{R}^\kappa \cap \prod_{\alpha<\kappa} \sigma_\alpha(U, s)$ is a winning coding strategy. $\square$

Theorem 14. *Let κ be a cardinal such that there exists a function $f : \kappa \to [\kappa]^{\leq\omega}$ where for every $W \in [\kappa]^{\leq\omega}$ there exists $\alpha_W < \kappa$ with $W \subseteq f(\alpha_W)$. (That is, $cf([\kappa]^{\leq\omega}) = \kappa$.) Then $F \uparrow_{code} PF_{F,C}(\kappa)$ and $O \uparrow_{code} Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$.*

Proof. Let $W \restriction n \in [\kappa]^n$ be a subset of $W \in [\kappa]^\omega$ such that $W \restriction n \subset W \restriction (n+1)$ and $\bigcup_{n<\omega} W \restriction n = W$.

Define

$$\sigma(N, W) = N \cup (|N| + 1) \cup \{\alpha_W\} \cup \bigcup_{\alpha\in N} f(\alpha) \restriction |N|$$

Consider the play $\langle \emptyset, W_0, N_1, W_1, N_2, W_2, \dots \rangle$ with F following the strategy σ. Let $\gamma \in W_i$, and note $\gamma \in f(\alpha_{W_i})$ (and $\gamma \in f(\alpha_{W_i}) \upharpoonright |N_n|$ for sufficiently large n).

$$N_{i+1} = \sigma(N_i, W_i) \supseteq \{\alpha_{W_i}\}$$

and thus

$$N_{n+1} = \sigma(N_n, W_n) \supseteq \bigcup_{\alpha \in N_n} f(\alpha) \upharpoonright |N_n| \supseteq \bigcup_{\alpha \in N_{i+1}} f(\alpha) \upharpoonright |N_n| \supseteq f(\alpha_{W_i}) \upharpoonright |N_n|$$

showing $\gamma \in N_{n+1}$. Since γ is forbidden in round $n+1$, γ appears in finitely many sets chosen by C.

We turn our attention to $Con_{O,P}(\Sigma\mathbb{R}^\kappa)$. We define the winning strategy $\tau(U,p)$ for O as follows: let $N(U)$ be the non-$\mathbb{R}$ coordinates in the basic open set U and $W(p)$ be the non-0 coordinates in p. Then $\tau(U,p) = \left(\prod_{\alpha<\kappa} U_\alpha\right) \cap \Sigma\mathbb{R}^\kappa$ where if $\alpha \in \sigma(N(U), W(p))$ then $U_\alpha = (-\frac{1}{|N(U)|}, \frac{1}{|N(U)|})$ and $U_\alpha = \mathbb{R}$ otherwise.

Consider the play $\langle \emptyset, p_0, U_1, p_1, U_2, p_2, \dots \rangle$ with O following the strategy τ. Observe that $N(\tau(U,p)) = \sigma(N(U), W(p))$. Thus $p_i(\gamma) \neq 0$ is equivalent to $\gamma \in W(p_i)$, and by the above argument, for sufficiently large n, $\gamma \in \sigma(N(U_n), W(p_n))$. Therefore from round n onward the γ-coordinates of points chosen by P must lay in $(-\frac{1}{|N(U)|}, \frac{1}{|N(U)|})$ and converge to 0. $\square$

Theorem 15. *Let κ be the limit of cardinals κ_n such that $cf([\kappa_n]^{\leq\omega}, \subseteq) = \kappa_n$. Then $F \uparrow_{code} PF_{F,C}(\kappa)$ and $O \uparrow_{code} Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$.*

Proof. Let $f_n : \kappa_n \to [\kappa_n]^{\leq\omega}$ be such that for every $W \in [\kappa_n]^{\leq\omega}$ there exists $\alpha_{W,n} < \kappa_n$ such that $f_n(\alpha_{W,n}) \supseteq W$.

Define

$$\sigma(N,W) = N \cup (|N|+1) \cup \{\alpha_{W \cap \kappa_{|N|}, |N|}\} \cup \bigcup_{n \leq |N|} \bigcup_{\alpha \in N} f_n(\alpha) \upharpoonright |N|$$

We claim that σ is a winning coding strategy.

Consider the play $\langle N_0, W_0, N_1, W_1, \dots \rangle$ where O follows the strategy σ. For σ to be a winning strategy for $F \uparrow_{\text{code}} PF_{F,C}(\kappa)$, it must follow that for each $\gamma \in \bigcup_{i<\omega} W_i$, γ is forbidden by some $\sigma(N_j, W_j)$.

Let $\gamma \in W_i \cap \kappa_{|N_i|}$. For all $j > i$, $\alpha_{W \cap \kappa_{N_i}, |N_i|} \in N_j$. Also, $\gamma \in f_{N_i}(\alpha_{W \cap \kappa_{N_i}, |N_i|}) \upharpoonright |N_j|$ for some sufficiently large j. So we observe that $\gamma \in \bigcup_{n \leq |N_j|} \bigcup_{\alpha \in N_j} f_n(\alpha) \upharpoonright |N_j| \subseteq \sigma(N_j, W_j)$.

We turn our attention to $Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$. We define the winning strategy $\tau(U,p)$ for O as follows: let $N(U)$ be the non-$\mathbb{R}$ coordinates in the basic open set U and $W(p)$ be the non-0 coordinates in p. Then $\tau(U,p) = \Sigma\mathbb{R}^\kappa \cap \prod_{\alpha<\kappa} U_\alpha$ where if $\alpha \in \sigma(N(U), W(p))$ then $U_\alpha = (-\frac{1}{|N(U)|}, \frac{1}{|N(U)|})$ and $U_\alpha = \mathbb{R}$ otherwise.

Consider the play $\langle \Sigma\mathbb{R}^\kappa, p_0, U_1, p_1, U_2, p_2, \dots \rangle$ with O following the strategy τ. Observe that $N(\tau(U,p)) = \sigma(N(U), W(p))$. Thus $p_i(\gamma) \neq 0$ is equivalent to $\gamma \in W(p_i)$, and by the above argument, for sufficiently large n, $\gamma \in \sigma(N(U_n), W(p_n))$. Therefore from round n onward the γ-coordinates of points chosen by P must lay in $(-\frac{1}{|N(U)|}, \frac{1}{|N(U)|})$ and converge to 0. □

Theorem 16. *$F \uparrow_{code} PF_{F,C}(\kappa)$ for all cardinals κ.*

Proof. Let κ be the limit of cardinals κ_n such that $F \uparrow_{\text{code}} PF_{F,C}(\kappa_n)$ using the strategy $\sigma_n(N,W)$ such that for $M \subseteq N$, $\sigma_n(M,W) \subseteq \sigma_n(N,W)$. Define

$$\sigma(N,W) = (|N|+1) \cup \bigcup_{n<|N|} \sigma_n(N \cap \kappa_n, W \cap \kappa_n)$$

Let $\langle N_0, W_0, N_1, W_1, \dots \rangle$ be a legal play of the game with $N_{i+1} = \sigma(N_i, W_i)$. Suppose $\gamma \in W_i$ for infinitely-many i. $\gamma \in \kappa_n$ for some n, so observe the play $\langle M_0, W_0 \cap \kappa_n, M_1, W_1 \cap \kappa_n, \dots \rangle$ with $M_0 = N_0 \cap \kappa_n$ and $M_{i+1} = \sigma_n(M_i, W_i \cap \kappa_n) \subseteq \sigma_n(N_i \cap \kappa_n, W_i \cap \kappa_n)$ which is $\subseteq \sigma(N_i, W_i) = N_{i+1}$ for sufficiently large i.

Since σ_n is a winning strategy, $\gamma \in M_{m+1} \subseteq N_{m+1}$ for sufficiently large i, making $\langle N_0, W_0, N_1, W_1, \dots \rangle$ illegal, contradiction.

Now suppose $F \uparrow_{\text{code}} PF_{F,C}(\kappa)$. For each $\alpha < \kappa^+$, let $\sigma_\alpha(N,W)$ be a winning coding strategy for $PF_{F,C}(\alpha)$ such that for $M \subseteq N$, $\sigma_\alpha(M,W) \subseteq \sigma_\alpha(N,W)$. We define the following strategy for F in $PF_{F,C}(\kappa^+)$:

$$\sigma(N,W) = N \cup \bigcup_{\alpha \in N} \sigma_{\alpha+1}(N \cap (\alpha+1), W \cap (\alpha+1))$$

Let $\langle N_0, W_0, N_1, W_1, \dots \rangle$ be a legal play of the game with $N_{i+1} = \sigma(N_i, W_i)$. Suppose $\gamma \in W_i$ for infinitely-many i. Observe the play $\langle M_0, W_0 \cap (\gamma+1), M_1, W_1 \cap (\gamma+1), \dots \rangle$ with $M_0 = N_0 \cap (\gamma+1)$ and $M_{i+1} = \sigma_{\gamma+1}(M_i, W_i \cap (\gamma+1)) \subseteq \sigma_{\gamma+1}(N_i \cap (\gamma+1), W_i \cap (\gamma+1))$.

Since $\sigma_{\gamma+1}$ is a winning strategy, $\gamma \in M_{m+1} \subseteq \sigma_{\gamma+1}(N_i \cap (\gamma+1), W_i \cap (\gamma+1)) \subseteq \sigma(N_i, W_i) = N_{i+1}$ for some sufficiently large m, making $\langle N_0, W_0, N_1, W_1, \dots \rangle$ illegal, contradiction. □

Corollary 17. *$O \uparrow_{code} Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$ for all cardinals κ.*

Proof. Let $\tau(N,W)$ be the winning coding strategy for F in $PF_{F,C}(\kappa)$, $N(U) \in [\kappa]^{<\omega}$ represent the non-$\mathbb{R}$ coordinates of a basic open set U of $\Sigma\mathbb{R}^\kappa$, and $W(p) \in [\kappa]^{\leq\omega}$ represent the non-0 coordinates of a point p in $\Sigma\mathbb{R}^\kappa$. For each $\alpha < \kappa$, let

$$\sigma_\alpha(U,p) = \begin{cases} (-\frac{1}{|N(U)|}, \frac{1}{|N(U)|}) & \text{if } \alpha \in \tau(N(U), W(p)) \\ \mathbb{R} & \text{otherwise} \end{cases}$$

and $\sigma(U,p) = \Sigma\mathbb{R}^\kappa \cap \prod_{\alpha<\kappa} \sigma_\alpha(U,p)$. $\square$