(joint work with Alan Dow)

Definition 1. Two functions f, g are almost compatible if $\{a \in \mathrm{dom} f \cap \mathrm{dom} g : f(a) \neq g(a)\}$ is finite.

Definition 2. $S'(\theta)$ states that there exists a cofinal family $\mathcal{S} \subseteq [\theta]^\omega$ and a collection of pairwise almost compatible finite-to-one functions $\{f_S \in \omega^S : S \in \mathcal{S}\}$

Definition 3. $S(\theta)$ strengthens $S'(\theta)$ by requiring the collection to contain one-to-one functions.

We wish to show that Scheeper's original $S(\theta)$ is strictly stronger than $S'(\theta)$.

Definition 4. A topological space is said to be ω-bounded if each countable subset of the space has compact closure.

Theorem 5. *For each $n \in \omega$, there is a locally countable, ω-bounded topology on ω_n. Note that this means that the closure of any set has the same cardinality and weight as the set.*

To prove the theorem, we must actually prove a stronger lemma.

Lemma 6. *Assume that X has cardinality at most ω_n (for any $n \in \omega$), and is locally countable, locally compact, and the closure of each set has the same cardinality as the set. Then X has an ω-bounded extension with the same properties.*

Proof. We prove this by induction on n. In fact, we make our inductive statement that if $\tilde{X}$ is the extension of X, then $\tilde{X} \setminus X$ also has cardinality ω_n. If $n = 0$, then we can just take the free union of two copies of X and then the one-point compactification. So suppose $n > 0$ and that X is such a topology on the ordinal ω_n. For each $\alpha < \omega_n$, the closure of the initial segment α is bounded by some γ_α. Also, because X is locally countable, γ_α can be chosen so that α is contained in the interior of γ_α. There is a cub $C \subset \omega_n$ with the property that for each $\delta \in C$ and $\alpha < \delta$, γ_α is also less than δ. This implies that for each $\delta \in C$, the initial segment δ is open, and if δ has uncountable cofinality, then δ is clopen.

The proof will be easier to visualize if we now identify the points of X with the point set $\omega_n \times \{0\}$ and we will add the points $\omega_n \times \{1\}$ to create the extension. By induction on $\lambda \in C$ we define a topology on $\omega_n \times \{0\} \cup \lambda \times \{1\}$ so that $\omega_n \times \{0\}$ is an open subset. We also ensure, by induction, for each $\alpha < \lambda$, the closure of $\alpha \times 2$ is an ω-bounded subset of $\lambda \times 2$.

In the case that $n = 1$, then choose any sequence $\lambda_n : n \in \omega$ increasing cofinal in λ. If λ is a limit in C, then we simply take the topology we have constructed so far on $\lambda \times 2$ and there's nothing more needs to be done. Otherwise we may assume that λ_0 is the predecessor of $\lambda \in C$ and we set Y_λ to equal the countable set $\overline{\lambda} \setminus \lambda$. For convenience, and with no loss, we assume that λ itself is a limit of limits. And we have a topology on

$$\lambda_0 \times 2 \cup (\lambda \cup Y_\lambda) \times \{0\} .$$

Recursively choose clopen sets U_n in this topology so that $\lambda_0 \times 2 \subset U_0$, $U_n \cup \lambda_{n+1} \times \{0\}$ is contained U_{n+1} while U_{n+1} is disjoint from Y_λ. It is easy to see that we can have all the points in $(\lambda \setminus \{\lambda_n : n \in \omega\}) \times \{1\}$ be isolated, and arrange that $(\lambda_n, 1)$ is the point at infinity in the one-point compactification $U_n \cup (\lambda_n \times \{1\})$.

Now we handle the case $n > 1$ and we can shrink C and now assume that C is the closure of $\{\lambda \in C : \mathrm{cf}(\lambda) > \omega\}$. We again proceed by induction on $\lambda \in C$. If λ is a limit in C, then there is nothing to do: we simply have defined an appropriate topology on $\omega_n \times \{0\} \cup \lambda \times \{1\}$ so that for each $\mu \in C \cap \lambda$ with $\mathrm{cf}(\mu) > \omega$, $\mu \times 2$ is a clopen ω-bounded subspace. In case λ is not a limit of C, then λ has uncountable cofinality and a predecessor $\mu \in C$. We therefore have that $\lambda \times \{0\}$ is clopen in $\omega_n \times \{0\}$. We apply the induction hypothesis to the space $\lambda \times \{0\} \cup \mu \times 2$ to choose the topology on $\lambda \times 2$.

□

Definition 7. A Kurepa family $\mathcal{K} \subseteq [\theta]^\omega$ on θ satisfies that $\mathcal{K} \restriction A = \{K \cap A : K \in \mathcal{K}\}$ is countable for each $A \in [\theta]^\omega$.

Corollary 8. *There exists a Kurepa family cofinal in $[\omega_k]^\omega$ for each $k < \omega$.*

Proof. This is actually a corollary of an observation of Todorcevic communicated by Dow in [TODO cite Gen Prog in Top I]: if every Kurepa family of size at most θ extends to a cofinal Kurepa family, then the same is true of θ^+. So the result follows as every Kurepa family $\mathcal{K}$ of size ω extends to the cofinal Kurepa family $[\bigcup \mathcal{K}]^\omega$.

We may alternatively obtain the result from the previous topological argument by using the family $\mathcal{K}$ of compact sets in the constructed topology on ω_k as our witness. Of course, every Lindelöf set in a locally countable space is countable. Thus $\mathcal{K}$ is cofinal in $[\omega_k]^\omega$ since for every countable set A, $\overline{A}$ is compact and countable. It is Kurepa since for every countable set A, let (TODO) □

Theorem 9. *$S'(\theta)$ holds whenever there exists a cofinal Kurepa family on θ.*

Proof. Let $k < \omega$, and $\mathcal{K} = \{K_\alpha : \alpha < \kappa\}$ be a cofinal Kurepa family on θ. We should define $f_\alpha : K_\alpha \to \omega$ for each $\alpha < \kappa$.

Suppose we've defined pairwise almost compatible $\{f_\beta : \beta < \alpha\}$. To define f_α, we first recall that $\mathcal{K} \restriction K_\alpha$ is countable, so we may choose $\beta_n < \alpha$ for $n < \omega$ such that $\{K_\beta : \beta < \alpha\} \restriction K_\alpha \setminus \{\emptyset\} = \{K_\alpha \cap K_{\beta_n} : n < \omega\}$. Let $K_\alpha = \{\delta_{i,j} : i \leq \omega, j < w_i\}$ where $w_i \leq \omega$ for each $i \leq \omega$, $K_\alpha \cap \left(K_{\beta_n} \setminus \bigcup_{m<n} K_{\beta_m}\right) = \{\delta_{n,j} : j < w_n\}$, and $K_\alpha \setminus \bigcup_{n<\omega} K_{\beta_n} = \{\delta_{\omega,j} : j < w_\omega\}$. Then let $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ for $n < \omega$ and $f_\alpha(\delta_{\omega,j}) = j$ otherwise.

We should show that f_α is finite-to-one. Let $n < \omega$. We need only worry about $\delta_{m,j}$ for $m \leq n$ since $f_\alpha(\delta_{m,j}) \geq m$. Since each f_{β_m} is finite-to-one, $f_{\beta_m}(\delta_{m,j}) \leq n$ for only finitely many j. Thus f_α maps to n only finitely often.

We now want to demonstrate that $f_\alpha \sim f_{\beta_n}$ for all $n < \omega$. We again need only concern ourselves with $\delta_{m,j}$ for $m \leq n$ since otherwise $\delta_{m,j} \notin K_{\beta_n}$. For $m = n$, we have $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ which differs from $f_{\beta_n}(\delta_{n,j})$ for only the finitely many j which are mapped below n by f_{β_n}. For $m < n$ and $\delta_{m,j} \in K_{\beta_n}$,we have $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$ which can only differ from $f_{\beta_n}(\delta_{m,j})$ for only the finitely many j which are mapped below m by f_{β_m} or the finitely many j for which the almost compatible $f_{\beta_n} \sim f_{\beta_m}$ differ. □

Corollary 10. *$S'(\omega_k)$ holds for all $k < \omega$.*

As noted in [TODO cite Dow], Jensen's one-gap two-cardinal theorem under $V = L$ [TODO cite] can be used to show that there exist cofinal Kurepa families on every cardinal.

Corollary 11 ($V = L$)**.** *$S'(\theta)$ holds for all cardinals.*

In particular, $S(\omega_2)$ fails under CH, showing the two are distinct. Actually, CH is not required to have $S(\omega_2)$ fail.

We are going to need a technical lemma (available in Kunen).

Lemma 12. *Assume that $G \subset \mathrm{Fn}(\omega_2, 2)$ is a generic filter, and let $\mu \in \omega_2$. Then the final model $V[G]$ can be regarded as forcing with $\mathrm{Fn}(\omega_2 \setminus \mu, 2)$ over the model $V[G_\mu]$. In addition, for each $\mathrm{Fn}(\omega_2, 2)$-name $\dot{A}$ of a subset of ω (treat as a subset of $\omega \times \mathrm{Fn}(\omega_2, 2)$), there is a canonical name $\dot{A}(G_\mu)$ where,*

$$\dot{A}(G_\mu) = \{(n, p \restriction [\mu, \omega_2)) : (n, p) \in \dot{A} \quad and \quad p \restriction \mu \in G_\mu\}$$

and we get that the valuation of $\dot{A}(G_\mu)$ by the tail of the generic, $G_{\omega_2 \setminus \mu}$, is the same as the valuation of $\dot{A}$ by the full generic.

Theorem 13. *If we add ω_2 Cohen reals to a model of CH we get that Scheepers' $S(\omega_2)$ (still) fails.*

Proof. The forcing poset is $\mathrm{Fn}(\omega_2, 2)$. Let $\{\dot{f}_A : A \in [\omega_2]^\omega\}$ be a family of names such that $\dot{f}_A$ is a one-to-one function from A into ω. It suffices to only consider sets A from the ground model.

Put all the lemma stuff in an elementary submodel M of the universe (technically of $H(\kappa)$, or of V_κ, for some large κ). Standard methods says that we can assume that $|M| = \omega_1 = \mathfrak{c}$ and that $M^\omega \subset M$ (which means that every countable subset of M is a member of M).

Let $\lambda = M \cap \omega_2$ (same as the supremum of $M \cap \omega_2$). Consider the name $\dot{f}_{[\lambda, \lambda+\omega)}$. What is such a name? We can assume that it is a set of pairs of the form $((\lambda + k, m), p)$ where

$p \in Fn(\omega_2, 2)$ and, of course, $k, m \in \omega$. This is (almost) equivalent to saying that p forces that $\dot{f}_{[\lambda,\lambda+\omega)}(\lambda+k) = m$. We don't take all such p, in fact for each k, m it is enough to take a countable set of such p to get an equivalent name (Kunen calls it a nice name if we take, for each k, m an antichain that is maximal among such conditions). Given any such name $\dot{f}$, let $\text{supp}(\dot{f})$ denote the union of the domains of conditions p appearing in the name.

Also let Y equal $\text{supp}(\dot{f}_{[\lambda,\lambda+\omega)}) \setminus \lambda$. Let δ denote the order type of Y and let the 2-parameter notation $\varphi_{\mu,\lambda}$ be the order-preserving function from $\mu \cup Y$ onto the ordinal $\mu+\delta$. This lifts canonically to an order-preserving bijection $\varphi_{\mu,\lambda} : \text{Fn}(\mu \cup Y, 2) \mapsto \text{Fn}(\mu+\delta, 2)$. Similarly, we make sense of the name $\varphi_{\mu,\lambda}(\dot{f}_{[\lambda,\lambda+\omega)})$, call it F_M. Here simply, for each tuple $(\ (k,m)\ , p) \in \dot{f}_{[\lambda,\lambda+\omega)}$, we have that $(\ (k,m)\ , \varphi_{\mu,\lambda}(p))$ is in F_M. Again, let $\varphi_{\mu,\lambda}(\dot{f}_{[\lambda,\lambda+\omega)})$ be interpreted in the above sense as giving F_M (which is an element of M). Note that we do not regard δ as fixed here, but rather simply depending on the $\text{supp}(\dot{f}_{[\lambda,\lambda+\omega)})$ described above. Other values replacing $\lambda > \mu$ will result in their own set Y and canonical map $\varphi_{\mu,\lambda}$; but one thing we do have to assume (or arrange) for other values α replacing λ is that $\text{supp}(\dot{f}_{[\alpha,\alpha+\omega)})$ intersected with α is contained in μ.

Now the object F_M is an element of M, and M believes this statement is true:

$$(\forall \beta \in \omega_2)\ (\exists \beta < \lambda \in \omega_2) \quad \text{supp}(\dot{f}_{[\lambda,\lambda+\omega)}) \cap \lambda \subset \mu \ \text{ and } \ F_M = \varphi_{\mu,\lambda}(\dot{f}_{[\lambda,\lambda+\omega)})$$

But now, this means that, not only is there an $\alpha \in M$, $F_M = \varphi_{\mu,\alpha}(\dot{f}_{[\alpha,\alpha+\omega)})$ but also that there is an increasing sequence $\{\alpha_\xi : \xi \in \omega_1\} \subset \lambda$ of such α's satisfying that, for each ξ we have that $\text{supp}(\dot{f}_{[\alpha_\xi,\alpha_\xi+\omega)})$ is contained in $\alpha_{\xi+1}$.

Choose such a sequence. This means that if we let $A = \bigcup_{n>0}[\alpha_n, \alpha_n+\omega)$ we have the name $\dot{f}_A$ in M. This then means that all the $((\beta,m),p)$ appearing in $\dot{f}_A$ have the property that $dom(p)$ is contained in M. There is, within M, a name $\dot{g}$ satisfying that $\dot{f}_A(\alpha_n+k) = \dot{f}_{[\alpha_n,\alpha_n+\omega)}(\alpha_n+k)$ for all $k > \dot{g}(n)$.

We now apply the above Lemma using $\mu = \mu_0$ and we are now working in the extension $V[G_\mu]$. We will abuse the notation and use $\dot{f}_{[\alpha_n,\alpha_n+\omega)}$ instead of $\dot{f}_{[\alpha_n,\alpha_n+\omega)}(G_\mu)$ as defined in the Lemma. We work for a contradiction. Something special has now happened, namely, the supports of the names $\{\dot{f}_{[\alpha_n,\alpha_n+\omega)} : 0 < n < \omega\}$ are pairwise disjoint and also disjoint from the support of the name $\dot{f}_{[\lambda,\lambda+\omega)}$ (under the same convention about G_μ. And not only that, these names are pairwise isomorphic (in the way that they all map to F_M).

Since A is disjoint from $[\lambda, \lambda+\omega)$, there must be an integer ℓ together with a condition $q \in Fn(\omega_2 \setminus \mu, 2)$ satisfying that for all $n > \ell$, q forces that
"if $k > \dot{g}(n)$ (since $\alpha_n + k \in A$) then $\dot{f}_{[\alpha_n,\alpha_n+\omega)}(\alpha_n+k) \neq \dot{f}_{[\lambda,\lambda+\omega)}(\lambda+k)$".

Choose n large enough so that $dom(q) \cap [\alpha_n, \mu_{n+1})$ is empty. Choose $q_1 < q \restriction \lambda$ (in M) so that

$$\varphi_{\mu,\alpha_n}(q_1 \restriction \text{supp}(\dot{f}_{[\alpha_n,\alpha_n+\omega)}) = \varphi_{\mu,\lambda}(q \restriction \text{supp}(\dot{f}_{[\lambda,\lambda+\omega)})$$

and then (again in M) choose $q_2 < q_1$ so that it both forces a value L on $\ell + \dot{g}(n)$ and subsequently forces a value m on $\dot{f}_{[\alpha_n,\alpha_n+\omega)}(\alpha_n + L + 1)$. But now, again calculate

$$q_3 = \varphi^{-1}_{\mu,\lambda} \circ \varphi_{\mu,\alpha_n}(q_2 \restriction \mathrm{supp}(\dot{f}_{[\alpha_n,\alpha_n+\omega)}))$$

and, by the isomorphisms, we have that q_3 forces that $\dot{f}_{[\lambda,\lambda+\omega)}(\lambda + L + 1) = m$.

Technically (or with more care) all of this is taking place in the poset $\mathrm{Fn}(\omega_2 \setminus \mu, 2)$ and this means that q_3 and q are all compatible with each other.

Follow the bouncing ball: it suffices to consider $q(\beta) = e$ and to assume that $q_3(\beta)$ is defined. Since $q_3(\beta)$ is defined, we have that there is a $\beta' \in dom(q_2)$ such that $\varphi_{\mu,\lambda}(\beta) = \varphi_{\mu,\alpha_n}(\beta')$, and that $q_3(\beta) = q_2(\beta')$. But, by definition of q_1, $\beta' \in dom(q_1)$ and even that $q_1(\beta') = q(\beta)$. Then, since $q_2 < q_1$, we have that $q_2(\beta') = q_1(\beta') = q(\beta)$. This completes the circle that $q_3(\beta) = q(\beta)$.

Finally, our contradiction is that $q_3 \cup q_2 \cup q$ forces that $k = L + 1$ violates the quoted statement above. □

On the other hand, it's also consistent that $S'(\theta)$ can fail.

Theorem 14. *There's a model where $S'(\omega_\omega)$ fails? (TODO: get Alan to send this argument.)*

Question 15. *Is $S'(\theta)$ equivalent to having a Kurepa family on θ?*

Applications!

Theorem 16. *Figure 1 holds. (Proven in [TODO cite]) (Actually, TODO double-check that it works with just S', particularly the strict game)*

It was left open if these implications can be reversed. The answer is consistently no.

Theorem 17. *Let α be the limit of increasing ordinals β_n for $n < \omega$. If $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Sch^{\cap}_{C,F}(\omega_{\beta_n})$ for all $n < \omega$, then $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Sch^{\cap}_{C,F}(\omega_\alpha)$.*

Proof. Let σ_n be a winning 2-mark for $\mathscr{F}$ in $Sch^{\cap}_{C,F}(\omega_{\beta_n})$. Define the 2-mark σ for $\mathscr{F}$ in $Sch^{\cap}_{C,F}(\omega_\alpha)$ as follows:

$$\sigma(\langle C\rangle, 0) = \sigma_0(\langle C \cap \omega_{\beta_0}\rangle, 0)$$

$$\sigma(\langle C, D\rangle, n+1) = \sigma_{n+1}(\langle D \cap \omega_{\beta_{n+1}}\rangle, 0) \cup \bigcup_{m \leq n} \sigma_m(\langle C \cap \omega_{\beta_m}, D \cap \omega_{\beta_m}\rangle, n - m + 1)$$

$$\begin{tikzcd}
 & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}\left(\kappa^{\dagger}\right) \arrow[r, leftrightarrow] \arrow[d] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}\left(\kappa\right) \arrow[d] \\
S'(\kappa) \arrow[ur] \arrow[r] \arrow[drr] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{1,\subseteq}_{C,F}\left(\kappa\right) \arrow[r, leftrightarrow] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cup,\subseteq}_{C,F}\left(\kappa\right) \\
 & & \mathscr{F} \underset{\text{2-tact}}{\uparrow} Sch^{\cup,\subseteq}_{C,F}\left(\kappa\right) \arrow[u]
\end{tikzcd}$$

Figure 1: Diagram of Scheeper/Menger game implications with $S'(\kappa)$

Let $\langle C_0, C_1, \ldots\rangle$ be an attack by $\mathscr{C}$ in $Sch^{\cap}_{C,F}\left(\omega_\alpha\right)$, and $\alpha \in \bigcap_{n<\omega} C_n$. Choose $N < \omega$ with $\alpha < \omega_{\beta_{N+1}}$. Consider the attack $\langle C_{N+1} \cap \omega_{\beta_{N+1}}, C_{N+2} \cap \omega_{\beta_{N+1}}, \ldots\rangle$ by $\mathscr{C}$ in $Sch^{\cap}_{C,F}\left(\omega_{\beta_{N+1}}\right)$. Since σ_{N+1} is a winning strategy and $\alpha \in \bigcap_{n<\omega} C_{N+n+1} \cap \omega_{\beta_{N+1}}$, either $\alpha \in \sigma_{N+1}(\langle C_{N+1} \cap \omega_{\beta_{N+1}}\rangle, 0)$ and thus $\alpha \in \sigma(\langle C_N, C_{N+1}\rangle, N+1)$, or $\alpha \in \sigma_{N+1}(\langle C_{N+M+1} \cap \omega_{\beta_{N+1}}, C_{N+M+2} \cap \omega_{\beta_{N+1}}\rangle, M+1)$ for some $M < \omega$ and thus $\alpha \in \sigma(\langle C_{N+M+1}, C_{N+M+2}\rangle, N+M+2)$. Thus σ is a winning strategy. $\square$

Theorem 18. *Let α be the limit of increasing ordinals β_n for $n < \omega$. If $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{1,\subseteq}_{C,F}\left(\omega_{\beta_n}\right)$ for all $n < \omega$, then $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{1,\subseteq}_{C,F}\left(\omega_\alpha\right)$.*

Proof. Let σ_n be a winning 2-mark for $\mathscr{F}$ in $Sch^{1,\subseteq}_{C,F}\left(\omega_{\beta_n}\right)$. Define the 2-mark σ for $\mathscr{F}$ in $Sch^{1,\subseteq}_{C,F}\left(\omega_\alpha\right)$ as follows:

$$\sigma(\langle C\rangle, 0) = \sigma_0(\langle C \cap \omega_{\beta_0}\rangle, 0)$$

$$\sigma(\langle C, D\rangle, n+1) = \sigma_{n+1}(\langle D \cap \omega_{\beta_{n+1}}\rangle, 0) \cup \bigcup_{m \leq n} \sigma_m(\langle C \cap \omega_{\beta_m}, D \cap \omega_{\beta_m}\rangle, n-m+1)$$

Let $\langle C_0, C_1, \ldots\rangle$ be an attack by $\mathscr{C}$ in $Sch^{1,\subseteq}_{C,F}\left(\omega_\alpha\right)$, and $\alpha \in C_0$. Choose $N < \omega$ with $\alpha < \omega_{\beta_{N+1}}$. Consider the attack $\langle C_{N+1} \cap \omega_{\beta_{N+1}}, C_{N+2} \cap \omega_{\beta_{N+1}}, \ldots\rangle$ by $\mathscr{C}$ in $Sch^{1,\subseteq}_{C,F}\left(\omega_{\beta_{N+1}}\right)$. Since σ_{N+1} is a winning strategy and $\alpha \in C_{N+1} \cap \omega_{\beta_{N+1}}$, either $\alpha \in \sigma_{N+1}(\langle C_{N+1} \cap \omega_{\beta_{N+1}}\rangle, 0)$ and thus $\alpha \in \sigma(\langle C_N, C_{N+1}\rangle, N+1)$, or $\alpha \in \sigma_{N+1}(\langle C_{N+M+1} \cap \omega_{\beta_{N+1}}, C_{N+M+2} \cap \omega_{\beta_{N+1}}\rangle, M+1)$ for some $M < \omega$ and thus $\alpha \in \sigma(\langle C_{N+M+1}, C_{N+M+2}\rangle, N+M+2)$. Thus σ is a winning strategy. $\square$

Corollary 19. *It is consistent that $S'(\omega_\omega)$ fails, but as $S'(\omega_k)$ holds for all $k < \omega$, we have $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}\left(\omega_\omega\right)$ and $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{1,\subseteq}_{C,F}\left(\omega_\omega\right)$.*

A tricky topological question: does $\mathscr{F} \uparrow Men_{C,F}(X)$ imply $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$? (C showed that) Under $V = L$, we cannot hope to find a counterexample using $X = \kappa^\dagger$ since $S'(\kappa)$ and thus $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}(\kappa)$ always holds.

Definition 20. Let R_ω be the real numbers with the topology of the usual open intervals with countably many elements removed.

Theorem 21. *$\mathscr{F} \uparrow Men_{C,F}(R_\omega)$. If there exists a Kurepa family on the reals, then $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(R_\omega)$.*