

**Definition 1.** Let a V-map be a u.s.c. idempotent surjection.

**Definition 2.** For any LOS  $\langle L, \leq \rangle$ , let  $\check{L}$  be the collection of leftward subsets of  $L$  (subsets for which  $b \in L, a \leq b \Rightarrow a \in L$ ) linearly ordered by  $\subseteq$ , and let  $\hat{L}$  be the collection of left-closed subsets of  $L$  (leftward subsets which are closed) linearly ordered by  $\subseteq$ .

**Proposition 3.**  $\check{L}, \hat{L}$  are compact.

*Proof.* Each subset  $S$  has an infimum  $\cap S$  and a supremum  $\cup S$  (or  $\text{cl}(\cap S)$ ).  $\square$

Note that  $\check{L}$  is not a “compactification” as  $L$  does not necessarily embed as a dense subspace of  $\check{L}$ : if  $L = I$ , we might attempt to embed  $t \mapsto [0, t]$ , but then note that the subspace topology induces the reverse Sorgenfrey interval as  $([0, s), [0, t]) = ([0, s), [0, t])$  is open. However  $\hat{L}$  is the typical way of compactifying a linearly ordered space  $L$ , provided  $L$  lacks a least element (otherwise the empty set is an [easily removable] isolated point in  $\hat{L}$ ). Note that we **always** assume that  $\emptyset \in \hat{L}$ :

**Example 4.**  $\hat{I} \cong \{-\infty\} \cup I$  where  $\emptyset \mapsto -\infty$  and  $[0, t] \mapsto t$ .

**Example 5.** For limit ordinals  $\alpha$ ,  $\hat{\alpha} \cong \alpha + 1$ , and for all other infinite ordinals,  $\hat{\alpha} \cong \alpha$ . (The addition of a new least isolated point is of course irrelevant).

**Definition 6.** For any compact LOTS  $K$  with minimum 0 and maximum 1, let  $\gamma$  be the V-map on  $K$  where  $\gamma(0) = K$  and  $\gamma(t) = \{1\}$  for  $t > 0$ .

**Theorem 7.**  $X = \varprojlim \{2, \gamma, L\} \cong \check{L}$

*Proof.* We start by placing an order on  $X$ . Let  $\vec{x} < \vec{y}$  if there exists  $a \in L$  with  $\vec{x}(a) = 0, \vec{y}(a) = 1$ . We claim this is a total order inducing the topology on  $X$ .

We first observe that if  $\vec{x}(b) = 1$ , then for all  $a \leq b$ ,  $\vec{x}(a) \in \gamma(1) = \{1\}$ . If  $\vec{x} \neq \vec{y}$ , then assume without loss of generality that  $\vec{x}(a) = 0, \vec{y}(a) = 1$ , so  $\vec{x} < \vec{y}$ . Also, whenever  $\vec{x}(b) = 1$ , we have that  $b < a$ , so  $\vec{y}(b) = 1$ , preventing  $\vec{y} < \vec{x}$ . Finally if  $\vec{x} < \vec{y}$  and  $\vec{y} < \vec{z}$ , take  $a, b$  with  $\vec{x}(a) = 0, \vec{y}(a) = 1, \vec{y}(b) = 0, \vec{z}(b) = 1$ . It follows that  $a < b$  so  $\vec{z}(a) = 1$  and  $\vec{x} < \vec{z}$ .

Consider the basic open set  $B(\vec{x}, F)$  for a finite set  $F \in [L]^{<\omega}$  about the sequence  $\vec{x} \in X$  which contains all sequences  $\vec{y}$  agreeing with  $\vec{x}$  on  $F$ . If  $\vec{x}(a) = 1$  for all  $a \in F$ , then let  $\vec{w} \in X$  be 0 on the maximum of  $F$ , and 1 for anything less. It follows that  $B(\vec{x}, F) = (\vec{w}, \rightarrow)$ . If  $\vec{x}(a) = 0$  for all  $a \in F$ , then let  $\vec{y} \in X$  be 1 on the minimum of  $F$ , and 0 for anything greater. It follows that  $B(\vec{x}, F) = (\leftarrow, \vec{y})$ . Finally if  $\vec{x}(a) = 1$  and  $\vec{x}(b) = 0$  for  $a < b$  in  $F$  and nothing between  $a, b$  is in  $F$ , then let  $\vec{w} \in X$  be 0 on  $a$  and 1 for anything less, and let  $\vec{y} \in X$  be 1 on  $b$  and 0 for anything greater. It follows that  $B(\vec{x}, F) = (\vec{w}, \vec{y})$ .

Let  $\phi$  evaluate each  $\vec{x} \in X \subseteq 2^L$  as the characteristic function for a subset of  $L$ . It's easy to see that  $\phi$  is an order isomorphism between  $\langle X, \leq \rangle$  and  $\langle \check{L}, \subseteq \rangle$ .  $\square$

**Corollary 8.**  $\varprojlim\{2, \gamma, \alpha\} \cong \alpha + 1$  for every ordinal  $\alpha$ .

*Proof.* Since  $\tilde{\alpha} = \alpha + 1$  (actually equals, not just homeomorphic!), we get  $\varprojlim^*\{2, \gamma, \alpha\} \cong \tilde{\alpha} = \alpha + 1$  for free. Note that C and Varagona used this in (TODO create citation) to break metrizable in uncountable-ordinal-indexed inverse limits (for any V-map there exists a two-point set  $2$  such that  $f \upharpoonright 2 \supseteq \gamma$ , that is, “ $f$  has condition  $\Gamma$ ”).  $\square$

We may generalize theorem 8 as follows:

**Theorem 9.** If  $M$  is a LOTS with minimum  $0$  and maximum  $1$ , then  $\varprojlim\{M, \gamma, L\} \cong \hat{L} \times_{\text{lex}} M / \sim$ , where  $\langle \langle \leftarrow, l_0 \rangle, 1 \rangle \sim \langle \langle \leftarrow, l_1 \rangle, 0 \rangle$  if  $l_0 < l_1$  and  $(l_0, l_1) = \emptyset$ , and where  $\langle A, m \rangle \sim \langle A, m' \rangle$  if  $A \in \hat{L} \setminus L$ .

*Proof.* Let  $\rho(\vec{x}) = \text{cl}\{l \in L : \vec{x}(l) > 0\}$ ,  $v(\vec{0}) = 0$ , and  $v(\vec{x}) = \min\{\vec{x}(l) : l \in \rho(\vec{x})\}$  otherwise. Say  $\vec{x} < \vec{y}$  if  $\rho(\vec{x}) \subsetneq \rho(\vec{y})$  or both  $\rho(\vec{x}) = \rho(\vec{y})$  and  $v(\vec{x}) < v(\vec{y})$ . The reader may verify that this is a linear order on  $\varprojlim\{M, \gamma, L\}$ , and  $\theta(\vec{x}) = \langle \rho(\vec{x}), v(\vec{x}) \rangle \in \hat{L} \times_{\text{lex}} M / \sim$  preserves order. For each left-closed set  $A$  and  $m \in M$ , let  $\vec{x}_{A,m}(l) = 1$  for  $l \in A$  unless  $l$  is the supremum element of  $A$ ,  $\vec{x}_{A,m}(l) = m$  if  $l$  is the supremum of  $A$ , and  $\vec{x}_{A,m}(l) = 0$  for  $l \notin A$ . To complete the proof, we should demonstrate that the linear order we defined induces the topology of the inverse limit, and that  $\theta$  is a surjection.

A basic open set in  $\varprojlim\{M, \gamma, L\} \subseteq L^M$  is of the form  $[U, F]$  where  $U(l)$  is an open interval in  $M$  for each  $l \in F \in [L]^{<\omega}$ , and  $[U, F] = \{\vec{x} : l \in F \Rightarrow \vec{x}(l) \in U(l)\}$ . If we assume that  $[U, F]$  is non-empty, one of the following must hold:

- $U[l_0] = (a, b)$  for some  $l_0 \in F$ . Then  $[U, F] = [U, \{l_0\}]$ , and note that  $[U, \{l_0\}] = (\vec{x}_{\langle \leftarrow, l_0 \rangle, a}, \vec{x}_{\langle \leftarrow, l_0 \rangle, b})$ .
- $U(l_0) = (a, 1]$  and  $U(l_1) = [0, b)$  for some  $l_0 < l_1 \in L$  and  $[U, F] = [U, \{l_0, l_1\}]$ . Then  $[U, \{l_0, l_1\}] = (\vec{x}_{l_0, a}, \vec{x}_{l_1, b})$ .

In the other direction, consider  $\vec{y} \in (\vec{x}, \vec{z})$ .

- In the case that  $l_0 \in \rho(\vec{y}) \setminus \rho(\vec{x})$  and  $l_1 \in \rho(\vec{z}) \setminus \rho(\vec{y})$ , let  $U(l_0) = (0, 1]$ ,  $U(l_1) = [0, v(\vec{z}))$  and note  $\vec{y} \in [U, \{l_0, l_1\}] \subseteq (\vec{x}, \vec{z})$ .
- In the case that  $l_0 \in \rho(\vec{y}) \setminus \rho(\vec{x})$ ,  $\rho(\vec{y}) = \rho(\vec{z})$ , and  $v(\vec{y}) < v(\vec{z})$ , it follows that  $\rho(\vec{y}) = \rho(\vec{z}) = \langle \leftarrow, l_1 \rangle$ , so let  $U(l_0) = (0, 1]$ ,  $U(l_1) = [0, v(\vec{z}))$  and note  $\vec{y} \in [U, \{l_0, l_1\}] \subseteq (\vec{x}, \vec{z})$ .
- In the case that  $\rho(\vec{x}) = \rho(\vec{y})$ ,  $v(\vec{x}) < v(\vec{y})$ , and  $l_1 \in \rho(\vec{z}) \setminus \rho(\vec{y})$ , it follows that  $\rho(\vec{x}) = \rho(\vec{y}) = \langle \leftarrow, l_0 \rangle$ , so let  $U(l_0) = (v(\vec{x}), 1]$ ,  $U(l_1) = [0, v(\vec{z}))$  and note  $\vec{y} \in [U, \{l_0, l_1\}] \subseteq (\vec{x}, \vec{z})$ .

- In the case that  $\rho(\vec{x}) = \rho(\vec{y}) = \rho(\vec{z})$  and  $v(\vec{x}) < v(\vec{y}) < v(\vec{z})$ , it follows that  $\rho(\vec{x}) = \rho(\vec{y}) = \rho(\vec{z}) = (\leftarrow, l_0]$ , so let  $U(l_0) = (v(\vec{x}), v(\vec{z}))$  and note  $\vec{y} \in [U, \{l_0\}] = (\vec{x}, \vec{z})$ .

We conclude by showing that  $\theta$  is a surjection. If  $B \in \hat{L} \setminus L$  and  $m \in M$ , consider  $\langle B, m \rangle$ .  $B$  lacks a supremum in  $L$ , so  $\vec{x}_{B,0}(l) = 1$  for  $l \in B$  and  $\vec{x}_{B,0}(l) = 0$  otherwise. So  $\theta(\vec{x}_{B,0}) = \langle \text{cl}B, 1 \rangle = \langle B, 1 \rangle \sim \langle B, m \rangle$  for all  $m \in M$ . Otherwise,  $B = (\leftarrow, l_1]$  for some  $l_1 \in L$ . Let  $m > 0$ . Then  $\theta(\vec{x}_{(\leftarrow, l_1], m}) = \langle \text{cl}(\leftarrow, l_1], v(\vec{x}_{(\leftarrow, l_1], m}) \rangle = \langle (\leftarrow, l_1], m \rangle$ . Finally, we want to map onto  $\langle (\leftarrow, l_1], 0 \rangle$ . If there exists  $l_0 < l_1$  with  $(l_0, l_1) = \emptyset$ , then  $\theta(\vec{x}_{(\leftarrow, l_1], 0}) = \theta(\vec{x}_{(\leftarrow, l_0], 1}) = \langle (\leftarrow, l_0], 1 \rangle$  will suffice. Otherwise,  $\theta(\vec{x}_{(\leftarrow, l_1], 0}) = \langle \text{cl}(\leftarrow, l_1), v(\vec{x}_{(\leftarrow, l_1], 0}) \rangle = \langle (\leftarrow, l_1], 0 \rangle$ .  $\square$

Here are some applications:

**Example 10.**  $\varprojlim \{2, \gamma, I\} \cong (\hat{I} \setminus \emptyset) \times_{\text{lex}} 2 \cong I \times_{\text{lex}} 2 \cong \check{I}$  (of course, this could be found quicker with theorem 8).

**Example 11.**  $\varprojlim \{I, \gamma, I\} \cong (\hat{I} \setminus \emptyset) \times_{\text{lex}} I \cong I \times_{\text{lex}} I$ .

**Example 12.** For infinite ordinals  $\alpha$ ,  $\varprojlim \{I, \gamma, \alpha\} \cong (\alpha \times_{\text{lex}} [0, 1)) \cup \{\infty\}$ . In particular,  $\alpha = \kappa$  for an infinite cardinal  $\kappa$  gives the closed long ray of length  $\kappa$ .

**Definition 13.** For any  $M$  containing a point 0, let  $\nu$  be the V-map on  $M$  where  $\nu(0) = M$  and  $\nu(t) = \{t\}$  for  $t > 0$ .

Note for  $M = 2$  that  $\nu = \gamma$ .

**Corollary 14.**  $\varprojlim\{2, \nu, L\} \cong \check{L}$ .

**Theorem 15.** If  $M$  is  $T_2$  or regular, then  $\varprojlim\{M, \nu, L\} \setminus \{\vec{0}\} \cong (\check{L} \setminus \{\emptyset\}) \times (M \setminus \{0\})$  with the usual product topology.

*Proof.* Each point in  $\varprojlim\{M, \nu, L\} \setminus \{\vec{0}\}$  is of the form  $\vec{x}_{A,m}$  where  $A \in \check{L} \setminus \{\emptyset\}$  and  $m \in M \setminus \{0\}$  defined by  $\vec{x}_{A,m}(l) = m$  for  $l \in A$  and  $x_{A,m}(l) = 0$  otherwise.

We claim that the bijection  $\theta(\vec{x}_{A,m}) = \langle A, m \rangle$  is a homeomorphism. Note that basic open sets of  $\varprojlim\{M, \nu, L\}$  are of the form  $[U, F]$  where  $U(l)$  is an open subset of  $M$  for each  $l \in F \in [L]^{<\omega}$ .

Consider the point  $\langle C, m \rangle$  in the basic open set  $V \times W$  in  $(\check{L} \setminus \{\emptyset\}) \times (M \setminus \{0\})$ . Note that  $V$  is either of the form  $(A, L]$  or  $(A, B)$ , and we may choose  $l_0 \in C \setminus A$ . We also may assume that  $W$  misses an open neighborhood  $Z$  of 0 as  $M$  is  $T_2$  or regular. In the case that  $V = (A, L]$  we let  $U(l_0) = W$ . Then since  $l_0 \in C$  and  $m \in W = U(l_0)$ , it follows that  $\vec{x}_{C,m} \in [U, \{l_0\}]$ . For any  $\vec{x}_{D,n} \in [U, \{l_0\}]$  we have that  $\vec{x}_{D,n}(l_0) = n \in W$ ; in particular, it's nonzero. So  $A \subsetneq (\leftarrow, l_0] \subseteq D$ , putting  $D \in (A, L] = V$ . Thus  $\theta(\vec{x}_{D,n}) = \langle D, n \rangle \in V \times W$ .

In the case that  $V = (A, B)$ , we may also choose  $l_1 \in B \setminus C$ . We again let  $U(l_0) = W$ , and we also let  $U(l_1) = Z$ . Then as  $\vec{x}_{C,m}(l_0) = m \in W = U(l_0)$  and  $\vec{x}_{C,m}(l_1) = 0 \in Z = U(l_1)$ , we have shown  $\vec{x}_{C,m} \in [U, \{l_0, l_1\}]$ . For any  $\vec{x}_{D,n} \in [U, \{l_0, l_1\}]$ ,  $\vec{x}_{D,n}(l_0) \in W$  and  $\vec{x}_{D,n}(l_1) \in Z$ . This shows that  $\vec{x}_{D,n}(l_0) = n \in W$  and  $\vec{x}_{D,n}(l_1) = 0$ , so  $A \subsetneq (\leftarrow, l_0] \subseteq D \subsetneq (\leftarrow, l_1] \subseteq B$ , putting  $D \in (A, B) = V$ . Thus  $\theta(\vec{x}_{D,n}) = \langle D, n \rangle \in V \times W$ .

On the other hand, consider the point  $\vec{x}_{C,m}$  in the basic open set  $[U, F]$ . It follows that  $m \in U(l)$  for all  $l \in C \cap F$ , and  $0 \in U(l)$  for all  $l \in F \setminus C$ .

TODO finish

□

We introduce an alternate definition of an arbitrarily indexed inverse limit.

**Definition 16.** Let  $\varprojlim^* \{X, f, L\} \subseteq \varprojlim \{X, f, L\}$  satisfy that  $\vec{x}(a) = \lim_{t \rightarrow a} \vec{x}(t)$  for all  $a \in L$  (for any open neighborhood  $U$  of  $\vec{x}(a)$  there is  $b < a$  where  $\vec{x}(t) \in U$  for all  $t \in (b, a]$ ).

**Theorem 17.**  $Y = \varprojlim^* \{2, \gamma, L\} \cong \hat{L}$ .

*Proof.* Consider  $Y$  as a subspace of  $X = \varprojlim \{2, \gamma, L\}$  with the linear order described above. We claim that if  $\phi$  is the characteristic function for a subset of  $L$ , then  $\phi$  is an order isomorphism between  $\langle Y, \leq \rangle$  and  $\langle \hat{L}, \subseteq \rangle$ .

Let  $A$  be a left-closed subset of  $L$ . Let  $\vec{x}(a) = 1$  when  $a \in A$  and  $\vec{x}(a) = 0$  otherwise. Then  $\vec{x} \in Y$  and  $\phi(\vec{x}) = A$ .

Let  $\vec{x}, \vec{y} \in Y$ . If  $\phi(\vec{x}) = \phi(\vec{y}) = A$ , then  $A$  is a left-closed set where  $\vec{x}(a) = \vec{y}(a) = 1$  for  $a \in A$  and  $\vec{x}(a) = \vec{y}(a) = 0$  otherwise, so  $\vec{x} = \vec{y}$ .

Finally let  $\vec{x} < \vec{y}$ , so there exists  $a \in L$  with  $\vec{x}(a) = 0$ ,  $\vec{y}(a) = 1$ . Then  $\phi(\vec{x}) \subseteq (\leftarrow, a) \subseteq \phi(\vec{y})$ . Thus  $\phi$  preserves order.  $\square$

**Corollary 18.**  $\varprojlim^* \{2, \gamma, \alpha\} \cong \alpha + 1$  for every infinite limit or finite ordinal  $\alpha$ .

*Proof.* If  $\alpha$  is finite, then of course all (leftward) sets are closed and we get  $\hat{\alpha} = \check{\alpha} = \alpha + 1$  for free. Otherwise, as observed previously  $\hat{\alpha}$  is homeomorphic to its usual compactification  $\alpha + 1$  for limit ordinals.  $\square$

In fact,  $\hat{\alpha} = \alpha + 1 \setminus L(\alpha)$  where  $L(\alpha)$  is the collection of all limit ordinals less than  $\alpha$ , which also shows  $\hat{\alpha} \cong \alpha$  for infinite successor ordinals  $\alpha$ .

## References