

Game-theoretic strengthenings of Menger's property

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Abstract

A certain topological game introduced by Hurewicz characterizes Menger's covering property whenever the first player lacks a winning strategy. It was later shown by Telgarsky (and later by Scheepers using a different argument) that for metric spaces, the second player having a winning strategy characterizes the stronger property of σ -compactness.

We factor out Scheepers' proof to show that for regular spaces, the second player having a winning 1-Marköf strategy characterizes σ -compactness; also, for second-countable spaces, the presence of a winning perfect information strategy for the second player implies the existence of a winning 1-Marköf strategy for that player. We also show that for all k , the existence of a winning k -Marköf strategy for the second player implies the existence of a winning 2-Marköf strategy, and give a space which permits a winning 2-Marköf strategy but not a winning 1-Marköf strategy.

The Menger property

Definition

A space X is Menger if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ of open covers of X there exists a sequence $\langle \mathcal{F}_0, \mathcal{F}_1, \dots \rangle$ such that $\mathcal{F}_n \subseteq \mathcal{U}_n$, $|\mathcal{F}_n| < \omega$, and $\bigcup_{n < \omega} \mathcal{F}_n$ is a cover of X .

Proposition

X is σ -compact $\Rightarrow X$ is Menger $\Rightarrow X$ is Lindelöf.

The Menger game

Game

Let $\text{Cov}_{\mathcal{C}, \mathcal{F}}(X)$ denote the *Menger game* with players \mathcal{C} , \mathcal{F} . In round n , \mathcal{C} chooses an open cover \mathcal{C}_n , followed by \mathcal{F} choosing a finite subcollection $\mathcal{F}_n \subseteq \mathcal{C}_n$.

\mathcal{F} wins the game, that is, $\mathcal{F} \uparrow \text{Cov}_{\mathcal{C}, \mathcal{F}}(X)$ if $\bigcup_{n < \omega} \mathcal{F}_n$ is a cover for the space X , and \mathcal{C} wins otherwise.

Theorem

X is Menger if and only if $\mathcal{C} \nmid \text{Cov}_{\mathcal{C}, \mathcal{F}}(X)$. (Hurewicz 1926, effectively)

Menger suspected that the subsets of the real line with his property were exactly the σ -compact spaces; however:

Theorem

There are ZFC examples of non- σ -compact subsets of the real line which are Menger. (Fremlin, Miller 1988)

But metrizable non- σ -compact Menger spaces will be *undetermined* for the Menger game.

Theorem

Let X be metrizable. $\mathcal{F} \uparrow \text{Cov}_{C,F}(X)$ if and only if X is σ -compact. (Telgarsky 1984, Scheepers 1995)

Note that for Lindelöf spaces, metrizability is characterized by regularity and second countability.

Sketch of Scheeper's proof:

- Using second-countability and the winning strategy for \mathcal{F} , construct certain subsets K_s for $s \in \omega^{<\omega}$ such that $X = \bigcup_{s \in \omega^{<\omega}} K_s$.
- Using regularity, show that each K_s is compact.
- The result follows since $|\omega^{<\omega}| = \omega$.

By considering winning *limited-information strategies*, we'll be able to factor out this proof a bit.

Limited information strategies

Definition

A (*perfect information*) *strategy* has knowledge of all the past moves of the opponent.

Definition

A *k-tactical strategy* has knowledge of only the past k moves of the opponent.

Definition

A *k-Marköf strategy* has knowledge of only the past k moves of the opponent and the round number.

Obviously,

$$\mathcal{A} \xrightarrow[k\text{-tact}} G \Rightarrow \mathcal{A} \xrightarrow[k\text{-mark}} G \Rightarrow \mathcal{A} \xrightarrow[\text{(perfect)}]} G$$

But tactical strategies aren't interesting for the Menger game.

Proposition

For any $k < \omega$, $\mathcal{F} \xrightarrow[k\text{-tact}} \text{Cov}_{C,F}(X)$ if and only if X is compact.

Effectively, \mathcal{F} needs some sort of seed to prevent from being stuck in a loop: there's nothing stopping \mathcal{C} from playing the same open cover during every round of the game.

Comparitively, Marköf strategies are very powerful.

Proposition

If X is σ -compact, then $\mathcal{F} \underset{1\text{-mark}}{\uparrow} \text{Cov}_{C,F}(X)$.

Proof.

Let $X = \bigcup_{n < \omega} K_n$. During round n , \mathcal{F} picks a finite subcollection of the last open cover played by \mathcal{C} (the only one \mathcal{F} remembers) which covers K_n . □

Without assuming regularity, we can't quite reverse the implication, but we can get close.

Definition

A subset Y of X is *relatively compact* if for every open cover for X , there exists a finite subcollection which covers Y .

Proposition

If X is σ -relatively-compact, then $\mathcal{F} \xrightarrow[1\text{-mark}]{\uparrow} \text{Cov}_{C,F}(X)$.

Proposition

For regular spaces, $Y \subseteq X$ is relatively compact if and only if \overline{Y} is compact. So σ -relatively-compact regular spaces are exactly the σ -compact regular spaces.

Theorem

$\mathcal{F} \xrightarrow[1\text{-mark}]{} \text{Cov}_{C,F}(X)$ if and only if X is σ -relatively-compact.

Proof.

Let $\sigma(\mathcal{U}, n)$ represent a 1-Marköv strategy. For every open cover $\mathcal{U} \in \mathfrak{C}$, $\sigma(\mathcal{U}, n)$ witnesses relative compactness for the set

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

If X is not σ -relatively compact, fix $x \notin R_n$ for any $n < \omega$. Then \mathcal{C} can beat σ by choosing $\mathcal{U}_n \in \mathfrak{C}$ during each round such that $x \notin \bigcup \sigma(\mathcal{U}_n, n)$. □

So for regular spaces, a winning strategy for \mathcal{F} in the Menger game isn't sufficient to characterize σ -compactness, but a winning 1-Marköf strategy does the trick.

We can complete Telgarsky's/Scheeper's result by showing the following:

Theorem

For second countable spaces X , $\mathcal{F} \uparrow \text{Cov}_{C,F}(X)$ if and only if $\mathcal{F} \uparrow_{1\text{-mark}} \text{Cov}_{C,F}(X)$.

Proof

Let σ be a perfect information strategy. Since X is a second-countable space, we may pretend that there are only countably many finite collections of open sets. Thus for $s \in \omega^{<\omega}$, we may define open covers $\mathcal{U}_{s \smallfrown \langle n \rangle}$ such that for each open cover \mathcal{U} , there is some $n < \omega$ where

$$\sigma(\mathcal{U}_{s \smallfrown 1}, \dots, \mathcal{U}_s, \mathcal{U}) = \sigma(\mathcal{U}_{s \smallfrown 1}, \dots, \mathcal{U}_s, \mathcal{U}_{s \smallfrown \langle n \rangle})$$

Let $t : \omega \rightarrow \omega^{<\omega}$ be a bijection. During round n and seeing only the latest open cover \mathcal{U} , \mathcal{F} may play the finite subcollection

$$\tau(\mathcal{U}, n) = \sigma(\mathcal{U}_{t(n) \smallfrown 1}, \dots, \mathcal{U}_{t(n)}, \mathcal{U})$$

Unlike the Banach-Mazur game, we can immediately see that knowledge of more than two previous moves of \mathcal{F} 's opponent must be infinite to be of any use.

Theorem

If $\mathcal{F} \uparrow_{k\text{-mark}} \text{Cov}_{C,F}(X)$, then $\mathcal{F} \uparrow_{2\text{-mark}} \text{Cov}_{C,F}(X)$.

Proof.

$$\tau(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) = \bigcup_{m < k+2} \sigma(\underbrace{\langle \mathcal{U}, \dots, \mathcal{U} \rangle}_{k+1-m}, \underbrace{\langle \mathcal{V}, \dots, \mathcal{V} \rangle}_{m+1}, (n+1)(k+2)+m)$$



Knowledge of two previous moves versus one is an important distinction: in the former case, the player is able to react to change by the opponent.

Definition

Let $\kappa^\dagger = \kappa \cup \{\infty\}$ be the *one point Lindelöf-ication* of discrete κ : neighborhoods of ∞ are exactly the co-countable sets containing it.

κ^\dagger is a simple space which is a regular and Lindelöf, but not second-countable space or σ -compact. Thus

$\mathcal{F} \not\uparrow \text{Cov}_{C,F}(\kappa^\dagger)$, but it's easy to see that $\mathcal{F} \uparrow \text{Cov}_{C,F}(\kappa^\dagger)$.
1-mark

What about 2-Marköf strategies?

In 1991, Scheepers introduced the statement $S(\kappa, \omega, \omega)$ to study an infinite game involving the countable and finite subsets of κ .

Game

Let $\text{Fill}_{C,F}^{\cup, \subset}(\kappa)$ denote the *strict union filling game* with two players \mathcal{C} , \mathcal{F} . In round 0, \mathcal{C} chooses $C_0 \in [\kappa]^{\leq \omega}$, followed by \mathcal{F} choosing $F_0 \in [\kappa]^{< \omega}$. In round $n+1$, \mathcal{C} chooses $C_{n+1} \in [\kappa]^{\leq \omega}$ such that $C_{n+1} \supset C_n$, followed by \mathcal{F} choosing $F_{n+1} \in [\kappa]^{< \omega}$. \mathcal{F} wins the game if $\bigcup_{n < \omega} F_n \supseteq \bigcup_{n < \omega} C_n$; otherwise, \mathcal{C} wins.

Definition

For two functions f, g we say f is *almost compatible* with g ($f \wr g$) if $|\{x \in \text{dom}(f) \cap \text{dom}(g) : f(x) \neq g(x)\}| < \omega$.

Definition

$S(\kappa, \omega, \omega)$ states that there exist functions $f_A : A \rightarrow \omega$ for each $A \in [\kappa]^{\leq \omega}$ such that $|f_A^{-1}(n)| < \omega$ for all $n < \omega$ and $f_A \wr f_B$ for all $A, B \in [\kappa]^\omega$.

Theorem

$S(\omega_1, \omega, \omega)$; $\text{Con}(S(2^\omega, \omega, \omega) + \neg CH)$; $\neg S(\kappa, \omega, \omega)$ for $\kappa > 2^\omega$.

The round number is unnecessary in Scheeper's game, since \mathcal{C} must choose strictly increasing sets.

Theorem

If $S(\kappa, \omega, \omega)$, then $\mathcal{F} \underset{2\text{-tact}}{\uparrow} \text{Fill}_{C,F}^{\cup, \subset}(\kappa)$.

As it turns out, a related game characterizes $\text{Cov}_{C,F}(\kappa^+)$.

Definition

Let $\text{Fill}_{C,F}^{\cap}(\kappa)$ denote the *intersection filling game* analogous to $\text{Fill}_{C,F}^{\cup, \subset}(\kappa)$, except that \mathcal{C} has no restriction on the countable sets she chooses, but \mathcal{F} need only ensure that $\bigcup_{n < \omega} \mathcal{F}_n \supseteq \bigcap_{n < \omega} \mathcal{U}_n$ to win the game.

Theorem

$\mathcal{F} \uparrow_{k\text{-mark}} \text{Cov}_{C,F}(\kappa^\dagger)$ if and only if $\mathcal{F} \uparrow_{k\text{-mark}} \text{Fill}_{C,F}^\cap(\kappa)$.

Theorem

If $S(\kappa, \omega, \omega)$, then $\mathcal{F} \uparrow_{2\text{-mark}} \text{Fill}_{C,F}^\cap(\kappa)$.

Proof.

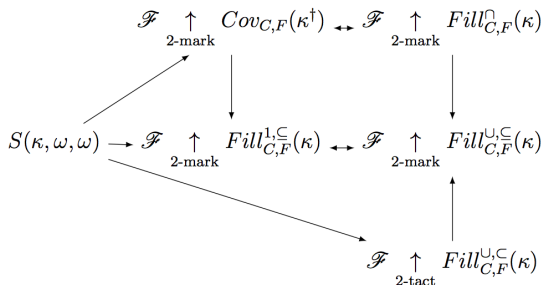
Let $f_A : A \rightarrow \omega$ witness $S(\kappa, \omega, \omega)$. Then we define the winning 2-Marköv strategy σ as follows:

$$\sigma(\langle A \rangle, 0) = \{\alpha \in A : f_A(\alpha) = 0\}$$

$$\sigma(\langle A, B \rangle, n+1) = \{\alpha \in A \cap B : f_B(\alpha) \leq n+1 \text{ or } f_A(\alpha) \neq f_B(\alpha)\}$$

Corollary

$\mathcal{F} \uparrow_{2\text{-mark}} \text{Cov}_{C,F}(\omega_1^\dagger)$, but $\mathcal{F} \nuparrow_{1\text{-mark}} \text{Cov}_{C,F}(\omega_1^\dagger)$.



Question

Does $\mathcal{F} \uparrow_{2\text{-mark}} \text{Cov}_{C,F}(\kappa^\dagger)$ imply $S(\kappa, \omega, \omega)$?

Question

Are $\mathcal{F} \uparrow \text{Cov}_{C,F}(X)$ and $\mathcal{F} \uparrow_{2\text{-mark}} \text{Cov}_{C,F}(X)$ distinct?

An affirmative answer to the first question answers this since $\neg S(\kappa, \omega, \omega)$ for $\kappa > 2^\omega$.

Question

Where does $\mathcal{F} \uparrow_{2\text{-mark}} \text{Cov}_{C,F}(X)$ fit in with other properties between σ -(relatively-)compact and Menger?

$\mathcal{F} \uparrow_{2\text{-mark}} \text{Cov}_{C,F}(X)$ seems to characterize an “almost- σ -(relative-)compactness”.

Any sufficient property must imply $\mathcal{F} \uparrow \text{Cov}_{C,F}(X)$ (ruling out Hurewicz[?]), and any (interesting) necessary property shouldn't be implied by $\mathcal{F} \uparrow \text{Cov}_{C,F}(X)$ (ruling out Alster and productively Lindelöf for regular spaces).

Questions? Thanks for listening!