k-Limited Strategies in Banach Mazur Games

Steven Clontz

December 6, 2017

Definition 1. Let $\mathbb{P}$ be partially ordered by $\leq$. Let $\mathbb{P}^{\downarrow} = \{f \in \mathbb{P}^{\omega} : f(n) \geq f(n+1)\}$. Then for $f, g \in \mathbb{P}^{\downarrow}$, we say that f, g zip into each other if for all $m < \omega$ there exists $n < \omega$ such that $f(m) \geq g(n)$ and $g(m) \geq f(n)$.

Definition 2. $BM_{po}(\mathbb{P}, W)$ is a game defined for all non-empty partial orders $\mathbb{P}$ and all subsets $W \subseteq \mathbb{P}^{\downarrow}$ closed under zipping. During round 0, I chooses $a_0 \in \mathbb{P}$, and then II chooses $b_0 \leq a_0$; during around $n+1$, I chooses $a_{n+1} \leq b_n$, and then II chooses $b_{n+1} \leq a_{n+1}$. II wins this game if $\langle a_0, a_1, \dots \rangle \in W$.

Theorem 3. *II* $\underset{(k+1)\text{-}mark}{\uparrow} BM_{po}(\mathbb{P}, W)$ *if and only if* II $\underset{(k+1)\text{-}tact}{\uparrow} BM_{po}(\mathbb{P}, W)$.

Proof. Let $\tau(\vec{p}, n)$ be a winning $(k+1)$-mark for II. Let $\preceq$ well-order $\mathbb{P}^{k+1}$. For $f \in \mathbb{P}^{\omega}$, let $f_n = \langle f(n), \dots, f(n+k) \rangle \in \mathbb{P}^{k+1}$. For $\vec{p}, \vec{r} \in \mathbb{P}^{k+1}$ say that $\vec{p}$ improves $\vec{r}$ if for each $m \leq k$, there exists $n \leq k$ such that $\vec{r}(m) \geq \vec{p}(n)$.

For $\vec{p} \in \mathbb{P}^{k+1}$ and $q \in \mathbb{P}$, say $\vec{p}$ is n-above q if there exists $s_n(\vec{p}) \in \mathbb{P}^{k+1}$ improving $\vec{p}$ such that

$$\vec{p}(k) \geq s_n(\vec{p})(k) \geq \tau(s_n(\vec{p}), n+k) \geq q$$

(noting $\vec{p}(k) \geq s_n(\vec{p})(k)$ is just a consequence of $s_n(\vec{p})$ improving $\vec{p}$).

Say $\vec{p}$ is ω-above q if $\vec{p}$ is n-above q for all $n < \omega$. If $\vec{p}$ is ω-above some $l(\vec{p})$, then say $\vec{p}$ is long; otherwise call $\vec{p}$ short. Note any $\vec{p}$ improved by a long vector is itself long, so any $\vec{p}$ improving a short vector is itself short.

For $\vec{p}$ short, let

$$\mu(\vec{p}) = \min_{\preceq}\{\vec{r} \text{ short} : \vec{p} \text{ improves } \vec{r}\}$$

and since $\mu(\vec{p})$ is not n-above $\vec{p}$ for some n, let

$$N(\vec{p}) = \min\{n < \omega : \mu(\vec{p}) \text{ is not } n\text{-above } \vec{p}\}.$$

We define

$$\sigma(\vec{p}) = \begin{cases} \tau(\vec{p}, |\vec{p}| - 1) & 0 < |\vec{p}| \leq k \\ l(\vec{p}) & \vec{p} \text{ is long} \\ \tau^{N(\vec{p})+1}(\vec{p}) & \vec{p} \text{ is short} \end{cases}.$$

Suppose σ is legally attacked by $a \in \mathbb{P}^{\omega}$. Thus for $n < k$,

$$a(n) \geq \tau(a \restriction (n+1), n) = \sigma(a \restriction (n+1)) \geq a(n+1).$$

For $n \leq \omega$, if a_n is long, then a_n is n-above $l(a_n)$. Therefore,

$$a(n+k) = a_n(k) \geq s_n(a_n)(k) \geq \tau(s_n(a_n), n+k) \geq l(a_n) = \sigma(a_n) \geq a(n+k+1).$$

Thus if a_n is long for $n < \omega$, it follows that $\langle a(0), \ldots, a(k-1), s_0(a_0)(k), s_1(a_1)(k), \ldots \rangle$ is a legal attack against τ. Since τ is winning, this attack belongs to W. Since this attack zips into a, a also belongs to W.

Otherwise, we may choose $k < N < \omega$ such that

- a_{n+N} is short for all $n < \omega$, since a_m short implies a_n short for all $m \leq n$.
- $\mu(a_{n+N}) = \vec{m}$ is fixed for all $n < \omega$, since there cannot be an infinite $\preceq$-decreasing sequence.

As a result, a_{n+N} is... Thus for $n < k$,

$$b(n) \geq \tau(b \upharpoonright (n+1), n) = \sigma(a \upharpoonright (n+1)) \geq a(n+1).$$

$\square$