Definition 1. The simplified proximal game can be defined as follows:

In round 0, $\mathscr{D}$ chooses a symmetric entourage D_0, followed by $\mathscr{P}$ choosing a point $p_0 \in X$.

In round $n+1$, $\mathscr{D}$ chooses a symmetric entourage D_{n+1}, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in \bigcap_{m \leq n} D_n[p_n]$.

Player $\mathscr{D}$ wins if either $\bigcap_{n<\omega} \left(\bigcap_{m \leq n} D_n \right) [p_n] = \emptyset$ or p_n converges.

Theorem 2. *The simplified proximal game is perfect-info equivalent to Bell's game for the entourage-picker $\mathscr{D}$.*

Proof. Let σ be a winning perfect information strategy for $\mathscr{D}$ in Bell's game. We define a perfect information strategy τ in the simplified game to be simply $\tau(p \restriction n) = \sigma(p \restriction n)$ (so $\tau(p \restriction n) = \bigcap_{m \leq n} \tau(p \restriction m)$) for all partial attacks $p \restriction n$.

If p attacks τ in the simplified game, $p(n+1) \in \bigcap_{m \leq n} \tau(p \restriction m)[p(n)] = \tau(p \restriction n)[p(n)] = \sigma(p \restriction n)[p(n)] \subseteq 4\sigma(p \restriction n)[p(n)]$, so p attacks σ in Bell's game. Thus either p converges, or

$$\emptyset = \bigcap_{n<\omega} 4\sigma(p \restriction n)[p(n)] \supseteq \bigcap_{n<\omega} \tau(p \restriction n)[p(n)] = \bigcap_{n<\omega} \left(\bigcap_{m \leq n} \tau(p \restriction n) \right) [p(n)]$$

For the other direction, let σ be a winning perfect information strategy for $\mathscr{D}$ in the simplified game such that $\sigma(p \restriction n) = \bigcap_{m \leq n} \sigma(p \restriction m)$. Define the perfect information strategy τ in Bell's Game such that $4\tau(p \restriction n) \subseteq \sigma(p \restriction n)$ and $\tau(p \restriction n) = \bigcap_{m \leq n} \tau(p \restriction m)$ for all partial attacks $p \restriction n$.

If p attacks τ in Bell's game, $p(n) \in 4\tau(p \restriction n) \subseteq \sigma(p \restriction n) = \bigcap_{m \leq n} \sigma(p \restriction m)$, so p attacks σ in the simplified game. Thus either p converges, or

$$\emptyset = \bigcap_{n<\omega} \left(\bigcap_{m \leq n} \sigma(p \restriction n) \right) [p(n)] = \bigcap_{n<\omega} \sigma(p \restriction n)[p(n)] \supseteq \bigcap_{n<\omega} 4\tau(p \restriction n)[p(n)] \supseteq \bigcap_{n<\omega} \tau(p \restriction n)[p(n)]$$

□

Definition 3. A **uniform space** $\langle X, \mathcal{D}\rangle$ is a set X paired with a filter $\mathcal{D}$ (called its **uniformity**) of relations (called **entourages**) on X such that for each entourage $D \in \mathcal{D}$:

- D is reflexive, i.e., the diagonal $\Delta \subseteq D$.
- Its inverse $D^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in D\} \in \mathcal{D}$.
- There exists $E \in \mathcal{D}$ such that $2E = E \circ E = \{\langle x, z\rangle : \exists y(\langle x, y\rangle, \langle y, z\rangle \in E)\} \subseteq D$

Note that since $\mathcal{D}$ is a filter, for each $D \in \mathcal{D}$, the symmetric relation $D \cap D^{-1} \in \mathcal{D}$.

Definition 4. For an entourage $D \in \mathcal{D}$, let $D[x] = \{y : (x, y) \in D\}$ be the D**-neighborhood** of x. The uniform topology for a uniform space $\langle X, \mathcal{D}\rangle$ is generated by the base $\{D[x] : x \in X, D \in \mathcal{D}\}$.

Theorem 5. *A space X is uniformizable (its topology is the uniform topology for some uniformity) if and only if X is completely regular ($T_{3\frac{1}{2}}$).*

Definition 6. For a uniform space X, the proximity game $Prox_{D,P}(X)$ proceeds as follows.

In round 0, $\mathscr{D}$ chooses a symmetric entourage D_0, followed by $\mathscr{P}$ choosing a point $p_0 \in X$.

Let $E_n = \bigcap_{m \leq n} D_n$. In round $n+1$, $\mathscr{D}$ chooses a symmetric entourage D_{n+1}, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in 4E_n[p_n]$.

Player $\mathscr{D}$ wins if either $\bigcap_{n<\omega} 4E_n[p_n] = \emptyset$ or p_n converges.

Definition 7. A uniform space is **proximal** if $\mathscr{D} \uparrow Prox_{D,P}(X)$.

Definition 8. For a space X and a point $x \in X$, the W**-convergence-game** $Con_{O,P}(X, x)$ proceeds as follows.

Let $V_n = \bigcap_{m \leq n} U_m$. In round 0, $\mathscr{O}$ chooses a neighborhood U_n of x, followed by $\mathscr{P}$ choosing a point $p_n \in V_n$.

Player $\mathscr{O}$ wins if p_n converges.

Definition 9. A space is W if $\mathscr{O} \uparrow Con_{O,P}(X, x)$ for all $x \in X$.

Definition 10. For each finite tuple $(m_0, \ldots, m_{n-1})$, we define the k**-tactical fog-of-war**

$$T_k(\langle m_0, \ldots, m_{n-1}\rangle) = \langle m_{n-k}, \ldots, m_{n-1}\rangle$$

and the k**-Markȯv fog-of-war**

$$M_k(\langle m_0, \ldots, m_{n-1}\rangle) = \langle\langle m_{n-k}, \ldots, m_{n-1}\rangle, n\rangle$$

So $P \uparrow_{k\text{-tact}} G$ if and only if there exists a winning strategy for P of the form $\sigma \circ T_k$, and $P \uparrow_{k\text{-mark}} G$ if and only if there exists a winning strategy of the form $\sigma \circ M_k$.

Theorem 11. *For all $x \in X$:*

- $\mathscr{D} \uparrow Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow Con_{O,P}(X,x)$
- $\mathscr{D} \uparrow_{2k\text{-}tact} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(X,x)$
- $\mathscr{D} \uparrow_{2k\text{-}mark} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(X,x)$

Proof. Let σ witness $\mathscr{D} \uparrow_{2k\text{-tact}} Prox_{D,P}(X)$ (resp. $\mathscr{D} \uparrow_{2k\text{-mark}} Prox_{D,P}(X)$, $\mathscr{D} \uparrow Prox_{D,P}(X)$). We define the k-tactical (resp. k-Marköv, perfect info) strategy τ such that

$$\tau \circ L_k(p) = \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(|p|-1)\rangle)[x] \cap \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(|p|-1), x\rangle)[x]$$

where L_{2k} is the $2k$-tactical fog-of-war (resp. $2k$-Marköv fog-of-war, identity) and L_k is the k-tactical fog-of-war (resp. k-Marköv fog-of-war, identity).

Let p attack τ. Consider the attack q against the winning strategy σ such that $q(2n) = x$ and $q(2n+1) = p(n)$, and let $D_n = \sigma \circ L_{2k}(q)$ and $E_n = \bigcap_{m \leq n} D_n$.

Certainly, $x \in E_{2n}[x] = E_{2n}[q(2n)]$ for any $n < \omega$. Note also for any $n < \omega$ that

$$p(n) \in \bigcap_{m \leq n} \tau \circ L_k(p \upharpoonright n)$$

$$= \bigcap_{m \leq n} (\sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(m-1)\rangle)[x] \cap \sigma \circ L_{2k}(\langle x, p(0), \dots, x, p(m-1), x\rangle)[x])$$

$$= \bigcap_{m \leq n} (D_{2m}[x] \cap D_{2m+1}[x]) = \bigcap_{m \leq 2n+1} D_m[x] = E_{2n+1}[x]$$

so by the symmetry of E_{2n+1}, $x \in E_{2n+1}[p(n)] = E_{2n+1}[q(2n+1)]$. Thus $x \in \bigcap_{n<\omega} E_n[q(n)] \subseteq \bigcap_{n<\omega} 4E_n[q(n)]$, and since σ is a winning strategy, the attack q converges, and since $q(2n) = x$, q must converge to x. Thus its subsequence p converges to x, and τ is a winning strategy. □

Corollary 12. *For all $x \in X$:*

- $\mathscr{D} \uparrow_{k\text{-}tact} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(X,x)$
- $\mathscr{D} \uparrow_{k\text{-}mark} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(X,x)$

Corollary 13. *All proximal spaces are W-spaces.*

Theorem 14. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete. Then*

- $\mathscr{O} \uparrow Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathscr{D} \uparrow Prox_{D,P}(X \cup \{\infty\})$

- $\mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathscr{D} \uparrow_{k\text{-}tact} Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathscr{D} \uparrow_{k\text{-}mark} Prox_{D,P}(X \cup \{\infty\})$

Proof. Note that the topology on $X \cup \{\infty\}$ is induced by the uniformity with equivalence relation entourages $D(U) = \Delta \cup U^2$ for each open neighborhood U of ∞.

Let σ witness $\mathscr{D} \uparrow_{k\text{-tact}} Con_{O,P}(X \cap \{\infty\}, \infty)$ (resp. $\mathscr{D} \uparrow_{k\text{-mark}} Con_{O,P}(X \cap \{\infty\}, \infty)$, $\mathscr{D} \uparrow Con_{O,P}(X \cap \{\infty\}, \infty)$). We define the k-tactical (resp. k-Markӧv, perfect info) strategy τ such that

$$\tau \circ L(p) = D(\sigma \circ L(p))$$

where L is the k-tactical fog-of-war (resp. k-Markӧv fog-of-war, identity).

Let $p \in (X \cup \{\infty\})^\omega$ attack τ such that $\infty \in \bigcap_{n<\omega} \tau(p \upharpoonright n)[p(n)]$. It follows then that p is an attack on σ. Since σ is a winning strategy, it follows that q and its subsequence p must coverge to ∞.

Otherwise, $\infty \notin \tau(p \upharpoonright N)[p(N)]$ for some $N < \omega$, and then $\tau(p \upharpoonright N)[p(N)] = 4\tau(p \upharpoonright N)[p(N)] = \{p(N)\}$ implies $p \to p(N)$.

Thus $\tau \circ L$ is a winning strategy. □

Corollary 15. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete. Then*

- $\mathscr{O} \uparrow Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \uparrow Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \uparrow_{k\text{-}tact} Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \uparrow_{k\text{-}mark} Prox_{D,P}(X \cup \{\infty\})$

Proposition 16. *For any $x \in X$,*

- $\mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(X, x) \Rightarrow \mathscr{O} \uparrow_{tact} Con_{O,P}(X, x)$
- $\mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(X, x) \Rightarrow \mathscr{O} \uparrow_{mark} Con_{O,P}(X, x)$

Proof. If σ witnesses $\mathscr{O} \uparrow_{k\text{-tact}} Con_{O,P}(X, x)$, let $\tau(\emptyset) = \sigma(\emptyset)$ and

$$\tau(\langle q \rangle) = \bigcap_{i<k} \sigma(\langle \underbrace{x, \dots, x}_{k-i-1}, q, \underbrace{x, \dots, x}_{i} \rangle)$$

This is easily verified to be a winning strategy. The proof for $\mathscr{O} \uparrow_{k\text{-mark}} Con_{O,P}(X, x)$ is analogous. □

Corollary 17. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete. Then*

- $\mathscr{D} \uparrow_{(k+1)\text{-}tact} Prox_{D,P}(X \cup \{\infty\}) \Leftrightarrow O \uparrow_{tact} Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{D} \uparrow_{(k+1)\text{-}mark} Prox_{D,P}(X \cup \{\infty\}) \Leftrightarrow O \uparrow_{mark} Prox_{D,P}(X \cup \{\infty\})$

Proposition 18. *For any uniform space X,*

- $\mathscr{O} \uparrow_{k\text{-}tact} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{2\text{-}tact} Prox_{D,P}(X)$
- $\mathscr{O} \uparrow_{k\text{-}mark} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{2\text{-}mark} Prox_{D,P}(X)$

Proof. If σ witnesses $\mathscr{O} \uparrow_{k\text{-tact}} Con_{O,P}(X, x)$, let $\tau(\emptyset) = \sigma(\emptyset)$ and

$$\tau(\langle q \rangle) = \bigcap_{i<k} \sigma(\langle \underbrace{q, \dots, q}_{i} \rangle)$$

$$\tau(\langle q, q' \rangle) = \bigcap_{i<k} \sigma(\langle \underbrace{q, \dots, q}_{k-i}, \underbrace{q', \dots, q'}_{i} \rangle)$$

This is easily verified to be a winning strategy. The proof for $\mathscr{O} \uparrow_{k\text{-mark}} Con_{O,P}(X, x)$ is analogous. $\square$

Theorem 19. *If $\mathscr{D} \uparrow Prox_{D,P}(X)$, then $\mathscr{O} \uparrow Clus_{O,P}(X, H)$ for all compact $H \subseteq X$.*

Proof. Let σ witness $\mathscr{D} \uparrow Prox_{D,P}(X)$ such that $q \subseteq p$ implies $\sigma(q) \supseteq \sigma(p)$. Then for certain $p \in X^{<\omega}$, we define a tree $T(p)$ and finite number $m(p)$ as follows:

- For $T(\emptyset)$:
  - Let $\emptyset \in T(\emptyset)$.
  -

$\square$

Corollary 20. *If $\mathscr{D} \uparrow Prox_{D,P}(X)$, then $\mathscr{O} \uparrow Con_{O,P}(X, H)$ for all compact $H \subseteq X$.*

Definition 21. A filter $\mathcal{F}$ on a uniform space X is **Cauchy** if for every entourage D, there exists $A \in \mathcal{F}$ such that $A^2 \subseteq D$.

Definition 22. A fitler $\mathcal{F}$ **converges** to x ($\mathcal{F} \to x$) if for every neighborhood U of x, there exists $A \in \mathcal{F}$ such that $x \in A \subseteq U$.

Definition 23. A uniform space X is **completely uniform** if every Cauchy filter converges.

Proposition 24. *Completely uniform metrizable spaces are completely metrizable.*

Proof. ?? □

Theorem 25. *For all completely uniform X, $\mathscr{O} \uparrow_{pre} Prox_{D,P}(X)$ if and only if X is metrizable.*

Proof. Assume X is metrizable, and thus completely metrizable. Define the predetermined strategy σ such that if $D_n = \{(x, y) : d(x, y) < \frac{1}{4^n}\}$ then $\sigma(n) = D_{n+1}$. Note that $\sigma(n+1) = D_{n+2} \subseteq 4D_{n+2} = D_{n+1} = \sigma(n)$, so $\bigcap_{m \leq n} \sigma(m) = \sigma(n)$.

Let p attack σ. We have $p(n + 1) \in 4\sigma(n)[p(n)] = 4D_{n+1}[p(n)] = D_n[p(n)]$, so $d(p(n), p(n + 1)) < \frac{1}{4^n}$. Thus p is Cauchy and converges.

Let σ witness $\mathscr{O} \uparrow_{\text{pre}} Prox_{D,P}(X)$. Claim: $\Delta = \bigcap_{n<\omega} \sigma(n)$. □

Clopen partition version

Definition 26. For any partition $\mathcal{R}$ of a space X and $x \in X$, let $\mathcal{R}[x]$ be such that $x \in \mathcal{R}[x] \in \mathcal{R}$.

For partitions $\mathcal{R}_0, \dots, \mathcal{R}_n$, let $\mathcal{H}_n = \bigwedge_{m \leq n} \mathcal{R}_m$ be the coarsest partition which refines each $\mathcal{R}_m$.

For partitions $\mathcal{R}, \mathcal{S}$ let $\mathcal{R} \otimes \mathcal{S} = \{r \times s : r \in R, s \in S\}$.

Proposition 27. $x \in \mathcal{R}[y] \Leftrightarrow y \in \mathcal{R}[x]$.

$\mathcal{H}_n[x] = \left(\bigwedge_{m \leq n} \mathcal{R}_m\right)[x] = \bigcap_{m \leq n} \mathcal{R}_m[x]$.

Definition 28. For zero-dimensional X, the proximity game $Prox_{D,P}(X)$ proceeds as follows: in round n, $\mathscr{R}$ chooses a clopen partition $\mathcal{R}_n$ of X, followed by $\mathscr{P}$ choosing a point $p_n \in X$.

Player $\mathscr{R}$ wins if either $\bigcap_{n<\omega} \mathcal{H}_n[p_n] = \emptyset$ or p_n converges.

Proposition 29. *This game is perfect-information equivalent to the analogous game studied by Bell, requiring $\mathscr{P}$'s play p_{n+1} to be in $\mathcal{H}_n[p_n]$ in rounds $n+1$, and requiring $\mathscr{O}$ choose refinements.*

Proof. Allowing $\mathscr{P}$ to play $p_{n+1} \notin \mathcal{H}_n[p_n] \Rightarrow \mathcal{H}_n[p_{n+1}] \neq \mathcal{H}_n[p_n]$ does not introduce any new winning plays for $\mathscr{P}$ as for any such move, $\bigcap_{m<\omega} \mathcal{H}_n[p_n] \subseteq \mathcal{H}_{n+1}[p_{n+1}] \cap \mathcal{H}_n[p_n] \subseteq \mathcal{H}_n[p_{n+1}] \cap \mathcal{H}_n[p_n] = \emptyset$.

Allowing $\mathscr{R}$ to play non-refining clopen partitions does not introduce any new winning plays for $\mathscr{R}$ as the winning condition relies on the refinement of all $\mathcal{R}_n$ anyway. $\square$

Definition 30. A space X is **proximal** iff X is zero-dimensional and $\mathscr{R} \uparrow Prox_{D,P}(X)$.

Definition 31. A space X is **Markӧv proximal** iff X is zero-dimensional and $\mathscr{R} \uparrow_{\text{mark}} Prox_{D,P}(X)$.

Definition 32. For any space X and a point $x \in X$, the W**-convergence-game** $Con_{O,P}(X,x)$ proceeds as follows: in round n, $\mathscr{O}$ chooses a neighborhood U_n of x, followed by $\mathscr{P}$ choosing a point $p_n \in X$.

For open sets $U_0, \dots, U_n$, let $V_n = \bigcap_{m \leq n} U_m$. Player $\mathscr{O}$ wins if either $p_n \notin V_n$ for some $n < \omega$, or if p_n converges.

Definition 33. A space X is a W**-space** iff $\mathscr{O} \uparrow Con_{O,P}(X,x)$ for all $x \in X$.

Definition 34. For each finite tuple $(m_0, \dots, m_{n-1})$, we define the k**-tactical fog-of-war**

$$T_k(m_0, \dots, m_{n-1}) = (m_{n-k}, \dots, m_{n-1})$$

and the k-**Markӧv fog-of-war**

$$M_k(m_0, \dots, m_{n-1}) = (m_{n-k}, \dots, m_{n-1}, n)$$

So $P \uparrow_{k\text{-tact}} G$ if and only if there exists a winning strategy for P of the form $\sigma \circ T_k$, and $P \uparrow_{k\text{-mark}} G$ if and only if there exists a winning strategy of the form $\sigma \circ M_k$.

Theorem 35. *For all $x \in X$:*

- $\mathscr{R} \uparrow Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow Con_{O,P}(X, x)$
- $\mathscr{R} \uparrow_{pre} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{pre} Con_{O,P}(X, x)$
- $\mathscr{R} \uparrow_{2k\text{-}tact} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(X, x)$
- $\mathscr{R} \uparrow_{2k\text{-}mark} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(X, x)$

Proof. Let σ witness $\mathscr{R} \uparrow_{2k\text{-tact}} Prox_{D,P}(X)$ (resp. $\mathscr{R} \uparrow_{2k\text{-mark}} Prox_{D,P}(X)$, $\mathscr{R} \uparrow Prox_{D,P}(X)$). We define the k-tactical (resp. k-Markӧv, perfect info) strategy τ such that

$$\tau \circ L_k(p_0, \dots, p_{n-1}) = \sigma \circ L_{2k}(x, p_0, \dots, x, p_{n-1})[x] \cap \sigma \circ L_{2k}(x, p_0, \dots, x, p_{n-1}, x)[x]$$

where L_{2k} is the $2k$-tactical fog-of-war (resp. $2k$-Markӧv fog-of-war, identity) and L_k is the k-tactical fog-of-war (resp. k-Markӧv fog-of-war, identity).

Let $p_0, p_1, \dots$ attack τ such that $p_n \in V_n = \bigcap_{m \leq n} \tau \circ L_k(p_0, \dots, p_{m-1})$ for all $n < \omega$. Consider the attack $q_0, q_1, \dots$ against the winning strategy σ such that $q_{2n} = x$ and $q_{2n+1} = p_n$.

Certainly, $x \in \mathcal{H}_{2n}[x] = \mathcal{H}_{2n}[q_{2n}]$ for any $n < \omega$. Note also for any $n < \omega$ that

$$p_n \in V_n = \bigcap_{m \leq n} \tau \circ L_k(p_0, \dots, p_{m-1})$$

$$= \bigcap_{m \leq n} (\sigma \circ L_{2k}(x, p_0, \dots, x, p_{m-1})[x] \cap \sigma \circ L_{2k}(x, p_0, \dots, x, p_{m-1}, x)[x])$$

$$= \bigcap_{m \leq n} (\sigma \circ L_{2k}(q_0, q_1, \dots, q_{2m-2}, q_{2m-1})[x] \cap \sigma \circ L_{2k}(q_0, q_1, \dots, q_{2m-2}, q_{2m-1}, q_{2m})[x])$$

$$\bigcap_{m \leq n} \mathcal{R}_{2m}[x] \cap R_{2m+1}[x] = \mathcal{H}_{2n+1}[x]$$

so $x \in \mathcal{H}_{2n+1}[p_n] = \mathcal{H}_{2n+1}[q_{2n+1}]$. Thus $x \in \bigcap_{n < \omega} \mathcal{H}_n[q_n]$, and since σ is a winning strategy, the attack $q_0, q_1, \dots$ converges, and must converge to x. Thus $p_0, p_1, \dots$ converges to x, and τ is also a winning strategy. $\square$

Corollary 36. *For all $x \in X$:*

- $\mathscr{R} \uparrow_{k\text{-}tact} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(X,x)$
- $\mathscr{R} \uparrow_{k\text{-}mark} Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(X,x)$

Corollary 37. *All proximal spaces are W-spaces.*

Definition 38. In the one-point compactification $\kappa^* = \kappa \cup \{\infty\}$ of discrete κ, define the clopen partition $\mathcal{C}(F) = [F]^1 \cup \{\kappa^* \setminus F\}$.

Theorem 39. $\mathscr{R} \uparrow_{code} Prox_{D,P}(\kappa^*)$

Proof. Use the coding strategy $\sigma() = \mathcal{C}(\emptyset) = \{\kappa^*\}$, $\sigma(\mathcal{C}(F), \alpha) = \mathcal{C}(F \cup \{\alpha\})$ for $\alpha < \kappa$ and $\sigma(\mathcal{C}(F), \infty) = \mathcal{C}(F)$. Note $\mathcal{R}_n = \mathcal{H}_n$. For any attack $p_0, p_1, \dots$ against σ such that $\bigcap_{n<\omega} \mathcal{H}_n[p_n] \neq \emptyset$, suppose

- $\infty \in \bigcap_{n<\omega} \mathcal{H}_n[p_n]$. Then $p_n \in \kappa^* \setminus \{p_m : m < n\}$ shows that the non-∞ p_n are all distinct. If co-finite $p_n = \infty$, we have $p_n \to \infty$. Otherwise, there are infinite distinct p_n, and since neighorhoods of ∞ are co-finite, we have $p_n \to \infty$.
- $\infty \not\in \mathcal{H}_N[p_N]$ for some $N < \omega$, so $\alpha \in \bigcap_{n<\omega} \mathcal{H}_n[p_n]$ for some $\alpha < \kappa$. Then $\mathcal{H}_n[p_n] = \{\alpha\}$ for all $n \geq N$, and thus $p_n \to \alpha$.

Thus σ is a winning coding strategy. □

Theorem 40. $\mathscr{O} \uparrow Con_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow Prox_{D,P}(\kappa^*)$

$\mathscr{O} \uparrow_{pre} Con_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{pre} Prox_{D,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{k\text{-}tact} Prox_{D,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{k\text{-}mark} Prox_{D,P}(\kappa^*)$

Proof. Let $\sigma \circ L$ be a winning strategy where L is the identify (resp. a k-tactical fog-of-war, a k-Marköv fog-of-war).

Define $\tau \circ L$ such that

$$\tau \circ L(p_0, \dots, p_{n-1}) = \mathcal{R}(\kappa^* \setminus (\sigma \circ L(p_0, \dots, p_{n-1})))$$

For any attack $p_0, p_1, \dots$ against τ such that $\bigcap_{n<\omega} \mathcal{H}_n[p_n] \neq \emptyset$, suppose

- $\mathcal{H}_n[p_n] = \mathcal{H}_n[\infty] = \bigcap_{m \leq n} \sigma \circ L(p_0, \dots, p_{m-1}) = \bigcap_{m \leq n} U_m = V_n$ for all $n < \omega$. Since σ is a winning strategy, the p_n converge at ∞.

- $\mathcal{H}_N[p_N] \neq \mathcal{H}_N[\infty]$ for some $N < \omega$. Then $\mathcal{H}_N[p_N] = \{p_N\}$, and since $\bigcap_{n<\omega} \mathcal{H}_n[p_n] \neq \emptyset$, we have $\mathcal{H}_n[p_n] = \mathcal{H}_N[p_N] = \{p_N\} \Rightarrow p_n = p_N$ for all $n \geq N$, and the p_n converge at p_N.

$\square$

Corollary 41. $\mathscr{O} \uparrow Con_{O,P}(\kappa^*, \infty) \Leftrightarrow \mathscr{R} \uparrow Prox_{D,P}(\kappa^*)$

$\mathscr{O} \uparrow_{pre} Con_{O,P}(\kappa^*, \infty) \Leftrightarrow \mathscr{R} \uparrow_{pre} Prox_{D,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(\kappa^*, \infty) \Leftrightarrow \mathscr{R} \uparrow_{k\text{-}tact} Prox_{D,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(\kappa^*, \infty) \Leftrightarrow \mathscr{R} \uparrow_{k\text{-}mark} Prox_{D,P}(\kappa^*)$

Corollary 42. $O \uparrow_{pre} Prox_{D,P}(\omega^*)$.

$O \uparrow_{tact} Prox_{D,P}(\omega^*)$.

$O \not\uparrow_{k\text{-}mark} Prox_{D,P}(\kappa^*)$ *for* $\kappa \geq \omega_1$.

Proof. Results hold for $\mathscr{O}$ and $Con_{O,P}(\kappa^*, \infty)$. $\square$

Definition 43. The **almost-proximal game** $aProx_{D,P}(X)$ is analogous to $Prox_{D,P}(X)$ except that the points p_n need only cluster for $\mathscr{R}$ to win the game.

Definition 44. The W**-clustering game** $Clus_{O,P}(X, x)$ is analogous to $Con_{O,P}(X, x)$ except that the points p_n need only cluster at x for $\mathscr{O}$ to win the game.

Proposition 45. $\mathscr{O} \uparrow Clus_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow aProx_{D,P}(\kappa^*)$

$\mathscr{O} \uparrow_{pre} Clus_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{pre} aProx_{D,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}tact} Clus_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{k\text{-}tact} aProx_{D,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}mark} Clus_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{k\text{-}mark} aProx_{D,P}(\kappa^*)$

Proof. Same proof as before, replacing “converge” with “cluster”. $\square$

Corollary 46. $\mathscr{R} \uparrow_{mark} aProx_{D,P}(\omega_1^*)$.

Proof. Holds for $\mathscr{O}$ and $Clus_{O,P}(\omega_1^*, \infty)$. $\square$

Proposition 47. *If* $\sigma \circ L$ *is a winning strategy for* $\mathscr{R}$ *in* $Prox_{D,P}(X)$ *(resp.* $aProx_{D,P}(X)$*) where* L *is the identity (or a* k*-tactical fog-of-war or a* k*-Marköv fog-of-war), and* C *is a closed subspace of* X*, then*

$$\tau \circ L(p_0, \ldots, p_{n-1}) = C \cap \sigma \circ L(p_0, \ldots, p_{n-1})$$

defines a winning strategy $\tau \circ L$ *for* $\mathscr{R}$ *in* $Prox_{D,P}(X)$ *(resp.* $aProx_{D,P}(X)$*).*

Proof. For any attack $p_0, p_1, \ldots$ against $\tau \circ L$ in $Prox_{D,P}(C)$ (resp. $aProx_{D,P}(C)$), note $p_0, p_1, \ldots$ is also an attack against $\sigma \circ L$ in $Prox_{D,P}(X)$ (resp. $aProx_{D,P}(X)$).

If $\mathscr{R}$ wins in $Prox_{D,P}(X)$ (resp. $aProx_{D,P}(X)$) by $\mathcal{H}_n^\sigma[p_n] = \emptyset$, then note that $\mathcal{H}_n^\tau[p_n] \subseteq \mathcal{H}_n^\sigma[p_n] = \emptyset$.

If If $\mathscr{R}$ wins in $Prox_{D,P}(X)$ (resp. $aProx_{D,P}(X)$) because the p_n converge (resp. cluster), then they converge (resp. cluster) in the closed set C.

Either way, $\tau \circ L$ defeats the arbitrary attack and is thus a winning strategy. □

Proposition 48. *If for any $i < m < \omega$, $\sigma_i \circ L$ is a winning strategy for $\mathscr{R}$ in $Prox_{D,P}(X_i)$ (resp. $aProx_{D,P}(X_i)$) where L is the identity (or a k-tactical fog-of-war or a k-Marköv fog-of-war), then*

$$\tau \circ L(p_0, \ldots, p_{n-1}) = \bigotimes_{i<m} \sigma_i \circ L(p_0(i), \ldots, p_{n-1}(i))$$

defines a winning strategy $\tau \circ L$ for $\mathscr{R}$ in $Prox_{D,P}(\prod_{i<m} X_i)$ (resp. $aProx_{D,P}(\prod_{i<m} X_i)$).

Proof. For any attack $p_0, p_1, \ldots$ against $\tau \circ L$ in $Prox_{D,P}(\prod_{i<m} X_i)$ (resp. $aProx_{D,P}(\prod_{i<m} X_i)$), note that for any $i < m$, $p_0(i), p_1(i), \ldots$ is an attack against $\sigma_i \circ L$ in $Prox_{D,P}(X_i)$ (resp. $aProx_{D,P}(X)$).

If for some $i < m$, $\mathscr{R}$ defeats the attack $p_0(i), p_1(i), \ldots$ because $\bigcap_{n<\omega} \mathcal{H}_n^i[p_n(i)] = \emptyset$, then we see immediately that $\bigcap_{n<\omega} \mathcal{H}_n[p_n] = \emptyset$ and τ defeats the attack $p_0, p_1, \ldots$.

Otherwise for all $i < m$, we have $p_n(i)$ converging (resp. clustering) at some $x_i \in X$. It follows then that $p_0, p_1, \ldots$ converges (resp. clusters) at $x = \langle x_i : i < m \rangle$ and τ defeats the attack $p_0, p_1, \ldots$. □

Definition 49. For $H \subseteq X$, the W**-subset-convergence-game** $Con_{O,P}(X, H)$ is analogous to $Con_{O,P}(X, x)$: $\mathscr{O}$ chooses open neighborhoods of H and tries to force $p_n \to H$.

Theorem 50. *For all compact $H \subseteq X$, $\mathscr{R} \uparrow Prox_{D,P}(X)$ implies $\mathscr{O} \uparrow Con_{O,P}(X, H)$.*

Proof. Adapted from G's proof.

Let σ witness $\mathscr{R} \uparrow Prox_{D,P}(X)$, assuming $\sigma(p)$ refines $\sigma(q)$ whenever $q \subseteq p$.

For certain finite sequences of points $p \in X^{<\omega}$, we define a tree of finite sequences $\langle T(p), \subseteq \rangle$ as follows:

- $T(\emptyset)$ contains the empty sequence, and for each of the finite nonempty

  $$V \in \{U \cap H : U \in \sigma(\emptyset)\}$$

  choose a unique $h_V \in V$ and include $\langle h_V \rangle$ in $T(\emptyset)$.

- Assume that whenever $T(p)$ is defined, it satisfies the following:
  - $T(p)$ is finite
  - $p' \subseteq p \Rightarrow T(p') \subseteq T(p)$
  - If $\langle h_0, q_0, \dots, h_n \rangle \in T(p)$ then $\langle q_0, \dots, q_{n-1} \rangle$ is a subsequence of p and $q_i \in \sigma(h_0, q_0, \dots, h_{i-1}, q_{i-1})[h_i]$ for all $i < n$
  - For each sequence $t^\frown \langle h, q \rangle \in T(p)$ and for each of the finite nonempty

    $$V \in \{U \cap H \cap \sigma(t)[h] : U \in \sigma(t^\frown \langle h, q \rangle)\}$$

    there is a unique $h_V \in V$ such that $t^\frown \langle h, q, h_V \rangle \in T(p)$.
  - $\{\sigma(t)[h] : t^\frown \langle h \rangle \text{ is maximal in } T(p)\}$ partitions $st\left(\bigwedge_{s \in T(p)} \sigma(s), H\right)$.

- Then when $T(p)$ is defined, we define $T(p^\frown \langle q \rangle)$ for each $q \in st\left(\bigwedge_{s \in T(p)} \sigma(s), H\right)$ as follows:
  - Assume $T(p) \subseteq T(p^\frown \langle q \rangle)$.
  - Find the maximal $t_q^\frown \langle h_q \rangle$ in $T(p)$ such that $q \in \sigma(t_q)[h_q]$. Include $t_q^\frown \langle h_q, q \rangle$ in $T(p^\frown \langle q \rangle)$.
  - For each of the finite nonempty

    $$V \in \mathcal{V}(t_q, h_q, q) = \{U \cap H \cap \sigma(t_q^\frown \langle h_q, q \rangle)[h] : U \in \sigma(t_q^\frown \langle h_q, q \rangle)\}$$

    choose a unique $h_V \in V$ and include $t_q^\frown \langle h_q, q, h_V \rangle$ in $T(p^\frown \langle q \rangle)$.
  - Note that

    $$\{\sigma(t)[h] : t^\frown \langle h \rangle \text{ is maximal in } T(p), h \neq h_q\}$$

    partitions

    $$st\left(\bigwedge_{s \in T(p)} \sigma(s), H\right) \setminus \sigma(t_q)[h_q] = st\left(\bigwedge_{s \in T(p^\frown \langle q \rangle)} \sigma(s), H\right) \setminus \sigma(t_q)[h_q]$$

    and that

    $$\{\sigma(t_q^\frown \langle h_q, q \rangle)[h_V] : \mathcal{V} \in V(t_q, h_q, q)\}$$

    partitions

    $$st\left(\bigwedge_{V \in \mathcal{V}(t_q, h_q, q)} \sigma(t_q^\frown \langle h_q, q, h_V \rangle), H\right) \cap \sigma(t_q)[h_q] = st\left(\bigwedge_{s \in T(p^\frown \langle q \rangle)} \sigma(s), H\right) \cap \sigma(t_q)[h_q]$$

    so our definition satisfies the recursion hypotheses.

We may define a strategy τ for $\mathscr{O}$ in $Con_{O,P}(X, H,)$ as follows. Let $\tau(\emptyset) = st\left(\bigwedge_{s\in T(\emptyset)} \sigma(s), H\right)$. If $T(p)$ is defined and $q \in st\left(\bigwedge_{s\in T(p)} \sigma(s), H\right)$, then let $\tau(p^\frown\langle q\rangle) = st\left(\bigwedge_{s\in T(p^\frown\langle q\rangle)} \sigma(s), H\right)$ (and $\tau(p^\frown\langle q\rangle) = X$ otherwise).

Let $p \in X^\omega$ attack τ such that $p(n) \in \tau(p \upharpoonright n)$ always. It follows that $T(p \upharpoonright n)$ is defined for all $n < \omega$, so let $T_p = \bigcup_{n<\omega} T(p \upharpoonright n)$. By definition, it is evident that T_p is an infinite tree with finite levels, so choose an infinite branch $p' = \langle h_0, q_0, \ldots\rangle$.

Since p' is an attack on σ, and $p'(n+1) \in \sigma(p \upharpoonright n+1)[p(n)]$ always, it follows that p' converges. Since $p(2n) = h_n \in H$, p' converges in H, and so does its subsequence $p'' = \langle q_0, q_1, \ldots\rangle$, which is also a subsequence of p.

We've shown p clusters in H, and since $\tau(p \upharpoonright n+1) \subseteq \tau(p)$, it follows analogously to a result of G that p converges in H. $\square$

Corollary 51. *If X is compact and $\mathscr{R} \uparrow Prox_{D,P}(X)$, then $\mathscr{O} \uparrow Con_{O,P}(X^2, \Delta)$, and thus X is Corson compact.*

Proof. Note $\mathscr{R} \uparrow Prox_{D,P}(X^2)$ and Δ is a compact subset of X^2, so $\mathscr{O} \uparrow Con_{O,P}(X^2, \Delta)$. By a result of G, X is Corson compact. $\square$