Limited information strategies for topological games

PhD Defense

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A *topological game* is a two-player game $G(X)$ of length $\omega = \{0, 1, 2, \ldots\}$ defined for certain topological spaces X.

During each round n, the first and second player take turns choosing certain topological objects from X (e.g. point, open set, open cover, etc.).

At the "end" of the game, a winner is declared by inspecting the sequences of choices made throughout the game.

The study of such games involves finding when a player has a *winning strategy* which defeats every possible counterattack by the opponent.

See Telgarsky's excellent survey on topological games for more details: [11]

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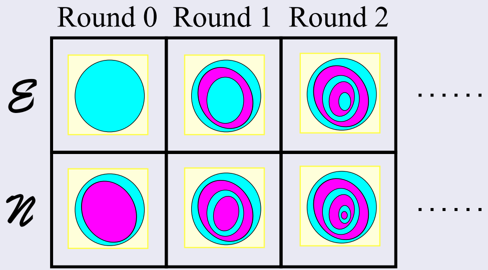
Game

The *Banach-Mazur Game* $BM_{E,N}(X)$ (1935) [5]



The first player $\mathscr{E}$ wins the game if the intersection of all the chosen open sets is empty.

Theorem

X is Baire if and only if $\mathscr{E}$ lacks a winning strategy in the Banach Mazur game ($\mathscr{E} \not\uparrow BM_{E,N}(X)$).

Thus the topological property of being a Baire space has a game-theoretic characterization using $BM_{E,N}(X)$.

By considering *limited information strategies*, we may characterize more properties.

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By considering *limited information strategies*, we may characterize more properties.

Consider the following:

Theorem

X is α-favorable $\Rightarrow$ X is Choquet $\Rightarrow$ X is Baire

α-favorability is characterized by $\mathscr{N} \underset{\text{tact}}{\uparrow} BM_{E,N}(X)$: player $\mathscr{E}$ has a *tactical* winning strategy which only considers the most recent move of the opponent.

This is stronger than the Choquet property [2], characterized by $\mathscr{N} \uparrow BM_{E,N}(X)$. In this case $\mathscr{N}$ still has a winning strategy, but it may rely on perfect information of the history of the game.

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By characterizing topological properties using the theory of topological games, we introduce new proof techniques for demonstrating the structure of given topological spaces.

In my dissertation I investigate four topological games from the literature to find new limited information characterizations.

In doing so I uncovered several new results in general topology, advancing research done by G. Gruenhage, P. Nyikos, R. Telgárksy, J. Bell, M. Scheepers, and others.

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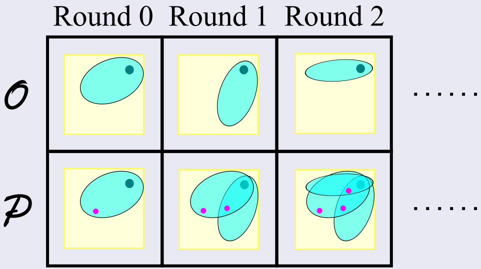
Game

Gruenhage's convergence game $Gru_{\vec{O},P}(X,x)$ and clustering game $Gru_{\overset{\rightsquigarrow}{O},P}(X,x)$ proceed as follows:



$\mathscr{O}$ wins the game if the points chosen by $\mathscr{P}$ converge/cluster to the given point $x \in X$. Otherwise, $\mathscr{P}$ wins.

Note that $\mathscr{O}$ need not know anything about the history of the game to play each round.

If $\mathscr{O} \uparrow Gru_{\vec{O},P}(X, x)$, then x is called a W-point in X. Obviously, all points of first-countablity are W-points, but $\mathscr{O} \uparrow Gru_{\vec{O},P}(\kappa^*, \infty)$ also, where ∞ is the added point in the one-point compactification κ^* of uncountable discrete κ.

Points of first-countability may in fact be characterized by this game as well:

Theorem

x has a countable local base in X if and only if $\mathscr{O} \underset{pre}{\uparrow} Gru_{\vec{O},P}(X, x)$ ($\mathscr{O}$ has a winning predetermined strategy using only the round number).

If every point in a space X is a W-point, then X is a W-space.

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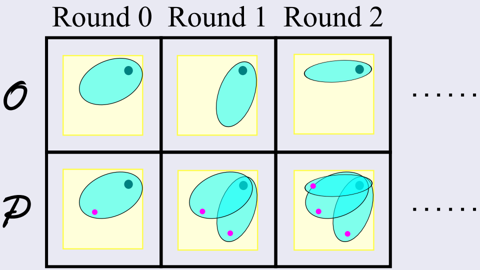
A variation of this game which is harder for $\mathscr{O}$ yields some difficult infinite combinatorial questions:

Game

Gruenhage's hard convergence game $Gru_{O,P}^{\rightarrow,\star}(X, x)$ and hard clustering game $Gru_{O,P}^{\rightsquigarrow,\star}(X, x)$ proceed as follows:



Nyikos observed in [6] that:

Theorem

$\mathscr{O} \underset{mark}{\not\uparrow} Gru^{\rightarrow,\star}_{O,P}(\omega_1^*, \infty)$.

($\mathscr{O}$ cannot guarantee a win using a Markov strategy which considers only the round number and most recent move.)

Some more work shows that in fact

Theorem

$\mathscr{O} \underset{k\text{-}mark}{\not\uparrow} Gru^{\rightarrow,\star}_{O,P}(\omega_1^*, \infty)$.

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Interestingly, the strategy which prevents convergence won't prevent clustering as well unless the cardinality of the space is sufficiently large.

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Proof that $\mathscr{O} \underset{\text{mark}}{\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$

For each $\alpha < \omega_1$, let $f_\alpha : \alpha \to \omega$ be injective.

Note that for each $F \in [\omega_1]^{<\omega}$, there is some $n_F < \omega$ such that $f_{\alpha+1}(\beta) < n_F$ for all $\beta \leq \alpha \in F$.

Let σ be a Markov strategy for $\mathscr{O}$ such that

$$\sigma(\langle \alpha \rangle, n) \subseteq \omega_1^* \setminus \{\beta < \omega_1 : f_{\alpha+1}(\beta) < n\}$$

for $\alpha < \omega_1$.

Thus as any legal counterattack cannot have finite range in ω_1 without repeating ∞ co-finitely, σ is a winning Markov strategy. $\square$

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Proof that $\mathscr{O} \underset{\text{tact}}{\not\uparrow} Gru^{\rightsquigarrow,\star}_{O,P}(\omega_1^*, \infty)$

Let σ be a tactic for $\mathscr{O}$ in $Gru^{\rightsquigarrow,\star}_{O,P}(\omega_1^*, \infty)$.

Then this set is closed and unbounded in ω_1:

$$C_\sigma = \{\alpha < \omega_1 : \beta < \alpha \Rightarrow \omega_1^* \setminus \sigma(\langle \beta \rangle) \subseteq \alpha\}$$

If $a_\sigma : \omega_1 \to C_\sigma$ is an order isomorphism, then there is $n < \omega$ such that $a_\sigma(n) \in \sigma(\langle a_\sigma(\omega) \rangle)$.

Then since $a_\sigma(\omega) > a_\sigma(n)$ implies $a_\sigma(\omega) \in \sigma(\langle a_\sigma(n) \rangle)$ (by the definition of C_σ), it follows that $\langle a_\sigma(n), a_\sigma(\omega), a_\sigma(n), a_\sigma(\omega), \dots \rangle$ is a legal counterattack against σ which does not cluster at ∞. □

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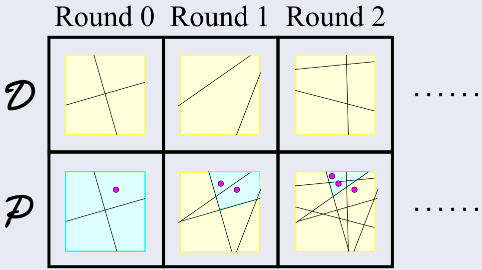
Game

Bell's proximal game $Bell_{\vec{D},P}(X)$ for compact zero-dimensional X:



$\mathscr{D}$ wins the game if the points chosen by $\mathscr{P}$ converge. Otherwise, $\mathscr{P}$ wins.

If $\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$, then X is called a proximal compact. This game was brought to my attention due to this result: [1]

Theorem

Every proximal space is a W-space. So $\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(X) \Rightarrow \mathscr{O} \uparrow Gru_{O,P}^{\rightarrow}(X, x)$ for all $x \in X$.

Proximal spaces have strong preservation properties, as any closed subset or Σ-product of proximal spaces is proximal. Since any proximal space is collectionwise normal, Bell's game gives an elegant proof of the classic result:

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Nyikos [7] observed that

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Corson compact spaces are proximal.

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With Gruenhage, I showed that the answer is yes:

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A winning limited information strategy may always be passed down to win in a closed subspace, but Bell's result that winning strategies are preserved for Σ-products does not quite generalize as well:

Theorem

For $k < \omega$, if $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell^{\rightarrow}_{D,P}(X_i)$ for all $i < \omega$, then $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell^{\rightarrow}_{D,P}\left(\prod_{i<\omega} X_i\right)$.

Other limited information lemmas proved in my dissertation allowed me to prove a game-theoretic characterization of another compactness property (paper in preparation):

Theorem

A compact space is strong Eberlein compact if and only if $\mathscr{D} \underset{tact}{\uparrow} Bell^{\rightarrow}_{D,P}(X)$.

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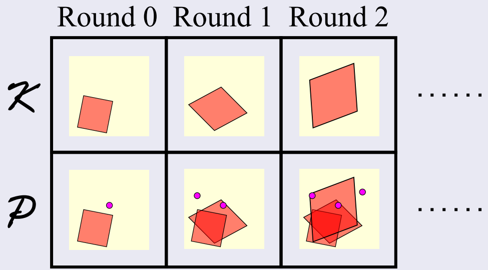

$\mathscr{K}$ wins the game if the points/sets chosen by $\mathscr{P}/\mathscr{L}$ are locally finite in the space. Otherwise, $\mathscr{P}/\mathscr{L}$ wins.

Gruenhage used these games in [3] to characterize metacompactness and σ-metacompactness amongst locally compact spaces:

Theorem

For locally compact spaces, $\mathscr{K} \underset{tact}{\uparrow} Gru_{K,P}(X)$ if and only if X is metacompact.

Theorem

For locally compact spaces, $\mathscr{K} \underset{mark}{\uparrow} Gru_{K,P}(X)$ if and only if X is σ-metacompact.

By removing knowledge of the round number, an analogous result is unsurfaced:

Theorem

For locally compact spaces, $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(X)$ if and only if X is σ-compact.

Actually, for locally compact or even compactly-generated spaces, $\mathscr{K} \underset{\text{pre}}{\uparrow} Gru_{K,P}(X)$ if and only if $\mathscr{K} \underset{\text{pre}}{\uparrow} Gru_{K,L}(X)$.

However, there is a non-compactly-generated counterexample:

Theorem

There exists a free ultrafilter $\mathcal{F}$ such that $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(\omega \cup \{\mathcal{F}\})$, but $\mathscr{K} \underset{pre}{\not\uparrow} Gru_{K,L}(\omega \cup \{\mathcal{F}\})$ for any free ultrafilter $\mathcal{F}$.

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For the related game $Gru_{\vec{O},P}(X,x)$, a $(k+1)$-tactic/mark may always be improved to only use the most recent move of the opponent. If this also holds for $Gru_{K,P}(X)$, then metacompactness and σ-metacompactness may would be characterized by the existence of any winning $(k+1)$-tactic/mark.

Due to a technical difference in the games, it's unclear if this is true. However, for a tricky non-σ-metacompact example $\boldsymbol{X}$:

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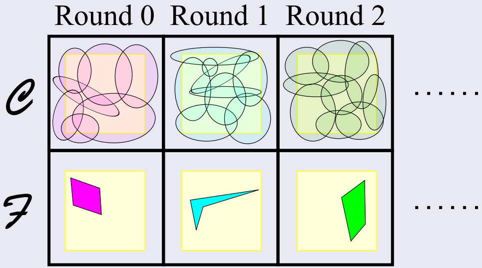
Menger's game $Men_{C,F}(X)$ proceeds as follows:



$\mathscr{F}$ wins the game if her finitely coverable subsets union to the space. Otherwise, $\mathscr{C}$ wins.

A covering property generalizing σ-compactness is characterized by this game, demonstrated by Hurewicz in the 1920's. [4]

Theorem

A space is Menger if and only if $\mathscr{C} \not\uparrow Men_{C,F}(X)$.

It was originally suspected that Menger subspaces of the real line were exactly the σ-compact subspaces, but as was shown by Telgarsky and Scheepers, σ-compact spaces have slightly more structure. [10] [9]

Theorem

A metrizable space X is σ*-compact if and only if* $\mathscr{F} \uparrow Men_{C,F}(X)$.

A covering property generalizing σ-compactness is characterized by this game, demonstrated by Hurewicz in the 1920's. [4]

Theorem

A space is Menger if and only if $\mathscr{C} \not\uparrow Men_{C,F}(X)$.

It was originally suspected that Menger subspaces of the real line were exactly the σ-compact subspaces, but as was shown by Telgarsky and Scheepers, σ-compact spaces have slightly more structure. [10] [9]

Theorem

A metrizable space X *is* σ*-compact if and only if* $\mathscr{F} \uparrow Men_{C,F}(X)$.

By considering Markov strategies, the previous theorem may be factored into two subresults.

Theorem

A regular space X is σ-compact if and only if $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$.

Theorem

For a second-countable space X, $\mathscr{F} \uparrow Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$.

Note that since the spaces we are considering are all Lindelöf, metrizability is characterized by regularity and second-countability.

Proof that σ-compact $\Leftrightarrow \mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$

Assume X is regular.

If $X = \bigcup_{n<\omega} K_n$, then $\mathscr{F}$ may choose K_n each round, which is a winning predetermined (and therefore Markov) strategy.

If X is not σ-compact, let σ be any Markov strategy for $\mathscr{F}$. Then for

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \sigma(\langle \mathcal{U} \rangle, n)$$

and any open cover $\mathcal{U}$ of the space X, $\sigma(\langle \mathcal{U} \rangle, n)$ is a finite subcover of R_n. Thus $\overline{R_n}$ is compact by the regularity of X.

Since X is not σ-compact, choose a point x not in any R_n. Then there is a sequence of open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$ for which $x \notin \sigma(\langle \mathcal{U}_n \rangle, n)$ for all n, and thus σ is not a winning Markov strategy.

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Proof that $\mathscr{F} \uparrow Men_{C,F}(X) \Leftrightarrow \mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$

Assume X is second-countable. Without loss of generality, assume $\mathscr{C}$ only chooses coverings using open sets from the countable base $\{U_n : n < \omega\}$.

If $\mathscr{F}$ has a winning Markov strategy, then she has a winning strategy.

Let σ be a winning strategy, and assume $\mathcal{U}_t$ is an open cover of basic open sets for each $t \leq s \in \omega^{<\omega}$. By exploiting the countable base, we may choose $\mathcal{U}_{s^\frown\langle n\rangle}$ for each $n < \omega$, such that for every open cover $\mathcal{U}$ there exists $n < \omega$ where $\sigma(\langle \mathcal{U}_{s\restriction 1}, \ldots, \mathcal{U}_s, \mathcal{U}\rangle) \subseteq \sigma(\langle \mathcal{U}_{s\restriction 1}, \ldots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle n\rangle}\rangle)$.

Then $\tau(\langle \mathcal{U}\rangle, n) = \sigma(\langle \mathcal{U}_{f(n)\restriction 1}, \ldots, \mathcal{U}_{f(n)}, \mathcal{U}\rangle$ where $f : \omega \to \omega^{<\omega}$ is a bijection is a winning strategy.

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Limited information strategies in topological games such as $Men_{C,F}(X)$ often have set-theoretic consequences. The statement $S(\kappa)$ due to M. Scheepers [8] says that there exist almost-compatible functions $f_A : A \to \omega$ for each $A \in [\kappa]^\omega$.

Theorem

$S(\omega_1)$ and $\neg S((2^\omega)^+)$ are theorems of ZFC, but $S(\kappa)$ is independent of ZFC for $\omega_1 < \kappa \leq 2^\omega$.

Theorem

If $S(\kappa)$ holds, then $\mathscr{F} \underset{2\text{-tact}}{\uparrow} Fill_{C,F}^{\cup,\subset}(\kappa)$.

Let $\kappa^\dagger$ be the one-point "Lindelöf-ication" of discrete κ.

Theorem

$$
\begin{tikzcd}
 & \mathscr{F} \underset{2\text{-mark}}{\uparrow} Cov_{C,F}(\kappa^\dagger) \arrow[r, leftrightarrow] \arrow[d] & \mathscr{F} \underset{2\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa) \arrow[d] \\
S(\kappa,\omega,\omega) \arrow[ur] \arrow[r] \arrow[drr] & \mathscr{F} \underset{2\text{-mark}}{\uparrow} Fill^{1,\subseteq}_{C,F}(\kappa) \arrow[r, leftrightarrow] & \mathscr{F} \underset{2\text{-mark}}{\uparrow} Fill^{\cup,\subseteq}_{C,F}(\kappa) \\
 & & \mathscr{F} \underset{2\text{-tact}}{\uparrow} Fill^{\cup,\subseteq}_{C,F}(\kappa) \arrow[u]
\end{tikzcd}
$$

For most of the games in the previous chart, including the Menger game $Men_{C,F}(X)$, there's no need to consider larger amounts of limited information.

Theorem

For each $k < \omega$, $\mathscr{F} \underset{k+2\text{-mark}}{\uparrow} Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men_{C,F}(X)$

The topological property $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men_{C,F}(X)$ seems to depend on the set-theoretic axioms at play.

Theorem

If $S(2^\omega)$, then $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men_{C,F}(R_\omega)$.

Any questions?

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