DUAL SELECTION GAMES

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Abstract. (an investigation of dual selection games)

1. Introduction

Definition 1. The *selection game* $G_1(\mathcal{A}, \mathcal{B})$ is an ω-length game involving Players I and II. During round n, I chooses $A_n \in \mathcal{A}$, followed by II choosing $B_n \in A_n$. Player II wins in the case that $\{B_n : n < \omega\} \in \mathcal{B}$, and Player I wins otherwise.

For brevity, let

$$G_1(\mathcal{A}, \neg\mathcal{B}) = G_1(\mathcal{A}, \mathcal{P}\left(\bigcup \mathcal{A}\right) \setminus \mathcal{B}).$$

That is, II wins in the case that $\{B_n : n < \omega\} \notin \mathcal{B}$, and I wins otherwise.

Definition 2. For a set X, let $\mathbf{C}(X)$ be the collection of all choice functions on X, functions $f : X \to \bigcup X$ such that $f(x) \in x$ for all $x \in X$.

Definition 3. The set $\mathcal{A}'$ is said to be a *reflection* of the set $\mathcal{A}$ if

$$\{\mathrm{range}(f) : f \in \mathbf{C}(\mathcal{A}')\} = \mathcal{A}.$$

For example, a reflection of the collection $\mathcal{O}_X$ of basic open covers of X would be $\mathcal{P}_X = \{\mathcal{T}_{X,x} : x \in X\}$, where $\mathcal{T}_{X,x}$ is the corresponding point-base at $x \in X$. Likewise for the collection $\Omega_{X,x}$ of sets with $x \in X$ as a limit point, $\mathcal{T}_{X,x}$ is itself a reflection.

Theorem 4. *Let $\mathcal{A}'$ be a reflection of $\mathcal{A}$.*
Then I $\underset{pre}{\uparrow}$ $G_1(\mathcal{A}, \mathcal{B})$ *if and only if* II $\underset{mark}{\uparrow}$ $G_1(\mathcal{A}', \neg\mathcal{B})$.

Proof. Let σ witness I $\underset{\text{pre}}{\uparrow}$ $G_1(\mathcal{A}, \mathcal{B})$. Since $\sigma(n) \in \mathcal{A}$, $\sigma(n) = \mathrm{range}(f_n)$ for some $f_n \in \mathbf{C}(\mathcal{A}')$. So let $\tau(A, n) = f_n(A)$ for all $A \in \mathcal{A}'$ and $n < \omega$. Suppose $A_n \in \mathcal{A}'$ for all $n < \omega$. Note that since σ is winning and $\tau(A_n, n) = f_n(A_n) \in \mathrm{range}(f_n) = \sigma(n)$, $\{\tau(A_n, n) : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses II $\underset{\text{mark}}{\uparrow}$ $G_1(\mathcal{A}', \neg\mathcal{B})$.

Now let σ witness II $\underset{\text{mark}}{\uparrow}$ $G_1(\mathcal{A}', \neg\mathcal{B})$. Let $f_n \in \mathbf{C}(\mathcal{A}')$ be defined by $f_n(A) = \sigma(A, n)$. Since $\mathcal{A} = \{\mathrm{range}(f) : f \in \mathbf{C}(\mathcal{A}')\}$, let $\tau(n) = \mathrm{range}(f_n)$. Suppose that $B_n \in \tau(n) = \mathrm{range}(f_n)$ for all $n < \omega$. Choose $A_n \in \mathcal{A}'$ such that $B_n = f_n(A_n) = \sigma(A_n, n)$. Since σ is winning, $\{B_n : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses I $\underset{\text{pre}}{\uparrow}$ $G_1(\mathcal{A}, \mathcal{B})$. ∎

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References

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