

Applications of almost compatible functions for limited information strategies in infinite length games

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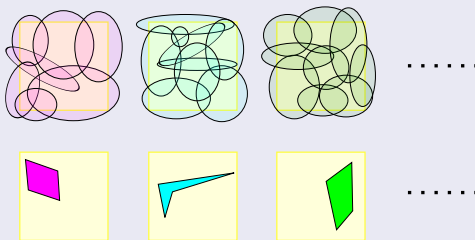
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Definition

A topological space X is Menger if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ of open covers of X there exists a sequence $\langle F_0, F_1, \dots \rangle$ such that F_n is covered by some finite subcollection of \mathcal{U}_n and $X = \bigcup_{n < \omega} F_n$.

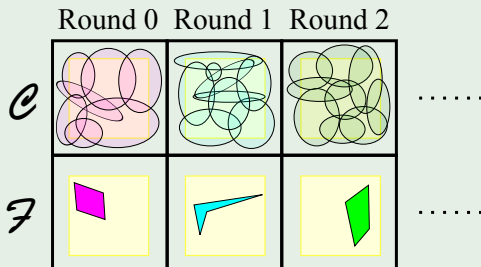


Proposition

X is σ -relatively-compact $\Rightarrow X$ is Menger $\Rightarrow X$ is Lindelöf.

Game

Let $Men_{C,F}(X)$ denote the *Menger game* with players \mathcal{C} , \mathcal{F} .



\mathcal{F} wins the game if $X = \bigcup_{n < \omega} F_n$, and \mathcal{C} wins otherwise.

Theorem (Hurewicz 1926 [1])

X is Menger if and only if $\mathcal{C} \nVdash \text{Men}_{C,F}(X)$.

Theorem (Telgarsky 1984 [5], Scheepers 1995 [4])

Let X be metrizable. $\mathcal{F} \uparrow \text{Men}_{C,F}(X)$ if and only if X is σ -compact.

Theorem (Fremlin, Miller 1988 [2])

There are ZFC examples of non- σ -compact subsets of the real line which are Menger.

Assume κ is an uncountable cardinal.

Example

Let $\kappa^\dagger = \kappa \cup \{\infty\}$, with κ discrete and neighborhoods of ∞ being co-countable. Then $\mathcal{F} \uparrow \text{Men}_{C,F}(\kappa^\dagger)$ but κ^\dagger is not σ -compact.

Definition

A *perfect information strategy* uses full information of the previous moves of the opponent. ($\mathcal{A} \uparrow G$)

Definition

A *k-tactical strategy* only uses the last k previous moves of the opponent. ($\mathcal{A} \underset{k\text{-tact}}{\uparrow} G$)

Definition

A *k-Markov strategy* only uses the last k previous moves of the opponent and the round number. ($\mathcal{A} \underset{k\text{-mark}}{\uparrow} G$)

If omitted, assume $k = 1$.

Considering such strategies allows us to factor out Scheepers's proof characterizing σ -compact metrizable spaces with the Menger game.

Lemma

$\mathcal{F} \uparrow_{\text{mark}} \text{Men}_{C,F}(X)$ if and only if X is σ -relatively-compact.

Lemma

Let X be second-countable. $\mathcal{F} \uparrow_{\text{mark}} \text{Men}_{C,F}(X)$ if and only if $\mathcal{F} \uparrow_{\text{mark}} \text{Men}_{C,F}(X)$

Since metrizable + Lindelöf \Leftrightarrow regular + second countable, we again have Telgarsky/Scheepers's result for metrizable spaces.

Example

$$\mathcal{F} \uparrow Men_{C,F}(\kappa^\dagger), \text{ but } \mathcal{F} \not\uparrow_{\text{mark}} Men_{C,F}(\kappa^\dagger).$$

Proposition

$$\mathcal{F} \uparrow_{(k+2)\text{-mark}} Men_{C,F}(X) \text{ if and only if } \mathcal{F} \uparrow_{2\text{-mark}} Men_{C,F}(X).$$

Example

$$\mathcal{F} \uparrow_{2\text{-mark}} Men_{C,F}(\omega_1^\dagger)$$

What about for $\kappa > \omega_1$? As we'll see, this question may not be answerable in *ZFC*.

The game $Men_{C,F}(\kappa^\dagger)$ essentially involves choosing countable and finite subsets of κ , such as in this game due to Scheepers [3]:

Game

Let $Sch_{C,F}^{\cup,\subset}(\kappa)$ denote Scheepers's *strict countable-finite game* in which each round \mathcal{C} chooses $C_n \in [\kappa]^{\leq \omega}$ such that $C_n \not\supseteq \bigcup_{i < n} C_i$, followed by \mathcal{F} choosing $F_n \in [C_n]^{< \omega}$. \mathcal{F} wins if $\bigcup_{n < \omega} F_n = \bigcup_{n < \omega} C_n$, and \mathcal{C} wins otherwise.

$Sch_{C,F}^{U,\subseteq}(\kappa)$ is more restrictive than the Menger game, but this is easily remedied.

Game

Let $Sch_{C,F}^{\cap}(\kappa)$ denote the *intersection countable-finite game* in which each round \mathcal{C} chooses $C_n \in [\kappa]^{\leq \omega}$, followed by \mathcal{F} choosing $F_n \in [C_n]^{< \omega}$.

\mathcal{F} wins if $\bigcup_{n < \omega} F_n \supseteq \bigcap_{n < \omega} C_n$, and \mathcal{C} wins otherwise.

Theorem

$\mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^{\cap}(\kappa)$ if and only if $\mathcal{F} \uparrow_{2\text{-mark}} Men_{C,F}(\kappa^{\dagger})$.

Perhaps this game is too dissimilar to the original. One may prefer to investigate either of these variants as well:

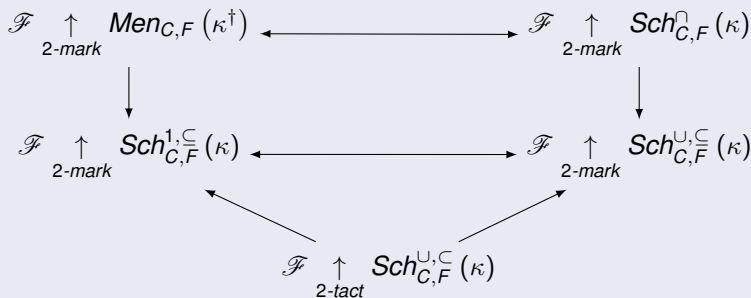
Game

Let $Sch_{C,F}^{U,\subseteq}(\kappa)$ denote the *nonstrict countable-finite game* in which each round \mathcal{C} chooses $C_n \in [\kappa]^{\leq \omega}$ such that $C_n \supseteq \bigcup_{i < n} C_i$, followed by \mathcal{F} choosing $F_n \in [C_n]^{< \omega}$. \mathcal{F} wins if $\bigcup_{n < \omega} F_n \supseteq \bigcup_{n < \omega} C_n$, and \mathcal{C} wins otherwise.

Game

Let $Sch_{C,F}^{1,\subseteq}(\kappa)$ denote the *initial countable-finite game* in which each round \mathcal{C} chooses $C_n \in [\kappa]^{\leq \omega}$ such that $C_n \supseteq \bigcup_{i < n} C_i$, followed by \mathcal{F} choosing $F_n \in [C_n]^{< \omega}$. \mathcal{F} wins if $\bigcup_{n < \omega} F_n \supseteq C_0$, and \mathcal{C} wins otherwise.

Theorem



Observe that there is no direct implication connecting

$$\mathcal{F} \uparrow_{2\text{-mark}} Men_{C,F}(\kappa^\dagger) \text{ and } \mathcal{F} \uparrow_{2\text{-tact}} Sch^{U,\subseteq}_{C,F}(\kappa).$$

The following was introduced by Scheepers to study k -tactics in his original countable-finite game.

Definition

For two functions f, g we say f is *almost compatible* with g ($f \parallel^* g$) if $|\{x \in \text{dom}(f) \cap \text{dom}(g) : f(x) \neq g(x)\}| < \omega$.

Definition

$S(\kappa)$ states that there exist functions $f_A : A \rightarrow \omega$ for each $A \in [\kappa]^{\leq \omega}$ such that $|\{\alpha \in A : f_A(\alpha) \leq n\}| < \omega$ for all $n < \omega$ and $f_A \parallel^* f_B$ for all $A, B \in [\kappa]^\omega$.

Proposition

$S(\omega_1)$.

Proposition

$\kappa > 2^\omega$ implies $\neg S(\kappa)$.

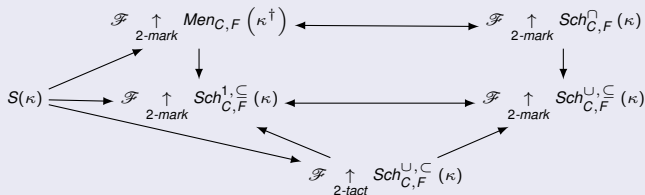
Theorem

$S(2^\omega)$ is a theorem of $ZFC + CH$ and consistent with $ZFC + \neg CH$.

Question

Let \mathfrak{s} be the smallest cardinal such that $S(\mathfrak{s})$. What is the relationship of \mathfrak{s} to other small cardinals such as \mathfrak{t} , \mathfrak{b} , \mathfrak{d} , etc.?

Theorem



Question

Should all the arrows be two-sided?

Definition

A subspace Y of X is *relatively robustly Menger* if there exist functions $r_{\mathcal{V}} : Y \rightarrow \omega$ for each open cover \mathcal{V} of X such that for all open covers \mathcal{U}, \mathcal{V} and numbers $n < \omega$, the following sets are finitely coverable by \mathcal{V} :

$$c(\mathcal{V}, n) = \{x \in Y : r_{\mathcal{V}}(x) \leq n\}$$

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{x \in Y : n < r_{\mathcal{U}}(x) < r_{\mathcal{V}}(x)\}$$

Definition

A space X is *robustly Menger* if it is relatively robustly Menger to itself.

Theorem

$\mathcal{F} \xrightarrow[\text{2-mark}]{\text{mark}} \text{Men}_{C,F}(X)$ implies X is robustly Menger implies $\mathcal{F} \xrightarrow[\text{2-mark}]{\text{mark}} \text{Men}_{C,F}(X)$.

Theorem

$S(\kappa)$ implies κ^\dagger is robustly Menger.

Question

Does $\mathcal{F} \xrightarrow[\text{2-mark}]{\text{mark}} \text{Men}_{C,F}(X)$ imply X is robustly Menger?

Example

Let $R_{\mathbb{Q}}$ be the real line with the basis generated by open intervals with or without the rationals removed.

Theorem

$R_{\mathbb{Q}}$ is second countable and $\mathcal{F} \uparrow Men_{C,F}(R_{\mathbb{Q}})$.

Corollary

$\mathcal{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(R_{\mathbb{Q}})$, even though it isn't σ -compact.

Example

Let R_ω be the real line with the basis generated by open intervals with any countable set removed.

Theorem

$\mathcal{F} \uparrow Men_{C,F}(R_\omega)$, but $\mathcal{F} \not\uparrow_{\text{mark}} Men_{C,F}(R_\omega)$.

Theorem

$S(2^\omega)$ implies R_ω is robustly Menger.

Question

Does there exist a space such that $\mathcal{F} \uparrow Men_{C,F}(X)$ but X is not robustly Menger or $\mathcal{F} \not\uparrow_{\text{2-mark}} Men_{C,F}(X)$?



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Rastislav Telgársky.

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Questions?