# Tactics and Marks in Banach Mazur Games

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## marks and tactics

My notes on Galvin/Telgarsky's Theorem 5 from [3].

**Definition 1.** Let $\mathbb{P}$ be partially ordered by $\leq$. Let $\mathbb{P}^{\downarrow} = \{f \in \mathbb{P}^\omega : f(n) \geq f(n+1)\}$. Then for $f, g \in \mathbb{P}^{\downarrow}$, we say that $f, g$ zip into each other if for all $m < \omega$ there exists $n < \omega$ such that $f(m) \geq g(n)$ and $g(m) \geq f(n)$.

**Definition 2.** $BM_{po}(\mathbb{P}, W)$ is a game defined for all non-empty partial orders $\mathbb{P}$ and all subsets $W \subseteq \mathbb{P}^{\downarrow}$. During round 0, I chooses $a_0 \in \mathbb{P}$, and then II chooses $b_0 \leq a_0$; during around $n+1$, I chooses $a_{n+1} \leq b_n$, and then II chooses $b_{n+1} \leq a_{n+1}$. II wins this game if $\langle a_0, a_1, \ldots \rangle \in W$.

**Theorem 3.** *Let $W \subseteq \mathbb{P}^{\downarrow}$ be closed under zipping.* II $\underset{\text{mark}}{\uparrow}$ $BM_{po}(\mathbb{P}, W)$ *if and only if* II $\underset{\text{tact}}{\uparrow}$ $BM_{po}(\mathbb{P}, W)$.

*Proof.* Let $\tau(p, n+1)$ be a winning mark for II, where $p$ is the most recent move by I and $n+1$ is the number of moves made by I. Define $\tau^0(p) = p$ and $\tau^{n+1}(p) = \tau(\tau^n(p), n+1)$. Let $\preceq$ well-order $\mathbb{P}$.

For $p, q \in \mathbb{P}$, say $p \geq_n q$ if there exist $s_m(p) \in \mathbb{P}$ for $m \leq n$ such that

$$p \geq s_m(p) \geq \tau(s_m(p), n+1) \geq q.$$

Note that $p' \geq p \geq_n q \geq q'$ implies $p' \geq_n q'$, and $p \geq_n \tau^n(p)$.

Say $p \geq_\omega q$ whenever $p \geq_n q$ for all $n < \omega$. If $p \geq_\omega l(p)$ for some $l(p)$, then say $p$ is long; otherwise call $p$ short.

For $p$ short, let

$$\mu(p) = \min_{\preceq}\{r \text{ short} : r \geq p\}$$

and since $\mu(p) \not\geq_n p$ for some $n$, let

$$N(p) = \min\{n < \omega : \mu(p) \not\geq_n p\}.$$

Note that whenever $\mu(p) = \mu(q)$ for $p \geq_n q$, it follows that $\mu(p) \geq_n q$ and therefore $N(p) < N(q)$.

We define

$$\sigma(p) = \begin{cases} l(p) & p \text{ is long} \\ \tau^{N(p)+1}(p) & p \text{ is short} \end{cases}.$$

Suppose $\sigma$ is legally attacked by $a \in \mathbb{P}^\omega$. For $n \leq \omega$, if $a(n)$ is long, then $a(n) \geq_n l(a(n))$. Therefore,

$$a(n) \geq s_n(a(n)) \geq \tau(s_n(a(n)), n+1) \geq l(a(n)) = \sigma(a(n)) \geq a(n+1).$$

Thus if $a(n)$ is long for $n < \omega$, it follows that $c \in \mathbb{P}^{\downarrow}$ defined by $c(n) = s_n(a(n))$ is a legal attack against $\tau$. Since $\tau$ is winning, $c \in W$, and since $c$ zips into $a$, $a \in W$ as well.

Otherwise, we may choose a final subsequence $b$ of $a$ such that

- $b(n)$ is short for all $n < \omega$, since $a(m)$ short implies $a(n+m)$ short for all $n < \omega$.
- $\mu(b(n)) = \mu'$ is fixed for all $n < \omega$, since there cannot be an infinite $\preceq$-decreasing sequence.

As a result,
$$b(n) \geq_{N(b(n))} \tau^{N(b(n))+1}(b(n)) = \sigma(b(n)) \geq b(n+1)$$
and therefore $N(b(n)) < N(b(n+1))$. In particular, $N(b(n)) \geq n$.

Thus for $n < \omega$,
$$b(n) \geq \tau^n(b(n)) \geq \tau(\tau^n(b(n)), n+1) \geq \tau^{N(b(n))+1}(b(n)) = \sigma(b(n)) \geq b(n+1).$$
As a result, $c \in \mathbb{P}^{\downarrow}$ defined by $c(n) = \tau^n(b(n))$ is a legal attack against the winning strategy $\tau$. Therefore $c \in W$, and since $c$ zips into $b$ and $a$, we conclude $a \in W$. $\square$

**Observation 4.** When $\mathbb{P} = T(X)\setminus\{\emptyset\}$ is ordered by set-inclusion and $W = \{U \in \mathbb{P}^{\downarrow} : \bigcap_{n<\omega} U(n) \neq \emptyset\}$, then $BM_{po}(\mathbb{P}, W)$ is exactly the topological Banach Mazur game $BM_{E,N}(X)$. Note $W$ is closed under zipping.

**Corollary 5.** *II $\underset{\text{mark}}{\uparrow}$ $BM_{E,N}(X)$ if and only if II $\underset{\text{tact}}{\uparrow}$ $BM_{E,N}(X)$.*

## 2+ marks and tactics

And this stuff is based on section 4.5.1 of [1].

**Definition 6.** Let $f \in S^{\leq\omega}$. Then $f \restriction n \in S^n$ is defined by $(f \restriction n)(i) = f(i)$. ($f \restriction n$ gives the first $n$ terms of $f$.)

Let $t \in S^{<\omega}$. Then $t \downharpoonleft k \in S^k$ is defined by $(t \downharpoonleft k)(i) = t(i + |t| - k)$. ($t \downharpoonleft k$ gives the last $n$ terms of $t$.)

**Definition 7.** For every partial order $\mathbb{P}$ and compatible $p, q \in \mathbb{P}$, let $p \wedge q$ satisfy $p \wedge q \leq p, q$.

**Claim 8.** *$\mathbb{P}$ contains no infinite antichains if and only if every antichain in $\mathbb{P}$ is of size $n$ or less for some $n < \omega$.*

*Proof.* MAYBE? Apparently true for $\mathbb{P} = \tau \setminus \{\emptyset\}$ due to Lemma 2.10 of [2]. $\square$

**Proposition 9.** *Let $W \subseteq \mathbb{P}^{\downarrow}$ be closed under zipping. Suppose every antichain in $\mathbb{P}$ is of size $n < \omega$ or less, and II $\uparrow BM_{po}(\mathbb{P}, W)$. Then II $\underset{\text{auto}}{\uparrow}$ $BM_{po}(\mathbb{P}, W)$ (i.e. II wins every play of $BM_{po}(\mathbb{P}, W)$, i.e. $W = \mathbb{P}^{\downarrow}$).*

*Proof.* First, let $\{p_i : i < n\}$ be an antichain of size $n < \omega$, then let $\mathbb{P}_i$ be a maximal pairwise-compatible subset of $\mathbb{P}$ containing $p_i$. Note that if there existed $q \in \mathbb{P} \setminus \bigcup_{i<n} \mathbb{P}_i$, $q$ must be incompatible with some $q_i \in \mathbb{P}_i$ for $i < n$. Since $p_i, q_i \in \mathbb{P}_i$, they are compatible, so let $r_i = p_i \wedge q_i$. Since $q$ is incompatible with $q_i$ for $i < n$, $q$ is incompatible with $r_i$ for $i < n$. Since $p_i$ is incompatible with $p_j$ for $i < j < n$, $r_i$ is incompatible with $r_j$ for $i < j < n$. But that makes $\{q\} \cup \{r_i : i < n\}$ an antichain of size $n+1$, contradicting the assumption of the proposition. Thus $\mathbb{P} = \bigcup_{i<n} \mathbb{P}_i$.

We now show that if $s \in \mathbb{P}_i^{\downarrow}$ for some $i$, then $s \in W$. Let $\sigma$ be a winning strategy for II in $BM_{po}(\mathbb{P}, W)$, and attack $\sigma$ with $q(0) = s(0) \wedge p_i$ and $q(n+1) = s(n+1) \wedge \sigma(\langle q(0), \ldots, q(n)\rangle)$. Note that the choice of $q(0)$ is valid as $s(0), p_i \in \mathbb{P}_i$. Similarly, $\sigma(\langle q(0), \ldots, q(n)\rangle) \leq q(0) \leq p_i$, so $\sigma(\langle q(0), \ldots, q(n)\rangle)$ cannot be compatible with any $p_j$ where $j \neq i$. Thus $s(n+1), \sigma(\langle q(0), \ldots, q(n)\rangle) \in \mathbb{P}_i$, making the choice of $q(n+1)$ valid. Since $\sigma$ is winning for II, we see that $q \in W$, and therefore $s \in W$.

Finally, consider any play of $BM_{po}(\mathbb{P}, W)$. It must contain have a subsequence $s \in \mathbb{P}_i^{\downarrow}$ for some $i < n$, so $s \in W$ and therefore the play is also in $W$, securing a victory for II. □

**Lemma 10.** *Let $W \subseteq \mathbb{P}^{\downarrow}$ be closed under zipping. Suppose that for every $p \in \mathbb{P}$, there exists an infinite antichain $A_p = \{a_p(n) : n < \omega\} \subseteq \{q \in \mathbb{P} : q \leq p\}$. Then* II $\underset{(k+2)-\text{mark}}{\uparrow}$ $BM_{po}(\mathbb{P}, W)$ *if and only if* II $\underset{(k+2)-\text{tact}}{\uparrow}$ $BM_{po}(\mathbb{P}, W)$.

*Proof.* The intuition of the following proof is simple: consider the case $k = 0$. During the first round, I plays some $p_0 \in \mathbb{P}$, and II can store the round number 0 (known by II since they only have knowledge of one move) by pretending I chose $a_{p_0}(0) \leq p_0$ instead, and applying the winning 2-mark. Thus when I plays $p_1 \leq a_{p_0}(0)$, II will have knowledge of both $p_0$ and $p_1$, and thus can observe that as $p_1 \leq a_{p_0}(0)$, it must be round 1 rather than some future round, and can repeat this process by pretending I chose $a_{p_1}(1) \leq p_1$ and $a_{p_0}(0) \leq p_0$ instead.

We now proceed with a formal proof. Let $\sigma$ witness II $\underset{(k+2)-\text{mark}}{\uparrow}$ $BM_{po}(\mathbb{P}, W)$. Define $\tau(t) = \sigma(\langle a_{t(0)}(0)\rangle, 1)$ for $t \in \mathbb{P}^1$. Since $\tau(t) = \sigma(\langle a_{t(0)}(0)\rangle, 1) \leq a_{t(0)}(0) \leq t(0)$, this is a legal move.

Consider $t \in \mathbb{P}^{j+2}$ for $j \leq k$. If there exists $l_t < \omega$ such that $t(j+1) \leq a_{t(j)}(l_t + j)$, define $t' \in \mathbb{P}^{j+2}$ by $t'(i) = a_{t(i)}(l_t + i)$ and let $\tau(t) = \sigma(t', l_t + |t|)$. Note that since

$$\tau(t) = \sigma(t', l_t + |t|) \leq t'(j+1) = a_{t(j+1)}(l_t + j + 1) \leq t(j+1)$$

this is a legal move. (If $l_t$ failed to exist, we could arbitrarily let, say, $\tau(t) = t(|t| - 1)$; as we will see, this case will never occur for any legal attack against $\tau$.)

Let $f$ be a legal attack against $\tau$. We may quickly verify that $l_{f \restriction 2} = 0$ since

$$\begin{aligned}
(f \restriction 2)(1) &= f(1) \\
&\leq \tau(f \restriction 1) \\
&= \sigma(\langle a_{f(0)}(0)\rangle, 1) \\
&\leq a_{f(0)}(0) \\
&= a_{(f \restriction 2)(0)}(0 + 0)
\end{aligned}$$

We claim in general that $l_{f \restriction (j+2)} = 0$ for $j \leq k$. Assuming $l_{f \restriction (j+2)} = 0$ for $j < k$,

$$\begin{aligned}
(f \restriction (j+3))(j+2) &= f(j+2) \\
&\leq \tau(f \restriction (j+2)) \\
&= \sigma(f \restriction (j+2)', 0 + (j+2)) \\
&\leq f \restriction (j+2)'(j+1) \\
&= a_{(f \restriction (j+2))(j+1)}(0 + (j+1)) \\
&= a_{(f \restriction (j+3))(j+1)}(0 + (j+1))
\end{aligned}$$

proving $l_{f \restriction (j+3)} = 0$.

Now we show that $l_{f\restriction(n+2)\downharpoonleft(k+2)} = j-k$ for $n \geq k$. We've just shown that this is true for our base case $n = k$ since in that case $f \restriction (n+2) \downharpoonleft (k+2) = f \restriction (k+2)$. Now assuming $l_{f\restriction(n+2)\downharpoonleft(k+2)} = n-k$ for some $n \geq k$, we observe

$$\begin{aligned}
(f \restriction (n+3) \downharpoonleft (k+2))(k+1) &= f(n+2) \\
&\leq \tau(f \restriction (n+2) \downharpoonleft (k+2)) \\
&= \sigma((f \restriction (n+2) \downharpoonleft (k+2))', (n-k) + (k+2)) \\
&\leq (f \restriction (n+2) \downharpoonleft (k+2))'(k+1) \\
&= a_{(f\restriction(n+2)\downharpoonleft(k+2))(k+1)}((n-k) + (k+1)) \\
&= a_{(f\restriction(n+3)\downharpoonleft(k+2))(k)}((n+1-k) + (k))
\end{aligned}$$

and conclude $l_{f\restriction(n+3)\downharpoonleft(k+2)} = n+1-k$.

Define $g \in \mathbb{P}^{\downarrow}$ by $g(0) = f(0)$ and $g(j+1) = a_{f(j+1)}(j+1)$. Reviewing the above, the reader may confirm that we have shown for $n < k+2$

$$f(n+1) \leq \tau(f \restriction (n+1)) = \sigma(g \restriction (n+1), n+1) \leq g(n) \leq f(n)$$

and for $n \geq k+2$

$$f(n+1) \leq \tau(f \restriction (n+1) \downharpoonleft (k+2)) = \sigma(g \restriction (n+1) \downharpoonleft (k+2), n+1) \leq g(n) \leq f(n)$$

Thus $g$ is a legal attack against $\sigma$, and since $\sigma$ is winning, $g \in W$. Since $W$ is closed under zipping, $f \in W$, and therefore $\tau$ is also winning. $\square$

## References


[1] Tomek Bartoszynski, Winfried Just, and Marion Scheepers. Covering games and the banach-mazur game: k-tactics. *Canad. J. Math*, 45:897–929, 1993.

[2] W. W. Comfort and S. Negrepontis. *Chain Conditions in Topology.* Cambridge Tracts in Mathematics. Cambridge University Press, 1982.

[3] Fred Galvin and Ratislav Telgársky. Stationary strategies in topological games. *Topology Appl.*, 22(1):51–69, 1986.