Notes on an example due to Baker

In [1], the author defines a game played on the unit interval.

Game 1. Let $G(X, J)$ denote a game with players $\mathscr{A}$ and $\mathscr{B}$.

In round 0, $\mathscr{A}$ chooses a number a_0 such that $0 \leq a_0 \leq 1$, followed by $\mathscr{B}$ choosing a number b_0 such that $a_0 < b_0 \leq 1$.

In round $n+1$, $\mathscr{A}$ chooses a number a_{n+1} such that $a_n < a_{n+1} < b_n$, followed by $\mathscr{B}$ choosing a number b_{n+1} such that $a_{n+1} < b_{n+1} < b_n$.

$\mathscr{A}$ wins the game if $\lim_{n\to\infty} a_n \in X$, and $\mathscr{B}$ wins otherwise.

The game is strongly related to one formulation of the Banach-Mazur game played upon the unit interval, which has been extensively studied [2].

Game 2. Let $M(X, J)$ denote the Banach-Mazur interval game with players $\mathscr{A}$ and $\mathscr{B}$.

In round 0, $\mathscr{A}$ chooses a closed interval $I_0 \subseteq J = [0, 1]$, followed by $\mathscr{B}$ choosing a closed interval $J_0 \subseteq I_0$.

In round $n+1$, $\mathscr{A}$ chooses a closed interval $I_{n+1} \subseteq J_n$, followed by $\mathscr{B}$ choosing a closed interval $J_{n+1} \subseteq I_{n+1}$.

The author of [1] asks if there is a set X such that $G(X, J)$ is indetermined: neither $\mathscr{A}$ nor $\mathscr{B}$ have a winning strategy.

We show that such a set would also make $MB(X, J)$ indeteremined.

Theorem 3. *$\mathscr{A} \uparrow MB(X, J) \Rightarrow \mathscr{A} \uparrow G(X, J)$ and $\mathscr{B} \uparrow MB(X, J) \Rightarrow \mathscr{B} \uparrow G(X, J)$. (Thus if $MB(X, J)$ is determined, then $G(X, J)$ is determined.)*

Proof. First let σ witness $\mathscr{A} \uparrow MB(X, J)$. We define the strategy τ for $\mathscr{A}$ in $G(X, J)$ like so:

$$a_0 = \tau(\emptyset) = \inf(\sigma(\emptyset))$$

$$a_{n+1} = \tau(\langle b_0, \ldots, b_n \rangle) = \inf\left(\sigma\left(\left\langle \left[\frac{2a_0 + b_0}{3}, \frac{a_0 + 2b_0}{3}\right], \ldots, \left[\frac{2a_n + b_n}{3}, \frac{a_n + 2b_n}{3}\right] \right\rangle\right)\right)$$

It is easily seen that $\bigcap_{n<\omega} [\frac{2a_n+b_n}{3}, \frac{a_n+2b_n}{3}] = \{x\}$ and $x \in X$ since σ is a winning strategy. Thus $\lim_{n\to\infty} a_n = x$ and $\mathscr{A} \uparrow G(X, J)$.

If σ now witnesses $\mathscr{B} \uparrow MB(X, J)$, then a similar argument shows that

$$b_n = \tau(\langle b_0, \ldots, b_n \rangle) = \inf\left(\sigma\left(\left\langle \left[\frac{2a_0 + b_0}{3}, \frac{a_0 + 2b_0}{3}\right], \ldots, \left[\frac{2a_n + b_n}{3}, \frac{a_n + 2b_n}{3}\right] \right\rangle\right)\right)$$

defines a winning strategy for $\mathscr{B}$ in $G(X, J)$. $\square$

Corollary 4. *If X is Baire, then $G(X, J)$ is determined.*

Proof. If X is a Baire subset of a Polish space, then $MB(X, J)$ is determined. $\square$

References

[1] Baker, M. H., *Uncountable sets and an innite real number game*. `http://arxiv.org/pdf/math/0606253.pdf` 2006.

[2] Telgarksy, R., *Topological games: On the 50th anniversary of the Banach-Mazur game*. `http://www.telgarsky.com/1987-RMJM-Telgarsky-Topological-Games.pdf` 1987.