Definition 1. Let $Men_{C,F}(X)$ be the Menger game played on X. Let $BD_{B,F}(X,x)$ be the countable fan tightness game introduced by Barman and Dow in [2]: $\mathscr{B}$ chooses a set B_n with $x \in \overline{B_n}$, followed by $\mathscr{F}$ choosing $F_n \in [B_n]^{<\omega}$, where $\mathscr{F}$ wins if $x \in \overline{\bigcup_{n<\omega} F_n}$.

Theorem 2. *X^k is Menger for all $k<\omega$ if and only if $C_p(X)$ has countable fan tightness.*

Proof. Arhangel'skiĭ in [1]. $\square$

Theorem 3. *If $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$, then $\mathscr{F} \underset{mark}{\uparrow} BD_{B,F}\left(C_p(X),\vec{0}\right)$.*

Proof. Essentially [3, 2.6], once one observes that $\mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$ characterizes σ-relative-compactness (equivalent to σ-compactness in regular spaces). Note further that this property is preserved for finite powers. $\square$

Theorem 4. *If $\mathscr{F} \uparrow Men_{C,F}\left(X^k\right)$ for all $k<\omega$, then $\mathscr{F} \uparrow BD_{B,F}\left(C_p(X),\vec{0}\right)$.*

Proof. For a sequence of blades $s^\frown\langle B\rangle \in (\mathcal{P}(X))^{\leq\omega}$ satisfying $\vec{0} \in \overline{s(i)}$ for all $i<|s|$, let $t_k^m(s^\frown\langle B\rangle)$ be the sequence defined by

$$t_k^m(s^\frown\langle B\rangle)(i) = \{(f^{-1}[(-2^{-i-m},2^{-i-m})])^k : f \in s^\frown\langle B\rangle(i)\}$$

for each $i \leq |s|$. In the case $i=|s|$, note

$$t_k^m(s^\frown\langle B\rangle)(|s|) = \{(f^{-1}[(-2^{-|s|-m},2^{-|s|-m})])^k : f \in B\}$$

We claim that $t_k(s^\frown\langle B\rangle)(i)$ is an open cover of X^k for each $i \leq |s|$. Indeed, for $\alpha \in X^k$, $U(\vec{0},\mathrm{ran}\alpha,2^{-i-m})$ is an open neighborhood of $\vec{0}$, and thus must hit some $f \in s^\frown\langle B\rangle(i)$. Note then that $\alpha \in (f^{-1}[(-2^{-i-m},2^{-i-m})])^k$.

For a sequence s of length $\leq\omega$, define its offset by n, $s \downharpoonleft n$, as follows. Let $s \downharpoonleft n = \emptyset$ when $n \geq |s|$ and otherwise satsify $n + |s \downharpoonleft n| = |s|$ and $s \downharpoonleft n(i) = s(n+i)$.

If σ_k is the winning strategy for $\mathscr{F}$ in $Men_{C,F}\left(X^k\right)$, then let τ^m be a strategy for $\mathscr{F}$ in $BD_{B,F}\left(C_p(X),\vec{0}\right)$ satisfying

$$\sigma_k(t_k^m(s^\frown\langle B\rangle)) \subseteq \{(f^{-1}[(-2^{-|s|-m},2^{-|s|-m})])^k : f \in \tau^m(s^\frown\langle B\rangle)\}$$

and τ be a winning strategy satisfying

$$\tau(s^\frown\langle B\rangle) = \bigcup_{m\leq|s|} \tau^m((s \downharpoonleft m)^\frown\langle B\rangle)$$

Attack τ with a. Let $\alpha \in X^k$ and $m < \omega$. Since $t_k^m(a \downharpoonright m)$ is an attack on σ in $Men_{C,F}\left(X^k\right)$, there is some $n < \omega$ where $\alpha \in \bigcup \sigma_k(t_k^m((a \downharpoonright m) \upharpoonright (n+1)))$, and therefore $\alpha \in (f^{-1}[(-2^{-n-m}, 2^{-n-m})])^k$ for some $f \in \tau^m((a \downharpoonright m) \upharpoonright (n+1))$, so $f \in U(\vec{0}, \mathrm{ran}\alpha, 2^{-n}) \cap \tau(a \upharpoonright (n+m+1))$. $\square$

References

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[3] Doyel Barman and Alan Dow. Proper forcing axiom and selective separability. *Topology Appl.*, 159(3):806–813, 2012.