Limited information strategies for topological games

PhD Defense

Steven Clontz http://stevenclontz.com

Department of Mathematics and Statistics
Auburn University

March 11, 2015

A *topological game* is a two-player game $G(X)$ of length $\omega = \{0, 1, 2, \dots\}$ defined for certain topological spaces X.

During each round n, the first and second player take turns choosing certain topological objects from X (e.g. point, open set, open cover, etc.).

At the "end" of the game, a winner is declared by inspecting the sequences of choices made throughout the game.

The study of such games involves finding when a player has a *winning strategy* which defeats every possible counterattack by the opponent.

See Telgarsky's excellent survey on topological games for more details: [11]

A *topological game* is a two-player game $G(X)$ of length $\omega = \{0, 1, 2, \dots\}$ defined for certain topological spaces X.

During each round n, the first and second player take turns choosing certain topological objects from X (e.g. point, open set, open cover, etc.).

At the "end" of the game, a winner is declared by inspecting the sequences of choices made throughout the game.

The study of such games involves finding when a player has a *winning strategy* which defeats every possible counterattack by the opponent.

See Telgarsky's excellent survey on topological games for more details: [11]

A *topological game* is a two-player game $G(X)$ of length $\omega = \{0, 1, 2, \dots\}$ defined for certain topological spaces X.

During each round n, the first and second player take turns choosing certain topological objects from X (e.g. point, open set, open cover, etc.).

At the "end" of the game, a winner is declared by inspecting the sequences of choices made throughout the game.

The study of such games involves finding when a player has a *winning strategy* which defeats every possible counterattack by the opponent.

See Telgarsky's excellent survey on topological games for more details: [11]

A *topological game* is a two-player game $G(X)$ of length $\omega = \{0, 1, 2, \dots\}$ defined for certain topological spaces X.

During each round n, the first and second player take turns choosing certain topological objects from X (e.g. point, open set, open cover, etc.).

At the "end" of the game, a winner is declared by inspecting the sequences of choices made throughout the game.

The study of such games involves finding when a player has a *winning strategy* which defeats every possible counterattack by the opponent.

See Telgarsky's excellent survey on topological games for more details: [11]

A *topological game* is a two-player game $G(X)$ of length $\omega = \{0, 1, 2, \dots\}$ defined for certain topological spaces X.

During each round n, the first and second player take turns choosing certain topological objects from X (e.g. point, open set, open cover, etc.).

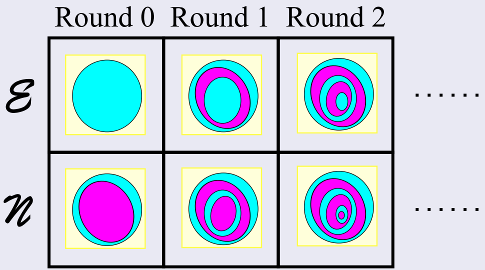
At the "end" of the game, a winner is declared by inspecting the sequences of choices made throughout the game.

The study of such games involves finding when a player has a *winning strategy* which defeats every possible counterattack by the opponent.

See Telgarsky's excellent survey on topological games for more details: [11]

Game

The *Banach-Mazur Game* $BM_{E,N}(X)$ (1935) [5]



The first player $\mathscr{E}$ wins the game if the intersection of all the chosen open sets is empty.

Theorem

X is Baire if and only if $\mathscr{E}$ lacks a winning strategy in the Banach Mazur game ($\mathscr{E} \not\uparrow BM_{E,N}(X)$).

Thus the topological property of being a Baire space has a game-theoretic characterization using $BM_{E,N}(X)$.

By considering *limited information strategies*, we may characterize more properties.

Theorem

X is Baire if and only if $\mathscr{E}$ lacks a winning strategy in the Banach Mazur game ($\mathscr{E} \not\uparrow BM_{E,N}(X)$).

Thus the topological property of being a Baire space has a game-theoretic characterization using $BM_{E,N}(X)$.

By considering *limited information strategies*, we may characterize more properties.

Theorem

X is Baire if and only if $\mathscr{E}$ lacks a winning strategy in the Banach Mazur game ($\mathscr{E} \not\uparrow BM_{E,N}(X)$).

Thus the topological property of being a Baire space has a game-theoretic characterization using $BM_{E,N}(X)$.

By considering *limited information strategies*, we may characterize more properties.

Consider the following:

Theorem

X is α-favorable $\Rightarrow$ X is Choquet $\Rightarrow$ X is Baire

α-favorability is characterized by $\mathscr{N} \underset{\text{tact}}{\uparrow} BM_{E,N}(X)$: player $\mathscr{E}$ has a *tactical* winning strategy which only considers the most recent move of the opponent.

This is stronger than the Choquet property [2], characterized by $\mathscr{N} \uparrow BM_{E,N}(X)$. In this case $\mathscr{N}$ still has a winning strategy, but it may rely on perfect information of the history of the game.

Consider the following:

Theorem

X is α-favorable $\Rightarrow$ X is Choquet $\Rightarrow$ X is Baire

α-favorability is characterized by $\mathscr{N} \underset{\text{tact}}{\uparrow} BM_{E,N}(X)$: player $\mathscr{E}$ has a *tactical* winning strategy which only considers the most recent move of the opponent.

This is stronger than the Choquet property [2], characterized by $\mathscr{N} \uparrow BM_{E,N}(X)$. In this case $\mathscr{N}$ still has a winning strategy, but it may rely on perfect information of the history of the game.

Consider the following:

Theorem

X is α-favorable $\Rightarrow$ X is Choquet $\Rightarrow$ X is Baire

α-favorability is characterized by $\mathscr{N} \underset{\text{tact}}{\uparrow} BM_{E,N}(X)$: player $\mathscr{E}$ has a *tactical* winning strategy which only considers the most recent move of the opponent.

This is stronger than the Choquet property [2], characterized by $\mathscr{N} \uparrow BM_{E,N}(X)$. In this case $\mathscr{N}$ still has a winning strategy, but it may rely on perfect information of the history of the game.

By characterizing topological properties using the theory of topological games, we introduce new proof techniques for demonstrating the structure of given topological spaces.

The aim of my dissertation was to investigate four topological games from the literature with unknown limited information implications.

In doing so I uncovered several new results in general topology, advancing research done by G. Gruenhage, P. Nyikos, R. Telgárksy, J. Bell, M. Scheepers, and others.

By characterizing topological properties using the theory of topological games, we introduce new proof techniques for demonstrating the structure of given topological spaces.

The aim of my dissertation was to investigate four topological games from the literature with unknown limited information implications.

In doing so I uncovered several new results in general topology, advancing research done by G. Gruenhage, P. Nyikos, R. Telgárksy, J. Bell, M. Scheepers, and others.

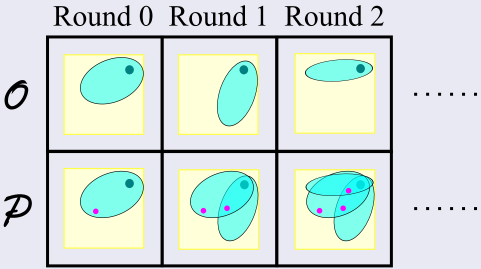
By characterizing topological properties using the theory of topological games, we introduce new proof techniques for demonstrating the structure of given topological spaces.

The aim of my dissertation was to investigate four topological games from the literature with unknown limited information implications.

In doing so I uncovered several new results in general topology, advancing research done by G. Gruenhage, P. Nyikos, R. Telgárksy, J. Bell, M. Scheepers, and others.

Game

Gruenhage's convergence game $Gru_{O,P}^{\rightarrow}(X, x)$ and clustering game $Gru_{O,P}^{\rightsquigarrow}(X, x)$ proceed as follows:



$\mathscr{O}$ wins the game if the points chosen by $\mathscr{P}$ converge/cluster to the given point $x \in X$. Otherwise, $\mathscr{P}$ wins.

Note that $\mathscr{O}$ need not know anything about the history of the game to play each round.

If $\mathscr{O} \uparrow Gru_{\vec{O},P}(X, x)$, then x is called a W-point in X. Obviously, all points of first-countablity are W-points, but $\mathscr{O} \uparrow Gru_{\vec{O},P}(\kappa^*, \infty)$ also, where ∞ is the added point in the one-point compactification κ^* of uncountable discrete κ.

Points of first-countability may in fact be characterized by this game as well:

Theorem

x has a countable local base in X if and only if $\mathscr{O} \underset{pre}{\uparrow} Gru_{\vec{O},P}(X, x)$ ($\mathscr{O}$ has a winning predetermined strategy using only the round number).

If every point in a space X is a W-point, then X is a W-space.

If $\mathscr{O} \uparrow Gru_{\vec{O},P}(X,x)$, then x is called a W-point in X. Obviously, all points of first-countablity are W-points, but $\mathscr{O} \uparrow Gru_{\vec{O},P}(\kappa^*,\infty)$ also, where ∞ is the added point in the one-point compactification κ^* of uncountable discrete κ.

Points of first-countability may in fact be characterized by this game as well:

Theorem

x has a countable local base in X if and only if $\mathscr{O} \underset{pre}{\uparrow} Gru_{\vec{O},P}(X,x)$ ($\mathscr{O}$ has a winning predetermined strategy using only the round number).

If every point in a space X is a W-point, then X is a W-space.

If $\mathscr{O} \uparrow Gru_{\vec{O},P}(X,x)$, then x is called a W-point in X. Obviously, all points of first-countablity are W-points, but $\mathscr{O} \uparrow Gru_{\vec{O},P}(\kappa^*,\infty)$ also, where ∞ is the added point in the one-point compactification κ^* of uncountable discrete κ.

Points of first-countability may in fact be characterized by this game as well:

Theorem

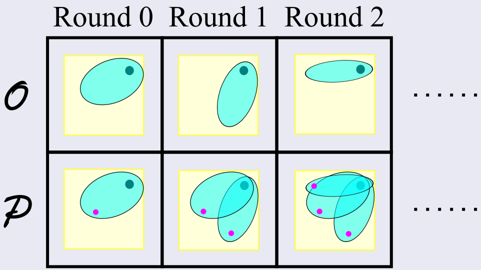
x has a countable local base in X if and only if $\mathscr{O} \underset{pre}{\uparrow} Gru_{\vec{O},P}(X,x)$ ($\mathscr{O}$ has a winning predetermined strategy using only the round number).

If every point in a space X is a W-point, then X is a W-space.

A variation of this game which is harder for $\mathscr{O}$ yields some difficult infinite combinatorial questions:

Game

Gruenhage's hard convergence game $Gru_{O,P}^{\rightarrow,\star}(X, x)$ and hard clustering game $Gru_{O,P}^{\rightsquigarrow,\star}(X, x)$ proceed as follows:



Nyikos observed in [6] that:

Theorem

$\mathscr{O} \underset{mark}{\not\uparrow} Gru^{\rightarrow,\star}_{O,P}(\omega_1^*, \infty)$.

($\mathscr{O}$ cannot guarantee a win using a Markov strategy which considers only the round number and most recent move.)

Some more work shows that in fact

Theorem

$\mathscr{O} \underset{k\text{-}mark}{\not\uparrow} Gru^{\rightarrow,\star}_{O,P}(\omega_1^*, \infty)$.

($\mathscr{O}$ cannot guarantee a win using a k-Markov strategy which considers only the round number and k most recent moves.)

Nyikos observed in [6] that:

Theorem

$\mathscr{O} \underset{mark}{\not\uparrow} Gru^{\rightarrow,\star}_{O,P}(\omega_1^*, \infty)$.

($\mathscr{O}$ cannot guarantee a win using a Markov strategy which considers only the round number and most recent move.)

Some more work shows that in fact

Theorem

$\mathscr{O} \underset{k\text{-}mark}{\not\uparrow} Gru^{\rightarrow,\star}_{O,P}(\omega_1^*, \infty)$.

($\mathscr{O}$ cannot guarantee a win using a k-Markov strategy which considers only the round number and k most recent moves.)

Interestingly, the strategy which prevents convergence won't prevent clustering as well unless the cardinality of the space is sufficiently large.

Theorem

$\mathscr{O} \underset{mark}{\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$, but $\mathscr{O} \underset{k\text{-}mark}{\not\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_2^*, \infty)$.

But knowledge of the round number is used non-trivially in doing so.

Theorem

$\mathscr{O} \underset{k\text{-}tact}{\not\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$.

Interestingly, the strategy which prevents convergence won't prevent clustering as well unless the cardinality of the space is sufficiently large.

Theorem

$\mathscr{O} \underset{mark}{\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$, *but* $\mathscr{O} \underset{k\text{-}mark}{\not\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_2^*, \infty)$.

But knowledge of the round number is used non-trivially in doing so.

Theorem

$\mathscr{O} \underset{k\text{-}tact}{\not\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$.

Proof that $\mathscr{O} \underset{\text{mark}}{\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$

For each $\alpha < \omega_1$, let $f_\alpha : \alpha \to \omega$ be injective.

Note that for each $F \in [\omega_1]^{<\omega}$, there is some $n_F < \omega$ such that $f_{\alpha+1}(\beta) < n_F$ for all $\beta, \alpha \in F$.

Let σ be a Markov strategy for $\mathscr{O}$ such that

$$\sigma(\langle\alpha\rangle, n) \subseteq \omega_1^* \setminus \{\beta < \omega_1 : f_{\alpha+1}(\beta) < n\}$$

for $\alpha < \omega_1$.

Thus as any legal counterattack cannot have finite range in ω_1 without repeating ∞ co-finitely, σ is a winning Markov strategy. $\square$

Proof that $\mathscr{O} \underset{\text{mark}}{\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$

For each $\alpha < \omega_1$, let $f_\alpha : \alpha \to \omega$ be injective.

Note that for each $F \in [\omega_1]^{<\omega}$, there is some $n_F < \omega$ such that $f_{\alpha+1}(\beta) < n_F$ for all $\beta, \alpha \in F$.

Let σ be a Markov strategy for $\mathscr{O}$ such that

$$\sigma(\langle \alpha \rangle, n) \subseteq \omega_1^* \setminus \{\beta < \omega_1 : f_{\alpha+1}(\beta) < n\}$$

for $\alpha < \omega_1$.

Thus as any legal counterattack cannot have finite range in ω_1 without repeating ∞ co-finitely, σ is a winning Markov strategy. $\square$

Proof that $\mathscr{O} \underset{\text{mark}}{\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$

For each $\alpha < \omega_1$, let $f_\alpha : \alpha \to \omega$ be injective.

Note that for each $F \in [\omega_1]^{<\omega}$, there is some $n_F < \omega$ such that $f_{\alpha+1}(\beta) < n_F$ for all $\beta, \alpha \in F$.

Let σ be a Markov strategy for $\mathscr{O}$ such that

$$\sigma(\langle\alpha\rangle, n) \subseteq \omega_1^* \setminus \{\beta < \omega_1 : f_{\alpha+1}(\beta) < n\}$$

for $\alpha < \omega_1$.

Thus as any legal counterattack cannot have finite range in ω_1 without repeating ∞ co-finitely, σ is a winning Markov strategy. $\square$

Proof that $\mathscr{O} \underset{\text{mark}}{\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$

For each $\alpha < \omega_1$, let $f_\alpha : \alpha \to \omega$ be injective.

Note that for each $F \in [\omega_1]^{<\omega}$, there is some $n_F < \omega$ such that $f_{\alpha+1}(\beta) < n_F$ for all $\beta, \alpha \in F$.

Let σ be a Markov strategy for $\mathscr{O}$ such that

$$\sigma(\langle\alpha\rangle, n) \subseteq \omega_1^* \setminus \{\beta < \omega_1 : f_{\alpha+1}(\beta) < n\}$$

for $\alpha < \omega_1$.

Thus as any legal counterattack cannot have finite range in ω_1 without repeating ∞ co-finitely, σ is a winning Markov strategy. $\square$

Proof that $\mathscr{O} \underset{\text{tact}}{\not\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$

Let σ be a tactic for $\mathscr{O}$ in $Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$.

Then this set is closed and unbounded in ω_1:

$$C_\sigma = \{\alpha < \omega_1 : \beta < \alpha \Rightarrow \omega_1^* \setminus \sigma(\langle\beta\rangle) \subseteq \alpha\}$$

If $a_\sigma : \omega_1 \to C_\sigma$ is an order isomorphism, then there is $n < \omega$ such that $a_\sigma(n) \in \sigma(\langle a_\sigma(\omega)\rangle)$.

Then since $a_\sigma(\omega) > a_\sigma(n)$ implies $a_\sigma(\omega) \in \sigma(\langle a_\sigma(n)\rangle)$ (by the definition of C_σ), it follows that $\langle a_\sigma(n), a_\sigma(\omega), a_\sigma(n), a_\sigma(\omega), \dots\rangle$ is a legal counterattack against σ which does not cluster at ∞. $\square$

Proof that $\mathscr{O} \underset{\text{tact}}{\not\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$

Let σ be a tactic for $\mathscr{O}$ in $Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$.

Then this set is closed and unbounded in ω_1:

$$C_\sigma = \{\alpha < \omega_1 : \beta < \alpha \Rightarrow \omega_1^* \setminus \sigma(\langle \beta \rangle) \subseteq \alpha\}$$

If $a_\sigma : \omega_1 \to C_\sigma$ is an order isomorphism, then there is $n < \omega$ such that $a_\sigma(n) \in \sigma(\langle a_\sigma(\omega) \rangle)$.

Then since $a_\sigma(\omega) > a_\sigma(n)$ implies $a_\sigma(\omega) \in \sigma(\langle a_\sigma(n) \rangle)$ (by the definition of C_σ), it follows that $\langle a_\sigma(n), a_\sigma(\omega), a_\sigma(n), a_\sigma(\omega), \ldots \rangle$ is a legal counterattack against σ which does not cluster at ∞. $\square$

Proof that $\mathscr{O} \underset{\text{tact}}{\not\uparrow} Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$

Let σ be a tactic for $\mathscr{O}$ in $Gru_{O,P}^{\rightsquigarrow,\star}(\omega_1^*, \infty)$.

Then this set is closed and unbounded in ω_1:

$$C_\sigma = \{\alpha < \omega_1 : \beta < \alpha \Rightarrow \omega_1^* \setminus \sigma(\langle \beta \rangle) \subseteq \alpha\}$$

If $a_\sigma : \omega_1 \to C_\sigma$ is an order isomorphism, then there is $n < \omega$ such that $a_\sigma(n) \in \sigma(\langle a_\sigma(\omega) \rangle)$.

Then since $a_\sigma(\omega) > a_\sigma(n)$ implies $a_\sigma(\omega) \in \sigma(\langle a_\sigma(n) \rangle)$ (by the definition of C_σ), it follows that $\langle a_\sigma(n), a_\sigma(\omega), a_\sigma(n), a_\sigma(\omega), \dots \rangle$ is a legal counterattack against σ which does not cluster at ∞. $\square$

Proof that $\mathscr{O} \underset{\text{tact}}{\not\uparrow} Gru^{\rightsquigarrow,\star}_{O,P}(\omega_1^*, \infty)$

Let σ be a tactic for $\mathscr{O}$ in $Gru^{\rightsquigarrow,\star}_{O,P}(\omega_1^*, \infty)$.

Then this set is closed and unbounded in ω_1:

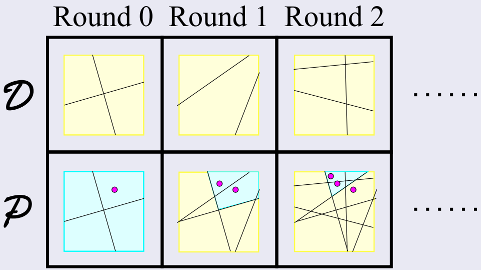
$$C_\sigma = \{\alpha < \omega_1 : \beta < \alpha \Rightarrow \omega_1^* \setminus \sigma(\langle\beta\rangle) \subseteq \alpha\}$$

If $a_\sigma : \omega_1 \to C_\sigma$ is an order isomorphism, then there is $n < \omega$ such that $a_\sigma(n) \in \sigma(\langle a_\sigma(\omega)\rangle)$.

Then since $a_\sigma(\omega) > a_\sigma(n)$ implies $a_\sigma(\omega) \in \sigma(\langle a_\sigma(n)\rangle)$ (by the definition of C_σ), it follows that $\langle a_\sigma(n), a_\sigma(\omega), a_\sigma(n), a_\sigma(\omega), \ldots\rangle$ is a legal counterattack against σ which does not cluster at ∞. $\square$

Game

Bell's proximal game $Bell_{\vec{D},P}(X)$ for compact zero-dimensional X:



$\mathscr{D}$ wins the game if the points chosen by $\mathscr{P}$ converge. Otherwise, $\mathscr{P}$ wins.

If $\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$, then X is called a proximal compact. This game was brought to my attention due to this result: [1]

Theorem

Every proximal space is a W-space. So $\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(X) \Rightarrow \mathscr{O} \uparrow Gru_{O,P}^{\rightarrow}(X, x)$ for all $x \in X$.

Proximal spaces have strong preservation properties, as any closed subset or Σ-product of proximal spaces is proximal. Since any proximal space is collectionwise normal, Bell's game gives an elegant proof of the classic result:

Theorem

A Σ-product of metrizable spaces is collection-wise normal.

If $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$, then X is called a proximal compact. This game was brought to my attention due to this result: [1]

Theorem

Every proximal space is a W-space. So $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X) \Rightarrow \mathscr{O} \uparrow Gru^{\rightarrow}_{O,P}(X, x)$ for all $x \in X$.

Proximal spaces have strong preservation properties, as any closed subset or Σ-product of proximal spaces is proximal. Since any proximal space is collectionwise normal, Bell's game gives an elegant proof of the classic result:

Theorem

A Σ-product of metrizable spaces is collection-wise normal.

Nyikos [7] observed that

Proposition

Corson compact spaces are proximal.

and asked if the converse holds as well.

With Gruenhage, I showed that the answer is yes:

Theorem

A compact space is Corson compact if and only if it is proximal.

Nyikos [7] observed that

Proposition

Corson compact spaces are proximal.

and asked if the converse holds as well.

With Gruenhage, I showed that the answer is yes:

Theorem

A compact space is Corson compact if and only if it is proximal.

A winning limited information strategy may always be passed down to win in a closed subspace, but Bell's result that winning strategies are preserved for Σ-products does not quite generalize as well:

Theorem

For $k < \omega$, if $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell^{\rightarrow}_{D,P}(X_i)$ for all $i < \omega$, then $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell^{\rightarrow}_{D,P}\left(\prod_{i<\omega} X_i\right)$.

Other limited information lemmas proved in my dissertation allowed me to prove a game-theoretic characterization of another compactness property (paper in preparation):

Theorem

A compact space is strong Eberlein compact if and only if $\mathscr{D} \underset{tact}{\uparrow} Bell^{\rightarrow}_{D,P}(X)$.

A winning limited information strategy may always be passed down to win in a closed subspace, but Bell's result that winning strategies are preserved for Σ-products does not quite generalize as well:

Theorem

For $k < \omega$, if $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell_{\vec{D},P}(X_i)$ for all $i < \omega$, then $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell_{\vec{D},P}(\prod_{i<\omega} X_i)$.

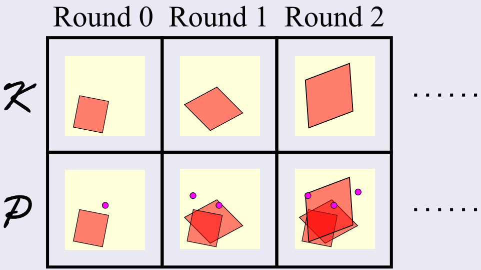
Other limited information lemmas proved in my dissertation allowed me to prove a game-theoretic characterization of another compactness property (paper in preparation):

Theorem

A compact space is strong Eberlein compact if and only if $\mathscr{D} \underset{tact}{\uparrow} Bell_{\vec{D},P}(X)$.

Game

Gruenhage's locally finite games $Gru_{K,P}(X)$ and $Gru_{K,L}(X)$ proceed as follows:



$\mathscr{K}$ wins the game if the points/sets chosen by $\mathscr{P}/\mathscr{L}$ are locally finite in the space. Otherwise, $\mathscr{P}/\mathscr{L}$ wins.

Gruenhage used these games in [3] to characterize metacompactness and σ-metacompactness amongst locally compact spaces:

Theorem

For locally compact spaces, $\mathscr{K} \underset{tact}{\uparrow} Gru_{K,P}(X)$ if and only if X is metacompact.

Theorem

For locally compact spaces, $\mathscr{K} \underset{mark}{\uparrow} Gru_{K,P}(X)$ if and only if X is σ-metacompact.

By removing knowledge of the round number, an analogous result is unsurfaced:

Theorem

For locally compact spaces, $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(X)$ if and only if X is σ-compact.

Actually, for locally compact or even compactly-generated spaces, $\mathscr{K} \underset{\text{pre}}{\uparrow} Gru_{K,P}(X)$ if and only if $\mathscr{K} \underset{\text{pre}}{\uparrow} Gru_{K,L}(X)$.

However, there is a non-compactly-generated counterexample:

Theorem

There exists a free ultrafilter $\mathcal{F}$ such that $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(\omega \cup \{\mathcal{F}\})$, but $\mathscr{K} \underset{pre}{\not\uparrow} Gru_{K,L}(\omega \cup \{\mathcal{F}\})$ for any free ultrafilter $\mathcal{F}$.

By removing knowledge of the round number, an analogous result is unsurfaced:

Theorem

For locally compact spaces, $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(X)$ if and only if X is σ-compact.

Actually, for locally compact or even compactly-generated spaces, $\mathscr{K} \underset{\text{pre}}{\uparrow} Gru_{K,P}(X)$ if and only if $\mathscr{K} \underset{\text{pre}}{\uparrow} Gru_{K,L}(X)$.

However, there is a non-compactly-generated counterexample:

Theorem

There exists a free ultrafilter $\mathcal{F}$ such that $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(\omega \cup \{\mathcal{F}\})$, but $\mathscr{K} \underset{pre}{\not\uparrow} Gru_{K,L}(\omega \cup \{\mathcal{F}\})$ for any free ultrafilter $\mathcal{F}$.

For the related game $Gru_{\vec{O},P}(X,x)$, a $(k+1)$-tactic/mark may always be improved to only use the most recent move of the opponent. If this also holds for $Gru_{K,P}(X)$, then metacompactness and σ-metacompactness may would be characterized by the existence of any winning $(k+1)$-tactic/mark.

Due to a technical difference in the games, it's unclear if this is true. However, for a tricky non-σ-metacompact example $\boldsymbol{X}$:

Theorem

$\mathscr{K} \uparrow Gru_{K,P}(\boldsymbol{X})$ but $\mathscr{K} \underset{k\text{-mark}}{\not\uparrow} Gru_{K,P}(\boldsymbol{X})$.

For the related game $Gru_{\vec{O},P}(X,x)$, a $(k+1)$-tactic/mark may always be improved to only use the most recent move of the opponent. If this also holds for $Gru_{K,P}(X)$, then metacompactness and σ-metacompactness may would be characterized by the existence of any winning $(k+1)$-tactic/mark.

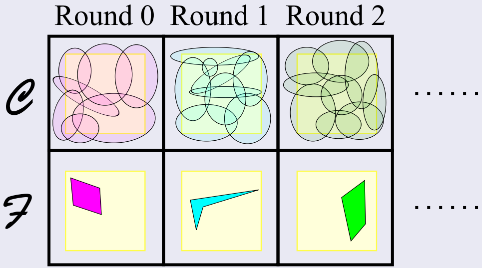
Due to a technical difference in the games, it's unclear if this is true. However, for a tricky non-σ-metacompact example $\boldsymbol{X}$:

Theorem

$\mathscr{K} \uparrow Gru_{K,P}(\boldsymbol{X})$ but $\mathscr{K} \underset{k\text{-mark}}{\not\uparrow} Gru_{K,P}(\boldsymbol{X})$.

Game

Menger's game $Men_{C,F}(X)$ proceeds as follows:



$\mathscr{F}$ wins the game if her finitely coverable subsets union to the space. Otherwise, $\mathscr{C}$ wins.

A covering property generalizing σ-compactness is characterized by this game, demonstrated by Hurewicz in the 1920's. [4]

Theorem

A space is Menger if and only if $\mathscr{C} \not\uparrow Men_{C,F}(X)$.

It was originally suspected that Menger subspaces of the real line were exactly the σ-compact subspaces, but as was shown by Telgarsky and Scheepers, σ-compact spaces have slightly more structure. [10] [9]

Theorem

A metrizable space X is σ-compact if and only if $\mathscr{F} \uparrow Men_{C,F}(X)$.

A covering property generalizing σ-compactness is characterized by this game, demonstrated by Hurewicz in the 1920's. [4]

Theorem

A space is Menger if and only if $\mathscr{C} \not\uparrow Men_{C,F}(X)$.

It was originally suspected that Menger subspaces of the real line were exactly the σ-compact subspaces, but as was shown by Telgarsky and Scheepers, σ-compact spaces have slightly more structure. [10] [9]

Theorem

A metrizable space X is σ-compact if and only if $\mathscr{F} \uparrow Men_{C,F}(X)$.

By considering Markov strategies, the previous theorem may be factored into two subresults.

Theorem

A regular space X is σ-compact if and only if $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$.

Theorem

For a second-countable space X, $\mathscr{F} \uparrow Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$.

Note that since the spaces we are considering are all Lindelöf, metrizability is characterized by regularity and second-countability.

Proof that σ-compact $\Leftrightarrow \mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$

Assume X is regular.

If $X = \bigcup_{n<\omega} K_n$, then $\mathscr{F}$ may choose K_n each round, which is a winning predetermined (and therefore Markov) strategy.

If X is not σ-compact, let σ be any Markov strategy for $\mathscr{F}$. Then for

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \sigma(\langle \mathcal{U} \rangle, n)$$

and any open cover $\mathcal{U}$ of the space X, $\sigma(\langle \mathcal{U} \rangle, n)$ is a finite subcover of R_n. Thus $\overline{R_n}$ is compact by the regularity of X.

Since X is not σ-compact, choose a point x not in any R_n. Then there is a sequence of open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$ for which $x \notin \sigma(\langle \mathcal{U}_n \rangle, n)$ for all n, and thus σ is not a winning Markov strategy.

Proof that σ-compact $\Leftrightarrow \mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$

Assume X is regular.

If $X = \bigcup_{n<\omega} K_n$, then $\mathscr{F}$ may choose K_n each round, which is a winning predetermined (and therefore Markov) strategy.

If X is not σ-compact, let σ be any Markov strategy for $\mathscr{F}$. Then for

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \sigma(\langle \mathcal{U} \rangle, n)$$

and any open cover $\mathcal{U}$ of the space X, $\sigma(\langle \mathcal{U} \rangle, n)$ is a finite subcover of R_n. Thus $\overline{R_n}$ is compact by the regularity of X.

Since X is not σ-compact, choose a point x not in any R_n. Then there is a sequence of open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$ for which $x \notin \sigma(\langle \mathcal{U}_n \rangle, n)$ for all n, and thus σ is not a winning Markov strategy.

Proof that σ-compact $\Leftrightarrow \mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$

Assume X is regular.

If $X = \bigcup_{n<\omega} K_n$, then $\mathscr{F}$ may choose K_n each round, which is a winning predetermined (and therefore Markov) strategy.

If X is not σ-compact, let σ be any Markov strategy for $\mathscr{F}$. Then for

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \sigma(\langle \mathcal{U} \rangle, n)$$

and any open cover $\mathcal{U}$ of the space X, $\sigma(\langle \mathcal{U} \rangle, n)$ is a finite subcover of R_n. Thus $\overline{R_n}$ is compact by the regularity of X.

Since X is not σ-compact, choose a point x not in any R_n. Then there is a sequence of open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ for which $x \notin \sigma(\langle \mathcal{U}_n \rangle, n)$ for all n, and thus σ is not a winning Markov strategy.

Proof that σ-compact $\Leftrightarrow \mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$

Assume X is regular.

If $X = \bigcup_{n<\omega} K_n$, then $\mathscr{F}$ may choose K_n each round, which is a winning predetermined (and therefore Markov) strategy.

If X is not σ-compact, let σ be any Markov strategy for $\mathscr{F}$. Then for

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \sigma(\langle \mathcal{U} \rangle, n)$$

and any open cover $\mathcal{U}$ of the space X, $\sigma(\langle \mathcal{U} \rangle, n)$ is a finite subcover of R_n. Thus $\overline{R_n}$ is compact by the regularity of X.

Since X is not σ-compact, choose a point x not in any R_n. Then there is a sequence of open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$ for which $x \notin \sigma(\langle \mathcal{U}_n \rangle, n)$ for all n, and thus σ is not a winning Markov strategy.

Proof that $\mathscr{F} \uparrow Men_{C,F}(X) \Leftrightarrow \mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$

Assume X is second-countable. Without loss of generality, assume $\mathscr{C}$ only chooses coverings using open sets from the countable base $\{U_n : n < \omega\}$.

If $\mathscr{F}$ has a winning Markov strategy, then she has a winning strategy.

Let σ be a winning strategy, and assume $\mathcal{U}_t$ is an open cover of basic open sets for each $t \leq s \in \omega^{<\omega}$. By exploiting the countable base, we may choose $\mathcal{U}_{s^\frown\langle n\rangle}$ for each $n < \omega$, such that for every open cover $\mathcal{U}$ there exists $n < \omega$ where $\sigma(\langle \mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{U}\rangle) \subseteq \sigma(\langle \mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle n\rangle}\rangle)$.

Then $\tau(\langle \mathcal{U}\rangle, n) = \sigma(\langle \mathcal{U}_{f(n)\restriction 1}, \dots, \mathcal{U}_{f(n)}, \mathcal{U}\rangle$ where $f : \omega \to \omega^{<\omega}$ is a bijection is a winning strategy.

Proof that $\mathscr{F} \uparrow Men_{C,F}(X) \Leftrightarrow \mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$

Assume X is second-countable. Without loss of generality, assume $\mathscr{C}$ only chooses coverings using open sets from the countable base $\{U_n : n < \omega\}$.

If $\mathscr{F}$ has a winning Markov strategy, then she has a winning strategy.

Let σ be a winning strategy, and assume $\mathcal{U}_t$ is an open cover of basic open sets for each $t \leq s \in \omega^{<\omega}$. By exploiting the countable base, we may choose $\mathcal{U}_{s^\frown\langle n\rangle}$ for each $n < \omega$, such that for every open cover $\mathcal{U}$ there exists $n < \omega$ where
$\sigma(\langle \mathcal{U}_{s\restriction 1}, \ldots, \mathcal{U}_s, \mathcal{U}\rangle) \subseteq \sigma(\langle \mathcal{U}_{s\restriction 1}, \ldots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle n\rangle}\rangle)$.

Then $\tau(\langle \mathcal{U}\rangle, n) = \sigma(\langle \mathcal{U}_{f(n)\restriction 1}, \ldots, \mathcal{U}_{f(n)}, \mathcal{U}\rangle$ where $f : \omega \to \omega^{<\omega}$ is a bijection is a winning strategy.

Proof that $\mathscr{F} \uparrow Men_{C,F}(X) \Leftrightarrow \mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$

Assume X is second-countable. Without loss of generality, assume $\mathscr{C}$ only chooses coverings using open sets from the countable base $\{U_n : n < \omega\}$.

If $\mathscr{F}$ has a winning Markov strategy, then she has a winning strategy.

Let σ be a winning strategy, and assume $\mathcal{U}_t$ is an open cover of basic open sets for each $t \leq s \in \omega^{<\omega}$. By exploiting the countable base, we may choose $\mathcal{U}_{s^\frown\langle n\rangle}$ for each $n < \omega$, such that for every open cover $\mathcal{U}$ there exists $n < \omega$ where $\sigma(\langle \mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{U}\rangle) \subseteq \sigma(\langle \mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle n\rangle}\rangle)$.

Then $\tau(\langle \mathcal{U}\rangle, n) = \sigma(\langle \mathcal{U}_{f(n)\restriction 1}, \dots, \mathcal{U}_{f(n)}, \mathcal{U}\rangle$ where $f : \omega \to \omega^{<\omega}$ is a bijection is a winning strategy.

Proof that $\mathscr{F} \uparrow Men_{C,F}(X) \Leftrightarrow \mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$

Assume X is second-countable. Without loss of generality, assume $\mathscr{C}$ only chooses coverings using open sets from the countable base $\{U_n : n < \omega\}$.

If $\mathscr{F}$ has a winning Markov strategy, then she has a winning strategy.

Let σ be a winning strategy, and assume $\mathcal{U}_t$ is an open cover of basic open sets for each $t \leq s \in \omega^{<\omega}$. By exploiting the countable base, we may choose $\mathcal{U}_{s^\frown\langle n\rangle}$ for each $n < \omega$, such that for every open cover $\mathcal{U}$ there exists $n < \omega$ where $\sigma(\langle \mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{U}\rangle) \subseteq \sigma(\langle \mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle n\rangle}\rangle)$.

Then $\tau(\langle \mathcal{U}\rangle, n) = \sigma(\langle \mathcal{U}_{f(n)\restriction 1}, \dots, \mathcal{U}_{f(n)}, \mathcal{U}\rangle$ where $f : \omega \to \omega^{<\omega}$ is a bijection is a winning strategy.

Limited information strategies in topological games such as $Men_{C,F}(X)$ often have set-theoretic consequences. The statement $S(\kappa)$ due to M. Scheepers [8] says that there exist almost-compatible functions $f_A : A \to \omega$ for each $A \in [\kappa]^\omega$.

Theorem

$S(\omega_1)$ and $\neg S((2^\omega)^+)$ are theorems of ZFC, but $S(\kappa)$ is independent of ZFC for $\omega_1 < \kappa \leq 2^\omega$.

Theorem

If $S(\kappa)$ holds, then $\mathscr{F} \underset{2\text{-tact}}{\uparrow} Fill_{C,F}^{\cup,\subset}(\kappa)$.

Let $\kappa^\dagger$ be the one-point "Lindelöf-ication" of discrete κ.

Theorem

$$
\begin{tikzcd}
 & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(\kappa^\dagger) \arrow[r, leftrightarrow] \arrow[d] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa) \arrow[d] \\
S(\kappa,\omega,\omega) \arrow[ur] \arrow[r] \arrow[drr] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill^{1,\subseteq}_{C,F}(\kappa) \arrow[r, leftrightarrow] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill^{\cup,\subseteq}_{C,F}(\kappa) \\
 & & \mathscr{F} \underset{\text{2-tact}}{\uparrow} Fill^{\cup,\subset}_{C,F}(\kappa) \arrow[u]
\end{tikzcd}
$$

For most of the games in the previous chart, including the Menger game $Men_{C,F}(X)$, there's no need to consider larger amounts of limited information.

Theorem

For each $k < \omega$, $\mathscr{F} \underset{k+2\text{-mark}}{\uparrow} Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men_{C,F}(X)$

The topological property $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men_{C,F}(X)$ seems to depend on the set-theoretic axioms at play.

Theorem

If $S(2^\omega)$, then $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men_{C,F}(R_\omega)$.

Any questions?

Jocelyn R. Bell.
An infinite game with topological consequences.
Topology Appl., 175:1–14, 2014.

Gabriel Debs.
Stratégies gagnantes dans certains jeux topologiques.
Fund. Math., 126(1):93–105, 1985.

Gary Gruenhage.
Covering properties on $X^2 \setminus \Delta$, W-sets, and compact subsets of Σ-products.
Topology Appl., 17(3):287–304, 1984.

Witold Hurewicz.
Über eine Verallgemeinerung des Borelschen Theorems.
Math. Z., 24(1):401–421, 1926.

R. Daniel Mauldin, editor.
The Scottish Book.
Birkhäuser, Boston, Mass., 1981.
Mathematics from the Scottish Café, Including selected papers presented at the Scottish Book Conference held at North Texas State University, Denton, Tex., May 1979.

Peter J. Nyikos.
Classes of compact sequential spaces.
In *Set theory and its applications (Toronto, ON, 1987)*, volume 1401 of *Lecture Notes in Math.*, pages 135–159. Springer, Berlin, 1989.

Peter J. Nyikos.
Proximal and semi-proximal spaces (preprint).
2013.

Marion Scheepers.
Concerning *n*-tactics in the countable-finite game.
J. Symbolic Logic, 56(3):786–794, 1991.

Marion Scheepers.
A direct proof of a theorem of Telgársky.
Proc. Amer. Math. Soc., 123(11):3483–3485, 1995.

Rastislav Telgársky.
On games of Topsøe.
Math. Scand., 54(1):170–176, 1984.

Rastislav Telgársky.
Topological games: on the 50th anniversary of the Banach-Mazur game.
Rocky Mountain J. Math., 17(2):227–276, 1987.