Let X_n be a continuum for each $n < \omega$, and $f_n : X_{n+1} \to X_n$ be a continuous function for each $n < \omega$, and $L = \varprojlim(X_n, f_n)$ be the inverse limit space induced by the spaces X_n and bonding maps f_n.

Lemma 1. *Let*

$$\mathcal{B}_N = \{L \cap \prod_{n<\omega} B_n : B_n = X_n \text{ for all } n \neq N \text{ and } B_N \text{ is open in } X_N\}$$

Then $\mathcal{B} = \bigcup_{N<\omega} \mathcal{B}_N$ is a basis for L.

Proof. Let α be a member of a basic open set induced by the product topology:

$$L \cap \prod_{n<\omega} A_n$$

where $A_n = X_n$ for all $n \neq N_i$ where $i < m$ and $N_i < N_{i+1}$, and A_{N_i} is open in X_{N_i}.

Let $g_{a\to b} : 2^{X_a} \to 2^{X_b}$ for $a \leq b$ be defined by

$$g_{a\to a} = id_{X_a}$$

$$g_{a\to(b+1)} = \underbrace{f_b^{-1} \circ \cdots \circ f_{a+1}^{-1} \circ f_a^{-1}}_{b-a+1 \text{ times}},$$

Let $N = N_{m-1}$ and

$$B_N = \bigcap_{i<m} g_{N_i \to N}(A_{N_i})$$

noting that B_N is open. Note that $\alpha(N_i) \in A_i$ for all $i < m$ implies $\alpha(N) \in B_N$, and thus $\alpha \in L \cap \prod_{n<\omega} B_n$ where $B_n = X_n$ for all $n \neq N$.

Finally, let $\beta \in L \cap \prod_{n<\omega} B_n$. Since $\beta(N) \in B_N = \bigcap_{i<m} g_{N_i\to N}(A_{N_i})$, we may easily see that $\beta(N_i) \in A_{N_i}$ for each $i < m$ and thus $\beta \in L \cap \prod_{n<\omega} A_n$. $\square$

Lemma 2. *For each subcontinuum $K \subseteq L$, there are minimal subcontinuua $K_n \subseteq X_n$ such that*

$$K = L \cap \prod_{n<\omega} K_n$$

Proof. For each $N < \omega$, let $\mathcal{B}'_N$ contain all basic open sets in $\mathcal{B}_N$ whose intersection with K is empty. Then let

$$K_N = X_N \setminus \bigcup_{B \in \mathcal{B}'_N} \pi_N(B)$$

K_N is a closed subset of a compact space, and is trivially compact. It is also connected: suppose $K_N \subseteq G \cup H$ with G, H disjoint open in X_N. Then $K \subseteq \pi_N^{-1}(G) \cup \pi_N^{-1}(H)$ with $\pi_N^{-1}(G)$,$\pi_N^{-1}(H)$ disjoint open, disconnecting K and showing the contradiction.

Let $\alpha \in K$. Then as $\alpha \notin \bigcup_{B \in \mathcal{B}'_N} B$, we know $\alpha(N) \notin \bigcup_{B \in \mathcal{B}'_N} \pi_N(B)$ for any $N < \omega$, so $\alpha \in L \cap \prod_{n<\omega} K_n$.

Let $\alpha \in L \setminus K$. Then $\alpha \in B \in \mathcal{B}_N$ for some N, and thus $\alpha(N) \in \bigcup_{B \in \mathcal{B}'_N} \pi_N(B)$. This shows $\alpha(N) \notin K_N$ and thus $\alpha \notin L \cap \prod_{n<\omega} K_n$.

This shows $K = L \cap \prod_{n<\omega} K_n$. To see that minimal candidates for K_n exist, observe that that if

$$K = L \cap \prod_{n<\omega} K_{n,\lambda}$$

for all λ in some indexing set I, then if $K_n^* = \bigcap_{\lambda \in I} K_{n,\lambda}$ we may see

$$K = L \cap \prod_{n<\omega} K_n^*$$

and thus K_n^* is the minimal subcontinuum for each n. (K_n^* is obviously compact, and observe that if it weren't connected, K wouldn't be connected either.) □

Example 3. Let L be the inverse limit space induced by $X_n = [0,1]$ and $f_n = f$ where

$$f(x) = \begin{cases} 2x & : x \leq 0.5 \\ 2 - 2x & : x \geq 0.5 \end{cases}$$

Then the following hold:

1. The subspaces $C_t = \{\alpha \in L : \alpha(0) = t\}$ are each homeomorphic to the Cantor Set.
2. All proper subcontinuua K are homeomorphic to the unit interval.
3. All proper subcontinuua K are nowhere dense in the space.

Proof. The reader may easily prove the first item by considering the Cantor tree produced by the branching sequences with a fixed initial coordinate.

By Lemma 2, we may write any proper subcontinuum K as

$$K = L \cap \prod_{n<\omega} [a_n, b_n]$$

for $0 \leq a_n \leq b_n \leq 1$ with $[a_n, b_n]$ minimal.

It's easily seen that $a_n < b_n$ must actually be strict (otherwise K is a single point).

We proceed to show that if each $[a_n, b_n]$ is minimal, then there must exist some N such that $0 \leq a_N < b_N < 1$.

If $a_n = 0$ and $b_n = 1$ always, then $K = L$ and is not a proper subcontinuum, so either:

- We assume $0 < a_N < b_N \leq 1$ and observe by the minimality of $[a_{N+1}, b_{N+1}]$ and $f(1 - \frac{a_N}{2}) = a_N$ that $[a_{N+1}, b_{N+1}] \subseteq [a_{N+1}, 1 - \frac{a_N}{2}]$, which implies $0 \leq a_{N+1} < b_{N+1} \leq 1 - \frac{a_N}{2} < 1$.
- We have $0 \leq a_N < b_N < 1$ for free.

Then one of the following occurs:

- Suppose $b_{N+1} = 1$; then $a_{N+2} > 0$. In this case, the reader may check that each sequence in K has a unique value in the $(N+2)^{th}$ coordinate - if not, then the sequences must "branch" in some $M > N + 2$ coordinate, but either $b_M < \frac{1}{2}$ or $a_M > \frac{1}{2}$ prevents such branching.
- It may be that $0 = a_n$ for all $n < \omega$. Then the reader may again check that each sequence in K has a unique value in the $(N+2)^{th}$ coordinate where $0 = a_{N+2} < b_{N+2} < 1$, as $M > N + 2 \Rightarrow b_M < \frac{1}{2}$ prevents branching.

It may be easily verified then that the projection π_{N+2} is a homeomorphism from K to $[a_{N+2}, b_{N+2}]$. $\square$