PhD Research Results

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Definition 1. Some notation on games of length ω:

- A **game of length** ω consists of two **Players** I and II. On **round** n of ω, Player I first takes a **turn** choosing an element of some set X, followed by Player II taking a turn choosing an element from some set Y.
- A **move** by a player is the selection that player makes during a particular round.
- A **rule** for a Player in a game is a condition on that player's move during each round. A move is said to be **legal** if it doesn't violate the rule.
- A **play** by Player I is a sequence $\langle x_0, x_1, \dots \rangle \in X^\omega$ (and similar for Player II). It is said to be **legal** if each move is legal. A finite initial sequence of a play is called a **partial play**.
- A **strategy** for a Player is a function which has all possible partial plays by the opposing player as its domain. It denotes the choice for that Player on each turn. It is said to be **legal** if it yields only legal moves when the opposing player makes only legal moves.
- A **counter** to a legal strategy by a Player is a function which has as its domain the turn number and the other player's strategy. It is said to be **legal** if it only yields legal moves.
- A **winning condition** is some condition on the plays made by both players.
- At the conclusion of a game, if all moves are legal, then Player I **wins** the game if the winning condition is satisfied, and Player II wins the game otherwise. (Should any move be illegal, then the first player to make an illegal move loses the game to the other player.)

When the sets X, Y are related to a topological space, it is said to be a **topological game**. The presence or absense of a "winning" strategy for one player or another characterizes a property of the space.

Definition 2. A strategy for Player A is said to be a **winning strategy** if there does not exist a counter which allows the other player to win the game. If Player Z has a winning strategy for the game G, this can be denoted $Z \uparrow G$.

Proofs showing the existance of a winning strategy typically define the winning strategy, and then show that it defeats every possible play by the opponent. Proofs showing the nonexistance of a winning strategy typically define the counter to any arbitrary strategy.

In the mid 1980s, Dr. Gary Gruenhage defined the following topological games.

Definition 3. The **compact-point game on a topological space** X is denoted $G_{K,P}(X)$. During round n, Player I (called K) chooses a compact set $K_n \in K[X]$, and Player II (called P) chooses a point $x_n \in X$. K must follow the rule that $K_{n+1} \supseteq K_n$, while P must follow the rule that $x_n \notin K_n$. The winning condition for K is that the collection of singletons chosen by P, $\{\{x_n\} : n < \omega\}$, must be locally finite everywhere in the space. (This is equivalent to the set $\{x_n : n < \omega\}$ lacking a cluster point.)

Definition 4. The **compact-compact game on a topological space** X is denoted $G_{K,L}(X)$. During round n, Player I (called K) chooses a compact set $K_n \in K[X]$, and Player II (called L) also chooses a compact set $L_n \in K[X]$. K must follow the rule that $K_{n+1} \supseteq K_n$, while P must follow the rule that $L_n \cap K_n = \emptyset$. The winning condition for K is that the collection of compact sets chosen by L, $\{L_n : n < \omega\}$, must be locally finite everywhere in the space.

An interesting property of the game $G_{K,L}(X)$ is the following result, proven by Gruenhage in his paper *Games Covering Properties and Eberlein Compacts.*

Theorem 5. *The following are equivalent for a locally compact space X:*

- *X is paracompact*
- *$K \uparrow G_{K,L}(X)$.*

However, often it is the presence of "limited information" strategies which can characterize interesting properties of a space.

Definition 6. A **limited information strategy** for a game is a function whose domain is restricted to less information than all previous moves by the opposing player.

In the above mentioned paper, Gruenhage used the following limited information strategies to prove some interesting characterizations based on the game $G_{K,P}(X)$.

Definition 7. A **tactical strategy** considers only the most recent move by the opposing player. If Player Z has a winning tactical strategy for a game G, this may be denoted $Z \uparrow_{\text{tact}} G$.

Definition 8. A **Markov strategy** considers only the most recent move by the opposing player and the turn number. If Player Z has a winning Markov strategy for a game G, this may be denoted $Z \uparrow_{\text{mark}} G$.

Theorem 9. *The following are equivalent for a locally compact space X:*

- *X is metacompact*
- *$K \uparrow_{tact} G_{K,P}(X)$.*

Theorem 10. *The following are equivalent for a locally compact space X:*

- *X is σ-metacompact*
- *$K \uparrow_{mark} G_{K,P}(X)$.*

Upon learning these results, one might wonder the consequences of the existence of this type of limited information strategy:

Definition 11. A **predetermined strategy** considers only the turn number. If Player Z has a winning predetermined strategy for a game G, this may be denoted $Z \uparrow_{\text{pre}} G$.

Intuitively, if a player is using a predetermined strategy, then that player decides every move he or she will make before the game even begins, ignoring the other player's moves.

Consider the following trivial result:

Definition 12. A **button-mashing strategy** is a constant function. If Player Z has a winning button-mashing strategy for a game G, this may be denoted $Z \uparrow_{\text{mash}} G$.

Proposition 13. *The following are equivalent for any space X:*

- *X is compact*
- *$K \uparrow_{mash} G_{K,P}(X)$.*

Observing that giving K the added information of turn number to a tactical strategy (making it Markov) changed the characterization of a metacompact space into a σ-metacompact space, it would be very convenient if adding that same information to a button-mashing strategy (making it predetermined) would similarly change the characterization of a compact space into σ-compact.

Proposition 14. *If $K \uparrow_{pre} G_{K,P}(X)$, then X is σ-compact.*

Proof. Let K_n be the sets given by the winning predetermined strategy. If they did not union to X, then the counter play $p_n = p$ for some $p \in X \setminus \bigcup_n K_n$ would defeat the "winning" strategy. $\square$

Theorem 15. *If Y is a locally compact, Lindelöf space, then $K \uparrow_{pre} G_{K,P}(X)$.*

Proof. Let $\mathcal{K}$ be a collection of compact neighborhoods whose interiors cover X. By Lindelöf, let $\{K_n : n < \omega\}$ be a countable subcollection whose interiors cover X. We then define the predetermined strategy $\sigma(n) = \bigcup_{m \leq n} K_n$.

Let p_n give a play by P. If p is a cluster point of the p_n, then every open set about p contains infinitely many p_n. Let K_N be some compact neighborhood in $\{K_n : n < \omega\}$ which covers p. Then K_N contains infinitely many p_n, which means sometime after round N, P played in a set already covered by the strategy σ, which is an illegal move. Thus σ is a winning predetermined strategy. □

Corollary 16. *The following are equivalent for a locally compact space X:*

- *X is σ-compact*
- *X is Lindelöf*
- *$K \uparrow_{pre} G_{K,P}(X)$.*

We now turn our attention to an example of a σ-compact space for which no predetermined strategy exists (which must, of course, not be locally compact). In fact, P will instead have a winning tactical strategy.

Definition 17. Let $M = \omega^2 \cup \{\infty\}$ denote the **metric fan space** with the topology generated by the singletons in ω^2 and sets of the form $((\omega \setminus n) \times \omega) \cup \{\infty\}$ for $n < \omega$.

Proposition 18. *For each compact set C in M, there exists a minimal dominating function f_C such that for each $(x, y) \in C \setminus \{\infty\}$, $f(x) > y$.*

Lemma 19. *$P \uparrow_{mark} G_{K,P}(M)$ where M is the metric fan space. (This implies $K \not\uparrow G_{K,P}(M)$.)*

Proof. Let P respond to the move $C \in K[X]$ by K on round n with the point $p = (n, s_C)$ such that $s_C = \min(\{y < \omega : f_C(n) < y\}$. It is easy to see that either $p_n \to \infty$, so P has a winning tactical strategy. □

Furthermore, by a theorem due to Eric van Douwen...

Theorem 20. *Every first-countable non-locally countably compact space has the metric fan space M as a closed subspace.*

... we have the following corollary:

Corollary 21. *$P \uparrow_{mark} G_{K,P}(X)$ where X is a first-countable non-locally countably compact space. (This implies $K \not\uparrow G_{K,P}(X)$.)*

(Open question: does $P \uparrow_{\text{tact}} G_{K,P}(X)$?)

Theorem 22. *$P \not\uparrow_{tact} D_{K,P}(M)$ where M is the metric fan space.*

Proof. Give P the tactic σ. Suppose that for all $n < \omega$, there is an upper bound $m < \omega$ so that for each $C \in K[M]$, if $\pi_1(\sigma(C)) = n$, then $\pi_2(\sigma(C)) < m$. We may then define $f(n) = m$, and let $C_f = \{(x, y) : f(x) < y\} \in K[M]$. $\sigma(C_f)$ must show a contradiction if σ is legal.

So it follows that there is some $n < \omega$ such that there are compact sets $C_i \in K[M]$ with $\pi_1(\sigma(C_i)) = n$ and $\pi_2(\sigma(C_i)) < \pi_2(\sigma(C_{i+1}))$. The play $\langle C_0, \sigma(C_0), C_1, \sigma(C_1), \dots \rangle$ is a counter to σ. $\square$

While $K \uparrow_{\text{pre}} G_{K,P}(X)$ implies X is σ-compact, it in fact implies something stronger.

Definition 23. A space X is **hemicompact** if there exists a chain of increasing compact sets $K_0 \subseteq K_1 \subseteq \dots$ such that every compact set in X is a subset of some K_n.

Lemma 24. *If $K \uparrow_{pre} G_{K,P}(X)$, then X is hemicompact. Furthermore, any predetermined winning strategy for K witnesses hemicompactness.*

Proof. Let σ be a predetermined strategy for K in the game $G_{K,P}(Y)$ such that there exists a compact set C with $C \not\subseteq \sigma(n)$ for all n. On each turn, have P play some $y_n \in C \setminus \sigma(n)$. Then the y_n are an infinite subset of the compact set C and must have a cluster point in C, showing σ is not a winning strategy.

Thus if K has a winning predetermined strategy, it witnesses that Y is hemicompact. $\square$

In fact, for locally compact spaces, finding winning predetermined strategies for $G_{K,P}(X)$ and $G_{K,L}(X)$ are equivalent problems.

Theorem 25. *The following are equivalent for any locally compact space X:*

- *X is hemicompact.*
- *$K \uparrow_{pre} G_{K,L}(X)$.*
- *$K \uparrow_{pre} G_{K,P}(X)$.*

Proof. Let Y be hemicompact, witnessed by $K_n = \sigma(n)$. Let $L_0, L_1, \dots$ be a play by L in $G_{K,L}(X)$. Suppose that this play defeats σ. Then let $x \in X$ be the point such that for all neighborhoods U of x, U hits infinite L_n. Let C be a compact neighborhood of x, which must hit infinite L_n. As K_n witnesses hemicompactness, $C \subseteq K_N = \sigma(N)$ for some N. But then $C \subset K_N$ intersects infinitely many L_n, which shows that the play $L_0, L_1, \dots$ was illegal. Thus σ defeats every legal play by L and is thus a winning predetermined strategy for K in $G_{K,L}(X)$.

We conclude by noting that any winning strategy for $G_{K,L}(X)$ is a winning strategy for $G_{K,P}(X)$, and the existence of a winning predetermined strategy for $G_{K,P}(X)$ implies hemicompact by the previous lemma. $\square$

Corollary 26. *The following are equivalent for any locally compact space X:*

- *X is Lindelöf.*
- *X is σ-compact.*
- *X is hemicompact.*
- *$K \uparrow_{pre} G_{K,L}(X)$.*
- *$K \uparrow_{pre} G_{K,P}(X)$.*

The compact-point and compact-compact games are also useful in inspecting compactly generated "k"-spaces.

Definition 27. A topological space is called a k**-space** if the following condition is satisfied:

$$C \subseteq X \text{ is closed in } X \Leftrightarrow C \cap K \text{ is closed in } K \text{ for all compact sets } K \in K[X]$$

Definition 28. A topological space is called a k_ω**-space** if there exist $K_0, K_1, \cdots \in K[X]$ that satisfy the following condition:

$$C \subseteq X \text{ is closed in } X \Leftrightarrow C \cap K_n \text{ is closed in } K_n \text{ for all } n$$

Theorem 29. *The following are equivalent for any Hausdorff k-space X:*

- *X is hemicompact.*
- *X is k_ω.*
- *$K \uparrow_{pre} G_{K,P}(X)$.*

Furthermore, all predetermined strategies for K witness hemicompact and k_ω, and any witness to hemicompact/k_ω witnesses the other and serves as a predetermined strategy for K.

Proof. If X is hemicompact, then let it be witnessed by K_n. We claim K_n also witnesses k_ω. Note that the forward implication of k_ω always holds for T_1 spaces as $C \cap K_n$ is closed in X, and thus in every K_n. So assume $C \cap K_n$ is closed in K_n for all n. Let H be any compact set. As X is hemicompact, $H \subseteq K_n$ for some n. Note $C \cap H = (C \cap K_n) \cap H$. As both $C \cap K_n$ and H are closed in K_n, $C \cap H$ is closed in K_n, and thus $C \cap H$ is closed in H. As Y is k and $C \cap H$ is closed in H for all compact H, C is closed, showing the backwards implication.

Now if Y is k_ω, let it be witnessed by K_n. Give K the predeterined strategy $\sigma(n) = K_n$ for the game $G_{K,P}(X)$, and let p_n be the result of a legal counter by P. Suppose by way of contradiction that p is a cluster point of the p_n. Note $p \in \sigma(N)$ for some N. p is a cluster point of $\{p_n : n \geq N\}$ but $p \notin \{p_n : n \geq N\}$. Also, $\{p_n : n \geq N\} \cap \sigma(m)$ is finite for all m, and thus closed, so as $\sigma(n)$ witnesses k_ω, $\{p_n : n \geq N\}$ is closed and must contain its cluster point p, which is a contradiction. Thus σ is a winning predetermined strategy for K in $G_{K,P}(Y)$.

Finally, if $K \uparrow_{\text{pre}} G_{K,P}(X)$, X is hemicompact by the previous lemma. □

For k-spaces, it turns out that finding winning predetermined strategies for $G_{K,P}(X)$ and $G_{K,L}(X)$ are also equivalent problems.

Theorem 30. *For any hemicompact Hausdorff k-space X, $K \uparrow_{pre} G_{K,L}(X)$.*

Proof. Let X's hemicompactness be witnessed by $K_n = \sigma(n)$. Note that this also witnesses k_ω by the proof of the previous theorem. Let $H_0, H_1, \ldots$ be a counter by H for the game $G_{K,L}(X)$ in response to σ. Suppose by way of contradiction the counter was legal and defeats σ. Then there is a point x such that every neighborhood of x hits infinitely many of the H_n.

Now, $x \in \sigma(N)$ for some N, and since the play $H_0, H_1, \ldots$ is legal, $x \notin H_n$ for all $n \geq N$. Consider the set $H_\omega = \bigcup_{n \geq N} H_n$. Note that as the K_n witness k_ω, H_ω is closed if and only if $H_\omega \cap \sigma(m)$ is closed in $\sigma(m)$ for all m. In fact, since every H_n is a subset of some $\sigma(m)$ (by hemicompactness), $H_\omega \cap \sigma(m)$ is a finite union of some H_n, and is thus closed in Y.

We thus have that H_ω is a closed set not containing x. But since every neighborhood of x intersects H_ω, x is a limit point of the closed set H_ω and should be included, demonstrating our contradiction. Thus σ is a winning predetermined strategy for K in the game $G_{K,L}(X)$. □

Corollary 31. *The following are equivalent for any Hausdorff k-space X:*

- *X is hemicompact.*
- *X is k_ω.*
- *K has a winning predetermined strategy in $G_{K,L}(X)$.*
- *K has a winning predetermined strategy in $G_{K,P}(X)$.*

It's natural to question whethere there is ever any difference between finding winning predetermined strategies for $G_{K,P}(X)$ and $G_{K,L}(X)$. We now look to a (non-locally compact, non-k) Hausdorff space where the distinction arises:

Definition 32. Given a set X, an ultrafilter on X is a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ such that

1. $\emptyset \notin \mathcal{F}$

2. $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

3. $A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$

4. $\forall A \subseteq X (A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F})$

As a result, ultrafilters which contain a finite set contain only one singleton (and are called **principal**). Otherwise, ultrafilters which contain no finite sets are called **free**.

Definition 33. The **Stone-Cech compactification** $\beta\omega$ of ω is the collection of ultrafilters on ω. The principal ultrafilters containing a singleton $\{n\}$ are each identified with n itself and are isolated. Free ultrafilters $\mathcal{F}$ are given neighborhoods of the form

$$\{\mathcal{G} : \mathcal{G} \text{ is an ultrafilter on } \omega \text{ and } A \in \mathcal{G}\} = A \cup \{\mathcal{G} : \mathcal{G} \text{ is a free ultrafilter on } \omega \text{ and } A \in \mathcal{G}\}$$

for each $A \in \mathcal{F}$.

Alternately $\beta\omega = \omega \cup \{\mathcal{F} : \mathcal{F}$ is a free ultrafilter on $\omega\}$ where ω is discrete and the free ultrafilters have the local base described above.

Definition 34. A **single-ultrafilter space** is a subset of $\beta\omega$ containing all elements of ω and a single ultrafilter $\mathcal{F}$.

Proposition 35. *The compact sets of a single-ultrafilter space are exactly the finite subsets of the space. Thus a single-ultrafilter space is neither locally compact nor k.*

Regardless of the ultrafilter chosen, we can see that K has no hope of having a winning predetermined strategy for $G_{K,L}$ played on a single-ultrafilter space.

Proposition 36. *If X is any single-ultrafilter space with the ultrafilter $\mathcal{F}$, then $K \not\uparrow_{pre} G_{K,L}(X)$.*

Proof. Compact sets are exactly finite sets in this space. Therefore, the difference of any two compact sets is compact.

Give K the predetermined strategy $\sigma(n)$. H counters with

$$H_n = (n \cup \sigma(n+1)) \setminus \sigma(n)$$

on turn n. Since any free ultrafilter contains only unbounded sets, every neighborhood $A \cup \{\mathcal{F}\}$ of $\mathcal{F}$ must intersect infinitely many H_n, defeating σ. □

However, while it is consistant that there is an ultrafilter which defies the existance of a predetermined winning strategy for K in $G_{K,P}$...

Proposition 37. *If a selective ultrafilter $\mathcal{F}$ exists (this is independent of ZFC), then K has no winning predetermined strategy in the compact-point game $G_{K,P}(Y)$ for the single selective ultrafilter space $Y = \omega \cup \{\mathcal{F}\}$.*

Proof. Let σ be a predetermined strategy for K. By the definition of a selective ultrafilter, for every partition $\{B_n : n < \omega\}$ of subsets of ω such that $B_n \notin \mathcal{F}$ for all n, there exists $A \in \mathcal{F}$ such that $|A \cap B_n| = 1$ for all n. So then let

$$B_n = \omega \cap \sigma(n) \setminus \sigma(n-1)$$

Note that B_n is finite and thus $B_n \notin \mathcal{F}$, so there exists $A \in \mathcal{F}$ such that $|A \cap B_n| = 1$. Let p_n be the singleton in $A \cap B_{n+1}$, so $\{p_n : n < \omega\}$ is cofinite in A, and thus is also a member of $\mathcal{F}$. Thus p_n converges to $\mathcal{F}$, and counters the strategy σ. □

... in general we can find many ultrafilters for which $K \uparrow_{\text{pre}} G_{K,P}$.

Theorem 38. *Let a_n be a sequence such that the sequence $\frac{a_n}{n}$ is unbounded above. Then there is an ultrafilter $\mathcal{F}$ such that $\sigma(n) = (\sum_{m\leq n} a_m) \cup \{\mathcal{F}\}$ is a winning predetermined strategy for K in $G_{K,P}(\omega \cup \{\mathcal{F}\})$.*

Proof. Let $\mathcal{P}$ be the collection of all legal plays by P against the strategy σ. Consider a finite collection of plays $P_0, \ldots, P_{n-1} \in \mathcal{P}$. As $\frac{a_m}{m}$ is unbounded above, we may find infinitely many m such that $\frac{a_m}{m} > n \Rightarrow mn < a_m$. As the a_m partition ω such that P may only play at most m points in each part, there are infinitely many parts which are not filled, and thus $\bigcup_{m<n} P_m$ is not cofinite.

It then follows that the closure of $\mathcal{P}$ under finite unions and subsets, along with all finite sets, is an ideal. Its dual filter may then be extended to an ultrafilter $\mathcal{F}$ such that every possible play by P is the complement of some member of $\mathcal{F}$. □

So we can see that there are non-k spaces X for which $K \uparrow_{\text{pre}} G_{K,P}(X)$. However, we have found no such spaces for the game $G_{K,L}(X)$. So we conclude with this open question:

Question 39. *$K \uparrow_{pre} G_{K,L}(X) \Rightarrow X$ is a k-space?*

Another game: $Con_{O,P}(X, x)$

Definition 40. Gruenhage's open-point convergence game $Con_{O,P}(X, x)$ has O choosing nested open sets and P choosing a point within the last chosen open set by O. O wins if the points chosen by P converge to x.

Definition 41. The one-point compactification of a space X is $X \cup \{\infty\}$, where neighborhoods of points in X are the same as they were originally, and neighborhoods of ∞ are sets $X \cup \{\infty\} \setminus K$ for compact K. If X is discrete then neighborhoods of ∞ are cofinite sets containing ∞.

Proposition 42. *$O \not\uparrow_{pre} Con_{O,P}(\kappa \cup \{\infty\}, \infty)$, where $\kappa \cup \{\infty\}$ is the one-point compactification of discrete $\kappa \geq \omega_1$.*

Proof. Given a predetermined strategy $\sigma(n)$ for O, P simply chooses any ordinal in $\bigcap_{n<\omega}\sigma(n)$ to play on every turn. $\square$

Definition 43. A **coding strategy** considers only the most recent move by each player. If Player Z has a winning coding strategy for a game G, this may be denoted $Z \uparrow_{\text{code}} G$.

Proposition 44. *$O \uparrow_{code} Con_{O,P}(\kappa \cup \{\infty\}, \infty)$ for any cardinal κ.*

Proof. Define $\sigma(U, p) = U \setminus \{p\}$. A legal play by P must never repeat the same point, so legal plays by P converge to ∞. $\square$

Proposition 45. *$O \not\uparrow_{tact} Con_{O,P}(\kappa \cup \{\infty\}, \infty)$ for $\kappa \geq \omega_1$.*

Proof. Let $F(\alpha)$ be the complement of a tactical strategy for O. There exists some $n < \omega$ such that we may find an infinite $B \subseteq \omega_1$ with $n \notin F(\beta)$ for $\beta \in B$ - if not, we may find uncountable ordinals α such that $\omega \subseteq F(\alpha)$, a contradiction. Choose $\beta \in B \setminus F(n)$, and note that repeating n, β is a successful counter to O's strategy. $\square$

Definition 46. A **k-tactical strategy** considers only the last k moves by the opposing player. If Player Z has a winning k-tactical strategy for a game G, this may be denoted $Z \uparrow_{k\text{-tact}} G$.

Theorem 47. *$O \not\uparrow_{k\text{-}tact} Con_{O,P}(\kappa \cup \{\infty\}, \infty)$ for $\kappa \geq \omega_1$.*

Proof. Let $\sigma : [\omega_1]^{\leq k} \to [\omega_1]^{<\omega}$ be a k-tactical strategy for O restricted to ω_1 and $F(S) = \omega_1 \setminus \sigma(S)$.

Let $W_0 = \omega_1$. We define W_α recursively as follows:

- For successor ordinals $\alpha + 1$, let β be the least element of W_α such that $[\beta + 1, \omega_1) \cap \bigcup_{S \leq \beta} F(S)$ is nonempty, where $S \leq \beta$ is shorthand for $\{S \in [\omega_1]^{\leq k} : \forall \gamma \in S(\gamma < \beta)\}$. If no such β exists, let $W_{\alpha+1} = W_\alpha$ and otherwise let $W_{\alpha+1} = W_\alpha \setminus \left([\beta + 1, \omega_1) \cap \bigcup_{S \leq \beta} F(S)\right)$.
- For limit ordinals α, let $W_\alpha = \bigcap_{\beta<\alpha} W_\beta$.

Finally let $W = \bigcap_{\alpha<\omega_1} W_\alpha$ and observe that it is unbounded. Let R collect all ordinals $\alpha \in W$ such that there is an ordinal β where for all $S \in [W \cap (\beta, \omega_1)]^{\leq k}$, $\alpha \in F(S)$. It is easily seen that R is finite. Let 0^* be the least element of $W \setminus R$.

Now, define a strictly increasing sequence of ordinals $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots \rangle$ such that $\alpha_i \in W$ and $0^* \notin F(\{\alpha_{2i+1}, \ldots, \alpha_{2i+k}\})$ for all i. The play $\langle 0^*, \alpha_1, \ldots, \alpha_k, 0^*, \alpha_{k+1}, \ldots, \alpha_{2k}, 0^*, \ldots \rangle$ then defeats the strategy σ. $\square$

Peter J. Nyikos has shown the following:

Theorem 48. *$O \not\uparrow_{mark} Con_{O,P}(\omega_1 \cup \{\infty\}, \infty)$.*

This can be improved:

Theorem 49. *$O \not\uparrow_{k\text{-}mark} Con_{O,P}(\kappa \cup \{\infty\}, \infty)$ for $\kappa \geq \omega_1$.*

Proof. Let $F : [\omega_1]^{\leq k} \times \omega \to [\omega_1]^{<\omega}$ be the complement of a k-Markov strategy for O restricted to ω_1.

Let $W_0 = \omega_1$. We define W_α recursively as follows:

- For successor ordinals $\alpha + 1$, let β be the least element of W_α such that

$$(\beta, \omega_1) \cap \bigcup_{S \in [\beta]^{\leq k}, n < \omega} F(S, n)$$

  is nonempty. If no such β exists, let $W_{\alpha+1} = W_\alpha$ and otherwise let

$$W_{\alpha+1} = W_\alpha \setminus \left((\beta, \omega_1) \cap \bigcup_{S \in [\beta]^{\leq k}, n < \omega} F(S, n) \right)$$

- For limit ordinals α, let $W_\alpha = \bigcap_{\beta < \alpha} W_\beta$.

Finally let $W = \bigcap_{\alpha < \omega_1} W_\alpha$ and observe that it is unbounded. Let R collect all ordinals $\alpha \in W$ such that there is an ordinal β_α and number $n_\alpha < \omega$ where for all $S \in [W \cap (\beta_\alpha, \omega_1)]^{\leq k}$, $\alpha \in F(S, n_\alpha)$.

If R was infinite, then there is $\alpha_i \in R$ for each $i < \omega$ and some $N < \omega$ where for all $S \in [W \cap (\beta_{\alpha_i}, \omega_1)]^{\leq k}$, $\alpha_i \in F(S, N)$, and thus for all $S \in [W \cap (\sup \beta_{\alpha_i}, \omega_1)]^{\leq k}$, $\{\alpha_i : i < \omega\} \subset F(S, N)$, a contradiction. So let 0^* be the least element of $W \setminus R$.

Now, define a strictly increasing sequence of ordinals $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots \rangle$ such that $\alpha_i \in W$ and $0^* \notin F(\{\alpha_{2i+1}, \dots, \alpha_{2i+k}\})$ for all i. The play $\langle 0^*, \alpha_1, \dots, \alpha_k, 0^*, \alpha_{k+1}, \dots, \alpha_{2k}, 0^*, \dots \rangle$ then defeats the Markov strategy. $\square$

While these counters prevent convergence to ∞, they do not prohibit clustering, as any sequence of points with infinite range clusters at ∞ in this space.

Definition 50. The open-point clustering game $Clus_{O,P}(X, x)$ has O choosing nested open sets and P choosing a point within the last chosen open set by O. O wins if the points chosen by P cluster to x.

Any strategy which forces convergence also forces clustering, so a coding strategy exists for player O. And since the counter used above to foil any tactical strategy had only finite range...

Proposition 51. *$O \not\uparrow_{tact} Clus_{O,P}(X, \infty)$.*

The other counters only attempt to repeat a point infinitely often, and may have infinite range.

Example 52. *For $\omega \leq \alpha < \omega_1$ let $\leq_\alpha$ be a well-ordering of ω_1 such that its restriction to α is isomorphic to ω with isomorphism $f(n) : \omega \to \alpha$. For $n < \omega$ let $\leq_n$ simply be $\leq$ and f be the identity.*

The 2-tactical strategy for O in $Clus_{O,P}(\omega_1 \cup \{\infty\}), \infty)$ whose complement is, for $\alpha < \beta$, $F(\alpha, \beta) = \{\gamma : \gamma \leq_\beta \alpha\} \cup \{\beta\}$, forces the point-picker to chose more than three distinct points.

Proof. Suppose Player P started with $\alpha_0 < \alpha_1$. Player O forbids

$$F(\alpha_0, \alpha_1) = \{\gamma : \gamma \leq_{\alpha_1} \alpha_0\} \cup \{\alpha_1\}$$

so Player P must respond with α_2 such that either $\alpha_0 < \alpha_1 < \alpha_2$ or $\alpha_0 <_{\alpha_1} \alpha_2 < \alpha_1$.

In the first case that $\alpha_0 < \alpha_1 < \alpha_2$, Player O now forbids

$$F(\alpha_1, \alpha_2) = \{\gamma : \gamma \leq_{\alpha_2} \alpha_1\} \cup \{\alpha_2\}$$

If $\alpha_0 <_{\alpha_2} \alpha_1$ we're done. Otherwise, Player P chooses $\alpha_0 >_{\alpha_2} \alpha_1$ again, and Player O responds with

$$F(\alpha_0, \alpha_2) = \{\gamma : \gamma \leq_{\alpha_2} \alpha_0\} \cup \{\alpha_2\}$$

which successfully forbids α_1.

In the second case that $\alpha_0 <_{\alpha_1} \alpha_2 < \alpha_1$, Player O forbids

$$F(\alpha_1, \alpha_2) = \{\gamma : \gamma \leq_{\alpha_1} \alpha_2\} \cup \{\alpha_1\}$$

which successfully forbids α_0.

Thus, a fourth ordinal must be selected. □

But the fourth ordinal is all that is needed.

Theorem 53. *$O \not\uparrow_{2\text{-}tact} Clus_{O,P}(\omega_1 \cup \{\infty\}, \infty)$.*

Proof. Let $F(\alpha, \beta)$ be the complement of a k-tactical strategy for O, and assume without loss of generality that it does not forbid any ordinals larger than its inputs.

Let $n_0 = \min(\omega \setminus F(\emptyset) \setminus F(\omega, \omega + 1))$ and $n_1 = \min(\omega \setminus F(n_0) \setminus F(n_0, \omega + 1))$.

It follows that $\langle n_0, n_1, \omega, \omega + 1, n_0, n_1, \omega, \omega + 1, \dots \rangle$ is a legal counter. □

But, in fact, a similar counter can be used for any k-tactical strategy.

Theorem 54. *$O \not\uparrow_{k\text{-}tact} Clus_{O,P}(\omega_1 \cup \{\infty\}, \infty)$.*

Proof. Let $F(\alpha_0, \ldots, \alpha_{k-1})$ be the complement of a k-tactical strategy for O, and assume without loss of generality that it does not forbid any ordinals larger than its inputs.

For $0 \leq i < k$, we define

$$n_i = \min(\omega \setminus F(n_0, \ldots, n_{i-1}, \omega + i, \ldots, \omega + k - 1) \setminus F(n_0, \ldots, n_{i-1}))$$

It follows that $\langle n_0, \ldots, n_{k-1}, \omega, \ldots, \omega + k - 1, n_0, \ldots, n_{k-1}, \omega, \ldots, \omega + k - 1, \ldots \rangle$ is a legal counter. □

However, a Markov strategy is sufficient to defeat P in the clustering game for ω_1.

Theorem 55. *$O \uparrow_{mark} Clus_{O,P}(\omega_1 \cup \{\infty\}, \infty)$.*

Proof. For $\omega \leq \alpha < \omega_1$ let $\leq_\alpha$ be a well-ordering of ω_1 such that its restriction to α is isomorphic to ω with isomorphism $f(n) : \omega \to \alpha$. For $n < \omega$ let $\leq_n$ simply be $\leq$ and f be the identity.

Let $F(\alpha, n) = \{\gamma : \gamma \leq_\alpha f(n)\}$ be the complement of O's Markov strategy in response to move α in round n.

If a legal counter by P does not repeat some largest ordinal infinitely often, it does not result in a win for P. Otherwise let β be largest ordinal repeated infinitely often in some legal counter by P. We note that the forbidden sets responded by O are of the form $F(\beta, n)$ where n increases as the game progresses. It is impossible to repeat any ordinal less than β infinitely often due to the increasing round number, so the range of the counter is infinite, and does not result in a win for P. □

...while it similarly fails for $\kappa > \omega_1$.

Theorem 56. *$O \not\uparrow_{k\text{-}mark} Clus_{O,P}(\kappa \cup \{\infty\}, \infty)$ for $\kappa > \omega_1$.*

Proof. Let $F : \omega \times [\kappa]^{\leq k} \to [\kappa]^{\omega}$ be the complement of a k-Markov strategy for O, and assume without loss of generality that it does not forbid any ordinals larger than its inputs.

For $0 \leq i < k$, we define

$$\beta_i = \min(\omega \setminus \bigcup_{n<\omega} F(n, \{\beta_0, \ldots, \beta_{i-1}, \omega_1 + i, \ldots, \omega_1 + k - 1\}) \setminus \bigcup_{n<\omega} F(n, \{\beta_0, \ldots, \beta_{i-1}\}))$$

It follows that $\langle \beta_0, \ldots, \beta_{k-1}, \omega_1, \ldots, \omega_1 + k - 1, \beta_0, \ldots, \beta_{k-1}, \omega_1, \ldots, \omega_1 + k - 1, \ldots \rangle$ is a legal counter. □

$Con_{O,P}(X,x)$ for Sigma Product

Proposition 57. *The player O has a winning strategy for $Con_{O,P}(X,x)$ where*

$$X = \Sigma\mathbb{R}^{\omega_1} = \{x \in \mathbb{R}^{\omega_1} : x_\alpha = 0 \text{ for all but countably many } \alpha < \omega_1\}$$

Proof. Without loss of generality we may assume that x is the zero function. Let y_n be the move by P on turn n. For each y_n, let $l_n \in \omega_1^\omega$ enumerate the nonzero coordinates in y_n. Give O the strategy $\sigma(y_0, \ldots, y_{n-1}) = \prod_{\alpha<\omega_1} O_\alpha$ where if α is listed in the first n coordinates of any term in $\langle l_k \rangle_{k<n}$, then $O_\alpha = (-2^{-n}, 2^{-n})$, and $O_\alpha = \mathbb{R}$ otherwise. □

Theorem 58. *For all cardinals $\kappa \le 2^\omega$, the player O has a winning coding strategy for $Con_{O,P}(X,x)$ where*

$$X = \Sigma\mathbb{R}^{\kappa} = \{x \in \mathbb{R}^{\kappa} : x_\alpha = 0 \text{ for all but countably many } \alpha < \kappa\}$$

Proof. Note that $|\Sigma\mathbb{R}^\kappa| \le 2^\omega = |\mathbb{R}|$. Define the following:

- Encode every finite sequence s of functions in $\Sigma\mathbb{R}^\kappa$ as a real number $0 < r(s) < 1$.
- Let $\gamma(U)$ be the function which, for basic open sets $U = \prod_{\alpha<\kappa} U_\alpha$ where $U_0 = (-\frac{1}{R}, \frac{1}{R})$ for some positive noninteger real number R, returns the sequence $r^{-1}(R - \lfloor R \rfloor)$.
- Let $n(U)$ be the length of $\epsilon^{-1}(\gamma(U))$.
- Let $\psi(U,y) = \epsilon^{-1}(\gamma(U))^\frown \langle y \rangle$.
- Define $\sigma_0(U,y)$ to be $\left(-\frac{1}{n(U)+r(\psi(U,y))}, \frac{1}{n(U)+r(\psi(U,y))}\right)$.
- For each $0 < \alpha < \kappa$, define the interval $\sigma_\alpha(U,y)$ about 0 as follows:
  - If α is listed in the first $n(U)$ coordinates of any term in the finite sequence $\psi(U,y)$, then $\sigma_\alpha(U,y) = (-\frac{1}{n(U)+1}, \frac{1}{n(U)+1})$.
  - Otherwise, then $\sigma_\alpha(U,y) = \mathbb{R}$.

It follows that $\sigma(U,y) = \prod_{\alpha<\kappa} \sigma_\alpha(U,y)$ is a winning coding strategy. □

Theorem 59. *For all cardinals $\kappa > 2^\omega$ such that $\kappa^\omega = \kappa$, the player O has a winning coding strategy for $Con_{O,P}(X,x)$ where*

$$X = \Sigma\mathbb{R}^{\kappa} = \{x \in \mathbb{R}^{\kappa} : x_\alpha = 0 \text{ for all but countably many } \alpha < \omega_1\}$$

Proof. Note that $|\Sigma\mathbb{R}^\kappa| = \kappa^\omega = \kappa$. Define the following:

- Encode every finite sequence s of functions in $\Sigma\mathbb{R}^\kappa$ as an ordinal $\epsilon(s) < \kappa$.
- Let $\gamma(U)$ be the function which, for basic open sets $U = \prod_{\alpha<\kappa} U_\alpha$ with a unique factor U_{α^*}, returns α^*.
- Let $n(U)$ be the length of $\epsilon^{-1}(\gamma(U))$.
- Let $\psi(U,y) = \epsilon^{-1}(\gamma(U))^\frown \langle y\rangle$.
- For each $\alpha < \kappa$, define the interval $\sigma_\alpha(U,y)$ about 0 as follows:
  - If α is listed in the first $n(U)$ coordinates of any term in the finite sequence $\psi(U,y)$ and is not equal to $\epsilon(\psi(U,y))$, then $\sigma_\alpha(U,y) = (-\frac{1}{n(U)+1}, \frac{1}{n(U)+1})$.
  - If $\alpha = \epsilon(\psi(U,y))$, then $\sigma_\alpha(U,y) = (-\frac{1}{n(U)+2}, \frac{1}{n(U)+2})$.
  - Otherwise, then $\sigma_\alpha(U,y) = \mathbb{R}$.

It follows that $\sigma(U,y) = \prod_{\alpha<\kappa} \sigma_\alpha(U,y)$ is a winning coding strategy. □

Finite Countable Game

Theorem 60. *Let κ be a cardinal such that there exists a function $f : \kappa \to [\kappa]^{\leq\omega}$ where for every $W \in [\kappa]^{\leq\omega}$ there exists $\alpha_W < \kappa$ with $W \subseteq f(\alpha_W)$. Then $F \uparrow_{code} PF_{F,C}(\kappa)$ and $O \uparrow_{code} Con_{O,P}(\Sigma\mathbb{R}^\kappa)$.*

Proof. Long version.

We begin by defining a few helpful tools. Assume $N \in [\kappa]^{<\omega}$ and $W \in [\kappa]^{\leq\omega}$.

- Let $N^* = N \setminus \{\max(N)\}$.
- Noting that $\kappa \setminus \alpha$ is order-isomorphic to κ for all $\alpha < \kappa$, let $\gamma_N < \kappa$ be the ordinal mapping to $\max(N)$ in $\kappa \setminus (\max(N^*)+1)$ by the appropriate order isomorphism.
- Let $s : [\kappa]^{<\omega} \to \kappa$ be a bijection.
- Define $W \upharpoonright n \in [\kappa]^{<\omega}$ so that $W \upharpoonright n \subseteq W \upharpoonright (n+1)$ and $\bigcup_{n<\omega} W \upharpoonright n = W$.
- Let $t_N = \max(N \cap \omega) + 2$.

Let the (non-winning) coding strategy σ' be defined as follows:

$$\sigma'(N,W) = \bigcup_{\alpha \in s^{-1}(\gamma_N)} f(\alpha) \upharpoonright t_N$$

Let $\langle N_0, N_1, \dots\rangle$ be a play by F following the considered strategy in response to the play $\langle W_0, W_1, \dots\rangle$ by C in the game $PF_{F,C}(\kappa)$. (For technical reasons, assume N_0 contains at least two ordinals.) This coding strategy could have been a winning strategy for F if it had only satisfied two requirements:

1. First, $s^{-1}(\gamma_{N_k})$ provided ordinals $\{\alpha_{W_0}, \dots, \alpha_{W_{k-1}}\}$. (Thus $N_{k+1} = \bigcup_{\alpha \in s^{-1}(\gamma_{N_k})} f(\alpha) \upharpoonright t_{N_k} = \bigcup_{0 \leq m < k} f(\alpha_{W_m}) \upharpoonright t_{N_k}$.)
2. In addition, $t_{N_k} \to \infty$. (Thus $\bigcup_{k<\omega} N_{k+1} = \bigcup_{k<\omega} \bigcup_{0 \leq m < k-1} f(\alpha_{W_m}) \upharpoonright t_{N_k} = \bigcup_{m<\omega} f(\alpha_{W_m}) \supseteq \bigcup_{m<\omega} W_m$.)

We shall improve σ' so that it does satsify those requirements, beginning with the second. Define σ^*:

$$\sigma^*(N, W) = t_N \cup \sigma'(N, W)$$

Note then that $t_{N_{k+1}} \geq t_{t_{N_k}} = \max(t_{N_k}) + 2 = t_{N_k} - 1 + 2 = t_{N_k} + 1$.

Lastly, to cover the first requirement, we define σ:

$$\sigma(N, W) = \sigma^*(N, W) \cup \{\max(\sigma^*(N, W)) + 1 + s(s^{-1}(\gamma_N) \cup \{\alpha_W\})\}$$

By induction

$$s^{-1}(\gamma_{N_{k+1}}) = s^{-1}(s(s^{-1}(\gamma_{N_k}) \cup \{\alpha_{W_k}\})) = s^{-1}(\gamma_{N_k}) \cup \{\alpha_{W_k}\} = \{\alpha_{W_0}, \dots, \alpha_{W_{k-1}}\} \cup \{\alpha_{W_k}\}$$

we see the first requirement is satisfied.

Finally, we turn our attention to $Con_{O,P}(\Sigma\mathbb{R}^\kappa)$. We define the winning strategy $\tau(U, p)$ for O as follows: let $N(U)$ be the non-$\mathbb{R}$ coordinates in the basic open set U and $W(p)$ be the non-0 coordinates in p. Then $\tau(U, p) = \left(\prod_{\alpha<\kappa} U_\alpha\right) \cap \Sigma\mathbb{R}^\kappa$ where if $\alpha \in \sigma(N(U), W(p))$ then $U_\alpha = (-1/t_{N(U)}, 1/t_{N(U)})$ and $U_\alpha = \mathbb{R}$ otherwise. $\square$

Proof. Short version, per GG's suggestion.

Let $W \upharpoonright n \in [\kappa]^n$ be a subset of $W \in [\kappa]^\omega$ such that $W \upharpoonright n \subset W \upharpoonright (n+1)$ and $\bigcup_{n<\omega} W \upharpoonright n = W$.

Define

$$\sigma(N, W) = N \cup (|N| + 1) \cup \{\alpha_W\} \cup \bigcup_{\alpha \in N} f(\alpha) \upharpoonright |N|$$

Consider the play $\langle \emptyset, W_0, N_1, W_1, N_2, W_2, \dots \rangle$ with F following the strategy σ. Let $\gamma \in W_i$, and note $\gamma \in f(\alpha_{W_i})$ (and $\gamma \in f(\alpha_{W_i}) \upharpoonright |N_n|$ for sufficiently large n).

$$N_{i+1} = \sigma(N_i, W_i) \supseteq \{\alpha_{W_i}\}$$

and thus

$$N_{n+1} = \sigma(N_n, W_n) \supseteq \bigcup_{\alpha \in N_n} f(\alpha) \upharpoonright |N_n| \supseteq \bigcup_{\alpha \in N_{i+1}} f(\alpha) \upharpoonright |N_n| \supseteq f(\alpha_{W_i}) \upharpoonright |N_n|$$

showing $\gamma \in N_{n+1}$. Since γ is forbidden in round $n+1$, γ appears in finitely many sets chosen by C.

We turn our attention to $Con_{O,P}(\Sigma\mathbb{R}^\kappa)$. We define the winning strategy $\tau(U,p)$ for O as follows: let $N(U)$ be the non-$\mathbb{R}$ coordinates in the basic open set U and $W(p)$ be the non-0 coordinates in p. Then $\tau(U,p) = \left(\prod_{\alpha<\kappa} U_\alpha\right) \cap \Sigma\mathbb{R}^\kappa$ where if $\alpha \in \sigma(N(U), W(p))$ then $U_\alpha = (-\frac{1}{|N(U)|}, \frac{1}{|N(U)|})$ and $U_\alpha = \mathbb{R}$ otherwise.

Consider the play $\langle \emptyset, p_0, U_1, p_1, U_2, p_2, \dots \rangle$ with O following the strategy τ. Observe that $N(\tau(U,p)) = \sigma(N(U), W(p))$. Thus $p_i(\gamma) \neq 0$ is equivalent to $\gamma \in W(p_i)$, and by the above argument, for sufficiently large n, $\gamma \in \sigma(N(U_n), W(p_n))$. Therefore from round n onward the γ-coordinates of points chosen by P must lay in $(-\frac{1}{|N(U)|}, \frac{1}{|N(U)|})$ and converge to 0. $\square$