Remark 1. Scheeper's $S(\kappa)$ requiring injections is stronger than my $S'(\kappa)$ requiring finite-to-one maps. Dow suggests that $S'(\omega_\omega)$ holds in ZFC by the following.

Definition 2. A topological space is said to be ω-bounded if each countable subset of the space has compact closure.

Theorem 3. *For each $k < \omega$ there exists a topology on ω_k which is ω-bounded and locally countable.*

Proof. Assume we've defined an ω-bounded topology for ω_k such that for each $\gamma \in \omega_k$ there exists a decreasing countable base $\{W_{\gamma,n} : n < \omega\}$ such that each $W_{\gamma,n}$ is countable, compact, and contains no ordinals greater than γ. Note that the usual linear order on ω_1 satisfies this requirement.

Let $\alpha < \omega_{k+1}$, and suppose we've defined compatible topologies satisfying the induction hypothesis on $\omega_k \cdot (\beta+1)$ for all $0 \leq \beta < \alpha$. If $\alpha = \beta+1$, then let $\omega_k \cdot (\alpha+1) = \omega_k \cdot (\beta+2)$ be the topological sum of the previously defined $\omega_k \cdot (\beta+1)$ and the previously defined $\omega_k \cdot (\beta+2) \setminus \omega_k \cdot (\beta+1) \cong \omega_k$. Similarly, if $cf(\alpha) > \omega$, then let $\omega_k \cdot (\alpha+1)$ be the topological sum of $\bigcup_{\beta<\alpha} \omega_k \cdot (\beta+1)$ and the previously defined $\omega_k \cdot (\alpha+1) \setminus \omega_k \cdot \alpha \cong \omega_k$. In either case, $\omega_k \cdot (\alpha+1) \setminus \omega_k \cdot \alpha$ is a clopen copy of ω_k under the homeomorphism $\gamma \mapsto \omega_k \cdot \alpha + \gamma$. Then for each $\gamma \in \omega_k$ we may define the witnessing decreasing countable base $\{W_{\omega_k \cdot \alpha+\gamma,n} : n < \omega\}$ for $\omega_k \cdot \alpha + \gamma$ by $W_{\omega_k \cdot \alpha+\gamma,n} = \{\omega_k \cdot \alpha + \delta : \delta \in W_{\gamma,n}\}$. Note that for $cf(\alpha) > \omega$, $\bigcup_{\beta<\alpha} \omega_k \cdot (\beta+1)$ is ω-bounded since any countable set is contained in some $\omega_k \cdot (\beta+1)$; thus in either case $\omega_k \cdot (\alpha+1)$ is the topological sum of two ω-bounded spaces and therefore itself ω-bounded.

The remaining case is where α is the limit of increasing α_n for $n < \omega$. Fix a bijection $f_\alpha : \omega_k \to \omega_k \cdot \alpha$. Let $\gamma \in \omega_k$ and define

$$W_{\omega_k \cdot \alpha+\gamma,n} = \{\omega_k \cdot \alpha + \delta : \delta \in W_{\gamma,n}\} \cup f_\alpha[W_{\gamma,0}] \setminus \omega_k \cdot (\alpha_n + 1)$$

to be the countable decreasing base $\{W_{\omega_k \cdot \alpha+\gamma,n} : n < \omega\}$ for $\omega_k \cdot \alpha + \gamma$. Note that $\omega_k \cdot (\alpha+1) \setminus \omega_k \cdot \alpha$ is then a closed (but not open) copy of ω_k under the homeomorphism $\gamma \mapsto \omega_k \cdot \alpha + \gamma$.

We wish to demonstrate the induction hypothesis for $\omega_k \cdot (\alpha+1)$. Each $W_{\omega_k \cdot \alpha+\gamma,n}$ is countable and contains no ordinals greater than $\omega_k \cdot \alpha + \gamma$. To see that it is compact, note that $\{\omega_k \cdot \alpha + \delta : \delta \in W_{\gamma,n}\}$ is a copy of compact $W_{\gamma,n}$, and for any basic neighborhood $W_{\omega_k \cdot \alpha+\gamma,m}$ of $\omega_k \cdot \alpha + \gamma$, the set $f_\alpha[W_{\gamma,0}] \cap \omega_k \cdot (\alpha_m+1)$ is a countable subspace of $\omega_k \cdot (\alpha_m+2)$

□

Theorem. $S'(\omega_\omega)$.