

**An Example of Gruenhage's Compact-Point Game for which K has
a winning strategy, but no winning k -Markov strategy**

We construct a ZFC example given by Gary Gruenhage, inspired by a ZFC+¬SH example due to Stephen Watson [5]. We require a result of Gruenhage and Michael [3]:

Theorem 1. *Let X be a meta-Lindelöf, locally Lindelöf regular space, and let \mathcal{B} be a base for X . Then X has a cover $\mathcal{B}' \subseteq \mathcal{B}$ such that $\{\overline{B} : B \in \mathcal{B}'\}$ is point-countable.*

Theorem 2. *There exists a compact, zero-dimensional topological space X and an uncountable subspace $C \subseteq 2^\omega$ of the Cantor Set, such that X has a point-countable cover $\mathcal{U} = \{U_c : c \in C\}$ of clopen sets which is not σ -point-finite (the union of countably-many point-finite collections).*

Proof. Take a zero-dimensional Corson compact Y of weight 2^ω , which is not Eberlein compact. (See Todercevic [4] for an example.) It follows by [1] that Y^2 is hereditarily metaLindelöf, but not hereditarily σ -metacompact. Note $X = Y^2 \setminus \Delta$ is an open non- σ -metacompact subspace of Y^2 , and thus locally compact, and let \mathcal{V} be an open cover of X such that no σ -point-finite open refinement exists.

Let \mathcal{B} be a clopen base for X of cardinality 2^ω which is a refinement of \mathcal{V} . Then as X is metaLindelöf, locally Lindelöf, and regular, by Thm 1, let $\mathcal{U} \subseteq \mathcal{B}$ be a point-countable clopen refinement of \mathcal{V} of cardinality $\leq 2^\omega$, which cannot be σ -point-finite (and is thus uncountable). \square

Definition 3. Using the X from Theorem 2, let

$$\mathbb{X} = (X \times 2^{<\omega}) \cup 2^\omega$$

compose a topological sum of $2^{<\omega}$ copies of X along with a discrete copy of a subspace $C \subseteq 2^\omega$ of the Cantor Set, and add open (in fact, compact) neighborhoods of the form:

$$B_c = c \cup (U_c \times \{c \restriction n : n < \omega\})$$

as seen in Figure 1.

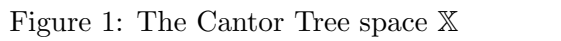
Definition 4. Let $S \in [C]^{<\omega}$ and $m < \omega$. Define

$$K_S = \bigcup_{c \in S} B_c$$

$$A = \{z \frown \langle 1 \rangle : z \in 1^{<\omega}\}$$

$$K_S^* = K_S \setminus (X \times A)$$

$$L_m = X \times 2^{<m}$$



and observe that every compact set is dominated by the compact set $K_S^* \cup L_m$ for some S, m .

Intuitively, K_S^* collects the branches of U_c converging up to each $c \in S$ while avoiding the copy of X for each s in an antichain A , and L_m collects the copies of X with $|s| < m$ at the base of the tree. (See Figure 2)

Definition 5. $LF_{K,P}(\mathbb{X})$ is a topological game consisting of players K and P . During each round, K chooses a compact subset of \mathbb{X} , and P chooses a point outside of any compact set previously played by K . K wins the game if the set of points chosen by P throughout all rounds of the game are locally finite in the space.

Definition 6. We say $I \uparrow G$ if Player I has a winning strategy in the game G .

We say $I \uparrow_{\text{tact}} G$ (resp. $I \uparrow_{k\text{-tact}} G$) if Player I has a winning tactical (resp. k -tactical) strategy in the game G , a strategy depending on only the (k) most recent move(s) of the opponent.

We say $I \uparrow_{\text{mark}} G$ (resp. $I \uparrow_{k\text{-mark}} G$) if Player I has a winning Markov (resp. k -Markov) strategy in the game G , a strategy depending on only the (k) most recent move(s) of the opponent and the current round number.

Proposition 7. *Without loss of generality, we may assume P always plays points in $X \times 2^{<\omega}$ throughout $LF_{K,P}(\mathbb{X})$.*

Proposition 8. $K \uparrow LF_{K,P}(\mathbb{X})$

Proof. Let $x \in X$. $C^x = \{c \in C : x \in U_c\}$ is a countable collection by the point-countability of \mathcal{U} , so label its elements as $\{c_n^x : n < \omega\}$.

K may use the strategy

$$\sigma(\langle x_0, s_0 \rangle, \dots, \langle x_{n-1}, s_{n-1} \rangle) = \bigcup_{i < n} K_{\{c_0^{x_i}, \dots, c_{n-1}^{x_i}\}} \cup L_{|s_i|+1}$$

This is a winning strategy because each move $\langle x_i, s_i \rangle$ by P cannot be a part of a subsequence of the play converging to any $c_n^{x_i}$, since $K_{\{c_0^{x_i}, \dots, c_n^{x_i}\}} \supseteq B_{c_n^{x_i}}$ was forbidden during round n . \square

Theorem 9. $K \not\uparrow_{\text{tact}} LF_{K,P}(\mathbb{X})$.

Proof. This is actually a corollary of Gruenhage's theorem in [2]: \mathbb{X} is locally compact (each point is either in some X or in some B_c) but not metacompact (any cover of C necessarily will have infinite overlap at some $X \times \{s\}$). However, we proceed with a direct game-theoretic proof.

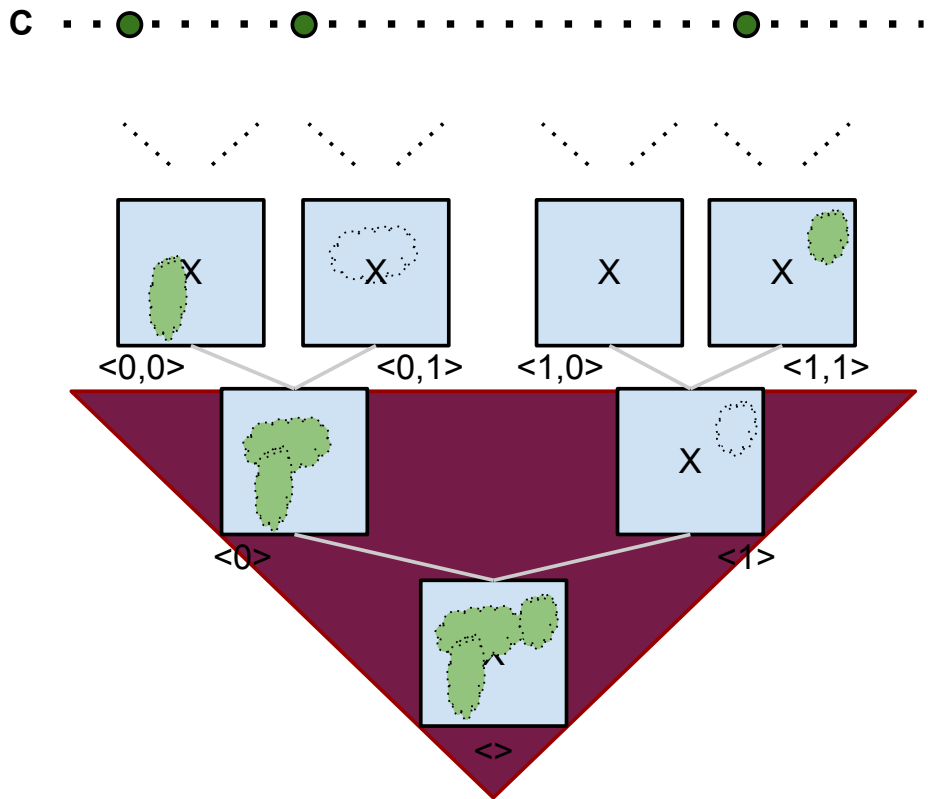


Figure 2: K_S^* and L_m

Suppose that $\sigma(\langle x, s \rangle)$ was a winning tactical strategy for K and define the compact set

$$\sigma'(x, n) = \bigcup_{|s| \leq n} \sigma(\langle x, s \rangle)$$

There exists some $f : C \rightarrow \omega$ such that for all $x \in U_c$, $\sigma'(x, f(c))$ covers some $B_c \setminus L_m$. (If not, P counters by simply always playing in $B_c \setminus L_m$.)

Recall that \mathcal{U} is not the union of the countably-many point-finite collections, and

$$\mathcal{U} = \bigcup_{n < \omega} \{U_c : f(c) = n\}$$

so we may choose n where $\mathcal{U}_n = \{U_c : f(c) = n\}$ is not point-finite. Fix x so that x belongs to each of $\{U_{c_0}, U_{c_1}, \dots\} \subseteq \mathcal{U}_n$.

For each c_i , $\sigma'(x, f(c_i)) = \sigma'(x, n)$ covers c_i . Thus $\sigma'(x, n) \supseteq \{c_0, c_1, \dots\}$ is a compact set covering a closed discrete subset, a contradiction. \square

Theorem 10. $K \not\mathcal{V}_{2\text{-tact}} LF_{K,P}(\mathbb{X})$.

Proof. Suppose $\sigma(\langle x, s \rangle, \langle y, t \rangle)$ was a winning 2-tactical strategy. Without loss of generality we assume it ignores order. We may define $S(x, y, n) \in [C]^{<\omega}$ (increasing on n) and $n < m(x, y, n) < \omega$ such that for each (x, y) ,

$$\bigcup_{s, t \in 2^{\leq n}} \sigma(\langle x, s \rangle, \langle y, t \rangle) \subseteq K_{S(x, y, n)}^* \cup L_{m(x, y, n)}$$

and so we assume

$$\sigma(\langle x, s \rangle, \langle y, t \rangle) = K_{S(x, y, \max(|s|, |t|))}^* \cup L_{m(x, y, \max(|s|, |t|))}$$

Select an arbitrary point $x' \in X$. We define a tactical strategy

$$\tau(x, s) = K_{S(x, x', m(x, x', |s|)+1)}^* \cup L_{m(x, x', m(x, x', |s|)+1)}$$

We complete the proof by showing τ is a winning tactical strategy (a contradiction).

Suppose

$$\langle x_0, s_0 \rangle, \langle x_1, s_1 \rangle, \langle x_2, s_2 \rangle, \dots$$

successfully countered τ by clustering at $c \in C$ (the strategy trivially prevents clustering elsewhere). Let $z_n = \langle 0, \dots, 0 \rangle$ with n zeros. We claim

$$\langle x_0, s_0 \rangle, \langle x', z_{m(x_0, x', |s_0|)} \frown \langle 1 \rangle \rangle, \langle x_1, s_1 \rangle, \langle x', z_{m(x_1, x', |s_1|)} \frown \langle 1 \rangle \rangle, \langle x_2, s_2 \rangle, \langle x', z_{m(x_2, x', |s_2|)} \frown \langle 1 \rangle \rangle, \dots$$

is a successful counter to σ .

We will need the fact that, as $\langle x_{i+1}, s_{i+1} \rangle$ was legal against τ :

$$\begin{aligned} |s_i| &< m(x_i, x', |s_i|) + 1 = |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle| \\ &< m(x_i, x', m(x_i, x', |s_i|) + 1) = m(x_i, x', |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|) \leq |s_{i+1}| \end{aligned}$$

Note that $m(x, y, \max(|s|, |t|))$ is increasing throughout this play of the game versus σ :

$$\begin{aligned} &m(x_i, x', \max(|s_i|, |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|)) \\ &= m(x_i, x', |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|) \\ &\leq |s_{i+1}| \\ &< m(x_{i+1}, x', |s_{i+1}|) \\ &= m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle|)) \\ &= |z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle| \\ &< |z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle| \\ &< m(x_{i+1}, x', |z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle|) \\ &= m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle|)) \end{aligned}$$

We turn to showing that $\langle x', z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle$ is always a legal move. Since $z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle$ is on the antichain avoided by any K^* , the problem is reduced to showing that this move isn't forbidden by

$$L_{m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle|))}$$

which we can see here:

$$m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle|)) = m(x_{i+1}, x', |s_{i+1}|) < |z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle|$$

We can conclude by showing that $\langle x_{i+1}, s_{i+1} \rangle$ is always a legal move. We can see it avoids

$$L_{m(x_i, x', \max(|s_i|, |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|))}$$

since

$$m(x_i, x', \max(|s_i|, |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|)) = m(x_i, x', |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|) \leq |s_{i+1}|$$

Since $\langle x_{i+1}, s_{i+1} \rangle$ was legal against τ , for $h \leq i$ it avoided

$$K_{S(x_h, x', m(x_h, x', |s_h|)+1)}^* = K_{S(x_h, x', \max(|s_h|, |z_{m(x_h, x', |s_h|)} \frown \langle 1 \rangle|))}^*$$

and when $h < i$, we see it avoids:

$$\begin{aligned} &K_{S(x_{h+1}, x', \max(|s_{h+1}|, |z_{m(x_{h+1}, x', |s_{h+1}|)} \frown \langle 1 \rangle|))}^* = K_{S(x_{h+1}, x', |s_{h+1}|)}^* \\ &\subseteq K_{S(x_{h+1}, x', m(x_{h+1}, x', |s_{h+1}|)+1)}^* \end{aligned}$$

This concludes the proof. □

Theorem 11. $K \not\upharpoonright_{k\text{-tact}} LF_{K,P}(\mathbb{X})$.

Proof. The proof proceeds in parallel to the proof of $K \not\upharpoonright_{2\text{-tact}} LF_{K,P}(\mathbb{X})$, and in fact is just a generalization of said proof (at the cost of simplicity).

Suppose $\sigma(\langle x_0, s_0 \rangle, \dots, \langle x_k, s_k \rangle)$ was a winning $(k+1)$ -tactical strategy. Without loss of generality we assume it ignores order. We may define $S(x_0, \dots, x_k, n) \in [C]^{<\omega}$ (increasing on n) and $n < m(x_0, \dots, x_k, n) < \omega$ such that for each (x_0, \dots, x_k) ,

$$\bigcup_{s_0, \dots, s_k \in 2^{\leq n}} \sigma(\langle x_0, s_0 \rangle, \dots, \langle x_k, s_k \rangle) \subseteq K_{S(x_0, \dots, x_k, n)}^* \cup L_{m(x_0, \dots, x_k, n)}$$

and so we assume

$$\sigma(\langle x_0, s_0 \rangle, \dots, \langle x_k, s_k \rangle) = K_{S(x_0, \dots, x_k, \max(|s_0|, \dots, |s_k|))}^* \cup L_{m(x_0, \dots, x_k, \max(|s_0|, \dots, |s_k|))}$$

Select an arbitrary point $x' \in X$. Let $M^0(x, n) = m(x, x', \dots, x', n)$ and $M^{i+1}(x, n) = M^0(x, M^i(x, n) + 1)$. We define a tactical strategy

$$\tau(x, s) = K_{S(x, x', \dots, x', M^{k-1}(x, |s|) + 1)}^* \cup L_{m(x, x', \dots, x', M^{k-1}(x, |s|) + 1)}$$

We complete the proof by showing τ is a winning tactical strategy (a contradiction).

Suppose

$$\langle x_0, s_0 \rangle, \langle x_1, s_1 \rangle, \langle x_2, s_2 \rangle, \dots$$

successfully countered τ by clustering at $c \in C$ (the strategy trivially prevents clustering elsewhere). Let $z_n = \langle 0, \dots, 0 \rangle$ with n zeros. We claim

$$\begin{aligned} & \langle x_0, s_0 \rangle, \langle x', z_{M^0(x_0, |s_0|)} \frown \langle 1 \rangle \rangle, \langle x', z_{M^1(x_0, |s_0|)} \frown \langle 1 \rangle \rangle, \dots, \langle x', z_{M^{k-1}(x_0, |s_0|)} \frown \langle 1 \rangle \rangle, \\ & \langle x_1, s_1 \rangle, \langle x', z_{M^0(x_1, |s_1|)} \frown \langle 1 \rangle \rangle, \langle x', z_{M^1(x_1, |s_1|)} \frown \langle 1 \rangle \rangle, \dots, \langle x', z_{M^{k-1}(x_1, |s_1|)} \frown \langle 1 \rangle \rangle, \dots \end{aligned}$$

is a successful counter to σ .

We will need the fact that, as $\langle x_{i+1}, s_{i+1} \rangle$ was legal against τ :

$$\begin{aligned} |s_i| & < M^0(x_i, |s_i|) + 1 = |z_{M^0(x_i, |s_i|)} \frown \langle 1 \rangle| < M^0(x_i, M^0(x_i, |s_i|) + 1) + 1 \\ & = M^1(x_i, |s_i|) + 1 = |z_{M^1(x_i, |s_i|)} \frown \langle 1 \rangle| < \dots < |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle| \\ & = M^{k-1}(x_i, |s_i|) + 1 < m(x_i, x', \dots, x', M^{k-1}(x_i, |s_i|) + 1) \leq |s_{i+1}| \end{aligned}$$

Note that $m(x_0, \dots, x_k, \max(|s_0|, \dots, |s_k|))$ is increasing throughout this play of the game versus σ :

$$m(x_i, x', \dots, x', \max(|s_i|, |z_{M^0(x_i, |s_i|)} \frown \langle 1 \rangle|, \dots, |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle|))$$

$$\begin{aligned}
&= m(x_i, x', \dots, x', |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle|) \\
&= m(x_i, x', \dots, x', M^{k-1}(x_i, |s_i|) + 1) \\
&\leq |s_{i+1}| \\
&< M^0(x_{i+1}, |s_{i+1}|) \\
&= m(x_{i+1}, x', \dots, x', |s_{i+1}|) \\
&= m(x_{i+1}, x', \dots, x', \max(|s_{i+1}|, |z_{M^0(x_i, |s_i|)} \frown \langle 1 \rangle|, \dots, |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle|)) \\
&= |z_{m(x_{i+1}, x', \dots, x', |s_{i+1}|)}| \\
&= |z_{M^0(x_{i+1}, |s_{i+1}|)}| \\
&< |z_{M^0(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle| \\
&< m(x_{i+1}, x', \dots, x', |z_{M^0(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|) \\
&= m(x_{i+1}, x', \dots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|, |z_{M^1(x_i, |s_i|)} \frown \langle 1 \rangle|, \dots, |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle|)) \\
&\quad \vdots \\
&< m(x_{i+1}, x', \dots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|, \dots, |z_{M^{k-1}(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|))
\end{aligned}$$

We turn to showing that $\langle x', z_{M^j(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle \rangle$ is always a legal move. Since $z_{M^j(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle$ is on the antichain avoided by any K^* , the problem is reduced to showing that this move isn't forbidden by

$$\begin{aligned}
&L_{m(x_{i+1}, x', \dots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|, \dots, |z_{M^{j-1}(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|, |z_{M^j(x_i, |s_i|)} \frown \langle 1 \rangle|, \dots, |z_{M^k(x_i, |s_i|)} \frown \langle 1 \rangle|))} \\
&= L_{m(x_{i+1}, x', \dots, x', |z_{M^{j-1}(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|)}
\end{aligned}$$

which we can see here:

$$\begin{aligned}
&m(x_{i+1}, x', \dots, x', |z_{M^{j-1}(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|) \\
&= m(x_{i+1}, x', \dots, x', M^{j-1}(x_{i+1}, |s_{i+1}|) + 1) \\
&= M^0(x_{i+1}, M^{j-1}(x_{i+1}, |s_{i+1}|) + 1) \\
&= M^j(x_{i+1}, s_{i+1}) \\
&< |z_{M^j(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|
\end{aligned}$$

We can conclude by showing that $\langle x_{i+1}, s_{i+1} \rangle$ is always a legal move. We can see it avoids

$$L_{m(x_i, x', \dots, x', \max(|s_i|, |z_{M^0(x_i, |s_i|)} \frown \langle 1 \rangle|, \dots, |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle|))}$$

since

$$\begin{aligned}
& m(x_i, x', \dots, x', \max(|s_i|, |z_{M^0(x_i, |s_i|)} \frown \langle 1 \rangle|, \dots, |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle|)) \\
&= m(x_i, x', \dots, x', |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle|) \\
&= m(x_i, x', \dots, x', M^{k-1}(x_i, |s_i|) + 1) \\
&\leq |s_{i+1}|
\end{aligned}$$

Since $\langle x_{i+1}, s_{i+1} \rangle$ was legal against τ , for $h \leq i$ it avoided

$$\begin{aligned}
& K_{S(x_h, x', \dots, x', M^{k-1}(x_h, |s_h|) + 1)}^* \\
&= K_{S(x_h, x', \dots, x', \max(|s_h|, |z_{M^0(x_h, |s_h|)} \frown \langle 1 \rangle|, \dots, |z_{M^{k-1}(x_h, |s_h|)} \frown \langle 1 \rangle|))}^*
\end{aligned}$$

and when $h < i$, we see it avoids both:

$$\begin{aligned}
& K_{S(x_{h+1}, x', \dots, x', \max(|s_{h+1}|, |z_{M^0(x_{h+1}, |s_{h+1}|)} \frown \langle 1 \rangle|, \dots, |z_{M^{j-1}(x_{h+1}, |s_{h+1}|)} \frown \langle 1 \rangle|, |z_{M^j(x_{h+1}, |s_{h+1}|)} \frown \langle 1 \rangle|, \dots, |z_{M^k(x_{h+1}, |s_{h+1}|)} \frown \langle 1 \rangle|))}^* \\
&= K_{S(x_{h+1}, x', \dots, x', |z_{M^{j-1}(x_{h+1}, |s_{h+1}|)} \frown \langle 1 \rangle|)}^* \\
&= K_{S(x_{h+1}, x', \dots, x', M^{j-1}(x_{h+1}, |s_{h+1}|) + 1)}^* \\
&\subseteq K_{S(x_{h+1}, x', \dots, x', M^{k-1}(x_{h+1}, |s_{h+1}|) + 1)}^*
\end{aligned}$$

and:

$$\begin{aligned}
& K_{S(x_{h+1}, x', \dots, x', \max(|s_{h+1}|, |z_{M^0(x_h, |s_h|)} \frown \langle 1 \rangle|, \dots, |z_{M^k(x_h, |s_h|)} \frown \langle 1 \rangle|))}^* \\
&= K_{S(x_{h+1}, x', \dots, x', |s_{k+1}|)}^* \\
&\subseteq K_{S(x_{h+1}, x', \dots, x', M^{k-1}(x_{h+1}, |s_{h+1}|) + 1)}^*
\end{aligned}$$

This concludes the proof. \square

Corollary 12. $K \not\models_{k\text{-mark}} LF_{K,P}(\mathbb{X})$.

Proof. Let $\sigma(\langle x_0, s_0 \rangle, \dots, \langle x_{k-1}, s_{k-1} \rangle, n)$ be a winning k -Markov strategy for K increasing on n . We define a k -tactical strategy

$$\tau(\langle x_0, s_0 \rangle, \dots, \langle x_{k-1}, s_{k-1} \rangle) = \sigma(\langle x_0, s_0 \rangle, \dots, \langle x_{k-1}, s_{k-1} \rangle, \max_{i < k}(|s_i|)) \cup L_{\max_{i < k}(|s_i|) + 1}$$

and observe that since for any legal play of the game, the round number $n \leq \max_{i < k}(|s_i|)$, we know τ always yields supersets of σ , and is thus also a winning strategy, contradiction. \square

Ladder Space Stuff

Proposition 13. *It is consistent that there exists a compact, zero-dimensional topological space X such that X has a point-countable cover $\mathcal{U} = \{U_\alpha : \alpha \in \omega_1^L\}$ of clopen sets such that no uncountable subset of \mathcal{U} is point-finite.*

Definition 14. For each $\alpha \in \omega_1^L$, define successor ordinals $\alpha_0 < \alpha_1 < \dots \rightarrow \alpha$ such that each successor ordinal β equals α_n for some α, n . Using the X from the previous lemma, let

$$\mathbb{X} = (X \times \omega_1^S) \cup \omega_1^L$$

compose a topological sum of copies of X for each countable successor ordinal, along with a discrete copy of the countable limit ordinals, and add open (in fact, compact) neighborhoods of the form:

$$B(\alpha) = \{\alpha\} \cup (U_\alpha \times \{\alpha_n : n < \omega\})$$

Definition 15. Define these things as I need them:

Proposition 16. *Without loss of generality, we may assume P always plays points in $X \times \omega_1^S$ throughout $LF_{K,P}(\mathbb{X})$.*

Proposition 17. $K \uparrow LF_{K,P}(\mathbb{X})$

Proof. Sketch of proof: For each point $\langle x, \beta \rangle$ played by P , K makes a promise to eventually cover $B(\alpha)$ for each of the countably-many α where $x \in U_\alpha$. \square

Theorem 18. $K \not\Uparrow_{tact} LF_{K,P}(\mathbb{X})$

Proof. Follows from G: \mathbb{X} is locally compact, but not metacompact. A game-theoretic proof follows.

Suppose $\sigma(x, \beta)$ was a winning strategy for K . We claim that for each $\alpha \in \omega_1^L$, there exists some $f(\alpha) < \omega$ such that $\bigcap_{x \in U_\alpha} \sigma(x, \alpha_{f(\alpha)}) \supseteq B(\alpha)$. If not, choose α for which $f(\alpha)$ cannot exist. Then there exists some $x_0 \in U_\alpha$ where $\sigma(x, \alpha_{m_0})$ misses $B(\alpha)$ for sufficiently large m_0 . In general, choose $x_n \in U_\alpha$ in round n where $\sigma(x, \alpha_{m_n})$ misses $B(\alpha)$ for sufficiently large m_n , and the play

$$\langle x_0, \alpha_{m_0} \rangle, \langle x_1, \alpha_{m_1} \rangle, \langle x_2, \alpha_{m_2} \rangle, \dots$$

counters σ by converging to α , contradiction.

So, define a function $f : \omega_1^L \rightarrow \omega$ where $\bigcap_{x \in U_\alpha} \sigma(x, \alpha_{f(\alpha)}) \supseteq B(\alpha)$, and let $g : \omega_1^L \rightarrow \omega_1^S$ be defined such that $g(\alpha) = \alpha_{f(\alpha)}$. By the pressing down lemma, there exists an uncountable $A \subseteq \omega_1^L$ and $\beta \in \omega_1^S$ such that $g(\alpha) = \alpha_{f(\alpha)} = \beta$ for all $\alpha \in A$.

Since no uncountable subset of $\mathcal{U} = \{U_\alpha : \alpha \in \omega_1^L\}$ is point-finite, there exists a point $x \in X$ such that x belongs to U_α for each α in some infinite $A' \subseteq A$. Then note that $\sigma(x, g(\alpha)) = \sigma(x, \beta) \supseteq B(\alpha)$ for each $\alpha \in A'$, and thus $\sigma(x, \beta)$ is not compact, contradiction. \square

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