DUAL SELECTION GAMES

STEVEN CLONTZ

Abstract. (an investigation of dual selection games)

1. Introduction

Definition 1. The *selection game* $G_1(\mathcal{A}, \mathcal{B})$ is an ω-length game involving Players I and II. During round n, I chooses $A_n \in \mathcal{A}$, followed by II choosing $B_n \in A_n$. Player II wins in the case that $\{B_n : n < \omega\} \in \mathcal{B}$, and Player I wins otherwise.

For brevity, let

$$G_1(\mathcal{A}, \neg\mathcal{B}) = G_1(\mathcal{A}, \mathcal{P}\left(\bigcup \mathcal{A}\right) \setminus \mathcal{B}).$$

That is, II wins in the case that $\{B_n : n < \omega\} \notin \mathcal{B}$, and I wins otherwise.

Definition 2. For a set X, let $\mathbf{C}(X)$ be the collection of all choice functions on X, functions $f : X \to \bigcup X$ such that $f(x) \in x$ for all $x \in X$.

Definition 3. The set $\mathcal{R}$ is said to be a *reflection* of the set $\mathcal{A}$ if

$$\mathcal{A} = \{\text{range}(f) : f \in \mathbf{C}(\mathcal{R})\}.$$

For example, a reflection of the collection $\mathcal{O}_X$ of basic open covers of X would be $\mathcal{P}_X = \{\mathcal{T}_{X,x} : x \in X\}$, where $\mathcal{T}_{X,x}$ is the corresponding point-base at $x \in X$. Likewise for the collection $\Omega_{X,x}$ of sets with $x \in X$ as a limit point, $\mathcal{T}_{X,x}$ is itself a reflection.

Lemma 4. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$. Then $\bigcup \mathcal{R} = \bigcup \mathcal{A}$.*

Proof. If $x \in \bigcup \mathcal{A}$, then $x \in \text{range}(f)$ for some $f \in \mathbf{C}(\mathcal{R})$. Thus $x = f(R) \in R$ for some $R \in \mathcal{R}$, showing $x \in \bigcup \mathcal{R}$.

Likewise if $x \in \bigcup \mathcal{R}$, so $x \in R$ for some $R \in \mathcal{R}$. Let $f \in \mathbf{C}(\mathcal{R})$ satisfy $f(R) = x$, so $x \in \text{range}(f)$, showing $x \in \bigcup \mathcal{A}$. $\square$

Theorem 5. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*

Then $\text{I} \underset{pre}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$ *if and only if* $\text{II} \underset{mark}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let σ witness $\text{I} \underset{\text{pre}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. Since $\sigma(n) \in \mathcal{A} = \{\text{range}(f) : f \in \mathbf{C}(\mathcal{R})\}$, $\sigma(n) = \text{range}(f_n)$ for some $f_n \in \mathbf{C}(\mathcal{R})$. So let $\tau(R, n) = f_n(R)$ for all $R \in \mathcal{R}$ and $n < \omega$. Suppose $R_n \in \mathcal{R}$ for all $n < \omega$. Note that since σ is winning and $\tau(R_n, n) = f_n(R_n) \in \text{range}(f_n) = \sigma(n)$, $\{\tau(R_n, n) : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\text{II} \underset{\text{mark}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

2010 *Mathematics Subject Classification.* 54C30, 54D20, 54D45, 91A44.

Key words and phrases. Selection principle, selection game, limited information strategies.

Now let σ witness $\text{II} \underset{\text{mark}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$. Let $f_n \in \mathbf{C}(\mathcal{R})$ be defined by $f_n(R) = \sigma(R, n)$. Since $\tau(n) \in \mathcal{A} = \{\text{range}(f) : f \in \mathbf{C}(\mathcal{R})\}$, let $\tau(n) = \text{range}(f_n)$. Suppose that $B_n \in \tau(n) = \text{range}(f_n)$ for all $n < \omega$. Choose $R_n \in \mathcal{R}$ such that $B_n = f_n(R_n) = \sigma(R_n, n)$. Since σ is winning, $\{B_n : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\text{I} \underset{\text{pre}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. □

Theorem 6. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*
Then $\text{II} \underset{\textit{mark}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$ if and only if $\text{I} \underset{\textit{pre}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let σ witness $\text{II} \underset{\text{mark}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. Let $n < \omega$. Suppose that for each $R \in \mathcal{R}$, there was $g(R) \in R$ such that for all $A \in \mathcal{A}$, $\sigma(A, n) \neq g(R)$. Then $g \in \mathbf{C}(\mathcal{R})$, and $\sigma(\text{range}(g), n) \neq g(R)$ for all $R \in \mathcal{R}$, a contradiction.

So choose $\tau(n) \in \mathcal{R}$ such that for all $r \in \tau(n)$ there exists $A_{r,n} \in \mathcal{A}$ such that $\sigma(A_{r,n}, n) = r$. It follows that when $r_n \in \tau(n)$ for $n < \omega$, $\{r_n : n < \omega\} = \{\sigma(A_{r_n,n} : n < \omega\} \in B$, so τ witnesses $\text{I} \underset{\text{pre}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Now let σ witness $\text{I} \underset{\text{pre}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$. Then $\sigma(n) \in \mathcal{R}$, so for $A \in \mathcal{A}$, let $f_A \in \mathbf{C}(\mathcal{R})$ satisfy $A = \text{range}(f_A)$, and let $\tau(A, n) = f_A(\sigma(n))$. Then if $A_n \in \mathcal{A}$ for $n < \omega$, $\tau(A_n, n) \in \sigma(n)$, so $\{\tau(A_n, n) : n < \omega\} \in \mathcal{B}$. Thus τ witnesses $\text{II} \underset{\text{mark}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. □

Theorem 7. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*
Then $\text{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$ if and only if $\text{II} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let σ witness $\text{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$. Let $c(\emptyset) = \emptyset$. Suppose $c(s) \in (\bigcup A)^{<\omega} = (\bigcup R)^{<\omega}$ is defined for $s \in \mathcal{R}^{<\omega}$. Since $\sigma(c(s)) \in \mathcal{A}$, let $f_s \in \mathbf{C}(\mathcal{R})$ satisfy $\sigma(c(s)) = \text{range}(f_s)$, and let $c(s^\frown\langle R\rangle) = c(s)^\frown\langle f_s(R)\rangle$. Then let $c(\alpha) = \bigcup\{c(\alpha \restriction n) : n < \omega\}$ for $\alpha \in \mathcal{R}^\omega$, so

$$c(\alpha)(n) = f_{\alpha\restriction n}(\alpha(n)) \in \text{range}(f_{\alpha\restriction n}) = \sigma(c(\alpha \restriction n))$$

demonstrating that $c(\alpha)$ is a legal attack against σ.

Let $\tau(s^\frown\langle R\rangle) = f_s(R)$. Consider the attack $\alpha \in \mathcal{R}^\omega$ against τ. Then since σ is winning and $\tau(\alpha \restriction n+1) = f_{\alpha\restriction n}(\alpha(n)) \in \text{range}(f_{\alpha\restriction n}) = \sigma(c(\alpha \restriction n))$, it follows that $\{\tau(\alpha \restriction n+1) : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\text{II} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$.

Now let σ witness $\text{II} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$. For $s \in \mathcal{R}^{<\omega}$, define $f_s \in \mathbf{C}(\mathcal{R})$ by $f_s(R) = \sigma(s^\frown\langle R\rangle)$. Let $\tau(\emptyset) = \text{range}(f_\emptyset)$, and for $x \in \tau(\emptyset)$, choose $R_{\langle x\rangle} \in \mathcal{R}$ such that $x = f_\emptyset(R_{\langle x\rangle})$ (for other $x \in \bigcup A$, choose $R_{\langle x\rangle}$ arbitrarily as it won't be used). Now let $s \in (\bigcup A)^{<\omega} \setminus \emptyset$, and suppose $\tau(s \restriction n) \in \mathcal{A}$ and $R_{s\restriction n+1} \in \mathcal{R}$ have been defined for $n < |s|$. Then let $\tau(s) = \text{range}(f_{\langle R_{s\restriction 0}, \ldots, R_s\rangle})$ and for $x \in \tau(s)$ choose $R_{s^\frown\langle x\rangle}$ such that $x = f_{\langle R_{s\restriction 0}, \ldots, R_s\rangle}(R_{s^\frown\langle x\rangle})$ (and again, choose $R_{s^\frown\langle x\rangle}$ arbitrarily for other $x \in \bigcup\mathcal{A}$ as it won't be used).

Then let α attack τ, so $\alpha(n) \in \tau(\alpha \restriction n)$ and thus $\alpha(n) = f_{\langle R_{\alpha\restriction 0}, \ldots, R_{\alpha\restriction n}\rangle}(R_{\alpha\restriction n+1}) = \sigma(\langle R_{\alpha\restriction 0}, \ldots, R_{\alpha\restriction n+1}\rangle)$. Since σ is winning, $\{\sigma(\langle R_{\alpha\restriction 0}, \ldots, R_{\alpha\restriction n+1}\rangle) : n < \omega\} = \{\alpha(n) : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\text{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$. □

Theorem 8. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*
Then $\text{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$ if and only if $\text{I} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let σ witness $\text{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$. Let $s \in (\bigcup R)^{<\omega}$ and assume $a(s) \in \mathcal{A}^{|s|}$ is defined (of course, $a(\emptyset) = \emptyset$). Suppose for all $R \in \mathcal{R}$ there existed $f(R) \in R$ such that for all $A \in \mathcal{A}$, $\sigma(a(s)^\frown\langle A\rangle) \neq f(R)$. Then $\sigma(a(s)^\frown\langle \mathrm{range}(f)\rangle) \neq f(R)$ for all $R \in \mathcal{R}$, a contradiction. So let $\tau(s) \in \mathcal{R}$ satisfy for all $x \in \tau(s)$ there exists $a(s^\frown\langle x\rangle) \in \mathcal{A}^{|s|+1}$ extending $a(s)$ such that $x = \sigma(a(s^\frown\langle x\rangle))$.

If τ is attacked by $\alpha \in (\bigcup R)^\omega$, then $\alpha(n) \in \tau(\alpha \restriction n)$. So $\alpha(n) = \sigma(a(\alpha \restriction n+1))$, and since σ is winning, $\{\sigma(a(\alpha \restriction n+1)) : n < \omega\} = \{\alpha(n) : n < \omega\} \in \mathcal{B}$. Therefore τ witnesses $\mathrm{I} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$.

Now let σ witness $\mathrm{I} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$. Let $s \in \mathcal{A}^{<\omega}$, and suppose $c(s) \in (\bigcup \mathcal{R})^{|s|}$ is defined (again, $c(\emptyset) = \emptyset$). Let $\tau(s^\frown\langle \mathrm{range}(f)\rangle) = f(\sigma(c(s)))$, and let $c(s^\frown\langle \mathrm{range}(f)\rangle)$ extend $c(s)$ by letting $c(s^\frown\langle \mathrm{range}(f)\rangle)(|s|) = \tau(s^\frown\langle \mathrm{range}(f)\rangle)$.

If τ is attacked by $\alpha \in \mathcal{A}^\omega$, where $\alpha(n) = \mathrm{range}(f_n)$ for $n < \omega$, then since $\tau(\alpha \restriction n+1) \in \sigma(c(\alpha \restriction n))$ and σ is winning, we conclude that $\{\tau(\alpha \restriction n+1) : n < \omega\} \in \mathcal{B}$. Therefore τ witnesses $\mathrm{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$. $\square$

References

Department of Mathematics and Statistics, The University of South Alabama, Mobile, AL 36688

E-mail address: `sclontz@southalabama.edu`