Fun with Menger's Game

AU DMS 1st-Year Graduate Student Seminar

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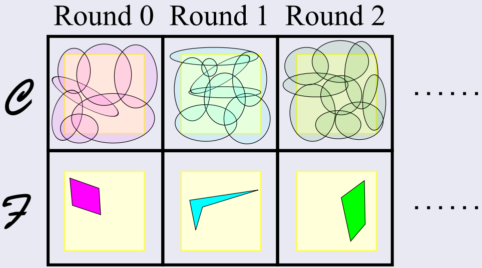
The Menger game

Game

Let $Men_{C,F}(X)$ denote the *Menger game* with players $\mathscr{C}$, $\mathscr{F}$.



Then $\mathscr{F}$ wins if her sets union to the space.

Who wins the Menger game for the following spaces?

- The reals $\mathbb{R}$
- The rationals $\mathbb{Q}$
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Theorem (C)

Let X be regular. X is σ-compact if and only if $\mathscr{F} \underset{pre}{\uparrow} Men_{C,F}(X)$.

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Proof that σ-compact $\Leftrightarrow \mathscr{F} \underset{\text{pre}}{\uparrow} Men_{C,F}(X)$

If $X = \bigcup_{n<\omega} X_n$, then $\tau(n) = X_n$ is a winning predetermined strategy.

Suppose X is not σ-compact. Let $\tau(n)$ be a predetermined strategy. Note $\tau(n)$ must be finitely coverable by every open cover of the *entire* space X. So let $\mathcal{U}$ be an open cover of just $\overline{\tau(n)}$. By regularity, let $V(x, U) \subseteq \overline{V(x, U)} \subseteq U$ for each open set U and point $x \in X$.

Let $\mathcal{V} = \{X \setminus \overline{\tau(n)}\} \cup \{V(x, U) : x \in \overline{\tau(n)} \cap U, U \in \mathcal{U}\}$, an open cover of the entire space. Choose a finite subcover, including $V(x_i, U_i)$ for some $x_i \in \overline{\tau(n)}$ and $U_i \in \mathcal{U}$ for $i < n$. Note $\{V(x_i, U_i) : i < n\}$ must be a cover of $\tau(n)$, so $\{\overline{V(x_i, U_i)} : i < n\}$ is a cover of $\overline{\tau(n)}$ (by finiteness). So there is a finite subcover $\{U_i : i < n\}$ for $\overline{\tau(n)}$, showing $\overline{\tau(n)}$ is compact.

Since X is not σ-compact, fix $x \notin \overline{\tau(n)}$ for any $n < \omega$. Then $x \notin \bigcup_{n<\omega} \tau(n)$, so τ is not a winning predetermined strategy.

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Another fun fact:

Theorem (C)

For second countable spaces X, $\mathscr{F} \uparrow Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{pre}{\uparrow} Men_{C,F}(X)$.

Sketchy proof that $\mathscr{F} \uparrow Men_{C,F}(X) \Leftrightarrow \mathscr{F} \underset{\text{pre}}{\uparrow} Men_{C,F}(X)$

Assume τ is a winning strategy for $\mathscr{F}$.

Since there are only countably many unions of finite collections of basic open sets in a secound countable space, let $\{C_n, n < \omega\}$ enumerate them all. It's okay to assume τ always yields some C_n.

Suppose the open covers $\mathcal{U}_s$ are defined for $t \leq s \in \omega^{<\omega}$. Then since there are only countably many C_n to equal $\tau(\mathcal{U}_{\langle s(0)\rangle}, \dots, \mathcal{U}_s, \mathcal{U})$, enumerate $\mathcal{U}_{s^\frown\langle n\rangle}$ to cover those cases.

Then this is a winning predetermined strategy (where $f : \omega \to \omega^{<\omega}$ is some bijection):

$$\tau(n) = \bigcap_{\mathcal{U} \in \mathfrak{C}} \tau(\mathcal{U}_{\langle f(n)(0)\rangle}, \dots, \mathcal{U}_{f(n)}, \mathcal{U})$$

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Since metrizable Lindelöf spaces are exactly the regular second-countable spaces:

Corollary (Telgarsky 1984, Scheepers 1995, C)

Let X be metrizable. $\mathscr{F} \uparrow Men_{C,F}(X)$ if and only if X is σ-compact.

Questions from any of you? Thanks!