RELATING GAMES OF MENGER, COUNTABLE FAN TIGHTNESS, AND SELECTIVE SEPARABILITY

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ABSTRACT. By adapting techniques of Arhangel'skii, Barman, and Dow, we may equate the existence of perfect-information, Markov, and tactical strategies between two interesting selection games. These results shed some light on Gruenhage's question asking whether all strategically selectively separable spaces are Markov selectively separable.

1. INTRODUCTION

Definition 1. The *selection principle* $S_{fin}(\mathcal{A}, \mathcal{B})$ states that given $A_n \in \mathcal{A}$ for $n < \omega$, there exist $B_n \in [A_n]^{<\omega}$ such that $\bigcup_{n<\omega} B_n \in \mathcal{B}$.

Definition 2. The *selection game* $G_{fin}(\mathcal{A}, \mathcal{B})$ is the analogous game to $S_{fin}(\mathcal{A}, \mathcal{B})$, where during each round $n < \omega$, Player I first chooses $A_n \in \mathcal{A}$, and then Player II chooses $B_n \in [A_n]^{<\omega}$. Player II wins in the case that $\bigcup_{n<\omega} B_n \in \mathcal{B}$, and Player I wins otherwise.

This game and property were first formally investigated by Scheepers in "Combinatorics of open covers" [10], which inspired a series of ten sequels with several co-authors. We may quickly observe that if II has a winning strategy for the game $G_{fin}(\mathcal{A}, \mathcal{B})$, then $S_{fin}(\mathcal{A}, \mathcal{B})$ will hold, but the converse need not follow.

The power of this selection principle and game comes from their ability to characterize several properties and games from the literature. Of interest to us are the following.

Definition 3. Let $\mathcal{O}_X$ be the collection of open covers for a topological space X. Then $S_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger property* for X (M for short), and $G_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger game*.

Definition 4. An ω-*cover* $\mathcal{U}$ for a topological space X is an open cover such that for every $F \in [X]^{<\omega}$, there exists some $U \in \mathcal{U}$ such that $F \subseteq U$.

Definition 5. Let Ω_X be the collection of ω-covers for a topological space X. Then $S_{fin}(\Omega_X, \Omega_X)$ is the Ω-*Menger property* for X (ΩM for short), and $G_{fin}(\Omega_X, \Omega_X)$ is the Ω-*Menger game*.

In [5, Theorem 3.9] it was shown that X is Ω-Menger if and only if X^n is Menger for all $n < \omega$.

Definition 6. Let $\Omega_{X,x}$ be the collection of subsets $A \subset X$ where $x \in \mathrm{cl}\, A$. (Call A a *blade* of x.) Then $S_{fin}(\Omega_{X,x}, \Omega_{X,x})$ is the *countable fan tightness property* for

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X at x (CFT_x for short), and $G_{fin}(\Omega_{X,x}, \Omega_{X,x})$ is the *countable fan tightness game* for X at x.

The similar notations for ω-covers Ω_X and blades $\Omega_{X,x}$ is well-established, see e.g. [7, Topological games and Ramsey theory] (wherein the notation is more succinctly given as Ω, Ω_x). The relationship between such sets becomes apparent in the study of C_p-theory; for this reason we choose to always denote the space explicitly to help distinguish the ambient spaces when discussing the relationship between Ω_X and $\Omega_{C_p(X),\mathbf{0}}$.

Definition 7. A space X has *countable fan tightness* (CFT for short) if it has countable fan tightness at each point $x \in X$.

Definition 8. Let $\mathcal{D}_X$ be the collection of dense subsets of a topological space X. Then $S_{fin}(\mathcal{D}_X, \Omega_{X,x})$ is the *countable dense fan tightness property* for X at x ($CDFT_x$ for short), and $G_{fin}(\mathcal{D}_X, \Omega_{X,x})$ is the *countable dense fan tightness game* for X at x.

Definition 9. A space X has *countable dense fan tightness* ($CDFT$ for short) if it has countable dense fan tightness at each point $x \in X$.

Note that $CFT \Rightarrow CDFT$ for any space X as $\mathcal{D}_X \subseteq \Omega_{X,x}$ for all $x \in X$.

The notion of countable fan tightness was first studied by by Arhangel'skii in [1]. A result of that paper showed that for $T_{3\frac{1}{2}}$ spaces X, the countable fan tightness of the space of real-vaued continuous functions with pointwise convergence $C_p(X)$ is characterized by the Ω-Menger property of X.

Definition 10. $S_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability property* for X (SS for short), and $G_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability game* for X.

Of course, one may easily observe that a selectively separable space is separable, and $SS \Rightarrow CDFT$. In [2] Barman and Dow demonstrated that all separable Frechet spaces are selectively separable. They were also able to produce a space which is selectively separable, but does not allow II a winning strategy in the selective separability game.

The object of this paper is to investigate the game-theoretic properties characterized by the presence of winning *limited information* strategies in these selection games.

Definition 11. A *strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function σ satisfying $\sigma(\langle A_0, \dots, A_n \rangle) \in [A_n]^{<\omega}$ for $\langle A_0 \dots, A_n \rangle \in \mathcal{A}^{n+1}$. We say this strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(\langle A_0, \dots, A_n \rangle)$ during each round $n < \omega$. If a winning strategy exists, then we write $\text{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$.

Definition 12. A *Markov strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function σ satisfying $\sigma(A, n) \in [A_n]^{<\omega}$ for $A \in \mathcal{A}$ and $n < \omega$. We say this Markov strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(A_n, n)$ during each round $n < \omega$. If a winning Markov strategy exists, then we write $\text{II} \underset{\text{mark}}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$.

Notation 13. If $S_{fin}(\mathcal{A}, \mathcal{B})$ characterizes the property P, then we say $\text{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes P^+ ("strategically P"), and $\text{II} \underset{\text{mark}}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes $P^{+\text{mark}}$ ("Markov P"). Of course, $P^{+\text{mark}} \Rightarrow P^+ \Rightarrow P$.

In this notation, Barman and Dow showed that SS does not imply SS^+. Our goal is to make progress on the following question attributed to Gary Gruenhage:

Question 14. *Does SS^+ imply $SS^{+\mathrm{mark}}$?*

The solution is already known to be "yes" in the context of countable spaces [2]. However in general, winning strategies in selection games cannot be improved to be winning Markov strategies. In [4] the author showed that M^+ implies $M^{+\mathrm{mark}}$ for second-countable spaces; however, the one-point Lindelöf-ication $L(\omega_1)$ of a discrete space of size ω_1 is a simple example of a regular non-second-countable space which is M^+ but not $M^{+\mathrm{mark}}$ (equivalently, not σ-compact). It will later be shown that it is no coincidence that countable SS^+ implies $SS^{+\mathrm{mark}}$ and second-countable M^+ implies $M^{+\mathrm{mark}}$.

2. CFT, $CDFT$ AND SS

We begin by generalizing the following result:

Theorem 15 (Lemma 2.7 of [2]). *The following are equivalent for any topological space X.*

- *X is SS.*
- *X is separable and $CDFT$.*
- *X has a countable dense subset D where $CDFT_x$ holds for all $x \in D$.*

Theorem 16. *The following are equivalent for any topological space X.*

- *X is SS (resp. SS^+, $SS^{+\mathrm{mark}}$).*
- *X is separable and $CDFT$ (resp. $CDFT^+$, $CDFT^{+\mathrm{mark}}$).*
- *X has a countable dense subset D where $CDFT_x$ (resp. $CDFT_x^+$, $CDFT_x^{+\mathrm{mark}}$) holds for all $x \in D$.*

Proof. We need only show that the final condition implies the first. Let $D = \{d_i : i < \omega\}$.

Let σ_i be a witness for $CDFT_{d_i}^+$ for each $i < \omega$. We define the strategy τ for the SS game by

$$\tau(\langle D_0, \ldots, D_n \rangle) = \bigcup_{i \leq n} \sigma_i(\langle D_i, \ldots, D_n \rangle).$$

Let $\langle D_0, D_1, \ldots \rangle \in \mathcal{D}_X^\omega$. By $CDFT_{d_i}^+$, we have

$$d_i \in \overline{\bigcup_{i \leq n < \omega} \sigma_i(\langle D_i, \ldots, D_n \rangle)} \subseteq \overline{\bigcup_{i \leq n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)} \subseteq \overline{\bigcup_{n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)}$$

and as $D \subseteq \overline{\bigcup_{n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)}$ it follows that

$$X \subseteq \overline{D} \subseteq \overline{\overline{\bigcup_{n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)}} = \overline{\bigcup_{n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)}.$$

Therefore τ witnesses SS^+.

Now let σ_i be a witness for $CDFT_{d_i}^{+\mathrm{mark}}$ for each $i < \omega$. We define the Markov strategy τ for the SS game by

$$\tau(D, n) = \bigcup_{i \leq n} \sigma_i(D, n - i).$$

Let $\langle D_0, D_1, \ldots \rangle \in \mathcal{D}_X^\omega$. By $CDFT_{d_i}^{+\mathrm{mark}}$, we have

$$d_i \in \overline{\bigcup_{i \leq n < \omega} \sigma_i(D_n, n - i)} \subseteq \overline{\bigcup_{i \leq n < \omega} \tau(D_n, n)} \subseteq \overline{\bigcup_{n < \omega} \tau(D_n, n)}$$

and as $D \subseteq \overline{\bigcup_{n < \omega} \tau(D_n, n)}$ it follows that

$$X \subseteq \overline{D} \subseteq \overline{\overline{\bigcup_{n < \omega} \tau(D, n)}} = \overline{\bigcup_{n < \omega} \tau(D, n)}.$$

Therefore τ witnesses $SS^{+\mathrm{mark}}$. □

So amongst separable spaces, we see that SS (resp. SS^+, $SS^{+\mathrm{mark}}$) and $CDFT$ (resp. $CDFT^+$, $CDFT^{+\mathrm{mark}}$) are equivalent. We now further bridge the gap between $CDFT$ and CFT in the context of function spaces. Consider the following result of Arhangel'skii.

Theorem 17 ([1])**.** *The following are equivalent for any $T_{3\frac{1}{2}}$ topological space X.*

- *X is ΩM.*
- *$C_p(X)$ is CFT.*

This result may similarly be generalized in a game-theoretic sense. In addition, this proof will demonstrate the equivalence of CFT and $CDFT$ in $C_p(X)$. It is unknown to the author whether Arhangel'skii used a strategy similar to the following proof in [1], but Sakai employed a similar technique in [9] to relate the Ω-Rothberger and countable strong fan tightness properties (and essentially, the countable strong dense fan tightness property). Due to the difficulty in obtaining an English translation of [1], we reprove Arhangel'skii's theorem above in our more general context below.

Proposition 18. *Any witness for ΩM (resp. $\Omega M^+, \Omega M^{+\mathrm{mark}}$) may be improved so that any final sequence of the chosen finite subcollections is an ω-cover.*

Proof. Consider any sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ of ω-covers. For each $n < \omega$, let $\langle \mathcal{F}_n^n, \mathcal{F}_{n+1}^n, \ldots \rangle$ witness ΩM for the final sequence $\langle \mathcal{U}_n, \mathcal{U}_{n+1}, \ldots \rangle$, and let $\mathcal{F}_n = \bigcup_{i \leq n} \mathcal{F}_n^i$; it follows that $\langle \mathcal{F}_0, \mathcal{F}_1, \ldots \rangle$ satisfies the proposition.

For ΩM^+, the winning strategy σ may be improved to τ where $\tau(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle) = \bigcup_{i \leq n} \sigma(\langle \mathcal{U}_i, \ldots, \mathcal{U}_n \rangle)$. For $\Omega M^{+\mathrm{mark}}$, the winning Markov strategy σ may be improved to τ where $\tau(\mathcal{U}, n) = \bigcup_{i \leq n} \sigma(\mathcal{U}, i)$. Both improved strategies are easily seen to satisfy the proposition. □

Definition 19. Let X be a $T_{3\frac{1}{2}}$ topological space. For $\mathbf{x} \in C_p(X)$, $F \in [X]^{<\omega}$, and $\epsilon > 0$, let

$$[\mathbf{x}, F, \epsilon] = \{\mathbf{y} \in C_p(x) : |\mathbf{y}(t) - \mathbf{x}(t)| < \epsilon \text{ for all } t \in F\}$$

give a basic open neighborhood of $\mathbf{x}$.

Lemma 20. *If X is ΩM (resp. ΩM^+, $\Omega M^{+\mathrm{mark}}$), then $C_p(X)$ is $CFT_{\mathbf{0}}$ (resp. $CFT_{\mathbf{0}}^+$, $CFT_{\mathbf{0}}^{+\mathrm{mark}}$).*

Proof. For each $B \in \Omega_{C_p(X), \mathbf{0}}$ define

$$\mathcal{U}_n(B) = \left\{ \mathbf{x}^{\leftarrow} \left[\left(-\frac{1}{2^n}, \frac{1}{2^n} \right) \right] : \mathbf{x} \in B \right\}.$$

Consider the finite set $F \in [X]^{<\omega}$. Since $\mathbf{0} \in \overline{B}$, choose $\mathbf{x} \in B \cap [\mathbf{0}, F, 1/2^n]$. It follows that $F \subseteq \mathbf{x}^{\leftarrow}[(-1/2^n, 1/2^n)] \in \mathcal{U}_n(B)$, so $\mathcal{U}_n(B)$ is an ω-cover of X.

Consider the sequence of blades $\langle B_0, B_1, \ldots \rangle \in \Omega^{\omega}_{C_p(X), \mathbf{0}}$, and the corresponding sequence of ω-covers $\langle \mathcal{U}_0(B_0), \mathcal{U}_1(B_1), \ldots \rangle \in \Omega^{\omega}_X$. For $U \in \mathcal{U}_n(B_n)$, choose $\mathbf{x}_U \in B_n$ such that $U = \mathbf{x}_U^{\leftarrow}\left[\left(-\frac{1}{2^n}, \frac{1}{2^n}\right)\right]$.

Assuming X is ΩM, choose a witness $\langle \mathcal{F}_0, \mathcal{F}_1, \ldots \rangle$ such that $\mathcal{F}_n \in [\mathcal{U}_n(B_n)]^{<\omega}$ and

$$\bigcup_{i \leq n < \omega} \mathcal{F}_n$$

is an ω-cover of X for all $i < \omega$. Let

$$F_n = \{\mathbf{x}_U : U \in \mathcal{F}_n\}.$$

We claim that $\mathbf{0} \in \overline{\bigcup_{n < \omega} F_n}$. Let $G \in [X]^{<\omega}, \epsilon > 0$. Choose $i < \omega$ such that $\frac{1}{2^i} < \epsilon$. Since $\bigcup_{i \leq n < \omega} \mathcal{F}_n$ is an ω-cover, we may choose $i \leq n < \omega$ and $U \in \mathcal{F}_n$ such that

$$G \subseteq U = \mathbf{x}_U^{\leftarrow}\left[\left(-\frac{1}{2^n}, \frac{1}{2^n}\right)\right] \subseteq \mathbf{x}_U^{\leftarrow}[(-\epsilon, \epsilon)].$$

It follows that $\mathbf{x}_U \in F_n \cap [\mathbf{0}, G, \epsilon]$, and thus every basic open neighborhood of $\mathbf{0}$ intersects $\bigcup_{n < \omega} F_n$.

Assuming X is ΩM^+, choose a witness σ such that

$$\bigcup_{i \leq n < \omega} \sigma(\langle \mathcal{U}_0(B_0), \ldots, \mathcal{U}_n(B_n) \rangle)$$

is an ω-cover of X for all $i < \omega$. Let

$$\tau(\langle B_0, \ldots, B_n \rangle) = \{\mathbf{x}_U : U \in \sigma(\langle \mathcal{U}_0(B_0), \ldots, \mathcal{U}_n(B_n) \rangle)\}.$$

We claim that $\mathbf{0} \in \overline{\bigcup_{n < \omega} \tau(\langle B_0, \ldots, B_n \rangle)}$. Let $G \in [X]^{<\omega}, \epsilon > 0$. Choose $i < \omega$ such that $\frac{1}{2^i} < \epsilon$. Since $\bigcup_{i \leq n < \omega} \sigma(\langle \mathcal{U}_0(B_0), \ldots, \mathcal{U}_n(B_n) \rangle)$ is an ω-cover, we may choose $i \leq n < \omega$ and $U \in \sigma(\langle \mathcal{U}_0(B_0), \ldots, \mathcal{U}_n(B_n) \rangle)$ such that

$$G \subseteq U = \mathbf{x}_U^{\leftarrow}\left[\left(-\frac{1}{2^n}, \frac{1}{2^n}\right)\right] \subseteq \mathbf{x}_U^{\leftarrow}[(-\epsilon, \epsilon)].$$

It follows that $\mathbf{x}_U \in \tau(\langle B_0, \ldots, B_n \rangle) \cap [\mathbf{0}, G, \epsilon]$, and thus every basic open neighborhood of $\mathbf{0}$ intersects $\bigcup_{n < \omega} \tau(\langle B_0, \ldots, B_n \rangle)$.

Assuming X is $\Omega M^{+\mathrm{mark}}$, choose a witness σ such that

$$\bigcup_{i \leq n < \omega} \sigma(\mathcal{U}_n(B_n), n)$$

is an ω-cover of X for all $i < \omega$. Let

$$\tau(B_n, n) = \{\mathbf{x}_U : U \in \sigma(\mathcal{U}_n(B_n), n)\}.$$

We claim that $\mathbf{0} \in \overline{\bigcup_{n < \omega} \tau(B_n, n)}$. Let $G \in [X]^{<\omega}, \epsilon > 0$. Choose $i < \omega$ such that $\frac{1}{2^i} < \epsilon$. Since $\bigcup_{i \leq n < \omega} \sigma(\mathcal{U}_n(B_n), n)$ is an ω-cover, we may choose $i \leq n < \omega$ and $U \in \sigma(\mathcal{U}_n(B_n), n)$ such that

$$G \subseteq U = \mathbf{x}_U^{\leftarrow}\left[\left(-\frac{1}{2^n}, \frac{1}{2^n}\right)\right] \subseteq \mathbf{x}_U^{\leftarrow}[(-\epsilon, \epsilon)].$$

It follows that $\mathbf{x}_U \in \tau(B_n, n) \cap [\mathbf{0}, G, \epsilon]$, and thus every basic open neighborhood of $\mathbf{0}$ intersects $\bigcup_{n < \omega} \tau(B_n, n)$. $\qquad \square$

Lemma 21. *Let X be a $T_{3\frac{1}{2}}$ topological space. If $C_p(X)$ is $CDFT_\mathbf{0}$ (resp. $CDFT_\mathbf{0}^+$, $CDFT_\mathbf{0}^{+\mathrm{mark}}$), then X is ΩM (resp. ΩM^+, $\Omega M^{+\mathrm{mark}}$).*

Proof. For each $\mathcal{U} \in \Omega_X$ define

$$D(\mathcal{U}) = \{\mathbf{y} \in C_p(X) : \mathbf{y}[X \setminus U_{\mathbf{y},\mathcal{U}}] = \{1\} \text{ for some } U_{\mathbf{y},\mathcal{U}} \in \mathcal{U}\}.$$

Consider the point $\mathbf{x} \in C_p(X)$ and its basic open neighborhood $[\mathbf{x}, G, \epsilon]$. If $\mathcal{U}$ is an ω-cover of X, $G \subseteq U$ for some $U \in \mathcal{U}$. Since X is $T_{3\frac{1}{2}}$, $X \setminus U$ is closed, and G is finite and disjoint from $X \setminus U$, we may choose some function $\mathbf{y} \in C_p(X)$ where $\mathbf{y}[X \setminus U] = \{1\}$ and $\mathbf{x}(t) = \mathbf{y}(t)$ for each $t \in G$. It follows $\mathbf{y} \in [\mathbf{x}, G, \epsilon] \cap D(\mathcal{U})$, so $D(\mathcal{U})$ is dense in $C_p(X)$.

Consider the sequence of ω-covers $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle \in \Omega_X^\omega$, and the corresponding sequence of dense subsets $\langle D(\mathcal{U}_0), D(\mathcal{U}_1), \ldots \rangle \in \mathcal{D}_{C_p(X)}^\omega$.

Assuming $C_p(X)$ is $CDFT_\mathbf{0}$, choose a witness $\langle F_0, F_1, \ldots \rangle$ such that

$$\mathbf{0} \in \overline{\bigcup_{n<\omega} F_n}.$$

Now let

$$\mathcal{F}_n = \{U_{\mathbf{y},\mathcal{U}_n} : \mathbf{y} \in F_n\} \in [\mathcal{U}_n]^{<\omega}.$$

We claim that $\bigcup_{n<\omega} \mathcal{F}_n$ is an ω-cover. Let $G \in [X]^{<\omega}$. The neighborhood $[\mathbf{0}, G, \frac{1}{2}]$ contains some point $\mathbf{y} \in F_n$ for some $n < \omega$. It follows that $U_{\mathbf{y},\mathcal{U}_n} \in \mathcal{U}_n$ and $\mathbf{y}[X \setminus U_{\mathbf{y},\mathcal{U}_n}] = \{1\}$. It follows that $G \cap (X \setminus U_{\mathbf{y},\mathcal{U}_n}) = \emptyset$, and therefore $G \subseteq U_{\mathbf{y},\mathcal{U}_n} \in \mathcal{F}_n$.

Assuming $C_p(X)$ is $CDFT_\mathbf{0}^+$, choose a witness σ such that

$$\mathbf{0} \in \overline{\bigcup_{n<\omega} \sigma(\langle D(\mathcal{U}_0), \ldots, D(\mathcal{U}_n)\rangle)}.$$

Now let

$$\tau(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle) = \{U_{\mathbf{y},\mathcal{U}_n} : \mathbf{y} \in \sigma(\langle D(\mathcal{U}_0), \ldots, D(\mathcal{U}_n)\rangle)\} \in [\mathcal{U}_n]^{<\omega}.$$

We claim that $\bigcup_{n<\omega} \tau(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$ is an ω-cover. Let $G \in [X]^{<\omega}$. The neighborhood $[\mathbf{0}, G, \frac{1}{2}]$ contains some point $\mathbf{y} \in \sigma(\langle D(\mathcal{U}_0), \ldots, D(\mathcal{U}_n)\rangle)$ for some $n < \omega$. It follows that $U_{\mathbf{y},\mathcal{U}_n} \in \mathcal{U}_n$ and $\mathbf{y}[X \setminus U_{\mathbf{y},\mathcal{U}_n}] = \{1\}$. As a result $G \cap (X \setminus U_{\mathbf{y},\mathcal{U}_n}) = \emptyset$, and therefore $G \subseteq U_{\mathbf{y},\mathcal{U}_n} \in \tau(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$.

Assuming $C_p(X)$ is $CDFT_\mathbf{0}^{+\mathrm{mark}}$, choose a witness σ such that

$$\mathbf{0} \in \overline{\bigcup_{n<\omega} \sigma(D(\mathcal{U}_n), n)}.$$

Now let

$$\tau(\mathcal{U}_n, n) = \{U_{\mathbf{y},\mathcal{U}_n} : \mathbf{y} \in \sigma(D(\mathcal{U}_n), n)\} \in [\mathcal{U}_n]^{<\omega}.$$

We claim that $\bigcup_{n<\omega} \tau(\mathcal{U}_n, n)$ is an ω-cover. Let $G \in [X]^{<\omega}$. The neighborhood $[\mathbf{0}, G, \frac{1}{2}]$ contains some point $\mathbf{y} \in \sigma(D(\mathcal{U}_n), n)$ for some $n < \omega$. It follows that $U_{\mathbf{y},\mathcal{U}_n} \in \mathcal{U}_n$ and $\mathbf{y}[X \setminus U_{\mathbf{y},\mathcal{U}_n}] = \{1\}$. As a result $G \cap (X \setminus U_{\mathbf{y},\mathcal{U}_n}) = \emptyset$, and therefore $G \subseteq U_{\mathbf{y},\mathcal{U}_n} \in \tau(\mathcal{U}_n, n)$. $\qquad\square$

Theorem 22. *The following are equivalent for any $T_{3\frac{1}{2}}$ topological space X.*

- *X is ΩM (resp. ΩM^+, $\Omega M^{+\mathrm{mark}}$).*
- *$C_p(X)$ is CFT (resp. CFT^+, $CFT^{+\mathrm{mark}}$).*
- *$C_p(X)$ is $CDFT$ (resp. $CDFT^+$, $CDFT^{+\mathrm{mark}}$).*

Proof. Since $\mathcal{D}_X \subseteq \Omega_{X,x}$, the second condition trivially implies the third. As $C_p(X)$ is homogeneous, the $C(D)FT$ properties follow from $C(D)FT_\mathbf{0}$. So the result follows from the previous lemmas. $\square$

3. A SPACE WHICH IS CFT^+ BUT NOT $CDFT^{+\mathrm{mark}}$

Our goal is to now demonstrate a space which is CFT^+, but not even $CDFT^{+\mathrm{mark}}$. To do so, we will find a space X which is ΩM^+ but not $\Omega M^{+\mathrm{mark}}$, yielding $C_p(X)$ as our example.

Proposition 23. *A space X is compact if and only if for each ω-cover $\mathcal{U}$ of X and $n < \omega$, there exists a finite subcollection $\mathcal{F} \in [\mathcal{U}]^{<\omega}$ such that for each $F \in [X]^{\leq n}$ there exists some $U \in \mathcal{F}$ where $F \subseteq U$.*

Proof. Let X and therefore X^n be compact. Let $\mathcal{F}_n$ be the finite subcover of the open cover $\mathcal{U}_n = \{U^n : U \in \mathcal{U}\}$. Then $\mathcal{F} = \{U : U^n \in \mathcal{F}_n\}$ satisfies that for each $F \in [X]^{\leq n}$ there exists some $U \in \mathcal{F}$ where $F \subseteq U$.

Conversely, assume that for each ω-cover $\mathcal{U}$ of X and $n < \omega$, there exists a finite subcollection $\mathcal{F} \in [\mathcal{U}]^{<\omega}$ such that for each $F \in [X]^{\leq n}$ there exists some $U \in \mathcal{F}$ where $F \subseteq U$. Given an arbitrary open cover $\mathcal{V}$, close it under finite unions to obtain an ω-cover $\mathcal{U}$. Then the finite subcollection $\mathcal{F}$ of $\mathcal{U}$ is of course a finite subcover, and since each member of $\mathcal{F}$ is a finite union of open sets in $\mathcal{V}$, there exists a finite subcover $\mathcal{G}$ of $\mathcal{V}$. $\square$

Lemma 24. *The following are equivalent for a regular topological space X:*
- *X is σ-compact*
- *X is $\Omega M^{+\mathrm{mark}}$*
- *X is $M^{+\mathrm{mark}}$*

Proof. The equivalence of σ-compact and $M^{+\mathrm{mark}}$ in regular spaces was shown in [4]. As $\Omega M^{+\mathrm{mark}}$ trivially implies $M^{+\mathrm{mark}}$, we need only demonstrate that if $X = \bigcup_{n<\omega} K_n$ for K_n compact and increasing, then X is $\Omega M^{+\mathrm{mark}}$.

We define $\sigma(\mathcal{U}, n)$ for each ω-cover $\mathcal{U}$ and $n < \omega$ to witness Proposition 23 for $\mathcal{U}$, K_n, and n. It follows that for every sequence of ω-covers $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ and $F \in [X]^{<\omega}$, $F \in [K_n]^{\leq n}$ for some $n \geq |F|$, and thus there exists some $U \in \sigma(\mathcal{U}_n, n)$ where $F \subseteq U$. Therefore $\bigcup_{n<\omega} \sigma(\mathcal{U}, n)$ is an ω-cover of X. $\square$

The reader may note that with this lemma, we may view Theorem 22 as a generalization of [3, Proposition 2.6].

Definition 25. Let X be a topological space such that all countable sets are closed. Then $L(X) = X \cup \{\infty\}$ is the *one-point Lindelöf-ication* of X, with a basis given by the usual open sets of X and the co-countable sets containing ∞.

Theorem 26. *Let ω_1 have the discrete topology. Then $L(\omega_1)$ is ΩM^+ but not $\Omega M^{+\mathrm{mark}}$.*

Proof. The proof that $X = L(\omega_1)$ is not σ-compact (and therefore not $\Omega M^{+\mathrm{mark}}$) follows from the fact that its countably infinite subsets are closed and discrete, so all of its compact subsets are finite.

To show ΩM^+, we define the strategy σ for II in $G_{fin}(\Omega_X, \Omega_X)$ as follows. For a given partial play $\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle$, suppose $\{\alpha_{i,m} : i < n, m < \omega\} \subseteq \omega_1$ has already been defined. Then as $\mathcal{U}_n$ is an ω-cover, we may choose $U(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle) \in \mathcal{U}_n$

that contains $\{\infty\} \cup \{\alpha_{i,j} : i, j < n\}$. Since $U(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$ is co-countable, define $\{\alpha_{n,m} : m < \omega\}$ such that $U(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle) = L(\omega_1) \setminus \{\alpha_{n,m} : m < \omega\}$. Then let $\sigma(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle) = \{U(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)\}$.

Consider the arbitrary sequence of moves $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ by I. For $F \in [L(\omega_1)]^{<\omega}$, choose $n < \omega$ such that

$$F \cap \{\alpha_{i,j} : i, j < \omega\} = F \cap \{\alpha_{i,j} : i, j < n\} \subseteq U(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$$

It follows that as

$$F \setminus \{\alpha_{i,j} : i, j < \omega\} \subseteq F \setminus \{\alpha_{n,m} : m < \omega\} \subseteq U(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$$

F is a subset of $U(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$, making $\bigcup_{n<\omega} \sigma(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$ an ω-cover. □

Note that the preceding result could be worded more strongly: $L(\omega_1)$ is strategically Ω-Rothberger, since the winning strategy only uses a single set of the ω-cover during each round. This topic will be discussed further in a later section.

Corollary 27. $C_p(L(\omega_1))$ *is* CFT^+ *but not* $CDFT^{+\mathrm{mark}}$.

If $C_p(L(\omega_1))$ were separable, it would be a negative solution to Question 14. However, it is not.

Lemma 28 ([6])**.** *For a* $T_{3\frac{1}{2}}$ *topological space* X, $C_p(X)$ *is separable if and only if* X *has a coarser separable metrizable topology.*

Corollary 29. $C_p(L(\omega_1))$ *is not separable.*

Proof. Consider a topology coarser than the topology of $L(\omega_1)$. Since open neighborhoods of ∞ in this topology are co-countable, a countable intersection of ∞ neighborhoods is not $\{\infty\}$. Since every metrizable topological space has points G_δ, this topology cannot be metrizable. □

An affirmative answer to either of these questions would answer Question 14 negatively.

Question 30. *Does there exist a separable subspace of* $C_p(L(\omega_1))$ *which is not* $CDFT^{+\mathrm{mark}}$?

Question 31. *Does there exist a non-σ-compact* ΩM^+ *space with a coarser separable metrizable topology?*

4. Equivalence of certain strategic and Markov selection properties

Barman and Dow previously demonstrated that SS^+ is equivalent to $SS^{+\mathrm{mark}}$ amongst countable spaces. A similar result by the author showed that M^+ is equivalent to $M^{+\mathrm{mark}}$ in second-countable spaces. The following result generalizes both.

Lemma 32. *Let* $\mathcal{A}$ *be a family of sets where* $|\bigcup \mathcal{A}| \leq \omega$ *and let* $\mathcal{B}$ *be a family of sets closed under supersets. Then* $\mathrm{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ *if and only if* $\mathrm{II} \underset{mark}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$.

Proof. Let σ witness $\mathrm{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$.

For $t \in \omega^{<\omega}$, suppose $Z_s \in \mathcal{A}$ has been defined for all $s \leq t$. Note then that $\{\sigma(\langle Z_{t \upharpoonright 1}, Z_{t \upharpoonright 2}, \ldots, Z_t, A \rangle) : A \in \mathcal{A}\} \subseteq [\bigcup \mathcal{A}]^{<\omega}$ and therefore is countable. So choose $Z_{t^\frown \langle n \rangle} \in \mathcal{A}$ for $n < \omega$ such that $\{\sigma(\langle Z_{t \upharpoonright 1}, Z_{t \upharpoonright 2}, \ldots, Z_t, A \rangle) : A \in \mathcal{A}\} = \{\sigma(\langle Z_{t \upharpoonright 1}, Z_{t \upharpoonright 2}, \ldots, Z_t, Z_{t^\frown \langle n \rangle} \rangle) : n < \omega\}$.

Let $b : \omega \to \omega^{<\omega}$ be a bijection, and define $\tau(A, n) = \sigma(\langle Z_{b(n)\restriction 1}, Z_{b(n)\restriction 2}, \ldots, Z_{b(n)}, A \rangle)$. Consider $\langle A_0, A_1, \ldots \rangle \in \mathcal{A}^\omega$.

Define $f \in \omega^\omega$ as follows. Let $n < \omega$ and suppose that $f \restriction n$ has already been defined. Then choose $f(n) < \omega$ such that $\sigma(\langle Z_{f\restriction 1}, Z_{f\restriction 2}, \ldots, Z_{f\restriction n}, A_{b\leftarrow(f\restriction n)} \rangle) = \sigma(\langle Z_{f\restriction 1}, Z_{f\restriction 2}, \ldots, Z_{f\restriction n}, Z_{f\restriction(n+1)} \rangle)$.

Since $\langle Z_{f\restriction 1}, Z_{f\restriction 2}, \ldots \rangle \in \mathcal{A}^\omega$, it follows that $\bigcup_{n<\omega} \sigma(\langle Z_{f\restriction 1}, Z_{f\restriction 2}, \ldots, Z_{f\restriction n} \rangle) \in \mathcal{B}$. Let $b(n) \prec f$ denote that $b(n) \in \omega^{<\omega}$ is a finite restriction of $f \in \omega^\omega$. The result then follows from

$$
\begin{aligned}
\bigcup_{n<\omega} \tau(A_n, n) &= \bigcup_{n<\omega} \sigma(\langle Z_{b(n)\restriction 1}, Z_{b(n)\restriction 2}, \ldots, Z_{b(n)}, A_n \rangle) \\
&\supseteq \bigcup_{n<\omega, b(n)\prec f} \sigma(\langle Z_{b(n)\restriction 1}, Z_{b(n)\restriction 2}, \ldots, Z_{b(n)}, A_n \rangle) \\
&= \bigcup_{n<\omega, b(n)\prec f} \sigma(\langle Z_{f\restriction 1}, Z_{f\restriction 2}, \ldots, Z_{f\restriction|b(n)|}, A_{b\leftarrow(f\restriction|b(n)|)} \rangle) \\
&= \bigcup_{n<\omega, b(n)\prec f} \sigma(\langle Z_{f\restriction 1}, Z_{f\restriction 2}, \ldots, Z_{f\restriction|b(n)|}, Z_{f\restriction(|b(n)|+1)} \rangle) \\
&= \bigcup_{n<\omega} \sigma(\langle Z_{f\restriction 1}, Z_{f\restriction 2}, \ldots, Z_{f\restriction n} \rangle) \\
&\in \mathcal{B}
\end{aligned}
$$

as $\mathcal{B}$ is closed under supersets. $\qquad\square$

Corollary 33 ([3]). *A countable space is SS^+ if and only if it is $SS^{+\mathrm{mark}}$.*

Proof. $\bigcup \mathcal{D}_X = X$ is countable, and any set containing a dense set is dense. $\qquad\square$

Corollary 34. *A second-countable space is ΩM^+ if and only if it is $\Omega M^{+\mathrm{mark}}$.*

Proof. Fix a countable base $\mathcal{B}$ for X closed under finite unions, and let Ω_X^* be the collection of ω-covers of X which only use basic open sets. One may quickly verify that ΩM^+ implies $\mathrm{II} \uparrow G_{fin}(\Omega_X^*, \Omega_X^*)$.

Note now that $\bigcup \Omega_X^* = \mathcal{B}$ is countable, and any set containing an ω-cover is an ω-cover; therefore $\mathrm{II} \uparrow G_{fin}(\Omega_X^*, \Omega_X^*)$ if and only if $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} G_{fin}(\Omega_X^*, \Omega_X^*)$.

So let σ witness $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} G_{fin}(\Omega_X^*, \Omega_X^*)$. For each ω-cover $\mathcal{U}$ and finite set $F \in [X]^{<\omega}$, choose $U_{\mathcal{U},F} \in \mathcal{U}$ such that $F \subseteq U$. Then choose a basic open set $U_{\mathcal{U},F}^*$ such that $F \subseteq U_{\mathcal{U},F}^* \subseteq U_{\mathcal{U},F}$. Finally, for each ω-cover $\mathcal{U}$, let

$$
\mathcal{U}^* = \{U_{\mathcal{U},F}^* : F \in [X]^{<\omega}\} \in \Omega_X^*.
$$

Define the strategy τ such that $\tau(\mathcal{U}, n) = \{U_{\mathcal{U},F} : U_{\mathcal{U},F}^* \in \sigma(\mathcal{U}^*, n)\}$. For any sequence of ω-covers $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$, it follows that $\bigcup_{n<\omega} \sigma(\mathcal{U}_n^*, n)$ is an ω-cover, and therefore $\bigcup_{n<\omega} \tau(\mathcal{U}_n, n)$ is an ω-cover also. $\qquad\square$

Corollary 35 ([4]). *A second-countable space is M^+ if and only if it is $M^{+\mathrm{mark}}$.*

Proof. An easy adaptation of the preceding proof, replacing ω-covers with open covers, and replacing $F \in [X]^{<\omega}$ with $x \in X$. $\qquad\square$

5. Strong variants of selection principles and games

Let $S_1(\mathcal{A}, \mathcal{B}), G_1(\mathcal{A}, \mathcal{B})$ be the natural variants of $S_{fin}(\mathcal{A}, \mathcal{B}), G_{fin}(\mathcal{A}, \mathcal{B})$ where each choice by II must either be a single element or singleton (whichever is more convenient for the proof at hand), rather than a finite set. Convention calls for denoting these as *strong* versions of the corresponding selection principles and games, although the "strong Menger" property is commonly known as "Rothberger" who first investigated the principle in [8]. We will thus call "strong Ω-Menger" "Ω-Rothberger" and shorten it with ΩR, and otherwise attach the prefix "s" when abbreviating to all other strong variants.

Theorem 36. *The following are equivalent for any topological space X.*
- *X is sSS (resp. sSS^+, $sSS^{+\mathrm{mark}}$).*
- *X is separable and sCDFT (resp. $sCDFT^+$, $sCDFT^{+\mathrm{mark}}$).*
- *X has a countable dense subset D where $sCDFT_x$ (resp. $sCDFT_x^+$, $sCDFT_x^{+\mathrm{mark}}$) holds for all $x \in D$.*

Proof. We need only show that the final condition implies the first. Let $D = \{d_i : i < \omega\}$.

Let $\{D_{m,n} \in \mathcal{D}_X : m, n < \omega\}$, and let $x_{i,n} \in D_{i,n}$ witness $sCDFT_{d_i}$, so

$$d_i \in \overline{\{x_{i,n} : n < \omega\}} \subseteq \overline{\{x_{m,n} : m, n < \omega\}}$$

and as $D \subseteq \overline{\{x_{m,n} : m, n < \omega\}}$ it follows that

$$X \subseteq \overline{D} \subseteq \overline{\overline{\{x_{m,n} : m, n < \omega\}}} = \overline{\{x_{m,n} : m, n < \omega\}}.$$

Therefore $x_{m,n} \in D_{m,n}$ witnesses sSS.

Let σ_i be a witness for $sCDFT_{d_i}^+$ for each $i < \omega$. Define $p : \omega \to \omega$ partitioning ω such that $|p^{\leftarrow}(i)| = \omega$ for all $i < \omega$. Essentially, we will partition each attack by I according to p into ω-many subattacks, and defeat each subattack to produce a blade of d_n.

For a nonempty finite sequence t, let t' be its subsequence deleting all terms of index n where $p(n) \neq p(|t| - 1)$. Note that the final term of t is the final term of t'. Thus the function τ defined by

$$\tau(t) = \sigma_{p(|t|-1)}(t')$$

is a legal strategy.

Let $\alpha \in \mathcal{D}_X^\omega$ be an attack by I, and let α_i be its subsequence deleting all terms of index n where $p(n) \neq i$. By $sCDFT_{d_i}^+$, we have

$$d_i \in \overline{\{\sigma_i(\alpha_i \upharpoonright (n+1)) : n < \omega\}} = \overline{\{\tau(\alpha \upharpoonright (n+1)) : n \in p^{\leftarrow}(i)\}} \subseteq \overline{\{\tau(\alpha \upharpoonright (n+1)) : n < \omega\}}$$

and as $D \subseteq \overline{\{\tau(\alpha \upharpoonright (n+1)) : n < \omega\}}$ it follows that

$$X \subseteq \overline{D} \subseteq \overline{\overline{\{\tau(\alpha \upharpoonright (n+1)) : n < \omega\}}} = \overline{\{\tau(\alpha \upharpoonright (n+1)) : n < \omega\}}.$$

Therefore τ witnesses SS^+.

Let σ_i be a witness for $sCDFT_{d_i}^{+\mathrm{mark}}$ for each $i < \omega$. Again, we will partition each attack by I according to p into ω-many subattacks, and defeat each subattack to produce a blade of d_n.

We define the Markov strategy τ by

$$\tau(D, n) = \sigma_{p(n)}(D, |\{m < n : p(m) = p(n)\}|).$$

Let $\alpha \in \mathcal{D}_X^\omega$, and let α_i be its subsequence removing all terms of index n such that $p(n) \neq i$. By $sCDFT_{d_i}^{+\mathrm{mark}}$, we have

$$d_i \in \overline{\{\sigma_i(\alpha_i(n), n) : n < \omega\}} = \overline{\{\tau(\alpha(n), n) : n \in p^\leftarrow(i)\}} \subseteq \overline{\{\tau(\alpha(n), n) : n < \omega\}}$$

and as $D \subseteq \overline{\{\tau(\alpha(n), n) : n < \omega\}}$ it follows that

$$X \subseteq \overline{D} \subseteq \overline{\overline{\{\tau(\alpha(n), n) : n < \omega\}}} = \overline{\{\tau(\alpha(n), n) : n < \omega\}}.$$

Therefore τ witnesses $SS^{+\mathrm{mark}}$. $\qquad\square$

As mentioned earlier, the following is a result of Sakai:

Theorem 37 ([9]). *The following are equivalent for any $T_{3\frac{1}{2}}$ topological space X.*

- *X is ΩR.*
- *$C_p(X)$ is $sCFT$.*
- *$C_p(X)$ is $sCDFT$.*

The corresponding game-theoretic results also hold.

Theorem 38. *The following are equivalent for any $T_{3\frac{1}{2}}$ topological space X.*

- *X is ΩR (resp. ΩR^+, $\Omega R^{+\mathrm{mark}}$).*
- *$C_p(X)$ is $sCFT$ (resp. $sCFT^+$, $sCFT^{+\mathrm{mark}}$).*
- *$C_p(X)$ is $sCDFT$ (resp. $sCDFT^+$, $sCDFT^{+\mathrm{mark}}$).*

Proof. In Lemmas 20 and 21, $|\tau(t)| = |\sigma(t)|$. Therefore they may be extended to the strong cases requiring $|\tau(t)| = |\sigma(t)| = 1$, which yields our result. $\qquad\square$

Theorem 39. *Let ω_1 have the discrete topology. Then $L(\omega_1)$ is ΩR^+ but not $\Omega M^{+\mathrm{mark}}$.*

Proof. The strategy constructed in Theorem 26 is a witness. $\qquad\square$

Corollary 40. *$C_p(L(\omega_1))$ is $sCFT^+$ but not $CDFT^{+\mathrm{mark}}$.*

Lemma 41. *Let $\mathcal{A}$ be a family of sets where $|\bigcup \mathcal{A}| \leq \omega$ and let $\mathcal{B}$ be a family of sets closed under supersets. Then $\mathrm{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$ if and only if $\mathrm{II} \underset{mark}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$.*

Proof. In the proof of Lemma 32, $|\tau(t)| = |\sigma(t)|$. $\qquad\square$

Corollary 42. *A countable space is sSS^+ if and only if it is $sSS^{+\mathrm{mark}}$.*

Corollary 43. *A second-countable space is ΩR^+ if and only if it is $\Omega R^{+\mathrm{mark}}$.*

Corollary 44. *A second-countable space is R^+ if and only if it is $R^{+\mathrm{mark}}$.*

6. ACKNOWLEDGEMENTS

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