ALMOST COMPATIBLE FUNCTIONS AND INFINITE LENGTH GAMES

STEVEN CLONTZ AND ALAN DOW

ABSTRACT. $\mathcal{A}'(\kappa)$ asserts the existence of pairwise almost compatible finite-to-one functions $A \to \omega$ for each countable subset A of κ. The existence of winning 2-Markov strategies in several infinite-length games, including the Menger game on the one-point Lindelöfication $\kappa^\dagger$ of κ, are guaranteed by $\mathcal{A}'(\kappa)$. $\mathcal{A}'(\kappa)$ is implied by the existence of cofinal Kurepa families of size κ, and thus holds for all cardinals less than $\aleph_\omega$. It's consistent that $\mathcal{A}'(\aleph_\omega)$ fails, but there must always be a winning 2-Markov strategy for the second player in the Menger game on $\omega_\omega^\dagger$.

1. Introduction

Definition 1. Two functions f, g are almost compatible, that is, $f \sim g$ when $\{a \in dom\, f \cap dom\, g : f(a) \neq g(a)\}$ is finite.

Marion Scheepers used almost compatible functions in [9] in order to study the existence of limited information strategies on a variation of the meager-nowhere dense game he introduced in [10].

Game 2. Let $Sch_{C,F}^{\cup,\subset}(\kappa)$ denote *Scheepers' strict countable-finite union game* with two players $\mathscr{C}$, $\mathscr{F}$. In round 0, $\mathscr{C}$ chooses $C_0 \in [\kappa]^{\leq\omega}$, followed by $\mathscr{F}$ choosing $F_0 \in [\kappa]^{<\omega}$. In round $n+1$, $\mathscr{C}$ chooses $C_{n+1} \in [\kappa]^{\leq\omega}$ such that $C_{n+1} \supset C_n$, followed by $\mathscr{F}$ choosing $F_{n+1} \in [\kappa]^{<\omega}$.

$\mathscr{F}$ wins the game if $\bigcup_{n<\omega} F_n \supseteq \bigcup_{n<\omega} C_n$; otherwise, $\mathscr{C}$ wins.

Of course, with perfect information this game is trivial: during round n player $\mathscr{F}$ simply chooses n ordinals from each of the n countable sets played by $\mathscr{C}$. However, if $\mathscr{F}$ is limited to using information from the last k moves by $\mathscr{C}$ during each round, the task becomes more difficult. Call such a strategy a *k-tactical strategy* or *k-tactic*; if using the round number is allowed, then the strategy is called a *k-Markov strategy* or a *k-mark*.

Definition 3. The statement $\mathcal{A}(\kappa)$ (given as $S(\kappa, \aleph_0, \omega)$ in [9] and $S(\kappa)$ in [1]) claims that there exist one-to-one functions $f_A : A \to \omega$ for each $A \in [\kappa]^{\leq\aleph_0}$ such that the collection $\{f_A : A \in [\kappa]^{\leq\aleph_0}\}$ is pairwise almost compatible.

In the same paper, Scheepers noted that $\mathcal{A}(\omega_1)$ holds in ZFC, and that it's possible to force $\mathfrak{c}$ to be arbitrarily large while preserving $\mathcal{A}(\mathfrak{c})$. However, $\mathcal{A}(\mathfrak{c}^+)$ always fails. This axiom may be applied to obtain a winning 2-tactic for $\mathscr{F}$ in the countable-finite game.

In [1], Clontz related this game to a game which may be used to characterize the Menger covering property of a topological space.

Game 4. Let $Men_{C,F}(X)$ denote the *Menger game* with players $\mathscr{C}$, $\mathscr{F}$. In round n, $\mathscr{C}$ chooses an open cover $\mathcal{U}_n$, followed by $\mathscr{F}$ choosing subset F_n of X which may be finitely covered by $\mathcal{U}_n$.

$\mathscr{F}$ wins the game if $X = \bigcup_{n<\omega} F_n$, and $\mathscr{C}$ wins otherwise.

This characterization is slightly different than the typical characterization in which the second player first chooses a specific finite subcollection $\mathcal{F}_n$ of the cover itself and lets $F_n = \bigcup \mathcal{F}_n$, denoted as $G_{fin}(\mathcal{O}, \mathcal{O})$ in [11]. However, it's easily seen that these games are equivalent for perfect information strategies (so both characterize the Menger property), and this characterization is more convenient for our concerns.

Definition 5. Let $\kappa^\dagger = \kappa \cup \{\infty\}$ where κ is discrete and ∞'s neighborhoods are the co-countable sets containing it.

The relationship between $Sch^{\cup,\subset}_{C,F}(\kappa)$ and $Men_{C,F}\left(\kappa^\dagger\right)$ is strong; in both games $\mathscr{C}$ essentially chooses a countable subset of κ followed by $\mathscr{F}$ choosing a finite subset of that choice, and it's easy to see the winning perfect information strategy for $\mathscr{F}$ in both games. In addition, it was shown in [1] that when $\mathcal{A}(\kappa)$ holds, $\mathscr{F}$ has a winning 2-Markov strategy in $Men_{C,F}\left(\kappa^\dagger\right)$.

One source of motivation is to make progress on the following open question:

Question 6. *Does there exist a topological space X for which $\mathscr{F} \uparrow Men_{C,F}(X)$ but $\mathscr{F} \underset{2\text{-}mark}{\not\uparrow} Men_{C,F}(X)$? (That is, the second player can win the Menger game on X with perfect information but not with 2-Markov information.)*

2. One-to-one and finite-to-one almost compatible functions

We may weaken Scheeper's $\mathcal{A}(\kappa)$ as follows:

Definition 7. The statement $\mathcal{A}'(\kappa)$ weakens $\mathcal{A}(\kappa)$ by only requiring the witnessing almost-compatible functions $f_A : A \to \omega$ to be finite-to-one.

Proposition 8. *$\mathcal{A}(\kappa)$ and $\mathcal{A}'(\kappa)$ need only be witnessed by functions $\{f_A : A \in \mathcal{S}\}$ for some family $\mathcal{S}$ cofinal in $[\kappa]^{\leq\aleph_0}$.*

Proof. For each $A \in [\kappa]^{\leq\aleph_0}$ choose $A' \supseteq A$ from $\mathcal{S}$ and let $g_A = f_{A'} \restriction A$. $\square$

In the next section we will show that $\mathcal{A}'(\kappa)$ is sufficient for the applications to the Scheepers and Menger games. In the meantime, we will demonstrate that $\mathcal{A}'(\kappa)$ is strictly weaker than $\mathcal{A}(\kappa)$.

Recall the following.

Definition 9. A Kurepa family $\mathcal{K} \subseteq [\kappa]^{\aleph_0}$ on κ satisfies that $\mathcal{K} \restriction A = \{K \cap A : K \in \mathcal{K}\}$ is countable for each $A \in [\kappa]^{\aleph_0}$. Let $\mathcal{K}(\kappa)$ be the statement claiming there exists a Kurepa family on κ cofinal in $[\kappa]^{\aleph_0}$.

Theorem 10. $\mathcal{K}(\kappa) \Rightarrow \mathcal{A}'(\kappa)$.

Proof. Let $\mathcal{K} = \{K_\alpha : \alpha < \theta\}$ be a cofinal Kurepa family on κ. We first define $f_\alpha : K_\alpha \to \omega$ for each $\alpha < \theta$.

Suppose we've already defined pairwise almost compatible finite-to-one functions $\{f_\beta : \beta < \alpha\}$. To define f_α, we first recall that $\mathcal{K} \restriction K_\alpha$ is countable, so we may choose $\beta_n < \alpha$ for $n < \omega$ such that $\{K_\beta : \beta < \alpha\} \restriction K_\alpha \setminus \{\emptyset\} = \{K_\alpha \cap K_{\beta_n} : n < \omega\}$. Let $K_\alpha = \{\delta_{i,j} : i \leq \omega, j < w_i\}$ where $w_i \leq \omega$ for each $i \leq \omega$, $K_\alpha \cap \big(K_{\beta_n} \setminus \bigcup_{m<n} K_{\beta_m}\big) = \{\delta_{n,j} : j < w_n\}$, and $K_\alpha \setminus \bigcup_{n<\omega} K_{\beta_n} = \{\delta_{\omega,j} : j < w_\omega\}$. Then let $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ for $n < \omega$ and $f_\alpha(\delta_{\omega,j}) = j$ otherwise.

We should show that f_α is finite-to-one. Let $n < \omega$. Since $f_\alpha(\delta_{m,j}) \geq m$, we only consider the finite cases where $m \leq n$. Since each f_{β_m} is finite-to-one, $f_{\beta_m}(\delta_{m,j}) \leq n$ for only finitely many j. Thus $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$ maps to n for only finitely many j.

We now want to demonstrate that $f_\alpha \sim f_{\beta_n}$ for all $n < \omega$. Note $\delta_{m,j} \in K_{\beta_n}$ implies $m \leq n$. For $m = n$, we have $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ which differs from $f_{\beta_n}(\delta_{n,j})$ for only the finitely many j which are mapped below n by f_{β_n}. For $m < n$ and $\delta_{m,j} \in K_{\beta_n}$,we have $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$ which can only differ from $f_{\beta_n}(\delta_{m,j})$ for only the finitely many j which are mapped below m by f_{β_m} or the finitely many j for which the almost compatible $f_{\beta_n} \sim f_{\beta_m}$ differ.

Finally for any $\beta < \alpha$, we may conclude $f_\alpha \sim f_\beta$ since there is some β_n with $K_\alpha \cap K_\beta = K_\alpha \cap K_{\beta_n}$, $f_\alpha \sim f_{\beta_n}$, and $f_{\beta_n} \sim f_\beta$. □

We now construct a topology on ω_n for each $n < \omega$ which will witness a Kurepa family of size $\aleph_n$. A similar construction was previously shown by Juhász et. al. in [5], and the relationship of Kurepa families and such spaces has also been investigated in a preprint of Nyikos [8].

Proposition 11. *Let X be a T_2 space with a base of countable and compact neighborhoods. Then X is locally metrizable with a base of compact open countable sets.*

Proof. For each point x let K be a countable and compact neighborhood of x, and it follows that it is contained in a countable, open, and locally compact neighborhood W of x, which in turn is zero-dimensional and metrizable. So choose V clopen in W such that $x \in V \subseteq K$; V is a compact open neighborhood of x in X. □

Definition 12. A topological space is said to be *ω-bounded* if each countable subset of the space has compact closure. As in [5] we call a T_2, locally countable, ω-bounded space *splendid*, and let $\mathcal{S}(\kappa)$ represent the claim that there exists a splendid space of cardinality κ.

Proposition 13. *Let X be a T_2 space with cardinality less than $\aleph_\omega$ which is locally countable and ω-bounded. Then the closure operation preserves cardinality and weight.*

Proof. Note that the closure of any countable neighborhood is compact, and any Lindelöf set is countable. This space is locally metrizable and thus first-countable, so cardinality and weight coincide for any subspace. The result is obvious if A is countable; otherwise let $A = \{a_\alpha : \alpha < \omega_{n+1}\}$ and since basic neighborhoods are countable note any limit point of A is a limit point of $A_\beta = \{a_\alpha : \alpha < \beta\}$ for some $\beta < \omega_{n+1}$. Thus $\overline{A} = \bigcup_{\beta<\omega_{n+1}} \overline{A_\beta}$ and by induction $|\overline{A}| = |A|$. □

Lemma 14. *Let X be a T_2 space with cardinality less than $\aleph_\omega$ which is locally countable and locally compact, and such that its closure operation preserves cardinalities. Then X has an ω-bounded extension $\tilde{X}$ with the same properties where $\tilde{X} \setminus X$ has the same cardinality as X.*

Proof. We prove this by induction on n. If $n = 0$, then we can just use the one-point compactification of two copies of X. So suppose $n > 0$ and that $X = \omega_n$ has an appropriate topology. Note that X has a base of countable and compact neighborhoods since the closure operation preserves cardinalities.

For each $\alpha < \omega_n$, γ_α may be chosen such that both the closure of the set α in X and a countable neighborhood of the point α are subsets of γ_α. Note that the set $\{\lambda < \omega_n : \alpha < \lambda \Rightarrow \gamma_\alpha < \lambda\}$ is a cub subset of ω_n containing a cub subset C of limit ordinals. Now for each $\lambda \in C$, the set λ is open as $\alpha < \lambda$ belongs to the neighborhood $\gamma_\alpha \subsetneq \lambda$. Also, if λ has uncountable cofinality, then for $\beta \geq \lambda$ and any countable neighborhood U of β, $U \cap \lambda = U \cap \alpha$ for some $\alpha < \lambda$; thus $U \setminus \overline{\alpha} = U \setminus \lambda$ is a neighborhood of β, showing that λ is clopen.

Let $\tilde{X} = \omega_n \times 2$. By induction on $\lambda \in C$ we will define compatible topologies for $\tilde{X}_\lambda = \omega_n \times \{0\} \cup \lambda \times \{1\}$ such that

- $\omega_n \times \{0\}$ is an open copy of X,
- $\lambda \times 2$ is open, and when $\mathrm{cf}(\lambda) > \omega$ also closed,
- the space has a base of countable and compact neighborhoods, and
- when λ is a successor, for each $\alpha < \lambda$ the closure of $\alpha \times 2$ is an ω-bounded subset of $\lambda \times 2$.

We first consider the case $n = 1$. If λ is a limit in C, then $\tilde{X}_\lambda = \bigcup_{\mu \in C \cap \lambda} \tilde{X}_\mu$ satisfies the induction requirements. Otherwise we choose an increasing sequence of ordinals $\{\alpha_k : k \in \omega\}$ with limit λ such that α_0 is the predecessor of λ in C, or $\alpha_0 = 0$ if λ is the least element of C.

The subspace $\overline{\lambda} \times \{0\} \cup \alpha_0 \times 2$ of X is countable and locally compact; therefore it is metrizable and zero-dimensional. So we may choose increasing sets U_k for $k < \omega$ which are clopen in this topology and satisfy

$$\overline{\alpha_k \times \{0\} \cup \alpha_0 \times 2} = \overline{\alpha_k} \times \{0\} \cup \alpha_0 \times 2 \subseteq U_k \subseteq \lambda \times \{0\} \cup \alpha_0 \times 2$$

Note that U_k is also clopen in $\tilde{X}_{\alpha_0}$ since it is closed in $\overline{\lambda} \times \{0\} \cup \alpha_0 \times 2$ and open in $\lambda \times \{0\} \cup \alpha_0 \times 2$.

We need only describe a base for the points $\langle \alpha, 1 \rangle \in (\lambda \setminus \alpha_0) \times \{1\}$. We do so by letting $\langle \alpha, 1 \rangle$ be isolated when $\alpha \notin \{\alpha_k : k < \omega\}$, and giving $\langle \alpha_k, 1 \rangle$ the open neighborhoods $(U_k \cup ((\alpha_k + 1) \times \{1\})) \setminus K$ for each compact subset K of $U_k \cup (\alpha_k \times \{1\})$; that is, $\langle \alpha_k, 1 \rangle$ is the one point compactifying $U_k \cup (\alpha_k \times \{1\})$.

The first two requirements of our inductive hypothesis are obviously satsified. Note points in $\lambda \times 2$ are covered by the compact countable neighborhood $U_k \cup ((\alpha_k + 1) \times \{1\})$ for some $k < \omega$, and for points in $(\omega_n \setminus \lambda) \times \{0\}$ we may use a compact countable neighborhood from X. For the final requirement, note that for $\alpha < \lambda$, we may choose $\alpha < \alpha_k < \lambda$ and note that $\alpha \times 2$ is contained in the compact subset $U_k \cup ((\alpha_k + 1) \times \{1\})$ of $\lambda \times 2$.

For the case $n > 1$, we may assume that the successors in C have uncountable cofinality. We again proceed by induction on $\lambda \in C$. Again when λ is a limit in C, $\tilde{X}_\lambda = \bigcup_{\mu \in C \cap \lambda} \tilde{X}_\mu$ satisfies the given requirements; in particular if $\alpha < \lambda$, then $\alpha < \mu < \lambda$ for some successor $\mu \in C$ with uncountable cofinality. As such, the closure of $\alpha \times 2$ is an ω-bounded subset of the clopen $\mu \times 2$ and therefore also of $\lambda \times 2$. In case λ is not a limit of C, then λ has uncountable cofinality and a predecessor $\mu \in C$. We therefore have that $\lambda \times \{0\}$ is clopen in $\omega_n \times \{0\}$. Since the cardinality of $\lambda \times \{0\} \cup \mu \times 2$ is less than $\aleph_n$, we may simply apply the induction hypothesis to choose an appropriate topology for $\lambda \times 2$.

As a result, $\tilde{X} = \bigcup_{\lambda \in C} \tilde{X}_\lambda$ is ω-bounded as any countable set is contained in some $\alpha \times 2$ for $\alpha < \lambda \in C$. □

Theorem 15. *For each $k < \omega$, there is a T_2, locally countable, ω-bounded topology on ω_k. That is, $\mathcal{S}(\aleph_k)$ for all $k < \omega$.*

Proof. Apply the previous lemma to ω_n with the discrete topology. □

Lemma 16. *The family of compact open sets in a locally countable, ω-bounded topological space X is a Kurepa family cofinal in $[X]^\omega$. That is, $\mathcal{S}(\kappa) \Rightarrow \mathcal{K}(\kappa)$.*

Proof. Let $\mathcal{K}$ collect all compact open subsets of X. Of course, every Lindelöf set in a locally countable space is countable, and the closure of every countable set is a compact countable set; thus $\mathcal{K}$ is cofinal in $[X]^\omega$. It is Kurepa since every countable set is contained in a countable compact open subspace of X; this subspace has a countable base of compact open sets, which closed under finite unions enumerates all compact open subsets of the subspace. □

Corollary 17. *$\mathcal{K}(\aleph_k)$ for all $k < \omega$.*

Alternatively, the previous corollary may be obtained via an observation of Todorcevic communicated by Dow in [2]: if every Kurepa family of size at most κ extends to a cofinal Kurepa family, then the same is true of κ^+.

Nyikos points out in [8] that a cofinal Kurepa family may be used to construct a locally metrizable, ω-bounded, zero-dimensional space with appropriate cardinality, but whether this can be strengthened to locally countable and ω-bounded (as asked in [5]) remains an open question.

Also left open is this extension of the question asked in [8] and [5] on the possible equivalence of $\mathcal{S}(\kappa)$ and $\mathcal{K}(\kappa)$.

Question 18. *May any of the implications in the theorem $\mathcal{S}(\kappa) \Rightarrow \mathcal{K}(\kappa) \Rightarrow \mathcal{A}'(\kappa)$ be reversed?*

Regardless, we have obtained our desired result.

Corollary 19. *$\mathcal{A}'(\aleph_k)$ for all $k < \omega$.*

3. Consistency results

As noted in [2], Jensen's one-gap two-cardinal theorem under $V = L$ introduced in [4] implies that $\mathcal{K}(\kappa)$ holds for all cardinals κ.

Corollary 20 ($V = L$)**.** *$\mathcal{A}'(\kappa)$ for all cardinals κ.*

Weakening to the continuum hypothesis, we see an obvious consequence.

Corollary 21 (CH)**.** *$\mathcal{A}'(\mathfrak{c}^+)$, but $\neg\mathcal{A}(\mathfrak{c}^+)$.*

But CH is not required to have $\mathcal{A}(\aleph_2)$ fail. We take the following lemma from Kunen.

Lemma 22 ([6])**.** *Assume that $G \subset \mathrm{Fn}(\omega_2, 2)$ is a generic filter, and let $\mu \in \omega_2$. Then the final model $V[G]$ can be regarded as forcing with $\mathrm{Fn}(\omega_2 \setminus \mu, 2)$ over the model $V[G_\mu]$. Let $\dot{n}, \ddot{n}$ be a $\mathrm{Fn}(\omega_2, 2)$-, $\mathrm{Fn}(\omega_2 \setminus \mu, 2)$-name respectively for each $n < \omega$. Then for each $\mathrm{Fn}(\omega_2, 2)$-name $\dot{A}$ for $A \subseteq \omega$, there is a canonical $\mathrm{Fn}(\omega_2 \setminus \mu, 2)$-name*

$$\ddot{A} = \{(\ddot{n}, p \restriction [\mu, \omega_2)) : (\dot{n}, p) \in \dot{A}\}$$

which satisfies $A = \mathrm{val}(\dot{A}, G) = \mathrm{val}(\ddot{A}, G_\mu)$.

Theorem 23. *It follows from the consistency of ZFC that there exists a model of ZFC for which $\mathfrak{c} = \aleph_2$ and $\neg\mathcal{A}(\aleph_2)$.*

Proof. The forcing poset is $\mathbb{P} = \mathrm{Fn}(\omega_2, 2)$. Let $\{\dot{f}_A : A \in [\omega_2]^\omega\}$ be a family of $\mathbb{P}$-names such that f_A is a one-to-one function from A into ω in the ground model. Let M be an elementary submodel of the universe containing $\mathbb{P}$ such that $|M| = \omega_1 = \mathfrak{c}$ and $M^\omega \subset M$.

Let λ be the ordinal $M \cap \omega_2$. We may assume $\dot{f}_{[\lambda,\lambda+\omega)} = \{((\lambda + k_p, m_p)^\cdot, p) : p \in C\}$ for some countable $C \subseteq \mathrm{Fn}(\omega_2, \omega)$. Let $\mathrm{supp}(\dot{f}_{[\lambda,\lambda+\omega)}) = \bigcup_{p\in C} \mathrm{dom}\, p$.

Let δ_λ denote the order type of $\mathrm{supp}(\dot{f}_{[\lambda,\lambda+\omega)}) \setminus \lambda$, and for $\mu \leq \lambda$ let $\varphi'_{\mu,\lambda}$ be the order-preserving bijection from $\mu \cup \left(\mathrm{supp}(\dot{f}_{[\lambda,\lambda+\omega)}) \setminus \lambda\right)$ onto the ordinal $\mu + \delta_\lambda$. This lifts canonically to an order-preserving bijection $\varphi_{\mu,\lambda} : \mathrm{Fn}(\mu \cup Y, 2) \to \mathrm{Fn}(\mu + \delta, 2)$.

Consider the $\mathbb{P}$-name $F_M = \{((\lambda + k_p, m_p)^\cdot, \varphi_{\mu,\lambda}(p)) : ((\lambda + k_p, m_p)^\cdot, p) \in \dot{f}_A\}$.

Now the object F_M is an element of M, and M believes this statement is true:

$$(\forall\beta \in \omega_2)\ (\exists\beta < \lambda \in \omega_2)\ \ \mathrm{supp}(\dot{f}_{[\lambda,\lambda+\omega)}) \cap \lambda \subset \mu \ \text{ and } \ F_M = \varphi_{\mu,\lambda}(\dot{f}_{[\lambda,\lambda+\omega)})$$

But now, this means that, not only is there an $\alpha \in M$, $F_M = \varphi_{\mu,\alpha}(\dot{f}_{[\alpha,\alpha+\omega)})$ but also that there is an increasing sequence $\{\alpha_\xi : \xi \in \omega_1\} \subset \lambda$ of such α's satisfying that, for each ξ we have that $\mathrm{supp}(\dot{f}_{[\alpha_\xi,\alpha_\xi+\omega)})$ is contained in $\alpha_{\xi+1}$.

Choose such a sequence. This means that if we let $A = \bigcup_{n>0}[\alpha_n, \alpha_n + \omega)$ we have the name $\dot{f}_A$ in M. This then means that all the $((\beta, m), p)$ appearing in $\dot{f}_A$ have the property that $dom(p)$ is contained in M. There is, within M, a name $\dot{g}$ satisfying that $\dot{f}_A(\alpha_n + k) = \dot{f}_{[\alpha_n,\alpha_n+\omega)}(\alpha_n + k)$ for all $k > \dot{g}(n)$.

We now apply the above Lemma using $\mu = \mu_0$ and we are now working in the extension $V[G_\mu]$. We will abuse the notation and use $\dot{f}_{[\alpha_n,\alpha_n+\omega)}$ instead of $\dot{f}_{[\alpha_n,\alpha_n+\omega)}(G_\mu)$ as defined in the Lemma. We work for a contradiction. Something special has now happened, namely, the supports of the names $\{\dot{f}_{[\alpha_n,\alpha_n+\omega)} : 0 <$

$n < \omega\}$ are pairwise disjoint and also disjoint from the support of the name $\dot{f}_{[\lambda,\lambda+\omega)}$ (under the same convention about G_μ. And not only that, these names are pairwise isomorphic (in the way that they all map to F_M).

Since A is disjoint from $[\lambda, \lambda + \omega)$, there must be an integer ℓ together with a condition $q \in Fn(\omega_2 \setminus \mu, 2)$ satisfying that for all $n > \ell$, q forces that

"if $k > \dot{g}(n)$ (since $\alpha_n + k \in A$) then $\dot{f}_{[\alpha_n,\alpha_n+\omega)}(\alpha_n + k) \neq \dot{f}_{[\lambda,\lambda+\omega)}(\lambda + k)$".

Choose n large enough so that $dom(q) \cap [\alpha_n, \mu_{n+1})$ is empty. Choose $q_1 < q \restriction \lambda$ (in M) so that

$$\varphi_{\mu,\alpha_n}(q_1 \restriction \mathrm{supp}(\dot{f}_{[\alpha_n,\alpha_n+\omega)}) = \varphi_{\mu,\lambda}(q \restriction \mathrm{supp}(\dot{f}_{[\lambda,\lambda+\omega)})$$

and then (again in M) choose $q_2 < q_1$ so that it both forces a value L on $\ell + \dot{g}(n)$ and subsequently forces a value m on $\dot{f}_{[\alpha_n,\alpha_n+\omega)}(\alpha_n + L + 1)$. But now, again calculate

$$q_3 = \varphi^{-1}_{\mu,\lambda} \circ \varphi_{\mu,\alpha_n}(q_2 \restriction \mathrm{supp}(\dot{f}_{[\alpha_n,\alpha_n+\omega)}))$$

and, by the isomorphisms, we have that q_3 forces that $\dot{f}_{[\lambda,\lambda+\omega)}(\lambda + L + 1) = m$.

Technically (or with more care) all of this is taking place in the poset $\mathrm{Fn}(\omega_2 \setminus \mu, 2)$ and this means that q_3 and q are all compatible with each other.

Follow the bouncing ball: it suffices to consider $q(\beta) = e$ and to assume that $q_3(\beta)$ is defined. Since $q_3(\beta)$ is defined, we have that there is a $\beta' \in dom(q_2)$ such that $\varphi_{\mu,\lambda}(\beta) = \varphi_{\mu,\alpha_n}(\beta')$, and that $q_3(\beta) = q_2(\beta')$. But, by definition of q_1, $\beta' \in dom(q_1)$ and even that $q_1(\beta') = q(\beta)$. Then, since $q_2 < q_1$, we have that $q_2(\beta') = q_1(\beta') = q(\beta)$. This completes the circle that $q_3(\beta) = q(\beta)$.

Finally, our contradiction is that $q_3 \cup q_2 \cup q$ forces that $k = L + 1$ violates the quoted statement above. $\square$

We are also able to force $\mathcal{A}'(\kappa)$ to fail for every cardinal other than the first ω-many we've already guaranteed.

Theorem 24. *It follows from the existence of a 2-huge cardinal that there is a model of ZFC for which* $\neg\mathcal{A}'(\aleph_\omega)$.

Proof. We will need the model constructed in [7] in which an instance of Chang's conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is shown to fail.

We can take as a given (as shown in [7, Theorem 5]) that we may assume that we have a model V of GCH in which there are regular limit cardinals $\kappa < \lambda$ satisfying that $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$.

What this says is that if L is a countable language with at least one unary relation symbol R and M is a model of L with base set $\lambda^{+\omega+1}$ in which the interpretation of R has cardinality $\lambda^{+\omega}$, then M has an elementary submodel N of cardinality $\kappa^{+\omega+1}$ in which $R \cap N$ has cardinality $\kappa^{+\omega}$ (of course $R \cap N$ is the interpretation of R in N because $N \prec M$).

The interested reader will want to know that it is shown in [7] that if κ is a 2-huge cardinal and j is the 2-huge embedding with critical point κ, then with $\lambda = j(\kappa)$

one has that $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$ holds. There is no loss of generality to also assume that GCH holds in this model.

Let $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$ be a scale in $\Pi\{\lambda^{+n+1} : n \in \omega\}$ ordered by the usual mod finite coordinatewise ordering. For convenience we may assume that $h_\xi(n) \geq \lambda^{+n}$ for all ξ and all n. For each integer m the cofinality of the mod finite ordering on $\Pi\{\lambda^{+n+1} : m < n \in \omega\}$ is the same as it is for the entire product $\Pi\{\lambda^{+n+1} : n \in \omega\}$. If P is any poset of cardinality less than λ^{+m} then, in the forcing extension by P, every function in $\Pi\{\lambda^{+n+1} : m < n \in \omega\}$ is bounded above by a ground model function. It therefore follows easily that in the forcing extension by P, the sequence $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$ remains cofinal in $\Pi\{\lambda^{+n+1} : n \in \omega\}$.

The forcing notion $\mathbb{P}_0$ is simply the finite condition collapse of $\kappa^{+\omega}$, i.e. $\mathbb{P}_0 = (\kappa^{+\omega})^{<\omega}$. In the forcing extension by $\mathbb{P}_0$, one now has that the ordinal $\kappa^{+\omega+1}$ from V is the first uncountable cardinal $\aleph_1$. Then in this forcing extension we let $\mathbb{P}_1$ be the countable condition Levy collapse, $Lv(\lambda, \omega_2)$, which collapses all cardinals less than λ to have cardinality at most $\aleph_1$. The poset $\mathbb{P}_1$ has cardinality λ. We treat $\mathbb{P} * \mathbb{P}_1$ as containing $\mathbb{P}_0$ as a subposet by identifying each $(p_0, 1)$ with p_0. After forcing with $\mathbb{P}_0 * \mathbb{P}_1$ we will have that ω_1 is the ordinal $\left(\kappa^{+\omega+1}\right)^V$, ω_2 is the ordinal λ, and ω_ω is the ordinal $\left(\lambda^{+\omega}\right)^V$.

Now we assume that we have an assignment $\dot{f}_{\dot{A}}$ of a $\mathbb{P}_0 * \mathbb{P}_1$-name of a finite-to-one function from $\dot{A}$ into ω for each $\mathbb{P}_0 * \mathbb{P}_1$-name of a countable subset of $\lambda^{+\omega+1}$. We will obtain a contradiction.

Let $\{\dot{A}_\xi : \xi \in \lambda^{+\omega+1}\}$ be an enumeration of all the nice $\mathbb{P}_0$-names of countable subsets of $\lambda^{+\omega}$. For each $\xi \in \lambda^{+\omega+1}$, let $\dot{f}_\xi$ be another notation for $\dot{f}_{\dot{A}_\xi}$. Since $\mathbb{P}_0$ forces that $\mathbb{P}_1$ is countably closed, the collection of all nice $\mathbb{P}_0$-names will produce all the countable sets in the extension by $\mathbb{P}_0 * \mathbb{P}_1$, but $\mathbb{P}_0 * \mathbb{P}_1$ can introduce new enumerations of these names. For each $\xi \in \lambda^{+\omega+1}$, there is a minimal ζ_ξ so that $\dot{A}_{\zeta_\xi}$ is the canonical name for the range of h_ξ. This means that $\dot{f}_{\zeta_\xi} \circ h_\xi$ is simply the $\mathbb{P}_0 * \mathbb{P}_1$-name of a finite-to-one function from ω to ω. For each $\xi \in \lambda^{+\omega+1}$, choose any $p_\xi \in \mathbb{P}_0 * \mathbb{P}_1$ so that there is a nice $\mathbb{P}_0$-name, $\dot{H}_\xi$, that is forced by p_ξ to equal $\dot{f}_{\zeta_\xi} \circ h_\xi$. Choose $\Lambda \subset \lambda^{+\omega+1}$ of cardinality $\lambda^{+\omega+1}$ and so that there is a pair $p, \dot{H}$ satisfying that $p_\xi = p$ and $\dot{H}_\xi = \dot{H}$ for all $\xi \in \Lambda$. We may assume that p is in a generic filter G.

Let $\{x_\xi : \xi \in \lambda^{+\omega+1}\}$ be any enumeration of $H(\lambda^{+\omega+1})$ such that $\{x_\xi : \xi \in \lambda^{+\omega}\}$ is also equal to $H(\lambda^{+\omega})$. We choose this enumeration in such a way that $x_\xi \in x_\eta$ implies $\xi < \eta$. We use relation symbol R_0 to code (and well order) $(H(\lambda^{+\omega+1}), \in)$ as follows: $(\xi, \eta) \in R_0$ if and only if $x_\xi \in x_\eta$. Let R_1 be a binary relation on $\kappa^{+\omega}$ so that $(\kappa^{+\omega}, R_1)$ is isomorphic to $\mathbb{P}_0$. Let R_2 be a binary relation on λ so that $R_2 \cap (\kappa^{+\omega} \times \kappa^{+\omega}) = R_1$ and (λ, R_2) is isomorphic to $\mathbb{P}_0 * \mathbb{P}_1$. Let ψ be the poset isomorphism from (λ, R_2) to $\mathbb{P}_0 * \mathbb{P}_1$.

We continue coding. We can code the sequence $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$ as another binary relation R_3 on $\lambda^{+\omega+1}$ where $R_3 \cap \big(\{\xi\} \times \lambda^{+\omega+1}\big) = \{(\xi, h_\xi(n)) : n \in \omega\}$ for each $\xi \in \lambda^{+\omega+1}$. The relation symbol R_4 can code the sequence $\{\dot{A}_\xi : \xi \in \lambda^{+\omega+1}\}$ where $(\xi, \alpha, \zeta) \in R_4$ if and only if $(\check{\alpha}, \psi(\zeta))$ is in the name $\dot{A}_\xi$. Let R_5 code this collection, i.e. $(\gamma, n, m, \eta) \in R_5$ if and only if $((n, m)\check{}, \psi(\eta)) \in \dot{H}_\gamma$. Also let R_6 code (equal) the set Λ. Finally we use the relation symbol R_7 to similarly code the sequence $\{\dot{f}_\xi : \xi \in \lambda^{+\omega+1}\}$: $(\xi, \alpha, n, \zeta) \in R_7$ if and only if $((\alpha, n)\check{}, \psi(\zeta))$ is in the name $\dot{f}_\xi$.

Needless to say, the unary relation symbol R is interpreted as the set $\lambda^{+\omega}$ for the application of $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$. Now we have defined our model M of the language $L = \{\in, R, R_0, \ldots, R_7\}$, and we choose an elementary submodel N witnessing $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$. Of course N is really just a $\kappa^{+\omega+1}$ sized subset of $\lambda^{+\omega+1}$ with the additional property that $N \cap \lambda^{+\omega}$ has cardinality $\kappa^{+\omega}$. In the forcing extension N has cardinality ω_1 and $A = N \cap \lambda^{+\omega}$ is countable.

We will need the following claim from [7]:

Claim. *We may assume that N satisfies that $N \cap \kappa^{+\omega+1}$ is transitive (i.e. an initial segment).*

Proof of Claim: Suppose our originally supplied N fails the conclusion of the claim. We know that $\kappa^{+\omega} \in N$, (via R_1) in which case so is $\kappa^{+\omega+1}$.

Then set $\beta_0 = \sup(N \cap \kappa^{+\omega+1})$ and consider the Skolem closure $Hull(N \cup \beta_0, M)$. A little informally (in that we have to formalize the enumeration of formulas as per Gödel coding) let $\{\varphi_n : n \in \omega\}$ be an enumeration of all formulas in the language

L, and let ℓ_n be the minimal integer such that the free variables of φ_n are among $\{v_0, \ldots, v_{\ell_n}\}$. Then, for each tuple $\langle \xi_1, \ldots, \xi_{\ell_n} \rangle$ of elements of $\lambda^{+\omega+1}$, we define $f_n(\xi_1, \ldots, \xi_{\ell_n})$ to be the minimal $\xi_0 \in \lambda^{+\omega+1}$ such that $M \models \varphi_n(\xi_0, \ldots, \xi_{\ell_n})$. If there is no such ξ_0, in other words if $M \models \neg\exists x\ \varphi_n(x, \xi_1, \ldots, \xi_{\ell_n})$, then set $f_n(\xi_1, \ldots, \xi_{\ell_n})$ to be 0. Now $Hull(N \cup \beta_0, M)$ is just the minimal superset X of $N \cup \beta_0$ that satisfies that $f_n[X^{\{1,\ldots,\ell_n\}}] \subset X$ for all n. Since this is simply a large algebra, we can generate all the terms t of the algebraic operations $\{f_n : n \in \omega\}$. It is easily seen that for each $\zeta \in X$, there is a term $t(v_1, \ldots, v_m)$ such that $\zeta = t(\delta_1, \ldots, \delta_m)$ for some sequence $\langle \delta_1, \ldots, \delta_m \rangle$ with each $\delta_i \in N \cup \beta_0$. Assume that $\zeta \in \kappa^{+\omega+1}$. By re-indexing the variables in the term we can assume that there is an $n \leq m$ so that $\delta_i < \beta_0$ for $1 \leq i \leq n$ and $\kappa^{+\omega+1} \leq \delta_i$ for $n < i \leq m$. Let $\vec{a}$ denote the tuple $\langle \delta_{n+1}, \ldots, \delta_m \rangle$. Choose $\eta \in N \cap \kappa^{+\omega+1}$ large enough so that $\{\delta_1, \ldots, \delta_n\}$ is contained in η. Since set-membership in M is coded by R_0 rather than $\in$ we have to argue a little less naturally. Consider the set $s_0(\eta, \vec{a}) = \{t(\gamma_1, \ldots, \gamma_n, \vec{a}) : \{\gamma_1, \cdots, \gamma_n\} \in [\eta]^{\leq n}\}$. Clearly $s_0(\eta, \vec{a})$ is a member of $H(\lambda^{+\omega+1})$. Now define $s_1(\eta, \vec{a})$ to be $\{x_\alpha : \alpha \in s_0(\eta, \vec{a})\}$, and choose the unique $\zeta_1 \in \lambda^{+\omega+1}$ such that $x_{\zeta_1} = s_1(\eta, \vec{a})$. We claim that $\zeta_1 \in N$. Note that $\alpha R_0 \zeta_1$ holds if and only if $\alpha \in s_0(\eta, \vec{a})$, and therefore

$$M \models (\forall \alpha)\, [\alpha R_0 \zeta_1 \text{ iff } (\exists \gamma_1 \in \eta) \cdots (\exists \gamma_n \in \eta)(\alpha = t(\gamma_1, \ldots, \gamma_n, \vec{a}))]\ .$$

By elementarity then we have that $\zeta_1 \in N$, and by similar reasoning the supremum, ζ_0, of $\zeta_1 \cap \kappa^{+\omega+1}$ is also in N. This of course means that $\zeta < \xi_0$. $\square$

We use the elementarity of N to deduce properties of the families $\{\dot{A}_\xi : \xi \in N\}$ and $\{\dot{f}_\xi : \xi \in N\}$. Actually the collection we are most interested in is the family $\{h_\xi : \xi \in \Lambda \cap N\}$.

Now we need a result from Shelah's pcf theory which is proven in Jech [3, 24.9]. Since $\aleph_1 = \mathfrak{c} < \kappa^{+\omega+1}$ there is a function $\langle \varrho_n : n \in \omega \rangle$ in $\Pi_n \lambda^{+\omega}$ such that the sequence $\{h_\xi : \xi \in N\}$ is unbounded mod finite in $\Pi_n \varrho_n$ For each n, $\varrho_n \leq \sup(N \cap \lambda^{+n+2})$. Since $\mathbb{P}_0$ has cardinality $\kappa^{+\omega}$, and so less than $|N| = \kappa^{+\omega+1}$, a standard argument (analogous to the fact that adding a Cohen real does not add a dominating real) shows that the sequence $\{h_\xi : \xi \in \Lambda \cap N\}$ remains unbounded mod finite in $\Pi_n \varrho_n$ (and in $\Pi_n(\varrho_n \cap N)$).

Now pass to the extension by $G \cap \mathbb{P}_0$ and let H be the function $\mathrm{val}_G(\dot{H})$, and we recall that $f_{\zeta_\xi}(h_\xi(n)) = H(n)$ for all $n \in \omega$ and $\xi \in \Lambda$. Now pass to the full extension $V[G]$ and again, since $\mathbb{P}_1$ was forced to be countably closed, the family $\{h_\xi : \xi \in \Lambda \cap N\}$ is still unbounded in $\Pi_n(\varrho_n \cap N)$ (no new elements were added). We let A be the countable set $N \cap \lambda^{+\omega}$, and for each $\xi \in \Lambda \cap N$, there is an n_ξ such that $f_\xi(h_\xi(m)) = f_A(h_\xi(m))$ for all $m > n_\xi$. There is a single n so that $\Lambda_n = \{\xi \in \Lambda \cap N : n_\xi = n\}$ has cardinality ω_1, and thus $\{h_\xi : \xi \in \Lambda_n \cap N\}$ is also unbounded in $\Pi_n(\rho_n \cap N)$. This certainly implies that there is an $m > n$ such that $\{h_\xi(m) : \xi \in \Lambda_n \cap N\}$ is infinite. This completes the proof since $f_A(h_\xi(m)) = H(m)$ for all $\xi \in \Lambda_n \cap N$. $\square$

4. Applications to infinite length games

We introduce three variations of Scheeper's game which we defined in the introduction.

Game 25. Let $Sch^{\cup,\subsetneq}_{C,F}(\kappa)$ denote the *Scheepers countable-finite union game* which proceeds analogously to $Sch^{\cup,\subset}_{C,F}(\kappa)$, except that $\mathscr{C}$'s restriction in round $n+1$ is weakened to $C_{n+1} \supseteq C_n$.

Game 26. Let $Sch^{1,\subsetneq}_{C,F}(\kappa)$ denote the *Scheepers countable-finite initial game* which proceeds analogously to $Sch^{\cup,\subsetneq}_{C,F}(\kappa)$, except that $\mathscr{F}$'s winning condition is weakened to $\bigcup_{n<\omega} F_n \supseteq C_0$.

Game 27. Let $Sch^{\cap}_{C,F}(\kappa)$ denote the *Scheepers countable-finite intersection game* which proceeds analogously to $Sch^{1,\subsetneq}_{C,F}(\kappa)$, except that $\mathscr{C}$ may choose any $C_n \in [\kappa]^{\leq\omega}$ each round, and $\mathscr{F}$'s winning condition is weakened to $\bigcup_{n<\omega} F_n \supseteq \bigcap_{n<\omega} C_n$.

In [1] Clontz extended Scheepers' application of almost-compatible injections to these game variants as well as $Men_{C,F}\left(\kappa^{\dagger}\right)$. However, when considering Markov strategies, finite-to-one functions suffice.

$$\begin{tikzcd}
 & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}\left(\kappa^{\dagger}\right) \arrow[r, leftrightarrow] \arrow[d] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}(\kappa) \arrow[d] \\
\mathcal{A}'(\kappa) \arrow[ur] \arrow[dr] & & \\
 & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{1,\subsetneq}_{C,F}(\kappa) \arrow[r, leftrightarrow] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cup,\subsetneq}_{C,F}(\kappa)
\end{tikzcd}$$

FIGURE 1. Diagram of Scheeper/Menger game implications with $\mathcal{A}'(\kappa)$

Theorem 28. *$\mathcal{A}'(\kappa)$ implies the game-theoretic conditions shown in Figure 1.*

Proof. The weaker claim $\mathcal{A}(\kappa) \Rightarrow \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}(\kappa)$ was proven in [1]; however, the strategy only required that the f_A be pairwise almost-compatible and that preimages of finite sets in each f_A are finite, which is also possible assuming $\mathcal{A}'(\kappa)$. Note that the relationships amongst the games were all shown in [1]. □

We include the following proof from [9] for the convenience of the reader.

Theorem 29. $\mathcal{A}(\kappa) \Rightarrow \mathscr{F} \underset{\textit{2-tact}}{\uparrow} Sch^{\cup,\subset}_{C,F}(\kappa)$

Proof. Let $\{f_A : A \in [\kappa]^{\leq\aleph_0}\}$ witness $\mathcal{A}(\kappa)$, and define $g_A : A \to \omega$ by $g_A(\alpha) = f_A(\alpha) - |\{\beta \in A : f_A(\beta) < f_A(\alpha)\}|$.

We claim that $\{\alpha \in A : g_A(\alpha) \leq g_B(\alpha)\}$ must be finite as it is bounded above by $\max\{M, f_A(\alpha), f_B(\alpha) : f_A(\alpha) \neq f_B(\alpha)\}$ where $M = f_B(\alpha)$ for some $\alpha \in B \setminus A$. To see this, let $f_A(\alpha) = f_B(\alpha) = N > M$ and assume $f_A(\beta) \neq f_B(\beta)$ implies $f_A(\beta), f_B(\beta) < N$. Then

$$g_A(\alpha) = N - |\{\beta \in A : f_A(\beta) < N\}| > N - |\{\beta \in B : f_B(\beta) < N\}| = g_B(\alpha)$$

with the strictness of the inequality witnessed by $f_B(\alpha) = M < N$ for some $\alpha \in B \setminus A$.

As a result,

$$\sigma(\langle A, B\rangle) = \{\alpha \in A : g_A(\alpha) \leq g_B(\alpha)\}$$

is a legal 2-tactic for $\mathscr{F}$. Let $C = \langle C_0, C_1, \dots\rangle$ be a strictly increasing sequence of countable sets and $\alpha \in C_n$. Noting that f_A is an injection and not just finite-to-one, $0 \leq g_{C_{n+m}}(\alpha)$ for all $m < \omega$, and it follows that $g_{C_{n+m}}(\alpha) \leq g_{C_{n+m+1}}(\alpha)$ for some $m < \omega$. Therefore $\alpha \in \sigma(\langle C_{n+m}, C_{n+m+1}\rangle)$. $\square$

So it would seem that $\mathcal{A}'(\kappa)$ is sufficient only when considering Markov strategies. (Of course, $\mathcal{A}'(\kappa) \Rightarrow \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cup,\subseteq}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cup,\subset}_{C,F}(\kappa)$.) We would like to demonstrate that $\mathcal{A}'(\kappa)$ is not necessary.

Theorem 30. *Let α be the limit of increasing ordinals β_n for $n < \omega$. If $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}(\aleph_{\beta_n})$ for all $n < \omega$, then $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}(\aleph_\alpha)$.*

Proof. Let σ_n be a winning 2-mark for $\mathscr{F}$ in $Sch^{\cap}_{C,F}(\aleph_{\beta_n})$. Define the 2-mark σ for $\mathscr{F}$ in $Sch^{\cap}_{C,F}(\aleph_\alpha)$ as follows:

$$\sigma(\langle C\rangle, 0) = \sigma_0(\langle C \cap \aleph_{\beta_0}\rangle, 0)$$

$$\sigma(\langle C, D\rangle, n+1) = \sigma_{n+1}(\langle D \cap \aleph_{\beta_{n+1}}\rangle, 0) \cup \bigcup_{m \leq n} \sigma_m(\langle C \cap \aleph_{\beta_m}, D \cap \aleph_{\beta_m}\rangle, n-m+1)$$

Let $\langle C_0, C_1, \dots\rangle$ be an attack by $\mathscr{C}$ in $Sch^{\cap}_{C,F}(\aleph_\alpha)$, and $\alpha \in \bigcap_{n<\omega} C_n$. Choose $N < \omega$ with $\alpha < \aleph_{\beta_{N+1}}$. Consider the attack $\langle C_{N+1} \cap \aleph_{\beta_{N+1}}, C_{N+2} \cap \aleph_{\beta_{N+1}}, \dots\rangle$ by $\mathscr{C}$ in $Sch^{\cap}_{C,F}\left(\aleph_{\beta_{N+1}}\right)$. Since σ_{N+1} is a winning strategy and $\alpha \in \bigcap_{n<\omega} C_{N+n+1} \cap \aleph_{\beta_{N+1}}$, either $\alpha \in \sigma_{N+1}(\langle C_{N+1} \cap \aleph_{\beta_{N+1}}\rangle, 0)$ and thus $\alpha \in \sigma(\langle C_N, C_{N+1}\rangle, N+1)$, or $\alpha \in \sigma_{N+1}(\langle C_{N+M+1} \cap \aleph_{\beta_{N+1}}, C_{N+M+2} \cap \aleph_{\beta_{N+1}}\rangle, M+1)$ for some $M < \omega$ and thus $\alpha \in \sigma(\langle C_{N+M+1}, C_{N+M+2}\rangle, N+M+2)$. Thus σ is a winning strategy. $\square$

Theorem 31. *Let α be the limit of increasing ordinals β_n for $n < \omega$. If $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{1,\subseteq}_{C,F}(\aleph_{\beta_n})$ for all $n < \omega$, then $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{1,\subseteq}_{C,F}(\aleph_\alpha)$.*

Proof. The proof proceeds nearly identically to the previous proof. $\square$

Corollary 32. *It is consistent that $\mathcal{A}'(\aleph_\omega)$ fails, but as $\mathcal{A}'(\aleph_k)$ holds in ZFC for all $k < \omega$, both $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}(\aleph_\omega)$ and $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{1,\subseteq}_{C,F}(\aleph_\omega)$ hold in ZFC.*

Note that Question 6 remains unsolved; however, our results have revealed that we cannot hope to find a ZFC counterexample where $X = \kappa^\dagger$. This is because if we also assume $V = L$, it follows that $\mathcal{A}'(\kappa)$ and therefore $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$ for every cardinal. Although, one may consider the following weaker question.

Question 33. *Is there a model of ZFC where both $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}\left(\aleph^\dagger_{\omega+1}\right)$ and $\mathcal{A}'(\aleph_\omega)$ fail?*

References

[1] Steven Clontz. Applications of limited information strategies in menger's game (preprint). 2015.

[2] Alan Dow. Set theory in topology. In *Recent progress in general topology (Prague, 1991)*, pages 167–197. North-Holland, Amsterdam, 1992.

[3] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[4] R. Björn Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4:229–308; erratum, ibid. 4 (1972), 443, 1972. With a section by Jack Silver.

[5] I. Juhász, Zs. Nagy, and W. Weiss. On countably compact, locally countable spaces. *Period. Math. Hungar.*, 10(2-3):193–206, 1979.

[6] Kenneth Kunen. *Set theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-New York, 1980. An introduction to independence proofs.

[7] Jean-Pierre Levinski, Menachem Magidor, and Saharon Shelah. Chang's conjecture for $\aleph_\omega$. *Israel J. Math.*, 69(2):161–172, 1990.

[8] Peter Nyikos. Generalied kurepa and mad families and topology.

[9] Marion Scheepers. Concerning n-tactics in the countable-finite game. *J. Symbolic Logic*, 56(3):786–794, 1991.

[10] Marion Scheepers. Meager-nowhere dense games. I. n-tactics. *Rocky Mountain J. Math.*, 22(3):1011–1055, 1992.

[11] Marion Scheepers. Combinatorics of open covers. I. Ramsey theory. *Topology Appl.*, 69(1):31–62, 1996.

Department of Mathematics and Statistics, UNC Charlotte, Charlotte, NC 28262

E-mail address: steven.clontz@gmail.com

Department of Mathematics and Statistics, UNC Charlotte, Charlotte, NC 28262

E-mail address: adow@uncc.edu