

# TACTIC-PROXIMAL COMPACT SPACES ARE STRONG EBERLEIN COMPACT

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ABSTRACT. The author and G. Gruenhage previously showed that J. Bell's proximal game may be used to characterize Corson compactness in compact Hausdorff spaces. Using limited information strategies, the proximal game may also be used to characterize the strong Eberlein compactness property. In doing so, a purely topological characterization of the proximal game is introduced, and several existing results on the proximal game have analogues considering limited information strategies.

Two papers published in 2014 introduced the *proximal uniform space game*  $Bell_{D,P}^{\text{uni}}(X)$  due to Jocelyn Bell. If  $X$  is a topological space, and there exists a uniform structure inducing its topology which gives the first player in this game has a winning strategy, then  $X$  is said to be a *proximal* space. Bell used this game as a tool in [1] for investigating uniform box products, and the author showed with Gary Gruenhage in [2] that this game characterizes Corson compactness amongst compact Hausdorff spaces, answering a question of Peter Nyikos in [4].

All spaces in this paper are assumed to be  $T_{3\frac{1}{2}}$ , so that they have a uniform structure inducing the topology on the space. Unlike many game-theoretic topological properties, the game  $Bell_{D,P}^{\text{uni}}(X)$  for which the proximal property was defined by is not itself a topological game. However, by considering entourages of the universal uniformity inducing the topology of a space, this original uniform space game may be easily modified to the purely topological games  $Bell_{D,P}^{\rightarrow}(X)$ ,  $Bell_{D,P}^{\leftarrow}(X)$ .

The aim of this paper is to use this topological interpretation of the proximal game to give a new game-theoretic characterization of the strong Eberlein compactness property. Strong Eberlein compacts are Corson compacts, and therefore proximal compact spaces; in fact, it will be shown that strong Eberlein compacts are exactly the compact spaces for which the first player has a *tactical* winning strategy for the proximal game, a strategy which relies on only the most recent move of the opponent.

## 1. TOPOLOGIZING $Bell_{D,P}^{\text{uni}}(X)$

We refer to [2] for definitions, notation, and basic theorems on uniform spaces and the proximal game  $Bell_{D,P}^{\text{uni}}(X)$  (denoted as  $Prox_{D,P}(X)$  in that paper). In particular recall that:

**Definition 1.1.**  $\mathcal{P} \uparrow G$  denotes that the player  $\mathcal{P}$  has a winning strategy in the  $\omega$ -length game  $G$ .

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2010 *Mathematics Subject Classification.* 54E15, 54D30, 54A20.

*Key words and phrases.* Proximal, tactic proximal, Corson compact, strong Eberlein compact, topological game.

**Definition 1.2.**  $X$  is a *proximal* space in the case that there exists a uniformity for  $X$  such that  $\mathcal{D} \uparrow \text{Bell}_{D,P}^{\text{uni}}(X)$ .

As it turns out, the search for such a uniformity is trivial.

**Definition 1.3.** The *universal uniformity* for a uniformizable topology is the union of all uniformities which induce the given topology.

**Theorem 1.4** ([5]). *The universal uniformity is itself a uniformity compatible with its given topology.*

**Definition 1.5.** For a uniformizable space  $X$ , a *universal entourage*  $D$  is a entourage of the universal uniformity.

**Theorem 1.6** ([5]). *For every uniformizable space, if  $D$  is a neighborhood of the diagonal  $\Delta$  such that there exist neighborhoods  $D_n$  of  $\Delta$  with  $D \supseteq D_0$  and  $D_n \supseteq D_{n+1} \circ D_{n+1}$ , then  $D$  is a universal entourage.*

**Definition 1.7.** For every entourage  $D$  and  $n < \omega$ , let  $\frac{1}{2^n}D$  denote an entourage such that  $\frac{1}{1}D = D$  and  $\frac{1}{2^{n+1}}D \circ \frac{1}{2^{n+1}}D \subseteq \frac{1}{2^n}D$ .

**Definition 1.8.** An *open symmetric entourage*  $D$  is a entourage which is open in the product topology induced by the uniformity and where  $D = D^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in D\}$ .

**Theorem 1.9.** *For every entourage  $D$ , there exists an open symmetric entourage  $U \subseteq D$ .*

Due to this theorem, we will simply use the word *entourage* to refer to open symmetric universal entourages. Note that if  $D$  is an entourage, then  $D[x] = \{y : \langle x, y \rangle \in D\}$  is an open neighborhood of  $x$ . One may consider  $D[x]$  to be an entourage-“ball” about  $x$ , generalizing the notion of an  $\epsilon$ -ball given by a metric structure.

In the case that the space is paracompact, entourages are even more easily found.

**Theorem 1.10** ([5]). *Every open neighborhood of the diagonal is a universal entourage for paracompact uniformizable spaces.*

### 1.1. Using universal entourages to characterize the proximal property.

The natural adaptation of the original uniform space game  $\text{Bell}_{D,P}^{\text{uni}}(X)$  to a topological game requires the use of the universal uniformity on  $X$ .

**Definition 1.11.** Let  $\text{Bell}_{D,P}^{\rightarrow,*}(X)$  denote the *hard Bell convergence game* with players  $\mathcal{D}$ ,  $\mathcal{P}$  which proceeds as follows for a uniformizable space  $X$ . In round 0,  $\mathcal{D}$  chooses an entourage  $D_0$ , followed by  $\mathcal{P}$  choosing a point  $p_0 \in X$ . In round  $n+1$ ,  $\mathcal{D}$  chooses an entourage  $D_{n+1}$ , followed by  $\mathcal{P}$  choosing a point  $p_{n+1} \in D_n[p_n]$ .

$\mathcal{D}$  wins in the case that either  $\langle p_0, p_1, \dots \rangle$  converges in  $X$ , or  $\bigcap_{n < \omega} D_n[p_n] = \emptyset$ .  $\mathcal{P}$  wins otherwise.

This game is considered “hard” due to the requirement that  $\mathcal{D}$  keep track of the history of the game to ensure that successive moves refine previous moves. This record-keeping may be eliminated by requiring that  $\mathcal{P}$  respect all moves made by  $\mathcal{D}$  rather than only the most recent move.

**Definition 1.12.** Let  $\text{Bell}_{D,P}^{\rightarrow}(X)$  denote the *Bell convergence game* with players  $\mathcal{D}$ ,  $\mathcal{P}$  which proceeds analogously to  $\text{Bell}_{D,P}^{\rightarrow,*}(X)$ , except for the following. Let

$E_n = \bigcap_{m \leq n} D_m$ , where  $D_n$  is the entourage played by  $\mathcal{D}$  in round  $n$ . Then  $\mathcal{D}$  must ensure that  $p_{n+1} \in E_n[p_n]$ , and  $\mathcal{D}$  wins when either  $\langle p_0, p_1, \dots \rangle$  converges in  $X$  or  $\bigcap_{n < \omega} E_n[p_n] = \emptyset$ .

These games are all essentially equivalent with respect to perfect information for  $\mathcal{D}$ .

**Theorem 1.13.**  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow,*}(X)$  if and only if  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$  if and only if  $X$  is proximal.

*Proof.* If  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow,*}(X)$ , then we immediately see that  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$ . If  $\sigma$  is a winning strategy for  $\mathcal{D}$  in  $Bell_{D,P}^{\rightarrow}(X)$ , then  $\tau$  defined by  $\tau(s) = \bigcap_{t \leq s} \sigma(t)$  is easily seen to be a winning strategy for  $\mathcal{D}$  in  $Bell_{D,P}^{\rightarrow,*}(X)$ .

If  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow,*}(X)$ , then  $\mathcal{D} \uparrow Bell_{D,P}^{\text{uni}}(X)$  for the universal uniformity, showing  $X$  is proximal. Finally, if  $X$  is proximal, then there exists a winning strategy  $\sigma$  for  $Bell_{D,P}^{\text{uni}}(X)$  for a uniformity inducing the topology on  $X$ . Then a winning strategy for  $\mathcal{D}$  in  $Bell_{D,P}^{\rightarrow,*}(X)$  may be constructed by converting every entourage in this uniformity into a smaller open symmetric universal entourage.  $\square$

The secondary winning condition in  $Bell_{D,P}^{\rightarrow}(X)$  allows for a space to be incomplete:  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow}(\mathbb{Q})$  by playing  $D_{2^{-n}} = \{(x, y) : d(x, y) < 2^{-n}\}$  in each round. This forces  $\mathcal{D}$ 's sequence to be Cauchy, and thus it either converges to a rational, or the sets  $D_{2^{-n}}[x_n]$  will have empty intersection (where the irrational point of convergence would be). Uniformly locally compact spaces (and in particular, compact spaces) lack such holes, so it will be convenient to eliminate this technicality when it is irrelevant.

**Definition 1.14.** Let  $Bell_{D,P}^{\rightarrow}(X)$  denote the *absolute Bell convergence game* which proceeds analogously to  $Bell_{D,P}^{\rightarrow}(X)$ , except that  $\mathcal{D}$  must always ensure that  $\langle p_0, p_1, \dots \rangle$  converges in  $X$  in order to win.

**Definition 1.15.** A uniformizable space  $X$  is *absolutely proximal* if  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$ .

As was shown in [2]:

**Definition 1.16.** A uniformizable space  $X$  is *uniformly locally compact* if there exists an entourage  $D$  such that  $\overline{D[x]}$  is compact for all  $x$ .

**Theorem 1.17.** If  $X$  is a uniformly locally compact space, then  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$  if and only if  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow,*}(X)$ .

So absolutely proximal compacts are simply proximal compacts (and therefore Corson compacts).

## 2. LIMITED INFORMATION ANALOGUES

First recall the definitions of the following limited information strategies.

**Definition 2.1.** A *k-tactical strategy* or *k-tactic* is a strategy which considers only the most recent move of the opponent.

If  $\mathcal{D}$  has a winning *k-tactic* for a game  $G$ , we write  $\mathcal{D} \uparrow_{k\text{-tact}} G$ . If omitted, assume  $k = 1$ .

**Definition 2.2.** A  $k$ -Markov strategy or  $k$ -mark is a strategy which considers only the round number and most recent move of the opponent.

If  $\mathcal{P}$  has a winning  $k$ -mark for a game  $G$ , we write  $\mathcal{P} \underset{k\text{-mark}}{\uparrow} G$ . If omitted, assume  $k = 1$ .

Limited information strategies may be used to strengthen game-theoretic topological properties.

**Definition 2.3.** A uniformizable space  $X$  is (*absolutely*) *tactic-proximal* if  $\mathcal{D} \underset{\text{tact}}{\uparrow} Bell_{\vec{D},P}^{\rightarrow}(X)$  ( $\mathcal{D} \underset{\text{tact}}{\uparrow} Bell_{\vec{D},P}^{\rightarrow}(X)$ ).

**Definition 2.4.** A uniformizable space  $X$  is (*absolutely*) *Markov-proximal* if  $\mathcal{D} \underset{\text{mark}}{\uparrow} Bell_{\vec{D},P}^{\rightarrow}(X)$  ( $\mathcal{D} \underset{\text{mark}}{\uparrow} Bell_{\vec{D},P}^{\rightarrow}(X)$ ).

As in Theorem 1.17, the “absolutely” is redundant in the above definitions when  $X$  is uniformly locally compact, so the absolute Bell convergence game will be used for convenience in the context of compact spaces.

Some results of Bell may be generalized to hold for such limited information strategies. The proofs of the following propositions are straight forward.

**Proposition 2.5.** Let  $X$  be a uniformizable space and  $H$  be a closed subset of  $X$ . If  $k < \omega$ , then

- $\mathcal{D} \underset{k\text{-tact}}{\uparrow} Bell_{\vec{D},P}^{\rightarrow}(X) \Rightarrow \mathcal{D} \underset{k\text{-tact}}{\uparrow} Bell_{\vec{D},P}^{\rightarrow}(H)$
- $\mathcal{D} \underset{k\text{-mark}}{\uparrow} Bell_{\vec{D},P}^{\rightarrow}(X) \Rightarrow \mathcal{D} \underset{k\text{-mark}}{\uparrow} Bell_{\vec{D},P}^{\rightarrow}(H)$
- $\mathcal{D} \uparrow Bell_{\vec{D},P}^{\rightarrow}(X) \Rightarrow \mathcal{D} \uparrow Bell_{\vec{D},P}^{\rightarrow}(H)$

**Proposition 2.6.** Let  $X$  be a uniformizable space and  $H$  be a closed subset of  $X$ . If  $k < \omega$ , then

- $\mathcal{D} \underset{k\text{-tact}}{\uparrow} Bell_{\vec{D},P}^{\rightarrow}(X) \Rightarrow \mathcal{D} \underset{k\text{-tact}}{\uparrow} Bell_{\vec{D},P}^{\rightarrow}(H)$
- $\mathcal{D} \underset{k\text{-mark}}{\uparrow} Bell_{\vec{D},P}^{\rightarrow}(X) \Rightarrow \mathcal{D} \underset{k\text{-mark}}{\uparrow} Bell_{\vec{D},P}^{\rightarrow}(H)$
- $\mathcal{D} \uparrow Bell_{\vec{D},P}^{\rightarrow}(X) \Rightarrow \mathcal{D} \uparrow Bell_{\vec{D},P}^{\rightarrow}(H)$

Less obvious is the following.

**Definition 2.7.** Let  $X_\alpha$  be a topological space for  $\alpha < \kappa$  and  $z \in \prod_{\alpha < \kappa} X_\alpha$ . The  $\Sigma$ -product  $\sum_{\alpha < \kappa}^z X_\alpha$  with base point  $z$  is given by

$$\sum_{\alpha < \kappa}^z X_\alpha = \left\{ x \in \prod_{\alpha < \kappa} X_\alpha : |\{\alpha < \kappa : x(\alpha) \neq z(\alpha)\}| \leq \omega \right\}$$

When  $X_\alpha = X$  and  $z(\alpha) = 0$  for all  $\alpha < \kappa$ , then we write  $\Sigma X^\kappa = \sum_{\alpha < \kappa}^z X$ .

**Theorem 2.8** ([1]). If  $X_\alpha$  is proximal for all  $\alpha < \kappa$ , then  $\sum_{\alpha < \kappa}^z X_\alpha$  is proximal for any base point  $z$ . If  $X_\alpha$  is absolutely proximal for all  $\alpha < \kappa$ , then  $\sum_{\alpha < \kappa}^z X_\alpha$  is absolutely proximal for any base point  $z$ .

**2.1.  $\Sigma^*$ - and  $\sigma$ -Products.** We may consider smaller subspaces of  $X^\kappa$  than given by  $\Sigma X^\kappa$ .

**Definition 2.9.** Let  $X$  be a metrizable space with compatible metric  $d$ , and let  $z \in X^\kappa$ . The  $\Sigma^*$ -product  $\Sigma_{\alpha < \kappa}^z X$  with base point  $z$  is given by

$$\Sigma_{\alpha < \kappa}^z X = \left\{ x \in \prod_{\alpha < \kappa} X : n < \omega \Rightarrow \left| \left\{ \alpha < \kappa : d(x(\alpha), z(\alpha)) > \frac{1}{2^n} \right\} \right| < \omega \right\}$$

When  $z(\alpha) = 0$  for all  $\alpha < \kappa$ , then we write  $\Sigma^* X^\kappa = \Sigma_{\alpha < \kappa}^z X$ .

**Definition 2.10.** Let  $X_\alpha$  be a topological space for  $\alpha < \kappa$  and  $z \in \prod_{\alpha < \kappa} X_\alpha$ . The  $\sigma$ -product  $\sigma_{\alpha < \kappa}^z X_\alpha$  with base point  $z$  is given by

$$\sigma_{\alpha < \kappa}^z X_\alpha = \left\{ x \in \prod_{\alpha < \kappa} X_\alpha : |\{\alpha < \kappa : x(\alpha) \neq z(\alpha)\}| < \omega \right\}$$

When  $X_\alpha = X$  and  $z(\alpha) = 0$  for all  $\alpha < \kappa$ , then we write  $\sigma X^\kappa = \sigma_{\alpha < \kappa}^z X$ .

Of course,

$$\sigma X^\kappa \subseteq \Sigma^* X^\kappa \subseteq \Sigma X^\kappa \subseteq X^\kappa$$

for metrizable  $X$ , and

$$\sigma_{\alpha < \kappa}^z X_\alpha \subseteq \sum_{\alpha < \kappa}^z X_\alpha \subseteq \prod_{\alpha < \kappa} X_\alpha$$

for all spaces  $X_\alpha$  and base points  $z$ .

Just as  $\Sigma$  products preserve winning perfect-information strategies, we will show that these product subspaces preserve certain winning limited-information strategies.

**Definition 2.11.** For a metric space  $\langle X, d \rangle$ , let  $E_\epsilon = \{\langle x, y \rangle : d(x, y) < \epsilon\}$ . Note that  $E_\epsilon$  is an (open symmetric) entourage on  $X$ .

**Proposition 2.12.** For any metrizable space  $X$ ,  $\mathcal{D} \uparrow_{0\text{-mark}} Bell_{D,P}^\rightarrow(X)$ . For any completely metrizable space  $X$ ,  $\mathcal{D} \uparrow_{0\text{-mark}} Bell_{D,P}^\rightarrow(X)$ .

*Proof.* Essentially shown by Bell in her paper: in either case,  $\mathcal{D}$  chooses  $E_{1/2^n}$  during round  $n$ , forcing legal attacks by  $\mathcal{P}$  to be Cauchy.  $\square$

As an aside, this next proposition may be proved similarly using  $E_{d(x,y)/2}$ .

**Proposition 2.13.** For any metrizable space  $X$ ,  $\mathcal{D} \uparrow_{2\text{-tact}} Bell_{D,P}^\rightarrow(X)$ . For any completely metrizable space  $X$ ,  $\mathcal{D} \uparrow_{2\text{-tact}} Bell_{D,P}^\rightarrow(X)$ .

We will exploit the last move of the opponent to obtain winning Markov strategies under  $\Sigma^*$  products.

**Proposition 2.14.** If  $X_\alpha$  is a uniformizable space for  $\alpha < \kappa$ , and  $D_\alpha$  is an entourage of  $X_\alpha$  for  $\alpha \in F \in [\kappa]^{<\omega}$ , then

$$P(\{\langle \alpha, D_\alpha \rangle : \alpha \in F\}) = \{\langle x, y \rangle \in \left( \prod_{\alpha < \kappa} X_\alpha \right)^2 : \alpha \in F \Rightarrow \langle x(\alpha), y(\alpha) \rangle \in D_\alpha\}$$

is an entourage of  $\prod_{\alpha < \kappa} X_\alpha$ .

**Theorem 2.15.** *For any metrizable space  $X$  and  $z \in X^\kappa$ ,  $\mathcal{D} \uparrow_{\text{mark}} \text{Bell}_{D,P}^\rightarrow (\sum_{\alpha < \kappa}^z X)$ . If  $X$  is completely metrizable, then  $\mathcal{D} \uparrow_{\text{mark}} \text{Bell}_{D,P}^\rightarrow (\sum_{\alpha < \kappa}^z \text{mark} X)$ .*

*Proof.* For  $x \in \sum_{\alpha < \kappa}^z X$ , let

$$\text{supp}_n(x) = \left\{ \alpha < \kappa : d(x(\alpha), z(\alpha)) > \frac{1}{2^n} \right\} \in \kappa^{<\omega}$$

where  $d$  is a (complete) metric compatible with the topology on  $X$ .

Define a strategy  $\sigma$  for  $\mathcal{D}$  by

$$\begin{aligned} \sigma(\emptyset, 0) &= \left( \sum_{\alpha < \kappa}^z X \right)^2 \\ \sigma(\langle p \rangle, n+1) &= \left( \sum_{\alpha < \kappa}^z X \right)^2 \cap P(\{\langle \alpha, E_{1/2^n} \rangle : \alpha \in \text{supp}_n(p)\}) \end{aligned}$$

□

**2.2. Eberlein and Strong Eberlein Compacts.** We recall some convenient definitions for a few strengthenings of compactness.

**Definition 2.16.** A *Corson compact* space is a compact space which may be embedded in

$$\Sigma \mathbb{R}^\kappa = \{x \in \mathbb{R}^\kappa : |\{\alpha < \kappa : x(\alpha) \neq 0\}| \leq \omega\}$$

**Definition 2.17.** An *Eberlein compact* space is a compact space which may be embedded in

$$\Sigma^* \mathbb{R}^\kappa = \{x \in \mathbb{R}^\kappa : n < \omega \Rightarrow |\{\alpha < \kappa : |x(\alpha)| > \frac{1}{2^n}\}| < \omega\}$$

**Definition 2.18.** A *strong Eberlein compact* space is a compact space which may be embedded in

$$\sigma 2^\kappa = \{x \in 2^\kappa : |\{\alpha < \kappa : x(\alpha) \neq 0\}| < \omega\}$$

Obviously, strong Eberlein compacts are Eberlein compact, and Eberlein compacts are Corson compact. Nyikos observed in [3] that as the  $\Sigma$ -product of proximal spaces are proximal and the closed subspaces of proximal spaces are proximal (both results of Bell in [1]), any compact subspace of  $\Sigma \mathbb{R}^\kappa$  (that is, any Corson compact) must itself be proximal. With some more work, this implication may also be reversed.

**Theorem 2.19** ([2]). *A compact space is proximal if and only if it is Corson compact.*

Let us first investigate limited information analogues of Nyikos's observation.

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