Limited information strategies for a topological proximal game

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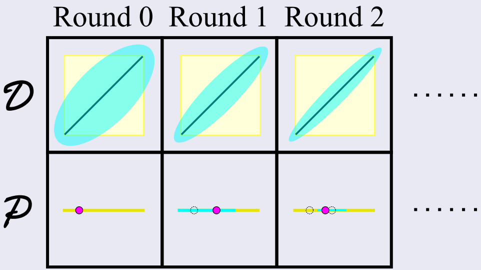
Game

Bell's absolutely proximal game $Bell_{\vec{D},P}(X)$ [1] (2014)



$\mathscr{D}$ wins the game if the points chosen by $\mathscr{P}$ converge. Otherwise, $\mathscr{P}$ wins.

If $\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$, then X is called an *absolutely proximal space*. "Absolutely proximal" is a strengthening of "proximal", but these concepts coincide for (uniformlly locally) compact spaces.

This game was brought to my attention due to this result: [1]

Theorem

Every proximal space is a W-space. So $\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(X) \Rightarrow \mathscr{O} \uparrow Gru_{O,P}^{\rightarrow}(X, x)$ *for all* $x \in X$.

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Proximal spaces have strong preservation properties, as any closed subset or Σ-product of proximal spaces is proximal.

Since any metrizable space is proximal, and any proximal space is collectionwise normal, Bell's game gives an elegant proof of the classic result of Rudin and Gulko:

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A Σ-product of metrizable spaces is collection-wise normal.

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and asked if the converse holds as well.

With Gruenhage, I showed that the answer is yes: [2]

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Player $\mathscr{D}$ chooses *entourages* of the diagonal: elements of a *uniformity* inducing the topology of the space.

A uniformity $\mathbb{D}$ on X is a filter of subsets of X^2 satisfying:

- $\bigcap \mathbb{D} = \Delta = \{\langle x, x\rangle : x \in X\}$
- $D \in \mathbb{D}$ implies $D^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in D\} \in \mathbb{D}$
- for each $D \in \mathbb{D}$ there is $\frac{1}{2}D \in \mathbb{D}$ such that $\frac{1}{2}D \circ \frac{1}{2}D \subseteq D$

The topology induced by a uniformity is the smallest topology such that $D[x] = \{y : \langle x, y\rangle \in D\}$ is a neighborhood of x for each $D \in \mathbb{D}$.

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Our goal is to obtain a purely topological characterization of the proximal properties.

As it turns out, the union of all uniformities inducing a topology is itself a uniformity inducing that topology, called the *universal uniformity*. Furthermore, $\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$ if and only if $\mathscr{D}$ is required to choose from the universal uniformity.

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From Willard's topology text [5], we have such a characterization:

Theorem

A neighborhood U of the diagonal is a universal uniformity if and only if there exist neighborhoods U_n for $n < \omega$ where $U = U_0$ and $U_{n+1} \circ U_{n+1} \subseteq U_n$.

As a bonus, for strongly collectionwise normal ($\Leftarrow$ paracompact) spaces, *all* neighborhoods of the diagonal have this property.

So we topologize Bell's game by saying an "entourage" is any (open symmetric) neighborhood of the diagonal with this property, and discard the need to consider a specific uniform structure.

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A *perfect information strategy* uses full information of the previous moves of the opponent. ($\mathscr{A} \uparrow G$)

A *k-tactical strategy* only uses the last k previous moves of the opponent. ($\mathscr{A} \underset{k\text{-tact}}{\uparrow} G$)

A *k-Markov strategy* only uses the round number and the last k previous moves of the opponent. ($\mathscr{A} \underset{k\text{-mark}}{\uparrow} G$)

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Bell showed that a winning perfect information strategy may always be passed down to win in a closed subspace.

Proposition

If $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X)$, then $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(H)$ for every closed subspace H of X.

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Bell's also showed that winning strategies are preserved for Σ-products.

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If $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}(X_\alpha)$ *for* $\alpha < \kappa$, *then* $\mathscr{D} \uparrow Bell^{\rightarrow}_{D,P}\left(\sum_{\alpha<\kappa} X_\alpha\right)$.

Idea of proof: during round n, consider the first n non-zero coordinates of the previous n moves by $\mathscr{P}$ and use the winning strategies for those finite coordinates. Note that this uses the round number and perfect information of all previous moves.

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We settle for considering a countable product instead (but still require knowledge of the round number).

Theorem

If $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell^{\rightarrow}_{D,P}(X_i)$ *for* $i < \omega$*, then* $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell^{\rightarrow}_{D,P}\left(\prod_{i<\omega} X_i\right)$*.*

Probable counter-example for generalization:

$\mathscr{D} \underset{\text{mark}}{\not\uparrow} Bell^{\rightarrow}_{D,P}\left(\sum_{\alpha<\omega_1} 2\right)$?

Existence of a winning limited information strategy characterizes a stronger topological property than the existence of a winning perfect information strategy.

As it turns out:

Theorem

A compact space X is strongly Eberlein compact if and only if $\mathscr{D} \underset{tact}{\uparrow} Bell^{\rightarrow}_{D,P}(X)$.

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Sketch of Proof

Easy direction:

Definition

Strong Eberlein compacts embed in $\sigma 2^\kappa$ for some κ.

Lemma

$\mathscr{D} \underset{tact}{\uparrow} Bell_{\overrightarrow{D},P}(\sigma 2^\kappa)$.

Sketch of Proof (cont.)

Lemmas which give the other direction:

Lemma (Gruenhage [3])

Scattered proximal compacts are strong Eberlein compact.

Lemma

Non-scattered proximal compacts contain copies of the Cantor space 2^ω.

Lemma

$\mathscr{D} \underset{tact}{\not\uparrow} Bell^{\rightarrow}_{D,P}(2^\omega)$.

Any questions?

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