Limited Information Strategies for Topological Games

by

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Abstract

I talk a lot about topological games.

TODO: Write this.

Acknowledgments

TODO: Thank people.

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Introduction

Basic overview of combinatorial games, topological games, limited info strategies, and applications in topology.

Chapter 2

Toplogical Games and Strategies

of Perfect and Limited Information

The goal of this paper is to explore the applications of limited information strategies in existing topological games. (TODO: History of combinatorical and topological games?)

2.1 Games

Intuitively, the games studied in this paper are two-player games for which each player takes turns making a choice from a set of possible moves. At the conclusion of the game, the choices made by both players are examined, and one of the players is declared the winner of that playthrough, with no ties allowed.

Games may be modeled mathematically in various ways (TODO: cite digraph model) but we will find it convenient to think of them in terms defined by Gale and Stewart. (TODO: cite)

Definition 1. A *game* is a tuple $\langle M, W \rangle$ such that $W \subseteq M^\omega$. M is set of *moves* for the game, and M^ω is the set of all possible *playthroughs* of the game.

W is the set of *winning playthroughs* or *victories* for the first player, and $M^\omega \setminus W$ is the set of victories for the second player. (W is often called the *payoff set* for the first player.)

To illustrate this definition, we may formally model the well-known game Tic-Tac-Toe. For convenince, we may label the game board as a "magic square" of the numbers 0 through 8 such that the combinations of three integers which sum to 12 are exactly the rows, columns, and diagonals of the grid.

(TODO: image)

Then writing X or O on the game board is equivalent to choosing a number 0 through 8, and getting a "tic-tac-toe" is equivalent to having chosen three numbers which sum to 12.

Game 2. Let $TTT_{X,O}$ denote the game *Tic-Tac-Toe*, with players $\mathscr{X}$ and $\mathscr{O}$.

The moves M for the game are the first nine non-negative integers $\{0, \ldots, 8\}$. The victories W for $\mathscr{X}$ consist of sequences $\langle X_0, O_0, X_1, O_1, \ldots \rangle$ such that either of the following hold:

- There exists some $k < 4$ such that $\langle X_0, O_0, \ldots, X_k \rangle$ contains no repeated integers, but $\langle X_0, O_0, \ldots, X_k, O_k \rangle$ does.

  ($\mathscr{O}$ drew her mark on a space already occupied.)

- There exists $i < j < k < 5$ such that $\langle X_0, O_0, \ldots, X_4 \rangle$ contains no repeated integers and $X_i + X_j + X_k = 12$, and for all $i' < j' < k' < k$ it follows that $O_{i'} + O_{j'} + O_{k'} \neq 12$.

  ($\mathscr{X}$ did not draw her mark on a space that was already occupied, and was able to secure a "tic-tac-toe" before $\mathscr{O}$.) [1]

Due to the tedium of defining games explicitly from the definition, we typically define games by declaring the *rules* that each player must follow (e.g. players may not make a mark on a space which was already marked) and the *winning condition* for the first player (e.g. $\mathscr{X}$ secures a "tic-tac-toe" before $\mathscr{O}$). Then a playthrough is in W if either the first player made only *legal moves* which observed the game's rules and the playthrough satisifies the winning condition, or the second player made an *illegal move* which contradicted the game's rules.

Often, players should not be able to choose from the same set of moves (e.g. player Black cannot move White's pieces in chess), in which case M may be partitioned into two sets M_0, M_1 and the rules for the game may be defined such that each player may only choose from one of those sets (e.g. Black will automatically lose if she moves one of White's pieces).

[1] The observant reader will note that in this formulation of Tic-Tac-Toe, player $\mathscr{O}$ wins "cat's games" where neither player secures a "tic-tac-toe" (since no ties are allowed).

An artifact of this game model is that games never technically end, since playthroughs of the game are infinite sequences. Nonetheless, Tic-Tac-Toe is always decided by $\mathscr{X}$'s fifth move, and in chess, there is also an upper bound for the length of a game due to the finite number of ways pieces may be positioned on the chessboard (and the "threefold rule" which declares the game finished if the same position is repeated three times).

But a two-player game need not have a bounded number of rounds, as is demonstrated by this game due to John Conway: [**?**]

Game 3. Let $Coin_{A,B}$ denote the game *Sylver Coinage* with players $\mathscr{A}$ and $\mathscr{B}$. During round n, $\mathscr{A}$ chooses an integer $a_n > 0$, followed by $\mathscr{B}$ choosing another integer $b_n > 0$, with the rule that neither player may choose an integer which is the sum of nonnegative integer multiples of previously chosen integers. $\mathscr{A}$ wins the game if $\mathscr{B}$ ever chooses the number 1, and $\mathscr{B}$ wins otherwise.

After some thought, the reader may confirm that there will be no legal moves except 1 remaining after a finite number of rounds. Specifically, if $\mathscr{A}$ chooses a_0 in the initial round, then all following legal moves must be of the form $i \bmod a_0$ for $0 < i < a_0$. $i \bmod a_0$ may be repeated, but all following repetitions must be strictly less than the previous iterations, which may only be done finitely many times. Thus after a finite number of moves, there will only be finitely many legal moves remaining. [2]

Sylver Coinage was defined as an example of an "unboundedly unboundedly bounded" game, the maximum length of which is not decided until at least the second move.

2.1.1 Infinite and Topological Games

(TODO: History of infinite games in topology / descriptive set theory)

Definition 4. A game is said to be an *infinite game* if there exists a playthrough $p \in M^\omega$ of the game such that for all $n < \omega$, $\{q : q \geq p \upharpoonright n\}$ is not a subset of either W or $M^\omega \setminus W$.

[2] A more involved argument due to Sylvester (for whom the game is named) shows that the number of legal moves is finite as soon as two coprime numbers have been played. [**?**]

Put another way, an infinite game need not be decided after a finite number of rounds. Games which are not infinite are of course called *finite games*.

As an illustration, we may consider an example due to Baker [**?**].

Game 5. Let $Con_{A,B}(A)$ denote a game with players $\mathscr{A}$ and $\mathscr{B}$, defined for each subset $A \subset \mathbb{R}$. In round 0, $\mathscr{A}$ chooses a number a_0, followed by $\mathscr{B}$ choosing a number b_0 such that $a_0 < b_0$. In round $n+1$, $\mathscr{A}$ chooses a number a_{n+1} such that $a_n < a_{n+1} < b_n$, followed by $\mathscr{B}$ choosing a number b_{n+1} such that $a_{n+1} < b_{n+1} < b_n$.

$\mathscr{A}$ wins the game if the sequence $\langle a_n : n < \omega \rangle$ converges to a point in A, and $\mathscr{B}$ wins otherwise.

Certainly, $\mathscr{A}$ and $\mathscr{B}$ will never be in a position without (infinitely many) legal moves available, so this is an infinite game. While the game could never be "completed" in reality, the winning condition considers the infinite sequence of moves made by the players and declares a victor at the "end" of the game.

As a simple example, if $A = \mathbb{R}$, then $W = M^\omega$. That is, all playthroughs of the game are victories for $\mathscr{A}$, since every bounded increasing sequence converges to some real number.

Definition 6. A *topological game* is a game defined in terms of an arbitrary topological space.

Topological games are usually infinite games. One of the earliest examples of a topological game is the Banach-Mazur game, proposed by Stanislaw Mazur as Problem 43 in Stefan Banach's Scottish Book (1935). A more comprehensive history of the Banacy-Mazur and other topological games may be found in Telgarsky's survey on the subject [**?**].

The original game was defined for subsets of the real line; however, we give a more general definition here.

Game 7. Let $Empty_{E,N}(X)$ denote the *Banach-Mazur game* with players $\mathscr{E}$, $\mathscr{N}$ defined for each topological space X. In round 0, $\mathscr{E}$ chooses a nonempty open set $E_0 \subseteq X$, followed by

$\mathscr{N}$ choosing a nonempty open subset $N_0 \subseteq E_0$. In round $n+1$, $\mathscr{E}$ chooses a nonempty open subset $E_{n+1} \subseteq N_n$, followed by $\mathscr{N}$ choosing a nonempty open subset $N_{n+1} \subseteq E_{n+1}$.

$\mathscr{E}$ wins the game if $\bigcap_{n<\omega} E_n = \emptyset$, and $\mathscr{N}$ wins otherwise.

For example, if X is a locally compact Hausdorff space, $\mathscr{N}$ can "force" a win by choosing N_0 such that $\overline{N_0}$ is compact, and choosing N_{n+1} such that $N_{n+1} \subseteq \overline{N_{n+1}} \subseteq O_{n+1} \subseteq N_n$ (possible since N_n is a compact Hausdorff $\Rightarrow$ normal space). Since $\bigcap_{n<\omega} E_n = \bigcap_{n<\omega} N_n$ is the decreasing intersection of compact sets, it cannot be empty.

This concept of when (and how) a player can "force" a win in certain topological games is the focus of this manuscript.

2.2 Strategies

We shall make the notion of forcing a win in a game rigorous by introducing "strategies" and "attacks" for games.

Definition 8. A *strategy* for a game $G = \langle M, W \rangle$ is a function from $M^{<\omega}$ to M.

Definition 9. An *attack* for a game $G = \langle M, W \rangle$ is a function from ω to M.

Intuitively, a strategy is a rule for one of the players on how to play the game based upon the previous moves of her opponent, while an attack is a fixed strike by an opponent.

Definition 10. The *result* of a game for a strategy σ for the first player and attack $\langle a_0, a_1, \ldots \rangle$ for the second player is the playthrough

$$\langle \sigma(\emptyset), a_0, \sigma(\langle a_0 \rangle), a_1, \sigma(\langle a_0, a_1 \rangle), \ldots \rangle$$

Likewise, if σ is a strategy for the second player, and $\langle a_0, a_1, \ldots \rangle$ is an attack by the first player, then the result is the playthrough

$$\langle a_0, \sigma(\langle a_0 \rangle), a_1, \sigma(\langle a_0, a_1 \rangle), \ldots \rangle$$

We now may rigorously define the notion of "forcing" a win in a game.

Definition 11. A strategy σ is a *winning strategy* for a player if for every attack by the opponent, the result of the game is a victory for that player.

If a winning strategy exists for player $\mathscr{A}$ in the game G, then we write $\mathscr{A} \uparrow G$. Otherwise, we write $\mathscr{A} \not\uparrow G$.

Of course, a strategy σ is not a winning strategy for a player if there exists some *counter-attack* by the opponent for which the result is a victory for the opponent. Typically this counter-attack is defined in terms of the strategy σ; else, the counter-attack is itself a winning strategy depending on only the round number (which we will investigate further in a later section).

Definition 12. A game G with players $\mathscr{A}$, $\mathscr{B}$ is said to be *determined* if either $\mathscr{A} \uparrow G$ or $\mathscr{B} \uparrow G$.

(TODO: cite Gale and Stewart)

Theorem 13. *If the move set M for a game $G = \langle M, W \rangle$ is given the discrete topology and W is either an open or closed subset of M^ω with the usual product topology, then G is determined.*

(TODO: outline proof?)

We get this important corollary:

Corollary 14. *Finite games are determined.*

Proof. For every victory $p \in W$ for the first player, there was some round $n < \omega$ for which that partial playthrough was guaranteed to result in a victory for the first player.

Put another way, there exists $n < \omega$ such that $\{q : q \geq p \upharpoonright n\}$ (a neighborhood of p) is contained in W. Thus W is open. $\square$

(TODO: find combinatorial proof and cite it)

(TODO: Cover Martin's Borel determinacy theorem?)

Non-trivial topological games are infinite, however, and many are indetermined for certain spaces.

Example 15. Let B be a Bernstein subset of the real line. Then $Empty_{E,N}(B)$ is undetermined.

Proof. It can be shown that B is a Baire space, and we will soon see that $\mathscr{E} \not\uparrow Empty_{E,N}(X)$ characterizes Baire spaces. Also, if σ is a winning strategy for $\mathscr{N}$ in $Empty_{E,N}(Y)$ for $Y \subseteq X$, it can be shown that Y contains a closed uncountable set in X. Thus $\mathscr{N} \not\uparrow Empty_{E,N}(B)$. □

2.2.1 Applications of Strategies

The presence or absence of a winning strategie for a player in a topological game characterizes a property of the topological space in question.

A classical result follows.

Theorem 16. *$\mathscr{E} \not\uparrow Empty_{E,N}(X)$ if and only if X is a Baire space. [**?**]*

2.2.2 Limited Information Strategies

2.3 Examples of Topological Games

(TODO:)