Proximal compact spaces are Corson compact

2015 Joint Mathematics Meetings at San Antonio

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A *topological game* is a two-player game $G(X)$ of length $\omega = \{0, 1, 2, \ldots\}$ defined for certain topological spaces X.

During each round n, the first and second player take turns choosing certain topological objects from X (e.g. point, open set, open cover, etc.).

At the "end" of the game, a winner is declared by inspecting the sequences of choices made throughout the game.

The study of such games involves finding when a player has a *winning strategy* which defeats every possible counterattack by the opponent.

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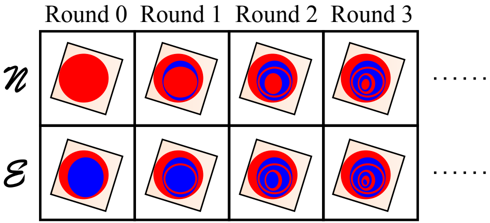
Canonical example: *Banach-Mazur Game* (1935) [5]



The first player $\mathscr{N}$ wins the game if the intersection of all the chosen open sets is nonempty.

Theorem

X is Baire if and only if $\mathscr{N}$ lacks a winning strategy in the Banach Mazur game.

See Telgarsky's excellent survey on topological games for more details: [8]

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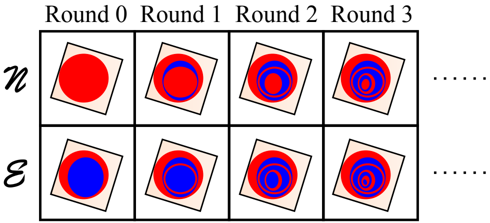

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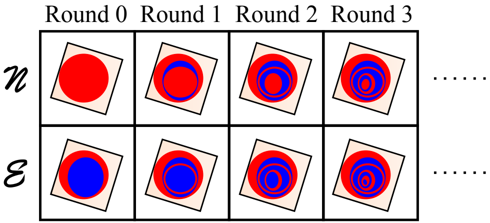

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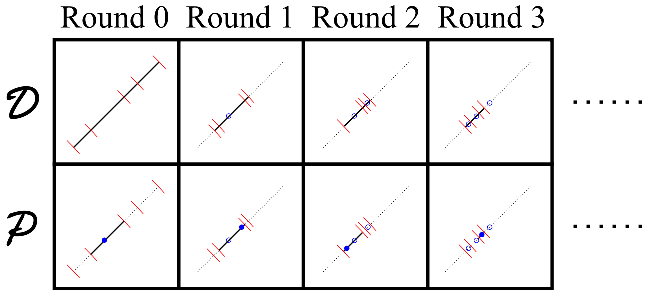

for compact T_1 0-dim spaces

The first player $\mathscr{D}$ wins the game if the points chosen by the second player $\mathscr{P}$ converge. If $\mathscr{D}$ has a winning strategy for this game, call X *proximal*.

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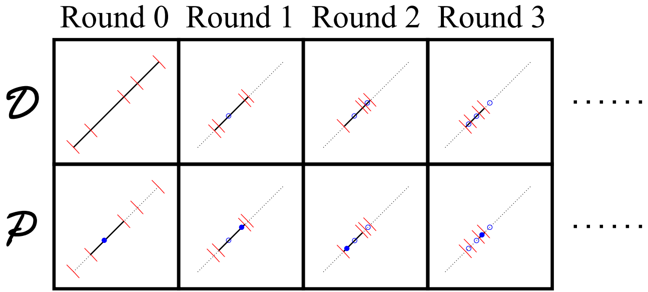

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Some results related to the Proximal Game due to Jocelyn Bell:

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If X is metrizable, then X is proximal.

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Σ-products and closed subspaces of proximal spaces are proximal.

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A *Corson compact* space is a space homeomorphic to a compact subset of the Σ-product of real lines.

Peter Nyikos observed:

Proposition

Every Corson compact space is proximal compact. [6]

C. and Gruenhage showed in [2] that any compact proximal space must be Corson compact, using another game-theoretic characterization of Corson compact due to Gruenhage:

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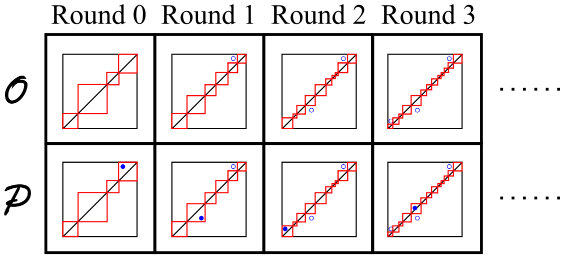
Diagonal Game (1984) [3]:



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The first player $\mathscr{O}$ wins the game if any open set containing the diagonal also contains infinitely many of $\mathscr{P}$'s chosen points.

Theorem

If $\mathscr{O}$ has a winning strategy for the diagonal game and X is compact, then X is Corson compact.

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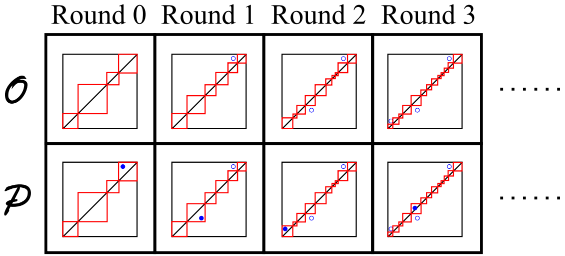

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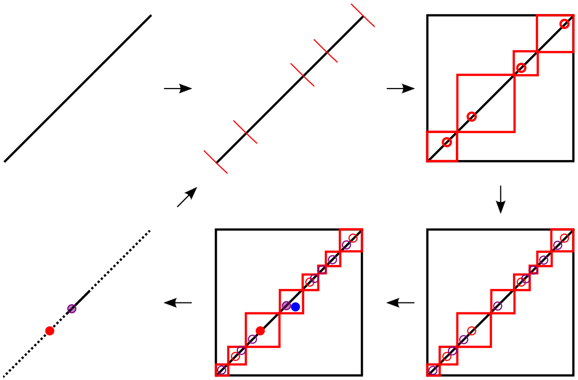
One may use a winning strategy σ for $\mathscr{D}$ in the proximal game to construct a strategy τ for $\mathscr{O}$ in the diagonal game.



In general:

$$\tau(a) = \bigcup_{s^\frown\langle i,h_{s,i},j\rangle\in\max(T(a))} \frac{1}{4}\sigma(o_s{}^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

Using the strategy τ defined for every proximal compact space, $\mathscr{O}$ cannot be defeated in the diagonal game, and therefore all proximal compacts are Corson compact. $\square$

Open questions:

- If compactness is dropped, does the proximal game characterize all copies of *closed* subspaces of a Σ-product of reals? (Nyikos)
- If the winning strategy for the proximal game is *Markov* (relies on only the latest move and round number) for a compact space, does that imply that the space is *Eberlein* compact? (This holds for the diagonal game.)

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Any questions?