Definition 1. X is **Menger** if for all open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ there exist finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Proposition 2. *σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelof*

Definition 3. In the two-player game $Cov_{C,F}(X)$ player C chooses open covers $\mathcal{U}_n$ of X, followed by player F choosing a finite subcollection $\mathcal{F}_n \subseteq \mathcal{U}_n$. F wins if $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Theorem 4. *X is Menger if and only if $C \not\uparrow Cov_{C,F}(X)$.*

Proof. First, suppose X wasn't Menger. Then there would exist open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ of X such that for any choice of finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$, $\bigcup_{n<\omega} \mathcal{F}_n$ isn't a cover of X. Thus $C \uparrow_{\text{pre}} Cov_{C,F}(X) \Rightarrow S \not\uparrow Cov_{C,F}(X)$.

The other direction is based upon Gruenhage's topological game presentation. Assume X is Menger, and consider a strategy for C in $Cov_{C,F}(X)$.

Since X is Lindelof, we can assume C plays only countable covers of X. Then, since F is choosing finite subsets, we may assume F chooses some initial segement of the countable cover. In turn, we can assume C plays an increasing open cover $\{U_0, U_1, \dots\}$ where $U_n \subseteq U_{n+1}$. And in that case, it's suffient to assume F simply chooses a singleton subset of each cover. And finally, since choices made by F are already covered, we can assume that every open set in a cover played by C covers the sets chosen by F previously.

As a result, we have the following figure of a tree of plays which I need to draw:

(Insert figure here.)

Note that for $a, b \in \omega^{<\omega}$ and $m \leq n$, we know:

(a) $U_{a^\frown m} \subseteq U_{a^\frown n}$
(for example, $U_{1627} \subseteq U_{1629}$ - increasing the final digit yields supersets)

(b) $U_a \subseteq U_{a^\frown b}$
(for example, $U_{1627} \subseteq U_{162789}$ - appending any sequence to the end yields supersets)

(c) $U_{a^\frown m} \subseteq U_{a^\frown n} \subseteq U_{a^\frown n^\frown b} \subseteq U_{a^\frown n^\frown b^\frown m}$
(for example: $U_{1627} \subseteq U_{1629283287}$ - injecting a subseqence with initial number larger than the original's final number, prior to the final number, yields supersets)

We may observe that if F can find an $f : \omega \to \omega$ such that $\bigcup_{n<\omega} U_{f\restriction(n+1)} = X$, she can use $\{U_{f\restriction 0}\}, \{U_{f\restriction 1}\}, \dots$ to counter C's strategy.

Let $V_k^n = \bigcap_{a \in \omega^{\leq n}} U_{a^\frown k}$. We claim that (1) V_k^n is open, (2) $\mathcal{V}^n = \{V_0^n, V_1^n, \dots\}$ is increasing, and (3) $\mathcal{V}^n$ is a cover. Proofs:

1. Since due to (c) for each $b \in \omega^{\leq n} \setminus k^{\leq n}$, there is an $a \in k^{\leq n}$ with $U_{a^\frown k} \subseteq U_{b^\frown k}$:

$$V_k^n = \bigcap_{a \in \omega^{\leq n}} U_{a^\frown k} = \bigcap_{a \in k^{\leq n}} U_{a^\frown k} \cap \bigcap_{b \in \omega^{\leq n} \setminus k^{\leq n}} U_{b^\frown k} = \bigcap_{a \in k^{\leq n}} U_{a^\frown k}$$

   making V_k^n a finite intersection of open sets.

2. We show $V_k^0 \subseteq V_{k+1}^0$:

$$V_k^0 = U_k \subseteq U_{k+1} = V_{k+1}^0$$

   and then assume $V_k^n \subseteq V_{k+1}^n$:

$$V_k^{n+1} = \bigcap_{a \in \omega^{\leq n+1}} U_{a^\frown k} = V_k^n \cap \bigcap_{a \in \omega^{n+1}} U_{a^\frown k} \subseteq V_{k+1}^n \cap \bigcap_{a \in \omega^{n+1}} U_{a^\frown (k+1)} = V_{k+1}^{n+1}$$

3. We easily see that $\mathcal{V}^0 = \{U_0, U_1, \dots\}$ is a cover, and then assume $\mathcal{V}^n$ is a cover.

   Let $x \in X$ and pick $l < \omega$ such that $x \in V_l^n$. For $a \in l^{n+1}$ choose l_a such that $x \in U_{a^\frown l_a}$, giving

$$x \in \bigcap_{a \in l^{n+1}} U_{a^\frown l_a}$$

   We will assume $k > l, l_a$ for all $a \in l^{\leq n+1}$.

   For any $a \in k^{n+1} \setminus l^{n+1}$ note that $a = b^\frown c$ where $b \in l^{\leq n}$ and c begins with a number l or greater:

$$V_l^n \subseteq U_{b^\frown l} \subseteq U_{b^\frown c} \subseteq U_{b^\frown c^\frown l_a} = U_{a^\frown l_a}$$

   Thus:

$$x \in V_l^n \cap \left(\bigcap_{a \in l^{n+1}} U_{a^\frown l_a} \right)$$

$$= V_l^n \cap \left(\bigcap_{a \in k^{n+1} \setminus l^{n+1}} U_{a^\frown l_a} \right) \cap \left(\bigcap_{a \in l^{n+1}} U_{a^\frown l_a} \right)$$

$$= V_l^n \cap \left(\bigcap_{a \in k^{n+1}} U_{a^\frown l_a} \right)$$

$$\subseteq V_k^n \cap \left(\bigcap_{a \in k^{n+1}} U_{a^\frown k} \right)$$

$$= V_k^{n+1}$$

Finally, apply Menger to $\mathcal{V}^n$, resulting in the cover $\{V^0_{f(0)}, V^1_{f(1)}, \dots\}$, noting

$$X = \bigcup_{n<\omega} V^n_{f(n)} \subseteq \bigcup_{n<\omega} U_{(f\restriction n)^\frown f(n)} = \bigcup_{n<\omega} U_{f\restriction(n+1)}$$

$\square$

Proposition 5. *X is compact if and only if $F \uparrow_{tact} Cov_{C,F}(X)$*

Proof. Assume X is compact. For each open cover played by C, pick the finite subcover.

Assume F has a winning tactical strategy. For any open cover, have C play only it during the entire game. F's only choice must be a finite subcover. $\square$

Proposition 6. *If X is σ-compact then $F \uparrow_{mark} Cov_{C,F}(X)$*

Proof. Let $X = \bigcup_{n<\omega} X_n$ for compact X_n. On round n, F picks the finite subcover of C's open cover of X_n. $\square$

Due to Telgarski in "On Games of Topsoe":

Theorem 7. *For metrizable X, X is σ-compact if and only if $F \uparrow Cov_{C,F}(X)$.*

In a question I posed to G, he answered:

Lemma 8. *For all $\alpha_0, \alpha_1, \dots < \omega_1$ and functions $\tau : \omega_1 \times \omega \to [\omega_1]^{<\omega}$, $\{\tau(\alpha_n, n) : n < \omega\}$ is not a cover for $\{\beta : \forall n < \omega(\beta < \alpha_n)\}$.*

Proof. Let $P_n = \{\beta : \beta < \alpha \Rightarrow \beta \in \tau(\alpha, n)\}$. Observe that each P_n is finite; else there is some α larger than every member of P_n such that $P_n \subseteq \tau(\alpha, n)$.

Choose $\beta \notin \bigcup_{n<\omega} P_n$. Then for each $n < \omega$, pick $\alpha_n > \beta$ such that $\beta \notin \tau(\alpha_n, n)$. $\square$

Note that the one-point Lindelöfication of discrete ω_1, $\omega_1^\dagger$, is not σ-compact. With the above lemma, we may see that:

Example 9. *$F \uparrow Cov_{C,F}(\omega_1^\dagger)$ but $F \not\uparrow_{mark} Cov_{C,F}(\omega_1^\dagger)$.*

Proof. First, we see F has a simple perfect information strategy: in response to the initial cover of $\omega_1^\dagger$, F chooses a co-countable neighborhood of ∞. On successive turns she may pick a single set from C's covers to cover the countable remainder.

Now, suppose that $\sigma(\mathcal{U}, n)$ was a winning Markov strategy and aim for a contradiction. Consider the covers

$$\mathcal{U}(\alpha) = \{[\alpha, \omega_1) \cup \{\infty\}\} \cup \{\{\beta\} : \beta < \alpha\}$$

and define $\tau(\alpha, n)$ to be the union of singletons chosen by $\sigma(\mathcal{U}(\alpha), n)$. As σ was a winning strategy, for all $\alpha_0, \alpha_1, \dots < \omega_1$, $\{\sigma(\mathcal{U}(\alpha_n), n) : n < \omega\}$ must cover $\omega_1^\dagger$, and thus $\{\tau(\alpha_n, n) : n < \omega\}$ must cover $\{\beta : \forall n < \omega(\beta < \alpha_n)\}$, contradiction. $\square$

The question remains:

Question 10. *Does $F \uparrow_{mark} Cov_{C,F}(X)$ imply X is σ-compact?*