ARHANGELSKII'S α-PRINCIPLES AND SELECTION GAMES

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Abstract. Arhangelskii's properties α_2 and α_4 defined for convergent sequences may be characterized in terms of Scheeper's selection principles. We generalize these results to hold for more general collections and consider these results in terms of selection games.

The following characterizations were given as Definition 1 by Kocinac in [7].

Definition 1. *Arhangelskii's α-principles* $\alpha_i(\mathcal{A},\mathcal{B})$ are defined as follows for $i \in \{1,2,3,4\}$. Let $A_n \in \mathcal{A}$ for all $n<\omega$; then there exists $B \in \mathcal{B}$ such that:

α_1: $A_n \cap B$ is cofinite in A_n for all $n<\omega$.
α_2: $A_n \cap B$ is infinite for all $n<\omega$.
α_3: $A_n \cap B$ is infinite for infinitely-many $n<\omega$.
α_4: $A_n \cap B$ is non-empty for infinitely-many $n<\omega$.

When $(\mathcal{A},\mathcal{B})$ is omitted, it is assumed that $\mathcal{A}=\mathcal{B}$ is the collection $\Gamma_{X,x}$ of sequences converging to some point $x \in X$, as introduced by Arhangelskii in [1]. Provided $\mathcal{A}$ only contains infinite sets, it's easy to see that $\alpha_n(\mathcal{A},\mathcal{B})$ implies $\alpha_{n+1}(\mathcal{A},\mathcal{B})$.

We aim to relate these to the following games.

Definition 2. The *selection game* $G_1(\mathcal{A},\mathcal{B})$ (resp. $G_{fin}(\mathcal{A},\mathcal{B})$) is an ω-length game involving Players I and II. During round n, I chooses $A_n \in \mathcal{A}$, followed by II choosing $a_n \in A_n$ (resp. $F_n \in [A_n]^{<\aleph_0}$). Player II wins in the case that $\{a_n : n<\omega\} \in \mathcal{B}$ (resp. $\bigcup\{F_n : n<\omega\} \in \mathcal{B}$), and Player I wins otherwise.

Such games are well-represented in the literature; see [12] for example. We will also consider the similarly-defined games $G_{<2}(\mathcal{A},\mathcal{B})$ (II chooses 0 or 1 points from each choice by I) and $G_{cf}(\mathcal{A},\mathcal{B})$ (II chooses cofinitely-many points). We use $G_\star(\mathcal{A},\mathcal{B})$ to denote an arbitrary selection game.

Definition 3. Let P be a player in a game G. P has a *winning strategy* for G, denoted $P \uparrow G$, if P has a strategy that defeats every possible counterplay by their opponent. If a strategy only relies on the round number and ignores the moves of the opponent, the strategy is said to be *predetermined*; the existence of a predetermined winning strategy is denoted $P \underset{\text{pre}}{\uparrow} G$.

We briefly note that the statement I $\underset{\text{pre}}{\not\uparrow} G_\star(\mathcal{A},\mathcal{B})$ is more often denoted as the *selection principle* $S_\star(\mathcal{A},\mathcal{B})$. However, we will generally characterize results in terms of selection games rather than selection principles in order to emphasize the commonalities between the statements I $\not\uparrow G_\star(\mathcal{A},\mathcal{B})$ and I $\underset{\text{pre}}{\not\uparrow} G_\star(\mathcal{A},\mathcal{B})$.

Key words and phrases. Selection principle, selection game, α_i property, convergence.

Definition 4. Let $\Gamma_{X,x}$ be the collection of non-trivial sequences $S \subseteq X$ converging to x, that is, infinite subsets of $X \setminus \{x\}$ such that for each neighborhood U of x, $S \cap U$ is cofinite in S.

Definition 5. Let Γ_X be the collection of open γ-*covers* $\mathcal{U}$ of X, that is, infinite open covers of X such that $X \notin \mathcal{U}$ and for each $x \in X$, $\{U \in \mathcal{U} : x \in U\}$ is cofinite in $\mathcal{U}$.

The similarity in nomenclature follows from the observation that every non-trivial sequence in $C_p(X)$ converging to the zero function $\mathbf{0}$ naturally defines a corresponding γ-cover in X, see e.g. Theorem 4 of [13].

The equivalence of $\alpha_2(\Gamma_{X,x}\Gamma_{X,x})$ and I $\underset{\text{pre}}{\not\uparrow}$ $G_1(\Gamma_{X,x}, \Gamma_{X,x})$ was briefly asserted by Sakai in the introduction of [11]; the similar equivalence of $\alpha_4(\Gamma_{X,x}\Gamma_{X,x})$ and I $\underset{\text{pre}}{\not\uparrow}$ $G_{fin}(\Gamma_{X,x}, \Gamma_{X,x})$ seems to be folklore. In fact, these relationships hold in more generality.

Note that by these definitions, convergent sequences (resp. γ-covers) may be uncountable, but any infinite subset of either would remain a convergent sequence (resp. γ-cover), in particular, countably infinite subsets. We capture this idea as follows.

Definition 6. Say a collection $\mathcal{A}$ is Γ-*like* if it satisfies the following for each $A \in \mathcal{A}$.

- $|A| \geq \aleph_0$.
- If $A' \subseteq A$ and $|A'| \geq \aleph_0$, then $A' \in \mathcal{A}$.

We also require the following.

Definition 7. Say a collection $\mathcal{A}$ is *almost*-Γ-*like* if for each $A \in \mathcal{A}$, there is $A' \subseteq A$ such that:

- $|A'| = \aleph_0$.
- If A'' is a cofinite subset of A', then $A'' \in \mathcal{A}$.

So all Γ-like sets are almost-Γ-like.

We are now able to prove a few general equivalences between α-princples and selection games.

1. On $\alpha_2(\mathcal{A}, \mathcal{B})$ and $G_1(\mathcal{A}, \mathcal{B})$

Theorem 8. *Let $\mathcal{A}$ be almost-Γ-like and $\mathcal{B}$ be Γ-like. Then $\alpha_2(\mathcal{A}, \mathcal{B})$ holds if and only if* I $\underset{pre}{\not\uparrow}$ $G_1(\mathcal{A}, \mathcal{B})$.

Proof. We first assume $\alpha_2(\mathcal{A}, \mathcal{B})$ and let $A_n \in \mathcal{A}$ for $n < \omega$ define a predetermined strategy for I. We may apply $\alpha_2(\mathcal{A}, \mathcal{B})$ to choose $B \in \mathcal{B}$ such that $|A_n \cap B| \geq \aleph_0$. We may then choose $a_n \in (A_n \cap B) \setminus \{a_i : i < n\}$ for each $n < \omega$. It follows that $B' = \{a_n : n < \omega\} \in \mathcal{B}$ since B' is an infinite subset of $B \in \mathcal{B}$; therefore A_n does not define a winning predetermined strategy for I.

Now suppose I $\underset{\text{pre}}{\not\uparrow}$ $G_1(\mathcal{A}, \mathcal{B})$. Given $A_n \in \mathcal{A}$ for $n < \omega$, first choose $A'_n \in \mathcal{A}$ such that $A'_n = \{a_{n,j} : j < \omega\} \subseteq A_n$, $j < k$ implies $a_{n,j} \neq a_{n,k}$, and $A_{n,m} = \{a_{n,j} : m \leq j < \omega\} \in \mathcal{A}$. Finally choose some $\theta : \omega \to \omega$ such that $|\theta^{\leftarrow}(n)| = \aleph_0$ for each $n < \omega$ (where $\theta^{\leftarrow}$ denotes the inverse set map.

Since playing $A_{\theta(m),m}$ during round m does not define a winning strategy for I in $G_1(\mathcal{A}, \mathcal{B})$, II may choose $x_m \in A_{\theta(m),m}$ such that $B = \{x_m : m < \omega\} \in \mathcal{B}$. Choose

$i_m < \omega$ for each $m < \omega$ such that $x_m = a_{\theta(m),i_m}$, noting $i_m \geq m$. It follows that $A_n \cap B \supseteq \{a_{\theta(m),i_m} : m \in \theta^{\leftarrow}(n)\}$. Since for each $m \in \theta^{\leftarrow}(n)$ there exists $M \in \theta^{\leftarrow}(n)$ such that $m \leq i_m < M \leq i_M$, and therefore $a_{\theta(m),i_m} \neq a_{\theta(m),i_M} = a_{\theta(M),i_M}$, we have shown that $A_n \cap B$ is infinite. Thus B witnesses $\alpha_2(\mathcal{A},\mathcal{B})$. □

While $\alpha_2(\mathcal{A},\mathcal{B})$ involves infinite intersection and $G_1(\mathcal{A},\mathcal{B})$ involves single selections, the previous result is made more intuitive given the following result, shown for $\mathcal{A} = \mathcal{B} = \Gamma_{X,x}$ by Nogura in [8].

Definition 9. $\alpha_2'(\mathcal{A},\mathcal{B})$ is the following claim: if $A_n \in \mathcal{A}$ for all $n < \omega$, then there exists $B \in \mathcal{B}$ such that $A_n \cap B$ is nonempty for all $n < \omega$.

(Note that α_5 is sometimes used in the literature in place of α_2'.)

Proposition 10. *If $\mathcal{A}$ is almost-Γ-like, then $\alpha_2(\mathcal{A},\mathcal{B})$ is equivalent to $\alpha_2'(\mathcal{A},\mathcal{B})$.*

Proof. The forward implication is immediate, so we assume $\alpha_2'(\mathcal{A},\mathcal{B})$. Given $A_n \in \mathcal{A}$, we apply the almost-Γ-like property to obtain $A_n' = \{a_{n,m} : m < \omega\} \subseteq A_n$ such that $A_{n,m} = A_n \setminus \{a_{i,j} : i,j < m\} \in \mathcal{A}$ for all $m < \omega$.

By applying $\alpha_2'(\mathcal{A},\mathcal{B})$ to $A_{n,m}$, we obtain $B \in \mathcal{B}$ such that $A_{n,m} \cap B$ is nonempty for all $n,m < \omega$. Since it follows that $A_n \cap B$ is infinite for all $n < \omega$, we have established $\alpha_2(\mathcal{A},\mathcal{B})$. □

2. On $\alpha_4(\mathcal{A},\mathcal{B})$ and $G_{fin}(\mathcal{A},\mathcal{B})$

A similar correspondence exists between $\alpha_4(\mathcal{A},\mathcal{B})$ and $G_{fin}(\mathcal{A},\mathcal{B})$.

Theorem 11. *Let $\mathcal{A}$ be almost-Γ-like and $\mathcal{B}$ be Γ-like. Then $\alpha_4(\mathcal{A},\mathcal{B})$ holds if and only if $\mathrm{I} \underset{pre}{\not\uparrow} G_{<2}(\mathcal{A},\mathcal{B})$ if and only if $\mathrm{I} \underset{pre}{\not\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$.*

Proof. We first assume $\alpha_4(\mathcal{A},\mathcal{B})$ and let $A_n \in \mathcal{A}$ for $n < \omega$ define a predetermined strategy for I in $G_{<2}(\mathcal{A},\mathcal{B})$. We then may choose $A_n' \in \mathcal{A}$ where $A_n' = \{a_{n,j} : j < \omega\} \subseteq A_n$, $j < k$ implies $a_{n,j} \neq a_{n,k}$, and $A_n'' = A_n' \setminus \{a_{i,j} : i,j < n\} \in \mathcal{A}$.

By applying $\alpha_4(\mathcal{A},\mathcal{B})$ to A_n'', we obtain $B \in \mathcal{B}$ such that $A_n'' \cap B \neq \emptyset$ for infintely-many $n < \omega$. We then let $F_n = \emptyset$ when $A_n'' \cap B = \emptyset$, and $F_n = \{x_n\}$ for some $x_n \in A_n'' \cap B$ otherwise. Then we will have that $B' = \bigcup\{F_n : n < \omega\} \subseteq B$ belongs to $\mathcal{B}$ once we show that B' is infinite. To see this, for $m \leq n < \omega$ note that either F_m is empty (and we let $j_m = 0$) or $F_m = \{a_{m,j_m}\}$ for some $j_m \geq m$; choose $N < \omega$ such that $j_m < N$ for all $m \leq n$ and $F_N = \{x_N\}$. Thus $F_m \neq F_N$ for all $m \leq n$ since $x_N \notin \{a_{i,j} : i,j < N\}$. Thus II may defeat the predetermined strategy A_n by playing F_n each round.

Since $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_{<2}(\mathcal{A},\mathcal{B})$ immediately implies $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$, we assume the latter. Given $A_n \in \mathcal{A}$ for $n < \omega$, we note this defines a (non-winning) predetermined strategy for I, so II may choose $F_n \in [A_n]^{<\aleph_0}$ such that $B = \bigcup\{F_n : n < \omega\} \in \mathcal{B}$. Since B is infinite, we note $F_n \neq \emptyset$ for infinitely-many $n < \omega$. Thus B witnesses $\alpha_4(\mathcal{A},\mathcal{B})$ since $A_n \cap B \supseteq F_n \neq \emptyset$ for infinitely-many $n < \omega$. □

This shows that II gains no advantage from picking more than one point per round. This in fact only depends on $\mathcal{B}$ being Γ-like, which we formalize in the following results.

Theorem 12. *Let $\mathcal{B}$ be Γ-like. Then $\mathrm{I} \underset{pre}{\uparrow} G_{<2}(\mathcal{A},\mathcal{B})$ if and only if $\mathrm{I} \underset{pre}{\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$.*

Proof. Assume $\bigcup \mathcal{A}$ is well-ordered. Given a winning predetermined strategy A_n for I in $G_{<2}(\mathcal{A}, \mathcal{B})$, consider $F_n \in [A_n]^{<\aleph_0}$. We set

$$F_n^* = \begin{cases} \emptyset & \text{if } F_n \setminus \bigcup\{F_m : m < n\} = \emptyset \\ \{\min(F_n \setminus \bigcup\{F_m : m < n\})\} & \text{otherwise} \end{cases}$$

Since $|F_n^*| < 2$, we have that $\bigcup\{F_n^* : n < \omega\} \notin \mathcal{B}$. In the case that $\bigcup\{F_n^* : n < \omega\}$ is finite, we immediately see that $\bigcup\{F_n : n < \omega\}$ is also finite and therefore not in $\mathcal{B}$. Otherwise $\bigcup\{F_n^* : n < \omega\} \notin \mathcal{B}$ is an infinite subset of $\bigcup\{F_n : n < \omega\}$, and thus $\bigcup\{F_n : n < \omega\} \notin \mathcal{B}$ too. Therefore A_n is a winning predetermined strategy for I in $G_{fin}(\mathcal{A}, \mathcal{B})$ as well. $\square$

Theorem 13. *Let $\mathcal{B}$ be Γ-like. Then $\mathrm{I} \uparrow G_{<2}(\mathcal{A}, \mathcal{B})$ if and only if $\mathrm{I} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$.*

Proof. Assume $\bigcup \mathcal{A}$ is well-ordered. Suppose $\mathrm{I} \uparrow G_{<2}(\mathcal{A}, \mathcal{B})$ is witnessed by the strategy σ. Let $\langle\rangle^\star = \langle\rangle$, and for $s^\frown \langle F\rangle \in ([\bigcup \mathcal{A}]^{<\aleph_0})^{<\omega} \setminus \{\langle\rangle\}$ let

$$(s^\frown \langle F\rangle)^\star = \begin{cases} s^{\star\frown} \langle\emptyset\rangle & \text{if } F \setminus \bigcup \mathrm{range}(s) = \emptyset \\ s^{\star\frown} \langle\{\min(F \setminus \bigcup \mathrm{range}(s))\}\rangle & \text{otherwise} \end{cases}$$

We then define the strategy τ for I in $G_{fin}(\mathcal{A}, \mathcal{B})$ by $\tau(s) = \sigma(s^\star)$. Then given any counterattack $\alpha \in ([\bigcup \mathcal{A}]^{<\aleph_0})^\omega$ by II played against τ, we note that $\alpha^* = \bigcup\{(\alpha \restriction n)^* : n < \omega\}$ is a counterattack to σ, and thus loses. This means $B = \bigcup \mathrm{range}(\alpha^*) \notin \mathcal{B}$.

We consider two cases. The first is the case that $\bigcup \mathrm{range}(\alpha^*)$ is finite. Noting that $\alpha^*(m) \cap \alpha^*(n) = \emptyset$ whenever $m \neq n$, there exists $N < \omega$ such that $\alpha^*(n) = \emptyset$ for all $n > N$. As a result, $\bigcup \mathrm{range}(\alpha) = \bigcup \mathrm{range}(\alpha \restriction n)$, and thus $\bigcup \mathrm{range}(\alpha)$ is finite, and therefore not in $\mathcal{B}$.

In the other case, $\bigcup \mathrm{range}(\alpha^*) \notin \mathcal{B}$ is an infinite subset of $\bigcup \mathrm{range}(\alpha)$, and therefore $\bigcup \mathrm{range}(\alpha) \notin \mathcal{B}$ as well. Thus we have shown that τ is a winning strategy for I in $G_{fin}(\mathcal{A}, \mathcal{B})$. $\square$

We note that the above proof technique could be used to establish that perfect-information and limited-information strategies for II in $G_{fin}(\mathcal{A}, \mathcal{B})$ may be improved to be valid in $G_{<2}(\mathcal{A}, \mathcal{B})$, provided $\mathcal{B}$ is Γ-like. As such, $G_{<2}(\mathcal{A}, \mathcal{B})$ and $G_{fin}(\mathcal{A}, \mathcal{B})$ are effectively equivalent games under this hypothesis, so we will no longer consider $G_{<2}(\mathcal{A}, \mathcal{B})$.

3. Perfect information and predetermined strategies

We now demonstrate the following, in the spirit of Pawlikowskii's celebrated result that a winning strategy for the first player in the Rothberger game may always be improved to a winning predetermined strategy [10].

Theorem 14. *Let $\mathcal{A}$ be almost-Γ-like and $\mathcal{B}$ be Γ-like. Then*

- $\mathrm{I} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ *if and only if* $\mathrm{I} \underset{pre}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$*, and*
- $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$ *if and only if* $\mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$.

Proof. We assume $\mathrm{I} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ and let the symbol $\dagger$ mean $< \aleph_0$ (respectively, $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$ and $\dagger = 1$, and for convenience we assume II plays singleton subsets of $\mathcal{A}$ rather than elements). As $\mathcal{A}$ is almost-Γ-like, there is a winning strategy σ

where $|\sigma(s)| = \aleph_0$ and $\sigma(s) \cap \bigcup \mathrm{range}(s) = \emptyset$ (that is, σ never replays the choices of II) for all partial plays s by II.

For each $s \in \omega^{<\omega}$, suppose $F_{s \restriction m} \in [\bigcup A]^\dagger$ is defined for each $0 < m \leq |s|$. Then let $s^\star : |s| \to [\bigcup \mathcal{A}]^\dagger$ be defined by $s^\star(m) = F_{s\restriction m+1}$, and define $\tau' : \omega^{<\omega} \to \mathcal{A}$ by $\tau'(s) = \sigma(s^\star)$. Finally, set $[\sigma(s^\star)]^\dagger = \{F_{s^\frown\langle n\rangle} : n < \omega\}$, and for some bijection $b : \omega^{<\omega} \to \omega$ let $\tau(n) = \tau'(b(n))$ be a predetermined strategy for I in $G_{fin}(\mathcal{A}, \mathcal{B})$ (resp. $G_1(\mathcal{A}, \mathcal{B})$).

Suppose α is a counterattack by II against τ, so

$$\alpha(n) \in [\tau(n)]^\dagger = [\tau'(b(n))]^\dagger = [\sigma(b(n)^\star)]^\dagger$$

It follows that $\alpha(n) = F_{b(n)^\frown\langle m\rangle}$ for some $m < \omega$. In particular, there is some infinite subset $W \subseteq \omega$ and $f \in \omega^\omega$ such that $\{\alpha(n) : n \in W\} = \{F_{f\restriction n+1} : n < \omega\}$. Note here that $(f \restriction n+1)^\star = (f \restriction n)^{\star\frown} \langle F_{f\restriction n+1}\rangle$. This shows that $F_{f\restriction n+1} \in [\sigma((f \restriction n)^\star)]^\dagger$ is an attempt by II to defeat σ, which fails. Thus $\bigcup\{F_{f\restriction n+1} : n < \omega\} = \bigcup\{\alpha(n) : n \in W\} \notin \mathcal{B}$, and since this set is infinite (as σ prevents II from repeating choices) we have $\bigcup\{\alpha(n) : n < \omega\} \notin \mathcal{B}$ too. Therefore τ is winning. □

Note that the assumption in Theorem 14 that $\mathcal{A}$ be almost-Γ-like cannot be omitted. In [2] an example of a space X^* and point $\infty \in X^*$ where I $\uparrow G_1(\mathcal{A}, \mathcal{B})$ but I $\underset{\mathrm{pre}}{\not\uparrow} G_1(\mathcal{A}, \mathcal{B})$ is given, where $\mathcal{A}$ is the set of open neighborhoods of ∞ (which are all uncountable), and $\mathcal{B}$ is the set $\Gamma_{X^*,\infty}$ of sequences converging to that point. (Note that $G_1(\mathcal{A}, \mathcal{B})$ is called $Gru_{O,P}(X^*, \infty)$ in that paper, and an equivalent game $Gru_{K,P}(X)$ is what is directly studied. In fact, more is shown: I has a winning perfect-information strategy, but for any natural number k, any strategy that only uses the most recent k moves of II and the round number can be defeated.)

While $\mathcal{A}$ is often not almost-Γ-like in general, it may satisfy that property in combination with the selection principles being considered.

Proposition 15. *Let $\mathcal{B}$ be Γ-like, $\mathcal{B} \subseteq \mathcal{A}$, and* I $\underset{pre}{\not\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$*. Then $\mathcal{A}$ is almost-Γ-like.*

Proof. Let $A \in \mathcal{A}$, and for all $n < \omega$ let $A_n = A$. Then A_n is not a winning predetermined strategy for I, so II may choose finite sets $B_n \subseteq A_n = A$ such that $A' = \bigcup\{B_n : n < \omega\} \in \mathcal{B} \subseteq \mathcal{A}$.

It follows that $A' \subseteq A$ and $|A'| = \aleph_0$, and for any infinite subset $A'' \subseteq A'$ (in particular, any cofinite subset), $A'' \in \mathcal{B} \subseteq \mathcal{A}$. Thus $\mathcal{A}$ is almost-Γ-like. □

Note that in the previous result, I $\underset{\mathrm{pre}}{\not\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$ could be weakened to the choice principle $\binom{\mathcal{A}}{\mathcal{B}}$: for every member of $\mathcal{A}$, there is some countable subset belonging to $\mathcal{B}$.

Corollary 16. *Let $\mathcal{B}$ be Γ-like and $\mathcal{B} \subseteq \mathcal{A}$. Then*

- I $\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ *if and only if* I $\underset{pre}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$*, and*
- I $\uparrow G_1(\mathcal{A}, \mathcal{B})$ *if and only if* I $\underset{pre}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$*.*

Proof. Assuming I $\underset{\mathrm{pre}}{\not\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$, we have I $\not\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ by Proposition 15 and Theorem 14.

Similarly, assuming $\text{I} \underset{\text{pre}}{\not\uparrow} G_1(\mathcal{A},\mathcal{B}) \Rightarrow \text{I} \underset{\text{pre}}{\not\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$, we have $\text{I} \not\uparrow G_1(\mathcal{A},\mathcal{B})$ by Proposition 15 and Theorem 14. $\square$

This corollary generalizes e.g. Theorems 26 and 30 of [12] Theorem 5 of [6], and Corollary 36 of [3].

In summary, using the selection principle notation $S_\star(\mathcal{A},\mathcal{B})$:

Corollary 17. *Let $\mathcal{B}$ be Γ-like and $\mathcal{B} \subseteq \mathcal{A}$. Then*

- $\text{I} \not\uparrow G_1(\mathcal{A},\mathcal{B})$ *if and only if* $S_1(\mathcal{A},\mathcal{B})$ *if and only if* $\alpha_2(\mathcal{A},\mathcal{B})$.
- $\text{I} \not\uparrow G_{fin}(\mathcal{A},\mathcal{B})$ *if and only if* $S_{fin}(\mathcal{A},\mathcal{B})$ *if and only if* $\alpha_4(\mathcal{A},\mathcal{B})$*, and*

4. Disjoint selections

In each $\alpha_i(\mathcal{A},\mathcal{B})$ principle, it is not required for the collection $\{A_n : n < \omega\}$ to be pairwise disjoint. However, in many cases it may as well be.

Definition 18. For $i \in \{1,2,3,4\}$ let $\alpha_{i.1}(\mathcal{A},\mathcal{B})$ denote the claim that $\alpha_i(\mathcal{A},\mathcal{B})$ holds provided the collection $\{A_n : n < \omega\}$ is pairwise disjoint.

Of course, $\alpha_i(\mathcal{A},\mathcal{B})$ implies $\alpha_{i.1}(\mathcal{A},\mathcal{B})$. It's also immediate that $\alpha_{i.1}(\mathcal{A},\mathcal{B})$ implies $\alpha_{i.1+1}(\mathcal{A},\mathcal{B})$ for the same reason that $\alpha_i(\mathcal{A},\mathcal{B})$ implies $\alpha_{i+1}(\mathcal{A},\mathcal{B})$.

We take advantage of the following lemma. The citation is given to Peter Nyikos who provides a nice proof. At a 2020 Fall meeting of the Carolinas Topology Seminar, it was suggested by Alan Dow that this lemma may be known as the "[Hausdorff] Disjoint Refinement Lemma", as found in e.g. [4, Lemma 3.4].

Lemma 19 (Lemma 1.2 of [9])**.** *Given a family $\{A_n : n < \omega\}$ of infinite sets, there exist infinite subsets $A'_n \subseteq A_n$ such that $\{A'_n : n < \omega\}$ is pairwise disjoint.*

Proposition 20. *Let $\mathcal{A}$ be Γ-like. For $i \in \{2,3,4\}$, $\alpha_i(\mathcal{A},\mathcal{B})$ is equivalent to $\alpha_{i.1}(\mathcal{A},\mathcal{B})$.*

Proof. Assume $\alpha_{i.1}(\mathcal{A},\mathcal{B})$. Let $A_n \in \mathcal{A}$. By applying the previous lemma, we have $\{A'_n : n < \omega\}$ pairwise disjoint with each A'_n being an infinite subset of A_n. Since $\mathcal{A}$ is Γ-like, $A'_n \in \mathcal{A}$, so we have a witness $B \in \mathcal{B}$ such that $A'_n \cap B$ satisfies $\alpha_{i.1}(\mathcal{A},\mathcal{B})$ for all $n < \omega$. Since $A'_n \subseteq A_n$, it follows that $A_n \cap B$ satisfies $\alpha_i(\mathcal{A},\mathcal{B})$ for all $n < \omega$. $\square$

It's also true that $\alpha_1(\Gamma_{X,x},\Gamma_{X,x})$ is equivalent to $\alpha_{1.1}(\Gamma_{X,x},\Gamma_{X,x})$, which is captured by the following theorem.

Theorem 21. *Let $\mathcal{A}$ be a Γ-like collection closed under finite unions and $\mathcal{A} \subseteq \mathcal{B}$. Then $\alpha_1(\mathcal{A},\mathcal{B})$ is equivalent to $\alpha_{1.1}(\mathcal{A},\mathcal{B})$.*

Proof. Let $A_n \in \mathcal{A}$ and assume $\alpha_{1.1}(\mathcal{A},\mathcal{B})$. To apply the assumption, we will define a pairwise disjoint collection $\{A'_n : n < \omega\}$. First let $0' = 0$ and $A'_0 = A_0$. Then suppose $m' \geq m$ and $A'_m \subseteq A_{m'} \subseteq \bigcup_{i \leq m} A'_i$ are defined for all $m \leq n$.

If $A_k \setminus \bigcup_{m \leq n} A'_m$ is finite for $k > n'$, let $B = \bigcup_{m \leq n'} A_m \in \mathcal{A} \subseteq \mathcal{B}$. This B then witnesses $\alpha_1(\mathcal{A},\mathcal{B})$ since $A_k \setminus B$ is finite for all $k < \omega$.

Otherwise pick the minimal $(n+1)' > n$ where $A'_{n+1} = A_{(n+1)'} \setminus \bigcup_{m \leq n} A'_m$ is infinite. It follows that $A'_{n+1} \subseteq A_{(n+1)'} \subseteq \bigcup_{m \leq n+1} A'_m$. By construction, $\{A'_n : n < \omega\}$ is a pairwise disjoint collection of members of $\mathcal{A}$, and we may apply $\alpha_{1.1}(\mathcal{A},\mathcal{B})$ to obtain $B \in \mathcal{B}$ where $A'_n \setminus B$ is finite for all $n < \omega$.

Finally let $k < \omega$. If $k = n'$ for some $n < \omega$, then $A_k \setminus B = A_{n'} \setminus B \subseteq (\bigcup_{m \leq n} A'_m) \setminus B$ is finite. Otherwise, $n' < k < (n+1)'$ for some $n < \omega$. Then $(A_k \setminus \bigcup_{m \leq n} A'_m) \setminus B \subseteq A_k \setminus \bigcup_{m \leq n} A'_m$ is finite, and $(A_k \cap \bigcup_{m \leq n} A'_m) \setminus B \subseteq (\bigcup_{m \leq n} A'_m) \setminus B$ is finite, showing $A_k \setminus B$ is finite. □

Another fractional version of these α-principles is given as $\alpha_{1.5}$ in [9], defined in general as follows.

Definition 22. Let $\alpha_{1.5}(\mathcal{A}, \mathcal{B})$ be the assertion that when $A_n \in \mathcal{A}$ and $\{A_n : n < \omega\}$ is pairwise disjoint, then there exists $B \in \mathcal{B}$ such that $A_n \cap B$ is cofinite in A_n for infinitely-many $n < \omega$.

It's immediate from their definitions that $\alpha_{1.1}(\mathcal{A}, \mathcal{B})$ implies $\alpha_{1.5}(\mathcal{A}, \mathcal{B})$, which implies $\alpha_{3.1}(\mathcal{A}, \mathcal{B})$. Nyikos originally showed that $\alpha_{1.5}(\Gamma_{X,x}, \Gamma_{X,x})$ implies $\alpha_2(\Gamma_{X,x}, \Gamma_{X,x})$; this result generalizes as follows.

Theorem 23. *Let $\mathcal{A}$ be a Γ-like collection closed under finite unions. Then $\alpha_{1.5}(\mathcal{A}, \mathcal{B})$ implies $\alpha_2(\mathcal{A}, \mathcal{B})$.*

Proof. We assume $\alpha_{1.5}(\mathcal{A}, \mathcal{B})$ and demonstrate $\alpha_{2.1}(\mathcal{A}, \mathcal{B})$, which is equivalent to $\alpha_2(\mathcal{A}, \mathcal{B})$ by Proposition 20. So let $A_n \in \mathcal{A}$ such that $\{A_n : n < \omega\}$ is pairwise-disjoint.

We may partition each A_n into $\{A_{n,m} : m < \omega\}$ with $A_{n,m} \in \mathcal{A}$ for all $m < \omega$. Let $A'_n = \bigcup\{A_{i,j} : i + j = n\} \in \mathcal{A}$; since $\{A'_n : n < \omega\}$ is pairwise disjoint, we may apply $\alpha_{1.5}(\mathcal{A}, \mathcal{B})$ to obtain $B \in \mathcal{B}$ where $A'_n \cap B$ is cofinite in A'_n for infinitely-many $n < \omega$.

Then for $n < \omega$, choose $N \geq n$ with $A'_N \cap B$ cofinite in A'_N. Then $A_{n,N-n} \subseteq A'_N$, so $A_{n,N-n} \cap B$ is cofinite in $A_{n,N-n}$, in particular, $A_{n,N-n} \cap B$ is infinite. Therefore $A_n \cap B$ is infinite, and we have shown $\alpha_{2.1}(\mathcal{A}, \mathcal{B})$. □

Corollary 24. *Let $\mathcal{A}$ be a Γ-like collection closed under finite unions. Then $\alpha_x(\mathcal{A}, \mathcal{B})$ implies $\alpha_y(\mathcal{A}, \mathcal{B})$ for $1 < x \leq y$. Additionally, if $\mathcal{A} \subseteq \mathcal{B}$, then $\alpha_x(\mathcal{A}, \mathcal{B})$ implies $\alpha_y(\mathcal{A}, \mathcal{B})$ for $1 \leq x \leq y$.*

For this paragraph we adopt the conventional assumption that $\Gamma_{X,x}$ is restricted to countable sets. Nyikos showed a consistent example where $\alpha_2(\Gamma_{X,x}, \Gamma_{X,x})$ fails to imply $\alpha_{1.5}(\Gamma_{X,x}, \Gamma_{X,x})$, and a consistent example where $\alpha_{1.5}(\Gamma_{X,x}, \Gamma_{X,x})$ fails to imply $\alpha_1(\Gamma_{X,x}, \Gamma_{X,x})$ [9]. On the other hand, Dow showed that $\alpha_2(\Gamma_{X,x}, \Gamma_{X,x})$ implies $\alpha_1(\Gamma_{X,x}, \Gamma_{X,x})$ in the Laver model for the Borel conjecture [5]; the author conjectures that this model (specifically, the fact that every ω-splitting family contains an ω-splitting family of size less than $\mathfrak{b}$ in this model) witnesses an affirmative answer to the following question.

Definition 25. A Γ-like collection is *strongly-Γ-like* if the collection is closed under finite unions and each member is countable.

Question 26. *Let $\mathcal{A}$ be strongly-Γ-like. Is it consistent that $\alpha_2(\mathcal{A}, \mathcal{A})$ implies $\alpha_1(\mathcal{A}, \mathcal{A})$?*

5. Conclusion

We conclude with the following easy result, and a couple questions.

Proposition 27. *Let $\mathcal{B}$ be Γ-like. Then $\alpha_1(\mathcal{A}, \mathcal{B})$ holds if and only if* $\mathrm{I} \underset{pre}{\not\uparrow} G_{cf}(\mathcal{A}, \mathcal{B})$.

Proof. We first assume $\alpha_1(\mathcal{A},\mathcal{B})$ and let $A_n \in \mathcal{A}$ for $n < \omega$ define a predetermined strategy for I. By $\alpha_1(\mathcal{A},\mathcal{B})$, we immediately obtain $B \in \mathcal{B}$ such that $|A_n \setminus B| < \aleph_0$. Thus $B_n = A_n \cap B$ is a cofinite choice from A_n, and $B' = \bigcup\{B_n : n < \omega\}$ is an infinite subset of B, so $B' \in \mathcal{B}$. Thus II may defeat I by choosing $B_n \subseteq A_n$ each round, witnessing $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_{cf}(\mathcal{A},\mathcal{B})$.

On the other hand, let $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_{cf}(\mathcal{A},\mathcal{B})$. Given $A_n \in \mathcal{A}$ for $n < \omega$, we note that II may choose a cofinite subset $B_n \subseteq A_n$ such that $B = \bigcup\{B_n : n < \omega\} \in \mathcal{B}$. Then B witnesses $\alpha_1(\mathcal{A},\mathcal{B})$ since $|A_n \setminus B| \leq |A_n \setminus B_n| \leq \aleph_0$. $\square$

Question 28. *Is there a game-theoretic characterization of* $\alpha_3(\mathcal{A},\mathcal{B})$*?*

Noting that $\mathrm{I} \uparrow G_1(\Gamma_X, \Gamma_X)$ if and only if $\mathrm{I} \uparrow G_{fin}(\Gamma_X, \Gamma_X)$ [7], but the same is not true of $G_\star(\Gamma_{X,x}, \Gamma_{X,x})$ (e.g. there are α_4 spaces that are not α_2 [14]), we also ask the following.

Question 29. *Is there a natural condition on* $\mathcal{A},\mathcal{B}$ *guaranteeing* $\mathrm{I} \uparrow G_1(\mathcal{A},\mathcal{B}) \Rightarrow \mathrm{I} \uparrow G_{fin}(\mathcal{A},\mathcal{B})$*?*

6. Acknowledgements and Dedication

The author would like to thank Alan Dow, Jared Holshouser, and Alexander Osipov for various discussions related to this paper.

This paper is dedicated to the memory of Dr. Phil Zenor. Dr. Zenor was the instructor of the author's undergraduate real analysis course, which was taught using inquiry-based learning. The author was first inspired to become a mathematician because Dr. Zenor's class modeled the experience of being a mathematician; rather than listening to lectures, students in Dr. Zenor's class were encouraged and rewarded for engaging with the material themselves and presenting their findings to their peers.

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