Example 1. *Let X be a compact, zero-dimensional space with a point-countable cover $\{U_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ which is not σ-point-finite. (Can we change this to, every uncountable subcollection is not point-finite, or every stationary subset S of $\mathrm{Lim}(\omega_1)$ doesn't yield a point-finite subcollection?)*

For $\alpha \in \mathrm{Lim}(\omega_1)$, pick $\beta_{\alpha,n}$ such that $\lim\limits_{n\to\infty} \beta_{\alpha,n} = \alpha$. Assume every $\beta \in \mathrm{Suc}(\omega_1)$ is $\beta_{\alpha,n}$ for some α, n. Let $\mathcal{B}_\alpha = \{\beta_{\alpha,0}, \beta_{\alpha,1}, \dots\}$

Let $\mathbb{X} = \mathrm{Lim}(\omega_1) \cup (X \times \mathrm{Suc}(\omega_1))$, with the topology induced by $X \times \{\beta\}$ being open and a topological copy of X, and each α given the open neighborhood $B_\alpha = \{\alpha\} \cup (U_\alpha \times \mathcal{B}_\alpha)$.

Theorem 2. *$K \uparrow LF_{K,P}(\mathbb{X})$.*

Proof. Idea: Seeing $\langle x_i, \beta_i \rangle$, K forbids $X \times \{\beta_i\}$, and promises to eventually forbid B_α for each α where $x_i \in U_\alpha$. □

Theorem 3. *$K \not\uparrow_{tact} LF_{K,P}(\mathbb{X})$.*

Proof. Locally compact, not metacompact.

Game idea: Suppose there exists $\alpha \in \mathrm{Lim}(\omega_1)$ such that one may find arbitrarily large $n < \omega$ and $x \in U_\alpha$ such that $\sigma(x, \beta_{\alpha,n})$ does not forbid a branch for α. Then P has a counter converging to α.

Otherwise define $f : \mathrm{Lim}(\omega_1) \to \omega$ such that for $n \geq f(\alpha)$, $\sigma(x, \beta_{\alpha,n})$ forbids a branch for α for all $x \in U_\alpha$. Then $g(\alpha) = \beta_{\alpha,f(\alpha)}$ is a downward function, and there exists a stationary subset $S \subseteq \mathrm{Lim}_{\omega_1}$ and some $\beta \in \mathrm{Suc}(\omega_1)$ such that $\beta = \beta_{\alpha,f(\alpha)}$ for all $\alpha \in S$.

Observing that $\{U_\alpha : \alpha \in S\}$ is dfggreat □