

# Limited Information Strategies for Topological Games

by

Steven Clontz

A dissertation submitted to the Graduate Faculty of  
Auburn University  
in partial fulfillment of the  
requirements for the Degree of  
Doctor of Philosophy

Auburn, Alabama  
May 4, 2015

Keywords: topology, uniform spaces, infinite games, limited information strategies

Copyright 2015 by Steven Clontz

Approved by

Gary Gruenhage, Chair, Professor of Mathematics  
Stewart Baldwin, Professor of Mathematics  
Chris Rodger, Professor of Mathematics  
Michel Smith, Professor of Mathematics  
George Flowers, Dean of the Graduate School

## Abstract

I talk a lot about topological games.

TODO: Write this.

## Acknowledgments

TODO: Thank people.

## Table of Contents

Abstract . . . . .	ii
Acknowledgments . . . . .	iii
List of Figures . . . . .	vi
List of Tables . . . . .	vii
1 Introduction . . . . .	1
2 Topological Games and Strategies of Perfect and Limited Information . . . . .	2
2.1 Games . . . . .	2
2.1.1 Infinite and Topological Games . . . . .	3
2.2 Strategies . . . . .	5
2.2.1 Applications of Strategies . . . . .	7
2.2.2 Limited Information Strategies . . . . .	9
3 Gruenhage's Convergence and Clustering Games . . . . .	11
3.1 Fort spaces . . . . .	14
3.2 Sigma-products . . . . .	18
4 Bell's Game . . . . .	24
4.1 Uniform Spaces . . . . .	24
4.2 Games on uniformizable spaces . . . . .	26
4.3 Characterizing Corson compactness . . . . .	29
4.4 Limited information results . . . . .	34
5 Gruenhage's Locally Finite Games . . . . .	38
6 Menger's Game and Rothberger's Game . . . . .	39
6.1 Markov strategies . . . . .	40
6.2 Menger game derived covering properties . . . . .	46

6.3	Robustly Lindelöf . . . . .	50
6.4	Rothberger game . . . . .	54
	Bibliography . . . . .	59

## List of Figures

6.1	Diagram of Filling/Menger game implications . . . . .	43
6.2	Diagram of Filling/Menger game implications with $S(\kappa, \omega, \omega)$ . . . . .	45
6.3	Diagram of covering properties related to the Menger game . . . . .	46

## List of Tables

## Chapter 1

### Introduction

Basic overview of combinatorial games, topological games, limited info strategies, and applications in topology.



Chapter 2  
Topological Games and Strategies  
of Perfect and Limited Information

The goal of this paper is to explore the applications of limited information strategies in existing topological games. There are a variety of frameworks for modeling such games, so we establish one within this chapter which we will use for this manuscript.

## 2.1 Games

Intuitively, the games studied in this paper are two-player games for which each player takes turns making a choice from a set of possible moves. At the conclusion of the game, the choices made by both players are examined, and one of the players is declared the winner of that playthrough.

Games may be modeled mathematically in various ways, but we will find it convenient to think of them in terms defined by Gale and Stewart. [6]

**Definition 2.1.1.** A *game* is a tuple  $\langle M, W \rangle$  such that  $W \subseteq M^\omega$ .  $M$  is set of *moves* for the game, and  $M^\omega$  is the set of all possible *playthroughs* of the game.

$W$  is the set of *winning playthroughs* or *victories* for the first player, and  $M^\omega \setminus W$  is the set of victories for the second player. ( $W$  is often called the *payoff set* for the first player.)

◇

Within this model, we may imagine two players  $\mathcal{A}$  and  $\mathcal{B}$  playing a game which consists of *rounds* enumerated for each  $n < \omega$ . During round  $n$ ,  $\mathcal{A}$  chooses  $a_n \in M$ , followed by  $\mathcal{B}$  choosing  $b_n \in M$ . The playthrough corresponding to those choices would be the sequence  $p = \langle a_0, b_0, a_1, b_1, \dots \rangle$ . If  $p \in W$ , then  $\mathcal{A}$  is the winner of that playthrough, and if  $p \notin W$ , then  $\mathcal{B}$  is the winner. Note that no ties are allowed.

Rather than explicitly defining  $W$ , we typically define games by declaring the *rules* that each player must follow and the *winning condition* for the first player. Then a playthrough is in  $W$  if either the first player made only *legal moves* which observed the game's rules and the playthrough satisfied the winning condition, or the second player made an *illegal move* which contradicted the game's rules. Often, we will consider *legal playthroughs* where both players only made legal moves, in which case only the winning condition need be considered.

As an illustration, we could model a game of chess (ignoring stalemates) by letting

$$M = \{ \langle p, s \rangle : p \text{ is a chess piece and } s \text{ is a space on the board} \}$$

representing moving a piece  $p$  to the space  $s$  on the board. Then the rules of chess restrict White from moving pieces which belong to Black, or moving a piece to an illegal space on the board.<sup>1</sup> The winning condition could then “inspect” the resulting positions of pieces on the board after each move to see if White attained a checkmate. This winning condition along with the rules implicitly define the set  $W$  of winning playthroughs for White.

### 2.1.1 Infinite and Topological Games

Games never technically end within this model, since playthroughs of the game are infinite sequences. However, for all practical purposes many games end after a finite number of turns.

**Definition 2.1.2.** A game is said to be a *finite game* if for every playthrough  $p \in M^\omega$  there exists a round  $n < \omega$  such that  $[p \upharpoonright n] = \{q \in M^\omega : q \supseteq p \upharpoonright n\}$  is a subset of either  $W$  or  $M^\omega \setminus W$ . ◇

---

<sup>1</sup>In practice,  $M$  is often defined as the union of two sets, such as white pieces and black pieces in chess. For example, the first player may choose open sets in a topology, while the second player chooses points within the topological space.

Put another way, a finite game is decided after a finite number of rounds, after which the game's winner could not change even if further rounds were played. Games which are not finite are called *infinite games*.

As an illustration of an infinite game, we may consider a simple example due to Baker [1].

**Game 2.1.3.** Let  $Bak_{A,B}(X)$  denote a game with players  $\mathcal{A}$  and  $\mathcal{B}$ , defined for each subset  $A \subset \mathbb{R}$ . In round 0,  $\mathcal{A}$  chooses a number  $a_0$ , followed by  $\mathcal{B}$  choosing a number  $b_0$  such that  $a_0 < b_0$ . In round  $n + 1$ ,  $\mathcal{A}$  chooses a number  $a_{n+1}$  such that  $a_n < a_{n+1} < b_n$ , followed by  $\mathcal{B}$  choosing a number  $b_{n+1}$  such that  $a_{n+1} < b_{n+1} < b_n$ .

$\mathcal{A}$  wins the game if the sequence  $\langle a_n : n < \omega \rangle$  limits to a point in  $X$ , and  $\mathcal{B}$  wins otherwise.  $\diamond$

Certainly,  $\mathcal{A}$  and  $\mathcal{B}$  will never be in a position without (infinitely many) legal moves available, and provided that  $A$  is non-trivial, there is a playthrough such that for all  $n < \omega$ , the segment  $(a_n, b_n)$  intersects both  $A$  and  $\mathbb{R} \setminus A$ . Such a playthrough could never be decided in a finite number of moves, so the winning condition considers the infinite sequence of moves made by the players and declares a victor at the “end” of the game.

**Definition 2.1.4.** A *topological game* is a game defined in terms of an arbitrary topological space.  $\diamond$

Topological games are usually infinite games, ignoring trivial examples. One of the earliest examples of a topological game is the Banach-Mazur game, proposed by Stanislaw Mazur as Problem 43 in Stefan Banach's Scottish Book (1935). A more comprehensive history of the Banach-Mazur and other topological games may be found in Telgarsky's survey on the subject [24].

The original game was defined for subsets of the real line; however, we give a more general definition here.

**Game 2.1.5.** Let  $BM_{E,N}(X)$  denote the *Banach-Mazur game* with players  $\mathcal{E}$ ,  $\mathcal{N}$  defined for each topological space  $X$ . In round 0,  $\mathcal{E}$  chooses a nonempty open set  $E_0 \subseteq X$ , followed by  $\mathcal{N}$  choosing a nonempty open subset  $N_0 \subseteq E_0$ . In round  $n + 1$ ,  $\mathcal{E}$  chooses a nonempty open subset  $E_{n+1} \subseteq N_n$ , followed by  $\mathcal{N}$  choosing a nonempty open subset  $N_{n+1} \subseteq E_{n+1}$ .

$\mathcal{E}$  wins the game if  $\bigcap_{n < \omega} E_n = \emptyset$ , and  $\mathcal{N}$  wins otherwise.  $\diamond$

For example, if  $X$  is a locally compact Hausdorff space,  $\mathcal{N}$  can “force” a win by choosing  $N_0$  such that  $\overline{N_0}$  is compact, and choosing  $N_{n+1}$  such that  $N_{n+1} \subseteq \overline{N_n} \subseteq O_{n+1} \subseteq N_n$  (possible since  $N_n$  is a compact Hausdorff  $\Rightarrow$  normal space). Since  $\bigcap_{n < \omega} E_n = \bigcap_{n < \omega} N_n$  is the decreasing intersection of compact sets, it cannot be empty.

This concept of when (and how) a player can “force” a win in certain topological games is the focus of this manuscript.

## 2.2 Strategies

We shall make the notion of forcing a win in a game rigorous by introducing “strategies” and “attacks” for games.

**Definition 2.2.1.** A *strategy* for a game  $G = \langle M, W \rangle$  is a function from  $M^{<\omega}$  to  $M$ .  $\diamond$

**Definition 2.2.2.** An *attack* for a game  $G = \langle M, W \rangle$  is a function from  $\omega$  to  $M$ .  $\diamond$

Intuitively, a strategy is a rule for one of the players on how to play the game based upon the previous (finite) moves of her opponent, while an attack is a fixed strike by an opponent indexed by round number.

**Definition 2.2.3.** The *result* of a game given a strategy  $\sigma$  for the first player and an attack  $\langle a_0, a_1, \dots \rangle$  by the second player is the playthrough

$$\langle \sigma(\emptyset), a_0, \sigma(\langle a_0 \rangle), a_1, \sigma(\langle a_0, a_1 \rangle), \dots \rangle$$

Likewise, if  $\sigma$  is a strategy for the second player, and  $\langle a_0, a_1, \dots \rangle$  is an attack by the first player, then the result is the playthrough

$$\langle a_0, \sigma(\langle a_0 \rangle), a_1, \sigma(\langle a_0, a_1 \rangle), \dots \rangle$$

◇

We now may rigorously define the notion of “forcing” a win in a game.

**Definition 2.2.4.** A strategy  $\sigma$  is a *winning strategy* for a player if for every attack by the opponent, the result of the game is a victory for that player.

If a winning strategy exists for a player  $\mathcal{A}$  in the game  $G$ , then we write  $\mathcal{A} \uparrow G$ . Otherwise, we write  $\mathcal{A} \nmid G$ . ◇

To show that a winning strategy exists for a player (i.e.  $\mathcal{A} \uparrow G$ ), we typically begin by defining it and showing that it is *legal*: it only yields moves which are legal according to the rules of the game. Then, we consider an arbitrary legal attack, and prove that the result of the game is a victory for that player.

If we wish to show that a winning strategy does not exist for a player (i.e.  $\mathcal{A} \nmid G$ ), we often consider an arbitrary legal strategy, and use it to define a legal *counter-attack* for the opponent. If we can prove that the result of the game for that strategy and counter-attack is a victory for the opponent, then a winning strategy does not exist.

Unlike finite games, is not the case that a winning strategy must exist for one of the players in an infinite game.

**Definition 2.2.5.** A game  $G$  with players  $\mathcal{A}, \mathcal{B}$  is said to be *determined* if either  $\mathcal{A} \uparrow G$  or  $\mathcal{B} \uparrow G$ . Otherwise, the game is *undetermined*. ◇

The Borel Determinacy Theorem states that  $G = \langle M, W \rangle$  is determined whenever  $W$  is a Borel subset of  $M^\omega$  [14]. It’s an easy corollary that all finite games are determined;  $W$  must be clopen.

However, as stated earlier, most topological games are infinite, and many are undetermined for certain spaces constructed using the Axiom of Choice.<sup>2</sup>

Often, we will build new strategies based on others.

**Definition 2.2.6.** A strategy  $\tau$  is a *strengthening* of another strategy  $\sigma$  for a player if whenever the result of the game for  $\sigma$  and an attack  $a$  by the opponent is a victory for the player, then the result of the game for  $\tau$  and  $a$  is also a victory for the player.  $\diamond$

**Proposition 2.2.7.** *If  $\sigma$  is a winning strategy, and  $\tau$  strengthens  $\sigma$ , then  $\tau$  is also a winning strategy.*  $\diamond$

## 2.2.1 Applications of Strategies

The power of studying these infinite-length games can be illustrated by considering the following proposition.

**Proposition 2.2.8.** *If  $X$  is countable, then  $\mathcal{B} \uparrow \text{Bak}_{A,B}(X)$ .*  $\diamond$

*Proof.* Adapted from [1]. Let  $X = \{x_0, x_1, \dots\}$ . Let  $i(a, b)$  be the least integer such that  $a < x_{i(a,b)} < b$ , if it exists. We define a strategy  $\sigma$  for  $\mathcal{B}$  such that:

- $\sigma(\langle a_0 \rangle) = x_{i(a_0, \infty)}$ . If  $i(a_0, \infty)$  does not exist, then the choice of  $\sigma(\langle a_0 \rangle)$  is arbitrary, say,  $a_0 + 1$ .
- $\sigma(\langle a_0, \dots, a_{n+1} \rangle) = x_{i(a_{n+1}, b_n)}$ , where  $b_n = \sigma(\langle a_0, \dots, a_n \rangle)$ . If  $i(a_{n+1}, b_n)$  does not exist, then the choice of  $\sigma(\langle a_0, \dots, a_{n+1} \rangle)$  is arbitrary, say,  $\sigma(\langle a_0, \dots, a_{n+1} \rangle) = \frac{a_{n+1} + b_n}{2}$ .

Observe that  $\sigma$  is a legal strategy according to the rules of the game since  $a_0 < \sigma(\langle a_0 \rangle)$  and  $a_{n+1} < \sigma(\langle a_0, \dots, a_{n+1} \rangle) < b_n$ . We claim this is a winning strategy for  $\mathcal{B}$ . Let  $a = \langle a_0, a_1, \dots \rangle$  be a legal attack by  $\mathcal{A}$  against  $\sigma$ : we will show that the resulting playthrough is

---

<sup>2</sup>These spaces cannot be constructed just only the axioms of ZF. In fact, mathematicians have studied an Axiom of Determinacy which declares that that all Gale-Stewart games are determined (and implies that the Axiom of Choice is false). [16]

a victory for  $\mathcal{B}$ , that is,  $\lim_{n \rightarrow \infty} a_n \notin X$ . Let  $b_n = \sigma(\langle a_0, \dots, a_n \rangle)$ . Note that

$$a_0 < a_1 < \dots < \lim_{n \rightarrow \infty} a_n < \dots < b_1 < b_0$$

If  $i(a_0, \infty)$  does not exist, then  $a_0$  is greater than every element of  $X$ , and thus  $\lim_{n \rightarrow \infty} a_n \notin X$ . A similar argument follows if some  $i(a_{n+1}, b_n)$  does not exist.

Otherwise,

$$i(a_0, \infty) < i(a_1, b_0) < i(a_2, b_1) < \dots$$

and for each  $i < \omega$ , one of the following must hold.

- $i < i(a_0, \infty)$ . Then  $x_i \leq a_0 < \lim_{n \rightarrow \infty} a_n$ .
- $i = i(a_0, \infty)$ . Then  $x_i = b_0 > \lim_{n \rightarrow \infty} a_n$ .
- $i(a_0, \infty) < i < i(a_1, b_0)$ . Then  $x_i \leq a_1 < \lim_{n \rightarrow \infty} a_n$  or  $x_i \geq b_0 > \lim_{n \rightarrow \infty} a_n$ .
- $i = i(a_{n+1}, b_n)$  for some  $n < \omega$ . Then  $x_i = b_{n+1} > \lim_{n \rightarrow \infty} a_n$ .
- $i(a_{n+1}, b_n) < i < i(a_{n+2}, b_{n+1})$  for some  $n < \omega$ . Then  $x_i \leq a_{n+2} < \lim_{n \rightarrow \infty} a_n$  or  $x_i \geq b_{n+1} > \lim_{n \rightarrow \infty} a_n$ .

In any case,  $x_i \neq \lim_{n \rightarrow \infty} a_n$ , and thus  $\lim_{n \rightarrow \infty} a_n \notin X$ . □

More informally,  $\mathcal{B}$  can force a win by enumerating the countable set  $X$  and playing every legal choice by the end of the game. This yields a classical result.

**Corollary 2.2.9.**  $\mathbb{R}$  is uncountable. ◇

*Proof.*  $\mathcal{A} \uparrow \text{Bak}_{A,B}(\mathbb{R})$ , since  $a_n$  must converge to some real number. This implies  $\mathcal{B} \nmid \text{Bak}_{A,B}(\mathbb{R})$ , and thus  $\mathbb{R}$  is not countable. □

Infinite games thus provide a rich framework for considering questions in set theory and topology. In general, the presence or absence of a winning strategy for a player in a topological game characterizes a property of the topological space in question.

**Theorem 2.2.10.**  $\mathcal{E} \not\preceq BM_{E,N}(X)$  if and only if  $X$  is a Baire space. [10]

◇

## 2.2.2 Limited Information Strategies

So far we have assumed both players enjoy *perfect information*, and may develop strategies which use all of the previous moves of the opponent as input.

**Definition 2.2.11.** For a game  $G = \langle M, W \rangle$ , the  $k$ -*tactical fog-of-war* is the function  $\nu_k : M^{<\omega} \rightarrow M^{\leq k}$  defined by

$$\nu_k(\langle m_0, \dots, m_{n-1} \rangle) = \langle m_{n-k}, \dots, m_{n-1} \rangle$$

and the  $k$ -*Markov fog-of-war* is the function  $\mu_k : M^{<\omega} \rightarrow (M^{\leq k} \times \omega)$  defined by

$$\mu_k(\langle m_0, \dots, m_{n-1} \rangle) = \langle \langle m_{n-k}, \dots, m_{n-1} \rangle, n \rangle$$

◇

Essentially, these fogs-of-war represent a limited memory:  $\nu_k$  filters out all but the last  $k$  moves of the opponent, and  $\mu_k$  filters out all but the last  $k$  moves of the opponent and the round number.

We call strategies which do not require full recollection of the opponent's moves *limited information strategies*.

**Definition 2.2.12.** A  $k$ -*tactical strategy* or  $k$ -*tactic* is a function  $\sigma : M^{\leq k} \rightarrow M$  yielding a corresponding strategy  $\sigma \circ \nu_k : M^{<\omega} \rightarrow M$ .

A  $k$ -*Markov strategy* or  $k$ -*mark* is a function  $\sigma : M^{\leq k} \times \omega \rightarrow M$  yielding a corresponding strategy  $\sigma \circ \mu_k : M^{<\omega} \rightarrow M$ .

◇

$k$ -tactics and  $k$ -marks may then only use the last  $k$  moves of the opponent, and in the latter case, also the round number.



The  $k$  is usually omitted when  $k = 1$ . A (1-)tactic is called a *stationary strategy* by some authors. 0-tactics are not usually interesting (such strategies would be constant functions); however, we will discuss 0-Markov strategies, called *predetermined strategies* since such a strategy only uses the round number and does not rely on knowing which moves the opponent will make. Of course, a limited information strategy  $\sigma$  is *winning* for a player if its corresponding strategy  $\sigma \circ \nu_k$  or  $\sigma \circ \mu_k$  is winning for that player.

**Definition 2.2.13.** If a winning  $k$ -tactical strategy exists for a player  $\mathcal{A}$  in the game  $G$ , then we write  $\mathcal{A} \underset{k\text{-tact}}{\uparrow} G$ . If  $k = 1$ , then  $\mathcal{A} \underset{\text{tact}}{\uparrow} G$ .

If a winning  $k$ -Markov strategy exists for a player  $\mathcal{A}$  in the game  $G$ , then we write  $\mathcal{A} \underset{k\text{-mark}}{\uparrow} G$ . If  $k = 1$ , then  $\mathcal{A} \underset{\text{mark}}{\uparrow} G$ , and if  $k = 0$ , then  $\mathcal{A} \underset{\text{pre}}{\uparrow} G$ .  $\diamond$

The existence of a winning limited information strategy can characterize a stronger property than the property characterized by a perfect information strategy.

**Definition 2.2.14.**  $X$  is an  $\alpha$ -favorable space when  $\mathcal{N} \underset{\text{tact}}{\uparrow} BM_{E,N}(X)$ .  $X$  is a *weakly  $\alpha$ -favorable space* when  $\mathcal{N} \uparrow BM_{E,N}(X)$ .  $\diamond$

**Proposition 2.2.15.**  $X$  is  $\alpha$ -favorable  $\Rightarrow X$  is weakly  $\alpha$ -favorable  $\Rightarrow X$  is Baire  $\diamond$

Those arrows may not be reversed. A Bernstein subset of the real line is an example of a Baire space which is not weakly  $\alpha$ -favorable, and Gabriel Debs constructed an example of a completely regular space for which  $\mathcal{N}$  has a winning 2-tactic, but lacks a winning 1-tactic.

[4]

## Chapter 3

### Gruenhage's Convergence and Clustering Games

We begin by investigating a game due to Gary Gruenhage.

**Game 3.0.16.** Let  $Gru_{\mathcal{O},P}^{\rightarrow}(X, S)$  denote the *W-convergence game* with players  $\mathcal{O}$ ,  $\mathcal{P}$ , for a topological space  $X$  and  $S \subseteq X$ .

In round  $n$ ,  $\mathcal{O}$  chooses an open neighborhood  $O_n \supseteq S$ , followed by  $\mathcal{P}$  choosing a point  $x_n \in \bigcap_{m \leq n} O_m$ .

$\mathcal{O}$  wins the game if the points  $x_n$  converge to the set  $S$ ; that is, for every open neighborhood  $U \supseteq S$ ,  $x_n \in U$  for all but finite  $n < \omega$ .

If  $S = \{x\}$  then we write  $Gru_{\mathcal{O},P}^{\rightarrow}(X, x)$  for short.  $\diamond$

The “W” in the name merely refers to  $\mathcal{O}$ ’s goal: to “win” the game. Gruenhage defined this game in his doctoral dissertation to define a class of spaces generalizing first-countability. [8]

**Definition 3.0.17.** The spaces  $X$  for which  $\mathcal{O} \uparrow Gru_{\mathcal{O},P}^{\rightarrow}(X, x)$  for all  $x \in X$  are called *W-spaces*.  $\diamond$

In fact, using limited information strategies, one may characterize the first-countable spaces using this game.

**Proposition 3.0.18.** *X is first countable if and only if  $\mathcal{O} \uparrow_{pre} Gru_{\mathcal{O},P}^{\rightarrow}(X, x)$  for all  $x \in X$ .*  $\diamond$

*Proof.* The forward implication shows that all first-countable spaces are *W* spaces, and was proven in [8]: if  $\{U_n : n < \omega\}$  is a countable base at  $x$ , let  $\sigma(n) = \bigcap_{m \leq n} U_m$ .  $\sigma$  is easily seen to be a winning predetermined strategy.

If  $X$  is not first countable at some  $x$ , let  $\sigma$  be a predetermined strategy for  $\mathcal{O}$  in  $\text{Gru}_{\mathcal{O},P}^{\rightarrow}(X, x)$ . There exists an open neighborhood  $U$  of  $x$  which does not contain any  $\bigcap_{m \leq n} \sigma(m)$  (otherwise  $\{\bigcap_{m \leq n} \sigma(m) : n < \omega\}$  would be a countable base at  $x$ ). Let  $x_n$  be an element of  $\bigcap_{m \leq n} \sigma(m) \setminus U$  for all  $n < \omega$ . Then  $\langle x_0, x_1, \dots \rangle$  is a winning counter-attack to  $\sigma$  for  $\mathcal{P}$ , so  $\mathcal{O}$  lacks a winning predetermined strategy.  $\square$

At first glance, the difficulty of  $\text{Gru}_{\mathcal{O},P}^{\rightarrow}(X, S)$  could be increased for  $\mathcal{O}$  by only restricting the choices for  $\mathcal{P}$  to be within the most recent open set played by  $\mathcal{O}$ , rather than all the previously played open sets.

**Definition 3.0.19.** Let  $\text{Gru}_{\mathcal{O},P}^{\rightarrow,*}(X, S)$  denote the *hard  $W$ -convergence game* which proceeds as  $\text{Gru}_{\mathcal{O},P}^{\rightarrow}(X, S)$ , except that  $\mathcal{P}$  need only choose  $x_n \in O_n$  rather than  $x_n \in \bigcap_{m \leq n} O_m$  during each round.  $\diamond$

This seemingly more difficult game for  $\mathcal{O}$  is Gruenhage's original formulation. But with perfect information, there is no real difference for  $\mathcal{O}$ .

**Proposition 3.0.20.**  $\mathcal{O} \uparrow_{\text{limit}} \text{Gru}_{\mathcal{O},P}^{\rightarrow}(X, S)$  if and only if  $\mathcal{O} \uparrow_{\text{limit}} \text{Gru}_{\mathcal{O},P}^{\rightarrow,*}(X, S)$ , where  $\uparrow_{\text{limit}}$  is either  $\uparrow$  or  $\uparrow_{\text{pre}}$ .  $\diamond$

*Proof.* The backwards implication is immediate.

For the forward implication, let  $\sigma$  be a winning predetermined (perfect information) strategy, and  $\lambda$  be the 0-Markov fog-of-war  $\mu_0$  (the identity).

We define a new predetermined (perfect information) strategy  $\tau$  by

$$\tau \circ \lambda(\langle x_0, \dots, x_{n-1} \rangle) = \bigcap_{m \leq n} \sigma \circ \lambda(\langle x_0, \dots, x_{m-1} \rangle)$$

so that each move by  $\mathcal{O}$  according to  $\tau \circ \lambda$  is the intersection of  $\mathcal{O}$ 's previous moves. Then any attack against  $\tau \circ \lambda$  is an attack against  $\sigma \circ \lambda$ , and since  $\sigma \circ \lambda$  is a winning strategy, so is  $\tau \circ \lambda$ .  $\square$

Put more simply,  $\tau(n) = \bigcap_{m \leq n} \sigma(m)$  in the predetermined case, and  $\tau(\langle x_0, \dots, x_{n-1} \rangle) = \bigcap_{m \leq n} \sigma(\langle x_0, \dots, x_{m-1} \rangle)$  in the perfect information case. The original proof would have been invalid if  $\lambda$  was required to be, say, the tactical fog-of-war  $\nu_1$ , since the value of  $\mathcal{O}$ 's own round 1 move  $\sigma \circ \nu_1(\langle x_0 \rangle) = \sigma(\langle x_0 \rangle)$  could not be determined from the information she has during round 2:  $\nu_1(\langle x_0, x_1 \rangle) = \langle x_1 \rangle$ .

Due to the equivalency of the “hard” and “normal” variations of the convergence game in the perfect information case, many authors use them interchangeably. However, it is possible to find spaces for which the games are not equivalent when considering  $k + 1$ -tactics and  $k + 1$ -marks, as we will soon see.

In addition to the  $W$ -convergence games, we will also investigate “clustering” analogs to both variations.

**Game 3.0.21.** Let  $Gru_{\mathcal{O},P}^{\rightsquigarrow}(X, S)$  ( $Gru_{\mathcal{O},P}^{\rightsquigarrow,*}(X, S)$ ) be a variation of  $Gru_{\mathcal{O},P}^{\rightarrow}(X, S)$  ( $Gru_{\mathcal{O},P}^{\rightarrow,*}(X, S)$ ) such that  $x_n$  need only cluster at  $S$ , that is, for every open neighborhood  $U$  of  $S$ ,  $x_n \in U$  for infinitely many  $n < \omega$ .  $\diamond$

This variation seems to make  $\mathcal{O}$ 's job easier, but Gruenhage noted that the clustering game is perfect-information equivalent to the convergence game for  $\mathcal{O}$ . This can easily be extended for some limited information cases as well.

**Proposition 3.0.22.**  $\mathcal{O} \uparrow_{\text{limit}} Gru_{\mathcal{O},P}^{\rightarrow}(X, S)$  if and only if  $\mathcal{O} \uparrow_{\text{limit}} Gru_{\mathcal{O},P}^{\rightsquigarrow}(X, S)$  where  $\uparrow_{\text{limit}}$  is any of  $\uparrow_{\text{pre}}$ ,  $\uparrow_{\text{tact}}$ ,  $\uparrow_{\text{mark}}$ , or  $\uparrow_{\text{mark}}$ .  $\diamond$

*Proof.* For the perfect information case we refer to [8].

In the predetermined (resp. tactical) case, suppose that  $\sigma$  is a winning predetermined (resp. tactical) strategy for  $\mathcal{O}$  in  $Gru_{\mathcal{O},P}^{\rightsquigarrow}(X, S)$ . Let  $p$  be a legal attack against  $\sigma$ , and  $q$  be a subsequence of  $p$ . It's easily seen that  $q$  is also a legal attack against  $\sigma$ , so  $q$  clusters at  $S$ . Since every subsequence of  $p$  clusters at  $S$ ,  $p$  converges to  $S$ , and  $\sigma$  is a winning predetermined (resp. tactical) strategy for  $\mathcal{O}$  in  $Gru_{\mathcal{O},P}^{\rightarrow}(X, S)$  as well.

In the final case, note that any Markov strategy  $\sigma'$  for  $\mathcal{O}$  may be strengthened to  $\sigma$  defined by  $\sigma(x, n) = \bigcap_{m \leq n} \sigma'(x, m)$ . So, suppose that  $\sigma$  is a winning Markov strategy for  $\mathcal{O}$  in  $Gru_{\mathcal{O}, P}^{\rightsquigarrow}(X, S)$  such that  $\sigma(x, m) \supseteq \sigma(x, n)$  for all  $m \leq n$ .

Let  $p$  be a legal attack against  $\sigma$ , and  $q$  be a subsequence of  $p$ . For  $m < \omega$ , there exists  $f(m) \geq m$  such that  $q(m) = p(f(m))$ . It follows that  $q(0) = p(f(0)) \in \sigma(\emptyset, 0) \cap \bigcap_{m \leq f(0)} \sigma(\langle p(m) \rangle, m) \subseteq \sigma(\emptyset, 0)$  and

$$\begin{aligned} q(n+1) = p(f(n+1)) &\in \sigma(\emptyset, 0) \cap \bigcap_{m < f(n+1)} \sigma(\langle p(m) \rangle, m+1) \\ &\subseteq \sigma(\emptyset, 0) \cap \bigcap_{m < n+1} \sigma(\langle p(f(m)) \rangle, f(m)+1) \\ &= \sigma(\emptyset, 0) \cap \bigcap_{m < n+1} \sigma(\langle q(m) \rangle, f(m)+1) \\ &\subseteq \sigma(\emptyset, 0) \cap \bigcap_{m < n+1} \sigma(\langle q(m) \rangle, m+1) \end{aligned}$$

so  $q$  is also a legal attack against  $\sigma$ . Since  $\sigma$  is a winning strategy,  $q$  clusters at  $S$ , and since every subsequence of  $p$  clusters at  $S$ ,  $p$  must converge to  $S$ . Thus  $\sigma$  is also a winning Markov strategy for  $\mathcal{O}$  in  $Gru_{\mathcal{O}, P}^{\rightarrow}(X, S)$  as well.  $\square$

Two types of questions emerge from these results.

**Question 3.0.23.** Does  $\mathcal{O} \uparrow_{2\text{-tact}} Gru_{\mathcal{O}, P}^{\rightsquigarrow}(X, S)$  imply  $\mathcal{O} \uparrow_{2\text{-tact}} Gru_{\mathcal{O}, P}^{\rightarrow}(X, S)$ ? What about for  $\uparrow_{2\text{-mark}} ?$   $\diamond$

**Question 3.0.24.** Could  $\mathcal{O} \uparrow_{k+1\text{-tact}} Gru_{\mathcal{O}, P}^{\rightarrow}(X, S)$  actually imply  $\mathcal{O} \uparrow_{\text{tact}} Gru_{\mathcal{O}, P}^{\rightarrow}(X, S)$ ? What about for  $Gru_{\mathcal{O}, P}^{\rightsquigarrow}(X, S)$ ?  $\diamond$

### 3.1 Fort spaces

In his original paper, Gruenhage suggested the one-point-compactification of a discrete space as an example of a  $W$ -space which is not first-countable.

**Definition 3.1.1.** A *Fort space*  $\kappa^* = \kappa \cup \{\infty\}$  is defined for each cardinal  $\kappa$ . Its subspace  $\kappa$  is discrete, and the neighborhoods of  $\infty$  are of the form  $\kappa^* \setminus F$  for each  $F \in [\kappa]^{<\omega}$ .  $\diamond$

**Proposition 3.1.2.**  $\mathcal{O} \uparrow_{tact} Gru_{\mathcal{O},P}^{\rightarrow}(\kappa^*, \infty)$  for all cardinals  $\kappa$   $\diamond$

*Proof.* Let  $\sigma(\emptyset) = \sigma(\langle \infty \rangle) = \kappa^*$  and  $\sigma(\langle \alpha \rangle) = \kappa^* \setminus \{\alpha\}$ . Any legal attack against the tactic  $\sigma$  could not repeat non- $\infty$  points, so it must converge to  $\infty$ .  $\square$

**Corollary 3.1.3.**  $\mathcal{O} \uparrow Gru_{\mathcal{O},P}^{\rightarrow,*}(\kappa^*, \infty)$  for all cardinals  $\kappa$   $\diamond$

*Proof.* Propositions 3.0.20 and 3.1.2.  $\square$

Since it's trivial to show that  $\mathcal{O} \uparrow_{pre} Gru_{\mathcal{O},P}^{\rightarrow}(\kappa^*, \infty)$  if and only if  $\kappa \leq \omega$ , this closes the question on limited information strategies for  $Gru_{\mathcal{O},P}^{\rightarrow}(\kappa^*, \infty)$ . However, limited information analysis of the harder  $Gru_{\mathcal{O},P}^{\rightarrow,*}(\kappa^*, \infty)$  is more interesting.

Peter Nyikos noted Proposition 3.1.2 and the following in [17].

**Theorem 3.1.4.**  $\mathcal{O} \not\uparrow_{mark} Gru_{\mathcal{O},P}^{\rightarrow,*}(\omega_1^*, \infty)$ .  $\diamond$

This actually can be generalized to any  $k$ -Markov strategy with just a little more book-keeping.

**Theorem 3.1.5.**  $\mathcal{O} \not\uparrow_{k-mark} Gru_{\mathcal{O},P}^{\rightarrow,*}(\omega_1^*, \infty)$ .  $\diamond$

*Proof.* Let  $\sigma$  be a  $k$ -mark for  $\mathcal{O}$ . Since the set

$$D_\sigma = \bigcap_{n < \omega, s \in \omega^{\leq k}} \sigma(s, n)$$

is co-countable, we may choose  $\alpha_\sigma \in D_\sigma \cap \omega_1$ . Thus, we may choose  $n_0 < n_1 < \dots < \omega$  such that

$$\langle n_0, \dots, n_{k-1}, \alpha_\sigma, n_k, \dots, n_{2k-1}, \alpha_\sigma, \dots \rangle$$

is a legal counterattack, which fails to converge to  $\infty$  since  $\alpha_\sigma$  is repeated infinitely often.  $\square$

However, while the clustering and convergence variants are equivalent for Markov strategies in the “normal” version of the  $W$  game, they are *not* equivalent in the “hard” version.

**Theorem 3.1.6.**  $\mathcal{O} \uparrow_{\text{mark}} Gru_{\mathcal{O},P}^{\sim,*}(\omega_1^*, \infty)$ .  $\diamond$

*Proof.* For each  $\alpha < \omega_1$  let  $A_\alpha = \langle A_\alpha(0), A_\alpha(1), \dots \rangle$  be a countable sequence of finite sets such that  $A_\alpha(n) \subset A_\alpha(n+1)$  and  $\bigcup_{n < \omega} A_\alpha(n) = \alpha + 1$ .

We define the Markov strategy  $\sigma$  by setting

$$\sigma(\emptyset, 0) = \sigma(\langle \infty \rangle, n) = \omega_1^*$$

and for all  $\alpha < \omega_1$  setting

$$\sigma(\langle \alpha \rangle, n) = \omega_1^* \setminus A_\alpha(n)$$

Note that for any  $\alpha_0 < \dots < \alpha_{k-1}$ , there is some  $n < \omega$  such that  $\{\alpha_0, \dots, \alpha_{k-1}\} \subseteq A_{\alpha_i}(n)$  for all  $i < k$ . Thus for any legal attack  $p$  against  $\sigma$ , the range of  $p$  cannot be finite. Since the range of  $p$  is infinite, every open neighborhood of  $\infty$  contains infinitely many points of  $p$ , so  $p$  clusters at  $\infty$ .  $\square$

However, knowledge of the round number is critical.

**Theorem 3.1.7.**  $\mathcal{O} \nearrow_{k\text{-tact}} Gru_{\mathcal{O},P}^{\sim,*}(\omega_1^*, \infty)$ .  $\diamond$

*Proof.* Let  $\sigma$  be a  $k$ -tactic for  $\mathcal{O}$  in  $Gru_{\mathcal{O},P}^{\sim,*}(\omega_1^*, \infty)$ . By the closing-up lemma, the set

$$C_\sigma = \{\alpha < \omega_1 : s \in \alpha^{\leq k} \Rightarrow \omega_1^* \setminus \sigma(s) \subset \alpha\}$$

is closed and unbounded. Let  $a_\sigma : \omega_1 \rightarrow C_\sigma$  be an order isomorphism.

Choose  $n_0 < \dots < n_{k-1} < \omega$  such that for each  $i < k$ :

$$a_\sigma(n_i) \in \sigma(\langle a_\sigma(n_0), \dots, a_\sigma(n_{i-1}), a_\sigma(\omega + i), \dots, a_\sigma(\omega + k - 1) \rangle)$$

Finally, observe that the legal counterattack

$$\langle a_\sigma(n_0), \dots, a_\sigma(n_{k-1}), a_\sigma(\omega), \dots, a_\sigma(\omega+k-1), a_\sigma(n_0), \dots, a_\sigma(n_{k-1}), a_\sigma(\omega), \dots, a_\sigma(\omega+k-1), \dots \rangle$$

has a range outside the open neighborhood

$$\omega_1^* \setminus \{a_\sigma(n_0), \dots, a_\sigma(n_{k-1}), a_\sigma(\omega), \dots, a_\sigma(\omega+k-1)\}$$

of  $\infty$ . Thus  $\sigma$  is not a winning  $k$ -tactic. □

Once the discrete space is larger than  $\omega_1$ , knowing the round number is not sufficient to construct a limited information strategy, due to a similar argument.

**Theorem 3.1.8.**  $O \not\uparrow_{k\text{-mark}} Gru_{O,P}^{\rightsquigarrow,*}(\omega_2^*, \infty)$ . ◇

*Proof.* Let  $\sigma$  be a  $k$ -mark for  $\mathcal{O}$  in  $Gru_{O,P}^{\rightsquigarrow}(\omega_2^*, \infty)$ . By the closing-up lemma, the set

$$C_\sigma = \{\alpha < \omega_2 : s \in \alpha^{<\omega} \Rightarrow \omega_2^* \setminus \sigma \circ \mu_k(s) \subset \alpha\}$$

is closed and unbounded. (Recall that  $\mu_k$  is the  $k$ -Markov fog-of-war which turns perfect information into the last  $k$  moves and the round number.) Let  $a_\sigma : \omega_2 \rightarrow C_\sigma$  be an order isomorphism.

Choose  $\beta_0 < \dots < \beta_{k-1} < \omega_1$  such that for each  $i < k$ :

$$a_\sigma(\beta_i) \in \bigcap_{n < \omega} \sigma(\langle a_\sigma(\beta_0), \dots, a_\sigma(\beta_{i-1}), a_\sigma(\omega_1 + i), \dots, a_\sigma(\omega_1 + k - 1) \rangle, n)$$

Finally, observe that the legal counterattack

$$\langle a_\sigma(\beta_0), \dots, a_\sigma(\beta_{k-1}), a_\sigma(\omega_1), \dots, a_\sigma(\omega_1+k-1), a_\sigma(\beta_0), \dots, a_\sigma(\beta_{k-1}), a_\sigma(\omega_1), \dots, a_\sigma(\omega_1+k-1), \dots \rangle$$



has a range outside the open neighborhood

$$\omega_2^* \setminus \{a_\sigma(\beta_0), \dots, a_\sigma(\beta_{k-1}), a_\sigma(\omega_1), \dots, a_\sigma(\omega_1 + k - 1)\}$$

of  $\infty$ . Thus  $\sigma$  is not a winning  $k$ -mark. □

### 3.2 Sigma-products

Knowing the status of  $W$ -games in simpler spaces yields insight to larger spaces.

**Proposition 3.2.1.** *Suppose  $S \subseteq Y \subseteq X$ ,  $\uparrow_{\text{limit}}$  is any of  $\uparrow$ ,  $\uparrow_{k\text{-tact}}$ , or  $\uparrow_{k\text{-mark}}$ , and  $G(X, S)$  is any of  $\text{Gru}_{O,P}^\rightarrow(X, S)$ ,  $\text{Gru}_{O,P}^{\rightarrow,*}(X, S)$ ,  $\text{Gru}_{O,P}^{\rightsquigarrow}(X, S)$ , or  $\text{Gru}_{O,P}^{\rightsquigarrow,*}(X, S)$ .*

*Then  $\mathcal{O} \uparrow_{\text{limit}} G(X, S)$  implies  $\mathcal{O} \uparrow_{\text{limit}} G(Y, S)$ .* ◇

*Proof.* Simply intersect the output of the winning strategy in  $G(X, S)$  with  $Y$ . □

A natural superspace of a Fort space is the sigma-product of a discrete cardinal.

**Definition 3.2.2.** Let  $\Sigma_y X^\kappa$  be a *sigma product* of  $X$  with dimension  $\kappa$  for each  $y \in X^\kappa$ , the subset of the usual Tychonoff product space  $X^\kappa$  such that  $x \in \Sigma_y X^\kappa$  if and only if  $\{\alpha < \kappa : x(\alpha) \neq y(\alpha)\}$  is countable.

For homogeneous spaces  $X$  containing 0,  $y$  is usually assumed to be the zero vector  $\vec{0}$  and the sigma product is written  $\Sigma X^\kappa$ . ◇

**Proposition 3.2.3.**  $\kappa^*$  is homeomorphic to the space

$$\{x \in \Sigma 2^\kappa : x(\alpha) = 0 \text{ for all but one } \alpha < \kappa\}$$

◇

*Proof.* Map  $\alpha < \kappa$  to  $x_\alpha$  such that

$$x_\alpha(\beta) = \begin{cases} 0 & \beta \neq \alpha \\ 1 & \beta = \alpha \end{cases}$$

and map  $\infty$  to the zero vector  $\vec{0}$ . □

**Corollary 3.2.4.**  $\mathcal{O} \uparrow_{k-tact}^{Gr_{O,P}^{\vec{\omega},*}(\Sigma\mathbb{R}^{\omega_1}, \vec{0})}$ ,  $\mathcal{O} \uparrow_{k-mark}^{Gr_{O,P}^{\vec{\omega},*}(\Sigma\mathbb{R}^{\omega_1}, \vec{0})}$ , and  $\mathcal{O} \uparrow_{k-mark}^{Gr_{O,P}^{\vec{\omega},*}(\Sigma\mathbb{R}^{\omega_2}, \vec{0})}$ . ◇

While this closes the question on tactics and marks for high dimensional sigma- (and Tychonoff-) products of the real line, there is another type of limited information strategy to investigate.

**Definition 3.2.5.** For a game  $G = \langle M, W \rangle$  and *coding strategy* or *code*  $\sigma : M^2 \rightarrow M$ , the  $\sigma$ -coding fog-of-war  $\gamma_\sigma : M^{<\omega} \rightarrow M^{\leq 2}$  is the function defined such that

$$\gamma_\sigma(\emptyset) = \emptyset$$

and

$$\gamma_\sigma(s \frown \langle x \rangle) = \langle \sigma \circ \gamma_\sigma(s), x \rangle$$

For a coding strategy  $\sigma$ , its corresponding strategy is  $\sigma \circ \gamma_\sigma$ . For a game  $G$ , if  $\sigma \circ \gamma_\sigma$  is a winning strategy for  $\mathcal{A}$ , then  $\sigma$  is a winning coding strategy and we write  $\mathcal{A} \uparrow_{code} G$ . ◇

Intuitively, a  $\sigma$ -coding fog-of-war converts perfect information of the game into the last moves of both the player and her opponent, so a player has a winning coding strategy when she only needs to know the move of her opponent and her own last move. The term “coding” comes from the fact that a player may encode information about the history of the game into her own moves, and use this encoded information in later rounds.

As an example, the existence of a winning coding strategy is necessary for the second player to force a win the Banach-Mazur game.

**Theorem 3.2.6.**  $\mathcal{N} \uparrow BM_{E,N}(X)$  if and only if  $\mathcal{N} \uparrow_{code} BM_{E,N}(X)$  [4] [7]. ◇

We are interested in whether the same holds for  $W$  games.

The hard and normal versions of the  $W$  games are all equivalent with regards to coding strategies since  $\mathcal{O}$  may always ensure her new move is a subset of her previous move. For Fort spaces, the question is immediately closed.

**Proposition 3.2.7.**  $\mathcal{O} \uparrow_{code} Gru_{\vec{O},P}^{\rightarrow}(\kappa^*, \infty)$ .  $\diamond$

*Proof.* Let  $\sigma(\emptyset) = \kappa^*$ ,  $\sigma(\langle U, \alpha \rangle) = U \setminus \{\alpha\}$  for  $\alpha < \kappa$ , and  $\sigma(\langle U, \infty \rangle) = U$ .  $\mathcal{P}$  cannot legally repeat non- $\infty$  points of the set, so her points converge to  $\infty$ .  $\square$

This trick does not simply extend to the  $\Sigma \mathbb{R}^\kappa$  case, however. An open set may only restrict finitely many coordinates of the product, and a point in  $\Sigma \mathbb{R}^\kappa$  may have countably infinite non-zero coordinates. Thus, information about the previous non-zero coordinates cannot be directly encoded into the open set.

Circumventing this takes a bit of extra machinery. We proceed by defining a simpler infinite game for each cardinal  $\kappa$ .

**Game 3.2.8.** Let  $PtFin_{F,C}(\kappa)$  denote the *point-finite game* with players  $\mathcal{F}$ ,  $\mathcal{C}$  for each cardinal  $\kappa$ .

In round  $n$ ,  $\mathcal{F}$  chooses  $F_n \in [\kappa]^{<\omega}$ , followed by  $\mathcal{C}$  choosing  $C_n \in [\kappa \setminus \bigcup_{m \leq n} F_m]^{<\omega}$ .

$\mathcal{F}$  wins the game if the collection  $\{C_n : n < \omega\}$  is a point-finite cover of its union  $\bigcup_{n < \omega} C_n$ , that is, each point in  $\bigcup_{n < \omega} C_n$  is in  $C_n$  only for finitely many  $n < \omega$ .  $\diamond$

This game has a strong resemblance to a game defined by Scheepers in [20] in relationship to the Banach-Mazur game and studied specifically with finite and countable sets in [21]. Scheeper's game and the results pertaining to it aren't of use here; however, they will be referenced in a later chapter in studying a different topological game.

This game of finite and countable sets is directly applicable to the  $W$  games played upon the sigma-product of real lines.

**Lemma 3.2.9.**  $\mathcal{F} \uparrow_{code} PtFin_{F,C}(\kappa)$  implies  $\mathcal{O} \uparrow_{code} Gru_{\vec{O},P}^{\rightarrow}(\Sigma \mathbb{R}^\kappa, \vec{0})$ .  $\diamond$

*Proof.* Let  $\sigma$  be a winning coding strategy for  $\mathcal{F}$  in  $PtFin_{F,C}(\kappa)$  such that  $\sigma(\emptyset) \supset \emptyset$  and  $\sigma(F, C) \supset F$ .

For  $F \in [\kappa]^{<\omega}$  and  $\epsilon > 0$  let  $U(F, \epsilon)$  be the basic open set in  $\mathbb{R}^\kappa$  such that each projection is of the form

$$\pi_\alpha(U(F, \epsilon)) = \begin{cases} (-\epsilon, \epsilon) & \alpha \in F \\ \mathbb{R} & \alpha \notin F \end{cases}$$

Note that  $F \supset \emptyset$  and  $\epsilon$  are uniquely identifiable given  $U(F, \epsilon) \cap \Sigma\mathbb{R}^\kappa$ .

For each point  $x \in \Sigma\mathbb{R}^\kappa$  and  $\epsilon > 0$ , let  $C_\epsilon(x) \in [\kappa]^{\leq \omega}$  such that  $\alpha \in C_\epsilon(x)$  if and only if  $|x(\alpha)| \geq \epsilon$ .

We define the coding strategy  $\tau$  for  $\mathcal{O}$  in  $Gru_{\vec{O}, P}(\Sigma\mathbb{R}^\kappa, \vec{0})$  as follows:

$$\tau(\emptyset) = U(\sigma(\emptyset), 1) \cap \Sigma\mathbb{R}^\kappa$$

$$\tau(\langle U(F, \epsilon) \cap \Sigma\mathbb{R}^\kappa, x \rangle) = U\left(\sigma(\langle F, C_\epsilon(x) \rangle), \frac{\epsilon}{2}\right) \cap \Sigma\mathbb{R}^\kappa$$

Let  $\langle a_0, a_1, a_2, \dots \rangle$  be a legal attack by  $\mathcal{P}$  against  $\tau$ . It then follows that

$$b = \langle C_1(a_0), C_{1/2}(a_1), C_{1/4}(a_2), \dots \rangle$$

is a legal attack by  $\mathcal{C}$  against  $\sigma$ . Since  $\sigma$  is a winning strategy, each ordinal in  $\bigcup_{n < \omega} C_{2^{-n}}(a_n)$  is in  $C_{2^{-n}}(a_n)$  only for finitely many  $n < \omega$ . Thus for every coordinate  $\alpha < \kappa$  it follows that there exists some  $n_\alpha < \omega$  such that  $a_n(\alpha) \leq 2^{-n}$  for  $n \geq n_\alpha$ . We conclude  $a_n \rightarrow \vec{0}$ , showing that  $\tau$  is a winning strategy.  $\square$

This lemma simplifies our notation in proving the main result. Intuitively, we aim to show that when  $\kappa$  has cofinality  $\omega$ ,  $\mathcal{F}$  can split up the game among  $\omega$ -many smaller cardinals converging to  $\kappa$ , and when  $\kappa$  has a larger cofinality,  $\mathcal{F}$  may exploit the fact that  $\mathcal{C}$  may only play within some ordinal smaller than  $\kappa$ .

**Theorem 3.2.10.**  $\mathcal{F} \uparrow_{code} PtFin_{F,C}(\kappa)$  for all cardinals  $\kappa$ .  $\diamond$

*Proof.* For each cardinal  $\kappa$  and  $\lambda < \kappa$ , assume  $\sigma_\lambda$  is a winning strategy for  $\mathcal{F}$  in  $PtFin_{F,C}(\lambda)$  such that  $\sigma_\lambda(\emptyset) \supset \emptyset$  and  $\sigma_\lambda(\langle F, C \rangle) \supset F$ .

In the case that  $\text{cf}(\kappa) = \omega$ , let  $\langle \kappa_0, \kappa_1, \dots \rangle$  be an increasing sequence of cardinals limiting to  $\kappa$ . Then we define the coding strategy  $\sigma$  for  $\mathcal{F}$  as follows:

$$\sigma(\emptyset) = \sigma_{\kappa_0}(\emptyset)$$

$$\sigma(\langle F, C \rangle) = \bigcup_{n \leq |F|} \sigma_{\kappa_n}(\langle F \cap \kappa_n, C \cap \kappa_n \rangle)$$

Then for each legal attack  $a = \langle a(0), a(1), \dots \rangle$  by  $\mathcal{C}$  against  $\sigma$  and each  $n < \omega$ , the sequence  $b_n = \langle a(n) \cap \kappa_n, a(n+1) \cap \kappa_n, \dots \rangle$  is a legal attack by  $\mathcal{C}$  against the winning coding strategy  $\sigma_{\kappa_n}$ . It follows then that  $\{a(i+n) \cap \kappa_n : i < \omega\}$  is a point-finite cover of  $\bigcup_{i < \omega} a(i+n) \cap \kappa_n$ . We conclude that  $\{a(i) : i < \omega\}$  is a point-finite cover of  $\bigcup_{i < \omega} a(i)$  and  $\sigma$  is a winning strategy.

It remains to consider the case where  $\text{cf}(\kappa) > \omega$ . Note that now, for each  $C \in [\kappa]^{\leq \omega}$ ,  $C$  is bounded above in  $\kappa$ . So we define the coding strategy  $\sigma$  for  $\mathcal{F}$  as follows:

$$\sigma(\emptyset) = \emptyset$$

$$\sigma(\langle F, C \rangle) = \{\sup(C)\} \cup \bigcup_{\alpha \in F} \sigma_{\alpha+1}(\langle F \cap (\alpha+1), C \cap (\alpha+1) \rangle)$$

Then for each legal attack  $a = \langle a(0), a(1), \dots \rangle$  by  $\mathcal{C}$  against  $\sigma$  and each  $n < \omega$ , the sequence  $b_n = \langle a(n) \cap (\sup(a(n)) + 1), a(n+1) \cap (\sup(a(n)) + 1), \dots \rangle$  is a legal attack by  $\mathcal{C}$  against the winning coding strategy  $\sigma_{\sup(a(n))+1}$ . It follows then that  $\{a(i+n) \cap (\sup(a(n)) + 1) : i < \omega\}$  is a point-finite cover of  $\bigcup_{i < \omega} a(i+n) \cap (\sup(a(n)) + 1)$ . We conclude that  $\{a(i) : i < \omega\}$  is a point-finite cover of  $\bigcup_{i < \omega} a(i)$  and  $\sigma$  is a winning strategy.  $\square$

**Corollary 3.2.11.**  $\mathcal{O} \uparrow_{code} Gru_{\vec{O},P}(\Sigma \mathbb{R}^\kappa, \vec{0})$  for all cardinals  $\kappa$ .  $\diamond$

This leaves open the question analogous to Theorem 3.2.6.

**Question 3.2.12.** Does  $\mathcal{O} \uparrow Gru_{\mathcal{O},P}^{\rightarrow}(X,x)$  imply  $\mathcal{O} \underset{\text{code}}{\uparrow} Gru_{\mathcal{O},P}^{\rightarrow}(X,x)$ ? ◇

## Chapter 4

### Bell's Game

A very recent development related to Gruenhage's convergence and clustering games comes from Jocelyn Bell, used to study the uniform box product of uniform spaces.

#### 4.1 Uniform Spaces

**Definition 4.1.1.** A *uniformity* on a set  $X$  is a filter  $\mathbb{D}$  of subsets of  $X^2$ , known as *entourages*, such that  $\bigcap \mathbb{D} = \Delta = \{\langle x, x \rangle : x \in X\}$  and, for each entourage  $D \in \mathbb{D}$ :

- There exists  $\frac{1}{2}D \in \mathbb{D}$  such that

$$\frac{1}{2}D \circ \frac{1}{2}D = \left\{ \langle x, z \rangle : \exists y \in X \left( \langle x, y \rangle, \langle y, z \rangle \in \frac{1}{2}D \right) \right\} \subseteq D$$

- $D^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in D\} \in \mathbb{D}$

◇

A set  $X$  with a uniformity is called a *uniform space*. As  $\mathbb{D}$  is a filter, we also have that  $D \cap E \in \mathbb{D}$  for all  $E \in \mathbb{D}$ , and  $F \in \mathbb{D}$  for all  $F \supseteq D$ . Note that if  $\mathbb{E}$  is a filter base satisfying the conditions for a uniformity, then we say  $\mathbb{E}$  is a *uniformity base* which may be extended to a uniformity by closing it under the superset operation.

A uniformity is a generalization of a metric.

**Definition 4.1.2.** For an entourage  $D \in \mathbb{D}$  and a point  $x \in X$ , the *D-ball around x* is the set  $D[x] = \{y : \langle x, y \rangle \in D\}$ .

◇

**Definition 4.1.3.** If  $d$  is a metric for the space  $X$ , then the *metric uniformity* for  $X$  is generated by the uniformity base  $\{D_\epsilon : \epsilon > 0\}$  where  $D_\epsilon = \{\langle x, y \rangle : d(x, y) < \epsilon\}$ .

◇

Like metrics, uniformities induce natural topological structures.

**Definition 4.1.4.** The *uniform topology* for a space  $X$  with uniformity  $\mathbb{D}$  is given by letting  $U$  be open if for each  $x \in U$ , there exists  $D \in \mathbb{D}$  such that  $D[x] \subseteq U$ .  $\diamond$

**Theorem 4.1.5.** *The uniform topology for a space  $X$  with uniformity  $\mathbb{D}$  is the coarsest topology such that for each  $x \in X$  and  $D \in \mathbb{D}$ ,  $D[x]$  is a neighborhood (not necessarily open) of  $x$ .*  $\diamond$

*Proof.* Let  $\mathcal{T}$  be a topology such that  $D[x]$  is a neighborhood of  $x$  for each  $x \in X$ , and let  $U$  be open in the uniform topology. For each  $x \in U$ , there exists  $D_x \in \mathbb{D}$  such that  $D_x[x] \subseteq U$ , and since  $D_x[x]$  is a neighborhood of  $x$ , there is  $U_x \in \mathcal{T}$  such that  $x \in U_x \subseteq D_x[x]$ . Since  $U = \bigcup_{x \in U} U_x \in \mathcal{T}$ , the result follows as  $\mathcal{T}$  contains the uniform topology.  $\square$

The uniform topology for a metric uniformity is simply the usual metric topology, and  $D_\epsilon[x]$  is the usual metric  $\epsilon$ -ball around  $x$ .

We require a few known results on uniform spaces. See e.g. [5] for proofs.

**Theorem 4.1.6.** *Every uniform topology is  $T_{3\frac{1}{2}}$ .*  $\diamond$

**Definition 4.1.7.** A topology on  $X$  is *uniformizable* if there exists a uniformity whose uniform topology equals the topology on  $X$ .  $\diamond$

**Theorem 4.1.8.** *Every  $T_{3\frac{1}{2}}$  topology is uniformizable.*  $\diamond$

**Definition 4.1.9.** The *universal uniformity* for a uniformizable topology is the uniformity finer than all uniformities which induce the given topology.  $\diamond$

**Theorem 4.1.10.** *Every uniformizable topology is induced by its universal uniformity.*  $\diamond$

**Definition 4.1.11.** For a uniformizable space  $X$ , a *symmetric open entourage*  $D$  is an entourage of the universal uniformity such that

- $D$  is open in the product topology on  $X^2$



- $D$  is symmetric; that is,  $D = D^{-1}$ .

◇

**Proposition 4.1.12.** *If  $X$  is a uniformizable space, then for all  $x \in X$  and symmetric open entourages  $D$ :*

$$x \in \frac{1}{2}D[y] \text{ and } y \in \frac{1}{2}D[z] \Rightarrow x \in D[z]$$

and

$$\frac{1}{2}D[x] \subseteq \overline{\frac{1}{2}D[x]} \subseteq D[x]$$

◇

**Theorem 4.1.13.** *For every entourage  $D$  in the universal uniformity on  $X$ , there exists a symmetric open entourage  $U \subseteq D$ .*

◇

If  $D$  is a symmetric open entourage, then we assume  $\frac{1}{2}D$  is also a symmetric open entourage. Note that if  $D$  is an open symmetric entourage, then  $D[x]$  is an open neighborhood of  $x$ .

## 4.2 Games on uniformizable spaces

**Game 4.2.1.** Let  $Bell_{D,P}^{\text{uni}}(X, \mathbb{D})$  denote the *Bell uniform space game* with players  $\mathcal{D}$ ,  $\mathcal{P}$  which proceeds as follows for a uniform space  $X$  with uniformity  $\mathbb{D}$ . In round 0,  $\mathcal{D}$  chooses an entourage  $D_0$ , followed by  $\mathcal{P}$  choosing a point  $p_0 \in X$ . In round  $n + 1$ ,  $\mathcal{D}$  chooses an entourage  $D_{n+1}$ , followed by  $\mathcal{P}$  choosing a point  $p_{n+1} \in D_n[p_n]$ .

$\mathcal{D}$  wins in the case that either  $\langle p_0, p_1, \dots \rangle$  converges with respect to the uniformity  $\mathbb{D}$ , or  $\bigcap_{n < \omega} D_n[p_n] = \emptyset$ .<sup>1</sup>  $\mathcal{P}$  wins otherwise.

◇

This original formulation was called the “proximal game” in [2]. The game was used to prove that the  $\sigma$ -product of spaces for which  $\mathcal{D}$  has a winning strategy is collectionwise normal, as well as to show the collectionwise normality of certain uniform box products.

---

<sup>1</sup>A sequence  $p$  converges to  $x$  with respect to a uniformity  $\mathbb{D}$  if for every entourage  $D \in \mathbb{D}$ , there exists  $N < \omega$  such that  $p(n) \in D[x]$  for  $n \geq N$ .

**Definition 4.2.2.** A uniformizable space  $X$  is *proximal* if it has a compatible uniformity  $\mathbb{D}$  such that  $\mathcal{D} \uparrow Bell_{D,P}^{\text{uni}}(X, \mathbb{D})$ .  $\diamond$

**Proposition 4.2.3.** *Every metric space is proximal.*  $\diamond$

Considering the universal uniformity, we may consider purely topological variations of this game.

**Game 4.2.4.** Let  $Bell_{D,P}^{\rightarrow,*}(X)$  denote the *hard Bell convergence game* with players  $\mathcal{D}$ ,  $\mathcal{P}$  which proceeds as follows for a uniformizable space  $X$ . In round 0,  $\mathcal{D}$  chooses a symmetric open entourage  $D_0$ , followed by  $\mathcal{P}$  choosing a point  $p_0 \in X$ . In round  $n + 1$ ,  $\mathcal{D}$  chooses a symmetric open entourage  $D_{n+1}$ , followed by  $\mathcal{P}$  choosing a point  $p_{n+1} \in D_n[p_n]$ .

$\mathcal{D}$  wins in the case that either  $\langle p_0, p_1, \dots \rangle$  converges in  $X$ , or  $\bigcap_{n < \omega} D_n[p_n] = \emptyset$ .  $\mathcal{P}$  wins otherwise.  $\diamond$

**Game 4.2.5.** Let  $Bell_{D,P}^{\rightarrow}(X)$  denote the *Bell convergence game* with players  $\mathcal{D}$ ,  $\mathcal{P}$  which proceeds analogously to  $Bell_{D,P}^{\rightarrow,*}(X)$ , except for the following. Let  $E_n = \bigcap_{m \leq n} D_m$ , where  $D_n$  is the entourage played by  $\mathcal{D}$  in round  $n$ . Then  $\mathcal{P}$  must ensure that  $p_{n+1} \in E_n[p_n]$ , and  $\mathcal{D}$  wins when either  $\langle p_0, p_1, \dots \rangle$  converges in  $X$  or  $\bigcap_{n < \omega} E_n[p_n] = \emptyset$ .  $\diamond$

**Theorem 4.2.6.**  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow,*}(X)$  if and only if  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$  if and only if  $X$  is proximal.  $\diamond$

*Proof.* If  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow,*}(X)$ , then we immediately see that  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$ . If  $\sigma$  is a winning strategy for  $\mathcal{D}$  in  $Bell_{D,P}^{\rightarrow}(X)$ , then  $\tau$  defined by  $\tau(s) = \bigcap_{t \leq s} \sigma(t)$  is easily seen to be a winning strategy for  $\mathcal{D}$  in  $Bell_{D,P}^{\rightarrow,*}(X)$ .

If  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow,*}(X)$ , then  $\mathcal{D} \uparrow Bell_{D,P}^{\text{uni}}(X, \mathbb{D})$  where  $\mathbb{D}$  is the universal uniformity, showing  $X$  is proximal. Finally, if  $X$  is proximal, then there exists a winning strategy  $\sigma$  for  $Bell_{D,P}^{\text{uni}}(X, \mathbb{D})$  where  $\mathbb{D}$  is a uniformity inducing the topology on  $X$ . Let  $\tau$  be defined by  $\tau(s) = O(\sigma(s))$  where  $O(D)$  is an open symmetric entourage which is a subset of  $D$ ;  $\tau$  is easily seen to be a winning strategy for  $\mathcal{D}$  in  $Bell_{D,P}^{\rightarrow,*}(X)$ .  $\square$

The hard and normal variants  $Bell_{D,P}^{\vec{\star}}(X)$ ,  $Bell_{D,P}^{\vec{\star}}(X)$  mirror the hard and normal variants of Gruenhage's convergence game. In fact, Bell showed a strong connection in [2].

**Theorem 4.2.7.** *If  $X$  is proximal, then  $X$  is a  $W$ -space.*  $\diamond$

We similarly may consider clustering versions of the Bell games. (TODO: will I actually use this game?)

**Definition 4.2.8.** Let  $Bell_{D,P}^{\vec{\star}}(X)$  ( $Bell_{D,P}^{\vec{\star}}(X)$ ) denote the *Bell clustering game* (hard Bell clustering game) which proceeds analogously to  $Bell_{D,P}^{\vec{\star}}(X)$  ( $Bell_{D,P}^{\vec{\star}}(X)$ ), except that  $\mathcal{D}$  need only ensure that  $\langle p_0, p_1, \dots \rangle$  clusters in  $X$  in order to win.  $\diamond$

**Definition 4.2.9.** A uniformizable space  $X$  is *almost proximal* if  $\mathcal{D} \uparrow Bell_{D,P}^{\vec{\star}}(X)$ .  $\diamond$

Or we eliminate the secondary winning condition to create a stronger variant of the game.

**Definition 4.2.10.** Let  $Bell_{D,P}^{\vec{\star}}(X)$  denote the *absolute Bell convergence game* which proceeds analogously to  $Bell_{D,P}^{\vec{\star}}(X)$ , except that  $\mathcal{D}$  must always ensure that  $\langle p_0, p_1, \dots \rangle$  converges in  $X$  in order to win.  $\diamond$

**Definition 4.2.11.** A uniformizable space  $X$  is *absolutely proximal* if  $\mathcal{D} \uparrow Bell_{D,P}^{\vec{\star}}(X)$ .  $\diamond$

Proximal spaces have strong preservation properties, demonstrated in [2].

**Theorem 4.2.12.** *Every closed subspace of a proximal space is proximal.*  $\diamond$

**Theorem 4.2.13.** *Every  $\Sigma$ -product of proximal spaces is proximal.*  $\diamond$

Gruenhage and the author showed in [3] that Bell's game yields an internal characterization of Corson compact spaces, answering a question of Nyikos in [18].

### 4.3 Characterizing Corson compactness

**Definition 4.3.1.** A compact space is *Corson compact* if it is homeomorphic to a compact subset of a  $\Sigma$ -product of real lines.  $\diamond$

Nyikos observed the following in [18].

**Proposition 4.3.2.** *Corson compact spaces are proximal.*  $\diamond$

*Proof.* The real line is proximal, and closed subsets of  $\Sigma$ -products of proximal spaces are proximal.  $\square$

Following this observation, Nyikos asked if Bell's game actually provided an internal characterization of Corson compactness. The author, working with Gary Gruenhage, answered this question in the affirmative in [3].

To see this, we require a result of Gruenhage which gives another game characterization of Corson compactness, found in [9].

**Theorem 4.3.3.** *A compact space is Corson compact if and only if  $\mathcal{O} \uparrow Gru_{\mathcal{O},P}^{\rightarrow}(X^2, \Delta)$ .*  $\diamond$

Thus our desired result will follow if we may show that a winning strategy in  $Bell_{D,P}^{\rightarrow}(X)$  may be used to construct a winning strategy in  $Gru_{\mathcal{O},P}^{\rightarrow}(X^2, \Delta)$ . However, due to the secondary winning condition for  $\mathcal{D}$  in  $Bell_{D,P}^{\rightarrow}(X)$  ( $\bigcap_{n < \omega} E_n[p_n] = \emptyset$ ), it will be more convenient if we may use a winning strategy for  $\mathcal{D}$  in  $Bell_{D,P}^{\rightarrow}(X)$  instead.

**Definition 4.3.4.** A uniformizable space  $X$  is *uniformly locally compact* if there exists an open symmetric entourage  $D$  such that  $\overline{D[x]}$  is compact for all  $x$ .  $\diamond$

Of course, any compact uniformizable space is uniformly locally compact. However, note that a space may be locally compact without being uniformly locally compact.

**Theorem 4.3.5.** *A uniformizable space is uniformly locally compact if and only if it is locally compact and paracompact. (See e.g. [13].)*  $\diamond$

As an example,  $\omega_1$  with the linear order topology is locally compact, but not paracompact.

**Theorem 4.3.6.** *If  $X$  is a uniformly locally compact space, then  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$  if and only if  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$ .*  $\diamond$

*Proof.* Let  $L$  be an open symmetric entourage such that  $\overline{L[x]}$  is compact for all  $x$ . Let  $\sigma$  be a strategy witnessing  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$ . Without loss of generality, we may assume such that  $\sigma(t) \subseteq L$  (so  $\overline{\sigma(t)[x]} \subseteq \overline{L[x]}$  is compact), and that  $t \supseteq s$  implies  $\sigma(t) \subseteq \frac{1}{4}\sigma(s)$ .

Let  $\tau(t) = \frac{1}{2}\sigma(t)$ . If  $p$  attacks  $\tau$  in  $Bell_{D,P}^{\rightarrow}(X)$ , then

$$p(n+1) \in \tau(p \upharpoonright n)[p(n)] = \frac{1}{2}\sigma(p \upharpoonright n)[p(n)]$$

and for

$$x \in \overline{\sigma(p \upharpoonright (n+1))[p(n+1)]} \subseteq \overline{\frac{1}{4}\sigma(p \upharpoonright n)[p(n+1)]} \subseteq \frac{1}{2}\sigma(p \upharpoonright n)[p(n+1)]$$

we can conclude  $x \in \sigma(p \upharpoonright n)[p(n)]$ . Thus

$$\sigma(p \upharpoonright (n+1))[p(n+1)] \subseteq \overline{\sigma(p \upharpoonright (n+1))[p(n+1)]} \subseteq \sigma(p \upharpoonright n)[p(n)]$$

Finally, note that since  $\tau$  yields subsets of  $\sigma$ ,  $p$  also attacks the winning strategy  $\sigma$  in  $Bell_{D,P}^{\rightarrow}(X)$ , but since the intersection of a descending chain of nonempty compact sets is nonempty, we have

$$\bigcap_{n < \omega} \sigma(p \upharpoonright n)[p(n)] = \bigcap_{n < \omega} \overline{\sigma(p \upharpoonright n)[p(n)]} \neq \emptyset$$

We conclude that  $p$  converges.  $\square$

**Theorem 4.3.7.** *If  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$ , then  $\mathcal{O} \uparrow Gru_{O,P}^{\rightarrow}(X, H)$  for all compact  $H$  in  $X$ .*  $\diamond$

*Proof.* Let  $\sigma$  be a winning strategy for  $\mathcal{D}$  in  $Bell_{D,P}^{\rightarrow}(X)$  game such that  $p \supsetneq q$  implies  $\sigma(p) \subseteq \frac{1}{4}\sigma(q)$ . For any sequence  $t$ , let  $o_t = \{\langle n, t(2n+1) \rangle : 2n+1 \in \text{dom}(t)\}$  be the subsequence of  $t$  consisting of its odd-indexed terms. We proceed by constructing a winning strategy for  $\mathcal{O}$  in  $Gru_{\mathcal{O},P}^{\rightsquigarrow}(X, H)$ . Since  $\mathcal{O} \uparrow Gru_{\mathcal{O},P}^{\rightsquigarrow}(X, H)$  if and only if  $\mathcal{O} \uparrow Gru_{\mathcal{O},P}^{\rightarrow}(X, H)$ , the result will follow.

We begin by defining a tree  $T(\emptyset)$ , during which we will define a number  $m_\emptyset < \omega$  and points  $h_{\emptyset,i}, h_{\emptyset,i,j}$  for  $i, j < m_\emptyset$  which yield an open set

$$\bigcup_{i,j < m_\emptyset} \frac{1}{4}\sigma(\langle h_{\emptyset,i} \rangle)[h_{\emptyset,i,j}] = \bigcup_{\emptyset \smallfrown \langle i, h_{\emptyset,i,j} \rangle \in \max(T(\emptyset))} \frac{1}{4}\sigma(o_\emptyset \smallfrown \langle h_{\emptyset,i} \rangle)[h_{\emptyset,i,j}]$$

containing  $H$ .  $\mathcal{O}$  will use this as the initial move in her winning strategy for  $Gru_{\mathcal{O},P}^{\rightsquigarrow}(X, H)$ .

- Choose  $m_\emptyset < \omega$ ,  $h_{\emptyset,i} \in H$  for  $i < m_\emptyset$ , and  $h_{\emptyset,i,j} \in H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$  for  $i, j < m_\emptyset$  such that

$$\left\{ \frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}] : i < m_\emptyset \right\}$$

is a cover for  $H$  and such that for each  $i < m_\emptyset$

$$\left\{ \frac{1}{4}\sigma(\langle h_{\emptyset,i} \rangle)[h_{\emptyset,i,j}] : j < m_\emptyset \right\}$$

is a cover for  $H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$ .

- Let  $\langle i, h_{\emptyset,i}, j \rangle$  and its initial segments be in  $T(\emptyset)$  for  $i, j < m_\emptyset$ .

It follows that

$$\left\{ \frac{1}{4}\sigma(o_\emptyset \smallfrown \langle h_{\emptyset,i} \rangle)[h_{\emptyset,i,j}] : \emptyset \smallfrown \langle i, h_{\emptyset,i}, j \rangle \in \max(T(a)) \right\}$$

covers  $H$ .

Now suppose that  $a$  is a partial attack by  $\mathcal{P}$  in  $Gru_{\mathcal{O},P}^{\rightsquigarrow}(X, H)$  for which we have defined a tree  $T(a)$ . We will define  $T(a \smallfrown \langle x \rangle) \supseteq T(a)$  for each  $x \in X$ , and whenever

$x \in \frac{1}{4}\sigma(o_s \frown \langle h_{s,i} \rangle)[h_{s,i,j}]$  for some  $s \frown \langle i, h_{s,i}, j \rangle \in \max(T(a))$ , we will set  $t = s \frown \langle i, h_{s,i}, j, x \rangle$  and define a number  $m_t < \omega$  and points  $h_{t,k}, h_{t,k,l}$  for  $k, l < m_t$  to yield an open set

$$\bigcup_{t \frown \langle k, h_{t,k}, l \rangle \in \max(T(a \frown \langle x \rangle))} \frac{1}{4}\sigma(o_t \frown \langle h_{t,k} \rangle)[h_{t,k,l}]$$

which will be the next open neighborhood of  $H$  in  $\mathcal{O}$ 's winning strategy for  $\text{Gru}_{O,P}^{\rightsquigarrow}(X, H)$ .

- We will extend the nodes  $s \frown \langle i, h_{s,i}, j \rangle \in \max(T(a))$  such that  $x \in \frac{1}{4}\sigma(o_s \frown \langle h_{s,i} \rangle)[h_{s,i,j}]$ . For each such  $s \frown \langle i, h_{s,i}, j \rangle$ , set  $t = s \frown \langle i, h_{s,i}, j, x \rangle$ .
- Note that whenever  $o_s \frown \langle h_{s,i} \rangle$  is a legal partial attack against  $\sigma$ , then

$$x \in \frac{1}{4}\sigma(o_s \frown \langle h_{s,i} \rangle)[h_{s,i,j}] \subseteq \frac{1}{4}\sigma(o_s)[h_{s,i,j}]$$

and

$$h_{s,i,j} \in \overline{\frac{1}{4}\sigma(o_s)[h_{s,i}]} \subseteq \overline{\frac{1}{2}\sigma(o_s)[h_{s,i}]}$$

implies

$$x \in \sigma(o_s)[h_{s,i}]$$

and thus  $o_s \frown \langle h_{s,i}, x \rangle = o_t$  is also a legal partial attack against  $\sigma$ .

- Choose  $m_t < \omega$ ,  $h_{t,k} \in H \cap \overline{\frac{1}{4}\sigma(o_s \frown \langle h_{s,i} \rangle)[h_{s,i,j}]}$  for  $k < m_t$ , and  $h_{t,k,l} \in H \cap \overline{\frac{1}{4}\sigma(o_t)[h_{t,k}]}$  for  $k, l < m_t$  such that

$$\left\{ \frac{1}{4}\sigma(o_t)[h_{t,k}] : k < m_t \right\}$$

is a cover for  $H \cap \overline{\frac{1}{4}\sigma(o_s \frown \langle h_{s,i} \rangle)[h_{s,i,j}]}$  and such that for each  $k < m_t$

$$\left\{ \frac{1}{4}\sigma(o_t \frown \langle h_{t,k} \rangle)[h_{t,k,l}] : l < m_t \right\}$$

is a cover for  $H \cap \overline{\frac{1}{4}\sigma(o_t)[h_{t,k}]}$ .

- Note that whenever  $o_t$  is a legal partial attack against  $\sigma$ , then

$$h_{t,k} \in \overline{\frac{1}{4}\sigma(o_s \frown \langle h_{s,i} \rangle)[h_{s,i,j}]} \subseteq \frac{1}{2}\sigma(o_s \frown \langle h_{s,i} \rangle)[h_{s,i,j}]$$

and

$$x \in \frac{1}{4}\sigma(o_s \frown \langle h_{s,i} \rangle)[h_{s,i,j}]$$

implies

$$h_{t,k} \in \sigma(o_s \frown \langle h_{s,i} \rangle)[x]$$

and thus  $o_t \frown \langle h_{t,k} \rangle$  is a legal partial attack against  $\sigma$ .

- Include all initial segments of  $t \frown \langle k, h_{t,k}, l \rangle$  in  $T(a \frown \langle x \rangle)$  for  $k, l < m_t$ .

This completes the construction of  $T(a \frown \langle x \rangle)$ . Note that since

$$\left\{ \frac{1}{4}\sigma(o_s \frown \langle h_{s,i} \rangle)[h_{s,i,j}] : s \frown \langle i, h_{s,i}, j \rangle \in \max(T(a)) \right\}$$

covers  $H$ , then since

$$\left\{ \frac{1}{4}\sigma(o_t \frown \langle h_{t,k} \rangle)[h_{t,k,l}] : s \frown \langle i, h_{s,i}, j, x, k, h_{t,k}, l \rangle \in \max(T(a \frown \langle x \rangle)) \setminus \max(T(a)) \right\}$$

covers  $H \cap \frac{1}{4}\sigma(o_s \frown \langle h_{s,i} \rangle)[h_{s,i,j}]$ , we have that

$$\left\{ \frac{1}{4}\sigma(o_t \frown \langle h_{t,k} \rangle)[h_{t,k,l}] : t \frown \langle k, h_{t,k}, l \rangle \in \max(T(a \frown \langle x \rangle)) \right\}$$

covers  $H$ .

We define a strategy  $\tau$  for  $\mathcal{O}$  in  $\text{Gru}_{\mathcal{O},P}^{\sim}(X, H)$  such that:

$$\tau(a) = \bigcup_{s \frown \langle i, h_{s,i}, j \rangle \in \max(T(a))} \frac{1}{4}\sigma(o_s \frown \langle h_{s,i} \rangle)[h_{s,i,j}]$$



If  $p$  is a legal attack by  $\mathcal{P}$  against  $\tau$ , then let  $T(p) = \bigcup_{n < \omega} T(p \upharpoonright n)$ . We note  $T(p)$  is an infinite tree (for each  $n < \omega$ , a node of  $T(p \upharpoonright n)$  was extended) with finite levels:

- $\emptyset$  has exactly  $m_\emptyset$  successors  $\langle i \rangle$ .
- $s^\frown \langle i \rangle$  has exactly one successor  $s^\frown \langle i, h_{s,i} \rangle$
- $s^\frown \langle i, h_{s,i} \rangle$  has exactly  $m_s$  successors  $s^\frown \langle i, h_{s,i}, j \rangle$
- $s^\frown \langle i, h_{s,i}, j \rangle$  has either no successors or exactly one successor  $s^\frown \langle i, h_{s,i}, j, x \rangle$
- $t = s^\frown \langle i, h_{s,i}, j, x \rangle$  has exactly  $m_t$  successors  $t^\frown \langle k \rangle$

Hence  $T(p)$  has an infinite branch

$$q' = \langle i_0, h_0, j_0, x_0, i_1, h_1, j_1, x_1, \dots \rangle$$

Let  $q = o_{q'} = \langle h_0, x_0, h_1, x_1, \dots \rangle$ . Note that by the construction of  $T(p)$ ,  $q$  is a legal attack on the winning strategy  $\sigma$  in  $Bell_{D,P}^\rightarrow(X)$ , so it must converge. Since every other term of  $q$  is in  $H$ , it must converge to  $H$ . Then since  $o_q$  is a subsequence of  $p$ ,  $p$  must cluster at  $H$ .  $\square$

**Corollary 4.3.8.** *A compact space is Corson compact if and only if  $\mathcal{D} \uparrow Bell_{D,P}^\rightarrow(X)$ .*  $\diamond$

*Proof.* If  $\mathcal{D} \uparrow Bell_{D,P}^\rightarrow(X)$ , then  $\mathcal{D} \uparrow Bell_{D,P}^\rightarrow(X^2)$ . As  $X^2$  is (uniformly locally) compact,  $\mathcal{D} \uparrow Bell_{D,P}^\rightarrow(X^2)$ . Thus  $\mathcal{O} \uparrow Gru_{O,P}^\rightarrow(X^2, \Delta)$ , showing that  $X$  is Corson compact.  $\square$

#### 4.4 Limited information results

Many of the results shown in [2] and [18] have limited information analogs.

**Theorem 4.4.1.** *For all  $x \in X$ :*

- $\mathcal{D} \xrightarrow{2k\text{-tact}} Bell_{D,P}^\rightarrow(X) \Rightarrow \mathcal{O} \xrightarrow{k\text{-tact}} Gru_{O,P}^\rightarrow(X, x)$
- $\mathcal{D} \xrightarrow{2k\text{-mark}} Bell_{D,P}^\rightarrow(X) \Rightarrow \mathcal{O} \xrightarrow{k\text{-mark}} Gru_{O,P}^\rightarrow(X, x)$

- $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow}(X) \Rightarrow \mathcal{O} \uparrow Gru_{O,P}^{\rightarrow}(X, x)$

◇

*Proof.* The perfect-information result was originally shown by Bell.

Let  $\sigma \circ \nu_{2k}$  witness  $\mathcal{D} \uparrow_{2k\text{-tact}} Bell_{D,P}^{\rightarrow}(X)$ , where  $\nu_k$  is the  $k$ -tactical fog-of-war which removes all information except the final  $k$  moves of the opponent.

We define the  $k$ -tactical strategy  $\tau$  for  $Gru_{O,P}^{\rightarrow}(X, x)$  such that

$$\tau \circ \nu_k(t) = \sigma \circ \nu_{2k}(\langle x, t(0), \dots, x, t(|t| - 1) \rangle)[x] \cap \sigma \circ \nu_{2k}(\langle x, t(0), \dots, x, t(|t| - 1), x \rangle)[x]$$

Let  $p$  attack  $\tau$ . Consider the attack  $q$  against the winning strategy  $\sigma \circ \nu_{2k}$  such that  $q(2n) = x$  and  $q(2n + 1) = p(n)$ , and let  $D_n = \sigma \circ \nu_{2k}(q \upharpoonright n)$  and  $E_n = \bigcap_{m \leq n} D_m$ .

Certainly,  $x \in E_{2n}[x] = E_{2n}[q(2n)]$  for any  $n < \omega$ . Note also for any  $n < \omega$  that

$$\begin{aligned} p(n) &\in \bigcap_{m \leq n} \tau \circ \nu_k(p \upharpoonright m) \\ &= \bigcap_{m \leq n} (\sigma \circ \nu_{2k}(\langle x, p(0), \dots, x, p(m - 1) \rangle)[x] \cap \sigma \circ \nu_{2k}(\langle x, p(0), \dots, x, p(m - 1), x \rangle)[x]) \\ &= \bigcap_{m \leq n} (D_{2m}[x] \cap D_{2m+1}[x]) = \bigcap_{m \leq 2n+1} D_m[x] = E_{2n+1}[x] \end{aligned}$$

so by the symmetry of  $E_{2n+1}$ ,  $x \in E_{2n+1}[p(n)] = E_{2n+1}[q(2n + 1)]$ . Thus  $x \in \bigcap_{n < \omega} E_n[q(n)] \neq \emptyset$ , and since  $\sigma$  is a winning  $2k$ -tactic, the attack  $q$  converges. Since  $q(2n) = x$ ,  $q$  must converge to  $x$ . Thus its subsequence  $p$  converges to  $x$ , and  $\tau$  is a winning  $k$ -tactic for  $\mathcal{O}$  in  $Gru_{O,P}^{\rightarrow}(X, x)$ .

The proofs of the other two implications follow analogously by replacing  $\nu_i$  with the  $i$ -Markov fog-of-war  $\mu_i$  or the identity function, respectively.  $\square$

**Theorem 4.4.2.** *Let  $X \cup \{\infty\}$  be a uniformizable space such that  $X$  is discrete, and  $k < \omega$ . Then*

- $\mathcal{O} \uparrow_{k\text{-tact}} Gru_{\vec{O},P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathcal{D} \uparrow_{k\text{-tact}} Bell_{\vec{D},P}(X \cup \{\infty\})$
- $\mathcal{O} \uparrow_{k\text{-mark}} Gru_{\vec{O},P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathcal{D} \uparrow_{k\text{-mark}} Bell_{\vec{D},P}(X \cup \{\infty\})$
- $\mathcal{O} \uparrow Gru_{\vec{O},P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathcal{D} \uparrow Bell_{\vec{D},P}(X \cup \{\infty\})$

◇

*Proof.* The perfect-information result is due to Nyikos.

The right-to-left implications have already been shown. For any open neighborhood  $U$  of  $\infty$ ,  $D(U) = \Delta \cup U^2$  is an open symmetric entourage of  $X$ .

Let  $\sigma \circ \nu_k$  witness  $\mathcal{D} \uparrow_{k\text{-tact}} Gru_{\vec{O},P}(X \cap \{\infty\}, \infty)$ . We then define the  $k$ -tactic  $\tau$  such that

$$\tau \circ \nu_k(t) = D(\sigma \circ \nu_k(t))$$

Let  $p$  attack  $\tau$  such that  $\bigcap_{n < \omega} \tau \circ \nu_k(p \upharpoonright n)[p(n)] \neq \emptyset$ .

If  $\infty \in \bigcap_{n < \omega} \tau \circ \nu_k(p \upharpoonright n)[p(n)]$ , it follows that  $p$  is an attack on  $\sigma \circ \nu_k$ . Since  $\sigma \circ \nu_k$  is a winning strategy, it follows that  $q$  and its subsequence  $p$  must coverge to  $\infty$ .

Otherwise,  $\infty \notin \tau \circ \nu_k(p \upharpoonright N)[p(N)]$  for some  $N < \omega$ , and then  $\tau \circ \nu_k(p \upharpoonright N)[p(N)] = \{p(N)\}$  implies  $p \rightarrow p(N)$ .

Thus  $\tau$  is a winning  $k$ -tactic. The other implications may be proven analogously by replacing  $\nu_i$  with the  $i$ -Markov fog-of-war  $\mu_i$  or the identity function, repsectively.  $\square$

**Proposition 4.4.3.** *For any  $x \in X$  and  $k < \omega$ ,*

- $\mathcal{O} \uparrow_{k+1\text{-tact}} Gru_{\vec{O},P}(X, x) \Leftrightarrow \mathcal{O} \uparrow_{tact} Gru_{\vec{O},P}(X, x)$
- $\mathcal{O} \uparrow_{k+1\text{-mark}} Gru_{\vec{O},P}(X, x) \Leftrightarrow \mathcal{O} \uparrow_{mark} Gru_{\vec{O},P}(X, x)$

◇

*Proof.* If  $\sigma$  witnesses  $\mathcal{O} \uparrow_{k+1\text{-tact}} Gru_{\vec{O},P}(X, x)$ , let  $\tau(\emptyset) = \sigma(\emptyset)$  and

$$\tau(\langle q \rangle) = \bigcap_{i < k} \sigma(\langle \underbrace{x, \dots, x}_{k-i}, q, \underbrace{x, \dots, x}_{i+1} \rangle)$$

Then  $\tau$  is easily verified to be a winning tactic, and the proof for the second part is analogous.  $\square$

**Corollary 4.4.4.** *Let  $X \cup \{\infty\}$  be a uniformizable space such that  $X$  is discrete, and  $k < \omega$ .*

*Then*

- $\mathcal{D} \uparrow_{k+1\text{-tact}} Bell_{D,P}^{\rightarrow}(X \cup \{\infty\}) \Leftrightarrow O \uparrow_{tact} Bell_{D,P}^{\rightarrow}(X \cup \{\infty\})$
- $\mathcal{D} \uparrow_{k+1\text{-mark}} Bell_{D,P}^{\rightarrow}(X \cup \{\infty\}) \Leftrightarrow O \uparrow_{mark} Bell_{D,P}^{\rightarrow}(X \cup \{\infty\})$

$\diamond$

**Proposition 4.4.5.** *For any uniform space  $X$  and  $k < \omega$ ,*

- $\mathcal{D} \uparrow_{k+2\text{-tact}} Bell_{D,P}^{\rightarrow}(X) \Leftrightarrow \mathcal{D} \uparrow_{2\text{-tact}} Bell_{D,P}^{\rightarrow}(X)$
- $\mathcal{D} \uparrow_{k+2\text{-mark}} Bell_{D,P}^{\rightarrow}(X) \Leftrightarrow \mathcal{D} \uparrow_{2\text{-mark}} Bell_{D,P}^{\rightarrow}(X)$

$\diamond$

*Proof.* If  $\sigma$  witnesses  $\mathcal{O} \uparrow_{k\text{-tact}} Bell_{D,P}^{\rightarrow}(X)$ , let  $\tau(\emptyset) = \sigma(\emptyset)$  and

$$\tau(\langle q \rangle) = \bigcap_{i < k+2} \sigma(\underbrace{\langle q, \dots, q \rangle}_{i+1})$$

$$\tau(\langle q, q' \rangle) = \bigcap_{i < k+2} \sigma(\underbrace{\langle q, \dots, q \rangle}_{k+1-i}, \underbrace{\langle q', \dots, q' \rangle}_{i+1})$$

Then  $\tau$  is easily verified to be a winning strategy, and the proof for the second part is analogous.  $\square$

**Proposition 4.4.6.**  $\mathcal{D} \uparrow_{pre} Bell_{D,P}^{\rightarrow}(X)$  *implies  $X$  is first countable.*  $\diamond$

*Proof.* Result follows as  $\mathcal{O} \uparrow_{pre} Gru_{O,P}^{\rightarrow}(X, x)$  for all  $x \in X$ .  $\square$

**Theorem 4.4.7.** *If  $H$  is a closed subset of  $X$ , then  $\mathcal{D} \uparrow_{limit} Bell_{D,P}^{\rightarrow}(X) \Rightarrow \mathcal{D} \uparrow_{limit} Bell_{D,P}^{\rightarrow}(H)$*

*where  $\uparrow_{limit}$  is any of  $\uparrow$ ,  $\uparrow_{k\text{-tact}}$ , or  $\uparrow_{k\text{-mark}}$ .*  $\diamond$

## Chapter 5

### Gruenhage's Locally Finite Games

placeholder

## Chapter 6

### Menger's Game and Rothberger's Game

In 1924 Karl Menger introduced a covering property generalizing  $\sigma$ -compactness [15].

**Definition 6.0.8.** A space  $X$  is Menger if for every sequence  $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$  of open covers of  $X$  there exists a sequence  $\langle \mathcal{F}_0, \mathcal{F}_1, \dots \rangle$  such that  $\mathcal{F}_n \subseteq \mathcal{U}_n$ ,  $|\mathcal{F}_n| < \omega$ , and  $\bigcup_{n < \omega} \mathcal{F}_n$  is a cover of  $X$ .  $\diamond$

**Proposition 6.0.9.**  $X$  is  $\sigma$ -compact  $\Rightarrow X$  is Menger  $\Rightarrow X$  is Lindelöf.  $\diamond$

None of these implications may be reversed; the irrationals are a simple example of a Lindelöf space which is not Menger, and we'll see several examples of Menger spaces which are not  $\sigma$ -compact.

It can be shown via a non-trivial proof that the following game can be used to characterize the Menger property.

**Definition 6.0.10.** For each cover  $\mathcal{U}$  of  $X$ ,  $S \subseteq X$  is  $\mathcal{U}$ -finite if there exists a finite subcollection of  $\mathcal{U}$  which covers  $S$ .  $\diamond$

Of course, a compact space is  $\mathcal{U}$ -finite for all open covers  $\mathcal{U}$ .

**Game 6.0.11.** Let  $Men_{C,F}(X)$  denote the *Menger game* with players  $\mathcal{C}$ ,  $\mathcal{F}$ . In round  $n$ ,  $\mathcal{C}$  chooses an open cover  $\mathcal{U}_n$ , followed by  $\mathcal{F}$  choosing a  $\mathcal{U}_n$ -finite subset  $F_n$  of  $X$ .

$\mathcal{F}$  wins the game if  $X = \bigcup_{n < \omega} F_n$ , and  $\mathcal{C}$  wins otherwise.  $\diamond$

**Theorem 6.0.12.** A space  $X$  is Menger if and only if  $\mathcal{C} \nmid Men_{C,F}(X)$  [11].  $\diamond$

The typical characterization of the Menger game involves  $\mathcal{F}$  choosing a finite subcollection  $\mathcal{F}_n$  of  $\mathcal{U}_n$ , but it is easy to see that the characterization given above is equivalent, and will be convenient for use in our proofs.

## 6.1 Markov strategies

To the author's knowledge, no other direct work has been done on limited information strategies pertaining to the Menger game, although as we'll see there are results which can be sharpened when considering them. However, we immediately see that tactics are not of any real interest.

**Proposition 6.1.1.**  *$X$  is compact if and only if  $\mathcal{F} \uparrow_{tact} Men_{C,F}(X)$  if and only if  $\mathcal{F} \uparrow_{k-tact} Men_{C,F}(X)$ .*  $\diamond$

*Proof.* If  $\sigma$  is a winning  $k$ -tactic, then for each open cover  $\mathcal{U}$ ,  $\sigma$  defeats the attack  $\langle \mathcal{U}, \mathcal{U}, \dots \rangle$ . Then

$$\bigcup_{i \leq k} \sigma(\underbrace{\langle \mathcal{U}, \dots, \mathcal{U} \rangle}_i) = X$$

and  $X$  is  $\mathcal{U}$ -finite.  $\square$

Essentially, because  $\mathcal{C}$  may repeat the same finite sequence of open covers,  $\mathcal{F}$  needs to be seeded with knowledge of the round number to prevent being trapped in a loop.

If  $\mathcal{F}$ 's memory of  $\mathcal{C}$ 's past moves is bounded, then there is no need to consider more than the two most recent moves. The intuitive reason is that  $\mathcal{C}$  could simply play the same cover repeatedly until  $\mathcal{F}$ 's memory is exhausted, in which case  $\mathcal{F}$  would only ever see the change from one cover to another.

**Theorem 6.1.2.** *There exists  $k < \omega$  such that  $F \uparrow_{(k+2)\text{-mark}} Men_{C,F}(X)$  if and only if  $F \uparrow_{2\text{-mark}} Men_{C,F}(X)$ .*  $\diamond$

*Proof.* Let  $\sigma$  be a winning  $(k+2)$ -mark. We define the 2-mark  $\tau$  as follows:

$$\begin{aligned} \tau(\langle \mathcal{U} \rangle, 0) &= \bigcup_{m < k+2} \sigma(\underbrace{\langle \mathcal{U}, \dots, \mathcal{U} \rangle}_{m+1}, m) \\ \tau(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) &= \bigcup_{m < k+2} \sigma(\underbrace{\langle \mathcal{U}, \dots, \mathcal{U} \rangle}_{k+1-m}, \underbrace{\langle \mathcal{V}, \dots, \mathcal{V} \rangle}_{m+1}, (n+1)(k+2) + m) \end{aligned}$$

Let  $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$  be an attack by  $\mathcal{C}$  against  $\tau$ . Then consider the attack

$$\langle \underbrace{\mathcal{U}_0, \dots, \mathcal{U}_0}_{k+2}, \underbrace{\mathcal{U}_1, \dots, \mathcal{U}_1}_{k+2}, \dots \rangle$$

by  $\mathcal{C}$  against  $\sigma$ . Since  $\sigma$  is a winning  $(k+2)$ -mark,

$$\begin{aligned} X &= \bigcup_{m < k+2} \sigma(\langle \underbrace{\mathcal{U}_0, \dots, \mathcal{U}_0}_{m+1}, m \rangle) \cup \bigcup_{n < \omega} \bigcup_{m < k+2} \sigma(\langle \underbrace{\mathcal{U}_n, \dots, \mathcal{U}_n}_{k+1-m}, \underbrace{\mathcal{U}_{n+1}, \dots, \mathcal{U}_{n+1}}_{m+1}, (n+1)(k+2) + m \rangle) \\ &= \tau(\langle \mathcal{U}_0 \rangle, 0) \cup \bigcup_{n < \omega} \tau(\langle \mathcal{U}_n, \mathcal{U}_{n+1} \rangle, n+1) \end{aligned}$$

Thus  $\tau$  is a winning 2-mark.  $\square$

A natural question arises: is there an example of a space  $X$  for which  $\mathcal{F} \uparrow_{2\text{-mark}} \text{Men}_{C,F}(X)$  but  $\mathcal{F} \not\uparrow_{\text{mark}} \text{Men}_{C,F}(X)$ ? We quickly see that perhaps the simplest example of a Lindelöf non- $\sigma$ -compact space has this property.

**Definition 6.1.3.** For any cardinal  $\kappa$ , let  $\kappa^\dagger = \kappa \cup \{\infty\}$  denote the *one-point Lindelöfication* of discrete  $\kappa$ , where points in  $\kappa$  are isolated, and the neighborhoods of  $\infty$  are the co-countable sets containing it.  $\diamond$

**Theorem 6.1.4.**  $F \not\uparrow_{\text{mark}} \text{Men}_{C,F}(\omega_1^\dagger)$ .  $\diamond$

*Proof.* This result will later follow from the fact that  $\omega_1^\dagger$  is not a  $\sigma$ -compact space (all its compact subsets are finite).

For now, let  $\sigma$  be a Markov strategy for  $\mathcal{F}$ . For each  $\alpha < \omega_1$ , let  $\mathcal{U}_\alpha$  be the open cover  $\{\{\beta\} : \beta < \alpha\} \cup \{\omega_1^\dagger \setminus \alpha\}$  of  $\omega_1^\dagger$ , and set  $F(\alpha, n)$  to be the finite set  $\alpha \cap \sigma(\langle \mathcal{U}_\alpha \rangle, n)$ .

If  $P_n = \{\beta : \beta < \alpha < \omega_1 \Rightarrow \beta \in F(\alpha, n)\}$ , then  $P_n \subseteq F(\sup(P_n) + 1, n)$ . Thus  $P_n$  is finite for  $n < \omega$ . Choose  $\beta \in \omega_1 \setminus \bigcup_{n < \omega} P_n$  and  $\alpha_n > \beta$  such that  $\beta \notin F(\alpha_n, n)$ . Then  $\mathcal{C}$  may attack  $\sigma$  with  $\langle \mathcal{U}_{\alpha_0}, \mathcal{U}_{\alpha_1}, \dots \rangle$ , and it follows that  $\beta \notin \bigcup_{n < \omega} F(\alpha_n, n)$  and  $\beta \notin \bigcup_{n < \omega} \sigma(\langle \mathcal{U}_{\alpha_n} \rangle, n)$ .  $\square$

The greatest advantage of a strategy which has knowledge of two or more previous moves of the opponent, versus only knowledge of the most recent move, is the ability to react to



changes from one round to the next. It's this ability to react that will give  $\mathcal{F}$  her winning 2-Markov strategy in the Menger game on  $\omega_1^\dagger$ .

For inspiration, we turn to a game whose  $n$ -tactics were studied by Marion Scheepers in [20] which has similar goals to the Menger game when played upon  $\kappa^\dagger$ .

**Game 6.1.5.** Let  $Fill_{C,F}^{\cup,\subseteq}(\kappa)$  denote the *strict union filling game* with two players  $\mathcal{C}$ ,  $\mathcal{F}$ . In round 0,  $\mathcal{C}$  chooses  $C_0 \in [\kappa]^{\leq \omega}$ , followed by  $\mathcal{F}$  choosing  $F_0 \in [\kappa]^{< \omega}$ . In round  $n + 1$ ,  $\mathcal{C}$  chooses  $C_{n+1} \in [\kappa]^{\leq \omega}$  such that  $C_{n+1} \supset C_n$ , followed by  $\mathcal{F}$  choosing  $F_{n+1} \in [\kappa]^{< \omega}$ .

$\mathcal{F}$  wins the game if  $\bigcup_{n < \omega} F_n \supseteq \bigcup_{n < \omega} C_n$ ; otherwise,  $\mathcal{C}$  wins.  $\diamond$

In  $Men_{C,F}(\kappa^\dagger)$ ,  $\mathcal{C}$  essentially chooses a countable set to not include in her neighborhood of  $\infty$ , followed by  $\mathcal{F}$  choosing a finite subset of this complement to cover during each round. Thus,  $\mathcal{F}$  need only be concerned with the *intersection* of the countable sets chosen by  $\mathcal{C}$  in  $Men_{C,F}(\kappa^\dagger)$ , rather than the union as in  $Fill_{C,F}^{\cup,\subseteq}(\kappa)$ .

Another difference: Scheepers required that  $\mathcal{C}$  always choose strictly growing countable sets. The reasoning is clear: if the goal is to study tactics, then  $\mathcal{C}$  cannot be allowed to trap  $\mathcal{F}$  in a loop by repeating the same moves. But by eliminating this requirement, the study can then turn to Markov strategies, bringing the game further in line with the Menger game played upon  $\kappa^\dagger$ .

We introduce a few games to make the relationship between Scheepers's  $Fill_{C,F}^{\cup,\subseteq}(\kappa)$  and  $Men_{C,F}(\kappa^\dagger)$  more precise.

**Game 6.1.6.** Let  $Fill_{C,F}^{\cup,\subseteq}(\kappa)$  denote the *union filling game* which proceeds analogously to  $Fill_{C,F}^{\cup,\subseteq}(\kappa)$ , except that  $\mathcal{C}$ 's restriction in round  $n + 1$  is reduced to  $C_{n+1} \supseteq C_n$ .  $\diamond$

**Game 6.1.7.** Let  $Fill_{C,F}^{1,\subseteq}(\kappa)$  denote the *initial filling game* which proceeds analogously to  $Fill_{C,F}^{\cup,\subseteq}(\kappa)$ , except that  $\mathcal{F}$  wins whenever  $\bigcup_{n < \omega} F_n \supseteq C_0$ .  $\diamond$

**Game 6.1.8.** Let  $Fill_{C,F}^\cap(\kappa)$  denote the *intersection filling game* which proceeds analogously to  $Fill_{C,F}^{1,\subseteq}(\kappa)$ , except that  $\mathcal{C}$  may choose any  $C_n \in [\kappa]^{\leq \omega}$  each round, and  $\mathcal{F}$  wins whenever  $\bigcup_{n < \omega} F_n \supseteq \bigcap_{n < \omega} C_n$ .  $\diamond$

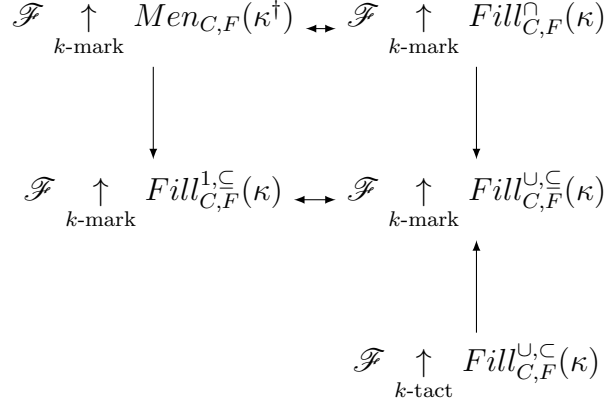


Figure 6.1: Diagram of Filling/Menger game implications

**Theorem 6.1.9.** *For any cardinal  $\kappa > \omega$  and integer  $k < \omega$ , Figure 6.1 holds.*  $\diamond$

*Proof.*  $\mathcal{F} \uparrow_{k\text{-mark}} Men_{C,F}(\kappa^\dagger) \Rightarrow \mathcal{F} \uparrow_{k\text{-mark}} Fill_{C,F}^\cap(\kappa)$ : Let  $\sigma$  be a winning  $k$ -mark for  $\mathcal{F}$  in  $Men_{C,F}(\kappa^\dagger)$ . Let  $\mathcal{U}(C)$  (resp.  $\mathcal{U}(s)$ ) convert each countable subset  $C$  of  $\kappa$  (resp. finite sequence  $s$  of such subsets) into the open cover  $[C]^1 \cup \{\kappa^\dagger \setminus C\}$  (resp. finite sequence of such open covers). Then  $\tau$  defined by

$$\tau(s^\frown \langle C \rangle, n) = C \cap \sigma(\mathcal{U}(s^\frown \langle C \rangle), n)$$

is a winning  $k$ -mark for  $\mathcal{F}$  in  $Fill_{C,F}^\cap(\kappa)$ .

$\mathcal{F} \uparrow_{k\text{-mark}} Fill_{C,F}^\cap(\kappa) \Rightarrow \mathcal{F} \uparrow_{k\text{-mark}} Men_{C,F}(\kappa^\dagger)$ : Let  $\sigma$  be a winning  $k$ -mark for  $\mathcal{F}$  in  $Fill_{C,F}^\cap(\kappa)$ . Let  $C(\mathcal{U})$  (resp.  $C(s)$ ) convert each open cover  $\mathcal{U}$  of  $\kappa^\dagger$  (resp. finite sequence  $s$  of such covers) into a countable set  $C$  which is the complement of some neighborhood of  $\infty$  in  $\mathcal{U}$  (resp. finite sequence of such countable sets). Then  $\tau$  defined by

$$\tau(s^\frown \langle \mathcal{U} \rangle, n) = (\kappa^\dagger \setminus C(\mathcal{U})) \cup \sigma(C(s^\frown \langle \mathcal{U} \rangle), n)$$

is a winning  $k$ -mark for  $\mathcal{F}$  in  $Men_{C,F}(\kappa^\dagger)$ .

$\mathcal{F} \uparrow_{k\text{-mark}} \text{Fill}_{C,F}^\cap(\kappa) \Rightarrow \mathcal{F} \uparrow_{k\text{-mark}} \text{Fill}_{C,F}^{1,\subseteq}(\kappa)$ : Let  $\sigma$  be a winning  $k$ -mark for  $\mathcal{F}$  in  $\text{Fill}_{C,F}^\cap(\kappa)$ .  $\sigma$  is also a winning  $k$ -mark for  $\mathcal{F}$  in  $\text{Fill}_{C,F}^{1,\subseteq}(\kappa)$ .

$\mathcal{F} \uparrow_{k\text{-mark}} \text{Fill}_{C,F}^{1,\subseteq}(\kappa) \Rightarrow \mathcal{F} \uparrow_{k\text{-mark}} \text{Fill}_{C,F}^{\cup,\subseteq}(\kappa)$ : Let  $\sigma$  be a winning  $k$ -mark for  $\mathcal{F}$  in  $\text{Fill}_{C,F}^{1,\subseteq}(\kappa)$ . For each finite sequence  $s$ , let  $t \preceq s$  mean  $t$  is a final subsequence of  $s$ . Then  $\tau$  defined by

$$\tau(s \smallfrown \langle C \rangle, n) = \bigcup_{t \preceq s, m \leq n} \sigma(t \smallfrown \langle C \rangle, m)$$

is a winning  $k$ -mark for  $\mathcal{F}$  in  $\text{Fill}_{C,F}^{\cup,\subseteq}(\kappa)$ .

$\mathcal{F} \uparrow_{k\text{-mark}} \text{Fill}_{C,F}^{\cup,\subseteq}(\kappa) \Rightarrow \mathcal{F} \uparrow_{k\text{-mark}} \text{Fill}_{C,F}^{1,\subseteq}(\kappa)$ : Let  $\sigma$  be a winning  $k$ -mark for  $\mathcal{F}$  in  $\text{Fill}_{C,F}^{\cup,\subseteq}(\kappa)$ .  $\sigma$  is also a winning  $k$ -mark for  $\mathcal{F}$  in  $\text{Fill}_{C,F}^{1,\subseteq}(\kappa)$ .

$\mathcal{F} \uparrow_{k\text{-tact}} \text{Fill}_{C,F}^{\cup,\subseteq}(\kappa) \Rightarrow \mathcal{F} \uparrow_{k\text{-mark}} \text{Fill}_{C,F}^{\cup,\subseteq}(\kappa)$ : Let  $\sigma$  be a winning  $k$ -tactic for  $\mathcal{F}$  in  $\text{Fill}_{C,F}^{\cup,\subseteq}(\kappa)$ . For each countable subset  $C$  of  $\kappa$ , let  $C + n$  be the union of  $C$  with the  $n$  least ordinals in  $\kappa \setminus C$ . Then  $\tau$  defined by

$$\tau(\langle C_0, \dots, C_i \rangle, n) = \sigma(\langle C_0 + (n - i), \dots, C_i + n \rangle)$$

is a winning  $k$ -mark for  $\mathcal{F}$  in  $\text{Fill}_{C,F}^{\cup,\subseteq}(\kappa)$ . □

While we have not proven a direct implication between the Menger game and Scheeper's original filling game, Scheepers introduced the statement  $S(\kappa, \omega, \omega)$  relating to the almost-compatibility of functions from countable subsets of  $\kappa$  into  $\omega$  which may be applied to both.

**Definition 6.1.10.** For two functions  $f, g$  we say  $f$  is  $\mu$ -**almost compatible** with  $g$  ( $f \parallel_\mu^* g$ ) if  $|\{x \in \text{dom}(f) \cap \text{dom}(g) : f(x) \neq g(x)\}| < \mu$ . If  $\mu = \omega$  then we say  $f, g$  are **almost compatible** ( $f \parallel^* g$ ). ◇

**Definition 6.1.11.**  $S(\kappa, \omega, \omega)$  states that there exist functions  $f_A : A \rightarrow \omega$  for each  $A \in [\kappa]^{\leq \omega}$  such that  $|\{\alpha \in A : f_A(\alpha) \leq n\}| < \omega$  for all  $n < \omega$  and  $f_A \parallel^* f_B$  for all  $A, B \in [\kappa]^\omega$ .<sup>1</sup> ◇

---

<sup>1</sup>This is equivalent to the original characterization given in [20]: there exist injections  $g_A : A \rightarrow \omega$  such that  $g_A \parallel^* g_B$  for all  $A, B \in [\kappa]^\omega$  and  $A \subset B$ .

Scheepers went on to show that  $S(\kappa, \omega, \omega)$  implies  $\mathcal{F} \uparrow_{2\text{-tact}}^{\text{U}, \subseteq} \text{Fill}_{C,F}^{\text{U}, \subseteq}(\kappa)$ . This proof, along with the following facts, give us inspiration for finding a winning 2-Markov strategy in the Menger game played on  $\kappa^\dagger$ .

**Theorem 6.1.12.**  $S(\omega_1, \omega, \omega)$  and  $\kappa > 2^\omega \Rightarrow \neg S(\kappa, \omega, \omega)$  are theorems of ZFC.  $S(2^\omega, \omega, \omega)$  is a theorem of ZFC + CH and consistent with ZFC +  $\neg CH$ .  $\diamond$

*Proof.* For  $S(\omega_1, \omega, \omega)$ , look at pg. 70 of [12]; this of course implies  $S(2^\omega, \omega, \omega)$  under CH.  $\neg S((2^\omega)^+, \omega, \omega)$  is shown by a cardinality argument in [20]. The consistency result under ZFC +  $\neg CH$  is a lemma for the main theorem in [20].  $\square$

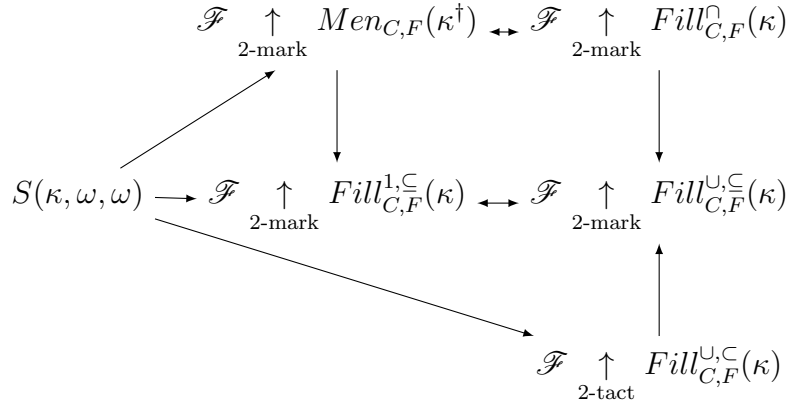


Figure 6.2: Diagram of Filling/Menger game implications with  $S(\kappa, \omega, \omega)$

**Theorem 6.1.13.**  $S(\kappa, \omega, \omega)$  implies the game-theoretic results in Figure 6.2.  $\diamond$

*Proof.* Since  $S(\kappa, \omega, \omega) \Rightarrow \mathcal{F} \uparrow_{2\text{-tact}}^{\text{U}, \subseteq} \text{Fill}_{C,F}^{\text{U}, \subseteq}(\kappa)$  was a main result of [20], it is sufficient to show that  $S(\kappa, \omega, \omega) \Rightarrow \mathcal{F} \uparrow_{2\text{-mark}}^\cap \text{Fill}_{C,F}^\cap(\kappa)$ .

Let  $f_A$  for  $A \in [\kappa]^{\leq \omega}$  witness  $S(\kappa, \omega, \omega)$ . We define the 2-mark  $\sigma$  as follows:

$$\sigma(\langle A \rangle, 0) = \{\alpha \in A : f_A(\alpha) \leq 0\}$$

$$\sigma(\langle A, B \rangle, n+1) = \{\alpha \in A \cap B : f_B(\alpha) \leq n+1 \text{ or } f_A(\alpha) \neq f_B(\alpha)\}$$

For any attack  $\langle A_0, A_1, \dots \rangle$  by  $\mathcal{C}$  and  $\alpha \in \bigcap_{n < \omega} A_n$ , either  $f_{A_n}(\alpha)$  is constant for all  $n$ , or  $f_{A_n}(\alpha) \neq f_{A_{n+1}}(\alpha)$  for some  $n$ ; either way,  $\alpha$  is covered.  $\square$

**Corollary 6.1.14.**  $\mathcal{F} \xrightarrow[2\text{-mark}]{\uparrow} Men_{C,F}(\omega_1^\dagger).$   $\diamond$

## 6.2 Menger game derived covering properties

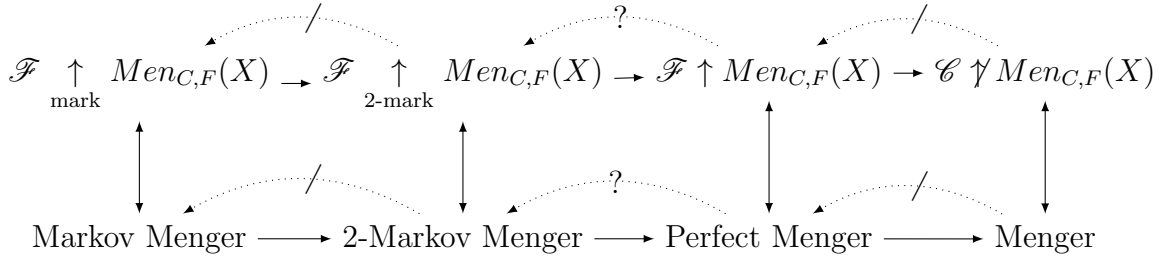


Figure 6.3: Diagram of covering properties related to the Menger game

Limited information strategies for the Menger game naturally define a spectrum of covering properties, see Figure 6.3. However, we do not know if the middle two properties are actually distinct.

**Question 6.2.1.** Does there exist a space  $X$  such that  $\mathcal{F} \xrightarrow{\uparrow} Men_{C,F}(X)$  but  $\mathcal{F} \not\xrightarrow[2\text{-mark}]{\uparrow} Men_{C,F}(X)$ ?  $\diamond$

We are also interested in non-game-theoretic characterizations of these covering properties. It has been known for some time that for metrizable spaces, winning Menger spaces are exactly the  $\sigma$ -compact spaces, shown first by Telgarsky in [23] and later directly by Scheepers in [22].

In the interest of generality, we will first characterize the Markov Menger spaces without any separation axioms.

**Definition 6.2.2.** A subset  $Y$  of  $X$  is *relatively compact* to  $X$  if for every open cover of  $X$ , there exists a finite subcollection which covers  $Y$ .  $\diamond$

For example, any bounded subset of Euclidean space is relatively compact whether it is closed or not. Actually, relative compactness can be thought of as an analogue of boundedness for regular spaces.

**Proposition 6.2.3.** *For regular spaces,  $Y$  is relatively compact to  $X$  if and only if  $\overline{Y}$  is compact in  $X$ .* <sup>2</sup>  $\diamond$

*Proof.* For any space, any subset of a compact set is relatively compact.

Assume  $Y$  is relatively compact, let  $\mathcal{U}$  be an open cover of  $\overline{Y}$ , and define  $x \in V_x \subseteq \overline{V_x} \subseteq U_x \in \mathcal{U}$  for each  $x \in X$ . Then if we take a subcollection  $\mathcal{F} = \{V_{x_i} : i < n\}$  covering  $Y$  by relative compactness, then  $\{U_{x_i} : i < n\}$  is a finite subcollection of  $\mathcal{U}$  covering  $\overline{Y}$ , showing compactness.  $\square$

We now begin the process of factoring out Scheeper's proof to reveal the limited information implications at work.

**Lemma 6.2.4.** *Let  $\sigma(\mathcal{U}, n)$  be a Markov strategy for  $F$  in  $\text{Men}_{C,F}(X)$ , and  $\mathfrak{C}$  collect all open covers of  $X$ . Then the set*

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \sigma(\mathcal{U}, n)$$

*is relatively compact to  $X$ . If  $\sigma$  is a winning Markov strategy, then  $\bigcup_{n < \omega} R_n = X$ .*  $\diamond$

*Proof.* First, for every open cover  $\mathcal{U} \in \mathfrak{C}$ ,  $R_n \subseteq \sigma(\mathcal{U}, n)$  is covered by a finite subcollection of  $\mathcal{U}$ .

Suppose that  $x \notin R_n$  for any  $n < \omega$ . Then for each  $n$ , pick  $\mathcal{U}_n \in \mathfrak{C}$  such that  $x \notin \sigma(\mathcal{U}_n, n)$ . Then  $\mathcal{C}$  may counter  $\sigma$  with the attack  $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ .  $\square$

**Definition 6.2.5.** A  $\sigma$ -relatively-compact space is the countable union of relatively compact subsets.  $\diamond$

---

<sup>2</sup>It should be noted that some authors define relative compactness in this way, but such a definition creates pathological implications for non-regular spaces. For example, the singleton containing the particular point of an infinite space with the particular point topology would not be relatively compact since its closure is not compact, even though it is finite.

**Corollary 6.2.6.** *The following are equivalent:*

- $X$  is  $\sigma$ -relatively-compact
- $F \uparrow_{pre} Men_{C,F}(X)$
- $F \uparrow_{mark} Men_{C,F}(X)$

◇

*Proof.* If  $X = \bigcup_{n < \omega} R_n$  for  $R_n$  relatively compact, then  $\sigma(n) = R_n$  is a winning predetermined strategy, which yields a winning Markov strategy. The previous lemma finishes the proof. □

**Corollary 6.2.7.** *Let  $X$  be a regular space. The following are equivalent:*

- $X$  is  $\sigma$ -compact
- $X$  is  $\sigma$ -relatively-compact
- $F \uparrow_{pre} Men_{C,F}(X)$
- $F \uparrow_{mark} Men_{C,F}(X)$

◇

For Lindelöf spaces, metrizability is characterized by regularity and second-countability, the latter of which was essentially used by Scheepers in this way:

**Lemma 6.2.8.** *Let  $X$  be a second-countable space.  $\mathcal{F} \uparrow Men_{C,F}(X)$  if and only if  $\mathcal{F} \uparrow_{mark} Men_{C,F}(X)$ .* ◇

*Proof.* Let  $\sigma$  be a strategy for  $\mathcal{F}$ , and note that it's sufficient to consider playthroughs with only basic open covers.

So if  $\mathcal{U}_t$  is a basic open cover for  $t < s \in \omega^{<\omega}$ , and  $\mathcal{V}$  is any basic open cover, we may choose a finite subcollection  $\mathcal{F}(s, \mathcal{V})$  of  $\mathcal{V}$  such that

$$\sigma(\langle \mathcal{U}_{s \upharpoonright 1}, \dots, \mathcal{U}_s, \mathcal{V} \rangle) \subseteq \bigcup \mathcal{F}(s, \mathcal{V})$$

Note that there are only countably-many finite collections of basic open sets. Thus we may choose basic open covers  $\mathcal{U}_{s \smallfrown \langle n \rangle}$  for  $n < \omega$  such that for any basic open cover  $\mathcal{V}$ , there exists  $n < \omega$  where  $\mathcal{F}(s, \mathcal{V}) = \mathcal{F}(s, \mathcal{U}_{s \smallfrown \langle n \rangle})$ .

Let  $t : \omega \rightarrow \omega^{<\omega}$  be a bijection. We define the Marköv strategy  $\tau$  as follows:

$$\tau(\langle \mathcal{V} \rangle, n) = \bigcup \mathcal{F}(t(n), \mathcal{V})$$

Suppose there exists a counter-attack  $\langle \mathcal{V}_0, \mathcal{V}_1, \dots \rangle$  of basic open covers which defeats  $\tau$ . Then there exists  $f : \omega \rightarrow \omega$  such that, letting  $t(m_n) = f \upharpoonright n$ :

$$\begin{aligned} x &\notin \tau(\langle \mathcal{V}_{m_n} \rangle, m_n) \\ &= \bigcup \mathcal{F}(f \upharpoonright n, \mathcal{V}_{m_n}) \\ &= \bigcup \mathcal{F}(f \upharpoonright n, \mathcal{U}_{f \upharpoonright (n+1)}) \\ &\supseteq \sigma(\langle \mathcal{U}_{f \upharpoonright 1}, \dots, \mathcal{U}_{f \upharpoonright (n+1)} \rangle) \end{aligned}$$

Thus  $\langle \mathcal{U}_{f \upharpoonright 1}, \mathcal{U}_{f \upharpoonright 2}, \dots \rangle$  is a successful counter-attack by  $\mathcal{C}$  against the perfect information strategy  $\sigma$ . □

**Corollary 6.2.9.** *Let  $X$  be a second-countable space. The following are equivalent:*

- $X$  is  $\sigma$ -relatively-compact
- $F \upharpoonright_{pre} Men_{C,F}(X)$
- $F \upharpoonright_{mark} Men_{C,F}(X)$
- $F \upharpoonright Men_{C,F}(X)$



◇

**Corollary 6.2.10.** *Let  $X$  be a metrizable space. The following are equivalent:*

- $X$  is  $\sigma$ -compact
- $X$  is  $\sigma$ -relatively-compact
- $F \uparrow_{pre} Men_{C,F}(X)$
- $F \uparrow_{mark} Men_{C,F}(X)$
- $F \uparrow Men_{C,F}(X)$

◇

*Proof.* Each property implies Lindelöf, so  $X$  may be assumed to be regular and second-countable. □

### 6.3 Robustly Lindelöf

To help describe  $\mathcal{F} \uparrow_{2\text{-mark}} Men_{C,F}(X)$  topologically, we introduce a subset variant of the Menger game and a related covering property.

**Game 6.3.1.** Let  $Men_{C,F}(X, Y)$  denote the *Menger subspace game* which proceeds analogously to the Menger game, except that  $\mathcal{F}$  wins whenever  $\bigcup_{n < \omega} \mathcal{F}_n$  is a cover for  $Y \subseteq X$ .

◇

Note of course that  $Men_{C,F}(X, X) = Men_{C,F}(X)$ .

**Definition 6.3.2.** A subset  $Y$  of  $X$  is *relatively robustly Menger* if there exist functions  $r_{\mathcal{V}} : Y \rightarrow \omega$  for each open cover  $\mathcal{V}$  of  $X$  such that for all open covers  $\mathcal{U}, \mathcal{V}$  and numbers  $n < \omega$ , the following sets are  $\mathcal{V}$ -finite:

$$c(\mathcal{V}, n) = \{x \in Y : r_{\mathcal{V}}(x) \leq n\}$$

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{x \in Y : n < r_{\mathcal{U}}(x) < r_{\mathcal{V}}(x)\}$$

◇

**Definition 6.3.3.** A space  $X$  is *robustly Menger* if it is relatively robustly Menger to itself.

◇

**Proposition 6.3.4.** All  $\sigma$ -relatively-compact spaces are robustly Menger.

◇

*Proof.* If  $X = \bigcup_{n < \omega} R_n$ , then for all  $\mathcal{U}$ , let  $r_{\mathcal{U}}(x)$  be the least  $n$  such that  $x \in R_n$ . Then  $c(\mathcal{V}, n) = \bigcup_{m \leq n} R_m$  and  $p(\mathcal{U}, \mathcal{V}) = \emptyset$ . □

**Theorem 6.3.5.** If  $Y \subseteq X$  is relatively robustly Menger, then  $\mathcal{F} \uparrow_{2\text{-mark}} \text{Men}_{C,F}(X, Y)$ . ◇

*Proof.* We define the Markov strategy  $\sigma$  as follows. Let  $\sigma(\langle \mathcal{U} \rangle, 0) = c(\mathcal{U}, 0)$ , and let  $\sigma(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) = c(\mathcal{V}, n+1) \cup p(\mathcal{U}, \mathcal{V}, n+1)$ .

For any attack  $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$  by  $\mathcal{C}$  and  $x \in Y$ , one of the following must occur:

- $r_{\mathcal{U}_0}(x) = 0$  and thus  $x \in c(\mathcal{U}_0, 0) \subseteq \sigma(\langle \mathcal{U}_0 \rangle, 0)$ .
- $r_{\mathcal{U}_0}(x) = N+1$  for some  $N \geq 0$  and:

- For all  $n \leq N$ ,

$$r_{\mathcal{U}_{n+1}}(x) \leq N+1$$

and thus  $x \in c(\mathcal{U}_{N+1}, N+1) \subseteq \sigma(\langle \mathcal{U}_N, \mathcal{U}_{N+1} \rangle, N+1)$ .

- For some  $n \leq N$ ,

$$r_{\mathcal{U}_n}(x) \leq n$$

and thus  $x \in c(\mathcal{U}_{n+1}, n+1) \subseteq \sigma(\langle \mathcal{U}_n, \mathcal{U}_{n+1} \rangle, n+1)$ .

- For some  $n \leq N$ ,

$$n < r_{\mathcal{U}_n}(x) \leq N+1 < r_{\mathcal{U}_{n+1}}(x)$$

and thus  $x \in p(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1) \subseteq \sigma(\langle \mathcal{U}_n, \mathcal{U}_{n+1} \rangle, n+1)$

□

**Theorem 6.3.6.**  $S(\kappa, \omega, \omega)$  implies  $\kappa^\dagger$  is robustly Menger, and thus  $\mathcal{F} \underset{2\text{-mark}}{\uparrow} \text{Men}_{C,F}(\kappa^\dagger)$ .

◇

*Proof.* Let  $f_A$  for  $A \in [\kappa]^{\leq \omega}$  witness  $S(\kappa, \omega, \omega)$  and fix  $A(\mathcal{U}) \in [\kappa]^{\leq \omega}$  for each open cover  $\mathcal{U}$  such that  $\kappa^\dagger \setminus A(\mathcal{U})$  is contained in some element of  $\mathcal{U}$ . Then let  $r_{\mathcal{U}}(x) = 0$  for  $x \in \kappa^\dagger \setminus A(\mathcal{U})$ , and  $r_{\mathcal{U}}(\alpha) = f_{A(\mathcal{U})}(\alpha)$  for  $\alpha \in A(\mathcal{U})$ .

It follows that

$$c(\mathcal{U}, n) = (\kappa^\dagger \setminus A(\mathcal{U})) \cup \{\alpha \in A(\mathcal{U}) : f_{A(\mathcal{U})}(\alpha) \leq n\}$$

is  $\mathcal{U}$ -finite,  $\bigcup_{n < \omega} c(\mathcal{U}, n) = X$ , and

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{\alpha \in A(\mathcal{U}) \cap A(\mathcal{V}) : n < f_{A(\mathcal{U})}(\alpha) < f_{A(\mathcal{V})}(\alpha)\}$$

is finite.

□

We may also consider common (non-regular) counterexamples which are finer than the usual Euclidean line.

**Definition 6.3.7.** Let  $R_{\mathbb{Q}}$  be the real line with the topology generated by open intervals with or without the rationals removed.

◇

**Theorem 6.3.8.**  $R_{\mathbb{Q}}$  is non-regular and non- $\sigma$ -compact, but is second-countable and  $\sigma$ -relatively-compact.

◇

*Proof.* Compact sets in  $R_{\mathbb{Q}}$  can be shown to not contain open intervals, and thus are nowhere dense in nonmeager  $\mathbb{R}$ , so  $R_{\mathbb{Q}}$  is not  $\sigma$ -compact. The usual base of intervals with rational endpoints (with or without rationals removed) witnesses second-countability.

To see that  $R_{\mathbb{Q}}$  is  $\sigma$ -relatively compact, consider  $[a, b] \setminus \mathbb{Q}$ . Let  $\mathcal{U}$  be a cover of  $R_{\mathbb{Q}}$ , and let  $\mathcal{U}'$  fill in the missing rationals for any open set in  $\mathcal{U}$ . There is a finite subcover  $\mathcal{V}' \subseteq \mathcal{U}'$  for

$[a, b]$  since  $\mathcal{U}'$  contains open sets from the Euclidean topology. Let  $\mathcal{V} = \{V \setminus \mathbb{Q} : V \in \mathcal{V}'\}$ : this is a finite refinement of  $\mathcal{U}$  covering  $[a, b] \setminus \mathbb{Q}$ , so  $[a, b] \setminus \mathbb{Q}$  is relatively compact. It follows then that  $R_{\mathbb{Q}} \setminus \mathbb{Q}$  is  $\sigma$ -relatively-compact, and since  $\mathbb{Q}$  is countable,  $R_{\mathbb{Q}}$  is  $\sigma$ -relatively-compact. Non-regularity follows since regular and  $\sigma$ -relatively-compact implies  $\sigma$ -compact.  $\square$

**Definition 6.3.9.** Let  $R_{\omega}$  be the real line with the topology generated by open intervals with countably many points removed.  $\diamond$

**Theorem 6.3.10.**  $R_{\omega}$  is non-regular, non-second-countable, and non- $\sigma$ -relatively-compact, but  $\mathcal{F} \uparrow Men_{C,F}(R_{\omega})$ .  $\diamond$

*Proof.* The closure of any open set is its closure in the usual Euclidean topology, so  $R_{\omega}$  is not regular. If  $S \supseteq \{s_n : n < \omega\}$  for  $s_n$  discrete, then  $U_m = R_{\omega} \setminus \{s_n : m < n < \omega\}$  yields an infinite cover  $\{U_m : m < \omega\}$  with no finite subcollection covering  $S$ , showing that all relatively compact sets are finite, and  $R_{\omega}$  is not  $\sigma$ -relatively-compact.

Define the winning strategy  $\sigma$  for  $\mathcal{F}$  in  $Men_{C,F}(R_{\omega})$  as follows: let  $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{2n}) = [-n, n] \setminus C_n$  for some countable  $C_n = \{c_{n,m} : m < \omega\}$ , and let  $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{2n+1}) = \{c_{i,j} : i, j < n\}$ . Non-second-countable follows since second-countable and  $\mathcal{F} \uparrow Men_{C,F}(X)$  implies  $\sigma$ -relatively-compact.  $\square$

We will soon see that, assuming  $S(2^{\omega}, \omega, \omega)$ ,  $\mathcal{F}$  has a winning 2-Marköv strategy for  $Men_{C,F}(R_{\omega})$  as well.

**Proposition 6.3.11.** Let  $\uparrow_{limit}$  be either  $\uparrow_{k\text{-mark}}$  or  $\uparrow$ . If  $X = \bigcup_{i < \omega} X_i$  and  $\mathcal{F} \uparrow_{limit} Men_{C,F}(X, X_i)$  for  $i < \omega$ , then  $\mathcal{F} \uparrow_{limit} Men_{C,F}(X)$   $\diamond$

*Proof.* Let  $L$  be the  $k$ -Markov fog-of-war  $\mu_k$  (resp. the identity), and let  $\sigma_i$  be a  $k$ -Markov strategy (resp. perfect information strategy) for  $\mathcal{F}$  in  $Men_{C,F}(X, X_i)$ .

We define the  $k$ -Markov strategy (resp. perfect information strategy)  $\sigma$  for  $Men_{C,F}(X)$  as follows:

$$\sigma \circ L(\langle \mathcal{U}_0, \dots, \mathcal{U}_n \rangle) = \bigcup_{i \leq n} \sigma_i \circ L(\langle \mathcal{U}_i, \dots, \mathcal{U}_n \rangle)$$

Let  $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$  be a successful counter-attack by  $\mathcal{C}$  against  $\sigma$ . Then there exists  $x \in X_i$  for some  $i < \omega$  such that  $x$  is not covered by  $\bigcup_{n < \omega} \sigma \circ L(\langle \mathcal{U}_0, \dots, \mathcal{U}_n \rangle)$ . It follows that  $x$  is not covered by  $\bigcup_{n < \omega} \sigma_i \circ L(\langle \mathcal{U}_i, \dots, \mathcal{U}_{i+n} \rangle)$ , and  $\langle \mathcal{U}_i, \mathcal{U}_{i+1}, \dots \rangle$  is a successful counter-attack by  $\mathcal{C}$  against  $\sigma_i$ .  $\square$

**Theorem 6.3.12.** *If  $S(2^\omega, \omega, \omega)$ , then  $\mathcal{F} \uparrow_{2\text{-mark}} \text{Men}_{C,F}(R_\omega)$ .*  $\diamond$

*Proof.* It's sufficient to show that  $[0, 1] \subseteq R_\omega$  is relatively robustly Menger. Let  $f_A$  witness  $S(2^\omega, \omega, \omega)$  for  $A \in [[a, b]]^{\leq \omega}$ . For each open cover  $\mathcal{U}$ , let  $A_\mathcal{U}$  be such that  $[0, 1] \setminus A_\mathcal{U}$  is  $\mathcal{U}$ -finite. Let  $r_\mathcal{U}(x) = 0$  if  $x \in [0, 1] \setminus A_\mathcal{U}$  and  $r_\mathcal{U}(x) = f_{A_\mathcal{U}}(x)$  otherwise.

It follows then that

$$c(\mathcal{U}, n) = [0, 1] \setminus \{x \in A_\mathcal{U} : f_{A_\mathcal{U}}(x) > n\}$$

is  $\mathcal{U}$ -finite and

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{x \in A_\mathcal{U} \cap A_\mathcal{V} : n < f_{A_\mathcal{U}}(x) < f_{A_\mathcal{V}}(x)\}$$

is finite.  $\square$

## 6.4 Rothberger game

We conclude this chapter with some results on another variation of the Menger property and game.

**Definition 6.4.1.** A space  $X$  is Rothberger if for every sequence  $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$  of open covers of  $X$  there exists a sequence  $\langle S_0, S_1, \dots \rangle$  such that  $S_n \in \mathcal{U}_n$  and  $\{S_n : n < \omega\}$  is a cover of  $X$ .  $\diamond$

**Game 6.4.2.** Let  $\text{Roth}_{C,S}(X)$  denote the Rothberger game with players  $\mathcal{C}$ ,  $\mathcal{S}$ . In round  $n$ ,  $\mathcal{C}$  chooses an open cover  $\mathcal{U}_n$ , followed by  $\mathcal{S}$  choosing a subset  $S_n$  of  $X$  contained in some open set in  $\mathcal{U}_n$ .  $\mathcal{S}$  wins the game if  $X = \bigcup_{n < \omega} S_n$ , and  $\mathcal{C}$  wins otherwise.  $\diamond$

**Theorem 6.4.3.**  *$X$  is Rothberger if and only if  $\mathcal{C} \nVdash \text{Roth}_{C,S}(X)$ . [19]*

◇

(TODO: cite)

**Theorem 6.4.4.** *Let  $X$  be a compact  $T_2$  space. The following are equivalent:*

- *$X$  is Rothberger*
- *$X$  is scattered*
- *$\mathcal{S} \uparrow \text{Roth}_{C,S}(X)$*
- *$\mathcal{C} \nVdash \text{Roth}_{C,S}(X)$*

◇

Another game exists which is “perfect information equivalent” to the Rothberger game.

**Game 6.4.5.** Let  $\text{Roth}_{P,O}(X)$  denote the *alternate Rothberger game* with players  $\mathcal{P}$ ,  $\mathcal{O}$ . In round  $n$ ,  $\mathcal{P}$  chooses a point  $p_n \in X$ , followed by  $\mathcal{O}$  choosing a neighborhood  $O_n$  of  $p_n$ .  $\mathcal{P}$  wins if  $X = \bigcup_{n < \omega} O_n$ ;  $\mathcal{O}$  wins otherwise.

◇

(TODO: cite Galvin (Indeterminacy of Point Open Games?) )

**Theorem 6.4.6.**  *$\mathcal{P} \uparrow \text{Roth}_{P,O}(X)$  if and only if  $\mathcal{S} \uparrow \text{Roth}_{C,S}(X)$ ;  $\mathcal{O} \uparrow \text{Roth}_{P,O}(X)$  if and only if  $\mathcal{C} \uparrow \text{Roth}_{C,S}(X)$ .*

◇

An analogous relationship exists for the winning limited information strategies.

**Theorem 6.4.7.**  $P \uparrow_{pre} Roth_{P,O}(X)$  if and only if  $S \uparrow_{mark} Roth_{C,S}(X)$ ;  $O \uparrow_{mark} Roth_{P,O}(X)$  if and only if  $C \uparrow_{pre} Roth_{C,S}(X)$ .  $\diamond$

*Proof.* TODO  $\square$

As it turns out,  $\mathcal{S}$  requires a lot of structure on  $X$  in order to guarantee victory with a Markov strategy in the Rothberger game.

**Definition 6.4.8.** TODO: almost countable  $\diamond$

**Theorem 6.4.9.** For any space  $X$ , the following are equivalent:

- $S \uparrow_{mark} Roth_{C,S}(X)$
- $P \uparrow_{pre} Roth_{P,O}(X)$
- $X$  is almost countable

$\diamond$

*Proof.* TODO  $\square$

**Theorem 6.4.10.** For any  $T_1$  space  $X$ , the following are equivalent:

- $S \uparrow_{mark} Roth_{C,S}(X)$
- $P \uparrow_{pre} Roth_{P,O}(X)$
- $X$  is almost countable
- $X$  is countable

$\diamond$

Let  $\kappa \cup \{\infty\}$  be a “weak Lindelöf-ication” of discrete  $\kappa > \omega$  such that open neighborhoods of  $\infty$  contain  $\kappa \setminus \omega$ . This space is an example of an almost countable space which is not countable.

**Theorem 6.4.11.** *The following are equivalent for points- $G_\delta$   $X$ :*

- $S \uparrow \text{Roth}_{C,S}(X)$
- $P \uparrow \text{Roth}_{P,O}(X)$
- $S \xrightarrow{\text{mark}} \text{Roth}_{C,S}(X)$
- $P \xrightarrow{\text{pre}} \text{Roth}_{P,O}(X)$
- $X$  is Rothberger
- $X$  is almost countable
- $X$  is countable

◇

*Proof.* TODO

□

**Corollary 6.4.12.** *The following are equivalent for compact points- $G_\delta$   $X$ :*

- $S \uparrow \text{Roth}_{C,S}(X)$
- $P \uparrow \text{Roth}_{P,O}(X)$
- $S \xrightarrow{\text{mark}} \text{Roth}_{C,S}(X)$
- $P \xrightarrow{\text{pre}} \text{Roth}_{P,O}(X)$
- $C \not\uparrow \text{Roth}_{C,S}(X)$
- $O \not\uparrow \text{Roth}_{P,O}(X)$
- $X$  is scattered
- $X$  is Rothberger
- $X$  is almost countable



- $X$  is countable

◊

## Bibliography

- [1] Matthew H. Baker. Uncountable sets and an infinite real number game. *Mathematics Magazine*, 80(5):377–380, December 2007.
- [2] Jocelyn R. Bell. An infinite game with topological consequences. *Topology Appl.*, 175:1–14, 2014.
- [3] Steven Clontz and Gary Gruenhage. Proximal compact spaces are Corson compact. *Topology Appl.*, 173:1–8, 2014.
- [4] Gabriel Debs. Stratégies gagnantes dans certains jeux topologiques. *Fund. Math.*, 126(1):93–105, 1985.
- [5] Ryszard Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.
- [6] David Gale and F. M. Stewart. Infinite games with perfect information. In *Contributions to the theory of games, vol. 2*, Annals of Mathematics Studies, no. 28, pages 245–266. Princeton University Press, Princeton, N. J., 1953.
- [7] Fred Galvin and Ratislav Telgársky. Stationary strategies in topological games. *Topology Appl.*, 22(1):51–69, 1986.
- [8] Gary Gruenhage. Infinite games and generalizations of first-countable spaces. *General Topology and Appl.*, 6(3):339–352, 1976.
- [9] Gary Gruenhage. Covering properties on  $X^2 \setminus \Delta$ ,  $W$ -sets, and compact subsets of  $\Sigma$ -products. *Topology Appl.*, 17(3):287–304, 1984.
- [10] R. C. Haworth and R. A. McCoy. Baire spaces. *Dissertationes Math. (Rozprawy Mat.)*, 141:73, 1977.
- [11] Witold Hurewicz. Über eine Verallgemeinerung des Borelschen Theorems. *Math. Z.*, 24(1):401–421, 1926.
- [12] Kenneth Kunen. *Set theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-New York, 1980. An introduction to independence proofs.
- [13] H. P. A. Künzi, S. Romaguera, and M. A. Sánchez-Granero. On uniformly locally compact quasi-uniform hyperspaces. *Czechoslovak Math. J.*, 54(129)(1):215–228, 2004.

- [14] Donald A. Martin. Borel determinacy. *Ann. of Math. (2)*, 102(2):363–371, 1975.
- [15] Karl Menger. Einige überdeckungssätze der punktmengenlehre. *Sitzungsberichte der Wiener Akademie*, 133:421–444, 1924.
- [16] Jan Mycielski and H. Steinhaus. A mathematical axiom contradicting the axiom of choice. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 10:1–3, 1962.
- [17] Peter J. Nyikos. Classes of compact sequential spaces. In *Set theory and its applications (Toronto, ON, 1987)*, volume 1401 of *Lecture Notes in Math.*, pages 135–159. Springer, Berlin, 1989.
- [18] Peter J. Nyikos. Proximal and semi-proximal spaces (preprint). 2013.
- [19] Janusz Pawlikowski. Undetermined sets of point-open games. *Fund. Math.*, 144(3):279–285, 1994.
- [20] Marion Scheepers. Concerning  $n$ -tactics in the countable-finite game. *J. Symbolic Logic*, 56(3):786–794, 1991.
- [21] Marion Scheepers. Meager-nowhere dense games. I.  $n$ -tactics. *Rocky Mountain J. Math.*, 22(3):1011–1055, 1992.
- [22] Marion Scheepers. A direct proof of a theorem of Telgársky. *Proc. Amer. Math. Soc.*, 123(11):3483–3485, 1995.
- [23] Rastislav Telgársky. On games of Topsøe. *Math. Scand.*, 54(1):170–176, 1984.
- [24] Rastislav Telgársky. Topological games: on the 50th anniversary of the Banach-Mazur game. *Rocky Mountain J. Math.*, 17(2):227–276, 1987.