

# TACTIC-PROXIMAL COMPACT SPACES ARE STRONG EBERLEIN COMPACT

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ABSTRACT. The author and G. Gruenhage previously showed that J. Bell's proximal game may be used to characterize Corson compactness in compact Hausdorff spaces. Using tactical strategies considering only the most recent move of the opponent, the proximal game may also be used to characterize the strong Eberlein compactness property. In doing so, a purely topological characterization of the proximal game is introduced, and several existing results on the proximal game are given analogues considering limited information strategies.

Two papers published in 2014 introduced the *proximal uniform space game*  $Bell_{D,P}^{\text{uni}}(X)$  due to Jocelyn Bell. If  $X$  is a topological space, and there exists a uniform structure inducing its topology which gives the first player in this game has a winning strategy, then  $X$  is said to be a *proximal* space. Bell used this game as a tool in [1] for investigating uniform box products, and the author showed with Gary Gruenhage in [2] that this game characterizes Corson compactness amongst compact Hausdorff spaces, answering a question of Peter Nyikos in [6].

All spaces in this paper are assumed to be  $T_{3\frac{1}{2}}$ , so that they have a uniform structure inducing the topology on the space. Unlike many game-theoretic topological properties, the game  $Bell_{D,P}^{\text{uni}}(X)$  for which the proximal property was defined by is not itself a topological game. However, by considering entourages of the universal uniformity inducing the topology of a space, this original uniform space game may be trivially modified to the purely topological game  $Bell_{D,P}^{\rightarrow}(X)$ .

The aim of this paper is to use limited information strategies in this topological interpretation of the proximal game to give a new game-theoretic characterization of the strong Eberlein compactness property. Strong Eberlein compacts are Corson compacts, and therefore proximal compact spaces; in fact, it will be shown that strong Eberlein compacts are exactly the compact spaces for which the first player has a *tactical* winning strategy for the proximal game, a strategy which relies on only the most recent move of the opponent. This is the only known purely game-theoretic internal characterization of strong Eberlein compactness among compact spaces.

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1. TOPOLOGIZING  $Bell_{D,P}^{\text{uni}}(X)$ 

We refer to [2] for definitions, notation, and basic theorems on uniform spaces and the proximal game  $Bell_{D,P}^{\text{uni}}(X)$  (denoted as  $Prox_{D,P}(X)$  in that paper). In particular recall that:

**Definition 1.1.**  $\mathcal{P} \uparrow G$  denotes that the player  $\mathcal{P}$  has a winning strategy in the  $\omega$ -length game  $G$ .

**Game 1.2.** Let  $Bell_{D,P}^{\text{uni}}(X, \mathbb{D})$  denote the *proximal uniform space game* with players  $\mathcal{D}$ ,  $\mathcal{P}$  which proceeds as follows for a space  $X$  with uniformity  $\mathbb{D}$ . In round 0,  $\mathcal{D}$  chooses an entourage  $D_0 \in \mathbb{D}$ , followed by  $\mathcal{P}$  choosing a point  $p_0 \in X$ . In round  $n+1$ ,  $\mathcal{D}$  chooses an entourage  $D_{n+1} \in \mathbb{D}$ , followed by  $\mathcal{P}$  choosing a point  $p_{n+1} \in D_n[p_n]$ .

$\mathcal{D}$  wins in the case that either  $\langle p_0, p_1, \dots \rangle$  converges in  $X$ , or  $\bigcap_{n < \omega} D_n[p_n] = \emptyset$ .  $\mathcal{P}$  wins otherwise.

**Definition 1.3.** A uniformizable space  $X$  is *proximal* in the case that there exists a compatible uniformity  $\mathbb{D}$  for  $X$  such that  $\mathcal{D} \uparrow Bell_{D,P}^{\text{uni}}(X, \mathbb{D})$ .

The aim of this section is to give an alternate characterization of proximal spaces using a slight variation of  $Bell_{D,P}^{\text{uni}}(X, \mathbb{D})$  which is purely topological. While the proof is trivial, the following definitions and theorems will be useful in investigating the main result.

**Definition 1.4.** The *universal uniformity* for a uniformizable topology is the union of all uniformities which induce the given topology.

**Theorem 1.5** ([7]). *The universal uniformity is itself a uniformity compatible with its given topology.*

**Theorem 1.6** ([7]). *For every uniformizable space, if  $D$  is a neighborhood of the diagonal  $\Delta$  such that there exist neighborhoods  $D_n$  of  $\Delta$  with  $D \supseteq D_0$  and  $D_n \supseteq D_{n+1} \circ D_{n+1}$ , then  $D$  is an entourage of the universal uniformity.*

**Definition 1.7.** For every entourage  $D$  and  $n < \omega$ , let  $\frac{1}{2^n}D$  denote an entourage such that  $\frac{1}{1}D = D$  and  $\frac{1}{2^{n+1}}D \circ \frac{1}{2^{n+1}}D \subseteq \frac{1}{2^n}D$ .

**Definition 1.8.** An *open symmetric entourage*  $D$  is a entourage which is open in the product topology induced by the uniformity and where  $D = D^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in D\}$ .

**Proposition 1.9.** *For every entourage  $D$ , there exists an open symmetric entourage  $U \subseteq D$ .*

Due to this theorem, we will simply use the word *entourage* to refer to open symmetric entourages of the universal uniformity for a given topological space. Note that if  $D$  is an entourage, then  $D[x] = \{y : \langle x, y \rangle \in D\}$  is an open neighborhood of  $x$ . One may consider  $D[x]$  to be an entourage-“ball” about  $x$ , generalizing the notion of an  $\epsilon$ -ball given by a metric structure.

In the case that the space is paracompact, entourages are very easily described.

**Theorem 1.10** ([7]). *Every open symmetric neighborhood of the diagonal is an entourage for paracompact uniformizable spaces.*

By replacing the entourages of  $\mathbb{D}$  in  $Bell_{D,P}^{\text{uni}}(X, \mathbb{D})$  with the (open symmetric) entourages of the universal uniformity, we obtain a purely topological game.

**Game 1.11.** Let  $Bell_{D,P}^{\rightarrow,*}(X)$  denote *Bell's hard convergence game* with players  $\mathcal{D}$ ,  $\mathcal{P}$  which proceeds as follows for a uniformizable space  $X$ . In round 0,  $\mathcal{D}$  chooses an entourage  $D_0$ , followed by  $\mathcal{P}$  choosing a point  $p_0 \in X$ . In round  $n+1$ ,  $\mathcal{D}$  chooses an entourage  $D_{n+1}$ , followed by  $\mathcal{P}$  choosing a point  $p_{n+1} \in D_n[p_n]$ .

$\mathcal{D}$  wins in the case that either  $\langle p_0, p_1, \dots \rangle$  converges in  $X$ , or  $\bigcap_{n < \omega} D_n[p_n] = \emptyset$ .  $\mathcal{P}$  wins otherwise.

To be clear, there's practically no difference between  $Bell_{D,P}^{\rightarrow,*}(X)$  and  $Bell_{D,P}^{\text{uni}}(X, \mathbb{D})$  where  $\mathbb{D}$  is the universal uniformity; any choice by  $\mathcal{D}$  in  $Bell_{D,P}^{\text{uni}}(X, \mathbb{D})$  may be improved to an open symmetric choice for  $\mathcal{D}$  in either game.

This game is labeled as “hard” due to the requirement that  $\mathcal{D}$  keep track of the entire history of the game if she wishes to ensure that successive moves refine previous moves. Put another way, any perfect information strategy  $\sigma$  may be improved to a better perfect information strategy  $\tau$  by letting  $\tau(s) = \bigcap_{t \leq s} \sigma(t)$  (where  $t \leq s$  means  $t$  is a subsequence of  $s$ ).

In the study of limited information strategies, it's preferable to consider games for which the opponent must respect all previous moves of the player restricted to limited information in the first place.

**Game 1.12.** Let  $Bell_{D,P}^{\rightarrow}(X)$  denote *Bell's convergence game* with players  $\mathcal{D}$ ,  $\mathcal{P}$  which proceeds analogously to  $Bell_{D,P}^{\rightarrow,*}(X)$ , except for the following. Let  $E_n = \bigcap_{m \leq n} D_m$ , where  $D_n$  is the entourage played by  $\mathcal{D}$  in round  $n$ . Then  $\mathcal{P}$  must ensure that  $p_{n+1} \in E_n[p_n]$ , and  $\mathcal{D}$  wins when either  $\langle p_0, p_1, \dots \rangle$  converges in  $X$  or  $\bigcap_{n < \omega} E_n[p_n] = \emptyset$ .

The proximal property may now be characterized using either game.

**Theorem 1.13.**  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow,*}(X)$  if and only if  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$  if and only if  $X$  is proximal.

*Proof.* If  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow,*}(X)$ , then we immediately see that  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$ . If  $\sigma$  is a winning strategy for  $\mathcal{D}$  in  $Bell_{D,P}^{\rightarrow}(X)$ , then  $\tau$  defined by  $\tau(s) = \bigcap_{t \leq s} \sigma(t)$  is easily seen to be a winning strategy for  $\mathcal{D}$  in  $Bell_{D,P}^{\rightarrow,*}(X)$ .

If  $\mathcal{D} \uparrow Bell_{D,P}^{\rightarrow,*}(X)$ , then  $\mathcal{D} \uparrow Bell_{D,P}^{\text{uni}}(X, \mathbb{D})$  where  $\mathbb{D}$  is the universal uniformity, showing  $X$  is proximal. Finally, if  $X$  is proximal, then there exists a winning strategy  $\sigma$  for  $Bell_{D,P}^{\text{uni}}(X)$  for a uniformity  $\mathbb{D}$  inducing the topology on  $X$ . Then a winning strategy for  $\mathcal{D}$  in  $Bell_{D,P}^{\rightarrow,*}(X)$  may be constructed by converting every entourage in this uniformity into a smaller open symmetric entourage of  $\mathbb{D}$ , and the result follows as  $\mathbb{D}$  is contained in the universal uniformity.  $\square$

The proximal property may now be considered without familiarity of the theory of uniform spaces: in either game,  $\mathcal{D}$  is simply choosing open symmetric neighborhoods of the diagonal with the property in Theorem 1.6 (a property always present when the space is paracompact).

**1.1. Absolutely proximal.** The secondary winning condition in  $Bell_{\vec{D},P}^\rightarrow(X)$  allows for a space to be incomplete:  $\mathcal{D} \uparrow Bell_{\vec{D},P}^\rightarrow(\mathbb{Q})$  by playing  $V_{1/2^n} = \{\langle x, y \rangle : d(x, y) < \frac{1}{2^n}\}$  in each round. This forces  $\mathcal{P}$ 's sequence to be Cauchy, and thus it either converges to a rational, or the sets  $V_{1/2^n}[x_n]$  will have empty intersection (where the irrational point of convergence would be). Uniformly locally compact spaces (and in particular, compact spaces) lack such holes, so it will be convenient to eliminate this technicality when it is irrelevant.

**Game 1.14.** Let  $Bell_{\vec{D},P}^\rightarrow(X)$  denote the *Bell absolute convergence game* which proceeds analogously to  $Bell_{\vec{D},P}^\rightarrow(X)$ , except that  $\mathcal{D}$  must always ensure that  $\langle p_0, p_1, \dots \rangle$  converges in  $X$  in order to win.

**Definition 1.15.** A uniformizable space  $X$  is *absolutely proximal* if  $\mathcal{D} \uparrow Bell_{\vec{D},P}^\rightarrow(X)$ .

As was shown in [2]:

**Definition 1.16.** A uniformizable space  $X$  is *uniformly locally compact* if there exists an entourage  $D$  such that  $\overline{D[x]}$  is compact for all  $x$ .

**Theorem 1.17.** If  $X$  is a uniformly locally compact space, then  $\mathcal{D} \uparrow Bell_{\vec{D},P}^\rightarrow(X)$  if and only if  $\mathcal{D} \uparrow Bell_{\vec{D},P}^\rightarrow(X)$ .

So all proximal compacts are absolutely proximal compacts.

## 2. LIMITED INFORMATION ANALOGUES

**Definition 2.1.** A *k-tactical strategy* or *k-tactic* is a strategy which considers only the most recent move of the opponent.

If  $\mathcal{P}$  has a winning *k-tactic* for a game  $G$ , we write  $\mathcal{P} \uparrow_{k\text{-tact}} G$ . If omitted, assume  $k = 1$ .

**Definition 2.2.** A *k-Markov strategy* or *k-mark* is a strategy which considers only the round number and most recent move of the opponent.

If  $\mathcal{P}$  has a winning *k-mark* for a game  $G$ , we write  $\mathcal{P} \uparrow_{k\text{-mark}} G$ . If omitted, assume  $k = 1$ .

Limited information strategies may be used to strengthen game-theoretic topological properties. Some results of Bell may be generalized to hold for such limited information strategies. The proofs of the following propositions are straightforward.

**Proposition 2.3.** Let  $X$  be a uniformizable space and  $H$  be a closed subset of  $X$ . If  $k < \omega$ , then

- $\mathcal{D} \uparrow_{k\text{-tact}} Bell_{\vec{D},P}^\rightarrow(X) \Rightarrow \mathcal{D} \uparrow_{k\text{-tact}} Bell_{\vec{D},P}^\rightarrow(H)$
- $\mathcal{D} \uparrow_{k\text{-mark}} Bell_{\vec{D},P}^\rightarrow(X) \Rightarrow \mathcal{D} \uparrow_{k\text{-mark}} Bell_{\vec{D},P}^\rightarrow(H)$
- $\mathcal{D} \uparrow Bell_{\vec{D},P}^\rightarrow(X) \Rightarrow \mathcal{D} \uparrow Bell_{\vec{D},P}^\rightarrow(H)$

**Proposition 2.4.** Let  $X$  be a uniformizable space and  $H$  be a closed subset of  $X$ . If  $k < \omega$ , then

- $\mathcal{D} \uparrow Bell_{\vec{D},P}(X) \Rightarrow \mathcal{D} \uparrow Bell_{\vec{D},P}(H)$
- $\mathcal{D} \xrightarrow{k-tact} \uparrow Bell_{\vec{D},P}(X) \Rightarrow \mathcal{D} \xrightarrow{k-tact} \uparrow Bell_{\vec{D},P}(H)$
- $\mathcal{D} \xrightarrow{k-mark} \uparrow Bell_{\vec{D},P}(X) \Rightarrow \mathcal{D} \xrightarrow{k-mark} \uparrow Bell_{\vec{D},P}(H)$

Bell observed a relationship between her game and another convergence game due to Gruenhage.

**Game 2.5.** Let  $Gru_{\vec{O},P}(X, S)$  denote *Gruenhage's convergence game* with players  $\mathcal{O}$ ,  $\mathcal{P}$  which proceeds as follows for a space  $X$  and a set  $S$ . In round  $n$ ,  $\mathcal{O}$  chooses an open set  $U_n$  containing  $S$ , followed by  $\mathcal{P}$  choosing a point  $p_n \in \bigcap_{m \leq n} U_m$ .

$\mathcal{O}$  wins in the case that  $\langle p_0, p_1, \dots \rangle$  converges to the set  $S$ ; that is, any open set containing  $S$  contains all but finite  $p_n$ .  $\mathcal{P}$  wins otherwise.

When  $S = \{x\}$ , we abuse notation and write  $Gru_{\vec{O},P}(X, x)$ .

**Definition 2.6.** Let  $A$  be any set,  $n < \omega$ , and  $f \in A^n$ .  $f \upharpoonright k \in A^{\min(k,n)}$  is the *restriction* of  $f$  to  $k$ , yielding the first  $k$  terms of  $f$  when  $k < n$  (and otherwise equaling  $f$  itself).  $f \downharpoonright k \in A^{\min(k,n)}$  is the *suffix* of  $f$  to  $k$ , yielding the last  $k$  terms of  $f$  when  $k < n$  (and otherwise equaling  $f$  itself).

**Definition 2.7.** Let  $A$  be any set,  $n \leq \omega$ , and  $f, g \in A^n$ .  $f \frown g$  is the *concatenation* of  $f$  and  $g$ , so  $f \frown g(i) = f(i)$  and  $f \frown g(n+i) = g(i)$  for  $i < n$ .  $f \bowtie g$  is the *zip* of  $f$  and  $g$ , so  $f \bowtie g(2i) = f(i)$  and  $f \bowtie g(2i+1) = g(i)$  for  $i < n$ .

**Theorem 2.8.** Let  $k < \omega$ . For all  $x \in X$ :

- $\mathcal{D} \uparrow Bell_{\vec{D},P}(X) \Rightarrow \mathcal{O} \uparrow Gru_{\vec{O},P}(X, x)$
- $\mathcal{D} \xrightarrow{2k-tact} \uparrow Bell_{\vec{D},P}(X) \Rightarrow \mathcal{O} \xrightarrow{k-tact} \uparrow Gru_{\vec{O},P}(X, x)$
- $\mathcal{D} \xrightarrow{2k-mark} \uparrow Bell_{\vec{D},P}(X) \Rightarrow \mathcal{O} \xrightarrow{k-mark} \uparrow Gru_{\vec{O},P}(X, x)$

*Proof.* The perfect-information result was proven in Bell's original paper [1]. We proceed by proving the Markov-information result, and note that the tactical-information result may be proven by simply dropping usage of the round number from the given proof.

Let  $\sigma$  be a winning  $2k$ -mark for  $\mathcal{D}$  in  $Bell_{\vec{D},P}(X)$ . Let  $\chi_n \in X^n$  have constant value  $x$  for  $n \leq \omega$ . We define the  $k$ -mark  $\tau$  for  $\mathcal{O}$  in  $Gru_{\vec{O},P}(X, x)$  such that

$$\tau(t, n) = \sigma(\chi_{|t|} \bowtie t, 2n)[x] \cap \sigma(\langle x \rangle \frown (t \bowtie \chi_{|t|}) \upharpoonright 2k, 2n+1)[x]$$

Let  $p$  be a legal attack against  $\tau$  in  $Gru_{\vec{O},P}(X, x)$ . Consider the attack  $q = \chi_\omega \bowtie p$  against the winning  $2k$ -mark  $\sigma$  in  $Bell_{\vec{D},P}(X)$ . Let  $D_n = \sigma(q \upharpoonright n \downharpoonright 2k, n)$  be the entourage played according to  $\sigma$  in round  $n$ , and  $E_n = \bigcap_{m \leq n} D_m$ .

Certainly,  $x \in E_{2n}[x] = E_{2n}[q(2n)]$  for any  $n < \omega$ . Note also for any  $n < \omega$  that

$$\begin{aligned} p(n) &\in \bigcap_{m \leq n} \tau(p \upharpoonright m \downharpoonright k, m) \\ &= \bigcap_{m \leq n} \left( \sigma(\chi_{\min(m,k)} \bowtie (p \upharpoonright m \downharpoonright k), 2m)[x] \cap \sigma(\langle x \rangle \frown ((p \upharpoonright m \downharpoonright k) \bowtie \chi_{\chi_{\min(m,k)}}) \upharpoonright 2k, 2m+1)[x] \right) \end{aligned}$$

$$\begin{aligned}
&= \bigcap_{m \leq n} \left( \sigma(q \upharpoonright 2m \downharpoonright 2k, 2m)[x] \cap \sigma(q \upharpoonright 2m+1 \downharpoonright 2k, 2m+1)[x] \right) \\
&= \bigcap_{m \leq n} (D_{2m}[x] \cap D_{2m+1}[x]) = \bigcap_{m \leq 2n+1} D_m[x] = E_{2n+1}[x]
\end{aligned}$$

so by the symmetry of  $E_{2n+1}$ ,

$$q(2n+2) = x \in E_{2n+1}[p(n)] = E_{2n+1}[q(2n+1)]$$

and

$$q(2n+1) = p(n) \in E_{2n+1}[x] \subseteq E_{2n}[x] = E_{2n}[q(2n)]$$

making  $q$  a legal attack against  $\sigma$ .

Then as  $x \in \bigcap_{n < \omega} E_n[q(n)] \neq \emptyset$ , and  $\sigma$  is a winning strategy, the attack  $q$  converges. Since  $q(2n) = x$ ,  $q$  must converge to  $x$ . Thus its subsequence  $p$  converges to  $x$ , and  $\tau$  is a winning  $k$ -mark for  $\mathcal{O}$  in  $\text{Gru}_{\mathcal{O},P}^{\rightarrow}(X, x)$ .  $\square$

### 2.1. Products and strengthenings of compactness.

**Definition 2.9.** Let  $X_\alpha$  be a topological space for  $\alpha < \kappa$  and  $z \in \prod_{\alpha < \kappa} X_\alpha$ . The  $\Sigma$ -product  $\sum_{\alpha < \kappa}^z X_\alpha$  with base point  $z$  is given by

$$\sum_{\alpha < \kappa}^z X_\alpha = \left\{ x \in \prod_{\alpha < \kappa} X_\alpha : |\{\alpha < \kappa : x(\alpha) \neq z(\alpha)\}| \leq \omega \right\}$$

When  $X_\alpha = X$  and  $z(\alpha) = 0$  for all  $\alpha < \kappa$ , then we write  $\Sigma X^\kappa = \sum_{\alpha < \kappa}^z X$ .

**Theorem 2.10** ([1]). *If  $X_\alpha$  is proximal for all  $\alpha < \kappa$ , then  $\sum_{\alpha < \kappa}^z X_\alpha$  is proximal for any base point  $z$ .*

It's this result which yielded the following observation by Nyikos in [5].

**Definition 2.11.** A *Corson compact* space is a compact space which may be embedded in  $\Sigma \mathbb{R}^\kappa$ .

**Corollary 2.12.** *Corson compacts are proximal.*

Nyikos asked if the result could be reversed, which the author answered with Gary Gruenhage in [2] in the affirmative.

**Theorem 2.13.** *A compact space  $X$  is Corson compact if and only if it is proximal.*

This result required an earlier game characterization of Corson compactness due to Gruenhage.

**Theorem 2.14** ([3]). *A compact space  $X$  is Corson compact if and only if  $\mathcal{O} \upharpoonright \text{Gru}_{\mathcal{O},P}^{\rightarrow}(X^2, \Delta)$ .*

So Corson compactness is characterized by both  $\mathcal{O} \upharpoonright \text{Gru}_{\mathcal{O},P}^{\rightarrow}(X^2, \Delta)$  and  $\mathcal{D} \upharpoonright \text{Bell}_{\mathcal{D},P}^{\rightarrow}(X)$  for compact spaces. However, we will see that these games are not equivalent with respect to limited information strategies.

In the study of limited information strategies in  $\text{Bell}_{\mathcal{D},P}^{\rightarrow}(X)$ , it is natural to consider  $\Sigma^*$ - and  $\sigma$ -products in place of  $\Sigma$ -products.

## 2.2. $\Sigma^*$ -products and Eberlein compacts.

**Definition 2.15.** Let  $X$  be a metrizable space with compatible metric  $d$ , and let  $z \in X^\kappa$ . The  $\Sigma^*$ -product  $\sum_{\alpha < \kappa}^z X$  with base point  $z$  is given by

$$\sum_{\alpha < \kappa}^z X = \left\{ x \in \prod_{\alpha < \kappa} X : n < \omega \Rightarrow \left| \left\{ \alpha < \kappa : d(x(\alpha), z(\alpha)) > \frac{1}{2^n} \right\} \right| < \omega \right\}$$

When  $z(\alpha) = 0$  for all  $\alpha < \kappa$ , then we write  $\Sigma^* X^\kappa = \sum_{\alpha < \kappa}^z X$ .

Just as  $\Sigma$  products preserve winning perfect-information strategies, we will show that  $\Sigma^*$  products preserve winning Markov strategies.

**Definition 2.16.** For a metric space  $\langle X, d \rangle$ , let  $V_\epsilon = \{\langle x, y \rangle : d(x, y) < \epsilon\}$ . Note that  $V_\epsilon$  is an (open symmetric) entourage on  $X$ .

Note that by definition, a 0-Markov strategy uses zero moves of the opponent, and only considers the round number.

**Proposition 2.17.** For any metrizable space  $X$ ,  $\mathcal{D} \uparrow_{0\text{-mark}} Bell_{D,P}^\rightarrow(X)$ . For any completely metrizable space  $X$ ,  $\mathcal{D} \uparrow_{0\text{-mark}} Bell_{D,P}^\rightarrow(X)$ .

*Proof.* Essentially shown by Bell in her paper: in either case,  $\mathcal{D}$  chooses  $V_{1/2^n}$  during round  $n$ , forcing legal attacks by  $\mathcal{P}$  to be Cauchy.  $\square$

As an aside, this next proposition may be proved similarly using  $V_{d(x,y)/2}$ .

**Proposition 2.18.** For any metrizable space  $X$ ,  $\mathcal{D} \uparrow_{2\text{-tact}} Bell_{D,P}^\rightarrow(X)$ . For any completely metrizable space  $X$ ,  $\mathcal{D} \uparrow_{2\text{-tact}} Bell_{D,P}^\rightarrow(X)$ .

We will exploit the last move of the opponent to obtain winning Markov strategies under  $\Sigma^*$  products.

**Proposition 2.19.** If  $X_\alpha$  is a uniformizable space for  $\alpha < \kappa$ , and  $D_\alpha$  is an entourage of  $X_\alpha$  for  $\alpha \in F \in [\kappa]^{<\omega}$ , then

$$P(\{\langle \alpha, D_\alpha \rangle : \alpha \in F\}) = \left\{ \langle x, y \rangle \in \left( \prod_{\alpha < \kappa} X_\alpha \right)^2 : \alpha \in F \Rightarrow \langle x(\alpha), y(\alpha) \rangle \in D_\alpha \right\}$$

is an entourage of  $\prod_{\alpha < \kappa} X_\alpha$ .

**Theorem 2.20.** For any metrizable space  $X$  and  $z \in X^\kappa$ ,  $\mathcal{D} \uparrow_{\text{mark}} Bell_{D,P}^\rightarrow(\sum_{\alpha < \kappa}^z X)$ .

*Proof.* For  $x \in \sum_{\alpha < \kappa}^z X$ , let

$$\text{supp}_n(x) = \left\{ \alpha < \kappa : d(x(\alpha), z(\alpha)) > \frac{1}{2^n} \right\} \in \kappa^{<\omega}$$

where  $d$  is a metric compatible with the topology on  $X$ .

Define a strategy  $\tau$  for  $\mathcal{D}$  by

$$\begin{aligned}\tau(\emptyset, 0) &= \left( \sum_{\alpha < \kappa}^z X \right)^2 \\ \tau(\langle p \rangle, n+1) &= \left( \sum_{\alpha < \kappa}^z X \right)^2 \cap P(\{\langle \alpha, V_{1/2^{n+1}} \rangle : \alpha \in \text{supp}_n(p)\})\end{aligned}$$

Let  $a : \omega \rightarrow \sum_{\alpha < \kappa}^z X$  be a legal attack by  $\mathcal{P}$  against  $\tau$ , so

$$\begin{aligned}\langle a(n+1), a(n+2) \rangle &\in \bigcap_{m \leq n} \tau(\langle a(m) \rangle, m+1) \\ &= \left( \sum_{\alpha < \kappa}^z X \right)^2 \cap \bigcap_{m \leq n} P(\{\langle \alpha, V_{1/2^{m+1}} \rangle : \alpha \in \text{supp}_m(a(m))\})\end{aligned}$$

and let  $\alpha < \kappa$ .

If  $d(a(m+1)(\alpha), z(\alpha)) \leq \frac{1}{2^m}$  for all but finite  $m < \omega$ , then  $\lim_{n < \omega} a(n)(\alpha) = z(\alpha)$ . Otherwise, we have  $\alpha \in \text{supp}_m(a(m))$  for infinitely many  $m < \omega$ .

If  $\bigcap_{n < \omega} \tau(\langle a(n) \rangle, n+1)[a(n+1)] = \emptyset$ , then  $\mathcal{D}$  has already won. Otherwise there exists a nonstrictly increasing unbounded function  $f \in \omega^\omega$  with  $\bigcap_{n < \omega} V_{1/2^{f(n)+1}}[a(n+1)(\alpha)] \neq \emptyset$ . Since the diameter of these sets approaches 0, the intersection is a singleton  $\{x\}$ ; furthermore, the  $\alpha$  coordinate of  $a$  must then converge to  $x$ . We conclude that  $\mathcal{D} \uparrow_{\text{mark}} \text{Bell}_{D,P}^{\rightarrow}(\sum_{\alpha < \kappa}^z X)$ .  $\square$

Note that we may obtain an analogous observation to Corollary 2.12.

**Definition 2.21.** An *Eberlein compact* space is a compact space which may be embedded in  $\Sigma^* \mathbb{R}^\kappa$ .

**Corollary 2.22.** *Eberlein compacts are Markov-proximal.*

It's natural to ask (particularly considering the results of the next section) whether this too may be reversed.

**Question 2.23.** Must all Markov-proximal compacts be Eberlein compact?

One might suspect that the answer is yes in light of the following fact:

**Theorem 2.24** ([4]). *A compact space  $X$  is Eberlein compact if and only if  $\mathcal{O} \uparrow_{\text{mark}} \text{Gru}_{O,P}^{\rightarrow}(X^2, \Delta)$ .*

### 2.3. $\sigma$ -products and strong Eberlein compacts.

**Definition 2.25.** Let  $X_\alpha$  be a topological space for  $\alpha < \kappa$  and  $z \in \prod_{\alpha < \kappa} X_\alpha$ . The  $\sigma$ -product  $\sigma_{\alpha < \kappa}^z X_\alpha$  with base point  $z$  is given by

$$\sigma_{\alpha < \kappa}^z X_\alpha = \left\{ x \in \prod_{\alpha < \kappa} X_\alpha : |\{\alpha < \kappa : x(\alpha) \neq z(\alpha)\}| < \omega \right\}$$

When  $X_\alpha = X$  and  $z(\alpha) = 0$  for all  $\alpha < \kappa$ , then we write  $\sigma X^\kappa = \sigma_{\alpha < \kappa}^z X$ .



The reader may anticipate our next goal:  $\Sigma$ -products preserve winning perfect information strategies, and  $\Sigma^*$ -products preserve winning Markov strategies, so we aim to show that  $\sigma$ -products preserve winning tactical strategies.

**Theorem 2.26.** *Let  $X_\alpha$  be a uniformizable space for  $\alpha < \kappa$  and  $z \in \prod_{\alpha < \kappa} X_\alpha$ . If  $\mathcal{D} \uparrow_{tact} Bell_{D,P}^{\rightarrow}(X_\alpha)$  for all  $\alpha < \kappa$ , then  $\mathcal{D} \uparrow_{tact} Bell_{D,P}^{\rightarrow}(\sigma_{\alpha < \kappa}^z X_\alpha)$ .*

*Proof.* For  $x \in \sigma_{\alpha < \kappa}^z X_\alpha$ , let

$$\text{supp}(x) = \{\alpha < \kappa : x(\alpha) \neq z(\alpha)\} \in \kappa^{<\omega}$$

and let  $\tau_\alpha$  be a winning strategy for  $\mathcal{D}$  in  $Bell_{D,P}^{\rightarrow}(X_\alpha)$  for  $\alpha < \kappa$ .

Define a strategy  $\tau$  for  $\mathcal{D}$  by

$$\begin{aligned} \tau(\emptyset) &= \left( \sigma_{\alpha < \kappa}^z X \right)^2 \\ \tau(\langle p \rangle) &= \left( \sigma_{\alpha < \kappa}^z X \right)^2 \cap P(\{\langle \alpha, \tau_\alpha(p(\alpha)) \cap \tau_\alpha(z(\alpha)) \rangle : \alpha \in \text{supp}(p)\}) \end{aligned}$$

Let  $a : \omega \rightarrow \sigma_{\alpha < \kappa}^z X$  be a legal attack by  $\mathcal{P}$  against  $\tau$ , so

$$\begin{aligned} \langle a(n+1), a(n+2) \rangle &\in \bigcap_{m \leq n} \tau(\langle a(m) \rangle) \\ &= \left( \sigma_{\alpha < \kappa}^z X \right)^2 \cap \bigcap_{m \leq n} P(\{\langle \alpha, \tau_\alpha(a(m)(\alpha)) \cap \tau_\alpha(z(\alpha)) \rangle : \alpha \in \text{supp}(a(m))\}) \end{aligned}$$

and let  $\alpha < \kappa$ . Let  $a_\alpha : \omega \rightarrow X_\alpha$  satisfy  $a_\alpha(n) = a(n)(\alpha)$ .

Of course if  $a_\alpha(n) = z(\alpha)$  for all  $n < \omega$ , then  $a_\alpha \rightarrow z(\alpha)$ . Otherwise, let  $N < \omega$  be minimal with  $a_\alpha(N) \neq z(\alpha)$ . Then for  $n \geq N$ ,

$$\begin{aligned} \langle a_\alpha(n+1), a_\alpha(n+2) \rangle &= \langle a(n+1)(\alpha), a(n+2)(\alpha) \rangle \\ &\in \bigcap_{m \leq n, \alpha \in \text{supp}(a(m))} \tau_\alpha(a(m)(\alpha)) \cap \tau_\alpha(z(\alpha)) \\ &\subseteq \bigcap_{N \leq m \leq n} \tau_\alpha(a_\alpha(m)) \end{aligned}$$

So the subsequence  $a'_\alpha$  obtained by omitting the terms of  $a_\alpha$  before  $N$  is a legal attack against the winning tactic  $\tau_\alpha$ , and therefore either  $a'_\alpha$  and  $a_\alpha$  converge, or

$$\bigcap_{N \leq n < \omega} \left( \bigcap_{N \leq m \leq n} \tau_\alpha(a_\alpha(m)) \right) [a_\alpha(n+1)] = \emptyset$$

We conclude that either  $a$  converges on each coordinate  $\alpha < \kappa$ , or the intersection

$$\bigcap_{n < \omega} \left( \bigcap_{m \leq n} \tau(\langle a(m) \rangle) \right) [a(n+1)]$$

is empty, verifying that  $\tau$  is a winning tactic.  $\square$

The proof of the following is similar.

**Theorem 2.27.** *Let  $X_\alpha$  be a uniformizable space for  $\alpha < \kappa$  and  $z \in \prod_{\alpha < \kappa} X_\alpha$ . If  $\mathcal{D} \uparrow_{tact} Bell_{D,P}^\rightarrow(X_\alpha)$  for all  $\alpha < \kappa$ , then  $\mathcal{D} \uparrow_{tact} Bell_{D,P}^\rightarrow(\sigma_{\alpha < \kappa}^z X_\alpha)$ .*

We again obtain an analogous observation to Corollary 2.12.

**Definition 2.28.** A *strong Eberlein compact* space is a compact space which may be embedded in  $\sigma 2^\kappa$ .

**Corollary 2.29.** *Strong Eberlein compacts are tactic-proximal.*

We can show that among compact spaces, the tactic-proximal property is necessary and sufficient for strong Eberlein compactness. However, the techniques used in showing this are different than those used to characterize Corson compactness with the proximal property in [2].

The presence of a Cantor set determines the success of  $\mathcal{D}$ 's tactical strategies in  $Bell_{D,P}^\rightarrow(X)$ .

**Lemma 2.30.** *Tactic-proximal spaces cannot contain a copy of the Cantor set.*

*Proof.* The result will follow once we show  $\mathcal{D} \not\uparrow_{tact} Bell_{D,P}^\rightarrow(2^\omega)$ . Let  $\sigma$  be a tactic for  $\mathcal{D}$  in  $Bell_{D,P}^\rightarrow(2^\omega)$  and let  $D_k = \{\langle f, g \rangle : f \upharpoonright k = g \upharpoonright k\}$ . Since  $\{D_k : k < \omega\}$  is a base for the universal uniformity on  $2^\omega$  (namely, all open neighborhoods of the diagonal), we may fix  $k(f) < \omega$  for each  $f \in 2^\omega$  such that  $D_{k(f)} \subseteq \sigma(\langle f \rangle)$ .

Then there exists  $k < \omega$  such that  $\{f : k(f) = k\}$  is uncountable, and therefore there exist distinct  $f, g \in 2^\omega$  such that  $k = k(f) = k(g)$  and  $f \upharpoonright k = g \upharpoonright k$ . Then  $p : \omega \rightarrow 2^\omega$  defined by  $p(2n) = f$  and  $p(2n+1) = g$  is an attack against  $\sigma$  which obviously doesn't converge. This attack is legal since  $f \in D_k[g] \subseteq \sigma(\langle g \rangle)[g]$  and  $g \in D_k[f] \subseteq \sigma(\langle f \rangle)[f]$ , so  $\sigma$  is not a winning tactic.  $\square$

**Lemma 2.31.** *Every non-scattered Corson compact space contains a homeomorphic copy of the Cantor set.*

*Proof.* Every non-scattered space contains a closed subspace without isolated points. Let  $X$  be such a subspace, and assume that this Corson compact is embedded in  $\Sigma \mathbb{R}^\kappa$ . Let  $B_{\alpha, \epsilon}(x) = \{y : d(x(\alpha), y(\alpha)) < \epsilon\}$ . For each  $x \in X$  and  $n < \omega$ , let  $\beta(x, n) < \kappa$  be defined such that  $\text{supp}(x) = \{\beta(x, n) : n < \omega\}$ .

Choose an arbitrary  $x_\emptyset \in X$  and  $\epsilon_0 > 0$ , and let  $A_0 = \emptyset$ .

Suppose then that for some  $n < \omega$ ,  $x_s \in X$  is defined for all  $s \in 2^n$ , and  $\epsilon_n > 0$  and  $A_n \in [\kappa]^{<\omega}$  are defined. Since each  $x_s$  is not isolated in  $X$ , let  $U_s$  be the open set

$$U_s = X \cap \bigcap_{\alpha \in A_{|s|}} B_{\alpha, \epsilon_{|s|}}(x_s)$$

and choose  $x_{s \smallfrown \langle 0 \rangle}, x_{s \smallfrown \langle 1 \rangle} \in U_s$  distinct. Then let  $\alpha_s < \kappa$  such that  $x_{s \smallfrown \langle 0 \rangle}(\alpha_s) \neq x_{s \smallfrown \langle 1 \rangle}(\alpha_s)$ . Let

$$A_{n+1} = \{\alpha_s : s \in 2^{\leq n}\} \cup \{\beta(x_s, i) : s \in 2^{\leq n}, i \leq n\}$$

Then choose  $0 < \epsilon_{n+1} < \frac{1}{2}\epsilon_n$  such that

$$B_{\alpha_s, \epsilon_{n+1}}(x_s \frown \langle 0 \rangle) \cap B_{\alpha_s, \epsilon_{n+1}}(x_s \frown \langle 1 \rangle) = \emptyset$$

and

$$\overline{\bigcap_{\alpha \in A_{n+1}} B_{\alpha, \epsilon_{n+1}}(x_s \frown \langle 0 \rangle)} \cup \overline{\bigcap_{\alpha \in A_{n+1}} B_{\alpha, \epsilon_{n+1}}(x_s \frown \langle 1 \rangle)} \subseteq \bigcap_{\alpha \in A_n} B_{\alpha, \epsilon_n}(x_s)$$

for all  $s \in 2^n$ .

Let  $x_f = \lim_{n < \omega} x_{f \upharpoonright n} \in X$  for each  $f \in 2^\omega$ . We claim  $C = \{x_f : f \in 2^\omega\}$  is a copy of the Cantor set. This will follow if we can show that  $\{U_s : s \in 2^{<\omega}\}$  is a base for  $C$ , since it has the structure of the Cantor tree.

Consider  $x_f$  for some  $f \in 2^\omega$ , and a subbasic open ball  $B_{\alpha, \epsilon}(x_f)$ . Observe that  $x_f \in \bigcap_{n < \omega} U_{f \upharpoonright n}$  since  $x_{f \upharpoonright n} \in U_{f \upharpoonright m}$  for all  $m < n < \omega$ .

If  $\alpha \in \{\beta(x_s, n) : s \in 2^{<\omega}, n < \omega\}$ , choose  $k < \omega$  with  $\alpha \in A_k$ . Then choose  $l < \omega$  such that  $\epsilon_l < \epsilon$ . Then  $U_{f \upharpoonright (l+k)} \subseteq B_{\alpha, \epsilon}(x_f)$ .

Otherwise,  $x_s(\alpha) = 0$  for all  $s \in 2^{<\omega}$ , so  $x_g(\alpha) = 0$  for all  $g \in 2^\omega$  and therefore  $C \subseteq B_{\alpha, \epsilon}(x_f)$ .  $\square$

A new game characterization of strong Eberlein compactness follows from the above and an earlier characterization by Gruenhage.

**Theorem 2.32** ([3]). *For compact spaces  $X$ ,  $X$  is strong Eberlein compact if and only if  $X$  is scattered and  $\mathcal{O} \uparrow \text{Gru}_{\vec{\mathcal{O}}, P}^\rightarrow(X, x)$  for all  $x \in X$ .*

**Corollary 2.33.** *For compact spaces  $X$ ,  $X$  is strong Eberlein compact if and only if  $X$  is tactic-proximal.*

*Proof.* Assume  $X$  is not strong Eberlein compact. One of two things must hold.

- (1) If  $\mathcal{O} \not\uparrow \text{Gru}_{\vec{\mathcal{O}}, P}^\rightarrow(X, x)$  for some  $x \in X$ , then  $\mathcal{D} \not\uparrow \text{Bell}_{\vec{D}, P}^\rightarrow(X)$  and therefore  $\mathcal{D} \not\uparrow_{\text{tact}} \text{Bell}_{\vec{D}, P}^\rightarrow(X)$ .
- (2) If  $X$  is not scattered, it contains a homeomorphic copy of the Cantor space  $2^\omega$ . Since  $\mathcal{D} \not\uparrow_{\text{tact}} \text{Bell}_{\vec{D}, P}^\rightarrow(2^\omega)$ , we have  $\mathcal{D} \not\uparrow_{\text{tact}} \text{Bell}_{\vec{D}, P}^\rightarrow(X)$ .

Thus  $X$  is not tactic-proximal.  $\square$

### 3. COMPARING $\text{Bell}_{\vec{D}, P}^\rightarrow(X)$ AND $\text{Gru}_{\vec{\mathcal{O}}, P}^\rightarrow(X^2, \Delta)$

As mentioned above, for compact spaces  $X$  it's true that  $\mathcal{D} \uparrow \text{Bell}_{\vec{D}, P}^\rightarrow(X)$  if and only if  $\mathcal{O} \uparrow \text{Gru}_{\vec{\mathcal{O}}, P}^\rightarrow(X^2, \Delta)$ . Indeed these games are very similar; since  $X$  is (para)compact, both  $\mathcal{D}$  and  $\mathcal{O}$  are simply choosing neighborhoods of the diagonal, aiming for some sort of convergence.

For tactical information, we may easily see how these games differ.

**Example 3.1.** Let  $X$  be compact and metrizable.

- (1)  $\mathcal{O} \uparrow_{\text{tact}} \text{Gru}_{\vec{\mathcal{O}}, P}^\rightarrow(X^2, \Delta)$
- (2)  $\mathcal{O} \not\uparrow_{\text{0-mark}} \text{Gru}_{\vec{\mathcal{O}}, P}^\rightarrow(X^2, \Delta)$

$$\begin{aligned}
(3) \quad \mathcal{D} &\uparrow_{2\text{-tact}} Bell_{D,P}^{\rightarrow}(X) \\
(4) \quad \mathcal{D} &\uparrow_{0\text{-mark}} Bell_{D,P}^{\rightarrow}(X)
\end{aligned}$$

However,  $\mathcal{D} \not\uparrow_{\text{tact}} Bell_{D,P}^{\rightarrow}(X)$  if  $X$  is non-scattered.

*Proof.* Let  $d$  be a compatible metric for the topology on  $X$ . Then  $\sigma$  as defined for each case below is a winning strategy.

$$\begin{aligned}
(1) \quad &\sigma(\langle x \rangle) = V_{d(x(0), x(1))/2} \\
(2) \quad &\sigma(n) = V_{1/2^n} \\
(3) \quad (\text{as noted earlier}) \quad &\sigma(\langle x, y \rangle) = V_{d(x, y)/2} \\
(4) \quad (\text{as noted earlier}) \quad &\sigma(n) = V_{1/2^n}
\end{aligned}$$

Finally, recall that  $\mathcal{D} \not\uparrow_{\text{tact}} Bell_{D,P}^{\rightarrow}(X)$  when  $X$  is non-scattered Corson compact.  $\square$

Effectively, the fact that  $\mathcal{O}$  can see two coordinates ( $\mathcal{P}$  chooses points in  $X^2$ ) gives her an extra edge to use in a game of limited information.

With respect to these games, question 2.23 could be restated as follows: is  $\mathcal{O} \uparrow_{\text{mark}} Gru_{O,P}^{\rightarrow}(X^2, \Delta)$  equivalent to  $\mathcal{D} \uparrow_{\text{mark}} Bell_{D,P}^{\rightarrow}(X)$  for compact spaces  $X$ ?

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