SELECTION GAMES AND ARHANGELSKII'S CONVERGENCE PRINCIPLES

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Abstract. We prove the things.

1. Clontz results

Definition 1. Say a collection $\mathcal{A}$ is *sequence-like* if it satisfies the following for each $A \in \mathcal{A}$.

- $|A| \geq \aleph_0$.
- If $A' \subseteq A$ and $|A'| \geq \aleph_0$, then $A' \in \mathcal{A}$.

Definition 2. Let Γ_X be the collection of open γ-*covers* $\mathcal{U}$ of X, that is, infinite open covers of X such that for each $x \in X$, $\{U \in \mathcal{U} : x \in U\}$ is cofinite in $\mathcal{U}$.

Definition 3. Let $\Gamma_{X,x}$ be the collection of non-trivial sequences $S \subseteq X$ converging to x, that is, infinite subsets of X such that for each neighborhood U of x, $S \cap U$ is cofinite in S.

It follows that $\Gamma_X, \Gamma_{X,x}$ are both sequence-like.

Theorem 4. *Let $\mathcal{B}$ be sequence-like. Then $\alpha_1(\mathcal{A}, \mathcal{B})$ holds if and only if* $\mathrm{I} \underset{pre}{\not\uparrow} G_{cf}(\mathcal{A}, \mathcal{B})$.

Proof. We first assume $\alpha_1(\mathcal{A}, \mathcal{B})$ and let $A_n \in \mathcal{A}$ for $n < \omega$ define a predetermined strategy for I. By $\alpha_1(\mathcal{A}, \mathcal{B})$, we immediately obtain $B \in \mathcal{B}$ such that $|A_n \setminus B| < \aleph_0$. Thus $B_n = A_n \cap B$ is a cofinite choice from A_n, and $B' = \bigcup\{B_n : n < \omega\}$ is an infinite subset of B, so $B' \in \mathcal{B}$. Thus II may defeat I by choosing $B_n \subseteq A_n$ each round, witnessing $\mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} G_{cf}(\mathcal{A}, \mathcal{B})$.

On the other hand, let $\mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} G_{cf}(\mathcal{A}, \mathcal{B})$. Given $A_n \in \mathcal{A}$ for $n < \omega$, we note that II may choose a cofinite subset $B_n \subseteq A_n$ such that $B = \bigcup\{B_n : n < \omega\} \in \mathcal{B}$. Then B witnesses $\alpha_1(\mathcal{A}, \mathcal{B})$ since $|A_n \setminus B| \leq |A_n \setminus B_n| \leq \aleph_0$. $\square$

Theorem 5. *Let $\mathcal{A}, \mathcal{B}$ be sequence-like. Then $\alpha_2(\mathcal{A}, \mathcal{B})$ holds if and only if* $\mathrm{I} \underset{pre}{\not\uparrow} G_1(\mathcal{A}, \mathcal{B})$.

Proof. We first assume $\alpha_2(\mathcal{A}, \mathcal{B})$ and let $A_n \in \mathcal{A}$ for $n < \omega$ define a predetermined strategy for $\mathscr{I}$. We may apply $\alpha_2(\mathcal{A}, \mathcal{B})$ to choose $B \in \mathcal{B}$ such that $|A_n \cap B| \geq \aleph_0$. We may then choose $a_n \in (A_n \cap B) \setminus \{a_i : i < n\}$ for each $n < \omega$. It follows that $B' = \{a_n : n < \omega\} \in \mathcal{B}$ since B' is an infinite subset of $B \in \mathcal{B}$; therefore A_n does not define a winning predetermined strategy for I.

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Now suppose I $\underset{\text{pre}}{\not\uparrow} G_1(\mathcal{A},\mathcal{B})$. Given $A_n \in \mathcal{A}$ for $n < \omega$, first choose $A'_n = \{a_{n,j} : j < \omega\} \subseteq A_n$ such that $j < k$ implies $a_{n,j} \neq a_{n,k}$, and then let $A_{n,m} = \{a_{n,j} : m \leq j < \omega\}$, noting $A_{n,m} \in \mathcal{A}$ since $A_{n,m}$ is an infinite subset of $A_n \in \mathcal{A}$. Finally choose some $\theta : \omega \to \omega$ such that $|\theta^{\leftarrow}(n)| = \aleph_0$ for each $n < \omega$.

Since playing $A_{\theta(m),m}$ during round m does not define a winning strategy for I in $G_1(\mathcal{A},\mathcal{B})$, II may choose $x_m \in A_{\theta(m),m}$ such that $B = \{x_m : m < \omega\} \in \mathcal{B}$. Choose $i_m < \omega$ for each $m < \omega$ such that $x_m = a_{\theta(m),i_m}$, noting $i_m \geq m$. It follows that $A_n \cap B \supseteq \{a_{\theta(m),i_m} : m \in \theta^{\leftarrow}(n)\}$. Since for each $m \in \theta^{\leftarrow}(n)$ there exists $M \in \theta^{\leftarrow}(n)$ such that $m \leq i_m < M \leq i_M$, and therefore $a_{\theta(m),i_m} \neq a_{\theta(m),i_M} = a_{\theta(M),i_M}$, we have shown that $A_n \cap B$ is infinite. Thus B witnesses $\alpha_2(\mathcal{A},\mathcal{B})$. $\square$

Theorem 6. *Let $\mathcal{A},\mathcal{B}$ be sequence-like. Then $\alpha_4(\mathcal{A},\mathcal{B})$ holds if and only if* I $\underset{pre}{\not\uparrow}$ $G_{<2}(\mathcal{A},\mathcal{B})$ *if and only if* I $\underset{pre}{\not\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$.

Proof. We first assume $\alpha_4(\mathcal{A},\mathcal{B})$ and let $A_n \in \mathcal{A}$ for $n < \omega$ define a predetermined strategy for I in $G_{<2}(\mathcal{A},\mathcal{B})$. We then choose $A'_n = \{a_{n,j} : j < \omega\} \subseteq A_n$ such that $j < k$ implies $a_{n,j} \neq a_{n,k}$, and then let $A''_n = A'_n \setminus \{a_{i,j} : i,j < n\}$, noting $A''_n \in \mathcal{A}$ since it is an infinite subset of A_n.

By applying $\alpha_4(\mathcal{A},\mathcal{B})$ to A''_n, we obtain $B \in \mathcal{B}$ such that $A''_n \cap B \neq \emptyset$ for infintely-many $n < \omega$. We then let $F_n = \emptyset$ when $A''_n \cap B = \emptyset$, and $F_n = \{x_n\}$ for some $x_n \in A''_n \cap B$ otherwise. Then we will have that $B' = \bigcup\{F_n : n < \omega\} \subseteq B$ belongs to $\mathcal{B}$ once we show that B' is infinite. To see this, for $m \leq n < \omega$ note that either F_m is empty (and we let $j_m = 0$) or $F_m = \{a_{m,j_m}\}$ for some $j_m \geq m$; choose $N < \omega$ such that $j_m < N$ for all $m \leq n$ and $F_N = \{x_N\}$. Thus $F_m \neq F_N$ for all $m \leq n$ since $x_N \notin \{a_{i,j} : i,j < N\}$. Thus II may defeat the predetermined strategy A_n by playing F_n each round.

Since I $\underset{\text{pre}}{\not\uparrow} G_{<2}(\mathcal{A},\mathcal{B})$ immediately implies I $\underset{\text{pre}}{\not\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$, we assume the latter. Given $A_n \in \mathcal{A}$ for $n < \omega$, we note this defines a (non-winning) predetermined strategy for I, so II may choose $F_n \in [A_n]^{<\aleph_0}$ such that $B = \bigcup\{F_n : n < \omega\} \in \mathcal{B}$. Since B is infinite, we note $F_n \neq \emptyset$ for infinitely-many $n < \omega$. Thus B witnesses $\alpha_4(\mathcal{A},\mathcal{B})$ since $A_n \cap B \supseteq F_n \neq \emptyset$ for infinitely-many $n < \omega$. $\square$

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