

Definition 1. X is **Menger** if for all open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ there exist finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\bigcup_{n < \omega} \mathcal{F}_n$ is a cover of X .

Proposition 2. $\sigma\text{-compact} \Rightarrow \text{Menger} \Rightarrow \text{Lindelof}$

Definition 3. In the two-player game $\text{Cov}_{C,F}(X)$ player C chooses open covers \mathcal{U}_n of X , followed by player F choosing a finite subcollection $\mathcal{F}_n \subseteq \mathcal{U}_n$. F wins if $\bigcup_{n < \omega} \mathcal{F}_n$ is a cover of X .

Theorem 4. X is Menger if and only if $C \nVdash \text{Cov}_{C,F}(X)$.

Proof. First, suppose X wasn't Menger. Then there would exist open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ of X such that for any choice of finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$, $\bigcup_{n < \omega} \mathcal{F}_n$ isn't a cover of X . Thus $C \uparrow_{\text{pre}} \text{Cov}_{C,F}(X) \Rightarrow S \nVdash \text{Cov}_{C,F}(X)$.

The other direction is based upon Gruenhage's topological game presentation. Assume X is Menger, and consider a strategy for C in $\text{Cov}_{C,F}(X)$.

Since X is Lindelof, we can assume C plays only countable covers of X . Then, since F is choosing finite subsets, we may assume F chooses some initial segment of the countable cover. In turn, we can assume C plays an increasing open cover $\{U_0, U_1, \dots\}$ where $U_n \subseteq U_{n+1}$. And in that case, it's sufficient to assume F simply chooses a singleton subset of each cover. And finally, since choices made by F are already covered, we can assume that every open set in a cover played by C covers the sets chosen by F previously.

As a result, we have the following figure of a tree of plays which I need to draw:

(Insert figure here.)

Note that for $a, b \in \omega^{<\omega}$ and $m \leq n$, we know:

- (a) $U_{a \smallfrown m} \subseteq U_{a \smallfrown n}$
(for example, $U_{1627} \subseteq U_{1629}$ - increasing the final digit yields supersets)
- (b) $U_a \subseteq U_{a \smallfrown b}$
(for example, $U_{1627} \subseteq U_{162789}$ - appending any sequence to the end yields supersets)
- (c) $U_{a \smallfrown m} \subseteq U_{a \smallfrown n} \subseteq U_{a \smallfrown n \smallfrown b} \subseteq U_{a \smallfrown n \smallfrown b \smallfrown m}$
(for example: $U_{1627} \subseteq U_{1629283287}$ - injecting a subsequence with initial number larger than the original's final number, prior to the final number, yields supersets)

We may observe that if F can find an $f : \omega \rightarrow \omega$ such that $\bigcup_{n < \omega} U_{f \upharpoonright (n+1)} = X$, she can use $\{U_{f \upharpoonright 0}\}, \{U_{f \upharpoonright 1}\}, \dots$ to counter C 's strategy.

Let $V_k^n = \bigcap_{a \in \omega^{<n}} U_{a \smallfrown k}$. We claim that (1) V_k^n is open, (2) $\mathcal{V}^n = \{V_0^n, V_1^n, \dots\}$ is increasing, and (3) \mathcal{V}^n is a cover. Proofs:

1. Since due to (c) for each $b \in \omega^{\leq n} \setminus k^{\leq n}$, there is an $a \in k^{\leq n}$ with $U_{a \smallfrown k} \subseteq U_{b \smallfrown k}$:

$$V_k^n = \bigcap_{a \in \omega^{\leq n}} U_{a \smallfrown k} = \bigcap_{a \in k^{\leq n}} U_{a \smallfrown k} \cap \bigcap_{b \in \omega^{\leq n} \setminus k^{\leq n}} U_{b \smallfrown k} = \bigcap_{a \in k^{\leq n}} U_{a \smallfrown k}$$

making V_k^n a finite intersection of open sets.

2. We show $V_k^0 \subseteq V_{k+1}^0$:

$$V_k^0 = U_k \subseteq U_{k+1} = V_{k+1}^0$$

and then assume $V_k^n \subseteq V_{k+1}^n$:

$$V_k^{n+1} = \bigcap_{a \in \omega^{\leq n+1}} U_{a \smallfrown k} = V_k^n \cap \bigcap_{a \in \omega^{n+1}} U_{a \smallfrown k} \subseteq V_{k+1}^n \cap \bigcap_{a \in \omega^{n+1}} U_{a \smallfrown (k+1)} = V_{k+1}^{n+1}$$

3. We easily see that $\mathcal{V}^0 = \{U_0, U_1, \dots\}$ is a cover, and then assume \mathcal{V}^n is a cover.

Let $x \in X$ and pick $l < \omega$ such that $x \in V_l^n$. For $a \in l^{n+1}$ choose l_a such that $x \in U_{a \smallfrown l_a}$, giving

$$x \in \bigcap_{a \in l^{n+1}} U_{a \smallfrown l_a}$$

We will assume $k > l, l_a$ for all $a \in l^{\leq n+1}$.

For any $a \in k^{n+1} \setminus l^{n+1}$ note that $a = b \smallfrown c$ where $b \in l^{\leq n}$ and c begins with a number l or greater:

$$V_l^n \subseteq U_{b \smallfrown l} \subseteq U_{b \smallfrown c} \subseteq U_{b \smallfrown c \smallfrown l_a} = U_{a \smallfrown l_a}$$

Thus:

$$\begin{aligned} x &\in V_l^n \cap \left(\bigcap_{a \in l^{n+1}} U_{a \smallfrown l_a} \right) \\ &= V_l^n \cap \left(\bigcap_{a \in k^{n+1} \setminus l^{n+1}} U_{a \smallfrown l_a} \right) \cap \left(\bigcap_{a \in l^{n+1}} U_{a \smallfrown l_a} \right) \\ &= V_l^n \cap \left(\bigcap_{a \in k^{n+1}} U_{a \smallfrown l_a} \right) \\ &\subseteq V_k^n \cap \left(\bigcap_{a \in k^{n+1}} U_{a \smallfrown k} \right) \\ &= V_k^{n+1} \end{aligned}$$

Finally, apply Menger to \mathcal{V}^n , resulting in the cover $\{V_{f(0)}^0, V_{f(1)}^1, \dots\}$, noting

$$X = \bigcup_{n < \omega} V_{f(n)}^n \subseteq \bigcup_{n < \omega} U_{(f \upharpoonright n) \cap f(n)} = \bigcup_{n < \omega} U_{f \upharpoonright (n+1)}$$

□

Proposition 5. *X is compact if and only if $F \uparrow_{tact} Cov_{C,F}(X)$*

Proof. Assume X is compact. For each open cover played by C , pick the finite subcover.

Assume F has a winning tactical strategy. For any open cover, have C play only it during the entire game. F 's only choice must be a finite subcover. □

Proposition 6. *If X is σ -compact then $F \uparrow_{mark} Cov_{C,F}(X)$*

Proof. Let $X = \bigcup_{n < \omega} X_n$ for compact X_n . On round n , F picks the finite subcover of C 's open cover of X_n . □

Due to Telgarski in “On Games of Topsoe”:

Theorem 7. *For metrizable X , X is σ -compact if and only if $F \uparrow Cov_{C,F}(X)$.*

In a question I posed to G, he answered:

Lemma 8. *For all $\alpha_0, \alpha_1, \dots < \omega_1$ and functions $\tau : \omega_1 \times \omega \rightarrow [\omega_1]^{<\omega}$, $\{\tau(\alpha_n, n) : n < \omega\}$ is not a cover for $\{\beta : \forall n < \omega (\beta < \alpha_n)\}$.*

Proof. Let $P_n = \{\beta : \beta < \alpha \Rightarrow \beta \in \tau(\alpha, n)\}$. Observe that each P_n is finite; else there is some α larger than every member of P_n such that $P_n \subseteq \tau(\alpha, n)$.

Choose $\beta \notin \bigcup_{n < \omega} P_n$. Then for each $n < \omega$, pick $\alpha_n > \beta$ such that $\beta \notin \tau(\alpha_n, n)$. □

Note that the one-point Lindelöfication of discrete $\omega_1, \omega_1^\dagger$, is not σ -compact. With the above lemma, we may see that:

Example 9. *$F \uparrow Cov_{C,F}(\omega_1^\dagger)$ but $F \not\uparrow_{mark} Cov_{C,F}(\omega_1^\dagger)$.*

Proof. First, we see F has a simple perfect information strategy: in response to the initial cover of ω_1^\dagger , F chooses a co-countable neighborhood of ∞ . On successive turns she may pick a single set from C 's covers to cover the countable remainder.

Now, suppose that $\sigma(\mathcal{U}, n)$ was a winning Markov strategy and aim for a contradiction. Consider the covers

$$\mathcal{U}(\alpha) = \{[\alpha, \omega_1) \cup \{\infty\}\} \cup \{\{\beta\} : \beta < \alpha\}$$

and define $\tau(\alpha, n)$ to be the union of singletons chosen by $\sigma(\mathcal{U}(\alpha), n)$. As σ was a winning strategy, for all $\alpha_0, \alpha_1, \dots < \omega_1$, $\{\sigma(\mathcal{U}(\alpha_n), n) : n < \omega\}$ must cover ω_1^\dagger , and thus $\{\tau(\alpha_n, n) : n < \omega\}$ must cover $\{\beta : \forall n < \omega (\beta < \alpha_n)\}$, contradiction. \square

Lemma 10. *Let $\sigma(\mathcal{U}, n)$ be a winning Markov strategy for F in $\text{Cov}_{C,F}(X)$, and \mathfrak{C} collect all open covers of X . Then for*

$$C_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \overline{\bigcup \sigma(\mathcal{U}, n)}$$

and

$$D_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

it follows that $\bigcup_{n < \omega} C_n = \bigcup_{n < \omega} D_n = X$.

Proof. Observe $D_n \subseteq C_n$. Suppose that $x \notin D_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$ for any $n < \omega$. Then for each n , pick $\mathcal{U}_n \in \mathfrak{C}$ such that $x \notin \bigcup \sigma(\mathcal{U}_n, n)$. Then σ does not defeat the play $\mathcal{U}_0, \mathcal{U}_1, \dots$ since the $\sigma(\mathcal{U}_n, n)$ do not cover x , contradiction. \square

Theorem 11. *For regular spaces X , $F \uparrow_{\text{mark}} \text{Cov}_{C,F}(X)$ if and only if X is σ -compact.*

Proof. The reverse implication has already been shown. To complete the proof, we look to Scheepers for inspiration.

Let $\sigma(\mathcal{U}, n)$ be a winning Markov strategy for F in $\text{Cov}_{C,F}(X)$. Let \mathfrak{C} collect all open covers of X . Define

$$C_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \overline{\bigcup \sigma(\mathcal{U}, n)}$$

as in the previous lemma. Note that $\bigcup_{n < \omega} C_n = X$, and we will show each C_n is compact as it is H -closed.

Let \mathcal{U} be an open cover of C_n , and \mathcal{V} be a cover of $X \setminus C_n$ by open sets whose closures are disjoint from C_n (possible by regularity).

Since $\mathcal{U} \cup \mathcal{V}$ covers X , $\overline{\bigcup \sigma(\mathcal{U} \cup \mathcal{V}, n)} \supseteq C_n$. Furthermore, if $\mathcal{F} = \sigma(\mathcal{U} \cup \mathcal{V}, n) \setminus \mathcal{V}$, then $\overline{\bigcup \mathcal{F}} \supseteq C_n$ (the closures of sets in \mathcal{V} missed C_n). Thus \mathcal{F} witnesses that C_n is H -closed. \square

Example 12. *Let R be given the topology from example 63 from Counterexamples in Topology, the topology generated by open intervals with countable sets removed. This space is non-regular, non- σ -compact, and Lindelöf. It is also Menger as $F \uparrow \text{Cov}_{C,F}(R)$, but $F \not\uparrow_{\text{mark}} \text{Cov}_{C,F}(R)$.*

Proof. From Counterexamples: The irrationals are open, but contain no closed neighborhood, showing non-regular. Compact subsets are exactly finite subsets, showing non- σ -compact.

Take open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$. Define $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{2n})$ to be a finite subcover of $[-n, n] \setminus C_n$ for some countable $C_n = \{c_{n,0}, c_{n,1}, \dots\}$. For $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{2n+1})$, use any subcover of $\{c_{i,j} : i, j < n\}$. It is easily seen that σ is a winning perfect information strategy.

There cannot be a winning Markov strategy $\sigma(\mathcal{U}, n)$, however. Define

$$D_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

where \mathfrak{C} is the collection of open covers of R . For any $x_0, x_1, \dots \in R$, we may define the open cover $\mathcal{U} = \{R \setminus \{x_i : i \neq n\} : n < \omega\}$, and observe that $\bigcup \sigma(\mathcal{U}, n) \supseteq D_n$ cannot contain every x_i . Thus D_n is finite, but since the previous lemma requires $\bigcup_{n < \omega} D_n = R$ if σ is a winning strategy, there exists a counter to σ . \square

Theorem 13. *For any topological space X and all $k \geq 2$, $F \uparrow_{k\text{-mark}} \text{Cov}_{C,F}(X)$ if and only if $F \uparrow_{2\text{-mark}} \text{Cov}_{C,F}(X)$.*

Proof. Assume $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{k-1}, n)$ is a winning k -Markov strategy. Define the 2-Markov strategy $\tau(\mathcal{U}, \mathcal{V}, n)$ so that it contains $\sigma(\mathcal{W}_0, \dots, \mathcal{W}_{k-1}, m)$ for the following conditions on $(\mathcal{W}_0, \dots, \mathcal{W}_{k-1}, m)$:

- Each $\mathcal{W}_i \in \{\mathcal{U}, \mathcal{V}\}$
- $m \leq (n+1)k$; in particular, for $i < k$,

$$\sigma(\mathcal{W}_0, \dots, \mathcal{W}_{k-1}, (n+1)k + i) \subseteq \tau(\mathcal{U}, \mathcal{V}, n+1)$$

Considering an arbitrary play $\mathcal{U}_0, \mathcal{U}_1, \dots$ by C versus τ , we note that σ defeats the play

$$\underbrace{\mathcal{U}_0, \mathcal{U}_0, \dots, \mathcal{U}_0}_k, \underbrace{\mathcal{U}_1, \mathcal{U}_1, \dots, \mathcal{U}_1}_k \dots$$

So we have that

$$\bigcup_{i < k, n < \omega} \sigma(\underbrace{\mathcal{U}_n, \dots, \mathcal{U}_n}_{k-i-1}, \underbrace{\mathcal{U}_{n+1}, \dots, \mathcal{U}_{n+1}}_{i+1}, (n+1)k + i)$$

is a cover for X , and as

$$\sigma(\underbrace{\mathcal{U}_n, \dots, \mathcal{U}_n}_{k-i-1}, \underbrace{\mathcal{U}_{n+1}, \dots, \mathcal{U}_{n+1}}_{i+1}, (n+1)k + i) \subseteq \tau(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1)$$

τ defeats the play $\mathcal{U}_0, \mathcal{U}_1, \dots$. \square

The question remains:

Question 14. *In general, does $F \upharpoonright_{\text{mark}} \text{Cov}_{C,F}(X)$ imply X is σ -compact?*