An Example of Gruenhage's Compact-Point Game for which K has a winning strategy, but no winning k-tactical strategy

We construct a ZFC example given by Gary Gruenhage, inpsired by a ZFC+¬SH example due to Stephen Watson [3].

Theorem 1. *There exists a compact, zero-dimensional topological space X which has a point-countable cover $\mathcal{U} = \{U_c : c \in 2^\omega\}$ which is not the union of countably-many point-finite collections.*

Proof. Take a zero-dimensional Corson compact Y of weight 2^ω, which is not Eberlein compact. It follows by [1] that $Y^2 \setminus \Delta = \{(y_1, y_2) : y_1, y_2 \in Y, y_1 \neq y_2\}$ is metalindelof, but not σ-metacompact. So every cover has a point-countable refinement of cardinality 2^ω, but no open refinement composed of the union of countably-many point-finite collections. □

Example 2. *Using the X from Theorem 1, let*

$$\mathbb{X} = (X \times 2^{<\omega}) \cup 2^\omega$$

compose a disjoint union of $2^{<\omega}$ copies of X along with a discrete copy of the Cantor Set 2^ω, and add open neighborhoods of the form:

$$B_c = c \cup (U_c \times \{c \restriction n : n < \omega\})$$

as seen in Figure 1.

Definition 3. Let $S \in [2^\omega]^{<\omega}$ and $m < \omega$. Define

$$K_S = \bigcup_{c \in S} B_c$$

$$A = \{z^\frown\langle 1\rangle : z \in 1^{<\omega}\}$$

$$K_S^* = K_S \setminus (X \times A)$$

$$L_m = X \times 2^{<m}$$

and observe that every compact set is dominated by $K_S^* \cup L_m$ for some S, m.

Intuitively, K_S^* collects the branches of U_c converging up to each $c \in S$ while avoiding the copy of X for each s in an antichain A, and L_m collects the copies of X with $|s| < m$ at the base of the tree. (See Figure 2)

Definition 4. $LF_{K,P}(\mathbb{X})$ is a topological game consisting of players K and P. During each round, K chooses a compact subset of $\mathbb{X}$, and P chooses a point outside of any compact set previously played by K. K wins the game if the set of points chosen by P throughout all rounds of the game are locally finite in the space.

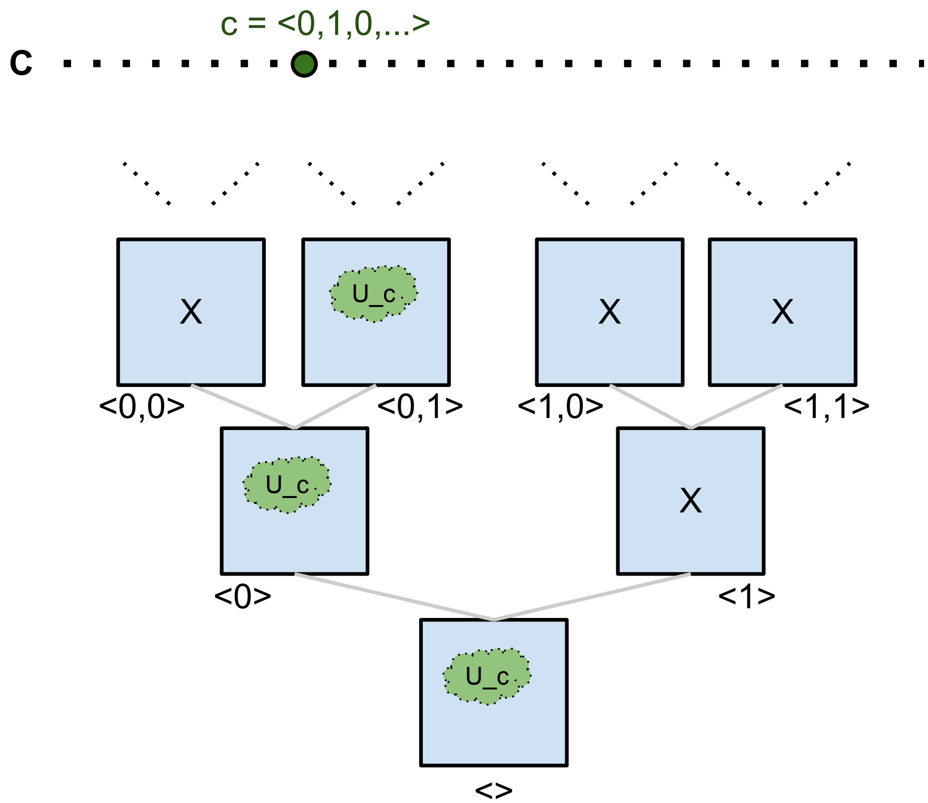

Figure 1: The Cantor Tree space $\mathbb{X}$

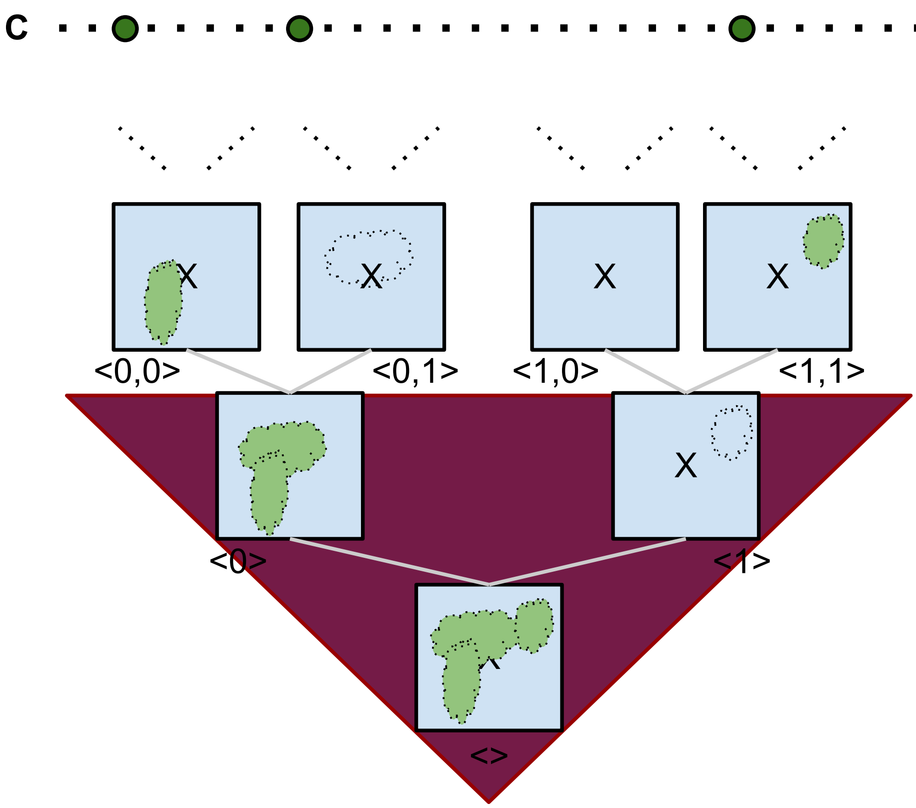

Figure 2: K_S^* and L_m

Proposition 5. *Without loss of generality, we may assume P always plays points in $X \times 2^{<\omega}$ throughout $LF_{K,P}(\mathbb{X})$.*

Proposition 6. $K \uparrow LF_{K,P}(\mathbb{X})$

Proof. Let $x \in X$. $C^x = \{c \in C : x \in U_c\}$ is a countable collection by the point-countability of $\mathcal{U}$, so label its elements as $\{c_n^x : n < \omega\}$.

K may use the strategy

$$\sigma(\langle x_0, s_0\rangle, \ldots, \langle x_{n-1}, s_{n-1}\rangle) = \bigcup_{i<n} K_{\{c_0^{x_i}, \ldots, c_{n-1}^{x_i}\}} \cup L_{|s_i|+1}$$

This is a winning strategy because each point $\langle x_i, s_i\rangle$ played by P cannot be a part of a sequence converging to some $c_n^{x_i}$, since a $K_{\{c_0^{x_i}, \ldots, c_n^{x_i}\}} \supseteq B_{c_n^{x_i}}$ was forbidden during round n. □

Theorem 7. $K \not\uparrow_{tact} LF_{K,P}(\mathbb{X})$.

Proof. This is actually a corollary of Gruenhage's theorem in [2] by proving $\mathbb{X}$ is not metacompact. The following is a direct game-theoretic proof.

Suppose that $\sigma(\langle x, s\rangle)$ was a winning tactical strategy for K and define the compact set

$$\sigma'(x, n) = \bigcup_{|s| \leq n} \sigma(\langle x, s\rangle)$$

There exists some $f : 2^\omega \to \omega$ such that for all $x \in U_c$, $\sigma'(x, f(c))$ covers some $B_c \setminus L_m$. (If not, P counters by simply always playing in $B_c \setminus L_m$.)

Recall that $\mathcal{U}$ is not the union of the countably-many point-finite collections, and

$$\mathcal{U} = \bigcup_{n<\omega} \{U_c : f(c) = n\}$$

so we may find choose n where $\mathcal{U}_n = \{U_c : f(c) = n\}$ is not point-finite. Fix x so that x belongs to each of $\{U_{c_0}, U_{c_1}, \ldots\} \subseteq \mathcal{U}_n$.

For each c_i, $\sigma'(x, f(c_i)) = \sigma'(x, n)$ covers c_i. Thus $\sigma'(x, n) \supseteq \{c_0, c_1, \ldots\}$ is not compact, a contradiction. □

Theorem 8. $K \not\uparrow_{2\text{-}tact} LF_{K,P}(\mathbb{X})$.

Proof. Suppose $\sigma(\langle x,s\rangle,\langle y,t\rangle)$ was a winning 2-tactical strategy. Without loss of generality we assume it ignores order. We may define $S(x,y,n) \in [2^\omega]^{<\omega}$ (increasing on n) and $n < m(x,y,n) < \omega$ such that for each (x,y),

$$\bigcup_{s,t\in 2^{\leq n}} \sigma(\langle x,s\rangle,\langle y,t\rangle) \subseteq K^*_{S(x,y,n)} \cup L_{m(x,y,n)}$$

and so we assume

$$\sigma(\langle x,s\rangle,\langle y,t\rangle) = K^*_{S(x,y,\max(|s|,|t|))} \cup L_{m(x,y,\max(|s|,|t|))}$$

Select an arbitrary point $x' \in X$. We define a tactical strategy

$$\tau(x,s) = K^*_{S(x,x',m(x,x',|s|)+1)} \cup L_{m(x,x',m(x,x',|s|)+1)}$$

We complete the proof by showing τ is a winning tactical strategy (a contradiction).

Suppose

$$\langle x_0,s_0\rangle,\langle x_1,s_1\rangle,\langle x_2,s_2\rangle,\dots$$

successfully countered τ by clustering at $c \in 2^\omega$ (the strategy trivially prevents clustering elsewhere). Let $z_n = \langle 0,\dots,0\rangle$ with n zeros. We claim

$$\langle x_0,s_0\rangle,\langle x',z_{m(x_0,x',|s_0|)}{}^\frown\langle 1\rangle\rangle,\langle x_1,s_1\rangle,\langle x',z_{m(x_1,x',|s_1|)}{}^\frown\langle 1\rangle\rangle,\langle x_2,s_2\rangle,\langle x',z_{m(x_2,x',|s_2|)}{}^\frown\langle 1\rangle\rangle,\dots$$

is a successful counter to σ.

We will need the fact that, as $\langle x_{i+1},s_{i+1}\rangle$ was legal against τ:

$$|s_i| < m(x_i,x',|s_i|)+1 = |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|$$

$$< m(x_i,x',m(x_i,x',|s_i|)+1) = m(x_i,x',|z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|) \leq |s_{i+1}|$$

Note that $m(x,y,\max(|s|,|t|))$ is increasing throughout this play of the game versus σ:

$$\begin{aligned} &m(x_i,x',\max(|s_i|,|z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|)) \\ &= m(x_i,x',|z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|) \\ &\leq |s_{i+1}| \\ &< m(x_{i+1},x',|s_{i+1}|) \\ &= m(x_{i+1},x',\max(|s_{i+1}|,|z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|)) \\ &= |z_{m(x_{i+1},x',|s_{i+1}|)}| \end{aligned}$$

$$
\begin{aligned}
&< |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle| \\
&< m(x_{i+1}, x', |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle|) \\
&= m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle|))
\end{aligned}
$$

We turn to showing that $\langle x', z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle\rangle$ is always a legal move. Since $z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle$ is on the antichain avoided by any K^*, the problem is reduced to showing that this move isn't forbidden by

$$L_{m(x_{i+1},x',\max(|s_{i+1}|,|z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|))}$$

which we can see here:

$$m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|)) = m(x_{i+1}, x', |s_{i+1}|) < |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle|$$

We can conclude by showing that $\langle x_{i+1}, s_{i+1}\rangle$ is always a legal move. We can see it avoids

$$L_{m(x_i,x',\max(|s_i|,|z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|))}$$

since

$$m(x_i, x', \max(|s_i|, |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|)) = m(x_i, x', |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|) \leq |s_{i+1}|$$

Since $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ, it avoided

$$K^*_{S(x_h,x',m(x_h,x',|s_h|)+1)} = K^*_{S(x_h,x',\max(|s_h|,|z_{m(x_h,x',|s_h|)}{}^\frown\langle 1\rangle|))}$$

for $h \leq i$. And when $h < i$, we see it avoids:

$$
\begin{aligned}
K^*_{S(x_{h+1},x',\max(|s_{h+1}|,|z_{m(x_h,x',|s_h|)}{}^\frown\langle 1\rangle|))} &= K^*_{S(x_{h+1},x',|s_{h+1}|)} \\
&\subseteq K^*_{S(x_{h+1},x',m(x_{h+1},x',|s_{h+1}|)+1)}
\end{aligned}
$$

This concludes the proof. □

Theorem 9. *$K \not\uparrow_{k\text{-tact}} LF_{K,P}(\mathbb{X})$.*

Proof. The proof proceeds in parallel to the proof of $K \not\uparrow_{\text{2-tact}} LF_{K,P}(\mathbb{X})$, and in fact is just a generalization of said proof (at the cost of simplicity).

Suppose $\sigma(\langle x_0, s_0\rangle, \ldots, \langle x_k, s_k\rangle)$ was a winning $(k+1)$-tactical strategy. Without loss of generality we assume it ignores order. We may define $S(x_0, \ldots, x_k, n) \in [2^\omega]^{<\omega}$ (increasing on n) and $n < m(x_0, \ldots, x_k, n) < \omega$ such that for each $(x_0, \ldots, x_k)$,

$$\bigcup_{s_0,\ldots,s_k \in 2^{\leq n}} \sigma(\langle x_0, s_0\rangle, \ldots, \langle x_k, s_k\rangle) \subseteq K^*_{S(x_0,\ldots,x_k,n)} \cup L_{m(x_0,\ldots,x_k,n)}$$

and so we assume

$$\sigma(\langle x_0, s_0\rangle, \ldots, \langle x_k, s_k\rangle) = K^*_{S(x_0,\ldots,x_k,\max(|s_0|,\ldots,|s_k|))} \cup L_{m(x_0,\ldots,x_k,\max(|s_0|,\ldots,|s_k|))}$$

Select an arbitrary point $x' \in X$. Let $M^0(x,n) = m(x, x', \ldots, x', n)$ and $M^{i+1}(x,n) = M^0(x, M^i(x,n)+1)$. We define a tactical strategy

$$\tau(x,s) = K^*_{S(x,x',\ldots,x',M^{k-1}(x,|s|)+1)} \cup L_{m(x,x',\ldots,x',M^{k-1}(x,|s|)+1)}$$

We complete the proof by showing τ is a winning tactical strategy (a contradiction).

Suppose

$$\langle x_0, s_0\rangle, \langle x_1, s_1\rangle, \langle x_2, s_2\rangle, \ldots$$

successfully countered τ by clustering at $c \in 2^\omega$ (the strategy trivially prevents clustering elsewhere). Let $z_n = \langle 0, \ldots, 0\rangle$ with n zeros. We claim

$$\langle x_0, s_0\rangle, \langle x', z_{M^0(x_0,|s_0|)}{}^\frown\langle 1\rangle\rangle, \langle x', z_{M^1(x_0,|s_0|)}{}^\frown\langle 1\rangle\rangle, \ldots, \langle x', z_{M^{k-1}(x_0,|s_0|)}{}^\frown\langle 1\rangle\rangle,$$

$$\langle x_1, s_1\rangle, \langle x', z_{M^0(x_1,|s_1|)}{}^\frown\langle 1\rangle\rangle, \langle x', z_{M^1(x_1,|s_1|)}{}^\frown\langle 1\rangle\rangle, \ldots, \langle x', z_{M^{k-1}(x_1,|s_1|)}{}^\frown\langle 1\rangle\rangle, \ldots$$

is a successful counter to σ.

We will need the fact that, as $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ:

$$|s_i| < M^0(x_i,|s_i|)+1 = |z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle| < M^0(x_i, M^0(x_i,|s_i|)+1)+1$$

$$= M^1(x_i,|s_i|)+1 = |z_{M^1(x_i,|s_i|)}{}^\frown\langle 1\rangle| < \cdots < |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|$$

$$= M^{k-1}(x_i,|s_i|)+1 < m(x_i, x', \ldots, x', M^{k-1}(x_i,|s_i|)+1) \le |s_{i+1}|$$

Note that $m(x_0, \ldots, x_k, \max(|s_0|, \ldots, |s_k|))$ is increasing throughout this play of the game versus σ:

$$\begin{aligned}
&m(x_i, x', \ldots, x', \max(|s_i|, |z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|))\\
&= m(x_i, x', \ldots, x', |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|)\\
&= m(x_i, x', \ldots, x', M^{k-1}(x_i,|s_i|)+1)\\
&\le |s_{i+1}|\\
&< M^0(x_{i+1},|s_{i+1}|)\\
&= m(x_{i+1}, x', \ldots, x', |s_{i+1}|)\\
&= m(x_{i+1}, x', \ldots, x', \max(|s_{i+1}|, |z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|))\\
&= |z_{m(x_{i+1},x',\ldots,x',|s_{i+1}|)}|
\end{aligned}$$

$$= |z_{M^0(x_{i+1},|s_{i+1}|)}|$$
$$< |z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|$$
$$< m(x_{i+1}, x', \dots, x', |z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|)$$
$$= m(x_{i+1}, x', \dots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|, |z_{M^1(x_i,|s_i|)}{}^\frown\langle 1\rangle|, \dots, |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|))$$
$$\vdots$$
$$< m(x_{i+1}, x', \dots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|, \dots, |z_{M^{k-1}(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|))$$

We turn to showing that $\langle x', z_{M^j(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle\rangle$ is always a legal move. Since $z_{M^j(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle$ is on the antichain avoided by any K^*, the problem is reduced to showing that this move isn't forbidden by

$$L_{m(x_{i+1},x',\dots,x',\max(|s_{i+1}|,|z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|,\dots,|z_{M^{j-1}(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|,|z_{M^j(x_i,|s_i|)}{}^\frown\langle 1\rangle|,\dots,|z_{M^k(x_i,|s_i|)}{}^\frown\langle 1\rangle|))}$$
$$= L_{m(x_{i+1},x',\dots,x',|z_{M^{j-1}(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|)}$$

which we can see here:

$$m(x_{i+1}, x', \dots, x', |z_{M^{j-1}(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|)$$
$$= m(x_{i+1}, x', \dots, x', M^{j-1}(x_{i+1}, |s_{i+1}|) + 1)$$
$$= M^0(x_{i+1}, M^{j-1}(x_{i+1}, |s_{i+1}|) + 1)$$
$$= M^j(x_{i+1}, s_{i+1})$$
$$< |z_{M^j(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|$$

We can conclude by showing that $\langle x_{i+1}, s_{i+1}\rangle$ is always a legal move. We can see it avoids

$$L_{m(x_i,x',\dots,x',\max(|s_i|,|z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle|,\dots,|z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|))}$$

since

$$m(x_i, x', \dots, x', \max(|s_i|, |z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle|, \dots, |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|))$$
$$= m(x_i, x', \dots, x', |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|)$$
$$= m(x_i, x', \dots, x', M^{k-1}(x_i, |s_i|) + 1)$$
$$\leq |s_{i+1}|$$

Since $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ, it avoided

$$K^*_{S(x_h,x',\dots,x',M^{k-1}(x_h,|s_h|)+1)}$$

$$= K^*_{S(x_h,x',\dots,x',\max(|s_h|,|z_{M^0(x_h,|s_h|)}{}^\frown\langle 1\rangle|,\dots,|z_{M^{k-1}(x_h,|s_h|)}{}^\frown\langle 1\rangle|))}$$

for $h \le i$. And when $h < i$, we see it avoids both:

$$K^*_{S(x_{h+1},x',\dots,x',\max(|s_{h+1}|,|z_{M^0(x_{h+1},|s_{h+1}|)}{}^\frown\langle 1\rangle|,\dots,|z_{M^{j-1}(x_{h+1},|s_{h+1}|)}{}^\frown\langle 1\rangle|,|z_{M^j(x_h,|s_h|)}{}^\frown\langle 1\rangle|,\dots,|z_{M^k(x_h,|s_h|)}{}^\frown\langle 1\rangle|))}$$

$$= K^*_{S(x_{h+1},x',\dots,x',|z_{M^{j-1}(x_{h+1},|s_{h+1}|)}{}^\frown\langle 1\rangle|)}$$

$$= K^*_{S(x_{h+1},x',\dots,x',M^{j-1}(x_{h+1},|s_{h+1}|)+1)}$$

$$\subseteq K^*_{S(x_{h+1},x',\dots,x',M^{k-1}(x_{h+1},|s_{h+1}|)+1)}$$

and:

$$K^*_{S(x_{h+1},x',\dots,x',\max(|s_{h+1}|,|z_{M^0(x_h,|s_h|)}{}^\frown\langle 1\rangle|,\dots,|z_{M^k(x_h,|s_h|)}{}^\frown\langle 1\rangle|))}$$

$$= K^*_{S(x_{h+1},x',\dots,x',|s_{k+1}|)}$$

$$\subseteq K^*_{S(x_{h+1},x',\dots,x',M^{k-1}(x_{h+1},|s_{h+1}|)+1)}$$

This conclues the proof. $\square$

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