

# 1 ARHANGELSKII'S $\alpha$ -PRINCIPLES AND SELECTION GAMES

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ABSTRACT. Arhangel'skii's properties  $\alpha_2$  and  $\alpha_4$  defined for convergent sequences may be characterized in terms of Scheeper's selection principles. We generalize these results to hold for more general collections and consider these results in terms of selection games.

3 The following characterizations were given as Definition 1 by Kocinac in [7].

4 **Definition 1.** *Arhangel'skii's  $\alpha$ -principles*  $\alpha_i(\mathcal{A}, \mathcal{B})$  are defined as follows for  $i \in$   
5  $\{1, 2, 3, 4\}$ . Let  $A_n \in \mathcal{A}$  for all  $n < \omega$ ; then there exists  $B \in \mathcal{B}$  such that:

- 6  $\alpha_1$ :  $A_n \cap B$  is cofinite in  $A_n$  for all  $n < \omega$ .
- 7  $\alpha_2$ :  $A_n \cap B$  is infinite for all  $n < \omega$ .
- 8  $\alpha_3$ :  $A_n \cap B$  is infinite for infinitely-many  $n < \omega$ .
- 9  $\alpha_4$ :  $A_n \cap B$  is non-empty for infinitely-many  $n < \omega$ .

10 When  $(\mathcal{A}, \mathcal{B})$  is omitted, it is assumed that  $\mathcal{A} = \mathcal{B}$  is the collection  $\Gamma_{X,x}$  of se-  
11 quences converging to some point  $x \in X$ , as introduced by Arhangel'skii in [1]. Pro-  
12 vided  $\mathcal{A}$  only contains infinite sets, it's easy to see that  $\alpha_n(\mathcal{A}, \mathcal{B})$  implies  $\alpha_{n+1}(\mathcal{A}, \mathcal{B})$ .

13 We aim to relate these to the following games.

14 **Definition 2.** The *selection game*  $G_1(\mathcal{A}, \mathcal{B})$  (resp.  $G_{fin}(\mathcal{A}, \mathcal{B})$ ) is an  $\omega$ -length  
15 game involving Players I and II. During round  $n$ , I chooses  $A_n \in \mathcal{A}$ , followed  
16 by II choosing  $a_n \in A_n$  (resp.  $F_n \in [A_n]^{<\aleph_0}$ ). Player II wins in the case that  
17  $\{a_n : n < \omega\} \in \mathcal{B}$  (resp.  $\bigcup\{F_n : n < \omega\} \in \mathcal{B}$ ), and Player I wins otherwise.

18 Such games are well-represented in the literature; see [12] for example. We will  
19 also consider the similarly-defined games  $G_{<2}(\mathcal{A}, \mathcal{B})$  (II chooses 0 or 1 points from  
20 each choice by I) and  $G_{cf}(\mathcal{A}, \mathcal{B})$  (II chooses cofinitely-many points).

21 **Definition 3.** Let  $P$  be a player in a game  $G$ .  $P$  has a *winning strategy* for  $G$ ,  
22 denoted  $P \uparrow G$ , if  $P$  has a strategy that defeats every possible counterplay by  
23 their opponent. If a strategy only relies on the round number and ignores the  
24 moves of the opponent, the strategy is said to be *predetermined*; the existence of a  
25 predetermined winning strategy is denoted  $P \uparrow_{\text{pre}} G$ .

26 We briefly note that the statement  $I \not\uparrow_{\text{pre}} G_\star(\mathcal{A}, \mathcal{B})$  is often denoted as the *selection*  
27 *principle*  $S_\star(\mathcal{A}, \mathcal{B})$ .

28 **Definition 4.** Let  $\Gamma_{X,x}$  be the collection of non-trivial sequences  $S \subseteq X$  converging  
29 to  $x$ , that is, infinite subsets of  $X \setminus \{x\}$  such that for each neighborhood  $U$  of  $x$ ,  
30  $S \cap U$  is cofinite in  $S$ .

**Definition 5.** Let  $\Gamma_X$  be the collection of open  $\gamma$ -covers  $\mathcal{U}$  of  $X$ , that is, infinite open covers of  $X$  such that  $X \notin \mathcal{U}$  and for each  $x \in X$ ,  $\{U \in \mathcal{U} : x \in U\}$  is cofinite in  $\mathcal{U}$ .

The similarity in nomenclature follows from the observation that every non-trivial sequence in  $C_p(X)$  converging to the zero function  $\mathbf{0}$  naturally defines a corresponding  $\gamma$ -cover in  $X$ , see e.g. Theorem 4 of [13].

The equivalence of  $\alpha_2(\Gamma_{X,x}\Gamma_{X,x})$  and  $\text{I} \nVdash_{\text{pre}} G_1(\Gamma_{X,x}, \Gamma_{X,x})$  was briefly asserted by Sakai in the introduction of [11]; the similar equivalence of  $\alpha_4(\Gamma_{X,x}\Gamma_{X,x})$  and  $\text{I} \nVdash_{\text{pre}} G_{fin}(\Gamma_{X,x}, \Gamma_{X,x})$  seems to be folklore. In fact, these relationships hold in more generality.

Note that by these definitions, convergent sequences (resp.  $\gamma$ -covers) may be uncountable, but any infinite subset of either would remain a convergent sequence (resp.  $\gamma$ -cover), in particular, countably infinite subsets. We capture this idea as follows.

**Definition 6.** Say a collection  $\mathcal{A}$  is  $\Gamma$ -like if it satisfies the following for each  $A \in \mathcal{A}$ .

- $|A| \geq \aleph_0$ .
- If  $A' \subseteq A$  and  $|A'| \geq \aleph_0$ , then  $A' \in \mathcal{A}$ .

We also require the following.

**Definition 7.** Say a collection  $\mathcal{A}$  is *almost- $\Gamma$ -like* if for each  $A \in \mathcal{A}$ , there is  $A' \subseteq A$  such that:

- $|A'| = \aleph_0$ .
- If  $A''$  is a cofinite subset of  $A'$ , then  $A'' \in \mathcal{A}$ .

So all  $\Gamma$ -like sets are almost- $\Gamma$ -like.

We are now able to prove a few general equivalences between  $\alpha$ -principles and selection games.

#### 1. ON $\alpha_2(\mathcal{A}, \mathcal{B})$ AND $G_1(\mathcal{A}, \mathcal{B})$

**Theorem 8.** Let  $\mathcal{A}$  be almost- $\Gamma$ -like and  $\mathcal{B}$  be  $\Gamma$ -like. Then  $\alpha_2(\mathcal{A}, \mathcal{B})$  holds if and only if  $\text{I} \nVdash_{\text{pre}} G_1(\mathcal{A}, \mathcal{B})$ .

*Proof.* We first assume  $\alpha_2(\mathcal{A}, \mathcal{B})$  and let  $A_n \in \mathcal{A}$  for  $n < \omega$  define a predetermined strategy for I. We may apply  $\alpha_2(\mathcal{A}, \mathcal{B})$  to choose  $B \in \mathcal{B}$  such that  $|A_n \cap B| \geq \aleph_0$ . We may then choose  $a_n \in (A_n \cap B) \setminus \{a_i : i < n\}$  for each  $n < \omega$ . It follows that  $B' = \{a_n : n < \omega\} \in \mathcal{B}$  since  $B'$  is an infinite subset of  $B \in \mathcal{B}$ ; therefore  $A_n$  does not define a winning predetermined strategy for I.

Now suppose  $\text{I} \nVdash_{\text{pre}} G_1(\mathcal{A}, \mathcal{B})$ . Given  $A_n \in \mathcal{A}$  for  $n < \omega$ , first choose  $A'_n \in \mathcal{A}$  such

that  $A'_n = \{a_{n,j} : j < \omega\} \subseteq A_n$ ,  $j < k$  implies  $a_{n,j} \neq a_{n,k}$ , and  $A_{n,m} = \{a_{n,j} : m \leq j < \omega\} \in \mathcal{A}$ . Finally choose some  $\theta : \omega \rightarrow \omega$  such that  $|\theta^{\leftarrow}(n)| = \aleph_0$  for each  $n < \omega$ .

Since playing  $A_{\theta(m),m}$  during round  $m$  does not define a winning strategy for I in  $G_1(\mathcal{A}, \mathcal{B})$ , II may choose  $x_m \in A_{\theta(m),m}$  such that  $B = \{x_m : m < \omega\} \in \mathcal{B}$ . Choose  $i_m < \omega$  for each  $m < \omega$  such that  $x_m = a_{\theta(m),i_m}$ , noting  $i_m \geq m$ . It follows that  $A_n \cap B \supseteq \{a_{\theta(m),i_m} : m \in \theta^{\leftarrow}(n)\}$ . Since for each  $m \in \theta^{\leftarrow}(n)$  there exists  $M \in \theta^{\leftarrow}(n)$  such that  $m \leq i_m < M \leq i_M$ , and therefore  $a_{\theta(m),i_m} \neq a_{\theta(m),i_M} = a_{\theta(M),i_M}$ , we have shown that  $A_n \cap B$  is infinite. Thus  $B$  witnesses  $\alpha_2(\mathcal{A}, \mathcal{B})$ .  $\square$

While  $\alpha_2(\mathcal{A}, \mathcal{B})$  involves infinite intersection and  $G_1(\mathcal{A}, \mathcal{B})$  involves single selections, the previous result is made more intuitive given the following result, shown for  $\mathcal{A} = \mathcal{B} = \Gamma_{X,x}$  by Nogura in [8].

**Definition 9.**  $\alpha'_2(\mathcal{A}, \mathcal{B})$  is the following claim: if  $A_n \in \mathcal{A}$  for all  $n < \omega$ , then there exists  $B \in \mathcal{B}$  such that  $A_n \cap B$  is nonempty for all  $n < \omega$ .

(Note that  $\alpha_5$  is sometimes used in the literature in place of  $\alpha'_2$ .)

**Proposition 10.** If  $\mathcal{A}$  is almost- $\Gamma$ -like, then  $\alpha_2(\mathcal{A}, \mathcal{B})$  is equivalent to  $\alpha'_2(\mathcal{A}, \mathcal{B})$ .

*Proof.* The forward implication is immediate, so we assume  $\alpha'_2(\mathcal{A}, \mathcal{B})$ . Given  $A_n \in \mathcal{A}$ , we apply the almost- $\Gamma$ -like property to obtain  $A'_n = \{a_{n,m} : m < \omega\} \subseteq A_n$  such that  $A_{n,m} = A_n \setminus \{a_{i,j} : i, j < m\} \in \mathcal{A}$  for all  $m < \omega$ .

By applying  $\alpha'_2(\mathcal{A}, \mathcal{B})$  to  $A_{n,m}$ , we obtain  $B \in \mathcal{B}$  such that  $A_{n,m} \cap B$  is nonempty for all  $n, m < \omega$ . Since it follows that  $A_n \cap B$  is infinite for all  $n < \omega$ , we have established  $\alpha_2(\mathcal{A}, \mathcal{B})$ .  $\square$

## 2. ON $\alpha_4(\mathcal{A}, \mathcal{B})$ AND $G_{fin}(\mathcal{A}, \mathcal{B})$

A similar correspondence exists between  $\alpha_4(\mathcal{A}, \mathcal{B})$  and  $G_{fin}(\mathcal{A}, \mathcal{B})$ .

**Theorem 11.** Let  $\mathcal{A}$  be almost- $\Gamma$ -like and  $\mathcal{B}$  be  $\Gamma$ -like. Then  $\alpha_4(\mathcal{A}, \mathcal{B})$  holds if and only if  $\text{I} \nVdash_{pre} G_{<2}(\mathcal{A}, \mathcal{B})$  if and only if  $\text{I} \nVdash_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$ .

*Proof.* We first assume  $\alpha_4(\mathcal{A}, \mathcal{B})$  and let  $A_n \in \mathcal{A}$  for  $n < \omega$  define a predetermined strategy for I in  $G_{<2}(\mathcal{A}, \mathcal{B})$ . We then may choose  $A'_n \in \mathcal{A}$  where  $A'_n = \{a_{n,j} : j < \omega\} \subseteq A_n$ ,  $j < k$  implies  $a_{n,j} \neq a_{n,k}$ , and  $A''_n = A'_n \setminus \{a_{i,j} : i, j < n\} \in \mathcal{A}$ .

By applying  $\alpha_4(\mathcal{A}, \mathcal{B})$  to  $A''_n$ , we obtain  $B \in \mathcal{B}$  such that  $A''_n \cap B \neq \emptyset$  for infinitely many  $n < \omega$ . We then let  $F_n = \emptyset$  when  $A''_n \cap B = \emptyset$ , and  $F_n = \{x_n\}$  for some  $x_n \in A''_n \cap B$  otherwise. Then we will have that  $B' = \bigcup \{F_n : n < \omega\} \subseteq B$  belongs to  $\mathcal{B}$  once we show that  $B'$  is infinite. To see this, for  $m \leq n < \omega$  note that either  $F_m$  is empty (and we let  $j_m = 0$ ) or  $F_m = \{a_{m,j_m}\}$  for some  $j_m \geq m$ ; choose  $N < \omega$  such that  $j_m < N$  for all  $m \leq n$  and  $F_N = \{x_N\}$ . Thus  $F_m \neq F_N$  for all  $m \leq n$  since  $x_N \notin \{a_{i,j} : i, j < N\}$ . Thus II may defeat the predetermined strategy  $A_n$  by playing  $F_n$  each round.

Since  $\text{I} \nVdash_{pre} G_{<2}(\mathcal{A}, \mathcal{B})$  immediately implies  $\text{I} \nVdash_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$ , we assume the latter.

Given  $A_n \in \mathcal{A}$  for  $n < \omega$ , we note this defines a (non-winning) predetermined strategy for I, so II may choose  $F_n \in [A_n]^{<\aleph_0}$  such that  $B = \bigcup \{F_n : n < \omega\} \in \mathcal{B}$ . Since  $B$  is infinite, we note  $F_n \neq \emptyset$  for infinitely-many  $n < \omega$ . Thus  $B$  witnesses  $\alpha_4(\mathcal{A}, \mathcal{B})$  since  $A_n \cap B \supseteq F_n \neq \emptyset$  for infinitely-many  $n < \omega$ .  $\square$

This shows that II gains no advantage from picking more than one point per round. This in fact only depends on  $\mathcal{B}$  being  $\Gamma$ -like, which we formalize in the following results.

**Theorem 12.** Let  $\mathcal{B}$  be  $\Gamma$ -like. Then  $\text{I} \uparrow_{pre} G_{<2}(\mathcal{A}, \mathcal{B})$  if and only if  $\text{I} \uparrow_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$ .

*Proof.* Assume  $\bigcup \mathcal{A}$  is well-ordered. Given a winning predetermined strategy  $A_n$  for I in  $G_{<2}(\mathcal{A}, \mathcal{B})$ , consider  $F_n \in [A_n]^{<\aleph_0}$ . We set

$$F_n^* = \begin{cases} \emptyset & \text{if } F_n \setminus \bigcup \{F_m : m < n\} = \emptyset \\ \{\min(F_n \setminus \bigcup \{F_m : m < n\})\} & \text{otherwise} \end{cases}$$

Since  $|F_n^*| < 2$ , we have that  $\bigcup\{F_n^* : n < \omega\} \notin \mathcal{B}$ . In the case that  $\bigcup\{F_n^* : n < \omega\}$  is finite, we immediately see that  $\bigcup\{F_n : n < \omega\}$  is also finite and therefore not in  $\mathcal{B}$ . Otherwise  $\bigcup\{F_n^* : n < \omega\} \notin \mathcal{B}$  is an infinite subset of  $\bigcup\{F_n : n < \omega\}$ , and thus  $\bigcup\{F_n : n < \omega\} \notin \mathcal{B}$  too. Therefore  $A_n$  is a winning predetermined strategy for I in  $G_{fin}(\mathcal{A}, \mathcal{B})$  as well.  $\square$

**Theorem 13.** *Let  $\mathcal{B}$  be  $\Gamma$ -like. Then  $I \uparrow G_{<2}(\mathcal{A}, \mathcal{B})$  if and only if  $I \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ .*

*Proof.* Assume  $\bigcup \mathcal{A}$  is well-ordered. Suppose  $I \uparrow G_{<2}(\mathcal{A}, \mathcal{B})$  is witnessed by the strategy  $\sigma$ . Let  $\langle \rangle^* = \langle \rangle$ , and for  $s \smallfrown \langle F \rangle \in ([\bigcup \mathcal{A}]^{<\aleph_0})^{<\omega} \setminus \{\langle \rangle\}$  let

$$(s \smallfrown \langle F \rangle)^* = \begin{cases} s^* \smallfrown \langle \emptyset \rangle & \text{if } F \setminus \bigcup \text{range}(s) = \emptyset \\ s^* \smallfrown \langle \{\min(F \setminus \bigcup \text{range}(s))\} \rangle & \text{otherwise} \end{cases}$$

We then define the strategy  $\tau$  for I in  $G_{fin}(\mathcal{A}, \mathcal{B})$  by  $\tau(s) = \sigma(s^*)$ . Then given any counterattack  $\alpha \in ([\bigcup \mathcal{A}]^{<\aleph_0})^\omega$  by II played against  $\tau$ , we note that  $\alpha^* = \bigcup\{(\alpha \upharpoonright n)^* : n < \omega\}$  is a counterattack to  $\sigma$ , and thus loses. This means  $B = \bigcup \text{range}(\alpha^*) \notin \mathcal{B}$ .

We consider two cases. The first is the case that  $\bigcup \text{range}(\alpha^*)$  is finite. Noting that  $\alpha^*(m) \cap \alpha^*(n) = \emptyset$  whenever  $m \neq n$ , there exists  $N < \omega$  such that  $\alpha^*(n) = \emptyset$  for all  $n > N$ . As a result,  $\bigcup \text{range}(\alpha) = \bigcup \text{range}(\alpha \upharpoonright n)$ , and thus  $\bigcup \text{range}(\alpha)$  is finite, and therefore not in  $\mathcal{B}$ .

In the other case,  $\bigcup \text{range}(\alpha^*) \notin \mathcal{B}$  is an infinite subset of  $\bigcup \text{range}(\alpha)$ , and therefore  $\bigcup \text{range}(\alpha) \notin \mathcal{B}$  as well. Thus we have shown that  $\tau$  is a winning strategy for I in  $G_{fin}(\mathcal{A}, \mathcal{B})$ .  $\square$

We note that the above proof technique could be used to establish that perfect-information and limited-information strategies for II in  $G_{fin}(\mathcal{A}, \mathcal{B})$  may be improved to be valid in  $G_{<2}(\mathcal{A}, \mathcal{B})$ , provided  $\mathcal{B}$  is  $\Gamma$ -like. As such,  $G_{<2}(\mathcal{A}, \mathcal{B})$  and  $G_{fin}(\mathcal{A}, \mathcal{B})$  are effectively equivalent games under this hypothesis, so we will no longer consider  $G_{<2}(\mathcal{A}, \mathcal{B})$ .

### 3. PERFECT INFORMATION AND PREDETERMINED STRATEGIES

We now demonstrate the following, in the spirit of Pawlikowski's celebrated result that a winning strategy for the first player in the Rothberger game may always be improved to a winning predetermined strategy [10].

**Theorem 14.** *Let  $\mathcal{A}$  be almost- $\Gamma$ -like and  $\mathcal{B}$  be  $\Gamma$ -like. Then*

- $I \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$  if and only if  $I \uparrow \overset{pre}{G_{fin}}(\mathcal{A}, \mathcal{B})$ , and
- $I \uparrow G_1(\mathcal{A}, \mathcal{B})$  if and only if  $I \uparrow \overset{pre}{G_1}(\mathcal{A}, \mathcal{B})$ .

*Proof.* We assume  $I \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$  and let the symbol  $\dagger$  mean  $< \aleph_0$  (respectively,  $I \uparrow G_1(\mathcal{A}, \mathcal{B})$  and  $\dagger = 1$ , and for convenience we assume II plays singleton subsets of  $\mathcal{A}$  rather than elements). As  $\mathcal{A}$  is almost- $\Gamma$ -like, there is a winning strategy  $\sigma$  where  $|\sigma(s)| = \aleph_0$  and  $\sigma(s) \cap \bigcup \text{range}(s) = \emptyset$  (that is,  $\sigma$  never replays the choices of II) for all partial plays  $s$  by II.

For each  $s \in \omega^{<\omega}$ , suppose  $F_{s \upharpoonright m} \in [\bigcup \mathcal{A}]^\dagger$  is defined for each  $0 < m \leq |s|$ . Then let  $s^* : |s| \rightarrow [\bigcup \mathcal{A}]^\dagger$  be defined by  $s^*(m) = F_{s \upharpoonright m+1}$ , and define  $\tau' : \omega^{<\omega} \rightarrow \mathcal{A}$  by  $\tau'(s) = \sigma(s^*)$ . Finally, set  $[\sigma(s^*)]^\dagger = \{F_{s \smallfrown \langle n \rangle} : n < \omega\}$ , and for some bijection

151  $b : \omega^{<\omega} \rightarrow \omega$  let  $\tau(n) = \tau'(b(n))$  be a predetermined strategy for I in  $G_{fin}(\mathcal{A}, \mathcal{B})$   
 152 (resp.  $G_1(\mathcal{A}, \mathcal{B})$ ).

153 Suppose  $\alpha$  is a counterattack by II against  $\tau$ , so

$$\alpha(n) \in [\tau(n)]^\dagger = [\tau'(b(n))]^\dagger = [\sigma(b(n)^*)]^\dagger$$

154 It follows that  $\alpha(n) = F_{b(n) \smallfrown \langle m \rangle}$  for some  $m < \omega$ . In particular, there is some  
 155 infinite subset  $W \subseteq \omega$  and  $f \in \omega^\omega$  such that  $\{\alpha(n) : n \in W\} = \{F_{f \upharpoonright n+1} : n < \omega\}$ .  
 156 Note here that  $(f \upharpoonright n+1)^* = (f \upharpoonright n)^* \smallfrown \langle F_{f \upharpoonright n+1} \rangle$ . This shows that  $F_{f \upharpoonright n+1} \in [\sigma((f \upharpoonright$   
 157  $n)^*)]^\dagger$  is an attempt by II to defeat  $\sigma$ , which fails. Thus  $\bigcup \{F_{f \upharpoonright n+1} : n < \omega\} =$   
 158  $\bigcup \{\alpha(n) : n \in W\} \notin \mathcal{B}$ , and since this set is infinite (as  $\sigma$  prevents II from repeating  
 159 choices) we have  $\bigcup \{\alpha(n) : n < \omega\} \notin \mathcal{B}$  too. Therefore  $\tau$  is winning.  $\square$

160 Note that the assumption in Theorem 14 that  $\mathcal{A}$  be almost- $\Gamma$ -like cannot be  
 161 omitted. In [2] an example of a space  $X^*$  and point  $\infty \in X^*$  where  $I \uparrow G_1(\mathcal{A}, \mathcal{B})$   
 162 but  $I \not\uparrow_{pre} G_1(\mathcal{A}, \mathcal{B})$  is given, where  $\mathcal{A}$  is the set of open neighborhoods of  $\infty$  (which  
 163 are all uncountable), and  $\mathcal{B}$  is the set  $\Gamma_{X^*, \infty}$  of sequences converging to that point.  
 164 (Note that  $G_1(\mathcal{A}, \mathcal{B})$  is called  $Gru_{O,P}(X^*, \infty)$  in that paper, and an equivalent game  
 165  $Gru_{K,P}(X)$  is what is directly studied. In fact, more is shown: I has a winning  
 166 perfect-information strategy, but for any natural number  $k$ , any strategy that only  
 167 uses the most recent  $k$  moves of II and the round number can be defeated.)

168 While  $\mathcal{A}$  is often not almost- $\Gamma$ -like in general, it may satisfy that property in  
 169 combination with the selection principles being considered.

170 **Proposition 15.** *Let  $\mathcal{B}$  be  $\Gamma$ -like,  $\mathcal{B} \subseteq \mathcal{A}$ , and  $I \not\uparrow_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$ . Then  $\mathcal{A}$  is almost-*  
 171  *$\Gamma$ -like.*

172 *Proof.* Let  $A \in \mathcal{A}$ , and for all  $n < \omega$  let  $A_n = A$ . Then  $A_n$  is not a winning  
 173 predetermined strategy for I, so II may choose finite sets  $B_n \subseteq A_n = A$  such that  
 174  $A' = \bigcup \{B_n : n < \omega\} \in \mathcal{B} \subseteq \mathcal{A}$ .

175 It follows that  $A' \subseteq A$  and  $|A'| = \aleph_0$ , and for any infinite subset  $A'' \subseteq A'$  (in  
 176 particular, any cofinite subset),  $A'' \in \mathcal{B} \subseteq \mathcal{A}$ . Thus  $\mathcal{A}$  is almost- $\Gamma$ -like.  $\square$

177 Note that in the previous result,  $I \not\uparrow_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$  could be weakened to the choice  
 178 principle  $(\mathcal{A} \smallfrown \mathcal{B})$ : for every member of  $\mathcal{A}$ , there is some countable subset belonging to  
 179  $\mathcal{B}$ .

180 **Corollary 16.** *Let  $\mathcal{B}$  be  $\Gamma$ -like and  $\mathcal{B} \subseteq \mathcal{A}$ . Then*

- 181  $\bullet$   $I \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$  if and only if  $I \uparrow_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$ , and
- 182  $\bullet$   $I \uparrow G_1(\mathcal{A}, \mathcal{B})$  if and only if  $I \uparrow_{pre} G_1(\mathcal{A}, \mathcal{B})$ .

183 *Proof.* Assuming  $I \not\uparrow_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$ , we have  $I \not\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$  by Proposition 15 and  
 184 Theorem 14.

185 Similarly, assuming  $I \not\uparrow_{pre} G_1(\mathcal{A}, \mathcal{B}) \Rightarrow I \not\uparrow_{pre} G_{fin}(\mathcal{A}, \mathcal{B})$ , we have  $I \not\uparrow G_1(\mathcal{A}, \mathcal{B})$  by  
 186 Proposition 15 and Theorem 14.  $\square$

187 This corollary generalizes e.g. Theorems 26 and 30 of [12] Theorem 5 of [6], and  
 188 Corollary 36 of [3].

189 In summary, using the selection principle notation  $S_\star(\mathcal{A}, \mathcal{B})$ :

190 **Corollary 17.** *Let  $\mathcal{B}$  be  $\Gamma$ -like and  $\mathcal{B} \subseteq \mathcal{A}$ . Then*

- 191 •  $I \nVdash G_1(\mathcal{A}, \mathcal{B})$  if and only if  $S_1(\mathcal{A}, \mathcal{B})$  if and only if  $\alpha_2(\mathcal{A}, \mathcal{B})$ .
- 192 •  $I \nVdash G_{fin}(\mathcal{A}, \mathcal{B})$  if and only if  $S_{fin}(\mathcal{A}, \mathcal{B})$  if and only if  $\alpha_4(\mathcal{A}, \mathcal{B})$ , and

#### 193 4. DISJOINT SELECTIONS

194 In each  $\alpha_i(\mathcal{A}, \mathcal{B})$  principle, it is not required for the collection  $\{A_n : n < \omega\}$  to  
 195 be pairwise disjoint. However, in many cases it may as well be.

196 **Definition 18.** For  $i \in \{1, 2, 3, 4\}$  let  $\alpha_{i,1}(\mathcal{A}, \mathcal{B})$  denote the claim that  $\alpha_i(\mathcal{A}, \mathcal{B})$   
 197 holds provided the collection  $\{A_n : n < \omega\}$  is pairwise disjoint.

198 Of course,  $\alpha_i(\mathcal{A}, \mathcal{B})$  implies  $\alpha_{i,1}(\mathcal{A}, \mathcal{B})$ . It's also immediate that  $\alpha_{i,1}(\mathcal{A}, \mathcal{B})$  implies  
 199  $\alpha_{i,1+1}(\mathcal{A}, \mathcal{B})$  for the same reason that  $\alpha_i(\mathcal{A}, \mathcal{B})$  implies  $\alpha_{i+1}(\mathcal{A}, \mathcal{B})$ .

200 We take advantage of the following lemma. The citation is given to Peter Nyikos  
 201 who provides a nice proof. At a 2020 Fall meeting of the Carolinas Topology  
 202 Seminar, it was suggested by Alan Dow that this lemma may be known as the  
 203 “[Hausdorff] Disjoint Refinement Lemma”, as found in e.g. [4, Lemma 3.4].

204 **Lemma 19** (Lemma 1.2 of [9]). *Given a family  $\{A_n : n < \omega\}$  of infinite sets, there  
 205 exist infinite subsets  $A'_n \subseteq A_n$  such that  $\{A'_n : n < \omega\}$  is pairwise disjoint.*

206 **Proposition 20.** *Let  $\mathcal{A}$  be  $\Gamma$ -like. For  $i \in \{2, 3, 4\}$ ,  $\alpha_i(\mathcal{A}, \mathcal{B})$  is equivalent to  
 207  $\alpha_{i,1}(\mathcal{A}, \mathcal{B})$ .*

208 *Proof.* Assume  $\alpha_{i,1}(\mathcal{A}, \mathcal{B})$ . Let  $A_n \in \mathcal{A}$ . By applying the previous lemma, we have  
 209  $\{A'_n : n < \omega\}$  pairwise disjoint with each  $A'_n$  being an infinite subset of  $A_n$ . Since  $\mathcal{A}$   
 210 is  $\Gamma$ -like,  $A'_n \in \mathcal{A}$ , so we have a witness  $B \in \mathcal{B}$  such that  $A'_n \cap B$  satisfies  $\alpha_{i,1}(\mathcal{A}, \mathcal{B})$   
 211 for all  $n < \omega$ . Since  $A'_n \subseteq A_n$ , it follows that  $A_n \cap B$  satisfies  $\alpha_i(\mathcal{A}, \mathcal{B})$  for all  
 212  $n < \omega$ .  $\square$

213 It's also true that  $\alpha_1(\Gamma_{X,x}, \Gamma_{X,x})$  is equivalent to  $\alpha_{1,1}(\Gamma_{X,x}, \Gamma_{X,x})$ , which is cap-  
 214 tured by the following theorem.

215 **Theorem 21.** *Let  $\mathcal{A}$  be a  $\Gamma$ -like collection closed under finite unions and  $\mathcal{A} \subseteq \mathcal{B}$ .  
 216 Then  $\alpha_1(\mathcal{A}, \mathcal{B})$  is equivalent to  $\alpha_{1,1}(\mathcal{A}, \mathcal{B})$ .*

217 *Proof.* Let  $A_n \in \mathcal{A}$  and assume  $\alpha_{1,1}(\mathcal{A}, \mathcal{B})$ . To apply the assumption, we will define  
 218 a pairwise disjoint collection  $\{A'_n : n < \omega\}$ . First let  $0' = 0$  and  $A'_0 = A_0$ . Then  
 219 suppose  $m' \geq m$  and  $A'_m \subseteq A_{m'} \subseteq \bigcup_{i \leq m} A'_i$  are defined for all  $m \leq n$ .

220 If  $A_k \setminus \bigcup_{m \leq n} A'_m$  is finite for  $k > n'$ , let  $B = \bigcup_{m \leq n'} A_m \in \mathcal{A} \subseteq \mathcal{B}$ . This  $B$  then  
 221 witnesses  $\alpha_1(\mathcal{A}, \mathcal{B})$  since  $A_k \setminus B$  is finite for all  $k < \omega$ .

222 Otherwise pick the minimal  $(n+1)' > n$  where  $A'_{n+1} = A_{(n+1)'} \setminus \bigcup_{m \leq n} A'_m$  is  
 223 infinite. It follows that  $A'_{n+1} \subseteq A_{(n+1)'} \subseteq \bigcup_{m \leq n+1} A'_m$ . By construction,  $\{A'_n : n < \omega\}$   
 224 is a pairwise disjoint collection of members of  $\mathcal{A}$ , and we may apply  $\alpha_{1,1}(\mathcal{A}, \mathcal{B})$   
 225 to obtain  $B \in \mathcal{B}$  where  $A'_n \setminus B$  is finite for all  $n < \omega$ .

226 Finally let  $k < \omega$ . If  $k = n'$  for some  $n < \omega$ , then  $A_k \setminus B = A_{n'} \setminus B \subseteq$   
 227  $(\bigcup_{m \leq n} A'_m) \setminus B$  is finite. Otherwise,  $n' < k < (n+1)'$  for some  $n < \omega$ . Then  
 228  $(A_k \setminus \bigcup_{m \leq n} A'_m) \setminus B \subseteq A_k \setminus \bigcup_{m \leq n} A'_m$  is finite, and  $(A_k \cap \bigcup_{m \leq n} A'_m) \setminus B \subseteq$   
 229  $(\bigcup_{m \leq n} A'_m) \setminus B$  is finite, showing  $A_k \setminus B$  is finite.  $\square$

230 Another fractional version of these  $\alpha$ -principles is given as  $\alpha_{1.5}$  in [9], defined in  
 231 general as follows.

**Definition 22.** Let  $\alpha_{1.5}(\mathcal{A}, \mathcal{B})$  be the assertion that when  $A_n \in \mathcal{A}$  and  $\{A_n : n < \omega\}$  is pairwise disjoint, then there exists  $B \in \mathcal{B}$  such that  $A_n \cap B$  is cofinite in  $A_n$  for infinitely-many  $n < \omega$ .

It's immediate from their definitions that  $\alpha_{1.1}(\mathcal{A}, \mathcal{B})$  implies  $\alpha_{1.5}(\mathcal{A}, \mathcal{B})$ , which implies  $\alpha_{3.1}(\mathcal{A}, \mathcal{B})$ . Nyikos originally showed that  $\alpha_{1.5}(\Gamma_{X,x}, \Gamma_{X,x})$  implies  $\alpha_2(\Gamma_{X,x}, \Gamma_{X,x})$ ; this result generalizes as follows.

**Theorem 23.** *Let  $\mathcal{A}$  be a  $\Gamma$ -like collection closed under finite unions. Then  $\alpha_{1.5}(\mathcal{A}, \mathcal{B})$  implies  $\alpha_2(\mathcal{A}, \mathcal{B})$ .*

*Proof.* We assume  $\alpha_{1.5}(\mathcal{A}, \mathcal{B})$  and demonstrate  $\alpha_{2.1}(\mathcal{A}, \mathcal{B})$ , which is equivalent to  $\alpha_2(\mathcal{A}, \mathcal{B})$  by Proposition 20. So let  $A_n \in \mathcal{A}$  such that  $\{A_n : n < \omega\}$  is pairwise-disjoint.

We may partition each  $A_n$  into  $\{A_{n,m} : m < \omega\}$  with  $A_{n,m} \in \mathcal{A}$  for all  $m < \omega$ . Let  $A'_n = \bigcup \{A_{i,j} : i + j = n\} \in \mathcal{A}$ ; since  $\{A'_n : n < \omega\}$  is pairwise disjoint, we may apply  $\alpha_{1.5}(\mathcal{A}, \mathcal{B})$  to obtain  $B \in \mathcal{B}$  where  $A'_n \cap B$  is cofinite in  $A'_n$  for infinitely-many  $n < \omega$ .

Then for  $n < \omega$ , choose  $N \geq n$  with  $A'_N \cap B$  cofinite in  $A'_N$ . Then  $A_{n,N-n} \subseteq A'_N$ , so  $A_{n,N-n} \cap B$  is cofinite in  $A_{n,N-n}$ , in particular,  $A_{n,N-n} \cap B$  is infinite. Therefore  $A_n \cap B$  is infinite, and we have shown  $\alpha_{2.1}(\mathcal{A}, \mathcal{B})$ .  $\square$

**Corollary 24.** *Let  $\mathcal{A}$  be a  $\Gamma$ -like collection closed under finite unions. Then  $\alpha_x(\mathcal{A}, \mathcal{B})$  implies  $\alpha_y(\mathcal{A}, \mathcal{B})$  for  $1 < x \leq y$ . Additionally, if  $\mathcal{A} \subseteq \mathcal{B}$ , then  $\alpha_x(\mathcal{A}, \mathcal{B})$  implies  $\alpha_y(\mathcal{A}, \mathcal{B})$  for  $1 \leq x \leq y$ .*

For this paragraph we adopt the conventional assumption that  $\Gamma_{X,x}$  is restricted to countable sets. Nyikos showed a consistent example where  $\alpha_2(\Gamma_{X,x}, \Gamma_{X,x})$  fails to imply  $\alpha_{1.5}(\Gamma_{X,x}, \Gamma_{X,x})$ , and a consistent example where  $\alpha_{1.5}(\Gamma_{X,x}, \Gamma_{X,x})$  fails to imply  $\alpha_1(\Gamma_{X,x}, \Gamma_{X,x})$  [9]. On the other hand, Dow showed that  $\alpha_2(\Gamma_{X,x}, \Gamma_{X,x})$  implies  $\alpha_1(\Gamma_{X,x}, \Gamma_{X,x})$  in the Laver model for the Borel conjecture [5]; the author conjectures that this model (specifically, the fact that every  $\omega$ -splitting family contains an  $\omega$ -splitting family of size less than  $\mathfrak{b}$  in this model) witnesses an affirmative answer to the following question.

**Definition 25.** A  $\Gamma$ -like collection is *strongly- $\Gamma$ -like* if the collection is closed under finite unions and each member is countable.

**Question 26.** *Let  $\mathcal{A}$  be strongly- $\Gamma$ -like. Is it consistent that  $\alpha_2(\mathcal{A}, \mathcal{A})$  implies  $\alpha_1(\mathcal{A}, \mathcal{A})$ ?*

## 5. CONCLUSION

We conclude with the following easy result, and a couple questions.

**Proposition 27.** *Let  $\mathcal{B}$  be  $\Gamma$ -like. Then  $\alpha_1(\mathcal{A}, \mathcal{B})$  holds if and only if  $\text{I} \not\Uparrow_{\text{pre}} G_{cf}(\mathcal{A}, \mathcal{B})$ .*

*Proof.* We first assume  $\alpha_1(\mathcal{A}, \mathcal{B})$  and let  $A_n \in \mathcal{A}$  for  $n < \omega$  define a predetermined strategy for I. By  $\alpha_1(\mathcal{A}, \mathcal{B})$ , we immediately obtain  $B \in \mathcal{B}$  such that  $|A_n \setminus B| < \aleph_0$ . Thus  $B_n = A_n \cap B$  is a cofinite choice from  $A_n$ , and  $B' = \bigcup \{B_n : n < \omega\}$  is an infinite subset of  $B$ , so  $B' \in \mathcal{B}$ . Thus II may defeat I by choosing  $B_n \subseteq A_n$  each round, witnessing  $\text{I} \not\Uparrow_{\text{pre}} G_{cf}(\mathcal{A}, \mathcal{B})$ .

On the other hand, let  $I \not\Uparrow_{\text{pre}} G_{cf}(\mathcal{A}, \mathcal{B})$ . Given  $A_n \in \mathcal{A}$  for  $n < \omega$ , we note that  
 II may choose a cofinite subset  $B_n \subseteq A_n$  such that  $B = \bigcup \{B_n : n < \omega\} \in \mathcal{B}$ . Then  
 $B$  witnesses  $\alpha_1(\mathcal{A}, \mathcal{B})$  since  $|A_n \setminus B| \leq |A_n \setminus B_n| \leq \aleph_0$ .  $\square$

**Question 28.** *Is there a game-theoretic characterization of  $\alpha_3(\mathcal{A}, \mathcal{B})$ ?*

Noting that  $I \uparrow G_1(\Gamma_X, \Gamma_X)$  if and only if  $I \uparrow G_{fin}(\Gamma_X, \Gamma_X)$  [7], but the same is  
 not true of  $G_\star(\Gamma_{X,x}, \Gamma_{X,x})$  (e.g. there are  $\alpha_4$  spaces that are not  $\alpha_2$  [14]), we also  
 ask the following.

**Question 29.** *Is there a natural condition on  $\mathcal{A}, \mathcal{B}$  guaranteeing  $I \uparrow G_1(\mathcal{A}, \mathcal{B}) \Rightarrow$   
 $I \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ ?*

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