Applications of almost compatible functions for limited information strategies in infinite length games

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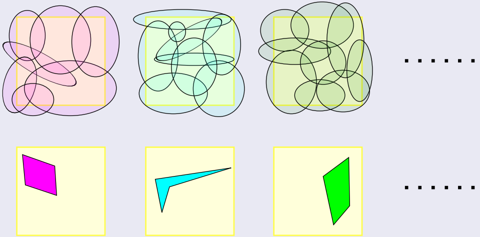
Definition

A topological space X is Menger if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ of open covers of X there exists a sequence $\langle F_0, F_1, \ldots \rangle$ such that F_n is covered by some finite subcollection of $\mathcal{U}_n$ and $X = \bigcup_{n<\omega} F_n$.

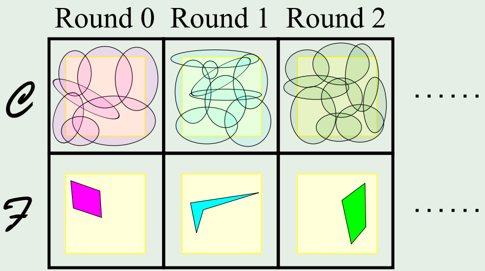


Proposition

X is σ-relatively-compact $\Rightarrow$ X is Menger $\Rightarrow$ X is Lindelöf.

Game

Let $Men_{C,F}(X)$ denote the *Menger game* with players $\mathscr{C}$, $\mathscr{F}$.



$\mathscr{F}$ wins the game if $X = \bigcup_{n<\omega} F_n$, and $\mathscr{C}$ wins otherwise.

Theorem (Hurewicz 1926 [1])

X is Menger if and only if $\mathscr{C} \not\uparrow Men_{C,F}(X)$.

Theorem (Telgarsky 1984 [5], Scheepers 1995 [4])

Let X be metrizable. $\mathscr{F} \uparrow Men_{C,F}(X)$ if and only if X is σ-compact.

Theorem (Fremlin, Miller 1988 [2])

There are ZFC examples of non-σ-compact subsets of the real line which are Menger.

Assume κ is an uncountable cardinal.

Example

Let $\kappa^\dagger = \kappa \cup \{\infty\}$, with κ discrete and neighborhoods of ∞ being co-countable. Then $\mathscr{F} \uparrow Men_{C,F}\left(\kappa^\dagger\right)$ but $\kappa^\dagger$ is not σ-compact.

Definition

A *perfect information strategy* uses full information of the previous moves of the opponent. ($\mathscr{A} \uparrow G$)

Definition

A *k-tactical strategy* only uses the last k previous moves of the opponent. ($\mathscr{A} \underset{k\text{-tact}}{\uparrow} G$)

Definition

A *k-Markov strategy* only uses the last k previous moves of the opponent and the round number. ($\mathscr{A} \underset{k\text{-mark}}{\uparrow} G$)

If omitted, assume $k = 1$.

Considering such strategies allows us to factor out Scheepers's proof characterizing σ-compact metrizable spaces with the Menger game.

Lemma

$\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$ if and only if X is σ-relatively-compact.

Lemma

Let X be second-countable. $\mathscr{F} \uparrow Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$

Since metrizable $+$ Lindelöf $\Leftrightarrow$ regular $+$ second countable, we again have Telgarsky/Scheepers's result for metrizable spaces.

Example

$\mathscr{F} \uparrow Men_{C,F}\left(\kappa^\dagger\right)$, but $\mathscr{F} \underset{\text{mark}}{\not\uparrow} Men_{C,F}\left(\kappa^\dagger\right)$.

Proposition

$\mathscr{F} \underset{(k+2)\text{-mark}}{\uparrow} Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men_{C,F}(X)$.

Example

$\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men_{C,F}\left(\omega_1^\dagger\right)$

What about for $\kappa > \omega_1$? As we'll see, this question is not answerable in *ZFC*.

The game $Men_{C,F}\left(\kappa^\dagger\right)$ essentially involves choosing countable and finite subsets of κ, such as in this game due to Scheepers [3]:

Game

Let $Sch^{\cup,\subset}_{C,F}(\kappa)$ denote Scheepers's *strict countable-finite game* in which each round $\mathscr{C}$ chooses $C_n \in [\kappa]^{\leq\omega}$ such that $C_n \supsetneq \bigcup_{i<n} C_i$, followed by $\mathscr{F}$ choosing $F_n \in [C_n]^{<\omega}$. $\mathscr{F}$ wins if $\bigcup_{n<\omega} F_n = \bigcup_{n<\omega} C_n$, and $\mathscr{C}$ wins otherwise.

$Sch^{\cup,\subseteq}_{C,F}(\kappa)$ is more restrictive than the Menger game, but this is easily remedied.

Game

Let $Sch^{\cap}_{C,F}(\kappa)$ denote the *intersection countable-finite game* in which each round $\mathscr{C}$ chooses $C_n \in [\kappa]^{\leq\omega}$, followed by $\mathscr{F}$ choosing $F_n \in [C_n]^{<\omega}$.
$\mathscr{F}$ wins if $\bigcup_{n<\omega} F_n \supseteq \bigcap_{n<\omega} C_n$, and $\mathscr{C}$ wins otherwise.

Theorem

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Sch^{1,\subseteq}_{C,F}(\kappa)$ if and only if $\mathscr{F} \underset{k\text{-mark}}{\uparrow} Men_{C,F}(\kappa^{\dagger})$.

Perhaps this game is too dissimilar to the original. One may prefer to investigate either of these variants as well:

Game

Let $Sch_{C,F}^{\cup,\subseteq}(\kappa)$ denote the *nonstrict countable-finite game* in which each round $\mathscr{C}$ chooses $C_n \in [\kappa]^{\leq\omega}$ such that $C_n \supseteq \bigcup_{i<n} C_i$, followed by $\mathscr{F}$ choosing $F_n \in [C_n]^{<\omega}$.
$\mathscr{F}$ wins if $\bigcup_{n<\omega} F_n \supseteq \bigcup_{n<\omega} C_n$, and $\mathscr{C}$ wins otherwise.

Game

Let $Sch_{C,F}^{1,\subseteq}(\kappa)$ denote the *initial countable-finite game* in which each round $\mathscr{C}$ chooses $C_n \in [\kappa]^{\leq\omega}$ such that $C_n \supseteq \bigcup_{i<n} C_i$, followed by $\mathscr{F}$ choosing $F_n \in [C_n]^{<\omega}$.
$\mathscr{F}$ wins if $\bigcup_{n<\omega} F_n \supseteq C_0$, and $\mathscr{C}$ wins otherwise.

Theorem

$$\begin{tikzcd}
\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}\left(\kappa^{\dagger}\right) \arrow[rr, leftrightarrow] \arrow[d] & & \mathscr{F} \underset{2\text{-}mark}{\uparrow} Sch^{\cap}_{C,F}\left(\kappa\right) \arrow[d] \\
\mathscr{F} \underset{2\text{-}mark}{\uparrow} Sch^{1,\subseteq}_{C,F}\left(\kappa\right) \arrow[rr, leftrightarrow] & & \mathscr{F} \underset{2\text{-}mark}{\uparrow} Sch^{\cup,\subseteq}_{C,F}\left(\kappa\right) \\
& \mathscr{F} \underset{2\text{-}tact}{\uparrow} Sch^{\cup,\subset}_{C,F}\left(\kappa\right) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Observe that there is no direct implication connecting $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}\left(\kappa^{\dagger}\right)$ and $\mathscr{F} \underset{\text{2-tact}}{\uparrow} Sch^{\cup,\subset}_{C,F}\left(\kappa\right)$.

The following was introduced by Scheepers to study k-tactics in his original countable-finite game.

Definition

For two functions f, g we say f is *almost compatible* with g ($f\|^* g$) if $|\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(x) \neq g(x)\}| < \omega$.

Definition

$S(\kappa)$ states that there exist functions $f_A : A \to \omega$ for each $A \in [\kappa]^{\leq\omega}$ such that $|\{\alpha \in A : f_A(\alpha) \leq n\}| < \omega$ for all $n < \omega$ and $f_A\|^* f_B$ for all $A, B \in [\kappa]^\omega$.

Proposition

$S(\omega_1)$.

Proposition

$\kappa > 2^\omega$ implies $\neg S(\kappa)$.

Theorem

$S(2^\omega)$ is a theorem of $ZFC + CH$ and consistent with $ZFC + \neg CH$.

Theorem

$$
\begin{tikzcd}
 & \mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}\left(\kappa^{\dagger}\right) \arrow[rr, leftrightarrow] \arrow[d] & & \mathscr{F} \underset{2\text{-}mark}{\uparrow} Sch^{\cap}_{C,F}(\kappa) \arrow[d] \\
S(\kappa) \arrow[ur] \arrow[r] \arrow[drr] & \mathscr{F} \underset{2\text{-}mark}{\uparrow} Sch^{1,\subseteq}_{C,F}(\kappa) \arrow[rr, leftrightarrow] & & \mathscr{F} \underset{2\text{-}mark}{\uparrow} Sch^{\cup,\subseteq}_{C,F}(\kappa) \\
 & & \mathscr{F} \underset{2\text{-}tact}{\uparrow} Sch^{\cup,\subset}_{C,F}(\kappa) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Question

Should all the arrows be two-sided?

Definition

A subspace Y of X is *relatively robustly Menger* if there exist functions $r_{\mathcal{V}} : Y \to \omega$ for each open cover $\mathcal{V}$ of X such that for all open covers $\mathcal{U}, \mathcal{V}$ and numbers $n < \omega$, the following sets are finitely coverable by $\mathcal{V}$:

$$c(\mathcal{V}, n) = \{x \in Y : r_{\mathcal{V}}(x) \leq n\}$$

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{x \in Y : n < r_{\mathcal{U}}(x) < r_{\mathcal{V}}(x)\}$$

Definition

A space X is *robustly Menger* if it is relatively robustly Menger to itself.

Theorem

$\mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$ *implies* X *is robustly Menger implies* $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men_{C,F}(X)$.

Theorem

$S(\kappa)$ *implies* $\kappa^\dagger$ *is robustly Menger.*

Question

Does $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men_{C,F}(X)$ imply X is robustly Menger?

Example

Let $R_{\mathbb{Q}}$ be the real line with the basis generated by open intervals with or without the rationals removed.

Theorem

$R_{\mathbb{Q}}$ is second countable and $\mathscr{F} \uparrow Men_{C,F}(R_{\mathbb{Q}})$.

Corollary

$\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(R_{\mathbb{Q}})$, even though it isn't σ-compact.

Example

Let R_ω be the real line with the basis generated by open intervals with any countable set removed.

Theorem

$\mathscr{F} \uparrow Men_{C,F}(R_\omega)$, but $\mathscr{F} \underset{mark}{\not\uparrow} Men_{C,F}(R_\omega)$.

Theorem

$S(2^\omega)$ *implies* R_ω *is robustly Menger.*

Question

Does there exist a space such that $\mathscr{F} \uparrow Men_{C,F}(X)$ but X is not robustly Menger or $\mathscr{F} \underset{\text{2-mark}}{\not\uparrow} Men_{C,F}(X)$?

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Questions?