Game-theoretic strengthenings of Menger's property

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Steven Clontz
http://stevenclontz.com

Department of Mathematics and Statistics
Auburn University

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Abstract

A certain topological game introduced by Hurewicz characterizes Menger's covering property whenever the first player lacks a winning strategy. It was later shown by Telgarsky (and later by Scheepers using a different argument) that for metric spaces, the second player having a winning strategy characterizes the stronger property of σ-compactness.

We factor out Scheepers' proof to show that for regular spaces, the second player having a winning 1-Marköv strategy characterizes σ-compactness; also, for second-countable spaces, the presence of a winning perfect information strategy for the second player implies the existence of a winning 1-Marköv strategy for that player. We also show the that for all k, the existence of a winning k-Marköv strategy for the second player implies the existence of a winning 2-Marköv strategy, and give a space which permits a winning 2-Marköv strategy but not a winning 1-Marköv strategy.

The Menger property

Definition

A space X is Menger if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ of open covers of X there exists a sequence $\langle \mathcal{F}_0, \mathcal{F}_1, \ldots \rangle$ such that $\mathcal{F}_n \subseteq \mathcal{U}_n$, $|\mathcal{F}_n| < \omega$, and $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Proposition

X is σ-compact $\Rightarrow$ X is Menger $\Rightarrow$ X is Lindelöf.

The Menger game

Game

Let $Cov_{C,F}(X)$ denote the *Menger game* with players $\mathscr{C}$, $\mathscr{F}$. In round n, $\mathscr{C}$ chooses an open cover $\mathcal{C}_n$, followed by $\mathscr{F}$ choosing a finite subcollection $\mathcal{F}_n \subseteq \mathcal{C}_n$.
$\mathscr{F}$ wins the game, that is, $\mathscr{F} \uparrow Cov_{C,F}(X)$ if $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover for the space X, and $\mathscr{C}$ wins otherwise.

Theorem

X is Menger if and only if $\mathscr{C} \not\uparrow Cov_{C,F}(X)$. (Hurewicz 1926, effectively)

Menger suspected that the subsets of the real line with his property were exactly the σ-compact spaces; however:

Theorem

There are ZFC examples of non-σ-compact subsets of the real line which are Menger. (Fremlin, Miller 1988)

But metrizable non-σ-compact Menger spaces will be *undetermined* for the Menger game.

Theorem

Let X be metrizable. $\mathscr{F} \uparrow Cov_{C,F}(X)$ if and only if X is σ-compact. (Telgarsky 1984, Scheepers 1995)

Note that for Lindelöf spaces, metrizability is characterized by regularity and secound countability.

Sketch of Scheeper's proof:

- Using second-countability and the winning strategy for $\mathscr{F}$, construct certain subsets R_s for $s \in \omega^{<\omega}$ such that $X = \bigcup_{s\in\omega^{<\omega}}$.
- Using regularity, show that each R_s is compact.
- The result follows since $|\omega^{<\omega}| = \omega$.

By considering winning *limited-information strategies*, we'll be able to factor out this proof a bit.

Limited information strategies

Definition

A *(perfect information) strategy* has knowledge of all the past moves of the opponent.

Definition

A *k-tactical strategy* has knowledge of only the past k moves of the opponent.

Definition

A *k-Markӧv strategy* has knowledge of only the past k moves of the opponent and the round number.

Obviously,

$$\mathscr{A} \underset{k\text{-tact}}{\uparrow} G \Rightarrow \mathscr{A} \underset{k\text{-mark}}{\uparrow} G \Rightarrow \mathscr{A} \underset{\text{(perfect)}}{\uparrow} G$$

But tactical strategies aren't interesting for the Menger game.

Proposition

For any $k < \omega$, $\mathscr{F} \underset{k\text{-tact}}{\uparrow} Cov_{C,F}(X)$ if and only if X is compact.

Effectively, $\mathscr{F}$ needs some sort of seed to prevent from being stuck in a loop: there's nothing stopping $\mathscr{C}$ from playing the same open cover during every round of the game.

Comparitively, Marköv strategies are very powerful.

Proposition

If X is σ-compact, then $\mathscr{F} \underset{1\text{-mark}}{\uparrow} Cov_{C,F}(X)$.

Proof.

Let $X = \bigcup_{n<\omega} K_n$. During round n, $\mathscr{F}$ picks a finite subcollection of the last open cover played by $\mathscr{C}$ (the only one $\mathscr{F}$ remembers) which covers K_n. $\square$

Without assuming regularity, we can't quite reverse the implication, but we can get close.

Definition

A subset Y of X is *relatively compact* if for every open cover for X, there exists a finite subcollection which covers Y.

Proposition

If X is σ-relatively-compact, then $\mathscr{F} \underset{1\text{-}mark}{\uparrow} Cov_{C,F}(X)$.

Proposition

For regular spaces, $Y \subseteq X$ is relatively compact if and only if $\overline{Y}$ is compact. So σ-relatively-compact regular spaces are exactly the σ-compact regular spaces.

Theorem

$\mathscr{F} \underset{1\text{-mark}}{\uparrow} Cov_{C,F}(X)$ if and only if X is σ-relatively-compact.

Proof.

Let $\sigma(\mathcal{U}, n)$ represent a 1-Marköv strategy. For every open cover $\mathcal{U} \in \mathfrak{C}$, $\sigma(\mathcal{U}, n)$ witnesses relative compactness for the set

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

If X is not σ-relatively compact, fix $x \notin R_n$ for any $n < \omega$. Then $\mathscr{C}$ can beat σ by choosing $\mathcal{U}_n \in \mathfrak{C}$ during each round such that $x \notin \bigcup \sigma(\mathcal{U}_n, n)$. $\square$

So for regular spaces, a winning strategy for $\mathscr{F}$ in the Menger game isn't sufficient to characterize σ-compactness, but a winning 1-Markóv strategy does the trick.

We can complete Telgarsky's/Scheeper's result by showing the following:

Theorem

For second countable spaces X, $\mathscr{F} \uparrow Cov_{C,F}(X)$ if and only if $\mathscr{F} \underset{1\text{-}mark}{\uparrow} Cov_{C,F}(X)$.

Proof

Let σ be a perfect information strategy. Since X is a second-countable space, we may pretend that there are only countably many finite collections of open sets. Thus for $s \in \omega^{<\omega}$, we may define open covers $\mathcal{U}_{s^\frown\langle n\rangle}$ such that for each open cover $\mathcal{U}$, there is some $n < \omega$ where

$$\sigma(\mathcal{U}_{s\restriction 1}, \ldots, \mathcal{U}_s, \mathcal{U}) = \sigma(\mathcal{U}_{s\restriction 1}, \ldots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle n\rangle})$$

Let $t : \omega \to \omega^{<\omega}$ be a bijection. During round n and seeing only the latest open cover $\mathcal{U}$, $\mathscr{F}$ may play the finite subcollection

$$\tau(\mathcal{U}, n) = \sigma(\mathcal{U}_{t(n)\restriction 1}, \ldots, \mathcal{U}_{t(n)}, \mathcal{U})$$

Proof (cont.)

Suppose there exists a counter-attack $\langle \mathcal{V}_0, \mathcal{V}_1, \ldots \rangle$ which defeats the 1-Markōv strategy τ. Then there exists $f : \omega \to \omega$ such that, if $\mathcal{V}^n = \mathcal{V}_{t^{-1}(f \restriction n)}$

$$\begin{aligned} x &\notin \bigcup \tau(\mathcal{V}^n, t^{-1}(f \restriction n)) \\ &= \bigcup \sigma(\mathcal{U}_{f \restriction 1}, \ldots, \mathcal{U}_{f \restriction n}, \mathcal{V}^n) \\ &= \bigcup \sigma(\mathcal{U}_{f \restriction 1}, \ldots, \mathcal{U}_{f \restriction n}, \mathcal{U}_{f \restriction (n+1)}) \end{aligned}$$

Thus $\langle \mathcal{U}_{f \restriction 1}, \mathcal{U}_{f \restriction 2}, \ldots \rangle$ is a successful counter-attack by $\mathscr{C}$ against the perfect information strategy σ. $\square$

Unlike the Banach-Mazur game, we can immediately see that knowledge of more than two previous moves of $\mathscr{F}$'s opponent must be infinite to be of any use.

Theorem

If $\mathscr{F} \underset{k\text{-mark}}{\uparrow} Cov_{C,F}(X)$, *then* $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Cov_{C,F}(X)$.

Proof.

$$\tau(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) = \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}, \dots, \mathcal{U}}_{k+1-m}, \underbrace{\mathcal{V}, \dots, \mathcal{V}}_{m+1} \rangle, (n+1)(k+2)+m)$$

$\square$

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Questions? Thanks for having me!