APPLICATIONS OF LIMITED INFORMATION STRATEGIES IN MENGER'S GAME

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Abstract. I need an abstract

I require an introduction.

1. The Menger property and game

Recall the following definition.

Definition 1.1. A space X is Menger if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ of open covers of X there exists a sequence $\langle \mathcal{F}_0, \mathcal{F}_1, \dots \rangle$ such that $\mathcal{F}_n \subseteq \mathcal{U}_n$, $|\mathcal{F}_n| < \omega$, and $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Note that many authors refer to this property as $S_{fin}(\mathcal{O}, \mathcal{O})$, where $\mathcal{O}$ is the collection of open covers of X, and $S_{fin}(A, B)$ denotes the selection property such that for each sequence in A^ω, there are finite subsets of each entry for which the union of these subsets belongs in B.

Proposition 1.2. *X is σ-compact $\Rightarrow$ X is Menger $\Rightarrow$ X is Lindelöf.*

None of these implications may be reversed; the irrationals are a simple example of a Lindelöf space which is not Menger, and we'll see several examples of Menger spaces which are not σ-compact.

It will be convenient to consider subsets of X rather than subsets of the open covers.

Definition 1.3. For each cover $\mathcal{U}$ of X, $S \subseteq X$ is *$\mathcal{U}$-finite* if there exists a finite subcollection of $\mathcal{U}$ which covers S.

Of course, a compact space is $\mathcal{U}$-finite for all open covers $\mathcal{U}$.

Proposition 1.4. *A space X is Menger if and only if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ of open covers of X there exists a sequence $\langle F_0, F_1, \dots \rangle$ such that $F_n \subseteq X$, F_n is $\mathcal{U}_n$-finite, and $X = \bigcup_{n<\omega} F_n$.*

This is the characterization we will use in this paper. A game version of the Menger property is also often considered.

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Game 1.5. Let $Men_{C,F}(X)$ denote the *Menger game* with players $\mathscr{C}$, $\mathscr{F}$. In round n, $\mathscr{C}$ chooses an open cover $\mathcal{U}_n$, followed by $\mathscr{F}$ choosing a $\mathcal{U}_n$-finite subset F_n of X.

$\mathscr{F}$ wins the game if $X = \bigcup_{n<\omega} F_n$, and $\mathscr{C}$ wins otherwise.

As with the Menger property, authors usually characterize this game using finite subsets $\mathcal{F}_n$ of $\mathcal{U}_n$ instead (and often refer to it as $G(\mathcal{O}, \mathcal{O})$ as with the selection property above). This is obviously equivalent in the case of perfect information, and is also equivalent in the case of limited information, provided $\mathscr{F}$ knows $\mathcal{U}_n$ during round n. However, we make this change as we will investigate 0-Markov strategies which consider no moves of the opponent, and instead rely only on the current round number.

This game may be used to characterize the Menger property.

Definition 1.6. If $\mathscr{A}$ has a winning strategy for a game G (which defeats every possible counterattack by her opponent), then we write $\mathscr{A} \uparrow G$.

Theorem 1.7 (Hurewicz [1])**.** *A space X is Menger if and only if $\mathscr{C} \not\uparrow Men_{C,F}(X)$.*

2. Limited information strategies

Recall the following definitions.

Definition 2.1. A k-tactical strategy for a game G with moveset M is a function $\sigma : M^{\leq k} \to M$; intuitively, it is a strategy which only considers the previous k moves of the opponent. If a winning k-tactical strategy exists for $\mathscr{P}$ in the game G, then we write $\mathscr{P} \underset{k\text{-tact}}{\uparrow} G$.

Definition 2.2. A k-Markov strategy for a game G with moveset M is a function $\sigma : M^{\leq k} \times \omega \to M$; intuitively, it is a strategy which only considers the previous k moves of the opponent and the round number. If a winning k-Markov strategy exists for $\mathscr{P}$ in the game G, then we write $\mathscr{P} \underset{k\text{-mark}}{\uparrow} G$.

We will call k-tactical strategies "k-tactics" and k-Markov strategies "k-marks". If the k is omitted then it is assumed that $k = 1$. In addition, note that some authors refer to tactics as *stationary strategies*.

Tactics will be of interest in a game discussed later; proving the following is an easy exercise.

Proposition 2.3. *X is compact if and only if $\mathscr{F} \underset{tact}{\uparrow} Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{k+1\text{-}tact}{\uparrow} Men_{C,F}(X)$ for some $k < \omega$.*

Essentially, because $\mathscr{C}$ may repeat the same finite sequence of open covers, $\mathscr{F}$ needs to be seeded with knowledge of the round number to prevent being trapped in a loop.

If $\mathscr{F}$'s memory of $\mathscr{C}$'s past moves is bounded, then there is no need to consider more than the two most recent moves. The intuitive reason is that $\mathscr{C}$ could simply play the same cover repeatedly until $\mathscr{F}$'s memory is exhausted, in which case $\mathscr{F}$ would only ever see the change from one cover to another.

Theorem 2.4. *For each $k < \omega$, $F \underset{(k+2)\text{-}mark}{\uparrow} Men_{C,F}(X)$ if and only if $F \underset{2\text{-}mark}{\uparrow} Men_{C,F}(X)$.*

Proof. Let σ be a winning $(k+2)$-mark. We define the 2-mark τ as follows:

$$\tau(\langle \mathcal{U} \rangle, 0) = \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}, \ldots, \mathcal{U}}_{m+1} \rangle, m)$$

$$\tau(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) = \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}, \ldots, \mathcal{U}}_{k+1-m}, \underbrace{\mathcal{V}, \ldots, \mathcal{V}}_{m+1} \rangle, (n+1)(k+2)+m)$$

Let $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ be an attack by $\mathscr{C}$ against τ. Then consider the attack

$$\langle \underbrace{\mathcal{U}_0, \ldots, \mathcal{U}_0}_{k+2}, \underbrace{\mathcal{U}_1, \ldots, \mathcal{U}_1}_{k+2}, \ldots \rangle$$

by $\mathscr{C}$ against σ. Since σ is a winning $(k+2)$-mark,

$$X = \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}_0, \ldots, \mathcal{U}_0}_{m+1} \rangle, m) \cup \bigcup_{n<\omega} \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}_n, \ldots, \mathcal{U}_n}_{k+1-m}, \underbrace{\mathcal{U}_{n+1}, \ldots, \mathcal{U}_{n+1}}_{m+1} \rangle, (n+1)(k+2)+m)$$

$$= \tau(\langle \mathcal{U}_0 \rangle, 0) \cup \bigcup_{n<\omega} \tau(\langle \mathcal{U}_n, \mathcal{U}_{n+1} \rangle, n+1)$$

Thus τ is a winning 2-mark. $\square$

A natural question arises: is there an example of a space X for which $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$ but $\mathscr{F} \underset{\text{mark}}{\not\uparrow} Men_{C,F}(X)$? We quickly see that perhaps the simplest example of a Lindelöf non-σ-compact space has this property.

Definition 2.5. For any cardinal κ, let $\kappa^\dagger = \kappa \cup \{\infty\}$ denote the *one-point Lindelöf-ication* of discrete κ, where points in κ are isolated, and the neighborhoods of ∞ are the co-countable sets containing it.

Proposition 2.6. *$\mathscr{F} \underset{mark}{\not\uparrow} Men_{C,F}\left(\omega_1^\dagger\right)$; in fact, $\omega_1^\dagger$ is not σ-compact.*

The greatest advantage of a strategy which has knowledge of two or more previous moves of the opponent, versus only knowledge of the most recent move, is the ability to react to changes from one round to the next. It's this ability to react that will give $\mathscr{F}$ her winning 2-Markov strategy in the Menger game on $\omega_1^\dagger$. This may be proven directly; however, we will postpone giving a proof so that we may take advantage of an equivalent game characterization of this property.

3. Scheepers' filling games

We now turn to a related game whose k-tactics were studied by Marion Scheepers in [3].

Game 3.1. Let $Sch_{C,F}^{\cup,\subset}(\kappa)$ denote *Scheepers' strict union filling game* with two players $\mathscr{C}$, $\mathscr{F}$. In round 0, $\mathscr{C}$ chooses $C_0 \in [\kappa]^{\leq\omega}$, followed by $\mathscr{F}$ choosing $F_0 \in [\kappa]^{<\omega}$. In round $n+1$, $\mathscr{C}$ chooses $C_{n+1} \in [\kappa]^{\leq\omega}$ such that $C_{n+1} \supset C_n$, followed by $\mathscr{F}$ choosing $F_{n+1} \in [\kappa]^{<\omega}$.

$\mathscr{F}$ wins the game if $\bigcup_{n<\omega} F_n \supseteq \bigcup_{n<\omega} C_n$; otherwise, $\mathscr{C}$ wins.

In $Men_{C,F}\left(\kappa^\dagger\right)$, $\mathscr{C}$ essentially chooses a countable set to not include in her neighborhood of ∞, followed by $\mathscr{F}$ choosing a finite subset of this complement to cover during each round. Thus, $\mathscr{F}$ need only be concerned with the *intersection* of the countable sets chosen by $\mathscr{C}$ in $Men_{C,F}\left(\kappa^\dagger\right)$, rather than the union as in $Sch_{C,F}^{\cup,\subset}(\kappa)$.

Another difference between these games: Scheepers required that $\mathscr{C}$ always choose strictly growing countable sets. If the goal is to study tactics, then $\mathscr{C}$ cannot be allowed to trap $\mathscr{F}$ in a loop by repeating the same moves. But by eliminating this requirement, the study can then turn to Markov strategies, bringing the game further in line with the Menger game played upon $\kappa^\dagger$.

We introduce a few games to make the relationship between Scheeper's $Sch_{C,F}^{\cup,\subset}(\kappa)$ and $Men_{C,F}\left(\kappa^\dagger\right)$ more precise.

Game 3.2. Let $Sch_{C,F}^{\cup,\subseteq}(\kappa)$ denote the *union filling game* which proceeds analogously to $Sch_{C,F}^{\cup,\subset}(\kappa)$, except that $\mathscr{C}$'s restriction in round $n+1$ is reduced to $C_{n+1} \supseteq C_n$.

Game 3.3. Let $Sch_{C,F}^{1,\subseteq}(\kappa)$ denote the *initial filling game* which proceeds analogously to $Sch_{C,F}^{\cup,\subseteq}(\kappa)$, except that $\mathscr{F}$ wins whenever $\bigcup_{n<\omega} F_n \supseteq C_0$.

Game 3.4. Let $Sch_{C,F}^{\cap}(\kappa)$ denote the *intersection filling game* which proceeds analogously to $Sch_{C,F}^{1,\subseteq}(\kappa)$, except that $\mathscr{C}$ may choose any $C_n \in [\kappa]^{\leq\omega}$ each round, and $\mathscr{F}$ wins whenever $\bigcup_{n<\omega} F_n \supseteq \bigcap_{n<\omega} C_n$.

$$
\begin{tikzcd}
\mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Men_{C,F}\left(\kappa^\dagger\right) \arrow[r, leftrightarrow] \arrow[d] & \mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Sch_{C,F}^{\cap}(\kappa) \arrow[d] \\
\mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Sch_{C,F}^{1,\subseteq}(\kappa) \arrow[r, leftrightarrow] & \mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Sch_{C,F}^{\cup,\subseteq}(\kappa) \\
 & \mathscr{F} \underset{k+1\text{-tact}}{\uparrow} Sch_{C,F}^{\cup,\subset}(\kappa) \arrow[u]
\end{tikzcd}
$$

FIGURE 1. Diagram of Scheepers/Menger game implications

Theorem 3.5. *For any cardinal $\kappa \geq \omega$ and integer $k < \omega$, Figure 1 holds.*

Proof. $\mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Men_{C,F}\left(\kappa^\dagger\right) \Rightarrow \mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Sch^{\cap}_{C,F}(\kappa)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Men_{C,F}\left(\kappa^\dagger\right)$. Let $\mathcal{U}(C)$ (resp. $\mathcal{U}(s)$) convert each countable subset C of κ (resp. finite sequence s of such subsets) into the open cover $[C]^1 \cup \{\kappa^\dagger \setminus C\}$ (resp. finite sequence of such open covers). Then τ defined by

$$\tau(s^\frown\langle C\rangle, n) = C \cap \sigma(\mathcal{U}(s^\frown\langle C\rangle), n)$$

is a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{\cap}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Sch^{\cap}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Men_{C,F}\left(\kappa^\dagger\right)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Sch^{\cap}_{C,F}(\kappa)$. Let $C(\mathcal{U})$ (resp. $C(s)$) convert each open cover $\mathcal{U}$ of $\kappa^\dagger$ (resp. finite sequence s of such covers) into a countable set C which is the complement of some neighborhood of ∞ in $\mathcal{U}$ (resp. finite sequence of such countable sets). Then τ defined by

$$\tau(s^\frown\langle\mathcal{U}\rangle, n) = (\kappa^\dagger \setminus C(\mathcal{U})) \cup \sigma(C(s^\frown\langle\mathcal{U}\rangle)), n)$$

is a winning $(k+1)$-mark for $\mathscr{F}$ in $Men_{C,F}\left(\kappa^\dagger\right)$.

$\mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Sch^{\cap}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Sch^{1,\subsetneq}_{C,F}(\kappa)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Sch^{\cap}_{C,F}(\kappa)$. σ is also a winning k-mark for $\mathscr{F}$ in $Sch^{1,\subsetneq}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Sch^{1,\subsetneq}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Sch^{\cup,\subsetneq}_{C,F}(\kappa)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Sch^{1,\subsetneq}_{C,F}(\kappa)$. For each finite sequence s, let $t \preceq s$ mean t is a final subsequence of s. Then τ defined by

$$\tau(s^\frown\langle C\rangle, n) = \bigcup_{t \preceq s, m \leq n} \sigma(t^\frown\langle C\rangle, m)$$

is a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{\cup,\subsetneq}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Sch^{\cup,\subsetneq}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Sch^{1,\subsetneq}_{C,F}(\kappa)$: Let σ be a winning k-mark for $\mathscr{F}$ in $Sch^{\cup,\subsetneq}_{C,F}(\kappa)$. σ is also a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{1,\subsetneq}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k+1\text{-tact}}{\uparrow} Sch^{\cup,\subset}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k+1\text{-mark}}{\uparrow} Sch^{\cup,\subsetneq}_{C,F}(\kappa)$: Let σ be a winning k-tactic for $\mathscr{F}$ in $Sch^{\cup,\subsetneq}_{C,F}(\kappa)$. For each countable subset C of κ, let $C+n$ be the union of C with the n least ordinals in $\kappa \setminus C$. Then τ defined by

$$\tau(\langle C_0, \ldots, C_i\rangle, n) = \sigma(\langle C_0 + (n-i), \ldots, C_i + n\rangle)$$

is a winning $(k+1)$-mark for $\mathscr{F}$ in $Sch^{\cup,\subsetneq}_{C,F}(\kappa)$. $\square$

While we have not shown a direct implication between the Menger game and Scheeper's original filling game, Scheepers introduced the statement $S(\kappa)$ relating to the almost-compatability of functions from countable subsets of κ into ω which may be applied to both.

Definition 3.6. For two functions f, g we say f is **almost compatible** with g ($f \|^* g$) if $|\{x \in \operatorname{dom}(f) \cap \operatorname{dom}(g) : f(x) \neq g(x)\}| < \omega$.

Definition 3.7. $S(\kappa)$ states that there exist functions $f_A : A \to \omega$ for each $A \in [\kappa]^{\leq\omega}$ such that $|\{\alpha \in A : f_A(\alpha) \leq n\}| < \omega$ for all $n < \omega$ and $f_A \|^* f_B$ for all $A, B \in [\kappa]^\omega$. [1]

Scheepers went on to show that $S(\kappa)$ implies $\mathscr{F} \underset{\text{2-tact}}{\uparrow} Sch^{\cup,\subseteq}_{C,F}(\kappa)$. This proof, along with the following facts, give us inspiration for finding a winning 2-Markov strategy in the Menger game played on $\kappa^\dagger$.

Theorem 3.8. *$S(\omega_1)$ and $\neg S(\mathfrak{c}^+)$ are theorems of ZFC. $S(\mathfrak{c})$ is a theorem of $ZFC + CH$ and consistent with $ZFC + \neg CH$.*

Proof. For $S(\omega_1)$, consider the construction of an Aronzajn tree as in [2]; this of course implies $S(\mathfrak{c})$ under CH. $\neg S(\mathfrak{c}^+)$ is shown by a cardinality argument in [3]. The consistency result under $ZFC + \neg CH$ is a lemma for the main theorem in [3]. $\square$

$$\begin{tikzcd}
 & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}\left(\kappa^\dagger\right) \arrow[r, leftrightarrow] \arrow[d] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}(\kappa) \arrow[d] \\
S(\kappa) \arrow[ur] \arrow[r] \arrow[drr] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{1,\subseteq}_{C,F}(\kappa) \arrow[r, leftrightarrow] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cup,\subseteq}_{C,F}(\kappa) \\
 & & \mathscr{F} \underset{\text{2-tact}}{\uparrow} Sch^{\cup,\subseteq}_{C,F}(\kappa) \arrow[u]
\end{tikzcd}$$

FIGURE 2. Diagram of Filling/Menger game implications with $S(\kappa)$

Theorem 3.9. *$S(\kappa)$ implies the game-theoretic results in Figure 2.*

Proof. Since $S(\kappa) \Rightarrow \mathscr{F} \underset{\text{2-tact}}{\uparrow} Sch^{\cup,\subseteq}_{C,F}(\kappa)$ was a main result of [3], it is sufficient to show that $S(\kappa) \Rightarrow \mathscr{F} \underset{\text{2-mark}}{\uparrow} Sch^{\cap}_{C,F}(\kappa)$.

Let f_A for $A \in [\kappa]^{\leq\omega}$ witness $S(\kappa)$. We define the 2-mark σ as follows:

$$\sigma(\langle A \rangle, 0) = \{\alpha \in A : f_A(\alpha) \leq 0\}$$

$$\sigma(\langle A, B \rangle, n+1) = \{\alpha \in A \cap B : f_B(\alpha) \leq n+1 \text{ or } f_A(\alpha) \neq f_B(\alpha)\}$$

For any attack $\langle A_0, A_1, \ldots \rangle$ by $\mathscr{C}$ and $\alpha \in \bigcap_{n<\omega} A_n$, either $f_{A_n}(\alpha)$ is constant for all n, or $f_{A_n}(\alpha) \neq f_{A_{n+1}}(\alpha)$ for some n; either way, α is covered. $\square$

Corollary 3.10. $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}\left(\omega_1^\dagger\right)$.

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[1] This is equivalent to the original characterization given in [3]: there exist injections $g_A : A \to \omega$ such that $g_A \|^* g_B$ for all $A, B \in [\kappa]^\omega$ and $A \subset B$.

4. Menger game derived covering properties

$$\begin{tikzcd}
\mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X) \arrow[r] \arrow[d, leftrightarrow] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X) \arrow[r] \arrow[d, leftrightarrow] \arrow[l, bend right, dotted, "/"'] & \mathscr{F} \uparrow Men_{C,F}(X) \arrow[r] \arrow[d, leftrightarrow] \arrow[l, bend right, dotted, "?"'] & \mathscr{C} \not\uparrow Men_{C,F}(X) \arrow[d, leftrightarrow] \arrow[l, bend right, dotted, "/"'] \\
\text{Markov Menger} \arrow[r] & \text{2-Markov Menger} \arrow[r] \arrow[l, bend right, dotted, "/"'] & \text{Perfect Menger} \arrow[r] \arrow[l, bend right, dotted, "?"'] & \text{Menger} \arrow[l, bend right, dotted, "/"']
\end{tikzcd}$$

Figure 3. Diagram of covering properties related to the Menger game

Limited information strategies for the Menger game naturally define a spectrum of covering properties, see Figure 3. However, we do not know if the middle two properties are actually distinct.

Question 4.1. Does there exist a space X such that $\mathscr{F} \uparrow Men_{C,F}(X)$ but $\mathscr{F} \underset{\text{2-mark}}{\not\uparrow} Men_{C,F}(X)$?

Note that while it's consistent that $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}\left(\mathfrak{c}^{\dagger}\right)$, $\kappa^{\dagger}$ for $\kappa > \mathfrak{c}$ is a candidate to answer the above question.

We are also interested in non-game-theoretic characterizations of these covering properties. It has been known for some time that metrizable perfect Menger spaces are exactly the metrizable σ-compact spaces, shown first by Telgarksy in [5] and later directly by Scheepers in [4].

In the interest of generality, we will first characterize the Markov Menger spaces without any separation axioms.

Definition 4.2. A subset Y of X is *relatively compact* to X if for every open cover of X, there exists a finite subcollection which covers Y.

Proposition 4.3. *For regular spaces, Y is relatively compact to X if and only if $\overline{Y}$ is compact in X.*

Lemma 4.4. *Let $\sigma(\mathcal{U}, n)$ be a Markov strategy for F in $Men_{C,F}(X)$, and $\mathfrak{C}$ collect all open covers of X. Then the set*

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \sigma(\mathcal{U}, n)$$

is relatively compact to X. If σ is a winning Markov startegy, then $\bigcup_{n<\omega} R_n = X$.

Proof. First, for every open cover $\mathcal{U} \in \mathfrak{C}$, $R_n \subseteq \sigma(\mathcal{U}, n)$ is covered by a finite subcollection of $\mathcal{U}$.

Suppose that $x \notin R_n$ for any $n < \omega$. Then for each n, pick $\mathcal{U}_n \in \mathfrak{C}$ such that $x \notin \sigma(\mathcal{U}_n, n)$. Then $\mathscr{C}$ may counter σ with the attack $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$. $\square$

Definition 4.5. A *σ-relatively-compact* space is the countable union of relatively compact subsets.

Corollary 4.6. *The following are equivalent:*

- X *is* σ*-relatively-compact*
- $\mathscr{F} \underset{0\text{-}mark}{\uparrow} Men_{C,F}(X)$
- $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$

Proof. If $X = \bigcup_{n<\omega} R_n$ for R_n relatively compact, then $\sigma(n) = R_n$ is a winning 0-mark, which of course gives a winning 1-mark. The previous lemma finishes the proof. □

Corollary 4.7. *Let* X *be a regular space. The following are equivalent:*

- X *is* σ*-compact*
- X *is* σ*-relatively-compact*
- $\mathscr{F} \underset{0\text{-}mark}{\uparrow} Men_{C,F}(X)$
- $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$

For Lindelöf spaces, metrizability is characterized by regularity and second-countability, the latter of which was essentially used by Scheepers in this way:

Lemma 4.8. *Let* X *be a second-countable space.* $\mathscr{F} \uparrow Men_{C,F}(X)$ *if and only if* $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$*.*

Proof. Let σ be a strategy for $\mathscr{F}$, and note that it's sufficient to consider playthroughs with only basic open covers.

So if $\mathcal{U}_t$ is a basic open cover for $t < s \in \omega^{<\omega}$, and $\mathcal{V}$ is any basic open cover, we may choose a finite subcollection $\mathcal{F}(s, \mathcal{V})$ of $\mathcal{V}$ such that

$$\sigma(\langle \mathcal{U}_{s\restriction 1}, \ldots, \mathcal{U}_s, \mathcal{V} \rangle) \subseteq \bigcup \mathcal{F}(s, \mathcal{V})$$

Note that there are only countably-many finite collections of basic open sets. Thus we may choose basic open covers $\mathcal{U}_{s^\frown\langle n\rangle}$ for $n < \omega$ such that for any basic open cover $\mathcal{V}$, there exists $n < \omega$ where $\mathcal{F}(s, \mathcal{V}) = \mathcal{F}(s, \mathcal{U}_{s^\frown\langle n\rangle})$.

Let $t : \omega \to \omega^{<\omega}$ be a bijection. We define the Markóv strategy τ as follows:

$$\tau(\langle \mathcal{V} \rangle, n) = \bigcup \mathcal{F}(t(n), \mathcal{V})$$

Suppose there exists a counter-attack $\langle \mathcal{V}_0, \mathcal{V}_1, \ldots \rangle$ of basic open covers which defeats τ. Then there exists $f : \omega \to \omega$ such that, letting $t(m_n) = f \restriction n$:

$$\begin{array}{rcl} x & \notin & \tau(\langle \mathcal{V}_{m_n} \rangle, m_n) \\ & = & \bigcup \mathcal{F}(f \restriction n, \mathcal{V}_{m_n}) \\ & = & \bigcup \mathcal{F}(f \restriction n, \mathcal{U}_{f\restriction(n+1)}) \\ & \supseteq & \sigma(\langle \mathcal{U}_{f\restriction 1}, \ldots, \mathcal{U}_{f\restriction(n+1)} \rangle) \end{array}$$

Thus $\langle \mathcal{U}_{f\restriction 1}, \mathcal{U}_{f\restriction 2}, \ldots \rangle$ is a successful counter-attack by $\mathscr{C}$ against the perfect information strategy σ. □

Corollary 4.9. *Let* X *be a second-countable space. The following are equivalent:*

- X *is* σ*-relatively-compact*
- $F \underset{0\text{-}mark}{\uparrow} Men_{C,F}(X)$
- $F \underset{mark}{\uparrow} Men_{C,F}(X)$
- $F \uparrow Men_{C,F}(X)$

Corollary 4.10. *Let X be a metrizable space. The following are equivalent:*

- X *is* σ*-compact*
- X *is* σ*-relatively-compact*
- $F \underset{0\text{-}mark}{\uparrow} Men_{C,F}(X)$
- $F \underset{mark}{\uparrow} Men_{C,F}(X)$
- $F \uparrow Men_{C,F}(X)$

Proof. Each bullet implies X is Lindelöf, so X may be assumed to be regular and second-countable. □

5. Robustly Lindelöf

To help describe $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$ topologically, we introduce a slight variant of the Menger game and a related covering property.

Game 5.1. Let $Men_{C,F}(X, Y)$ denote the *Menger subspace game* which proceeds analogously to the Menger game, except that $\mathscr{F}$ wins whenever $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover for $Y \subseteq X$.

Note of course that $Men_{C,F}(X, X) = Men_{C,F}(X)$.

Definition 5.2. A subset Y of X is *relatively robustly Menger* if there exist functions $r_\mathcal{V} : Y \to \omega$ for each open cover $\mathcal{V}$ of X such that for all open covers $\mathcal{U}, \mathcal{V}$ and numbers $n < \omega$, the following sets are $\mathcal{V}$-finite:

$$c(\mathcal{V}, n) = \{x \in Y : r_\mathcal{V}(x) \leq n\}$$
$$p(\mathcal{U}, \mathcal{V}, n + 1) = \{x \in Y : n < r_\mathcal{U}(x) < r_\mathcal{V}(x)\}$$

Definition 5.3. A space X is *robustly Menger* if it is relatively robustly Menger to itself.

Proposition 5.4. *All σ-relatively-compact spaces are robustly Menger.*

Proof. If $X = \bigcup_{n<\omega} R_n$, then for all $\mathcal{U}$, let $r_\mathcal{U}(x)$ be the least n such that $x \in R_n$. Then $c(\mathcal{V}, n) = \bigcup_{m\leq n} R_m$ and $p(\mathcal{U}, \mathcal{V}) = \emptyset$. □

Theorem 5.5. *If $Y \subseteq X$ is relatively robustly Menger, then $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}(X, Y)$.*

Proof. We define the Markov strategy σ as follows. Let $\sigma(\langle\mathcal{U}\rangle, 0) = c(\mathcal{U}, 0)$, and let $\sigma(\langle\mathcal{U}, \mathcal{V}\rangle, n + 1) = c(\mathcal{V}, n + 1) \cup p(\mathcal{U}, \mathcal{V}, n + 1)$.

For any attack $\langle\mathcal{U}_0, \mathcal{U}_1, \ldots\rangle$ by $\mathscr{C}$ and $x \in Y$, one of the following must occur:

- $r_{\mathcal{U}_0}(x) = 0$ and thus $x \in c(\mathcal{U}_0, 0) \subseteq \sigma(\langle\mathcal{U}_0\rangle, 0)$.
- $r_{\mathcal{U}_0}(x) = N + 1$ for some $N \geq 0$ and:

– For all $n \leq N$,
$$r_{\mathcal{U}_{n+1}}(x) \leq N+1$$
and thus $x \in c(\mathcal{U}_{N+1}, N+1) \subseteq \sigma(\langle U_N, U_{N+1}\rangle, N+1)$.
– For some $n \leq N$,
$$r_{\mathcal{U}_n}(x) \leq n$$
and thus $x \in c(\mathcal{U}_{n+1}, n+1) \subseteq \sigma(\langle U_n, U_{n+1}\rangle, n+1)$.
– For some $n \leq N$,
$$n < r_{\mathcal{U}_n}(x) \leq N+1 < r_{\mathcal{U}_{n+1}}(x)$$
and thus $x \in p(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1) \subseteq \sigma(\langle U_n, U_{n+1}\rangle, n+1)$

□

Theorem 5.6. *$S(\kappa)$ implies $\kappa^\dagger$ is robustly Menger, and thus $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}\left(\kappa^\dagger\right)$.*

Proof. Let f_A for $A \in [\kappa]^{\leq\omega}$ witness $S(\kappa)$ and fix $A(\mathcal{U}) \in [\kappa]^{\leq\omega}$ for each open cover $\mathcal{U}$ such that $\kappa^\dagger \setminus A(\mathcal{U})$ is contained in some element of $\mathcal{U}$. Then let $r_{\mathcal{U}}(x) = 0$ for $x \in \kappa^\dagger \setminus A(\mathcal{U})$, and $r_{\mathcal{U}}(\alpha) = f_{A(\mathcal{U})}(\alpha)$ for $\alpha \in A(\mathcal{U})$.

It follows that
$$c(\mathcal{U}, n) = \left(\kappa^\dagger \setminus A(\mathcal{U})\right) \cup \{\alpha \in A(\mathcal{U}) : f_{A(\mathcal{U})}(\alpha) \leq n\}$$
is $\mathcal{U}$-finite, $\bigcup_{n<\omega} c(\mathcal{U}, n) = X$, and
$$p(\mathcal{U}, \mathcal{V}, n+1) = \{\alpha \in A(\mathcal{U}) \cap A(\mathcal{V}) : n < f_{A(\mathcal{U})}(\alpha) < f_{A(\mathcal{V})}(\alpha)\}$$
is finite. □

We may also consider common counterexamples which are finer than the usual Euclidean line.

Definition 5.7. Let $R_{\mathbb{Q}}$ be the real line with the topology generated by open intervals with or without the rationals removed.

Example 5.8. $R_{\mathbb{Q}}$ is non-regular and non-σ-compact, but is second-countable and σ-relatively-compact.

Proof. Compact sets in $R_{\mathbb{Q}}$ cannot contain open intervals, and thus are nowhere dense in nonmeager $\mathbb{R}$, so $R_{\mathbb{Q}}$ is not σ-compact. The usual base of intervals with rational endpoints (with or without rationals removed) witnesses second-countability, and $[-n, n]$ is relatively compact. □

Definition 5.9. Let R_ω be the real line with the topology generated by open intervals with countably many points removed.

Example 5.10. R_ω is non-regular, non-second-countable, and non-σ-relatively-compact, but $\mathscr{F} \uparrow Men_{C,F}(R_\omega)$.

Proof. If $S \supseteq \{s_n : n < \omega\}$ for s_n discrete, then $U_m = R_\omega \setminus \{s_n : m < n < \omega\}$ yields an infinite cover $\{U_m : m < \omega\}$ with no finite subcollection covering S, showing that all relatively compact sets are finite, and R_ω is not σ-relatively-compact.

Define the winning strategy σ for $\mathscr{F}$ in $Men_{C,F}(R_\omega)$ as follows: let $\sigma(\langle\mathcal{U}_0,\ldots,\mathcal{U}_{2n}\rangle) = [-n,n]\setminus C_n$ for some countable $C_n = \{c_{n,m} : m < \omega\}$, and let $\sigma(\langle\mathcal{U}_0,\ldots,\mathcal{U}_{2n+1}\rangle) = \{c_{i,j} : i,j < n\}$. $\square$

We will soon see that, assuming $S(\mathfrak{c})$, $\mathscr{F}$ has a winning 2-Markö v strategy for $Men_{C,F}(R_\omega)$ as well.

Proposition 5.11. *If $X = \bigcup_{i<\omega} X_i$ and $\mathscr{F} \underset{k\text{-}mark}{\uparrow} Men_{C,F}(X, X_i)$ for $i < \omega$, then $\mathscr{F} \underset{k\text{-}mark}{\uparrow} Men_{C,F}(X)$*

Proof. Let σ_i be a k-Markov strategy for $\mathscr{F}$ in $Men_{C,F}(X, X_i)$.

We define the k-Markov strategy σ for $Men_{C,F}(X)$ as follows:

$$\sigma(\langle\mathcal{U}_0,\ldots,\mathcal{U}_n\rangle) = \bigcup_{i\leq n} \sigma_i(\langle\mathcal{U}_i,\ldots,\mathcal{U}_n\rangle)$$

Let $\langle\mathcal{U}_0,\mathcal{U}_1,\ldots\rangle$ be a successful counter-attack by $\mathscr{C}$ against σ. Then there exists $x \in X_i$ for some $i < \omega$ such that x is not covered by $\bigcup_{n<\omega}\sigma(\langle\mathcal{U}_0,\ldots,\mathcal{U}_n\rangle)$. It follows that x is not covered by $\bigcup_{n<\omega}\sigma_i(\langle\mathcal{U}_i,\ldots,\mathcal{U}_{i+n}\rangle)$, and $\langle\mathcal{U}_i,\mathcal{U}_{i+1},\ldots\rangle$ is a successful counter-attack by $\mathscr{C}$ against σ_i. $\square$

Theorem 5.12 ($S(\mathfrak{c})$)**.** $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}(R_\omega)$.

Proof. It's sufficient to show that $[0,1] \subseteq R_\omega$ is relatively robustly Menger. Let f_A witness $S(\mathfrak{c})$ for $A \in [[a,b]]^{\leq\omega}$. For each open cover $\mathcal{U}$, let $A_\mathcal{U}$ be such that $[0,1]\setminus A_\mathcal{U}$ is $\mathcal{U}$-finite. Let $r_\mathcal{U}(x) = 0$ if $x \in [0,1]\setminus A_\mathcal{U}$ and $r_\mathcal{U}(x) = f_{A_\mathcal{U}}(x)$ otherwise.

It follows then that

$$c(\mathcal{U}, n) = [0,1] \setminus \{x \in A_\mathcal{U} : f_{\mathcal{A}_\mathcal{U}}(x) > n\}$$

is $\mathcal{U}$-finite and

$$p(\mathcal{U},\mathcal{V},n+1) = \{x \in A_\mathcal{U} \cap A_\mathcal{V} : n < f_{\mathcal{A}_\mathcal{U}}(x) < f_{\mathcal{A}_\mathcal{V}}(x)\}$$

is finite. $\square$

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