Definition 1. Let a V-map be a u.s.c. idempotent surjection.

Definition 2. For any LOS $\langle L, \leq \rangle$, let $\check{L}$ be the collection of leftward subsets of L (subsets for which $b \in L, a \leq b \Rightarrow a \in L$) linearly ordered by $\subseteq$, and let $\hat{L}$ be the collection of left-closed subsets of L (leftward subsets which are closed) linearly ordered by $\subseteq$.

Proposition 3. *$\check{L}$, $\hat{L}$ are compact.*

Proof. Each subset S has an infimum $\cap S$ and a supremum $\cup S$ (or $\mathrm{cl}(\cap S)$). □

Note that $\check{L}$ is not a "compactification" as L does not necessarily embed as a dense subspace of $\check{L}$: if $L = I$, we might attempt to embed $t \mapsto [0, t)$, but then note that the subspace topology induces the reverse Sorgenfrey interval as $([0, s), [0, t]) = ([0, s), [0, t)]$ is open. However $\hat{L}$ is the typical way of compactifying a linearly ordered space L, provided L lacks a least element (otherwise the empty set is an [easily removable] isolated point in $\hat{L}$). Note that we **always** assume that $\emptyset \in \hat{L}$:

Example 4. $\hat{I} \cong \{-\infty\} \cup I$ where $\emptyset \mapsto -\infty$ and $[0, t] \mapsto t$.

Example 5. For limit ordinals α, $\hat{\alpha} \cong \alpha + 1$, and for all other infinite ordinals, $\hat{\alpha} \cong \alpha$. (The addition of a new least isolated point is of course irrelevant).

Definition 6. For any compact LOTS K with minimum 0 and maximum 1, let γ be the V-map on K where $\gamma(0) = K$ and $\gamma(t) = \{1\}$ for $t > 0$.

Theorem 7. $X = \varprojlim\{2, \gamma, L\} \cong \check{L}$

Proof. We start by placing an order on X. Let $\vec{x} < \vec{y}$ if there exists $a \in L$ with $\vec{x}(a) = 0, \vec{y}(a) = 1$. We claim this is a total order inducing the topology on X.

We first observe that if $\vec{x}(b) = 1$, then for all $a \leq b$, $\vec{x}(a) \in \gamma(1) = \{1\}$. If $\vec{x} \neq \vec{y}$, then assume without loss of generality that $\vec{x}(a) = 0, \vec{y}(a) = 1$, so $\vec{x} < \vec{y}$. Also, whenever $\vec{x}(b) = 1$, we have that $b < a$, so $\vec{y}(b) = 1$, preventing $\vec{y} < \vec{x}$. Finally if $\vec{x} < \vec{y}$ and $\vec{y} < \vec{z}$, take a, b with $\vec{x}(a) = 0$, $\vec{y}(a) = 1$,$\vec{y}(b) = 0$,$\vec{z}(b) = 1$. It follows that $a < b$ so $\vec{z}(a) = 1$ and $\vec{x} < \vec{z}$.

Consider the basic open set $B(\vec{x}, F)$ for a finite set $F \in [L]^{<\omega}$ about the sequence $\vec{x} \in X$ which contains all sequences $\vec{y}$ agreeing with $\vec{x}$ on F. If $\vec{x}(a) = 1$ for all $a \in F$, then let $\vec{w} \in X$ be 0 on the maximum of F, and 1 for anything less. It follows that $B(\vec{x}, F) = (\vec{w}, \rightarrow)$. If $\vec{x}(a) = 0$ for all $a \in F$, then let $\vec{y} \in X$ be 1 on the minimum of F, and 0 for anything greater. It follows that $B(\vec{x}, F) = (\leftarrow, \vec{y})$. Finally if $\vec{x}(a) = 1$ and $\vec{x}(b) = 0$ for $a < b$ in F and nothing between a, b is in F, then let $\vec{w} \in X$ be 0 on a and 1 for anything less, and let $\vec{y} \in X$ be 1 on b and 0 for anything greater. It follows that $B(\vec{x}, F) = (\vec{w}, \vec{y})$.

Let ϕ evaluate each $\vec{x} \in X \subseteq 2^L$ as the characteristic function for a subset of L. It's easy to see that ϕ is an order isomorphism between $\langle X, \leq \rangle$ and $\langle \check{L}, \subseteq \rangle$. □

Corollary 8. $\varprojlim\{2,\gamma,\alpha\} \cong \alpha+1$ *for every ordinal* α.

Proof. Since $\breve{\alpha} = \alpha+1$ (actually equals, not just homeomorphic!), we get $\varprojlim^\star\{2,\gamma,\alpha\} \cong \breve{\alpha} = \alpha+1$ for free. Note that C and Varagona used this in (TODO create citataion) to break metrizability in uncountable-ordinal-indexed inverse limits (for any V-map there exists a two-point set 2 such that $f \restriction 2 \supseteq \gamma$, that is, "$f$ has condition Γ"). □

We may generalize theorem 8 as follows:

Theorem 9. *If* M *is a LOTS with minimum* 0 *and maximum* 1*, then* $\varprojlim\{M,\gamma,L\} \cong \hat{L} \times_{lex} M/\sim$*, where* $\langle(\leftarrow,l_0],1\rangle \sim \langle(\leftarrow,l_1],0\rangle$ *if* $l_0 < l_1$ *and* $(l_0,l_1)=\emptyset$*, and where* $\langle A,m\rangle \sim \langle A,m'\rangle$ *if* $A \in \hat{L}\setminus L$.

Proof. Let $\rho(\vec{x}) = \mathrm{cl}\{l \in L : \vec{x}(l) > 0\}$, $v(\vec{0}) = 0$, and $v(\vec{x}) = \min\{\vec{x}(l) : l \in \rho(\vec{x})\}$ otherwise. Say $\vec{x} < \vec{y}$ if $\rho(\vec{x}) \subsetneq \rho(\vec{y})$ or both $\rho(\vec{x}) = \rho(\vec{y})$ and $v(\vec{x}) < v(\vec{y})$. The reader may verify that this is a linear order on $\varprojlim\{M,\gamma,L\}$, and $\theta(\vec{x}) = \langle\rho(\vec{x}), v(\vec{x})\rangle \in \hat{L}\times_{\mathrm{lex}} M/\sim$ preserves order. For each left-closed set A and $m \in M$, let $\vec{x}_{A,m}(l) = 1$ for $l \in A$ unless l is the supremum element of A, $\vec{x}_{A,m}(l) = m$ if l is the supremum of A, and $\vec{x}_{A,m}(l) = 0$ for $l \notin A$. To complete the proof, we should demonstrate that the linear order we defined induces the topology of the inverse limit, and that θ is a surjection.

A basic open set in $\varprojlim\{M,\gamma,L\} \subseteq L^m$ is of the form $[U,F]$ where $U(l)$ is an open interval in M for each $l \in F \in [L]^{<\omega}$, and $[U,F] = \{\vec{x} : l \in F \Rightarrow \vec{x}(l) \in U(l)\}$. If we assume that $[U,F]$ is non-empty, one of the following must hold:

- $U[l_0] = (a,b)$ for some $l_0 \in F$. Then $[U,F] = [U,\{l_0\}]$, and note that $[U,\{l_0\}] = (\vec{x}_{(\leftarrow,l_0],a}, \vec{x}_{(\leftarrow,l_0],b})$.
- $U(l_0) = (a,1]$ and $U(l_1) = [0,b)$ for some $l_0 < l_1 \in L$ and $[U,F] = [U,\{l_0,l_1\}]$. Then $[U,\{l_0,l_1\}] = (\vec{x}_{l_0,a}, \vec{x}_{l_1,b})$.

In the other direction, consider $\vec{y} \in (\vec{x},\vec{z})$.

- In the case that $l_0 \in \rho(\vec{y})\setminus\rho(\vec{x})$ and $l_1 \in \rho(\vec{z})\setminus\rho(\vec{y})$, let $U(l_0) = (0,1]$, $U(l_1) = [0,v(\vec{z}))$ and note $\vec{y} \in [U,\{l_0,l_1\}] \subseteq (\vec{x},\vec{z})$.
- In the case that $l_0 \in \rho(\vec{y})\setminus\rho(\vec{x})$, $\rho(\vec{y}) = \rho(\vec{z})$, and $v(\vec{y}) < v(\vec{z})$, it follows that $\rho(\vec{y}) = \rho(\vec{z}) = (\leftarrow,l_1]$, so let $U(l_0) = (0,1]$, $U(l_1) = [0,v(\vec{z}))$ and note $\vec{y} \in [U,\{l_0,l_1\}] \subseteq (\vec{x},\vec{z})$.
- In the case that $\rho(\vec{x}) = \rho(\vec{y})$, $v(\vec{x}) < v(\vec{y})$, and $l_1 \in \rho(\vec{z})\setminus\rho(\vec{y})$, it follows that $\rho(\vec{x}) = \rho(\vec{y}) = (\leftarrow,l_0]$, so let $U(l_0) = (v(\vec{x}),1]$, $U(l_1) = [0,v(\vec{z}))$ and note $\vec{y} \in [U,\{l_0,l_1\}] \subseteq (\vec{x},\vec{z})$.

- In the case that $\rho(\vec{x}) = \rho(\vec{y}) = \rho(\vec{z})$ and $v(\vec{x}) < v(\vec{y}) < v(\vec{z})$, it follows that $\rho(\vec{x}) = \rho(\vec{y}) = \rho(\vec{z}) = (\leftarrow, l_0]$, so let $U(l_0) = (v(\vec{x}), v(\vec{z}))$ and note $\vec{y} \in [U, \{l_0\}] = (\vec{x}, \vec{z})$.

We conclude by showing that θ is a surjection. If $B \in \hat{L} \setminus L$ and $m \in M$, consider $\langle B, m\rangle$. B lacks a supremum in L, so $\vec{x}_{B,0}(l) = 1$ for $l \in B$ and $\vec{x}_{B,0}(l) = 0$ otherwise. So $\theta(\vec{x}_{B,0}) = \langle \mathrm{cl} B, 1\rangle = \langle B, 1\rangle \sim \langle B, m\rangle$ for all $m \in M$. Otherwise, $B = (\leftarrow, l_1]$ for some $l_1 \in L$. Let $m > 0$. Then $\theta(\vec{x}_{(\leftarrow, l_1], m}) = \langle \mathrm{cl}(\leftarrow, l_1], v(\vec{x}_{(\leftarrow, l_1], m})\rangle = \langle (\leftarrow, l_1], m\rangle$. Finally, we want to map onto $\langle (\leftarrow, l_1], 0\rangle$. If there exists $l_0 < l_1$ with $(l_0, l_1) = \emptyset$, then $\theta(\vec{x}_{(\leftarrow, l_1], 0}) = \theta(\vec{x}_{(\leftarrow, l_0], 1}) = \langle (\leftarrow, l_0], 1\rangle$ will suffice. Otherwise, $\theta(\vec{x}_{(\leftarrow, l_1], 0}) = \langle \mathrm{cl}(\leftarrow, l_1), v(\vec{x}_{(\leftarrow, l_1), 0})\rangle = \langle (\leftarrow, l_1], 0\rangle$. □

Here are some applications:

Example 10. $\varprojlim\{2, \gamma, I\} \cong (\hat{I} \setminus \emptyset) \times_{\mathrm{lex}} 2 \cong I \times_{\mathrm{lex}} 2 \cong \check{I}$ (of course, this could be found quicker with theorem 8).

Example 11. $\varprojlim\{I, \gamma, I\} \cong (\hat{I} \setminus \emptyset) \times_{\mathrm{lex}} I \cong I \times_{\mathrm{lex}} I$.

Example 12. For infinite ordinals α, $\varprojlim\{I, \gamma, \alpha\} \cong (\alpha \times_{\mathrm{lex}} [0, 1)) \cup \{\infty\}$. In particular, $\alpha = \kappa$ for an infinite cardinal κ gives the closed long ray of length κ.

Definition 13. For any M containing a point 0, let ν be the V-map on M where $\nu(0) = M$ and $\nu(t) = \{t\}$ for $t > 0$.

Note for $M = 2$ that $\nu = \gamma$.

Corollary 14. $\varprojlim\{2, \nu, L\} \cong \check{L}$.

Theorem 15. *If M is T_2, then $\varprojlim\{M, \nu, L\} \setminus \{\vec{0}\} \cong (\check{L} \setminus \{\emptyset\}) \times (M \setminus \{0\})$ with the usual product topology.*

Proof. Each point in $\varprojlim\{M, \nu, L\} \setminus \{\vec{0}\}$ is of the form $\vec{x}_{C,m}$ where $C \in \check{L} \setminus \{\emptyset\}$ and $m \in M \setminus \{0\}$ defined by $\vec{x}_{C,m}(l) = m$ for $l \in C$ and $x_{C,m}(l) = 0$ otherwise.

We claim that the bijection $\theta(\vec{x}_{C,m}) = \langle C, m \rangle$ is a homeomorphism. Note that basic open sets of $\varprojlim\{M, \nu, L\}$ are of the form $[U, F]$ where $U(l)$ is an open subset of M for each $l \in F \in [L]^{<\omega}$.

Consider the point $\langle C, m \rangle$ in the basic open set $V \times W$ in $(\check{L} \setminus \{\emptyset\}) \times (M \setminus \{0\})$. Note that V is either of the form $(A, L]$ or (A, B), and we may choose $l_0 \in C \setminus A$. We also may assume that W misses an open neighborhood Z of 0 as M is T_2.

In the case that $V = (A, L]$ we let $U(l_0) = W$. Then since $l_0 \in C$ and $m \in W = U(l_0)$, it follows that $\vec{x}_{C,m} \in [U, \{l_0\}]$. For any $\vec{x}_{D,n} \in [U, \{l_0\}]$ we have that $\vec{x}_{D,n}(l_0) = n \in W$; in particular, it's nonzero. So $A \subsetneq (\leftarrow, l_0] \subseteq D$, putting $D \in (A, L] = V$. Thus $\theta(\vec{x}_{D,n}) = \langle D, n \rangle \in V \times W$.

In the case that $V = (A, B)$, we may also choose $l_1 \in B \setminus C$. We again let $U(l_0) = W$, and we also let $U(l_1) = Z$. Then as $\vec{x}_{C,m}(l_0) = m \in W = U(l_0)$ and $\vec{x}_{C,m}(l_1) = 0 \in Z = U(l_1)$, we have shown $\vec{x}_{C,m} \in [U, \{l_0, l_1\}]$. For any $\vec{x}_{D,n} \in [U, \{l_0, l_1\}]$, $\vec{x}_{D,n}(l_0) \in W$ and $\vec{x}_{D,n}(l_1) \in Z$. This shows that $\vec{x}_{D,n}(l_0) = n \in W$ and $\vec{x}_{D,n}(l_1) = 0$, so $A \subsetneq (\leftarrow, l_0] \subseteq D \subsetneq (\leftarrow, l_1] \subseteq B$, putting $D \in (A, B) = V$. Thus $\theta(\vec{x}_{D,n}) = \langle D, n \rangle \in V \times W$.

On the other hand, consider the point $\vec{x}_{C,m}$ in the basic open set $[U, F]$ of $\varprojlim\{M, \nu, L\} \setminus \{\vec{0}\}$. It follows that $m \in U(l)$ for all $l \in F \cap C$, and $0 \in U(l)$ for all $l \in F \setminus C$.

If $F \subseteq C$, then let l_0 be the maximum element of F and $U' = \bigcap_{l \in F} U(l)$. Note $\langle C, m \rangle \in ((\leftarrow, l_0), L] \times U'$. So let $\langle D, n \rangle \in ((\leftarrow, l_0), L] \times U'$. Since $l_0 \in D$, $\vec{x}_{D,n}(l_0) = n \in U'$. Since $n \neq 0$, we have $\vec{x}_{D,n}(l) = n \in U' \subseteq U(l)$ for all $l \in F$, so $\vec{x}_{D,n} \in [U, F]$.

If $F \cap C = \emptyset$, then let l_1 be the minimum element of $F \setminus C$. Note $\langle C, m \rangle \in (\emptyset, (\leftarrow, l_1]) \times (M \setminus \{0\})$. So let $\langle D, n \rangle \in (\emptyset, (\leftarrow, l_1]) \times (M \setminus \{0\})$. Since $l_1 \notin D$, $\vec{x}_{D,n}(l) = 0 \in U(l)$ for all $l \in F$, giving $\vec{x}_{D,n} \in [U, F]$.

Otherwise, let l_0 be the maximum element of $F \cap C$ and l_1 be the minimum element of $F \setminus C$. Let $U' \subseteq \bigcap_{l \in C \cap F} U(l)$ and $U'' \subseteq \bigcap_{l \in F \setminus C} U(l)$. Note $\langle C, m \rangle \in ((\leftarrow, l_0), (\leftarrow, l_1]) \times U'$. So let $\langle D, n \rangle \in ((\leftarrow, l_0), (\leftarrow, l_1]) \times U'$. Since $l_0 \in D$ and $l_1 \notin D$, we have $\vec{x}_{D,n}(l_0) = n \in U'$

and $\vec{x}_{D,n}(l_1) = 0 \in U''$. Furthermore, $\vec{x}_{D,n}(l) = n \in U' \subseteq U(l)$ for all $l \in F \cap C$, and $\vec{x}_{D,n}(l) = 0 \in U'' \subseteq U(l)$ for all $l \in F \setminus C$, so $\vec{x}_{D,n} \in [U, F]$. $\square$

We introduce an alternate definition of an arbitrarily indexed inverse limit.

Definition 16. Let $\varprojlim^\star\{X, f, L\} \subseteq \varprojlim\{X, f, L\}$ satisfy that $\vec{x}(a) = \lim_{t\to a} \vec{x}(t)$ for all $a \in L$ (for any open neighborhood $U$of $\vec{x}(a)$ there is $b < a$ where $\vec{x}(t) \in U$ for all $t \in (b, a]$).

Theorem 17. $Y = \varprojlim^\star\{2, \gamma, L\} \cong \hat{L}$.

Proof. Consider Y as a subspace of $X = \varprojlim\{2, \gamma, L\}$ with the linear order described above. We claim that if ϕ is the characteristic function for a subset of L, then ϕ is an order isomorphsim between $\langle Y, \leq\rangle$ and $\langle \hat{L}, \subseteq\rangle$.

Let A be a left-closed subset of L. Let $\vec{x}(a) = 1$ when $a \in A$ and $\vec{x}(a) = 0$ otherwise. Then $\vec{x} \in Y$ and $\phi(\vec{x}) = A$.

Let $\vec{x}, \vec{y} \in Y$. If $\phi(\vec{x}) = \phi(\vec{y}) = A$, then A is a left-closed set where $\vec{x}(a) = \vec{y}(a) = 1$ for $a \in A$ and $\vec{x}(a) = \vec{y}(a) = 0$ otherwise, so $\vec{x} = \vec{y}$.

Finally let $\vec{x} < \vec{y}$, so there exists $a \in L$ with $\vec{x}(a) = 0$, $\vec{y}(a) = 1$. Then $\phi(\vec{x}) \subseteq (\leftarrow, a) \subseteq \phi(\vec{y})$. Thus ϕ preserves order. $\square$

Corollary 18. $\varprojlim^\star\{2, \gamma, \alpha\} \cong \alpha + 1$ *for every infinite limit or finite ordinal* α.

Proof. If α is finite, then of course all (leftward) sets are closed and we get $\hat{\alpha} = \check{\alpha} = \alpha + 1$ for free. Otherwise, as observed previously $\hat{\alpha}$ is homeomorphic to its usual compactification $\alpha + 1$ for limit ordinals. $\square$

In fact, $\hat{\alpha} = \alpha + 1 \setminus L(\alpha)$ where $L(\alpha)$ is the colleciton of all limit ordinals less than α, which also shows $\hat{\alpha} \cong \alpha$ for infinite successor ordinals α.

References