Tactics and Marks in Banach Mazur Games

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marks and tactics

My notes on Galvin/Telgarsky's Theorem 5 from [3].

Definition 1. Let $\mathbb{P}$ be partially ordered by $\leq$. Let $\mathbb{P}^{\omega,\downarrow} = \{f \in \mathbb{P}^\omega : f(n) \geq f(n+1)\}$. Then for $f, g \in \mathbb{P}^{\omega,\downarrow}$, we say that f, g zip into each other if for all $m < \omega$ there exists $n < \omega$ such that $f(m) \geq g(n)$ and $g(m) \geq f(n)$.

Definition 2. $BM_{po}(\mathbb{P}, W)$ is a game defined for all non-empty partial orders $\mathbb{P}$ and all subsets $W \subseteq \mathbb{P}^{\omega,\downarrow}$. During round 0, I chooses $a_0 \in \mathbb{P}$, and then II chooses $b_0 \leq a_0$; during around $n+1$, I chooses $a_{n+1} \leq b_n$, and then II chooses $b_{n+1} \leq a_{n+1}$. II wins this game if $\langle a_0, a_1, \ldots \rangle \in W$.

Theorem 3. *Let $W \subseteq \mathbb{P}^{\omega,\downarrow}$ be closed under zipping.* II $\underset{\text{mark}}{\uparrow}$ $BM_{po}(\mathbb{P}, W)$ *if and only if* II $\underset{\text{tact}}{\uparrow}$ $BM_{po}(\mathbb{P}, W)$.

Proof. Let $\tau(p, n+1)$ be a winning mark for II, where p is the most recent move by I and $n+1$ is the number of moves made by I. Define $\tau^0(p) = p$ and $\tau^{n+1}(p) = \tau(\tau^n(p), n+1)$. Let $\preceq$ well-order $\mathbb{P}$.

For $p, q \in \mathbb{P}$, say $p \geq_n q$ if there exist $s_m(p) \in \mathbb{P}$ for $m \leq n$ such that

$$p \geq s_m(p) \geq \tau(s_m(p), n+1) \geq q.$$

Note that $p' \geq p \geq_n q \geq q'$ implies $p' \geq_n q'$, and $p \geq_n \tau^n(p)$.

Say $p \geq_\omega q$ whenever $p \geq_n q$ for all $n < \omega$. If $p \geq_\omega l(p)$ for some $l(p)$, then say p is long; otherwise call p short.

For p short, let

$$\mu(p) = \min_{\preceq}\{r \text{ short} : r \geq p\}$$

and since $\mu(p) \not\geq_n p$ for some n, let

$$N(p) = \min\{n < \omega : \mu(p) \not\geq_n p\}.$$

Note that whenever $\mu(p) = \mu(q)$ for $p \geq_n q$, it follows that $\mu(p) \geq_n q$ and therefore $N(p) < N(q)$.

We define

$$\sigma(p) = \begin{cases} l(p) & p \text{ is long} \\ \tau^{N(p)+1}(p) & p \text{ is short} \end{cases}.$$

Suppose σ is legally attacked by $a \in \mathbb{P}^\omega$. For $n \leq \omega$, if $a(n)$ is long, then $a(n) \geq_n l(a(n))$. Therefore,

$$a(n) \geq s_n(a(n)) \geq \tau(s_n(a(n)), n+1) \geq l(a(n)) = \sigma(a(n)) \geq a(n+1).$$

Thus if $a(n)$ is long for $n < \omega$, it follows that $c \in \mathbb{P}^{\omega,\downarrow}$ defined by $c(n) = s_n(a(n))$ is a legal attack against τ. Since τ is winning, $c \in W$, and since c zips into a, $a \in W$ as well.

Otherwise, we may choose a final subsequence b of a such that

- $b(n)$ is short for all $n < \omega$, since $a(m)$ short implies $a(n+m)$ short for all $n < \omega$.
- $\mu(b(n)) = \mu'$ is fixed for all $n < \omega$, since there cannot be an infinite $\preceq$-decreasing sequence.

As a result,

$$b(n) \geq_{N(b(n))} \tau^{N(b(n))+1}(b(n)) = \sigma(b(n)) \geq b(n+1)$$

and therefore $N(b(n)) < N(b(n+1))$. In particular, $N(b(n)) \geq n$.

Thus for $n < \omega$,

$$b(n) \geq \tau^n(b(n)) \geq \tau(\tau^n(b(n)), n+1) \geq \tau^{N(b(n))+1}(b(n)) = \sigma(b(n)) \geq b(n+1).$$

As a result, $c \in \mathbb{P}^{\omega,\downarrow}$ defined by $c(n) = \tau^n(b(n))$ is a legal attack against the winning strategy τ. Therefore $c \in W$, and since c zips into b and a, we conclude $a \in W$. $\square$

Observation 4. When $\mathbb{P} = T(X) \setminus \{\emptyset\}$ is ordered by set-inclusion and $W = \{U \in \mathbb{P}^{\omega,\downarrow} : \bigcap_{n<\omega} U(n) \neq \emptyset\}$, then $BM_{po}(\mathbb{P}, W)$ is exactly the topological Banach Mazur game $BM_{E,N}(X)$. Note W is closed under zipping.

Corollary 5. $\text{II} \underset{\text{mark}}{\uparrow} BM_{E,N}(X)$ *if and only if* $\text{II} \underset{\text{tact}}{\uparrow} BM_{E,N}(X)$.

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And this stuff is based on section 4.5.1 of [1].

Definition 6. Let $f \in S^{\leq\omega}$. Then $f \upharpoonright n \in S^n$ is defined by $(f \upharpoonright n)(i) = f(i)$. ($f \upharpoonright n$ gives the first n terms of f.)

Let $t \in S^{<\omega}$. Then $t \downharpoonleft k \in S^k$ is defined by $(t \downharpoonleft k)(i) = t(i + |t| - k)$. ($t \downharpoonleft k$ gives the last n terms of t.)

Definition 7. For every partial order $\mathbb{P}$ and compatible $p, q \in \mathbb{P}$, write $p \not\perp q$ and let $p \wedge q$ satisfy $p \wedge q \leq p, q$. If p, q are incompatible, write $p \perp q$.

Definition 8. For every partial order $\mathbb{P}$ and compatible $p \in \mathbb{P}$, let $p^\downarrow = \{q \in \mathbb{P} : q \leq p\}$.

Lemma 9. $\mathbb{P}$ *contains no infinite antichains if and only if every antichain in* $\mathbb{P}$ *is of size* n *or less for some* $n < \omega$.

Proof. This was shown to be true for $\mathbb{P} = \tau \setminus \{\emptyset\}$ in Lemma 2.10 of [2]. It's likely known for general $\mathbb{P}$, but I can't find a citation, so let's roll our own proof here. Assume $\mathbb{P}$ has antichains of size n for all $n < \omega$.

Say $p \in \mathbb{P}$ is bad if there exists $r_p \leq p$ such that $r_p^\downarrow$ is pairwise compatible. Let $\mathbb{P}_{bad}$ collect all bad points in $\mathbb{P}$, and say $p \sim q$ for $p, q \in \mathbb{P}_{bad}$ if $r_p \not\perp r_q$. This is obviously symmetric and reflexive, and if we assume $p \sim q, q \sim t$, then let $s_p = r_p \wedge r_q$ and $s_r = r_q \wedge r_t$. Since $r_q^\downarrow \in \mathbb{P}_{bad}$, $s_p \not\perp s_r$, so $r_p \not\perp r_t$ and thus $p \sim t$. Thus $\sim$ is an equivalence relation.

If $\mathbb{P}_{bad}/\sim$ is infinite, we may choose $p_i \in \mathbb{P}_{bad}$ such that $p_i \not\sim p_j$ for $i < j < \omega$. Thus $r_{p_i} \perp r_{p_j}$ for $i < j < \omega$, giving us an infinite antichain $\{r_{p_i} : i < \omega\}$.

Otherwise $|\mathbb{P}_{bad}/\sim| = n < \omega$, and choose an antichain $\{p_i : i \leq n\}$ in $\mathbb{P}$. If $\{p_i : i < n\} \subseteq \mathbb{P}_{bad}$, $p_i \perp p_j$ implies $r_{p_i} \perp r_{p_j}$ and $p_i \not\sim p_j$ for all $i \leq n$. Thus $\mathbb{P}_{bad} = \bigcup_{i<n} \widetilde{p_i}$, and $p_n \notin \mathbb{P}_{bad}$.

So we've found $b_0 \in \mathbb{P} \setminus \mathbb{P}_{bad}$. Given $b_n \in \mathbb{P} \setminus \mathbb{P}_{bad}$, we may choose $a_n, b_{n+1} \leq b_n$ such that $a_n \perp b_{n+1}$. Thus by construction, $a_n \perp a_{m+1}$ for all $n \leq m < \omega$. Therefore $\{a_n : n < \omega\}$ is an antichain. □

Proposition 10. *Let $W \subseteq \mathbb{P}^{\omega,\downarrow}$ be closed under zipping. Suppose every antichain in $\mathbb{P}$ is of size $n < \omega$ or less, and* II $\uparrow BM_{po}(\mathbb{P}, W)$*. Then* II $\underset{\text{auto}}{\uparrow} BM_{po}(\mathbb{P}, W)$ *(i.e.* II *wins every play of $BM_{po}(\mathbb{P}, W)$, i.e. $W = \mathbb{P}^{\omega,\downarrow}$).*

Proof. First, let $\{p_i : i < n\}$ be an antichain of size $n < \omega$, then let $\mathbb{P}_i$ be a maximal pairwise-compatible subset of $\mathbb{P}$ containing p_i. Note that if there existed $q \in \mathbb{P} \setminus \bigcup_{i<n} \mathbb{P}_i$, q must be incompatible with some $q_i \in \mathbb{P}_i$ for $i < n$. Since $p_i, q_i \in \mathbb{P}_i$, they are compatible, so let $r_i = p_i \wedge q_i$. Since q is incompatible with q_i for $i < n$, q is incompatible with r_i for $i < n$. Since p_i is incompatible with p_j for $i < j < n$, r_i is incompatible with r_j for $i < j < n$. But that makes $\{q\} \cup \{r_i : i < n\}$ an antichain of size $n + 1$, contradicting the assumption of the proposition. Thus $\mathbb{P} = \bigcup_{i<n} \mathbb{P}_i$.

We now show that if $s \in \mathbb{P}_i^{\downarrow}$ for some i, then $s \in W$. Let σ be a winning strategy for II in $BM_{po}(\mathbb{P}, W)$, and attack σ with $q(0) = s(0) \wedge p_i$ and $q(n + 1) = s(n + 1) \wedge \sigma(\langle q(0), \dots, q(n)\rangle)$. Note that the choice of $q(0)$ is valid as $s(0), p_i \in \mathbb{P}_i$. Similarly, $\sigma(\langle q(0), \dots, q(n)\rangle) \leq q(0) \leq p_i$, so $\sigma(\langle q(0), \dots, q(n)\rangle)$ cannot be compatible with any p_j where $j \neq i$. Thus $s(n+1), \sigma(\langle q(0), \dots, q(n)\rangle) \in \mathbb{P}_i$, making the choice of $q(n + 1)$ valid. Since σ is winning for II, we see that $q \in W$, and therefore $s \in W$.

Finally, consider any play of $BM_{po}(\mathbb{P}, W)$. It must contain have a subsequence $s \in \mathbb{P}_i^{\downarrow}$ for some $i < n$, so $s \in W$ and therefore the play is also in W, securing a victory for II. □

Lemma 11. *Let $W \subseteq \mathbb{P}^{\omega,\downarrow}$ be closed under zipping. Suppose that for every $p \in \mathbb{P}$, there exists an infinite antichain $A_p = \{a_p(n) : n < \omega\} \subseteq \{q \in \mathbb{P} : q \leq p\}$. Then* II $\underset{(k+2)-\text{mark}}{\uparrow} BM_{po}(\mathbb{P}, W)$ *if and only if* II $\underset{(k+2)-\text{tact}}{\uparrow} BM_{po}(\mathbb{P}, W)$*.*

Proof. The intuition of the following proof is simple: consider the case $k = 0$. During the first round, I plays some $p_0 \in \mathbb{P}$, and II can store the round number 0 (known by II since they only have knowledge of one move) by pretending I chose $a_{p_0}(0) \leq p_0$ instead, and applying the winning 2-mark. Thus when I plays $p_1 \leq a_{p_0}(0)$, II will have knowledge of both p_0 and p_1, and thus can observe that as $p_1 \leq a_{p_0}(0)$, it must be round 1 rather than some future round, and can repeat this process by pretending I chose $a_{p_1}(1) \leq p_1$ and $a_{p_0}(0) \leq p_0$ instead.

We now proceed with a formal proof. Let σ witness II $\underset{(k+2)-\text{mark}}{\uparrow} BM_{po}(\mathbb{P}, W)$. Define $\tau(t) = \sigma(\langle a_{t(0)}(0)\rangle, 1)$ for $t \in \mathbb{P}^1$. Since $\tau(t) = \sigma(\langle a_{t(0)}(0)\rangle, 1) \leq a_{t(0)}(0) \leq t(0)$, this is a legal move.

Consider $t \in \mathbb{P}^{j+2}$ for $j \leq k$. If there exists $l_t < \omega$ such that $t(j + 1) \leq a_{t(j)}(l_t + j)$, define $t' \in \mathbb{P}^{j+2}$ by $t'(i) = a_{t(i)}(l_t + i)$ and let $\tau(t) = \sigma(t', l_t + |t|)$. Note that since

$$\tau(t) = \sigma(t', l_t + |t|) \leq t'(j + 1) = a_{t(j+1)}(l_t + j + 1) \leq t(j + 1)$$

this is a legal move. (If l_t failed to exist, we could arbitrarily let, say, $\tau(t) = t(|t| - 1)$; as we will see, this case will never occur for any legal attack against τ.)

Let f be a legal attack against τ. We may quickly verify that $l_{f\restriction 2} = 0$ since

$$\begin{aligned}(f \restriction 2)(1) &= f(1) \\ &\leq \tau(f \restriction 1) \\ &= \sigma(\langle a_{f(0)}(0)\rangle, 1) \\ &\leq a_{f(0)}(0) \\ &= a_{(f\restriction 2)(0)}(0+0)\end{aligned}$$

We claim in general that $l_{f\restriction(j+2)} = 0$ for $j \leq k$. Assuming $l_{f\restriction(j+2)} = 0$ for $j < k$,

$$\begin{aligned}(f \restriction (j+3))(j+2) &= f(j+2) \\ &\leq \tau(f \restriction (j+2)) \\ &= \sigma(f \restriction (j+2)', 0 + (j+2)) \\ &\leq f \restriction (j+2)'(j+1) \\ &= a_{(f\restriction(j+2))(j+1)}(0 + (j+1)) \\ &= a_{(f\restriction(j+3))(j+1)}(0 + (j+1))\end{aligned}$$

proving $l_{f\restriction(j+3)} = 0$.

Now we show that $l_{f\restriction(n+2)\downharpoonleft(k+2)} = j - k$ for $n \geq k$. We've just shown that this is true for our base case $n = k$ since in that case $f \restriction (n+2) \downharpoonleft (k+2) = f \restriction (k+2)$. Now assuming $l_{f\restriction(n+2)\downharpoonleft(k+2)} = n - k$ for some $n \geq k$, we observe

$$\begin{aligned}(f \restriction (n+3) \downharpoonleft (k+2))(k+1) &= f(n+2) \\ &\leq \tau(f \restriction (n+2) \downharpoonleft (k+2)) \\ &= \sigma((f \restriction (n+2) \downharpoonleft (k+2))', (n-k) + (k+2)) \\ &\leq (f \restriction (n+2) \downharpoonleft (k+2))'(k+1) \\ &= a_{(f\restriction(n+2)\downharpoonleft(k+2))(k+1)}((n-k) + (k+1)) \\ &= a_{(f\restriction(n+3)\downharpoonleft(k+2))(k)}((n+1-k) + (k))\end{aligned}$$

and conclude $l_{f\restriction(n+3)\downharpoonleft(k+2)} = n + 1 - k$.

Define $g \in \mathbb{P}^{\omega,\downarrow}$ by $g(0) = f(0)$ and $g(j+1) = a_{f(j+1)}(j+1)$. Reviewing the above, the reader may confirm that we have shown for $n < k+2$

$$f(n+1) \leq \tau(f \restriction (n+1)) = \sigma(g \restriction (n+1), n+1) \leq g(n) \leq f(n)$$

and for $n \geq k+2$

$$f(n+1) \leq \tau(f \restriction (n+1) \downharpoonleft (k+2)) = \sigma(g \restriction (n+1) \downharpoonleft (k+2), n+1) \leq g(n) \leq f(n)$$

Thus g is a legal attack against σ, and since σ is winning, $g \in W$. Since W is closed under zipping, $f \in W$, and therefore τ is also winning. $\square$

References

[1] Tomek Bartoszynski, Winfried Just, and Marion Scheepers. Covering games and the banach-mazur game: k-tactics. *Canad. J. Math*, 45:897–929, 1993.

[2] W. W. Comfort and S. Negrepontis. *Chain Conditions in Topology.* Cambridge Tracts in Mathematics. Cambridge University Press, 1982.

[3] Fred Galvin and Ratislav Telgársky. Stationary strategies in topological games. *Topology Appl.*, 22(1):51–69, 1986.