

# ALMOST COMPATIBLE FUNCTIONS AND INFINITE LENGTH GAMES

STEVEN CLONTZ AND ALAN DOW

ABSTRACT.  $\mathcal{A}'(\kappa)$  asserts the existence of pairwise almost compatible finite-to-one functions  $A \rightarrow \omega$  for each countable subset  $A$  of  $\kappa$ . The existence of winning 2-Markov strategies in several infinite-length games, including the Menger game on the one-point Lindelöfication  $\kappa^\dagger$  of  $\kappa$ , are guaranteed by  $\mathcal{A}'(\kappa)$ .  $\mathcal{A}'(\kappa)$  is implied by the existence of cofinal Kurepa families of size  $\kappa$ , and thus holds for all cardinals less than  $\aleph_\omega$ . It's consistent that  $\mathcal{A}'(\aleph_\omega)$  fails, but there must always be a winning 2-Markov strategy for the second player in the Menger game on  $\omega_\omega^\dagger$ .

## 1. INTRODUCTION

**Definition 1.** Two functions  $f, g$  are almost compatible, that is,  $f \sim g$  when  $\{a \in \text{dom } f \cap \text{dom } g : f(a) \neq g(a)\}$  is finite.

Marion Scheepers used almost compatible functions in [9] in order to study the existence of limited information strategies on a variation of the meager-nowhere dense game he introduced in [10].

**Game 2.** Let  $Sch_{C,F}^{\cup, \subset}(\kappa)$  denote *Scheepers' strict countable-finite union game* with two players  $\mathcal{C}, \mathcal{F}$ . In round 0,  $\mathcal{C}$  chooses  $C_0 \in [\kappa]^{\leq \omega}$ , followed by  $\mathcal{F}$  choosing  $F_0 \in [\kappa]^{< \omega}$ . In round  $n + 1$ ,  $\mathcal{C}$  chooses  $C_{n+1} \in [\kappa]^{\leq \omega}$  such that  $C_{n+1} \supset C_n$ , followed by  $\mathcal{F}$  choosing  $F_{n+1} \in [\kappa]^{< \omega}$ .

$\mathcal{F}$  wins the game if  $\bigcup_{n < \omega} F_n \supseteq \bigcup_{n < \omega} C_n$ ; otherwise,  $\mathcal{C}$  wins.

Of course, with perfect information this game is trivial: during round  $n$  player  $\mathcal{F}$  simply chooses  $n$  ordinals from each of the  $n$  countable sets played by  $\mathcal{C}$ . However, if  $\mathcal{F}$  is limited to using information from the last  $k$  moves by  $\mathcal{C}$  during each round, the task becomes more difficult. Call such a strategy a *k-tactical strategy* or *k-tactic*; if using the round number is allowed, then the strategy is called a *k-Markov strategy* or a *k-mark*.

**Definition 3.** The statement  $\mathcal{A}(\kappa)$  (given as  $S(\kappa, \aleph_0, \omega)$  in [9] and  $S(\kappa)$  in [1]) claims that there exist one-to-one functions  $f_A : A \rightarrow \omega$  for each  $A \in [\kappa]^{\leq \aleph_0}$  such that the collection  $\{f_A : A \in [\kappa]^{\leq \aleph_0}\}$  is pairwise almost compatible.

In the same paper, Scheepers noted that  $\mathcal{A}(\omega_1)$  holds in *ZFC*, and that it's possible to force  $\mathfrak{c}$  to be arbitrarily large while preserving  $\mathcal{A}(\mathfrak{c})$ . However,  $\mathcal{A}(\mathfrak{c}^+)$  always fails. This axiom may be applied to obtain a winning 2-tactic for  $\mathcal{F}$  in the countable-finite game.

In [1], Clontz related this game to a game which may be used to characterize the Menger covering property of a topological space.

**Game 4.** Let  $Men_{C,F}(X)$  denote the *Menger game* with players  $\mathcal{C}$ ,  $\mathcal{F}$ . In round  $n$ ,  $\mathcal{C}$  chooses an open cover  $\mathcal{U}_n$ , followed by  $\mathcal{F}$  choosing subset  $F_n$  of  $X$  which may be finitely covered by  $\mathcal{U}_n$ .

$\mathcal{F}$  wins the game if  $X = \bigcup_{n < \omega} F_n$ , and  $\mathcal{C}$  wins otherwise.

This characterization is slightly different than the typical characterization in which the second player first chooses a specific finite subcollection  $\mathcal{F}_n$  of the cover itself and lets  $F_n = \bigcup \mathcal{F}_n$ , denoted as  $G_{fin}(\mathcal{O}, \mathcal{O})$  in [11]. However, it's easily seen that these games are equivalent for perfect information strategies (so both characterize the Menger property), and this characterization is more convenient for our concerns.

**Definition 5.** Let  $\kappa^\dagger = \kappa \cup \{\infty\}$  where  $\kappa$  is discrete and  $\infty$ 's neighborhoods are the co-countable sets containing it.

The relationship between  $Sch_{C,F}^{\cup, \subseteq}(\kappa)$  and  $Men_{C,F}(\kappa^\dagger)$  is strong; in both games  $\mathcal{C}$  essentially chooses a countable subset of  $\kappa$  followed by  $\mathcal{F}$  choosing a finite subset of that choice, and it's easy to see the winning perfect information strategy for  $\mathcal{F}$  in both games. In addition, it was shown in [1] that when  $\mathcal{A}(\kappa)$  holds,  $\mathcal{F}$  has a winning 2-Markov strategy in  $Men_{C,F}(\kappa^\dagger)$ .

One source of motivation is to make progress on the following open question:

**Question 6.** *Does there exist a topological space  $X$  for which  $\mathcal{F} \uparrow Men_{C,F}(X)$  but  $\mathcal{F} \not\uparrow_{2\text{-mark}} Men_{C,F}(X)$ ? (That is, the second player can win the Menger game on  $X$  with perfect information but not with 2-Markov information.)*

## 2. ONE-TO-ONE AND FINITE-TO-ONE ALMOST COMPATIBLE FUNCTIONS

We may weaken Scheeper's  $\mathcal{A}(\kappa)$  as follows:

**Definition 7.** The statement  $\mathcal{A}'(\kappa)$  weakens  $\mathcal{A}(\kappa)$  by only requiring the witnessing almost-compatible functions  $f_A : A \rightarrow \omega$  to be finite-to-one.

**Proposition 8.**  $\mathcal{A}(\kappa)$  and  $\mathcal{A}'(\kappa)$  need only be witnessed by functions  $\{f_A : A \in \mathcal{S}\}$  for some family  $\mathcal{S}$  cofinal in  $[\kappa]^{\leq \aleph_0}$ .

*Proof.* For each  $A \in [\kappa]^{\leq \aleph_0}$  choose  $A' \supseteq A$  from  $\mathcal{S}$  and let  $g_A = f_{A'} \upharpoonright A$ . □

In the next section we will show that  $\mathcal{A}'(\kappa)$  is sufficient for the applications to the Scheepers and Menger games. In the meantime, we will demonstrate that  $\mathcal{A}'(\kappa)$  is strictly weaker than  $\mathcal{A}(\kappa)$ .

Recall the following.

**Definition 9.** A Kurepa family  $\mathcal{K} \subseteq [\kappa]^{\aleph_0}$  on  $\kappa$  satisfies that  $\mathcal{K} \upharpoonright A = \{K \cap A : K \in \mathcal{K}\}$  is countable for each  $A \in [\kappa]^{\aleph_0}$ . Let  $\mathcal{K}(\kappa)$  be the statement claiming there exists a Kurepa family on  $\kappa$  cofinal in  $[\kappa]^{\aleph_0}$ .

**Theorem 10.**  $\mathcal{K}(\kappa) \Rightarrow \mathcal{A}'(\kappa)$ .

*Proof.* Let  $\mathcal{K} = \{K_\alpha : \alpha < \theta\}$  be a cofinal Kurepa family on  $\kappa$ . We first define  $f_\alpha : K_\alpha \rightarrow \omega$  for each  $\alpha < \theta$ .

Suppose we've already defined pairwise almost compatible finite-to-one functions  $\{f_\beta : \beta < \alpha\}$ . To define  $f_\alpha$ , we first recall that  $\mathcal{K} \restriction K_\alpha$  is countable, so we may choose  $\beta_n < \alpha$  for  $n < \omega$  such that  $\{K_\beta : \beta < \alpha\} \restriction K_\alpha \setminus \{\emptyset\} = \{K_\alpha \cap K_{\beta_n} : n < \omega\}$ . Let  $K_\alpha = \{\delta_{i,j} : i \leq \omega, j < w_i\}$  where  $w_i \leq \omega$  for each  $i \leq \omega$ ,  $K_\alpha \cap (K_{\beta_n} \setminus \bigcup_{m < n} K_{\beta_m}) = \{\delta_{n,j} : j < w_n\}$ , and  $K_\alpha \setminus \bigcup_{n < \omega} K_{\beta_n} = \{\delta_{\omega,j} : j < w_\omega\}$ . Then let  $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$  for  $n < \omega$  and  $f_\alpha(\delta_{\omega,j}) = j$  otherwise.

We should show that  $f_\alpha$  is finite-to-one. Let  $n < \omega$ . Since  $f_\alpha(\delta_{m,j}) \geq m$ , we only consider the finite cases where  $m \leq n$ . Since each  $f_{\beta_m}$  is finite-to-one,  $f_{\beta_m}(\delta_{m,j}) \leq n$  for only finitely many  $j$ . Thus  $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$  maps to  $n$  for only finitely many  $j$ .

We now want to demonstrate that  $f_\alpha \sim f_{\beta_n}$  for all  $n < \omega$ . Note  $\delta_{m,j} \in K_{\beta_n}$  implies  $m \leq n$ . For  $m = n$ , we have  $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$  which differs from  $f_{\beta_n}(\delta_{n,j})$  for only the finitely many  $j$  which are mapped below  $n$  by  $f_{\beta_n}$ . For  $m < n$  and  $\delta_{m,j} \in K_{\beta_n}$ , we have  $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$  which can only differ from  $f_{\beta_n}(\delta_{m,j})$  for only the finitely many  $j$  which are mapped below  $m$  by  $f_{\beta_m}$  or the finitely many  $j$  for which the almost compatible  $f_{\beta_n} \sim f_{\beta_m}$  differ.

Finally for any  $\beta < \alpha$ , we may conclude  $f_\alpha \sim f_\beta$  since there is some  $\beta_n$  with  $K_\alpha \cap K_\beta = K_\alpha \cap K_{\beta_n}$ ,  $f_\alpha \sim f_{\beta_n}$ , and  $f_{\beta_n} \sim f_\beta$ .  $\square$

We now construct a topology on  $\omega_n$  for each  $n < \omega$  which will witness a Kurepa family of size  $\aleph_n$ . A similar construction was previously shown by Juhász et. al. in [5], and the relationship of Kurepa families and such spaces has also been investigated in a preprint of Nyikos [8].

**Proposition 11.** *Let  $X$  be a  $T_2$  space with a base of countable and compact neighborhoods. Then  $X$  is locally metrizable with a base of compact open countable sets.*

*Proof.* For each point  $x$  let  $K$  be a countable and compact neighborhood of  $x$ , and it follows that it is contained in a countable, open, and locally compact neighborhood  $W$  of  $x$ , which in turn is zero-dimensional and metrizable. So choose  $V$  clopen in  $W$  such that  $x \in V \subseteq K$ ;  $V$  is a compact open neighborhood of  $x$  in  $X$ .  $\square$

**Definition 12.** A topological space is said to be  $\omega$ -bounded if each countable subset of the space has compact closure. As in [5] we call a  $T_2$ , locally countable,  $\omega$ -bounded space *splendid*, and let  $\mathcal{S}(\kappa)$  represent the claim that there exists a splendid space of cardinality  $\kappa$ .

**Proposition 13.** *Let  $X$  be a  $T_2$  space with cardinality less than  $\aleph_\omega$  which is locally countable and  $\omega$ -bounded. Then the closure operation preserves cardinality and weight.*

*Proof.* Note that the closure of any countable neighborhood is compact, and any Lindelöf set is countable. This space is locally metrizable and thus first-countable, so cardinality and weight coincide for any subspace. The result is obvious if  $A$  is countable; otherwise let  $A = \{a_\alpha : \alpha < \omega_{n+1}\}$  and since basic neighborhoods are countable note any limit point of  $A$  is a limit point of  $A_\beta = \{a_\alpha : \alpha < \beta\}$  for some  $\beta < \omega_{n+1}$ . Thus  $\overline{A} = \bigcup_{\beta < \omega_{n+1}} \overline{A_\beta}$  and by induction  $|\overline{A}| = |A|$ .  $\square$

**Lemma 14.** *Let  $X$  be a  $T_2$  space with cardinality less than  $\aleph_\omega$  which is locally countable and locally compact, and such that its closure operation preserves cardinalities. Then  $X$  has an  $\omega$ -bounded extension  $\tilde{X}$  with the same properties where  $\tilde{X} \setminus X$  has the same cardinality as  $X$ .*

*Proof.* We prove this by induction on  $n$ . If  $n = 0$ , then we can just use the one-point compactification of two copies of  $X$ . So suppose  $n > 0$  and that  $X = \omega_n$  has an appropriate topology. Note that  $X$  has a base of countable and compact neighborhoods since the closure operation preserves cardinalities.

For each  $\alpha < \omega_n$ ,  $\gamma_\alpha$  may be chosen such that both the closure of the set  $\alpha$  in  $X$  and a countable neighborhood of the point  $\alpha$  are subsets of  $\gamma_\alpha$ . Note that the set  $\{\lambda < \omega_n : \alpha < \lambda \Rightarrow \gamma_\alpha < \lambda\}$  is a cub subset of  $\omega_n$  containing a cub subset  $C$  of limit ordinals. Now for each  $\lambda \in C$ , the set  $\lambda$  is open as  $\alpha < \lambda$  belongs to the neighborhood  $\gamma_\alpha \subsetneq \lambda$ . Also, if  $\lambda$  has uncountable cofinality, then for  $\beta \geq \lambda$  and any countable neighborhood  $U$  of  $\beta$ ,  $U \cap \lambda = U \cap \alpha$  for some  $\alpha < \lambda$ ; thus  $U \setminus \bar{\alpha} = U \setminus \lambda$  is a neighborhood of  $\beta$ , showing that  $\lambda$  is clopen.

Let  $\tilde{X} = \omega_n \times 2$ . By induction on  $\lambda \in C$  we will define compatible topologies for  $\tilde{X}_\lambda = \omega_n \times \{0\} \cup \lambda \times \{1\}$  such that

- $\omega_n \times \{0\}$  is an open copy of  $X$ ,
- $\lambda \times 2$  is open, and when  $\text{cf}(\lambda) > \omega$  also closed,
- the space has a base of countable and compact neighborhoods, and
- when  $\lambda$  is a successor, for each  $\alpha < \lambda$  the closure of  $\alpha \times 2$  is an  $\omega$ -bounded subset of  $\lambda \times 2$ .

We first consider the case  $n = 1$ . If  $\lambda$  is a limit in  $C$ , then  $\tilde{X}_\lambda = \bigcup_{\mu \in C \cap \lambda} \tilde{X}_\mu$  satisfies the induction requirements. Otherwise we choose an increasing sequence of ordinals  $\{\alpha_k : k \in \omega\}$  with limit  $\lambda$  such that  $\alpha_0$  is the predecessor of  $\lambda$  in  $C$ , or  $\alpha_0 = 0$  if  $\lambda$  is the least element of  $C$ .

The subspace  $\bar{\lambda} \times \{0\} \cup \alpha_0 \times 2$  of  $X$  is countable and locally compact; therefore it is metrizable and zero-dimensional. So we may choose increasing sets  $U_k$  for  $k < \omega$  which are clopen in this topology and satisfy

$$\overline{\alpha_k \times \{0\} \cup \alpha_0 \times 2} = \bar{\alpha}_k \times \{0\} \cup \alpha_0 \times 2 \subseteq U_k \subseteq \lambda \times \{0\} \cup \alpha_0 \times 2$$

Note that  $U_k$  is also clopen in  $\tilde{X}_{\alpha_0}$  since it is closed in  $\bar{\lambda} \times \{0\} \cup \alpha_0 \times 2$  and open in  $\lambda \times \{0\} \cup \alpha_0 \times 2$ .

We need only describe a base for the points  $\langle \alpha, 1 \rangle \in (\lambda \setminus \alpha_0) \times \{1\}$ . We do so by letting  $\langle \alpha, 1 \rangle$  be isolated when  $\alpha \notin \{\alpha_k : k < \omega\}$ , and giving  $\langle \alpha_k, 1 \rangle$  the open neighborhoods  $(U_k \cup ((\alpha_k + 1) \times \{1\})) \setminus K$  for each compact subset  $K$  of  $U_k \cup (\alpha_k \times \{1\})$ ; that is,  $\langle \alpha_k, 1 \rangle$  is the one point compactifying  $U_k \cup (\alpha_k \times \{1\})$ .

The first two requirements of our inductive hypothesis are obviously satisfied. Note points in  $\lambda \times 2$  are covered by the compact countable neighborhood  $U_k \cup ((\alpha_k + 1) \times \{1\})$  for some  $k < \omega$ , and for points in  $(\omega_n \setminus \lambda) \times \{0\}$  we may use a compact countable neighborhood from  $X$ . For the final requirement, note that for  $\alpha < \lambda$ , we may choose  $\alpha < \alpha_k < \lambda$  and note that  $\alpha \times 2$  is contained in the compact subset  $U_k \cup ((\alpha_k + 1) \times \{1\})$  of  $\lambda \times 2$ .

For the case  $n > 1$ , we may assume that the successors in  $C$  have uncountable cofinality. We again proceed by induction on  $\lambda \in C$ . Again when  $\lambda$  is a limit in  $C$ ,  $\tilde{X}_\lambda = \bigcup_{\mu \in C \cap \lambda} \tilde{X}_\mu$  satisfies the given requirements; in particular if  $\alpha < \lambda$ , then  $\alpha < \mu < \lambda$  for some successor  $\mu \in C$  with uncountable cofinality. As such, the closure of  $\alpha \times 2$  is an  $\omega$ -bounded subset of the clopen  $\mu \times 2$  and therefore also of  $\lambda \times 2$ . In case  $\lambda$  is not a limit of  $C$ , then  $\lambda$  has uncountable cofinality and a predecessor  $\mu \in C$ . We therefore have that  $\lambda \times \{0\}$  is clopen in  $\omega_n \times \{0\}$ . Since the cardinality of  $\lambda \times \{0\} \cup \mu \times 2$  is less than  $\aleph_n$ , we may simply apply the induction hypothesis to choose an appropriate topology for  $\lambda \times 2$ .

As a result,  $\tilde{X} = \bigcup_{\lambda \in C} \tilde{X}_\lambda$  is  $\omega$ -bounded as any countable set is contained in some  $\alpha \times 2$  for  $\alpha < \lambda \in C$ .  $\square$

**Theorem 15.** *For each  $k < \omega$ , there is a  $T_2$ , locally countable,  $\omega$ -bounded topology on  $\omega_k$ . That is,  $\mathcal{S}(\aleph_k)$  for all  $k < \omega$ .*

*Proof.* Apply the previous lemma to  $\omega_n$  with the discrete topology.  $\square$

**Lemma 16.** *The family of compact open sets in a locally countable,  $\omega$ -bounded topological space  $X$  is a Kurepa family cofinal in  $[X]^\omega$ . That is,  $\mathcal{S}(\kappa) \Rightarrow \mathcal{K}(\kappa)$ .*

*Proof.* Let  $\mathcal{K}$  collect all compact open subsets of  $X$ . Of course, every Lindelöf set in a locally countable space is countable, and the closure of every countable set is a compact countable set; thus  $\mathcal{K}$  is cofinal in  $[X]^\omega$ . It is Kurepa since every countable set is contained in a countable compact open subspace of  $X$ ; this subspace has a countable base of compact open sets, which closed under finite unions enumerates all compact open subsets of the subspace.  $\square$

**Corollary 17.**  $\mathcal{K}(\aleph_k)$  for all  $k < \omega$ .

Alternatively, the previous corollary may be obtained via an observation of Todorćević communicated by Dow in [2]: if every Kurepa family of size at most  $\kappa$  extends to a cofinal Kurepa family, then the same is true of  $\kappa^+$ .

Nyikos points out in [8] that a cofinal Kurepa family may be used to construct a locally metrizable,  $\omega$ -bounded, zero-dimensional space with appropriate cardinality, but whether this can be strengthened to locally countable and  $\omega$ -bounded (as asked in [5]) remains an open question.

Also left open is this extension of the question asked in [8] and [5] on the possible equivalence of  $\mathcal{S}(\kappa)$  and  $\mathcal{K}(\kappa)$ .

**Question 18.** *May any of the implications in the theorem  $\mathcal{S}(\kappa) \Rightarrow \mathcal{K}(\kappa) \Rightarrow \mathcal{A}'(\kappa)$  be reversed?*

Regardless, we have obtained our desired result.

**Corollary 19.**  $\mathcal{A}'(\aleph_k)$  for all  $k < \omega$ .

### 3. CONSISTENCY RESULTS

As noted in [2], Jensen's one-gap two-cardinal theorem under  $V = L$  introduced in [4] implies that  $\mathcal{K}(\kappa)$  holds for all cardinals  $\kappa$ .

**Corollary 20** ( $V = L$ ).  $\mathcal{A}'(\kappa)$  for all cardinals  $\kappa$ .

Weakening to the continuum hypothesis, we see an obvious consequence.

**Corollary 21** ( $CH$ ).  $\mathcal{A}'(\mathfrak{c}^+)$ , but  $\neg\mathcal{A}(\mathfrak{c}^+)$ .

But  $CH$  is not required to have  $\mathcal{A}(\aleph_2)$  fail. We take the following lemma from Kunen.

**Lemma 22** ([6]). *Assume that  $G \subset \text{Fn}(\omega_2, 2)$  is a generic filter over a model  $M$ , and let  $\mu \in \omega_2$ . Then the generic extension  $M[G]$  can be regarded as forcing with  $\text{Fn}(\omega_2 \setminus \mu, 2)$  over the model  $M[G_\mu]$ . Let  $\dot{n}, \ddot{n}$  be a  $\text{Fn}(\omega_2, 2)$ -,  $\text{Fn}(\omega_2 \setminus \mu, 2)$ -name respectively for each  $n < \omega$ . Then for each  $\text{Fn}(\omega_2, 2)$ -name  $\dot{A}$  for  $A \subseteq \omega$ , there is a canonical  $\text{Fn}(\omega_2 \setminus \mu, 2)$ -name*

$$\ddot{A} = \{(\ddot{n}, p \restriction [\mu, \omega_2)) : (\dot{n}, p) \in \dot{A}\}$$

which satisfies  $A = \text{val}(\dot{A}, G) = \text{val}(\ddot{A}, G_\mu)$ .

**Theorem 23.** *There exists a model of ZFC for which  $\mathfrak{c} = \aleph_2$  and  $\neg\mathcal{A}(\aleph_2)$ .*

*Proof.* The forcing poset is  $\mathbb{P} = \text{Fn}(\omega_2, 2)$ . Let  $\{\dot{f}_A : A \in [\omega_2]^\omega\}$  be a family of  $\mathbb{P}$ -names such that  $f_A$  is a one-to-one function from  $A$  into  $\omega$  in the ground model. Let  $M$  be an elementary submodel of  $ZFC + CH$  containing all its countable subsets, containing  $\mathbb{P}$ , and satisfying  $|M| = \omega_1 = \mathfrak{c}$ .

Let  $\lambda$  be the ordinal  $M \cap \omega_2$ . For  $\alpha \leq \lambda$  we may assume  $\dot{f}_{[\alpha, \alpha+\omega)} = \{((\alpha + k_p, m_p), p) : p \in C_\alpha\}$  for some countable  $C_\alpha \subseteq \text{Fn}(\omega_2, \omega)$ . Let  $\text{supp}(\dot{f}_{[\alpha, \alpha+\omega)}) = \bigcup_{p \in C_\alpha} \text{dom } p$ .

For each  $\alpha \leq \lambda$ , let  $\delta_\alpha$  denote the order type of  $\text{supp}(\dot{f}_{[\alpha, \alpha+\omega)}) \setminus \alpha$ , and when  $\mu \leq \alpha$  and  $\text{supp}(\dot{f}_{[\alpha, \alpha+\omega)}) \cap \alpha \subseteq \mu$  let  $\varphi'_{\mu, \alpha}$  be the order-preserving bijection from  $\mu \cup (\text{supp}(\dot{f}_{[\alpha, \alpha+\omega)}) \setminus \alpha)$  onto the ordinal  $\mu + \delta_\alpha$ . This lifts canonically to an order-preserving bijection  $\varphi_{\mu, \alpha} : \text{Fn}(\mu \cup (\text{supp}(\dot{f}_{[\alpha, \alpha+\omega)}) \setminus \alpha), 2) \rightarrow \text{Fn}(\mu + \delta_\alpha, 2)$ .

We now fix  $\mu < \lambda$  such that

$$\bigcup_{\alpha \leq \lambda} \text{supp}(\dot{f}_{[\alpha, \alpha+\omega)}) \cap \alpha \subseteq \mu$$

(TODO how?) and define the  $\mathbb{P}$ -name

$$F = \{((\lambda + k_p, m_p), \varphi_{\mu, \lambda}(p)) : ((\lambda + k_p, m_p), p) \in \dot{f}_{[\lambda, \lambda+\omega)}\}.$$

So in  $M$ , we may reason (TODO why) that for each  $\beta < \omega_2$  there exists some  $\alpha \in \omega_2 \setminus \beta$  where

$$\text{supp}(\dot{f}_{[\alpha, \alpha+\omega)}) \cap \alpha \subseteq \mu$$

and

$$F = \{((\alpha + k_p, m_p), \varphi_{\mu, \alpha}(p)) : ((\alpha + k_p, m_p), p) \in \dot{f}_{[\alpha, \alpha+\omega)}\}.$$

But now, this means not only is there an  $\alpha < \lambda$  such that  $F = \varphi(\dot{f}_{[\alpha, \alpha+\omega)})$ , but also that there is an increasing sequence  $\{\alpha_\xi : \xi \in \omega_1\} \subseteq \lambda$  satisfying that  $F = \varphi(\dot{f}_{[\alpha_\xi, \alpha_\xi+\omega)})$  and  $\text{supp}(\dot{f}_{[\alpha_\xi, \alpha_\xi+\omega)})$  is contained in  $\alpha_{\xi+1}$  (TODO why not  $\mu$ ?).

Choose such a sequence, and let  $A = \bigcup_{0 < n < \omega} [\alpha_n, \alpha_n + \omega)$ . Consider the name  $\dot{f}_A$  in  $M$ . Note all the  $((\beta, m), p)$  appearing in  $\dot{f}_A$  have the property that  $\text{dom}(p)$  is contained in  $M$  (QUESTION: isn't the domain of any member of  $\mathbb{P}$  in the domain?). There is, within  $M$ , a name  $\dot{g}$  satisfying that  $\dot{f}_A(\alpha_n + k) = \dot{f}_{[\alpha_n, \alpha_n + \omega)}(\alpha_n + k)$  for all  $k > \dot{g}(n)$  (TODO: why?).

(TODO catch up to here)

We now apply the above Lemma and work in  $M[G_\mu]$ . We work for a contradiction. Something special has now happened, namely, the supports of the names  $\{\dot{f}_{[\alpha_n, \alpha_n + \omega)} : 0 < n < \omega\}$  are pairwise disjoint and also disjoint from the support of the name  $\dot{f}_{[\lambda, \lambda + \omega)}$  (under the same convention about  $G_\mu$ . And not only that, these names are pairwise isomorphic (in the way that they all map to  $F_M$ ).

Since  $A$  is disjoint from  $[\lambda, \lambda + \omega)$ , there must be an integer  $\ell$  together with a condition  $q \in \text{Fn}(\omega_2 \setminus \mu, 2)$  satisfying that for all  $n > \ell$ ,  $q$  forces that

“if  $k > \dot{g}(n)$  (since  $\alpha_n + k \in A$ ) then  $\dot{f}_{[\alpha_n, \alpha_n + \omega)}(\alpha_n + k) \neq \dot{f}_{[\lambda, \lambda + \omega)}(\lambda + k)$ ”.

Choose  $n$  large enough so that  $\text{dom}(q) \cap [\alpha_n, \mu_{n+1})$  is empty. Choose  $q_1 < q \restriction \lambda$  (in  $M$ ) so that

$$\varphi_{\mu, \alpha_n}(q_1 \restriction \text{supp}(\dot{f}_{[\alpha_n, \alpha_n + \omega)})) = \varphi(q \restriction \text{supp}(\dot{f}_{[\lambda, \lambda + \omega)}))$$

and then (again in  $M$ ) choose  $q_2 < q_1$  so that it both forces a value  $L$  on  $\ell + \dot{g}(n)$  and subsequently forces a value  $m$  on  $\dot{f}_{[\alpha_n, \alpha_n + \omega)}(\alpha_n + L + 1)$ . But now, again calculate

$$q_3 = \varphi^{-1} \circ \varphi_{\mu, \alpha_n}(q_2 \restriction \text{supp}(\dot{f}_{[\alpha_n, \alpha_n + \omega)}))$$

and, by the isomorphisms, we have that  $q_3$  forces that  $\dot{f}_{[\lambda, \lambda + \omega)}(\lambda + L + 1) = m$ .

Technically (or with more care) all of this is taking place in the poset  $\text{Fn}(\omega_2 \setminus \mu, 2)$  and this means that  $q_3$  and  $q$  are all compatible with each other.

Follow the bouncing ball: it suffices to consider  $q(\beta) = e$  and to assume that  $q_3(\beta)$  is defined. Since  $q_3(\beta)$  is defined, we have that there is a  $\beta' \in \text{dom}(q_2)$  such that  $\varphi(\beta) = \varphi_{\mu, \alpha_n}(\beta')$ , and that  $q_3(\beta) = q_2(\beta')$ . But, by definition of  $q_1$ ,  $\beta' \in \text{dom}(q_1)$  and even that  $q_1(\beta') = q(\beta)$ . Then, since  $q_2 < q_1$ , we have that  $q_2(\beta') = q_1(\beta') = q(\beta)$ . This completes the circle that  $q_3(\beta) = q(\beta)$ .

Finally, our contradiction is that  $q_3 \cup q_2 \cup q$  forces that  $k = L + 1$  violates the quoted statement above.  $\square$

We are also able to force  $\mathcal{A}'(\kappa)$  to fail for every cardinal other than the first  $\omega$ -many we've already guaranteed.

**Theorem 24.** *It follows from the existence of a 2-huge cardinal that there is a model of ZFC for which  $\neg \mathcal{A}'(\aleph_\omega)$ .*

*Proof.* We will need the model constructed in [7] in which an instance of Chang's conjecture  $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$  is shown to fail.

We can take as a given (as shown in [7, Theorem 5]) that we may assume that we have a model  $V$  of GCH in which there are regular limit cardinals  $\kappa < \lambda$  satisfying that  $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$ .

What this says is that if  $L$  is a countable language with at least one unary relation symbol  $R$  and  $M$  is a model of  $L$  with base set  $\lambda^{+\omega+1}$  in which the interpretation of  $R$  has cardinality  $\lambda^{+\omega}$ , then  $M$  has an elementary submodel  $N$  of cardinality  $\kappa^{+\omega+1}$  in which  $R \cap N$  has cardinality  $\kappa^{+\omega}$  (of course  $R \cap N$  is the interpretation of  $R$  in  $N$  because  $N \prec M$ ).

The interested reader will want to know that it is shown in [7] that if  $\kappa$  is a 2-huge cardinal and  $j$  is the 2-huge embedding with critical point  $\kappa$ , then with  $\lambda = j(\kappa)$  one has that  $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$  holds. There is no loss of generality to also assume that GCH holds in this model.

Let  $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$  be a scale in  $\Pi\{\lambda^{+n+1} : n \in \omega\}$  ordered by the usual mod finite coordinatewise ordering. For convenience we may assume that  $h_\xi(n) \geq \lambda^{+n}$  for all  $\xi$  and all  $n$ . For each integer  $m$  the cofinality of the mod finite ordering on  $\Pi\{\lambda^{+n+1} : m < n \in \omega\}$  is the same as it is for the entire product  $\Pi\{\lambda^{+n+1} : n \in \omega\}$ . If  $P$  is any poset of cardinality less than  $\lambda^{+m}$  then, in the forcing extension by  $P$ , every function in  $\Pi\{\lambda^{+n+1} : m < n \in \omega\}$  is bounded above by a ground model function. It therefore follows easily that in the forcing extension by  $P$ , the sequence  $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$  remains cofinal in  $\Pi\{\lambda^{+n+1} : n \in \omega\}$ .

The forcing notion  $\mathbb{P}_0$  is simply the finite condition collapse of  $\kappa^{+\omega}$ , i.e.  $\mathbb{P}_0 = (\kappa^{+\omega})^{<\omega}$ . In the forcing extension by  $\mathbb{P}_0$ , one now has that the ordinal  $\kappa^{+\omega+1}$  from  $V$  is the first uncountable cardinal  $\aleph_1$ . Then in this forcing extension we let  $\mathbb{P}_1$  be the countable condition Levy collapse,  $Lv(\lambda, \omega_2)$ , which collapses all cardinals less than  $\lambda$  to have cardinality at most  $\aleph_1$ . The poset  $\mathbb{P}_1$  has cardinality  $\lambda$ . We treat  $\mathbb{P} * \mathbb{P}_1$  as containing  $\mathbb{P}_0$  as a subposet by identifying each  $(p_0, 1)$  with  $p_0$ . After forcing with  $\mathbb{P}_0 * \mathbb{P}_1$  we will have that  $\omega_1$  is the ordinal  $(\kappa^{+\omega+1})^V$ ,  $\omega_2$  is the ordinal  $\lambda$ , and  $\omega_\omega$  is the ordinal  $(\lambda^{+\omega})^V$ .



Now we assume that we have an assignment  $\dot{f}_{\dot{A}}$  of a  $\mathbb{P}_0 * \mathbb{P}_1$ -name of a finite-to-one function from  $\dot{A}$  into  $\omega$  for each  $\mathbb{P}_0 * \mathbb{P}_1$ -name of a countable subset of  $\lambda^{+\omega+1}$ . We will obtain a contradiction.

Let  $\{\dot{A}_\xi : \xi \in \lambda^{+\omega+1}\}$  be an enumeration of all the nice  $\mathbb{P}_0$ -names of countable subsets of  $\lambda^{+\omega}$ . For each  $\xi \in \lambda^{+\omega+1}$ , let  $\dot{f}_\xi$  be another notation for  $\dot{f}_{\dot{A}_\xi}$ . Since  $\mathbb{P}_0$  forces that  $\mathbb{P}_1$  is countably closed, the collection of all nice  $\mathbb{P}_0$ -names will produce all the countable sets in the extension by  $\mathbb{P}_0 * \mathbb{P}_1$ , but  $\mathbb{P}_0 * \mathbb{P}_1$  can introduce new enumerations of these names. For each  $\xi \in \lambda^{+\omega+1}$ , there is a minimal  $\zeta_\xi$  so that  $\dot{A}_{\zeta_\xi}$  is the canonical name for the range of  $h_\xi$ . This means that  $\dot{f}_{\zeta_\xi} \circ h_\xi$  is simply the  $\mathbb{P}_0 * \mathbb{P}_1$ -name of a finite-to-one function from  $\omega$  to  $\omega$ . For each  $\xi \in \lambda^{+\omega+1}$ , choose any  $p_\xi \in \mathbb{P}_0 * \mathbb{P}_1$  so that there is a nice  $\mathbb{P}_0$ -name,  $\dot{H}_\xi$ , that is forced by  $p_\xi$  to equal  $\dot{f}_{\zeta_\xi} \circ h_\xi$ . Choose  $\Lambda \subset \lambda^{+\omega+1}$  of cardinality  $\lambda^{+\omega+1}$  and so that there is a pair  $p, \dot{H}$  satisfying that  $p_\xi = p$  and  $\dot{H}_\xi = \dot{H}$  for all  $\xi \in \Lambda$ . We may assume that  $p$  is in a generic filter  $G$ .

Let  $\{x_\xi : \xi \in \lambda^{+\omega+1}\}$  be any enumeration of  $H(\lambda^{+\omega+1})$  such that  $\{x_\xi : \xi \in \lambda^{+\omega}\}$  is also equal to  $H(\lambda^{+\omega})$ . We choose this enumeration in such a way that  $x_\xi \in x_\eta$  implies  $\xi < \eta$ . We use relation symbol  $R_0$  to code (and well order)  $(H(\lambda^{+\omega+1}), \in)$  as follows:  $(\xi, \eta) \in R_0$  if and only if  $x_\xi \in x_\eta$ . Let  $R_1$  be a binary relation on  $\kappa^{+\omega}$  so that  $(\kappa^{+\omega}, R_1)$  is isomorphic to  $\mathbb{P}_0$ . Let  $R_2$  be a binary relation on  $\lambda$  so that  $R_2 \cap (\kappa^{+\omega} \times \kappa^{+\omega}) = R_1$  and  $(\lambda, R_2)$  is isomorphic to  $\mathbb{P}_0 * \mathbb{P}_1$ . Let  $\psi$  be the poset isomorphism from  $(\lambda, R_2)$  to  $\mathbb{P}_0 * \mathbb{P}_1$ .

We continue coding. We can code the sequence  $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$  as another binary relation  $R_3$  on  $\lambda^{+\omega+1}$  where  $R_3 \cap (\{\xi\} \times \lambda^{+\omega+1}) = \{(\xi, h_\xi(n)) : n \in \omega\}$  for each  $\xi \in \lambda^{+\omega+1}$ . The relation symbol  $R_4$  can code the sequence  $\{\dot{A}_\xi : \xi \in \lambda^{+\omega+1}\}$  where  $(\xi, \alpha, \zeta) \in R_4$  if and only if  $(\dot{\alpha}, \psi(\zeta))$  is in the name  $\dot{A}_\xi$ . Let  $R_5$  code this collection, i.e.  $(\gamma, n, m, \eta) \in R_5$  if and only if  $((n, m), \psi(\eta)) \in \dot{H}_\gamma$ . Also let  $R_6$  code (equal) the set  $\Lambda$ . Finally we use the relation symbol  $R_7$  to similarly code the sequence  $\{\dot{f}_\xi : \xi \in \lambda^{+\omega+1}\}$ :  $(\xi, \alpha, n, \zeta) \in R_7$  if and only if  $((\alpha, n), \psi(\zeta))$  is in the name  $\dot{f}_\xi$ .

Needless to say, the unary relation symbol  $R$  is interpreted as the set  $\lambda^{+\omega}$  for the application of  $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$ . Now we have defined our model  $M$  of the language  $L = \{\in, R, R_0, \dots, R_7\}$ , and we choose an elementary submodel  $N$  witnessing  $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$ . Of course  $N$  is really just a  $\kappa^{+\omega+1}$  sized subset of  $\lambda^{+\omega+1}$  with the additional property that  $N \cap \lambda^{+\omega}$  has cardinality  $\kappa^{+\omega}$ . In the forcing extension  $N$  has cardinality  $\omega_1$  and  $A = N \cap \lambda^{+\omega}$  is countable.

We will need the following claim from [7]:

**Claim.** *We may assume that  $N$  satisfies that  $N \cap \kappa^{+\omega+1}$  is transitive (i.e. an initial segment).*

*Proof of Claim:* Suppose our originally supplied  $N$  fails the conclusion of the claim. We know that  $\kappa^{+\omega} \in N$ , (via  $R_1$ ) in which case so is  $\kappa^{+\omega+1}$ .

Then set  $\beta_0 = \sup(N \cap \kappa^{+\omega+1})$  and consider the Skolem closure  $Hull(N \cup \beta_0, M)$ . A little informally (in that we have to formalize the enumeration of formulas as per Gödel coding) let  $\{\varphi_n : n \in \omega\}$  be an enumeration of all formulas in the language

$L$ , and let  $\ell_n$  be the minimal integer such that the free variables of  $\varphi_n$  are among  $\{v_0, \dots, v_{\ell_n}\}$ . Then, for each tuple  $\langle \xi_1, \dots, \xi_{\ell_n} \rangle$  of elements of  $\lambda^{+\omega+1}$ , we define  $f_n(\xi_1, \dots, \xi_{\ell_n})$  to be the minimal  $\xi_0 \in \lambda^{+\omega+1}$  such that  $M \models \varphi_n(\xi_0, \dots, \xi_{\ell_n})$ . If there is no such  $\xi_0$ , in other words if  $M \models \neg \exists x \varphi_n(x, \xi_1, \dots, \xi_{\ell_n})$ , then set  $f_n(\xi_1, \dots, \xi_{\ell_n})$  to be 0. Now  $\text{Hull}(N \cup \beta_0, M)$  is just the minimal superset  $X$  of  $N \cup \beta_0$  that satisfies that  $f_n[X^{\{1, \dots, \ell_n\}}] \subset X$  for all  $n$ . Since this is simply a large algebra, we can generate all the terms  $t$  of the algebraic operations  $\{f_n : n \in \omega\}$ . It is easily seen that for each  $\zeta \in X$ , there is a term  $t(v_1, \dots, v_m)$  such that  $\zeta = t(\delta_1, \dots, \delta_m)$  for some sequence  $\langle \delta_1, \dots, \delta_m \rangle$  with each  $\delta_i \in N \cup \beta_0$ . Assume that  $\zeta \in \kappa^{+\omega+1}$ . By re-indexing the variables in the term we can assume that there is an  $n \leq m$  so that  $\delta_i < \beta_0$  for  $1 \leq i \leq n$  and  $\kappa^{+\omega+1} \leq \delta_i$  for  $n < i \leq m$ . Let  $\vec{a}$  denote the tuple  $\langle \delta_{n+1}, \dots, \delta_m \rangle$ . Choose  $\eta \in N \cap \kappa^{+\omega+1}$  large enough so that  $\{\delta_1, \dots, \delta_n\}$  is contained in  $\eta$ . Since set-membership in  $M$  is coded by  $R_0$  rather than  $\in$  we have to argue a little less naturally. Consider the set  $s_0(\eta, \vec{a}) = \{t(\gamma_1, \dots, \gamma_n, \vec{a}) : \{\gamma_1, \dots, \gamma_n\} \in [\eta]^{\leq n}\}$ . Clearly  $s_0(\eta, \vec{a})$  is a member of  $H(\lambda^{+\omega+1})$ . Now define  $s_1(\eta, \vec{a})$  to be  $\{x_\alpha : \alpha \in s_0(\eta, \vec{a})\}$ , and choose the unique  $\zeta_1 \in \lambda^{+\omega+1}$  such that  $x_{\zeta_1} = s_1(\eta, \vec{a})$ . We claim that  $\zeta_1 \in N$ . Note that  $\alpha R_0 \zeta_1$  holds if and only if  $\alpha \in s_0(\eta, \vec{a})$ , and therefore

$$M \models (\forall \alpha) [\alpha R_0 \zeta_1 \text{ iff } (\exists \gamma_1 \in \eta) \cdots (\exists \gamma_n \in \eta) (\alpha = t(\gamma_1, \dots, \gamma_n, \vec{a}))] .$$

By elementarity then we have that  $\zeta_1 \in N$ , and by similar reasoning the supremum,  $\zeta_0$ , of  $\zeta_1 \cap \kappa^{+\omega+1}$  is also in  $N$ . This of course means that  $\zeta < \xi_0$ .  $\square$

We use the elementarity of  $N$  to deduce properties of the families  $\{\dot{A}_\xi : \xi \in N\}$  and  $\{\dot{f}_\xi : \xi \in N\}$ . Actually the collection we are most interested in is the family  $\{h_\xi : \xi \in \Lambda \cap N\}$ .

Now we need a result from Shelah's pcf theory which is proven in Jech [3, 24.9]. Since  $\aleph_1 = \mathfrak{c} < \kappa^{+\omega+1}$  there is a function  $\langle \varrho_n : n \in \omega \rangle$  in  $\Pi_n \lambda^{+\omega}$  such that the sequence  $\{h_\xi : \xi \in N\}$  is unbounded mod finite in  $\Pi_n \varrho_n$ . For each  $n$ ,  $\varrho_n \leq \sup(N \cap \lambda^{+n+2})$ . Since  $\mathbb{P}_0$  has cardinality  $\kappa^{+\omega}$ , and so less than  $|N| = \kappa^{+\omega+1}$ , a standard argument (analogous to the fact that adding a Cohen real does not add a dominating real) shows that the sequence  $\{h_\xi : \xi \in \Lambda \cap N\}$  remains unbounded mod finite in  $\Pi_n \varrho_n$  (and in  $\Pi_n(\varrho_n \cap N)$ ).

Now pass to the extension by  $G \cap \mathbb{P}_0$  and let  $H$  be the function  $\text{val}_G(\dot{H})$ , and we recall that  $f_{\xi_\xi}(h_\xi(n)) = H(n)$  for all  $n \in \omega$  and  $\xi \in \Lambda$ . Now pass to the full extension  $V[G]$  and again, since  $\mathbb{P}_1$  was forced to be countably closed, the family  $\{h_\xi : \xi \in \Lambda \cap N\}$  is still unbounded in  $\Pi_n(\varrho_n \cap N)$  (no new elements were added). We let  $A$  be the countable set  $N \cap \lambda^{+\omega}$ , and for each  $\xi \in \Lambda \cap N$ , there is an  $n_\xi$  such that  $f_\xi(h_\xi(m)) = f_A(h_\xi(m))$  for all  $m > n_\xi$ . There is a single  $n$  so that  $\Lambda_n = \{\xi \in \Lambda \cap N : n_\xi = n\}$  has cardinality  $\omega_1$ , and thus  $\{h_\xi : \xi \in \Lambda_n \cap N\}$  is also unbounded in  $\Pi_n(\varrho_n \cap N)$ . This certainly implies that there is an  $m > n$  such that  $\{h_\xi(m) : \xi \in \Lambda_n \cap N\}$  is infinite. This completes the proof since  $f_A(h_\xi(m)) = H(m)$  for all  $\xi \in \Lambda_n \cap N$ .  $\square$

#### 4. APPLICATIONS TO INFINITE LENGTH GAMES

We introduce three variations of Scheeper's game which we defined in the introduction.

**Game 25.** Let  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$  denote the *Scheepers countable-finite union game* which proceeds analogously to  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$ , except that  $\mathcal{C}$ 's restriction in round  $n+1$  is weakened to  $C_{n+1} \supseteq C_n$ .

**Game 26.** Let  $Sch_{C,F}^{1,\subseteq}(\kappa)$  denote the *Scheepers countable-finite initial game* which proceeds analogously to  $Sch_{C,F}^{\cup,\subseteq}(\kappa)$ , except that  $\mathcal{F}$ 's winning condition is weakened to  $\bigcup_{n < \omega} F_n \supseteq C_0$ .

**Game 27.** Let  $Sch_{C,F}^{\cap}(\kappa)$  denote the *Scheepers countable-finite intersection game* which proceeds analogously to  $Sch_{C,F}^{1,\subseteq}(\kappa)$ , except that  $\mathcal{C}$  may choose any  $C_n \in [\kappa]^{\leq \omega}$  each round, and  $\mathcal{F}$ 's winning condition is weakened to  $\bigcup_{n < \omega} F_n \supseteq \bigcap_{n < \omega} C_n$ .

In [1] Clontz extended Scheepers' application of almost-compatible injections to these game variants as well as  $Men_{C,F}(\kappa^\dagger)$ . However, when considering Markov strategies, finite-to-one functions suffice.

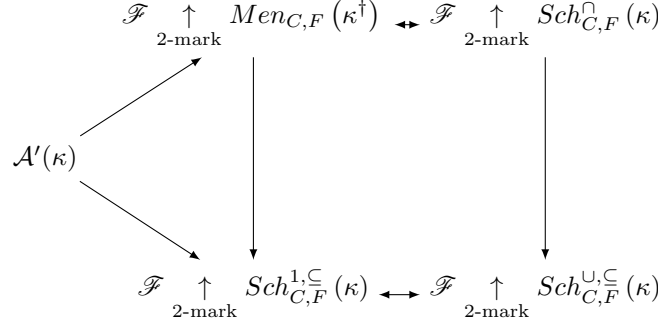


FIGURE 1. Diagram of Scheeper/Menger game implications with  $\mathcal{A}'(\kappa)$

**Theorem 28.**  $\mathcal{A}'(\kappa)$  implies the game-theoretic conditions shown in Figure 1.

*Proof.* The weaker claim  $\mathcal{A}(\kappa) \Rightarrow \mathcal{F} \uparrow_{2\text{-mark}} Sch_{C,F}^{\cap}(\kappa)$  was proven in [1]; however, the strategy only required that the  $f_A$  be pairwise almost-compatible and that preimages of finite sets in each  $f_A$  are finite, which is also possible assuming  $\mathcal{A}'(\kappa)$ . Note that the relationships amongst the games were all shown in [1].  $\square$

We include the following proof from [9] for the convenience of the reader.

**Theorem 29.**  $\mathcal{A}(\kappa) \Rightarrow \mathcal{F} \uparrow_{2\text{-tact}} Sch_{C,F}^{\cup,\subseteq}(\kappa)$

*Proof.* Let  $\{f_A : A \in [\kappa]^{\leq \aleph_0}\}$  witness  $\mathcal{A}(\kappa)$ , and define  $g_A : A \rightarrow \omega$  by  $g_A(\alpha) = f_A(\alpha) - |\{\beta \in A : f_A(\beta) < f_A(\alpha)\}|$ .

We claim that  $\{\alpha \in A : g_A(\alpha) \leq g_B(\alpha)\}$  must be finite as it is bounded above by  $\max\{M, f_A(\alpha), f_B(\alpha) : f_A(\alpha) \neq f_B(\alpha)\}$  where  $M = f_B(\alpha)$  for some  $\alpha \in B \setminus A$ . To see this, let  $f_A(\alpha) = f_B(\alpha) = N > M$  and assume  $f_A(\beta) \neq f_B(\beta)$  implies  $f_A(\beta), f_B(\beta) < N$ . Then

$$g_A(\alpha) = N - |\{\beta \in A : f_A(\beta) < N\}| > N - |\{\beta \in B : f_B(\beta) < N\}| = g_B(\alpha)$$

with the strictness of the inequality witnessed by  $f_B(\alpha) = M < N$  for some  $\alpha \in B \setminus A$ .

As a result,

$$\sigma(\langle A, B \rangle) = \{\alpha \in A : g_A(\alpha) \leq g_B(\alpha)\}$$

is a legal 2-tactic for  $\mathcal{F}$ . Let  $C = \langle C_0, C_1, \dots \rangle$  be a strictly increasing sequence of countable sets and  $\alpha \in C_n$ . Noting that  $f_A$  is an injection and not just finite-to-one,  $0 \leq g_{C_{n+m}}(\alpha)$  for all  $m < \omega$ , and it follows that  $g_{C_{n+m}}(\alpha) \leq g_{C_{n+m+1}}(\alpha)$  for some  $m < \omega$ . Therefore  $\alpha \in \sigma(\langle C_{n+m}, C_{n+m+1} \rangle)$ .  $\square$

So it would seem that  $\mathcal{A}'(\kappa)$  is sufficient only when considering Markov strategies. (Of course,  $\mathcal{A}'(\kappa) \Rightarrow \mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{\cup, \subseteq}(\kappa) \Rightarrow \mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{\cup, \subseteq}(\kappa)$ .) We would like to demonstrate that  $\mathcal{A}'(\kappa)$  is not necessary.

**Theorem 30.** *Let  $\alpha$  be the limit of increasing ordinals  $\beta_n$  for  $n < \omega$ . If  $\mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{\cap}(\aleph_{\beta_n})$  for all  $n < \omega$ , then  $\mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{\cap}(\aleph_{\alpha})$ .*

*Proof.* Let  $\sigma_n$  be a winning 2-mark for  $\mathcal{F}$  in  $\text{Sch}_{C,F}^{\cap}(\aleph_{\beta_n})$ . Define the 2-mark  $\sigma$  for  $\mathcal{F}$  in  $\text{Sch}_{C,F}^{\cap}(\aleph_{\alpha})$  as follows:

$$\sigma(\langle C \rangle, 0) = \sigma_0(\langle C \cap \aleph_{\beta_0} \rangle, 0)$$

$$\sigma(\langle C, D \rangle, n+1) = \sigma_{n+1}(\langle D \cap \aleph_{\beta_{n+1}} \rangle, 0) \cup \bigcup_{m \leq n} \sigma_m(\langle C \cap \aleph_{\beta_m}, D \cap \aleph_{\beta_m} \rangle, n-m+1)$$

Let  $\langle C_0, C_1, \dots \rangle$  be an attack by  $\mathcal{C}$  in  $\text{Sch}_{C,F}^{\cap}(\aleph_{\alpha})$ , and  $\alpha \in \bigcap_{n < \omega} C_n$ . Choose  $N < \omega$  with  $\alpha < \aleph_{\beta_{N+1}}$ . Consider the attack  $\langle C_{N+1} \cap \aleph_{\beta_{N+1}}, C_{N+2} \cap \aleph_{\beta_{N+1}}, \dots \rangle$  by  $\mathcal{C}$  in  $\text{Sch}_{C,F}^{\cap}(\aleph_{\beta_{N+1}})$ . Since  $\sigma_{N+1}$  is a winning strategy and  $\alpha \in \bigcap_{n < \omega} C_{N+n+1} \cap \aleph_{\beta_{N+1}}$ , either  $\alpha \in \sigma_{N+1}(\langle C_{N+1} \cap \aleph_{\beta_{N+1}} \rangle, 0)$  and thus  $\alpha \in \sigma(\langle C_N, C_{N+1} \rangle, N+1)$ , or  $\alpha \in \sigma_{N+1}(\langle C_{N+M+1} \cap \aleph_{\beta_{N+1}}, C_{N+M+2} \cap \aleph_{\beta_{N+1}} \rangle, M+1)$  for some  $M < \omega$  and thus  $\alpha \in \sigma(\langle C_{N+M+1}, C_{N+M+2} \rangle, N+M+2)$ . Thus  $\sigma$  is a winning strategy.  $\square$

**Theorem 31.** *Let  $\alpha$  be the limit of increasing ordinals  $\beta_n$  for  $n < \omega$ . If  $\mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{1, \subseteq}(\aleph_{\beta_n})$  for all  $n < \omega$ , then  $\mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{1, \subseteq}(\aleph_{\alpha})$ .*

*Proof.* The proof proceeds nearly identically to the previous proof.  $\square$

**Corollary 32.** *It is consistent that  $\mathcal{A}'(\aleph_{\omega})$  fails, but as  $\mathcal{A}'(\aleph_k)$  holds in ZFC for all  $k < \omega$ , both  $\mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{\cap}(\aleph_{\omega})$  and  $\mathcal{F} \uparrow_{2\text{-mark}} \text{Sch}_{C,F}^{1, \subseteq}(\aleph_{\omega})$  hold in ZFC.*

Note that Question 6 remains unsolved; however, our results have revealed that we cannot hope to find a ZFC counterexample where  $X = \kappa^{\dagger}$ . This is because if we also assume  $V = L$ , it follows that  $\mathcal{A}'(\kappa)$  and therefore  $\mathcal{F} \uparrow_{2\text{-mark}} \text{Men}_{C,F}(X)$  for every cardinal. Although, one may consider the following weaker question.

**Question 33.** *Is there a model of ZFC where both  $\mathcal{F} \uparrow_{2\text{-mark}} \text{Men}_{C,F}(\aleph_{\omega+1}^{\dagger})$  and  $\mathcal{A}'(\aleph_{\omega})$  fail?*

## REFERENCES

- [1] Steven Clontz. Applications of limited information strategies in menger's game (preprint). 2015.
- [2] Alan Dow. Set theory in topology. In *Recent progress in general topology (Prague, 1991)*, pages 167–197. North-Holland, Amsterdam, 1992.
- [3] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.
- [4] R. Björn Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4:229–308; erratum, *ibid.* 4 (1972), 443, 1972. With a section by Jack Silver.
- [5] I. Juhász, Zs. Nagy, and W. Weiss. On countably compact, locally countable spaces. *Period. Math. Hungar.*, 10(2-3):193–206, 1979.
- [6] Kenneth Kunen. *Set theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-New York, 1980. An introduction to independence proofs.
- [7] Jean-Pierre Levinski, Menachem Magidor, and Saharon Shelah. Chang's conjecture for  $\aleph_\omega$ . *Israel J. Math.*, 69(2):161–172, 1990.
- [8] Peter Nyikos. Generalized kurepa and mad families and topology.
- [9] Marion Scheepers. Concerning  $n$ -tactics in the countable-finite game. *J. Symbolic Logic*, 56(3):786–794, 1991.
- [10] Marion Scheepers. Meager-nowhere dense games. I.  $n$ -tactics. *Rocky Mountain J. Math.*, 22(3):1011–1055, 1992.
- [11] Marion Scheepers. Combinatorics of open covers. I. Ramsey theory. *Topology Appl.*, 69(1):31–62, 1996.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNC CHARLOTTE, CHARLOTTE, NC 28262

*E-mail address:* `steven.clontz@gmail.com`

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNC CHARLOTTE, CHARLOTTE, NC 28262

*E-mail address:* `adow@uncc.edu`