Limited information strategies for topological games

PhD Defense

Steven Clontz http://stevenclontz.com

Department of Mathematics and Statistics
Auburn University

March 11, 2015

A *topological game* is a two-player game $G(X)$ of length $\omega = \{0, 1, 2, \dots\}$ defined for certain topological spaces X.

During each round n, the first and second player take turns choosing certain topological objects from X (e.g. point, open set, open cover, etc.).

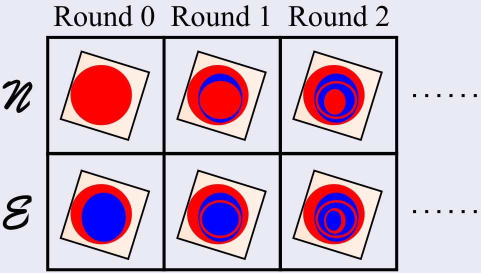
At the "end" of the game, a winner is declared by inspecting the sequences of choices made throughout the game.

The study of such games involves finding when a player has a *winning strategy* which defeats every possible counterattack by the opponent.

See Telgarsky's excellent survey on topological games for more details: [3]

Game

The *Banach-Mazur Game* $BM_{E,N}(X)$ (1935) [1]



The first player $\mathscr{N}$ wins the game if the intersection of all the chosen open sets is nonempty.

Theorem

X is Baire if and only if $\mathscr{N}$ lacks a winning strategy in the Banach Mazur game ($\mathscr{N} \underset{\uparrow}{\not} BM_{E,N}(X)$).

Thus the topological property of being a Baire space has a game-theoretic characterization using $BM_{E,N}(X)$.

By considering *limited information strategies*, we may characterize more properties.

Consider the following:

Theorem

X is α-favorable $\Rightarrow$ X is weakly α-favorable $\Rightarrow$ X is Baire

α-favorability is characterized by $\mathscr{E} \underset{\text{tact}}{\uparrow} BM_{E,N}(X)$: player $\mathscr{E}$ has a *tactical* winning strategy which only considers the most recent move of the opponent.

This is stronger than weak α-favorability, characterized by $\mathscr{E} \uparrow BM_{E,N}(X)$. In this case $\mathscr{E}$ still has a winning strategy, but it may rely on perfect information of the history of the game.

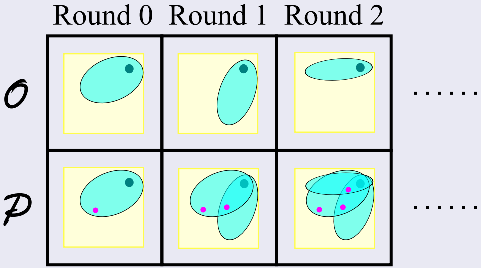
By characterizing topological properties using the theory of topological games, we introduce new proof techniques for demonstrating the structure of given topological spaces.

The aim of my dissertation was to investigate four topological games from the literature with unknown limited information implications.

In doing so I uncovered several new results in general topology, advancing research done by G. Gruenhage, P. Nyikos, F. Gavin, R. Telgárksy, J. Bell, M. Scheepers, and others.

Game

Gruenhage's convergence game $Gru_{\overrightarrow{O},P}(X,x)$ and clustering game $Gru_{\overset{\rightsquigarrow}{O},P}(X,x)$ proceed as follows:



$\mathscr{O}$ wins the game if the points chosen by $\mathscr{P}$ converge/cluster to the given point $x \in X$. Otherwise, $\mathscr{P}$ wins.

Note that $\mathscr{O}$ need not know anything about the history of the game to play each round.

If $\mathscr{O} \uparrow Gru_{\vec{O},P}(X, x)$, then x is called a W-point in X. Obviously, all points of first-countablity are W-points, but $\mathscr{O} \uparrow Gru_{\vec{O},P}(\kappa^*, \infty)$ also, where ∞ is the added point in the one-point compactification κ^* of uncountable discrete κ.

Points of first-countability may in fact be characterized by this game as well:

Theorem

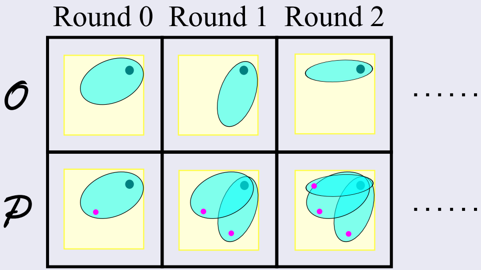
x has a countable local base in X if and only if $\mathscr{O} \underset{pre}{\uparrow} Gru_{\vec{O},P}(X, x)$ ($\mathscr{O}$ has a winning predetermined strategy using only the round number).

If every point in a space X is a W-point, then X is a W-space.

A variation of this game which is harder for $\mathscr{O}$ yields some difficult infinite combinatorial questions:

Game

Gruenhage's hard convergence game $Gru_{O,P}^{\rightarrow,\star}(X,x)$ and hard clustering game $Gru_{O,P}^{\rightsquigarrow,\star}(X,x)$ proceed as follows:



Nyikos observed in [2] that:

Theorem

$\mathscr{O} \underset{mark}{\not\uparrow} Gru^{\rightarrow,\star}_{O,P}(\omega_1^*, \infty)$.

($\mathscr{O}$ cannot guarantee a win using a Markov strategy which considers only the round number and most recent move.)

Some more work shows that in fact

Theorem

$\mathscr{O} \underset{k\text{-}mark}{\not\uparrow} Gru^{\rightarrow,\star}_{O,P}(\omega_1^*, \infty)$.

($\mathscr{O}$ cannot guarantee a win using a k-Markov strategy which considers only the round number and k most recent moves.)

Interestingly, the strategy which prevents convergence won't prevent clustering as well unless the cardinality of the space is sufficiently large.

Theorem

$\mathscr{O} \underset{mark}{\uparrow} Gru^{\rightsquigarrow,\star}_{O,P}(\omega_1^*,\infty)$, but $\mathscr{O} \underset{k\text{-}mark}{\not\uparrow} Gru^{\rightsquigarrow,\star}_{O,P}(\omega_2^*,\infty)$.

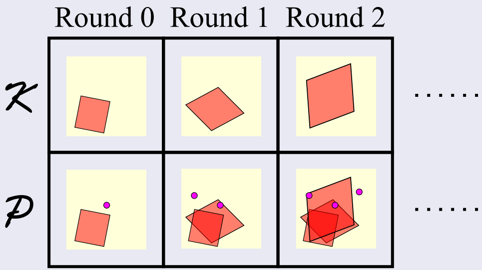
But knowledge of the round number is used non-trivially in doing so.

Theorem

$\mathscr{O} \underset{k\text{-}tact}{\not\uparrow} Gru^{\rightsquigarrow,\star}_{O,P}(\omega_1^*,\infty)$.

Game

Gruenhage's locally finite games $Gru_{K,P}(X)$ and $Gru_{K,L}(X)$ proceed as follows:



$\mathscr{K}$ wins the game if the points/sets chosen by $\mathscr{P}/\mathscr{L}$ are locally finite in the space. Otherwise, $\mathscr{P}/\mathscr{L}$ wins.

R. Daniel Mauldin, editor.
The Scottish Book.
Birkhäuser, Boston, Mass., 1981.
Mathematics from the Scottish Café, Including selected papers presented at the Scottish Book Conference held at North Texas State University, Denton, Tex., May 1979.

Peter J. Nyikos.
Classes of compact sequential spaces.
In *Set theory and its applications (Toronto, ON, 1987)*, volume 1401 of *Lecture Notes in Math.*, pages 135–159. Springer, Berlin, 1989.

Rastislav Telgársky.
Topological games: on the 50th anniversary of the Banach-Mazur game.
Rocky Mountain J. Math., 17(2):227–276, 1987.

Any questions?