

## An Example of Gruenhage's Compact-Point Game for which $K$ has a winning strategy, but no winning $k$ -Markov strategy

We construct a ZFC example given by Gary Gruenhage, inspired by a ZFC+¬SH example due to Stephen Watson [5]. We require a slight modification of a result of Gruenhage and Michael [3]:

**Theorem 1.** *Let  $X$  be a meta-Lindelöf, locally Lindelöf regular space, and let  $\mathcal{B}$  be a base of cardinality  $\kappa$  for  $X$ . Then  $X$  has a cover  $\mathcal{B}' \subseteq \mathcal{B}$  of cardinality  $\kappa$  such that  $\{\overline{B} : B \in \mathcal{B}'\}$  is point-countable.*

*Proof.* asdf? □

**Theorem 2.** *There exists a compact, zero-dimensional topological space  $X$  which has a point-countable cover  $\mathcal{U} = \{U_c : c \in 2^\omega\}$  of clopen sets which is not  $\sigma$ -point-finite (the union of countably-many point-finite collections).*

*Proof.* Take a zero-dimensional Corson compact  $Y$  of weight  $2^\omega$ , which is not Eberlein compact. (See Todorcevic [4] for an example.) It follows by [1] that  $Y^2$  is hereditarily metaLindelöf, but not hereditarily  $\sigma$ -metacompact. Let  $X$  be a non- $\sigma$ -metacompact subset of  $Y^2$ , and let  $\mathcal{V}$  be an open cover of  $X$  such that no  $\sigma$ -point-finite open refinement exists.

Let  $\mathcal{B}$  be a clopen base for  $X$  of cardinality  $2^\omega$  which is a refinement of  $\mathcal{V}$ . Then as  $X$  is metaLindelöf, locally Lindelöf, and regular, by Thm 1, let  $\mathcal{U} \subseteq \mathcal{B}$  be a point-countable clopen refinement of  $\mathcal{V}$  of cardinality  $2^\omega$ , which cannot be  $\sigma$ -point-finite. □

**Definition 3.** Using the  $X$  from Theorem 2, let

$$\mathbb{X} = (X \times 2^{<\omega}) \cup 2^\omega$$

compose a topological sum of  $2^{<\omega}$  copies of  $X$  along with a discrete copy of the Cantor Set  $2^\omega$ , and add open (in fact, compact) neighborhoods of the form:

$$B_c = c \cup (U_c \times \{c \upharpoonright n : n < \omega\})$$

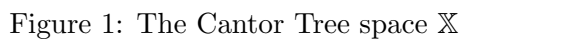
as seen in Figure 1.

**Definition 4.** Let  $S \in [2^\omega]^{<\omega}$  and  $m < \omega$ . Define

$$K_S = \bigcup_{c \in S} B_c$$

$$A = \{z \smallfrown \langle 1 \rangle : z \in 1^{<\omega}\}$$

$$K_S^* = K_S \setminus (X \times A)$$



$$L_m = X \times 2^{<m}$$

and observe that every compact set is dominated by the compact set  $K_S^* \cup L_m$  for some  $S, m$ .

Intuitively,  $K_S^*$  collects the branches of  $U_c$  converging up to each  $c \in S$  while avoiding the copy of  $X$  for each  $s$  in an antichain  $A$ , and  $L_m$  collects the copies of  $X$  with  $|s| < m$  at the base of the tree. (See Figure 2)

**Definition 5.**  $LF_{K,P}(\mathbb{X})$  is a topological game consisting of players  $K$  and  $P$ . During each round,  $K$  chooses a compact subset of  $\mathbb{X}$ , and  $P$  chooses a point outside of any compact set previously played by  $K$ .  $K$  wins the game if the set of points chosen by  $P$  throughout all rounds of the game are locally finite in the space.

**Definition 6.** We say  $I \uparrow G$  if Player  $I$  has a winning strategy in the game  $G$ .

We say  $I \uparrow_{\text{tact}} G$  (resp.  $I \uparrow_{k\text{-tact}} G$ ) if Player  $I$  has a winning tactical (resp.  $k$ -tactical) strategy in the game  $G$ , a strategy depending on only the  $(k)$  most recent move(s) of the opponent.

We say  $I \uparrow_{\text{mark}} G$  (resp.  $I \uparrow_{k\text{-mark}} G$ ) if Player  $I$  has a winning Markov (resp.  $k$ -Markov) strategy in the game  $G$ , a strategy depending on only the  $(k)$  most recent move(s) of the opponent and the current round number.

**Proposition 7.** *Without loss of generality, we may assume  $P$  always plays points in  $X \times 2^{<\omega}$  throughout  $LF_{K,P}(\mathbb{X})$ .*

**Proposition 8.**  $K \uparrow LF_{K,P}(\mathbb{X})$

*Proof.* Let  $x \in X$ .  $C^x = \{c \in C : x \in U_c\}$  is a countable collection by the point-countability of  $\mathcal{U}$ , so label its elements as  $\{c_n^x : n < \omega\}$ .

$K$  may use the strategy

$$\sigma(\langle x_0, s_0 \rangle, \dots, \langle x_{n-1}, s_{n-1} \rangle) = \bigcup_{i < n} K_{\{c_0^{x_i}, \dots, c_{n-1}^{x_i}\}} \cup L_{|s_i|+1}$$

This is a winning strategy because each move  $\langle x_i, s_i \rangle$  by  $P$  cannot be a part of a subsequence of the play converging to any  $c_n^{x_i}$ , since  $K_{\{c_0^{x_i}, \dots, c_n^{x_i}\}} \supseteq B_{c_n^{x_i}}$  was forbidden during round  $n$ .  $\square$

**Theorem 9.**  $K \not\uparrow_{\text{tact}} LF_{K,P}(\mathbb{X})$ .

*Proof.* This is actually a corollary of Gruenhage's theorem in [2]:  $\mathbb{X}$  is locally compact (each point is either in some  $X$  or in some  $B_c$ ) but not metacompact (any cover of  $2^\omega$

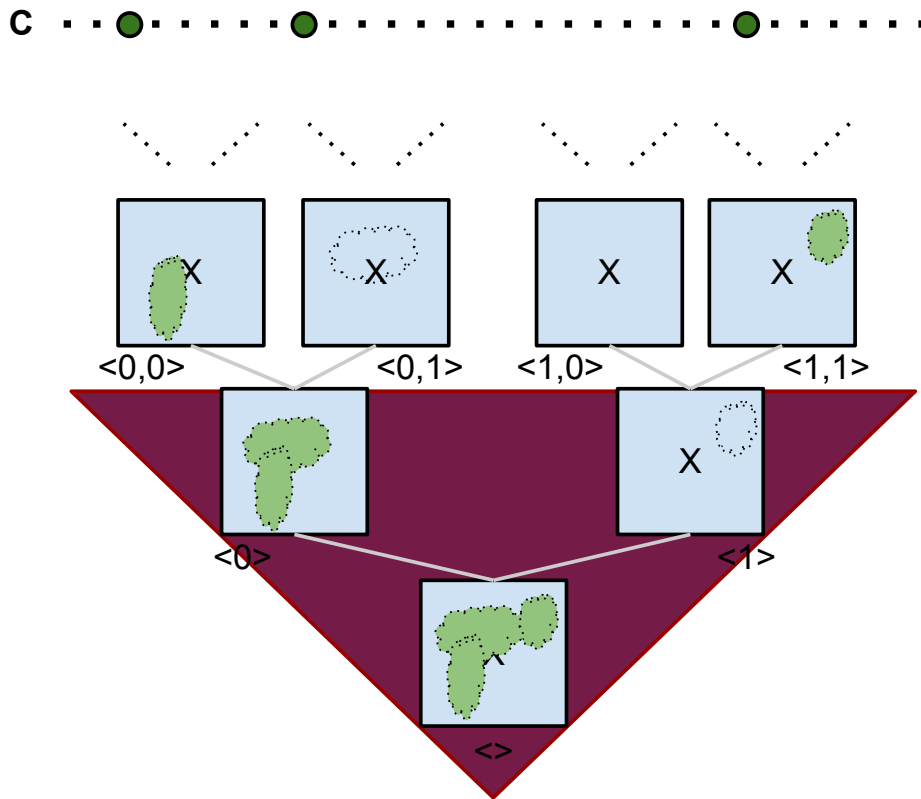


Figure 2:  $K_S^*$  and  $L_m$

necessarily will have infinite overlap at some  $X \times \{s\}$ ). However, we proceed with a direct game-theoretic proof.

Suppose that  $\sigma(\langle x, s \rangle)$  was a winning tactical strategy for  $K$  and define the compact set

$$\sigma'(x, n) = \bigcup_{|s| \leq n} \sigma(\langle x, s \rangle)$$

There exists some  $f : 2^\omega \rightarrow \omega$  such that for all  $x \in U_c$ ,  $\sigma'(x, f(c))$  covers some  $B_c \setminus L_m$ . (If not,  $P$  counters by simply always playing in  $B_c \setminus L_m$ .)

Recall that  $\mathcal{U}$  is not the union of the countably-many point-finite collections, and

$$\mathcal{U} = \bigcup_{n < \omega} \{U_c : f(c) = n\}$$

so we may choose  $n$  where  $\mathcal{U}_n = \{U_c : f(c) = n\}$  is not point-finite. Fix  $x$  so that  $x$  belongs to each of  $\{U_{c_0}, U_{c_1}, \dots\} \subseteq \mathcal{U}_n$ .

For each  $c_i$ ,  $\sigma'(x, f(c_i)) = \sigma'(x, n)$  covers  $c_i$ . Thus  $\sigma'(x, n) \supseteq \{c_0, c_1, \dots\}$  is a compact set covering a closed discrete subset, a contradiction.  $\square$

**Theorem 10.**  $K \not\Uparrow_{2\text{-tact}} LF_{K,P}(\mathbb{X})$ .

*Proof.* Suppose  $\sigma(\langle x, s \rangle, \langle y, t \rangle)$  was a winning 2-tactical strategy. Without loss of generality we assume it ignores order. We may define  $S(x, y, n) \in [2^\omega]^{<\omega}$  (increasing on  $n$ ) and  $n < m(x, y, n) < \omega$  such that for each  $(x, y)$ ,

$$\bigcup_{s, t \in 2^{\leq n}} \sigma(\langle x, s \rangle, \langle y, t \rangle) \subseteq K_{S(x, y, n)}^* \cup L_{m(x, y, n)}$$

and so we assume

$$\sigma(\langle x, s \rangle, \langle y, t \rangle) = K_{S(x, y, \max(|s|, |t|))}^* \cup L_{m(x, y, \max(|s|, |t|))}$$

Select an arbitrary point  $x' \in X$ . We define a tactical strategy

$$\tau(x, s) = K_{S(x, x', m(x, x', |s|)+1)}^* \cup L_{m(x, x', m(x, x', |s|)+1)}$$

We complete the proof by showing  $\tau$  is a winning tactical strategy (a contradiction).

Suppose

$$\langle x_0, s_0 \rangle, \langle x_1, s_1 \rangle, \langle x_2, s_2 \rangle, \dots$$

successfully countered  $\tau$  by clustering at  $c \in 2^\omega$  (the strategy trivially prevents clustering elsewhere). Let  $z_n = \langle 0, \dots, 0 \rangle$  with  $n$  zeros. We claim

$$\langle x_0, s_0 \rangle, \langle x', z_{m(x_0, x', |s_0|)} \frown \langle 1 \rangle \rangle, \langle x_1, s_1 \rangle, \langle x', z_{m(x_1, x', |s_1|)} \frown \langle 1 \rangle \rangle, \langle x_2, s_2 \rangle, \langle x', z_{m(x_2, x', |s_2|)} \frown \langle 1 \rangle \rangle, \dots$$

is a successful counter to  $\sigma$ .

We will need the fact that, as  $\langle x_{i+1}, s_{i+1} \rangle$  was legal against  $\tau$ :

$$\begin{aligned} |s_i| &< m(x_i, x', |s_i|) + 1 = |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle| \\ &< m(x_i, x', m(x_i, x', |s_i|) + 1) = m(x_i, x', |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|) \leq |s_{i+1}| \end{aligned}$$

Note that  $m(x, y, \max(|s|, |t|))$  is increasing throughout this play of the game versus  $\sigma$ :

$$\begin{aligned} &m(x_i, x', \max(|s_i|, |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|)) \\ &= m(x_i, x', |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|) \\ &\leq |s_{i+1}| \\ &< m(x_{i+1}, x', |s_{i+1}|) \\ &= m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|)) \\ &= |z_{m(x_{i+1}, x', |s_{i+1}|)}| \\ &< |z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle| \\ &< m(x_{i+1}, x', |z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle|) \\ &= m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle|)) \end{aligned}$$

We turn to showing that  $\langle x', z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle \rangle$  is always a legal move. Since  $z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle$  is on the antichain avoided by any  $K^*$ , the problem is reduced to showing that this move isn't forbidden by

$$L_{m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|))}$$

which we can see here:

$$m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|)) = m(x_{i+1}, x', |s_{i+1}|) < |z_{m(x_{i+1}, x', |s_{i+1}|)} \frown \langle 1 \rangle|$$

We can conclude by showing that  $\langle x_{i+1}, s_{i+1} \rangle$  is always a legal move. We can see it avoids

$$L_{m(x_i, x', \max(|s_i|, |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|))}$$

since

$$m(x_i, x', \max(|s_i|, |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|)) = m(x_i, x', |z_{m(x_i, x', |s_i|)} \frown \langle 1 \rangle|) \leq |s_{i+1}|$$

Since  $\langle x_{i+1}, s_{i+1} \rangle$  was legal against  $\tau$ , for  $h \leq i$  it avoided

$$K_{S(x_h, x', m(x_h, x', |s_h|)+1)}^* = K_{S(x_h, x', \max(|s_h|, |z_{m(x_h, x', |s_h|)} \frown \langle 1 \rangle|))}^*$$

and when  $h < i$ , we see it avoids:

$$\begin{aligned} K_{S(x_{h+1}, x', \max(|s_{h+1}|, |z_{m(x_h, x', |s_h|)} \frown \langle 1 \rangle))}^* &= K_{S(x_{h+1}, x', |s_{h+1}|)}^* \\ &\subseteq K_{S(x_{h+1}, x', m(x_{h+1}, x', |s_{h+1}|)+1)}^* \end{aligned}$$

This concludes the proof. □

**Theorem 11.**  $K \not\upharpoonright_{k\text{-tact}} LF_{K,P}(\mathbb{X})$ .

*Proof.* The proof proceeds in parallel to the proof of  $K \not\upharpoonright_{2\text{-tact}} LF_{K,P}(\mathbb{X})$ , and in fact is just a generalization of said proof (at the cost of simplicity).

Suppose  $\sigma(\langle x_0, s_0 \rangle, \dots, \langle x_k, s_k \rangle)$  was a winning  $(k+1)$ -tactical strategy. Without loss of generality we assume it ignores order. We may define  $S(x_0, \dots, x_k, n) \in [2^\omega]^{<\omega}$  (increasing on  $n$ ) and  $n < m(x_0, \dots, x_k, n) < \omega$  such that for each  $(x_0, \dots, x_k)$ ,

$$\bigcup_{s_0, \dots, s_k \in 2^{\leq n}} \sigma(\langle x_0, s_0 \rangle, \dots, \langle x_k, s_k \rangle) \subseteq K_{S(x_0, \dots, x_k, n)}^* \cup L_{m(x_0, \dots, x_k, n)}$$

and so we assume

$$\sigma(\langle x_0, s_0 \rangle, \dots, \langle x_k, s_k \rangle) = K_{S(x_0, \dots, x_k, \max(|s_0|, \dots, |s_k|))}^* \cup L_{m(x_0, \dots, x_k, \max(|s_0|, \dots, |s_k|))}$$

Select an arbitrary point  $x' \in X$ . Let  $M^0(x, n) = m(x, x', \dots, x', n)$  and  $M^{i+1}(x, n) = M^0(x, M^i(x, n) + 1)$ . We define a tactical strategy

$$\tau(x, s) = K_{S(x, x', \dots, x', M^{k-1}(x, |s|) + 1)}^* \cup L_{m(x, x', \dots, x', M^{k-1}(x, |s|) + 1)}$$

We complete the proof by showing  $\tau$  is a winning tactical strategy (a contradiction).

Suppose

$$\langle x_0, s_0 \rangle, \langle x_1, s_1 \rangle, \langle x_2, s_2 \rangle, \dots$$

successfully countered  $\tau$  by clustering at  $c \in 2^\omega$  (the strategy trivially prevents clustering elsewhere). Let  $z_n = \langle 0, \dots, 0 \rangle$  with  $n$  zeros. We claim

$$\begin{aligned} & \langle x_0, s_0 \rangle, \langle x', z_{M^0(x_0, |s_0|)} \frown \langle 1 \rangle \rangle, \langle x', z_{M^1(x_0, |s_0|)} \frown \langle 1 \rangle \rangle, \dots, \langle x', z_{M^{k-1}(x_0, |s_0|)} \frown \langle 1 \rangle \rangle, \\ & \langle x_1, s_1 \rangle, \langle x', z_{M^0(x_1, |s_1|)} \frown \langle 1 \rangle \rangle, \langle x', z_{M^1(x_1, |s_1|)} \frown \langle 1 \rangle \rangle, \dots, \langle x', z_{M^{k-1}(x_1, |s_1|)} \frown \langle 1 \rangle \rangle, \dots \end{aligned}$$

is a successful counter to  $\sigma$ .

We will need the fact that, as  $\langle x_{i+1}, s_{i+1} \rangle$  was legal against  $\tau$ :

$$\begin{aligned} |s_i| & < M^0(x_i, |s_i|) + 1 = |z_{M^0(x_i, |s_i|)} \frown \langle 1 \rangle| < M^0(x_i, M^0(x_i, |s_i|) + 1) + 1 \\ & = M^1(x_i, |s_i|) + 1 = |z_{M^1(x_i, |s_i|)} \frown \langle 1 \rangle| < \dots < |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle| \\ & = M^{k-1}(x_i, |s_i|) + 1 < m(x_i, x', \dots, x', M^{k-1}(x_i, |s_i|) + 1) \leq |s_{i+1}| \end{aligned}$$

Note that  $m(x_0, \dots, x_k, \max(|s_0|, \dots, |s_k|))$  is increasing throughout this play of the game versus  $\sigma$ :

$$m(x_i, x', \dots, x', \max(|s_i|, |z_{M^0(x_i, |s_i|)} \frown \langle 1 \rangle|, \dots, |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle|))$$



$$\begin{aligned}
&= m(x_i, x', \dots, x', |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle|) \\
&= m(x_i, x', \dots, x', M^{k-1}(x_i, |s_i|) + 1) \\
&\leq |s_{i+1}| \\
&< M^0(x_{i+1}, |s_{i+1}|) \\
&= m(x_{i+1}, x', \dots, x', |s_{i+1}|) \\
&= m(x_{i+1}, x', \dots, x', \max(|s_{i+1}|, |z_{M^0(x_i, |s_i|)} \frown \langle 1 \rangle|, \dots, |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle|)) \\
&= |z_{m(x_{i+1}, x', \dots, x', |s_{i+1}|)}| \\
&= |z_{M^0(x_{i+1}, |s_{i+1}|)}| \\
&< |z_{M^0(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle| \\
&< m(x_{i+1}, x', \dots, x', |z_{M^0(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|) \\
&= m(x_{i+1}, x', \dots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|, |z_{M^1(x_i, |s_i|)} \frown \langle 1 \rangle|, \dots, |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle|)) \\
&\quad \vdots \\
&< m(x_{i+1}, x', \dots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|, \dots, |z_{M^{k-1}(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|))
\end{aligned}$$

We turn to showing that  $\langle x', z_{M^j(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle \rangle$  is always a legal move. Since  $z_{M^j(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle$  is on the antichain avoided by any  $K^*$ , the problem is reduced to showing that this move isn't forbidden by

$$\begin{aligned}
&L_{m(x_{i+1}, x', \dots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|, \dots, |z_{M^{j-1}(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|, |z_{M^j(x_i, |s_i|)} \frown \langle 1 \rangle|, \dots, |z_{M^k(x_i, |s_i|)} \frown \langle 1 \rangle|))} \\
&= L_{m(x_{i+1}, x', \dots, x', |z_{M^{j-1}(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|)}
\end{aligned}$$

which we can see here:

$$\begin{aligned}
&m(x_{i+1}, x', \dots, x', |z_{M^{j-1}(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|) \\
&= m(x_{i+1}, x', \dots, x', M^{j-1}(x_{i+1}, |s_{i+1}|) + 1) \\
&= M^0(x_{i+1}, M^{j-1}(x_{i+1}, |s_{i+1}|) + 1) \\
&= M^j(x_{i+1}, s_{i+1}) \\
&< |z_{M^j(x_{i+1}, |s_{i+1}|)} \frown \langle 1 \rangle|
\end{aligned}$$

We can conclude by showing that  $\langle x_{i+1}, s_{i+1} \rangle$  is always a legal move. We can see it avoids

$$L_{m(x_i, x', \dots, x', \max(|s_i|, |z_{M^0(x_i, |s_i|)} \frown \langle 1 \rangle|, \dots, |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle|))}$$

since

$$\begin{aligned}
& m(x_i, x', \dots, x', \max(|s_i|, |z_{M^0(x_i, |s_i|)} \frown \langle 1 \rangle|, \dots, |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle|)) \\
&= m(x_i, x', \dots, x', |z_{M^{k-1}(x_i, |s_i|)} \frown \langle 1 \rangle|) \\
&= m(x_i, x', \dots, x', M^{k-1}(x_i, |s_i|) + 1) \\
&\leq |s_{i+1}|
\end{aligned}$$

Since  $\langle x_{i+1}, s_{i+1} \rangle$  was legal against  $\tau$ , for  $h \leq i$  it avoided

$$\begin{aligned}
& K_{S(x_h, x', \dots, x', M^{k-1}(x_h, |s_h|) + 1)}^* \\
&= K_{S(x_h, x', \dots, x', \max(|s_h|, |z_{M^0(x_h, |s_h|)} \frown \langle 1 \rangle|, \dots, |z_{M^{k-1}(x_h, |s_h|)} \frown \langle 1 \rangle|))}^*
\end{aligned}$$

and when  $h < i$ , we see it avoids both:

$$\begin{aligned}
& K_{S(x_{h+1}, x', \dots, x', \max(|s_{h+1}|, |z_{M^0(x_{h+1}, |s_{h+1}|)} \frown \langle 1 \rangle|, \dots, |z_{M^{j-1}(x_{h+1}, |s_{h+1}|)} \frown \langle 1 \rangle|, |z_{M^j(x_{h+1}, |s_{h+1}|)} \frown \langle 1 \rangle|, \dots, |z_{M^k(x_{h+1}, |s_{h+1}|)} \frown \langle 1 \rangle|))}^* \\
&= K_{S(x_{h+1}, x', \dots, x', |z_{M^{j-1}(x_{h+1}, |s_{h+1}|)} \frown \langle 1 \rangle|)}^* \\
&= K_{S(x_{h+1}, x', \dots, x', M^{j-1}(x_{h+1}, |s_{h+1}|) + 1)}^* \\
&\subseteq K_{S(x_{h+1}, x', \dots, x', M^{k-1}(x_{h+1}, |s_{h+1}|) + 1)}^*
\end{aligned}$$

and:

$$\begin{aligned}
& K_{S(x_{h+1}, x', \dots, x', \max(|s_{h+1}|, |z_{M^0(x_h, |s_h|)} \frown \langle 1 \rangle|, \dots, |z_{M^k(x_h, |s_h|)} \frown \langle 1 \rangle|))}^* \\
&= K_{S(x_{h+1}, x', \dots, x', |s_{k+1}|)}^* \\
&\subseteq K_{S(x_{h+1}, x', \dots, x', M^{k-1}(x_{h+1}, |s_{h+1}|) + 1)}^*
\end{aligned}$$

This concludes the proof.  $\square$

**Corollary 12.**  $K \not\models_{k\text{-mark}} LF_{K,P}(\mathbb{X})$ .

*Proof.* Let  $\sigma(\langle x_0, s_0 \rangle, \dots, \langle x_{k-1}, s_{k-1} \rangle, n)$  be a winning  $k$ -Markov strategy for  $K$  increasing on  $n$ . We define a  $k$ -tactical strategy

$$\tau(\langle x_0, s_0 \rangle, \dots, \langle x_{k-1}, s_{k-1} \rangle) = \sigma(\langle x_0, s_0 \rangle, \dots, \langle x_{k-1}, s_{k-1} \rangle, \max_{i < k}(|s_i|)) \cup L_{\max_{i < k}(|s_i|) + 1}$$

and observe that since for any legal play of the game, the round number  $n \leq \max_{i < k}(|s_i|)$ , we know  $\tau$  always yields supersets of  $\sigma$ , and is thus also a winning strategy, contradiction.  $\square$

## References

- [1] G. Gruenhage, *Covering Properties on  $X^2 \setminus \Delta$ ,  $W$ -sets, and Compact Subsets of  $\Sigma$ -Products*. Topology Appl. 17 (1984), 287-304
- [2] G. Gruenhage, *Games, Covering Properities, and Eberlein Compacts*. Topology Appl. 23 (1986), 291-297
- [3] G. Gruenhage and E. Michael, *A result on shrinkable open covers*. Proceedings of the 1983 topology conference (Houston, Tex., 1983), 1983, pp. 3743.
- [4] S. Todercevic, *Trees and linearly ordered sets*. Handbook of Set-theoretic Topology (North-Holland, Amsterdam, 1984), pp. 235-293
- [5] S. Watson, *Locally compact normal meta-Lindelöf spaces may not be paracompact: an application of uniformization and Suslin lines*. Proc. Amer. Math. Soc. 98 (1986), no. 4, 676680.