Assume all spaces are locally compact.

Proposition 1. *The following are all equivalent winning conditions for $Con_{O,P}(X^*, \infty)$:*

- *The points chosen by P converge to ∞.*
- *All compact subsets of X contain finitely many points chosen by P.*
- *No compact subset of X contains infinite points chosen by P.*

The following are all equivalent winning conditions for $Clus_{O,P}(X^*, \infty)$:

- *The points chosen by P cluster about ∞.*
- *All compact subsets of X miss infinitely many points chosen by P.*
- *No compact subset of X contains cofinite points chosen by P.*

Proposition 2. *The winning condition for $Con_{O,P}(X^*, \infty)$ is equivalent to the winning condition of $LF_{K,P}(X)$.*

Proof. First, suppose that the points chosen by P have a limit point l, the contradiction of $LF_{K,P}(X^*)$'s winning condition. Every open set about l contains infinitely many points, including a compact neighborhood of l. This contradicts the winning condition of $Con_{O,P}(X^*, \infty)$.

Then, suppose that there was a compact subset of X containing infinite points chosen by P, the contradiction of $Con_{O,P}(X^*, \infty)$'s winning condition. Every infinite subset of a compact set has a limit point, contradicting $LF_{K,P}(X^*)$'s winning condition. □

Theorem 3. *The following are all equivalent.*

- *X is metacompact.*
- *$O \uparrow_{tact} Con_{O,P}(X^*, \infty)$.*
- *$O \uparrow_{tact} Clus_{O,P}(X^*, \infty)$.*

Proof. Gruenhage has shown X is metacompact $\Rightarrow K \uparrow_{\text{tact}} LF_{K,P}(X)$ (which is equivalent to $O \uparrow_{\text{tact}} Con_{O,P}(X^*, \infty)$), and obviously $O \uparrow_{\text{tact}} Con_{O,P}(X^*, \infty) \Rightarrow O \uparrow_{\text{tact}} Clus_{O,P}(X^*, \infty)$. We proceed by modifying Gruenhage's proof that $K \uparrow_{\text{tact}} LF_{K,P}(X)$ implies X is metacompact to show $O \uparrow_{\text{tact}} Clus_{O,P}(X^*, \infty)$ does also.

Let $\mathcal{U}$ be a cover of X, and refine it to open F_σ sets with compact closures. Let $K : X^* \to K[X]$ be the complement of a winning clustering strategy for O such that $K(x)$ is a compact neighborhood of x for all $x \in X$.

Let $A(x) = \{p : x \notin K(p)\}$.

For each x, we claim x is not even a limit point of $A(x)$. To see this, suppose there was such an x, and choose any compact neighborhood N of x. If x was a limit of $A(x)$, then $N \cap A(x) \neq \emptyset$. We choose $x_0 \in N \cap A(x)$, and note $x \notin K(x_0)$ since $x_0 \in A(x)$.

This makes $N \setminus K(x_0)$ a neighborhood of x, which must then intersect $A(x)$. We may then pick an x_1 in $N \cap A(x) \setminus K(x_0)$. By continuing this process inductively we find x_n in $N \cap A(x) \setminus \bigcup_{0 \leq i < n} K(x_i)$. Since the x_n are all in the compact set N, the winning condition for $Clus_{O,P}(X^*, \infty)$ is not met for the play $\langle x_0, X^* \setminus K(x_0), x_1, X^* \setminus K(x_1), \dots \rangle$, contradicting the fact that K is the complement of a winning strategy.

Let $K'(x) = Int(K(x) \setminus A(x))$. We note that $x \in K'(x)$ since x was not a limit point or member of $A(x)$. So for each K let $\{K'(x) : x \in K\}$ be an open cover, and take a finite subset $F(K) \subset K$ which yields the subcover $\{K'(x) : x \in F(K)\}$.

Enumerate $\mathcal{U} = \{U_\alpha : \alpha < \lambda\}$. We define $\mathcal{U}_\alpha$ for $\alpha < \lambda$ to fulfill the following:

- $\mathcal{U}_\alpha$ is countable
- $\{U_\beta : \beta < \alpha\} \subseteq \bigcup_{\beta \leq \alpha} \mathcal{U}_\beta$
- If
$$N_\alpha = \left(\bigcup \mathcal{U}_\alpha\right) \setminus \bigcup_{\beta < \alpha} \left(\bigcup \mathcal{U}_\beta\right)$$
(that is, N_α contains the points covered by $\mathcal{U}_\alpha$ and not covered by a previous $\mathcal{U}_\beta$)
then there exists a countable $S_\alpha \subseteq N_\alpha$ where
$$N_\alpha \subseteq \bigcup_{x \in S_\alpha} K'(x) \subseteq \bigcup_{x \in S_\alpha} K(x) \subseteq \bigcup \mathcal{U}_\alpha$$

To start, let $\mathcal{U}_0$ and $\mathcal{U}_\alpha$ for every limit α be the empty set. For the successor ordinal $\alpha + 1$, let $\mathcal{U}_{\alpha+1,0} = \{U_\alpha\}$ and $S_{\alpha+1,0} = \emptyset$. Then let $O_{\alpha+1} = \bigcup_{\beta \leq \alpha} \bigcup \mathcal{U}_\beta$, the points covered by previous $\mathcal{U}_\beta$.

We then define
$$S_{\alpha+1,n+1} = \bigcup_{U \in \mathcal{U}_{\alpha+1,n}} F(Cl(U) \setminus O_{\alpha+1})$$
and let $\mathcal{U}_{\alpha+1,n+1} \subseteq \mathcal{U}$ be a finite cover of $\bigcup_{x \in S_{\alpha+1,n+1}} K(x)$. Note that $S_{\alpha,n}$ and $U_{\alpha,n}$ are finite at every step n, so we may define $S_\alpha = \bigcup_{n<\omega} S_{\alpha,n}$ and $U_\alpha = \bigcup_{n<\omega} U_{\alpha,n}$.

Such $\mathcal{U}_\alpha$ may be seen to fulfill the above requirements.

Let $W_\alpha = \bigcup_{x \in S_\alpha} K'(x)$. Note W_α contains everything in $\mathcal{U}_\alpha$ not covered by lower $\mathcal{U}_\beta$.

We now show the collection of W_α is point-finite. Suppose it wasn't: if $x \in W_{\alpha_n}$ for $\alpha_0 < \alpha_1 < \dots$, then for each n choose some $x_n \in S_{\alpha_n}$ where $x \in K'(x_n)$.

FINISH THIS

$\square$

Example 4. *Let X be a zero-dimensional, compact L-space (hereditarally Lindeloff and non-separable). It is a fact that there exists a point-countable collection $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ of clopen sets in X, and it is also true that any point-finite subcollection of $\mathcal{U}$ is countable.*

Let $C = \{c_\alpha : \alpha < \omega_1\}$ be any uncountable subset of the Cantor space 2^ω. Let $X_s = X \times \{s\}$ for each $s \in 2^{<\omega}$, and $U_{\alpha,s} = U_\alpha \times \{s\}$.

Finally, let

$$\mathbb{X} = C \cup \bigcup_{s \in 2^{<\omega}} X_s$$

be a tree of $2^{<\omega}$ copies of X, and where

$$c_\alpha \cup \bigcup_{n<\omega} U_{\alpha, x_\alpha \upharpoonright n}$$

is an open set about each c_α.

Definition 5. Let $S \in [\omega_1]^{<\omega}$ and $m < \omega$. Define

$$K_S = \bigcup_{\alpha \in S} \left(c_\alpha \cup \left(\bigcup_{s < c_\alpha} U_{\alpha,s} \right) \right)$$

$$A = \{z^\frown\langle 1\rangle : z \in 1^{<\omega}\}$$

$$K_S^* = K_S \setminus \bigcup_{s \in A} X_s$$

and

$$L_m = \bigcup_{s \in 2^{<m}} X_s$$

and observe that every compact set is dominated by $K_S^* \cup L_m$ for some S, m. Intuitively, K_S^* collects the branches of U_α converging up to c_α for each $\alpha \in S$ while avoiding copies X_s of X for each s in an antichain A, and L_m collects the copies X_s of X with $|s| < m$ at the base of the tree.

Proposition 6. *Without loss of generality, P always plays points in $\bigcup_{s \in 2^{<\omega}} X_s$.*

Proposition 7. *$K \uparrow LF_{K,P}(\mathbb{X})$.*

Proof. In response to a point $\langle x, s\rangle$, K observes that there are only countably many α such that $U_\alpha \times \{s\}$ contains $\langle x, s\rangle$ (by point-countability of X). Enumerate these as α_n. K makes a promise that during round m, K will forbid some superset of $K_{\{\alpha_n : n \leq m\}}$. Finally, K also always forbids a superset of $L_{|s|+1}$.

Suppose P's moves clustered at some point. Since K forbade $L_{|s|+1}$ during each round, that point must be c_α for some α. P's play then must have included a subsequence of points $\langle x_0, s_0\rangle, \langle x_1, s_1\rangle, \langle x_2, s_2\rangle \dots$ such that $x_n \in U_\alpha$ and $s_n \leq s_{n+1} \leq c_\alpha$. However, in response to $\langle x_0, s_0\rangle$, K made a promise to eventually forbid a superset of $K_{\{\alpha\}}$, making every $\langle x_n, t_n\rangle$ illegal after that round. $\square$

Theorem 8. $K \not\uparrow_{tact} LF_{K,P}(\mathbb{X})$.

Proof. (Vanished somewhere during my editing... looking for it now!) $\square$

Theorem 9. $K \not\uparrow_{2\text{-}tact} LF_{K,P}(\mathbb{X})$.

Proof. Suppose $\sigma(\langle x, s\rangle, \langle y, t\rangle)$ was a winning 2-tactical strategy. We may define $S(x, y, n) \in [\omega_1]^{<\omega}$ (increasing on n) and $n < m(x, y, n) < \omega$ such that for each (x, y),

$$\bigcup_{s,t \in 2^{\leq n}} \sigma(\langle x, s\rangle, \langle y, t\rangle) \subseteq K^*_{S(x,y,n)} \cup L_{m(x,y,n)}$$

and so we assume

$$\sigma(\langle x, s\rangle, \langle y, t\rangle) = K^*_{S(x,y,\max(|s|,|t|))} \cup L_{m(x,y,\max(|s|,|t|))}$$

Select an arbitrary point $x' \in X$. We define a tactical strategy

$$\tau(x, s) = K^*_{S(x,x',m(x,x',|s|)+1)} \cup L_{m(x,x',m(x,x',|s|)+1)}$$

We complete the proof by showing τ is a winning tactical strategy (a contradiction).

Suppose

$$\langle x_0, s_0\rangle, \langle x_1, s_1\rangle, \langle x_2, s_2\rangle, \dots$$

successfully countered τ by clustering at $c \in C$ (the strategy trivially prevents clustering elsewhere). Let $z_n = \langle 0, \dots, 0\rangle$ with n zeros. We claim

$$\langle x_0, s_0\rangle, \langle x', z_{m(x_0,x',|s_0|)}{}^\frown\langle 1\rangle\rangle, \langle x_1, s_1\rangle, \langle x', z_{m(x_1,x',|s_1|)}{}^\frown\langle 1\rangle\rangle, \langle x_2, s_2\rangle, \langle x', z_{m(x_2,x',|s_2|)}{}^\frown\langle 1\rangle\rangle, \dots$$

is a successful counter to σ.

We will need the fact that, as $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ:

$$|s_i| < m(x_i, x', |s_i|) + 1 = |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|$$

$$< m(x_i, x', m(x_i, x', |s_i|) + 1) = m(x_i, x', |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|) \leq |s_{i+1}|$$

Note that $m(x, y, \max(|s|, |t|))$ is increasing throughout this play of the game versus σ:

$$\begin{aligned}
&m(x_i, x', \max(|s_i|, |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|)) \\
&= m(x_i, x', |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|) \\
&\leq |s_{i+1}| \\
&< m(x_{i+1}, x', |s_{i+1}|) \\
&= m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|)) \\
&= |z_{m(x_{i+1},x',|s_{i+1}|)}| \\
&< |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle| \\
&< m(x_{i+1}, x', |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle|) \\
&= m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle|))
\end{aligned}$$

We turn to showing that $\langle x', z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle\rangle$ is always a legal move. Since $z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle$ is on the antichain avoided by any K^*, the problem is reduced to showing that this move isn't forbidden by

$$L_{m(x_{i+1},x',\max(|s_{i+1}|,|z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|))}$$

which we can see here:

$$m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|)) = m(x_{i+1}, x', |s_{i+1}|) < |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle|$$

We can conclude by showing that $\langle x_{i+1}, s_{i+1}\rangle$ is always a legal move. We can see it avoids

$$L_{m(x_i,x',\max(|s_i|,|z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|))}$$

since

$$m(x_i, x', \max(|s_i|, |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|)) = m(x_i, x', |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|) \leq |s_{i+1}|$$

Since $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ, it avoided

$$K^*_{S(x_h,x',m(x_h,x',|s_h|)+1)} = K^*_{S(x_h,x',\max(|s_h|,|z_{m(x_h,x',|s_h|)}{}^\frown\langle 1\rangle|))}$$

for $h \leq i$. And when $h < i$, we see it avoids:

$$\begin{aligned}
K^*_{S(x_{h+1},x',\max(|s_{h+1}|,|z_{m(x_h,x',|s_h|)}{}^\frown\langle 1\rangle|))} &= K^*_{S(x_{h+1},x',|s_{h+1}|)} \\
&\subseteq K^*_{S(x_{h+1},x',m(x_{h+1},x',|s_{h+1}|)+1)}
\end{aligned}$$

This concludes the proof. $\square$

Theorem 10. *$K \not\uparrow_{k\text{-}tact} LF_{K,P}(\mathbb{X})$.*

Proof. The proof proceeds in parallel to the proof of $K \not\uparrow_{\text{2-tact}} LF_{K,P}(\mathbb{X})$.

Suppose $\sigma(\langle x_0, s_0\rangle, \dots, \langle x_k, s_k\rangle)$ was a winning $(k+1)$-tactical strategy. We may define $S(x_0, \dots, x_k, n) \in [\omega_1]^{<\omega}$ (increasing on n) and $n < m(x_0, \dots, x_k, n) < \omega$ such that for each $(x_0, \dots, x_k)$,

$$\bigcup_{s_0,\dots,s_k \in 2^{\leq n}} \sigma(\langle x_0, s_0\rangle, \dots, \langle x_k, s_k\rangle) \subseteq K^*_{S(x_0,\dots,x_k,n)} \cup L_{m(x_0,\dots,x_k,n)}$$

and so we assume

$$\sigma(\langle x_0, s_0\rangle, \dots, \langle x_k, s_k\rangle) = K^*_{S(x_0,\dots,x_k,\max(|s_0|,\dots,|s_k|))} \cup L_{m(x_0,\dots,x_k,\max(|s_0|,\dots,|s_k|))}$$

Select an arbitrary point $x' \in X$. Let $M^0(x,n) = m(x, x', \dots, x', n)$ and $M^{i+1}(x,n) = M^0(x, M^i(x,n)+1)$. We define a tactical strategy

$$\tau(x,s) = K^*_{S(x,x',\dots,x',M^{k-1}(x,|s|)+1)} \cup L_{m(x,x',\dots,x',M^{k-1}(x,|s|)+1)}$$

We complete the proof by showing τ is a winning tactical strategy (a contradiction).

Suppose

$$\langle x_0, s_0\rangle, \langle x_1, s_1\rangle, \langle x_2, s_2\rangle, \dots$$

successfully countered τ by clustering at $c \in C$ (the strategy trivially prevents clustering elsewhere). Let $z_n = \langle 0, \dots, 0\rangle$ with n zeros. We claim

$$\langle x_0, s_0\rangle, \langle x', z_{M^0(x_0,|s_0|)}{}^\frown\langle 1\rangle\rangle, \langle x', z_{M^1(x_0,|s_0|)}{}^\frown\langle 1\rangle\rangle, \dots, \langle x', z_{M^{k-1}(x_0,|s_0|)}{}^\frown\langle 1\rangle\rangle,$$

$$\langle x_1, s_1\rangle, \langle x', z_{M^0(x_1,|s_1|)}{}^\frown\langle 1\rangle\rangle, \langle x', z_{M^1(x_1,|s_1|)}{}^\frown\langle 1\rangle\rangle, \dots, \langle x', z_{M^{k-1}(x_1,|s_1|)}{}^\frown\langle 1\rangle\rangle, \dots$$

is a successful counter to σ.

We will need the fact that, as $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ:

$$\begin{aligned} |s_i| < M^0(x_i,|s_i|)+1 = |z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle| < M^0(x_i, M^0(x_i,|s_i|)+1)+1 \\ = M^1(x_i,|s_i|)+1 = |z_{M^1(x_i,|s_i|)}{}^\frown\langle 1\rangle| < \dots < |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle| \\ = M^{k-1}(x_i,|s_i|)+1 < m(x_i, x', \dots, x', M^{k-1}(x_i,|s_i|)+1) \leq |s_{i+1}| \end{aligned}$$

Note that $m(x_0, \dots, x_k, \max(|s_0|, \dots, |s_k|))$ is increasing throughout this play of the game versus σ:

$$m(x_i, x', \dots, x', \max(|s_i|, |z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle|, \dots, |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|))$$

$$
\begin{aligned}
&= m(x_i, x', \ldots, x', |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|) \\
&= m(x_i, x', \ldots, x', M^{k-1}(x_i, |s_i|) + 1) \\
&\leq |s_{i+1}| \\
&< M^0(x_{i+1}, |s_{i+1}|) \\
&= m(x_{i+1}, x', \ldots, x', |s_{i+1}|) \\
&= m(x_{i+1}, x', \ldots, x', \max(|s_{i+1}|, |z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|)) \\
&= |z_{m(x_{i+1},x',\ldots,x',|s_{i+1}|)}| \\
&= |z_{M^0(x_{i+1},|s_{i+1}|)}| \\
&< |z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle| \\
&< m(x_{i+1}, x', \ldots, x', |z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|) \\
&= m(x_{i+1}, x', \ldots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|, |z_{M^1(x_i,|s_i|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|)) \\
&\vdots \\
&< m(x_{i+1}, x', \ldots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|))
\end{aligned}
$$

We turn to showing that $\langle x', z_{M^j(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle\rangle$ is always a legal move. Since $z_{M^j(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle$ is on the antichain avoided by any K^*, the problem is reduced to showing that this move isn't forbidden by

$$
\begin{aligned}
&L_{m(x_{i+1},x',\ldots,x',\max(|s_{i+1}|,|z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|,\ldots,|z_{M^{j-1}(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|,|z_{M^j(x_i,|s_i|)}{}^\frown\langle 1\rangle|,\ldots,|z_{M^k(x_i,|s_i|)}{}^\frown\langle 1\rangle|))} \\
&= L_{m(x_{i+1},x',\ldots,x',|z_{M^{j-1}(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|)}
\end{aligned}
$$

which we can see here:

$$
\begin{aligned}
&m(x_{i+1}, x', \ldots, x', |z_{M^{j-1}(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|) \\
&= m(x_{i+1}, x', \ldots, x', M^{j-1}(x_{i+1}, |s_{i+1}|) + 1) \\
&= M^0(x_{i+1}, M^{j-1}(x_{i+1}, |s_{i+1}|) + 1) \\
&= M^j(x_{i+1}, s_{i+1}) \\
&< |z_{M^j(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|
\end{aligned}
$$

We can conclude by showing that $\langle x_{i+1}, s_{i+1}\rangle$ is always a legal move. We can see it avoids

$$
L_{m(x_i,x',\ldots,x',\max(|s_i|,|z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle|,\ldots,|z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|))}
$$

since

$$\begin{aligned}
m(x_i, x', \ldots, x', \max(|s_i|, |z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|))\\
= m(x_i, x', \ldots, x', |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|)\\
= m(x_i, x', \ldots, x', M^{k-1}(x_i, |s_i|) + 1)\\
\leq |s_{i+1}|
\end{aligned}$$

Since $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ, it avoided

$$K^*_{S(x_h,x',\ldots,x',M^{k-1}(x_h,|s_h|)+1)}$$

$$= K^*_{S(x_h,x',\ldots,x',\max(|s_h|,|z_{M^0(x_h,|s_h|)}{}^\frown\langle 1\rangle|,\ldots,|z_{M^{k-1}(x_h,|s_h|)}{}^\frown\langle 1\rangle|))}$$

for $h \leq i$. And when $h < i$, we see it avoids both:

$$\begin{aligned}
K^*_{S(x_{h+1},x',\ldots,x',\max(|s_{h+1}|,|z_{M^0(x_{h+1},|s_{h+1}|)}{}^\frown\langle 1\rangle|,\ldots,|z_{M^{j-1}(x_{h+1},|s_{h+1}|)}{}^\frown\langle 1\rangle|,|z_{M^j(x_h,|s_h|)}{}^\frown\langle 1\rangle|,\ldots,|z_{M^k(x_h,|s_h|)}{}^\frown\langle 1\rangle|))}\\
= K^*_{S(x_{h+1},x',\ldots,x',|z_{M^{j-1}(x_{h+1},|s_{h+1}|)}{}^\frown\langle 1\rangle|)}\\
= K^*_{S(x_{h+1},x',\ldots,x',M^{j-1}(x_{h+1},|s_{h+1}|)+1)}\\
\subseteq K^*_{S(x_{h+1},x',\ldots,x',M^{k-1}(x_{h+1},|s_{h+1}|)+1)}
\end{aligned}$$

and:

$$\begin{aligned}
K^*_{S(x_{h+1},x',\ldots,x',\max(|s_{h+1}|,|z_{M^0(x_h,|s_h|)}{}^\frown\langle 1\rangle|,\ldots,|z_{M^k(x_h,|s_h|)}{}^\frown\langle 1\rangle|))}\\
= K^*_{S(x_{h+1},x',\ldots,x',|s_{k+1}|)}\\
\subseteq K^*_{S(x_{h+1},x',\ldots,x',M^{k-1}(x_{h+1},|s_{h+1}|)+1)}
\end{aligned}$$

This concludes the proof. $\square$