Discretely Selective and related games

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1 2-marks in CD(X)

Let $[f, F, \epsilon] = \{g \in C_p(X) : |g(x) - f(x)| < \epsilon \text{ for all } x \in F\}$.

Game 1. Let G be the following game. During round n, player I chooses $\beta_n < \omega_1$, and player II chooses $F_n \in [\omega_1]^{<\aleph_0}$. II wins if whenever $\gamma < \beta_n$ for co-finitely many $n < \omega$, $\gamma \in F_n$ for infinitely many $n < \omega$.

For $f \in \omega^\alpha$, let $f^{\leftarrow}[n] = \{\beta < \alpha : f(\beta) < n\}$.

Proposition 2. *II $\underset{2\text{-}mark}{\uparrow}$ G.*

Proof. Let $\{f_\alpha \in \omega^\alpha : \alpha < \omega_1\}$ be a collection of pairwise almost-compatible finite-to-one functions. Define a 2-mark σ for II by

$$\sigma(\langle \alpha \rangle, 0) = \emptyset$$

and

$$\sigma(\langle \alpha, \beta \rangle, n+1) \ = f_\beta^{\leftarrow}[n] \cup \{\gamma < \alpha \cap \beta : f_\alpha(\gamma) \neq f_\beta(\gamma)\}.$$

Let ν be an attack by I against σ, and let $\gamma < \nu(n)$ for $N \leq n < \omega$. If $f_{\nu(n)}(\gamma) \neq f_{\nu(n+1)}(\gamma)$ for infinitely-many $N \leq n < \omega$, then $\gamma \in \sigma(\langle \nu(n), \nu(n+1) \rangle, n+1)$ for infinitely-many $N \leq n < \omega$. Otherwise $f_{\nu(n)}(\gamma) = f_{\nu(n+1)}(\gamma) = M$ for cofinitely-many $N \leq n < \omega$, so $\gamma \in \sigma(\langle \nu(n), \nu(n+1) \rangle, n+1)$ for cofinitely-many $N \leq n < \omega$. Therefore σ is a winning 2-mark. ☐

Theorem 3. I $\underset{2\text{-}mark}{\uparrow}$ $CD(C_p(\omega_1 + 1))$

Proof. Let σ be a winning 2-mark for II in G.

Given a point $f \in C_p(\omega_1 + 1)$, let $\alpha_f < \omega_1$ satisfy $f(\beta) = f(\gamma)$ for all $\alpha_f \leq \beta \leq \gamma \leq \omega_1$.

Let $\tau(\emptyset, 0) = [\mathbf{0}, \{\omega_1\}, 4]$, $\tau(\langle f \rangle, 1) = [\mathbf{0}; \sigma(\langle \alpha_f \rangle, 0) \cup \{\omega_1\}; 2]$, and

$$\tau(\langle f, g \rangle, n+2) = [\mathbf{0}; \sigma(\langle \alpha_f, \alpha_g \rangle, n+1) \cup \{\omega_1\}; 2^{-n}].$$

Let ν be a legal attack by II against σ. For $\beta \leq \omega_1$, if $\beta < \alpha_{\nu(n)}$ for co-finitely many $n < \omega$, then $\beta \in \sigma(\langle \alpha_{\nu(n)}, \alpha_{\nu(n+1)} \rangle)$ for infinitely-many $n < \omega$, and thus $0 \in \text{cl}\{\nu(n)(\beta) : n < \omega\}$. Otherwise $\beta \geq \alpha_{\nu(n)}$ for infinitely many $n < \omega$, and thus $0 \in \text{cl}\{\nu(n)(\beta) : n < \omega\}$ as well. Thus $\mathbf{0} \in \text{cl}\{\nu(n) : n < \omega\}$. ☐

2 Combining game results

Theorem 4. *The following are equivalent for $T_{3.5}$ spaces X.*

a) X is R^+, that is, $\mathrm{II} \uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$.

b) $\mathrm{I} \uparrow PO(X)$.

c) $\mathrm{I} \uparrow FO(X)$.

d) $\mathrm{I} \uparrow Gru_{\overrightarrow{O,P}}(C_p(X), \mathbf{0})$.

e) $\mathrm{I} \uparrow CL(C_p(X), \mathbf{0})$.

f) $\mathrm{I} \uparrow CD(C_p(X))$.

g) X is ΩR^+, that is, $\mathrm{II} \uparrow G_1(\Omega_X, \Omega_X)$.

h) $C_p(X)$ is $sCFT^+$, that is, $\mathrm{II} \uparrow G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$.

i) $C_p(X)$ is $sCDFT^+$, that is, $\mathrm{II} \uparrow G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X),\mathbf{0}})$.

Proof. (a) $\Leftrightarrow$ (b) is a well-known result of Galvin.
(b) $\Leftrightarrow$ (c) is 4.3 of [Telgarksy 1975].
The equivalence of (b), (d), (e), and (f) are given as 3.8 of [Tkachuk 2017].
The equivalence of (g), (h), and (i) are due to Clontz.
(i) $\Leftrightarrow$ (e) follows from 3.18a of [Tkachuk 2017]. $\square$

Let $\Omega PO(X)$ be the point-open game where I wins if they force II to create an ω-cover. Likewise for $\Omega FO(X)$. In 3.9 of [Tkachuk 2017] it's shown that $\mathrm{II} \uparrow \Omega FO(X)$ implies $\mathrm{I} \uparrow CD(C_p(X))$.

Theorem 5. *Let X be $T_{3.5}$. Then $\mathrm{I} \uparrow \Omega FO(X)$ if and only if X is R^+.*

Proof. The forward implication holds because R^+ is equivalent to $\mathrm{I} \uparrow FO(X)$.

So assume X is R^+, which is equivalent to ΩR^+. Let σ be a winning strategy for II in $G_1(\Omega_X, \Omega_X)$. Let $T(X)$ be the non-empty open sets of X, and let $s \in T(X)^{<\omega}$. Assume $\tau(t) \in [X]^{<\omega}$ is defined for all $t < s$, and $\mathcal{U}_t \in \Omega_X$ is defined for all $\emptyset < t \leq s$.

Suppose that for all $F \in [X]^{<\omega}$, there existed $U_F \in T(X)$ containing F such that for all $\mathcal{U} \in \Omega_X$, $U_F \neq \sigma(\langle \mathcal{U}_{s\restriction 1}, \ldots, U_s, \mathcal{U} \rangle)$. Let $\mathcal{U} = \{U_F : F \in [X]^{<\omega}\} \in \Omega_X$. Then $\sigma(\langle \mathcal{U}_{s\restriction 1}, \ldots, U_s, \mathcal{U} \rangle)$ must equal some U_F, demonstrating a contradiction.

So there exists $\tau(s) \in [X]^{<\omega}$ such that for all $U \in T(X)$ containing $\tau(s)$, there exists $\mathcal{U}_{s^\frown\langle U \rangle} \in \Omega_X$ such that $U = \sigma(\langle \mathcal{U}_{s\restriction 1}, \ldots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle U \rangle} \rangle)$. (To complete the induction, $\mathcal{U}_{s^\frown\langle U \rangle}$ may be chosen arbitrarily for all other $U \in T(X)$.)

So τ is a strategy for I in $\Omega FO(X)$. Let ν legally attack τ, so $\tau(\nu \restriction n) \subseteq \nu(n)$ for all $n < \omega$. It follows that $\nu(n) = \sigma(\langle \mathcal{U}_{\nu\restriction 1}, \ldots, \mathcal{U}_{\nu\restriction n}, \mathcal{U}_{\nu\restriction n+1} \rangle)$. Since $\langle \mathcal{U}_{\nu\restriction 1}, \mathcal{U}_{n\restriction 2}, \ldots \rangle$ is a legal attack against σ, it follows that $\{\sigma(\langle \mathcal{U}_{\nu\restriction 1}, \ldots, \mathcal{U}_{\nu\restriction n+1} \rangle) : n < \omega\} = \{\nu(n) : n < \omega\}$ is an ω-cover. Therefore τ is a winning strategy, verifying $\mathrm{I} \uparrow \Omega FO(X)$. $\square$

In that paper, Tkachuk characterizes $\mathrm{II} \uparrow \Omega FO(X)$ as the second player having an "almost winning strategy" (II can prevent I from constructing an ω-cover but perhaps not an arbitrary open cover) in $PO(X)$, which he conflates with $FO(X)$ as they are equivalent for "completely" winning perfect information strategies.

But they cannot be interchanged in general. Note that II $\underset{\text{tact}}{\uparrow}$ $\Omega PO(2)$, where 2 is the two-point discrete space: let $\sigma(\langle x \rangle) = \{x\}$. Since every ω-cover of 2 includes 2, and σ never produces 2, this is a winning tactic. But for the same reason, 2 is R^+: it is legal to play 2 every round, which produces the ω-cover $\{2\}$. So $\Omega PO(X)$ is a very different game than those described previously.