Two mark in DS game

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Let $[f,F,\epsilon]=\{g\in C_p(X):|g(x)-f(x)|<\epsilon \text{ for all } x\in F\}$.

Game 1. Let G be the following game. During round n, player I chooses $\beta_n<\omega_1$, and player II chooses $F_n\in[\omega_1]^{<\aleph_0}$. II wins if whenever $\gamma<\beta_n$ for co-finitely many $n<\omega$, $\gamma\in F_n$ for infinitely many $n<\omega$.

For $f\in\omega^\alpha$, let $f^{\leftarrow}[n]=\{\beta<\alpha:f(\beta)<n\}$.

Proposition 2. II $\underset{2\text{-}mark}{\uparrow}$ G.

Proof. Let $\{f_\alpha\in\omega^\alpha:\alpha<\omega_1\}$ be a collection of pairwise almost-compatible finite-to-one functions. Define a 2-mark σ for II by

$$\sigma(\langle\alpha\rangle,0)=\emptyset$$

and

$$\sigma(\langle\alpha,\beta\rangle,n+1)\ =f_\beta^{\leftarrow}[n]\cup\{\gamma<\alpha\cap\beta:f_\alpha(\gamma)\neq f_\beta(\gamma)\}.$$

Let ν be an attack by I against σ, and let $\gamma<\nu(n)$ for $N\le n<\omega$. If $f_{\nu(n)}(\gamma)\neq f_{\nu(n+1)}(\gamma)$ for infinitely-many $N\le n<\omega$, then $\gamma\in\sigma(\langle\nu(n),\nu(n+1)\rangle,n+1)$ for infinitely-many $N\le n<\omega$. Otherwise $f_{\nu(n)}(\gamma)=f_{\nu(n+1)}(\gamma)=M$ for cofinitely-many $N\le n<\omega$, so $\gamma\in\sigma(\langle\nu(n),\nu(n+1)\rangle,n+1)$ for cofinitely-many $N\le n<\omega$. Therefore σ is a winning 2-mark. $\square$

Theorem 3. I $\underset{2\text{-}mark}{\uparrow}$ $DS(C_p(\omega_1+1))$

Proof. Let σ be a winning 2-mark for II in G.

Given a point $f\in C_p(\omega_1+1)$, let $\alpha_f<\omega_1$ satisfy $f(\beta)=f(\gamma)$ for all $\alpha_f\le\beta\le\gamma\le\omega_1$.

Let $\tau(\emptyset,0)=[\mathbf{0},\{\omega_1\},4]$, $\tau(\langle f\rangle,1)=[\mathbf{0};\sigma(\langle\alpha_f\rangle,0)\cup\{\omega_1\};2]$, and

$$\tau(\langle f,g\rangle,n+2)=[\mathbf{0};\sigma(\langle\alpha_f,\alpha_g\rangle,n+1)\cup\{\omega_1\};2^{-n}].$$

Let ν be a legal attack by II against σ. For $\beta\le\omega_1$, if $\beta<\alpha_{\nu(n)}$ for co-finitely many $n<\omega$, then $\beta\in\sigma(\langle\alpha_{\nu(n)},\alpha_{\nu(n+1)}\rangle)$ for infinitely-many $n<\omega$, and thus $0\in\mathrm{cl}\{\nu(n)(\beta):n<\omega\}$. Otherwise $\beta\ge\alpha_{\nu(n)}$ for infinitely many $n<\omega$, and thus $0\in\mathrm{cl}\{\nu(n)(\beta):n<\omega\}$ as well. Thus $\mathbf{0}\in\mathrm{cl}\{\nu(n):n<\omega\}$. $\square$