Definition 1. Let a V-map be a u.s.c. idempotent surjection.

Definition 2. For any LOS $\langle L, \leq \rangle$, let $\hat{L}$ be the collection of left-closed subsets of L (closed subsets for which $b \in L, a \leq b \Rightarrow a \in L$) linearly ordered by $\subseteq$.

It's well-known that $\hat{L}$ is compact. If L lacks a least element, L embeds as a dense subspace (consider the sets $(\leftarrow, l]$ for $l \in L$).

Definition 3. For any compact LOTS K with minimum 0 and maximum 1, let Γ be the V-map on K where $\Gamma(0) = K$ and $\Gamma(t) = \{1\}$ for $t > 0$.

Theorem 4. $X = \varprojlim\{2, \Gamma, L\} \cong \hat{L}$

Proof. We start by placing an order on X. Let $\vec{x} < \vec{y}$ if there exists $a \in L$ with $\vec{x}(a) = 0, \vec{y}(a) = 1$. We claim this is a total order inducing the topology on X.

We first observe that if $\vec{x}(b) = 1$, then for all $a \leq b$, $\vec{x}(a) \in \Gamma(1) = \{1\}$. If $\vec{x} \neq \vec{y}$, then assume without loss of generality that $\vec{x}(a) = 0, \vec{y}(a) = 1$, so $\vec{x} < \vec{y}$. Also, whenever $\vec{x}(b) = 1$, we have that $b < a$, so $\vec{y}(b) = 1$, preventing $\vec{y} < \vec{x}$. Finally if $\vec{x} < \vec{y}$ and $\vec{y} < \vec{z}$, take a, b with $\vec{x}(a) = 0$, $\vec{y}(a) = 1$,$\vec{y}(b) = 0$,$\vec{z}(b) = 1$. It follows that $a < b$ so $\vec{z}(a) = 1$ and $\vec{x} < \vec{z}$.

Consider the basic open set $B(\vec{x}, F)$ for a finite set $F \in [L]^{<\omega}$ about the sequence $\vec{x} \in X$ which contains all sequences $\vec{y}$ agreeing with $\vec{x}$ on F. If $\vec{x}(a) = 1$ for all $a \in F$, then let $\vec{w} \in X$ be 0 on the maximum of F, and 1 for anything less. It follows that $B(\vec{x}, F) = (\vec{w}, \rightarrow)$. If $\vec{x}(a) = 0$ for all $a \in F$, then let $\vec{y} \in X$ be 1 on the minimum of F, and 0 for anything greater. It follows that $B(\vec{x}, F) = (\leftarrow, \vec{y})$. Finally if $\vec{x}(a) = 1$ and $\vec{x}(b) = 0$ for $a < b$ in F and nothing between a, b is in F, then let $\vec{w} \in X$ be 0 on a and 1 for anything less, and let $\vec{y} \in X$ be 1 on b and 0 for anything greater. It follows that $B(\vec{x}, F) = (\vec{w}, \vec{y})$.

Let ϕ evaluate each $\vec{x} \in X \subseteq 2^L$ as the characteristic function for a subset of L. It's easy to see that ϕ is an order isomorphism between $(X, \leq)$ and $(\hat{L}, \subseteq)$. $\square$

References