Applications of $S(\kappa)$

BEST 2015 - San Francisco State University

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June 16, 2016

Definition

A topological space X is Menger if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ of open covers of X there exists a sequence $\langle \mathcal{F}_0, \mathcal{F}_1, \ldots \rangle$ such that $\mathcal{F}_n \in [\mathcal{U}_n]^{<\omega}$ and $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Proposition

X is σ-relatively-compact $\Rightarrow$ X is Menger $\Rightarrow$ X is Lindelöf.

Game

Let $Men_{C,F}(X)$ denote the *Menger game* with players $\mathscr{C}$, $\mathscr{F}$. In round n, $\mathscr{C}$ chooses an open cover $\mathcal{C}_n$, followed by $\mathscr{F}$ choosing $\mathcal{F}_n \in [\mathcal{C}_n]^{<\omega}$.
$\mathscr{F}$ wins the game if $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover for the space X, and $\mathscr{C}$ wins otherwise.

Theorem (Hurewicz 1926)

X is Menger if and only if $\mathscr{C} \not\uparrow Men_{C,F}(X)$.

Theorem (Telgarsky 1984, Scheepers 1995)

Let X be metrizable. $\mathscr{F} \uparrow Men_{C,F}(X)$ if and only if X is σ-compact.

Theorem (Fremlin, Miller 1988)

There are ZFC examples of non-σ-compact subsets of the real line which are Menger.

Assume κ is an uncountable cardinal.

Example

Let $\kappa^\dagger = \kappa \cup \{\infty\}$, with κ discrete and neighborhoods of ∞ being co-countable. Then $\mathscr{F} \uparrow Men_{C,F}\left(\kappa^\dagger\right)$ but $\kappa^\dagger$ is not σ-compact.

Definition

A *perfect information strategy* uses full information of the previous moves of the opponent. ($\mathscr{A} \uparrow G$)

Definition

A *k-tactical strategy* only uses the last k previous moves of the opponent. ($\mathscr{A} \underset{k\text{-tact}}{\uparrow} G$)

Definition

A *k-Markov strategy* only uses the last k previous moves of the opponent and the round number. ($\mathscr{A} \underset{k\text{-mark}}{\uparrow} G$)

If omitted, assume $k = 1$.

Considering such strategies allows us to factor out Scheeper's proof characterizing σ-compact metrizable spaces with the Menger game.

Lemma

$\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$ if and only if X is σ-relatively-compact.

Lemma

Let X be second-countable. $\mathscr{F} \uparrow Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{mark}{\uparrow} Men_{C,F}(X)$

The result follows since metrizable $+$ Lindelöf $\Leftrightarrow$ regular $+$ second countable.

Example

$\mathscr{F} \uparrow Men_{C,F}(\kappa^\dagger)$, but $\mathscr{F} \underset{\text{mark}}{\not\uparrow} Men_{C,F}(\kappa^\dagger)$.

Proposition

$\mathscr{F} \underset{(k+2)\text{-}mark}{\uparrow} Men_{C,F}(X)$ *if and only if* $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Men_{C,F}(X)$.

Example

$\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}\left(\omega_1^\dagger\right)$

What about for $\kappa > \omega_1$? As we'll see, this question is not answerable in *ZFC*.