Definition 1. For any partition $\mathcal{R}$ of a space X and $x \in X$, let $\mathcal{R}[x]$ be such that $x \in \mathcal{R}[x] \in \mathcal{R}$.

For partitions $\mathcal{R}_0, \dots, \mathcal{R}_n$, let $\mathcal{H}_n = \bigwedge_{m \leq n} \mathcal{R}_m$ be the coarsest partition which refines each $\mathcal{R}_m$.

For partitions $\mathcal{R}, \mathcal{S}$ let $\mathcal{R} \otimes \mathcal{S} = \{r \times s : r \in R, s \in S\}$.

Proposition 2. $x \in \mathcal{R}[y] \Leftrightarrow y \in \mathcal{R}[x]$.

$\mathcal{H}_n[x] = \left(\bigwedge_{m \leq n} \mathcal{R}_m\right)[x] = \bigcap_{m \leq n} \mathcal{R}_m[x]$.

Definition 3. For zero-dimensional X, the proximity game $Prox_{R,P}(X)$ proceeds as follows: in round n, $\mathscr{R}$ chooses a clopen partition $\mathcal{R}_n$ of X, followed by $\mathscr{P}$ choosing a point $p_n \in X$.

Player $\mathscr{R}$ wins if either $\bigcap_{n<\omega} \mathcal{H}_n[p_n] = \emptyset$ or p_n converges.

Proposition 4. *This game is perfect-information equivalent to the analogous game studied by Bell, requiring $\mathscr{P}$'s play p_{n+1} to be in $\mathcal{H}_n[p_n]$ in rounds $n+1$, and requiring $\mathscr{O}$ choose refinements.*

Proof. Allowing $\mathscr{P}$ to play $p_{n+1} \notin \mathcal{H}_n[p_n] \Rightarrow \mathcal{H}_n[p_{n+1}] \neq \mathcal{H}_n[p_n]$ does not introduce any new winning plays for $\mathscr{P}$ as for any such move, $\bigcap_{m<\omega} \mathcal{H}_n[p_n] \subseteq \mathcal{H}_{n+1}[p_{n+1}] \cap \mathcal{H}_n[p_n] \subseteq \mathcal{H}_n[p_{n+1}] \cap \mathcal{H}_n[p_n] = \emptyset$.

Allowing $\mathscr{R}$ to play non-refining clopen partitions does not introduce any new winning plays for $\mathscr{R}$ as the winning condition relies on the refinement of all $\mathcal{R}_n$ anyway. $\square$

Definition 5. A space X is **proximal** iff X is zero-dimensional and $\mathscr{R} \uparrow Prox_{R,P}(X)$.

Definition 6. A space X is **Markōv proximal** iff X is zero-dimensional and $\mathscr{R} \uparrow_{\text{mark}} Prox_{R,P}(X)$.

Definition 7. For any space X and a point $x \in X$, the W**-convergence-game** $Con_{O,P}(X, x)$ proceeds as follows: in round n, $\mathscr{O}$ chooses a neighborhood U_n of x, followed by $\mathscr{P}$ choosing a point $p_n \in X$.

For open sets $U_0, \dots, U_n$, let $V_n = \bigcap_{m \leq n} U_m$. Player $\mathscr{O}$ wins if either $p_n \notin V_n$ for some $n < \omega$, or if p_n converges.

Definition 8. A space X is a W**-space** iff $\mathscr{O} \uparrow Con_{O,P}(X, x)$ for all $x \in X$.

Definition 9. For each finite tuple $(m_0, \dots, m_{n-1})$, we define the k**-tactical fog-of-war**

$$T_k(m_0, \dots, m_{n-1}) = (m_{n-k}, \dots, m_{n-1})$$

and the k**-Markōv fog-of-war**

$$M_k(m_0, \dots, m_{n-1}) = (m_{n-k}, \dots, m_{n-1}, n)$$

So $P \uparrow_{k\text{-tact}} G$ if and only if there exists a winning strategy for P of the form $\sigma \circ T_k$, and $P \uparrow_{k\text{-mark}} G$ if and only if there exists a winning strategy of the form $\sigma \circ M_k$.

Theorem 10. *For all $x \in X$:*

- $\mathscr{R} \uparrow Prox_{R,P}(X) \Rightarrow \mathscr{O} \uparrow Con_{O,P}(X, x)$
- $\mathscr{R} \uparrow_{pre} Prox_{R,P}(X) \Rightarrow \mathscr{O} \uparrow_{pre} Con_{O,P}(X, x)$
- $\mathscr{R} \uparrow_{2k\text{-}tact} Prox_{R,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(X, x)$
- $\mathscr{R} \uparrow_{2k\text{-}mark} Prox_{R,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(X, x)$

Proof. Let σ witness $\mathscr{R} \uparrow_{2k\text{-tact}} Prox_{R,P}(X)$ (resp. $\mathscr{R} \uparrow_{2k\text{-mark}} Prox_{R,P}(X)$, $\mathscr{R} \uparrow Prox_{R,P}(X)$). We define the k-tactical (resp. k-Marköv, perfect info) strategy τ such that

$$\tau \circ L_k(p_0, \dots, p_{n-1}) = \sigma \circ L_{2k}(x, p_0, \dots, x, p_{n-1})[x] \cap \sigma \circ L_{2k}(x, p_0, \dots, x, p_{n-1}, x)[x]$$

where L_{2k} is the $2k$-tactical fog-of-war (resp. $2k$-Marköv fog-of-war, identity) and L_k is the k-tactical fog-of-war (resp. k-Marköv fog-of-war, identity).

Let $p_0, p_1, \dots$ attack τ such that $p_n \in V_n = \bigcap_{m \leq n} \tau \circ L_k(p_0, \dots, p_{m-1})$ for all $n < \omega$. Consider the attack $q_0, q_1, \dots$ against the winning strategy σ such that $q_{2n} = x$ and $q_{2n+1} = p_n$.

Certainly, $x \in \mathcal{H}_{2n}[x] = \mathcal{H}_{2n}[q_{2n}]$ for any $n < \omega$. Note also for any $n < \omega$ that

$$p_n \in V_n = \bigcap_{m \leq n} \tau \circ L_k(p_0, \dots, p_{m-1})$$

$$= \bigcap_{m \leq n} (\sigma \circ L_{2k}(x, p_0, \dots, x, p_{m-1})[x] \cap \sigma \circ L_{2k}(x, p_0, \dots, x, p_{m-1}, x)[x])$$

$$= \bigcap_{m \leq n} (\sigma \circ L_{2k}(q_0, q_1, \dots, q_{2m-2}, q_{2m-1})[x] \cap \sigma \circ L_{2k}(q_0, q_1, \dots, q_{2m-2}, q_{2m-1}, q_{2m})[x])$$

$$\bigcap_{m \leq n} \mathcal{R}_{2m}[x] \cap R_{2m+1}[x] = \mathcal{H}_{2n+1}[x]$$

so $x \in \mathcal{H}_{2n+1}[p_n] = \mathcal{H}_{2n+1}[q_{2n+1}]$. Thus $x \in \bigcap_{n < \omega} \mathcal{H}_n[q_n]$, and since σ is a winning strategy, the attack $q_0, q_1, \dots$ converges, and must converge to x. Thus $p_0, p_1, \dots$ converges to x, and τ is also a winning strategy. □

Corollary 11. *For all $x \in X$:*

- $\mathscr{R} \uparrow_{k\text{-}tact} Prox_{R,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(X, x)$
- $\mathscr{R} \uparrow_{k\text{-}mark} Prox_{R,P}(X) \Rightarrow \mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(X, x)$

Corollary 12. *All proximal spaces are W-spaces.*

Definition 13. In the one-point compactification $\kappa^* = \kappa \cup \{\infty\}$ of discrete κ, define the clopen partition $\mathcal{C}(F) = [F]^1 \cup \{\kappa^* \setminus F\}$.

Theorem 14. $\mathscr{R} \uparrow_{code} Prox_{R,P}(\kappa^*)$

Proof. Use the coding strategy $\sigma() = \mathcal{C}(\emptyset) = \{\kappa^*\}$, $\sigma(\mathcal{C}(F), \alpha) = \mathcal{C}(F \cup \{\alpha\})$ for $\alpha < \kappa$ and $\sigma(\mathcal{C}(F), \infty) = \mathcal{C}(F)$. Note $\mathcal{R}_n = \mathcal{H}_n$. For any attack $p_0, p_1, \dots$ against σ such that $\bigcap_{n<\omega} \mathcal{H}_n[p_n] \neq \emptyset$, suppose

- $\infty \in \bigcap_{n<\omega} \mathcal{H}_n[p_n]$. Then $p_n \in \kappa^* \setminus \{p_m : m < n\}$ shows that the non-∞ p_n are all distinct. If co-finite $p_n = \infty$, we have $p_n \to \infty$. Otherwise, there are infinite distinct p_n, and since neighorhoods of ∞ are co-finite, we have $p_n \to \infty$.
- $\infty \notin \mathcal{H}_N[p_N]$ for some $N < \omega$, so $\alpha \in \bigcap_{n<\omega} \mathcal{H}_n[p_n]$ for some $\alpha < \kappa$. Then $\mathcal{H}_n[p_n] = \{\alpha\}$ for all $n \geq N$, and thus $p_n \to \alpha$.

Thus σ is a winning coding strategy. □

Theorem 15. $\mathscr{O} \uparrow Con_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow Prox_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{pre} Con_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{pre} Prox_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{k\text{-}tact} Prox_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{k\text{-}mark} Prox_{R,P}(\kappa^*)$

Proof. Let $\sigma \circ L$ be a winning strategy where L is the identify (resp. a k-tactical fog-of-war, a k-Markӧv fog-of-war).

Define $\tau \circ L$ such that

$$\tau \circ L(p_0, \dots, p_{n-1}) = \mathcal{R}(\kappa^* \setminus (\sigma \circ L(p_0, \dots, p_{n-1})))$$

For any attack $p_0, p_1, \dots$ against τ such that $\bigcap_{n<\omega} \mathcal{H}_n[p_n] \neq \emptyset$, suppose

- $\mathcal{H}_n[p_n] = \mathcal{H}_n[\infty] = \bigcap_{m\leq n} \sigma \circ L(p_0, \dots, p_{m-1}) = \bigcap_{m\leq n} U_m = V_n$ for all $n < \omega$. Since σ is a winning strategy, the p_n converge at ∞.
- $\mathcal{H}_N[p_N] \neq \mathcal{H}_N[\infty]$ for some $N < \omega$. Then $\mathcal{H}_N[p_N] = \{p_N\}$, and since $\bigcap_{n<\omega} \mathcal{H}_n[p_n] \neq \emptyset$, we have $\mathcal{H}_n[p_n] = \mathcal{H}_N[p_N] = \{p_N\} \Rightarrow p_n = p_N$ for all $n \geq N$, and the p_n converge at p_N.

□

Corollary 16. $\mathscr{O} \uparrow Con_{O,P}(\kappa^*, \infty) \Leftrightarrow \mathscr{R} \uparrow Prox_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{pre} Con_{O,P}(\kappa^*, \infty) \Leftrightarrow \mathscr{R} \uparrow_{pre} Prox_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}tact} Con_{O,P}(\kappa^*, \infty) \Leftrightarrow \mathscr{R} \uparrow_{k\text{-}tact} Prox_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}mark} Con_{O,P}(\kappa^*, \infty) \Leftrightarrow \mathscr{R} \uparrow_{k\text{-}mark} Prox_{R,P}(\kappa^*)$

Corollary 17. *$O \uparrow_{pre} Prox_{R,P}(\omega^*)$.*

$O \uparrow_{tact} Prox_{R,P}(\omega^*)$.

$O \not\uparrow_{k\text{-}mark} Prox_{R,P}(\kappa^*)$ for $\kappa \geq \omega_1$.

Proof. Results hold for $\mathscr{O}$ and $Con_{O,P}(\kappa^*, \infty)$. □

Definition 18. The **almost-proximal game** $aProx_{R,P}(X)$ is analogous to $Prox_{R,P}(X)$ except that the points p_n need only cluster for $\mathscr{R}$ to win the game.

Definition 19. The **W-clustering game** $Clus_{O,P}(X, x)$ is analogous to $Con_{O,P}(X, x)$ except that the points p_n need only cluster at x for $\mathscr{O}$ to win the game.

Proposition 20. $\mathscr{O} \uparrow Clus_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow aProx_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{pre} Clus_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{pre} aProx_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}tact} Clus_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{k\text{-}tact} aProx_{R,P}(\kappa^*)$

$\mathscr{O} \uparrow_{k\text{-}mark} Clus_{O,P}(\kappa^*, \infty) \Rightarrow \mathscr{R} \uparrow_{k\text{-}mark} aProx_{R,P}(\kappa^*)$

Proof. Same proof as before, replacing "converge" with "cluster". □

Corollary 21. *$\mathscr{R} \uparrow_{mark} aProx_{R,P}(\omega_1^*)$.*

Proof. Holds for $\mathscr{O}$ and $Clus_{O,P}(\omega_1^*, \infty)$. □

Proposition 22. *If $\sigma \circ L$ is a winning strategy for $\mathscr{R}$ in $Prox_{R,P}(X)$ (resp. $aProx_{R,P}(X)$) where L is the identity (or a k-tactical fog-of-war or a k-Marköv fog-of-war), and C is a closed subspace of X, then*

$$\tau \circ L(p_0, \dots, p_{n-1}) = C \cap \sigma \circ L(p_0, \dots, p_{n-1})$$

defines a winning strategy $\tau \circ L$ for $\mathscr{R}$ in $Prox_{R,P}(X)$ (resp. $aProx_{R,P}(X)$).

Proof. For any attack $p_0, p_1, \dots$ against $\tau \circ L$ in $Prox_{R,P}(C)$ (resp. $aProx_{R,P}(C)$), note $p_0, p_1, \dots$ is also an attack against $\sigma \circ L$ in $Prox_{R,P}(X)$ (resp. $aProx_{R,P}(X)$).

If $\mathscr{R}$ wins in $Prox_{R,P}(X)$ (resp. $aProx_{R,P}(X)$) by $\mathcal{H}_n^\sigma[p_n] = \emptyset$, then note that $\mathcal{H}_n^\tau[p_n] \subseteq \mathcal{H}_n^\sigma[p_n] = \emptyset$.

If If $\mathscr{R}$ wins in $Prox_{R,P}(X)$ (resp. $aProx_{R,P}(X)$) because the p_n converge (resp. cluster), then they converge (resp. cluster) in the closed set C.

Either way, $\tau \circ L$ defeats the arbitrary attack and is thus a winning strategy. $\square$

Proposition 23. *If for any $i < m < \omega$, $\sigma_i \circ L$ is a winning strategy for $\mathscr{R}$ in $Prox_{R,P}(X_i)$ (resp. $aProx_{R,P}(X_i)$) where L is the identity (or a k-tactical fog-of-war or a k-Marköv fog-of-war), then*

$$\tau \circ L(p_0, \dots, p_{n-1}) = \bigotimes_{i<m} \sigma_i \circ L(p_0(i), \dots, p_{n-1}(i))$$

defines a winning strategy $\tau \circ L$ for $\mathscr{R}$ in $Prox_{R,P}(\prod_{i<m} X_i)$ (resp. $aProx_{R,P}(\prod_{i<m} X_i)$).

Proof. For any attack $p_0, p_1, \dots$ against $\tau \circ L$ in $Prox_{R,P}(\prod_{i<m} X_i)$ (resp. $aProx_{R,P}(\prod_{i<m} X_i)$), note that for any $i < m$, $p_0(i), p_1(i), \dots$ is an attack against $\sigma_i \circ L$ in $Prox_{R,P}(X_i)$ (resp. $aProx_{R,P}(X)$).

If for some $i < m$, $\mathscr{R}$ defeats the attack $p_0(i), p_1(i), \dots$ because $\bigcap_{n<\omega} \mathcal{H}_n^i[p_n(i)] = \emptyset$, then we see immediately that $\bigcap_{n<\omega} \mathcal{H}_n[p_n] = \emptyset$ and τ defeats the attack $p_0, p_1, \dots$.

Otherwise for all $i < m$, we have $p_n(i)$ converging (resp. clustering) at some $x_i \in X$. It follows then that $p_0, p_1, \dots$ converges (resp. clusters) at $x = \langle x_i : i < m \rangle$ and τ defeats the attack $p_0, p_1, \dots$. $\square$

Definition 24. For $H \subseteq X$, the **W-subset-convergence-game** $Con_{O,P}(X, H)$ is analogous to $Con_{O,P}(X, x)$: $\mathscr{O}$ chooses open neighborhoods of H and tries to force $p_n \to H$.

Theorem 25. *For all compact $H \subseteq X$, $\mathscr{R} \uparrow Prox_{R,P}(X)$ implies $\mathscr{O} \uparrow Con_{O,P}(X, H)$.*

Proof. Adapted from G's proof.

Let σ witness $\mathscr{R} \uparrow Prox_{R,P}(X)$, assuming $\sigma(p)$ refines $\sigma(q)$ whenever $q \subseteq p$.

For certain finite sequences of points $p \in X^{<\omega}$, we define a tree of finite sequences $\langle T(p), \subseteq \rangle$ as follows:

- $T(\emptyset)$ contains the empty sequence, and for each of the finite nonempty

  $$V \in \{U \cap H : U \in \sigma(\emptyset)\}$$

  choose a unique $h_V \in V$ and include $\langle h_V \rangle$ in $T(\emptyset)$.
- Assume that whenever $T(p)$ is defined, it satisfies the following:
  - $T(p)$ is finite
  - $p' \subseteq p \Rightarrow T(p') \subseteq T(p)$

- If $\langle h_0, q_0, \dots, h_n \rangle \in T(p)$ then $\langle q_0, \dots, q_{n-1} \rangle$ is a subsequence of p and $q_i \in \sigma(h_0, q_0, \dots, h_{i-1}, q_{i-1})[h_i]$ for all $i < n$
- For each sequence $t^\frown \langle h, q \rangle \in T(p)$ and for each of the finite nonempty

$$V \in \{U \cap H \cap \sigma(t)[h] : U \in \sigma(t^\frown \langle h, q \rangle)\}$$

there is a unique $h_V \in V$ such that $t^\frown \langle h, q, h_V \rangle \in T(p)$.
- $\{\sigma(t)[h] : t^\frown \langle h \rangle$ is maximal in $T(p)\}$ partitions $st\left(\bigwedge_{s \in T(p)} \sigma(s), H\right)$.

* Then when $T(p)$ is defined, we define $T(p^\frown \langle q \rangle)$ for each $q \in st\left(\bigwedge_{s \in T(p)} \sigma(s), H\right)$ as follows:

  - Assume $T(p) \subseteq T(p^\frown \langle q \rangle)$.
  - Find the maximal $t_q^\frown \langle h_q \rangle$ in $T(p)$ such that $q \in \sigma(t_q)[h_q]$. Include $t_q^\frown \langle h_q, q \rangle$ in $T(p^\frown \langle q \rangle)$.
  - For each of the finite nonempty

  $$V \in \mathcal{V}(t_q, h_q, q) = \{U \cap H \cap \sigma(t_q^\frown \langle h_q, q \rangle)[h] : U \in \sigma(t_q^\frown \langle h_q, q \rangle)\}$$

  choose a unique $h_V \in V$ and include $t_q^\frown \langle h_q, q, h_V \rangle$ in $T(p^\frown \langle q \rangle)$.
  - Note that

  $$\{\sigma(t)[h] : t^\frown \langle h \rangle \text{ is maximal in } T(p), h \neq h_q\}$$

  partitions

  $$st\left(\bigwedge_{s \in T(p)} \sigma(s), H\right) \setminus \sigma(t_q)[h_q] = st\left(\bigwedge_{s \in T(p^\frown \langle q \rangle)} \sigma(s), H\right) \setminus \sigma(t_q)[h_q]$$

  and that

  $$\{\sigma(t_q^\frown \langle h_q, q \rangle)[h_V] : \mathcal{V} \in V(t_q, h_q, q)\}$$

  partitions

  $$st\left(\bigwedge_{V \in \mathcal{V}(t_q, h_q, q)} \sigma(t_q^\frown \langle h_q, q, h_V \rangle), H\right) \cap \sigma(t_q)[h_q] = st\left(\bigwedge_{s \in T(p^\frown \langle q \rangle)} \sigma(s), H\right) \cap \sigma(t_q)[h_q]$$

  so our definition satisfies the recursion hypotheses.

We may define a strategy τ for $\mathscr{O}$ in $Con_{O,P}(X, H,)$ as follows. Let $\tau(\emptyset) = st\left(\bigwedge_{s \in T(\emptyset)} \sigma(s), H\right)$. If $T(p)$ is defined and $q \in st\left(\bigwedge_{s \in T(p)} \sigma(s), H\right)$, then let $\tau(p^\frown \langle q \rangle) = st\left(\bigwedge_{s \in T(p^\frown \langle q \rangle)} \sigma(s), H\right)$ (and $\tau(p^\frown \langle q \rangle) = X$ otherwise).

Let $p \in X^\omega$ attack τ such that $p(n) \in \tau(p \restriction n)$ always. It follows that $T(p \restriction n)$ is defined for all $n < \omega$, so let $T_p = \bigcup_{n<\omega} T(p \restriction n)$. By definition, it is evident that T_p is an infinite tree with finite levels, so choose an infinite branch $p' = \langle h_0, q_0, \dots \rangle$.

Since p' is an attack on σ, and $p'(n+1) \in \sigma(p \restriction n+1)[p(n)]$ always, it follows that p' converges. Since $p(2n) = h_n \in H$, p' converges in H, and so does its subsequence $p'' = \langle q_0, q_1, \dots \rangle$, which is also a subsequence of p.

We've shown p clusters in H, and since $\tau(p \restriction n+1) \subseteq \tau(p)$, it follows analogously to a result of G that p converges in H. $\square$

Corollary 26. *If X is compact and $\mathscr{R} \uparrow Prox_{R,P}(X)$, then $\mathscr{O} \uparrow Con_{O,P}(X^2, \Delta)$, and thus X is Corson compact.*

Proof. Note $\mathscr{R} \uparrow Prox_{R,P}(X^2)$ and Δ is a compact subset of X^2, so $\mathscr{O} \uparrow Con_{O,P}(X^2, \Delta)$. By a result of G, X is Corson compact. $\square$