

**Definition 1.** A space  $X$  is *strong Eberlein compact* if it embeds in  $\sigma 2^\kappa = \{x \in 2^\kappa : |\{\alpha : x(\alpha) = 1\}| < \omega\}$ .

**Theorem 2** (Gruenhage). *For compact spaces  $X$ ,  $X$  is strong Eberlein compact if and only if  $X$  is scattered and  $X$  is a  $W$ -space ( $\mathcal{O} \uparrow \text{Gru}_{\vec{O},P}(X, x)$  for all  $x \in X$ ).*

**Theorem 3.**  $\mathcal{D} \uparrow_{\text{tact}} \text{Bell}_{\vec{D},P}(\sigma 2^\kappa)$ .

*Proof.* Let  $\text{supp}(x) = \{\alpha : x(\alpha) = 1\} \in [\kappa]^{<\omega}$ .

Define the tactic  $\sigma$  for  $\mathcal{D}$  such that

$$\sigma(\langle x \rangle) = \bigcap \{P_\alpha(\Delta) : \alpha \in \text{supp}(x)\}$$

Fix a legal attack  $p : \omega \rightarrow \sigma 2^\kappa$ , and let  $\alpha < \kappa$ . If  $p_\alpha : \omega \rightarrow \sigma 2^\kappa$  defined by  $p_\alpha(n) = p(n)(\alpha)$  converges, then  $\sigma$  is a winning tactic. So assume  $p_\alpha(n) = 1$  for some  $n$ , and as  $\alpha \in \text{supp}(p(n))$ ,  $\sigma(p(n)) \subseteq P_\alpha(\Delta)$ . As  $p$  is a legal attack, it follows that  $p_\alpha(m) = p_\alpha(m+1)$  for all  $m > n$ , so  $p_\alpha$  converges. Otherwise  $p_\alpha(n) = 0$  for all  $n$  so  $p_\alpha$  converges.  $\square$

**Corollary 4.** *If  $X$  is strong Eberlein compact, then  $\mathcal{D} \uparrow_{\text{tact}} \text{Bell}_{\vec{D},P}(X)$ .*

**Theorem 5.** *If  $X$  contains a copy of the Cantor set, then  $\mathcal{D} \not\uparrow_{\text{tact}} \text{Bell}_{\vec{D},P}(X)$ .*

*Proof.* The result follows from showing that  $\mathcal{D} \not\uparrow_{\text{tact}} \text{Bell}_{\vec{D},P}(2^\omega)$  (any copy of the Cantor set within a Hausdorff space is a compact and thus closed subspace). Let  $\sigma$  be a tactic for  $\mathcal{D}$  in  $\text{Bell}_{\vec{D},P}(2^\omega)$  and let  $D_k = \{\langle f, g \rangle : f \upharpoonright k = g \upharpoonright k\}$ . Since  $\{D_k : k < \omega\}$  is a base for the uniformity on  $2^\omega$ , we may fix  $k(f) < \omega$  for each  $f \in 2^\omega$  such that  $D_{k(f)} \subseteq \sigma(\langle f \rangle)$ .

Then there exists  $k < \omega$  such that  $\{f : k = k(f)\}$  is uncountable, and therefore there exist distinct  $f, g$  such that  $k = k(f) = k(g)$  and  $f \upharpoonright k = g \upharpoonright k$ . Then  $p : \omega \rightarrow 2^\omega$  defined by  $p(2n) = f$  and  $p(2n+1) = g$  is an attack against  $\sigma$  which obviously doesn't converge. This attack is legal since  $f \in D_k[g] \subseteq \sigma(\langle g \rangle)[g]$  and  $g \in D_k[f] \subseteq \sigma(\langle f \rangle)[f]$ .  $\square$

**Lemma 6.** *Every non-scattered Corson compact space contains a homeomorphic copy of the Cantor set.*

*Proof.* Every non-scattered space contains a closed subspace without isolated points. Let  $X$  be such a subspace, and assume that this Corson compact is embedded in  $\Sigma \mathbb{R}^\kappa$ . Let  $B_{\alpha,\epsilon}(x) = \{y : d(x(\alpha), y(\alpha)) < \epsilon\}$ . For each  $x \in X$  and  $n < \omega$ , let  $\beta(x, n) < \kappa$  be defined such that  $\{\alpha : x(\alpha) \neq 0\} = \{\beta(x, n) : n < \omega\}$ .

Choose an arbitrary  $x_0 \in X$  and  $\epsilon_0 > 0$ , and let  $A_0 = \emptyset$ .

Suppose then that for some  $n < \omega$ ,  $x_s \in X$  is defined for all  $s \in 2^n$ , and  $\epsilon_n > 0$  and  $A_n \in [\kappa]^{<\omega}$  are defined. Since each  $x_s$  is not isolated in  $X$ , let  $U_s$  be the open set

$$U_s = X \cap \bigcap_{\alpha \in A_{|s|}} B_{\alpha, \epsilon_{|s|}}(x_s)$$

and choose  $x_{s \smallfrown \langle 0 \rangle}, x_{s \smallfrown \langle 1 \rangle} \in U_s$  distinct. Then let  $\alpha_s < \kappa$  such that  $x_{s \smallfrown \langle 0 \rangle}(\alpha_s) \neq x_{s \smallfrown \langle 1 \rangle}(\alpha_s)$ . Let

$$A_{n+1} = \{\alpha_s : s \in 2^{\leq n}\} \cup \{\beta(x_s, i) : s \in 2^{\leq n}, i \leq n\}$$

Then choose  $0 < \epsilon_{n+1} < \frac{1}{2}\epsilon_n$  such that

$$B_{\alpha_s, \epsilon_{n+1}}(x_{s \smallfrown \langle 0 \rangle}) \cap B_{\alpha_s, \epsilon_{n+1}}(x_{s \smallfrown \langle 1 \rangle}) = \emptyset$$

and

$$\overline{\bigcap_{\alpha \in A_{n+1}} B_{\alpha, \epsilon_{n+1}}(x_{s \smallfrown \langle 0 \rangle})} \cup \overline{\bigcap_{\alpha \in A_{n+1}} B_{\alpha, \epsilon_{n+1}}(x_{s \smallfrown \langle 1 \rangle})} \subseteq \bigcap_{\alpha \in A_n} B_{\alpha, \epsilon_n}(x_s)$$

for all  $s \in 2^n$ .

Let  $x_f = \lim_{n < \omega} x_{f \upharpoonright n} \in X$  for each  $f \in 2^\omega$ . We claim  $C = \{x_f : f \in 2^\omega\}$  is a copy of the Cantor set. This will follow if we can show that  $\{U_s : s \in 2^{<\omega}\}$  is a base for  $C$ , since it has the structure of the Cantor tree.

Consider  $x_f$  for some  $f \in 2^\omega$ , and a subbasic open ball  $B_{\alpha, \epsilon}(x_f)$ . Observe that  $x_f \in \bigcap_{n < \omega} U_{f \upharpoonright n}$  since  $x_{f \upharpoonright n} \in U_{f \upharpoonright m}$  for all  $m < n < \omega$ .

If  $\alpha \in \{\beta(x_s, n) : s \in 2^{<\omega}, n < \omega\}$ , choose  $k < \omega$  with  $\alpha \in A_k$ . Then choose  $l < \omega$  such that  $\epsilon_l < \epsilon$ . Then  $U_{f \upharpoonright (l+k)} \subseteq B_{\alpha, \epsilon}(x_f)$ .

Otherwise,  $x_s(\alpha) = 0$  for all  $s \in 2^{<\omega}$ , so  $x_g(\alpha) = 0$  for all  $g \in 2^\omega$  and therefore  $C \subseteq B_{\alpha, \epsilon}(x_f)$ .  $\square$

**Corollary 7.** *For compact spaces  $X$ ,  $X$  is strong Eberlein compact if and only if  $\mathcal{D} \uparrow_{tact} Bell_{D,P}^{\rightarrow}(X)$ .*

*Proof.* Suppose  $X$  is not strong Eberlein compact; then  $X$  is either not a  $W$ -space or not scattered. If  $\mathcal{D} \not\uparrow Bell_{D,P}^{\rightarrow}(X)$ , then the result follows immediately, which only leaves non-scattered proximal compact spaces to be considered. But non-scattered proximal compacts are non-scattered Corson compacts, and thus contain a copy of the Cantor set, so the result follows from Theorem 5.  $\square$

**Definition 8.** A space  $X$  is *Eberlein compact* if it embeds in  $\Sigma^*\mathbb{R}^\kappa = \{x \in 2^\kappa : |\{\alpha : |x(\alpha)| \geq \epsilon\}| < \omega \text{ for all } \epsilon > 0\}$ .

**Theorem 9.**  $\mathcal{D} \uparrow_{\text{mark}} \text{Bell}_{D,P}^\rightarrow(\Sigma^*\mathbb{R}^\kappa)$ .

*Proof.* Let  $\text{supp}_\epsilon(x) = \{\alpha : |x(\alpha)| \geq \epsilon\} \in [\kappa]^{<\omega}$ . Let  $D_\epsilon$  be the entourage of the diagonal formed by balls of radius  $\epsilon$ .

Define the tactic  $\sigma$  for  $\mathcal{D}$  such that

$$\sigma(\langle x \rangle, n) = \bigcap \{P_\alpha(D_{2^{-n}}) : \alpha \in \text{supp}_{2^{-n}}(x)\}$$

□

Miscellaneous:

**Example 10.**  $\mathcal{D} \uparrow_{\text{tact}} Bell_{D,P}^{\rightarrow}(\kappa^*)$ , so  $\mathcal{D} \uparrow_{\text{mark}} Bell_{D,P}^{\rightarrow}((\kappa^*)^\omega)$ .