

ON k -TACTICS IN GRUENHAGE'S COMPACT-POINT GAME

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ABSTRACT. Gary Gruenhage showed in [3] that metacompactness and σ -metacompactness may be characterized for locally compact spaces by way of a certain topological game using limited information strategies which consider only the most recent move of the opponent (tactical/stationary strategies) and possibly the round number (Markov strategies). This paper demonstrates a non-trivial example of a space for which a winning strategy exists but every limited information strategy considering a maximum of k moves of the opponent and the round number may be defeated. The question follows: are metacompactness and σ -metacompactness characterized by winning k -tactical and k -Markov strategies?

1. INTRODUCTION

Consider the following topological game.

Game 1.1. Let $Gru_{K,P}(X)$ denote the *Gruenhage compact/point game* with players \mathcal{K}, \mathcal{P} . During round n , \mathcal{K} chooses a compact subset K_n of X , followed by \mathcal{P} choosing a point $p_n \in X$ such that $p_n \notin \bigcup_{m \leq n} K_m$.

\mathcal{K} wins the game if the points p_n are locally finite in the space, and \mathcal{P} wins otherwise.

Definition 1.2. A *strategy* for a game G with moveset M is a function $\sigma : M^{<\omega} \rightarrow M$; intuitively, this is a fixed rule for one player's choices in consideration of her opponent's moves. If using such a strategy always results in a win for the player \mathcal{P} using it, then it is called a *winning strategy* and we write $\mathcal{P} \uparrow G$.

When $G(X)$ is a topological game played with space X , then $\mathcal{P} \uparrow G(X)$ and $\mathcal{P} \nmid G(X)$ are topological properties of the space X .

The above game was used by the author's adviser Gary Gruenhage in [3] to characterize metacompactness (every open cover of the space has a point-finite open refinement covering the space) and σ -metacompactness (every open cover of the space has a σ -point-finite open refinement covering the space) in locally compact spaces. However, the characterization considers so-called *limited information strategies* which do not use full information of the history of the game.

Definition 1.3. A k -tactical strategy for a game G with moveset M is a function $\sigma : M^{\leq k} \rightarrow M$; intuitively, it is a strategy which only considers the previous k moves of the opponent. If using such a strategy always results in a win for the player \mathcal{P} using it, then we write $\mathcal{P} \uparrow_{k\text{-tact}} G$.

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Definition 1.4. A k -Markov strategy for a game G with moveset M is a function $\sigma : M^{\leq k} \times \omega \rightarrow M$; intuitively, it is a strategy which only considers the previous k moves of the opponent and the round number. If using such a strategy always results in a win for the player \mathcal{P} using it, then we write $\mathcal{P} \underset{k\text{-mark}}{\uparrow} G$.

We will call k -tactical strategies “ k -tactics” and k -Markov strategies “ k -marks”. If the k is omitted then it is assumed that $k = 1$. In addition, note that some authors refer to tactics as *stationary strategies*.

In this language, one may write Gruenhage’s results as follows:

Theorem 1.5. *If X is a locally compact space, then*

- X is metacompact if and only if $\mathcal{K} \underset{tact}{\uparrow} Gru_{K,P}(X)$, and
- X is σ -metacompact if and only if $\mathcal{K} \underset{mark}{\uparrow} Gru_{K,P}(X)$

The main question essentially asks whether (at least for locally compact spaces) a winning $(k+2)$ -tactic or $(k+2)$ -mark may always be improved to a 1-tactic or 1-mark.

Question 1.6. Let X be a locally compact space and $k < \omega$. Does $\mathcal{K} \underset{(k+2)\text{-tact}}{\uparrow} Gru_{K,P}(X)$ imply X is metacompact? Does $\mathcal{K} \underset{(k+2)\text{-mark}}{\uparrow} Gru_{K,P}(X)$ imply X is σ -metacompact?

This question is very similar to the open question on the well-known Banach-Mazur (a.k.a. Choquet) game $BM_{E,N}(X)$: does there exist a space for which $\mathcal{N} \underset{3\text{-tact}}{\uparrow} BM_{E,N}(X)$ but $\mathcal{N} \not\underset{2\text{-tact}}{\uparrow} BM_{E,N}(X)$ (where \mathcal{N} is the player wishing for a nonempty intersection)?¹ For fans of such infinite combinatorial game theory puzzles, we may restate the main question as follows.

Question 1.7. Does there exist a locally compact space X such that $\mathcal{K} \underset{2\text{-tact}}{\uparrow} Gru_{K,P}(X)$ but $\mathcal{K} \not\underset{tact}{\uparrow} Gru_{K,P}(X)$? What about for Markov strategies?

2. RELATED RESULTS

The game $Gru_{K,P}(X)$ has its roots in another topological game due to Gruenhage.

Game 2.1. Let $Gru_{O,P}^{\rightarrow}(X, x)$ denote Gruenhage’s W -convergence game with players \mathcal{O} , \mathcal{P} , for a topological space X and point $x \in X$. In round n , \mathcal{O} chooses an open neighborhood O_n of x , followed by \mathcal{P} choosing a point $p_n \in \bigcap_{m \leq n} O_m$.

\mathcal{O} wins the game if the points p_n converge to x , and \mathcal{P} wins otherwise.

Let $X^* = X \cup \{\infty\}$ be the *one-point compactification* of a noncompact locally compact space X , where points in X have their usual neighborhoods, and neighborhoods of ∞ are complements of compact sets in X . Then convergence to ∞ in X^* corresponds to local finiteness in the subspace X . One may then assume that $Gru_{K,P}(X)$ and $Gru_{O,P}^{\rightarrow}(X^*, \infty)$ are equivalent games when X is locally compact. Considering perfect information, 1-tactical, and 1-Markov strategies, this is essentially true.

¹The analogous question for 2- and 1-tactics was answered in the affirmative by Gabriel Debs in [1].

Theorem 2.2. *If X is locally compact, then*

- $\mathcal{K} \uparrow Gru_{K,P}(X)$ if and only if $\mathcal{O} \uparrow Gru_{\vec{\mathcal{O}},P}(X^*, \infty)$.
- $\mathcal{K} \uparrow_{\text{mark}} Gru_{K,P}(X)$ if and only if $\mathcal{O} \uparrow_{\text{mark}} Gru_{\vec{\mathcal{O}},P}(X^*, \infty)$.
- $\mathcal{K} \uparrow_{\text{tact}} Gru_{K,P}(X)$ if and only if $\mathcal{O} \uparrow_{\text{tact}} Gru_{\vec{\mathcal{O}},P}(X^*, \infty)$.

Proof. Let σ be a winning mark for \mathcal{K} in $Gru_{K,P}(X)$. Define the tactic τ for \mathcal{O} in $Gru_{\vec{\mathcal{O}},P}(X^*, \infty)$ as follows:

$$\begin{aligned} \tau(\emptyset, 0) &= X^* \setminus \sigma(\emptyset, 0) \\ \tau(\langle x \rangle, n) &= \begin{cases} X^* & : x = \infty \\ X^* \setminus \bigcup_{m \leq n} \sigma(\langle x \rangle, m) & : x \neq \infty \end{cases} \end{aligned}$$

Then for any legal attack p against τ , consider its subsequence p' which removes all instances of ∞ . If p' is a finite sequence, then p contains ∞ co-finitely and therefore converges to ∞ .

Note that for each $n < \omega$, $p'(n) = p(f(n))$ for some $f(n) \geq n$. Since p is a legal attack against τ ,

$$\begin{aligned} p'(n) &= p(f(n)) \in \tau(\emptyset, 0) \cap \bigcap_{m < f(n)} \tau(\langle p(m) \rangle, m+1) \\ &= \tau(\emptyset, 0) \cap \bigcap_{m < n} \tau(\langle p'(m) \rangle, f(m)+1) = X^* \setminus \left(\sigma(\emptyset, 0) \cup \bigcup_{m < n, i \leq f(m)+1} \sigma(\langle p'(m) \rangle, i) \right) \\ &\subseteq X^* \setminus \left(\sigma(\emptyset, 0) \cup \bigcup_{m < n} \sigma(\langle p'(m) \rangle, m+1) \right) \end{aligned}$$

so p' is a legal attack against σ . Since σ is a winning strategy, the points $p'(n)$ are locally finite in X , so p' and therefore p converge to ∞ .

If σ is a winning mark for \mathcal{O} in $Gru_{\vec{\mathcal{O}},P}(X^*, \infty)$, let τ be a mark for \mathcal{K} in $Gru_{K,P}(X)$ such that

$$\tau(s, n) = X \setminus \sigma(s, n)$$

Then for any legal attack p against τ , p is a legal attack against σ . Since σ is a winning strategy, p converges to ∞ , and therefore the points $p(n)$ are locally finite in X .

The proofs of the first and third bullets are similar and are left to the reader. \square

The reason why the games are not entirely equivalent is related to the extra point ∞ in X^* , which gives \mathcal{P} an extra choice in $Gru_{\vec{\mathcal{O}},P}(X^*, \infty)$ unavailable in $Gru_{K,P}(X)$. In fact, a generalization of the above proof for a $(k+2)$ -mark would not hold.

For instance, suppose \mathcal{P} wants to use a winning 2-tactic σ from $Gru_{K,P}(X)$ to create a winning 2-tactic for $Gru_{\vec{\mathcal{O}},P}(X^*, \infty)$. If \mathcal{P} is attacked by $\langle x_0, \infty, x_2, \infty, \dots \rangle$, then \mathcal{P} must ensure that the x_{2n} converge without using the point $\infty \notin X$. It's difficult to see how, as \mathcal{P} may not take advantage of $\sigma(\langle x_{2n}, x_{2n+2} \rangle)$; x_{2n}, x_{2n+2} are not consecutive moves in the original game.

So we cannot (at least easily) find a result for $Gru_{K,P}(X)$ comparable to the following result for $Gru_{\vec{\mathcal{O}},P}(X, x)$:

Proposition 2.3. *For any $x \in X$ and $k < \omega$,*

- $\mathcal{O} \xrightarrow{(k+1)\text{-tact}} Gru_{\vec{\mathcal{O}},P}(X, x) \Leftrightarrow \mathcal{O} \xrightarrow{tact} Gru_{\vec{\mathcal{O}},P}(X, x)$
- $\mathcal{O} \xrightarrow{(k+1)\text{-mark}} Gru_{\vec{\mathcal{O}},P}(X, x) \Leftrightarrow \mathcal{O} \xrightarrow{mark} Gru_{\vec{\mathcal{O}},P}(X, x)$

Proof. If σ witnesses $\mathcal{O} \xrightarrow{(k+1)\text{-tact}} Gru_{\vec{\mathcal{O}},P}(X, x)$, let $\tau(\emptyset) = \sigma(\emptyset)$ and

$$\tau(\langle p \rangle) = \bigcap_{i < k} \sigma(\langle \underbrace{x, \dots, x}_{k-i}, p, \underbrace{x, \dots, x}_{i+1} \rangle)$$

Then τ is easily verified to be a winning tactic, and the proof for the second part is analogous. \square

Note that this proof ironically relies on \mathcal{P} playing the point x she wishes to avoid convergence to; a luxury not allowed to \mathcal{P} in $Gru_{K,P}(X)$ as that point (∞) is fictional.

So we do have this corollary at least.

Corollary 2.4. *If X is a locally compact space, then*

- *X is metacompact if and only if $\mathcal{K} \xrightarrow{tact} Gru_{\vec{\mathcal{O}},P}(X^*, \infty)$ if and only if $\mathcal{K} \xrightarrow{(k+1)\text{-tact}} Gru_{\vec{\mathcal{O}},P}(X^*, \infty)$ for some $k < \omega$, and*
- *X is σ -metacompact if and only if $\mathcal{K} \xrightarrow{mark} Gru_{\vec{\mathcal{O}},P}(X^*, \infty)$ if and only if $\mathcal{K} \xrightarrow{(k+1)\text{-mark}} Gru_{\vec{\mathcal{O}},P}(X^*, \infty)$ for some $k < \omega$.*

3. A NON-TRIVIAL EXAMPLE

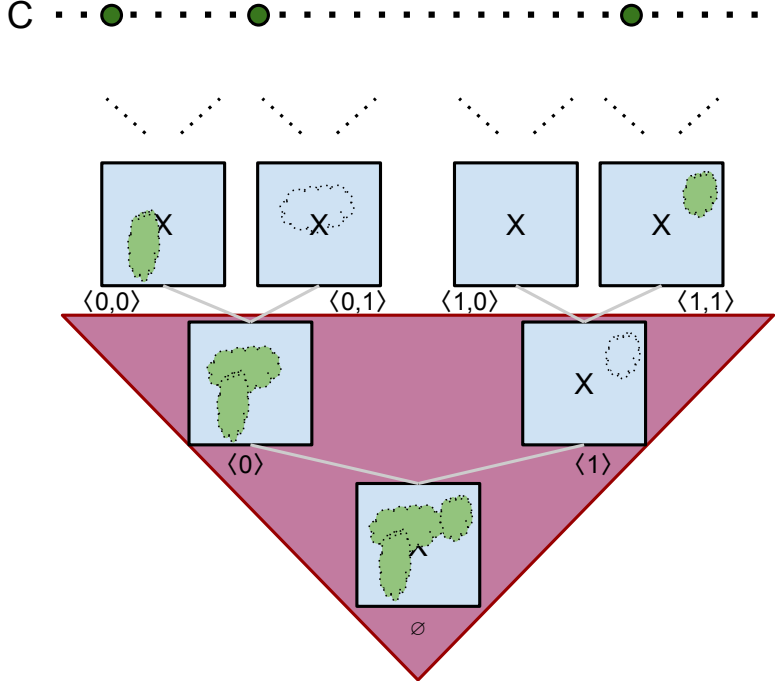
So if the analogous result does not hold for $Gru_{K,P}(X)$, then we should be able to find a counterexample. Gruenhage suggested the following class of spaces to the author:

Definition 3.1. Let $\mathbf{X} = (X \times 2^{<\omega}) \cup C$ denote a Cantor tree of copies of a zero-dimensional, compact space X with a point-countable cover $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ of distinct clopen sets, along with an uncountable subset of the Cantor set $C = \{c_\alpha : \alpha < \omega_1\} \in [2^\omega]^{\omega_1}$. The topology on \mathbf{X} is given by declaring $U \times \{s\}$ to be an open neighborhood of $\langle x, s \rangle \in X \times 2^{<\omega}$ for each open neighborhood U of x in X , and declaring $B_{\alpha,m} = (U_\alpha \times \{c_\alpha \upharpoonright n : m \leq n < \omega\}) \cup \{c_\alpha\}$ to be a clopen neighborhood of $c_\alpha \in C$ for each $\alpha < \omega_1$, $m < \omega$.

Definition 3.2. Let $F \in \omega_1^{<\omega}$ and $m, n < \omega$.

$$\begin{aligned} K_F &= \bigcup_{\alpha \in F} B_{\alpha,0} \\ a_n &= \{\langle i, 0 \rangle : i < n\} \cup \{\langle n, 1 \rangle\} \\ A &= \{a_n : n < \omega\} \\ K'_F &= K_F \setminus (X \times A) \\ L_m &= X \times 2^{<m} \end{aligned}$$

Figure 1 provides a rough illustration of \mathbf{X} , L_m (the triangle at the base), and K'_F (the branches of open sets descending from C).

FIGURE 1. \mathbf{X} , with L_m and K'_F

Lemma 3.3. K_F , K'_F , and L_m are compact in \mathbf{X} . Furthermore, every compact set is contained in a union of K'_F , L_m for some $F \in C^{<\omega}$ and $m < \omega$.

Proof. K_F contains $C_F = \{c_\alpha : \alpha \in F\} \subseteq C$, so any cover of basic open sets must include B_{α, n_α} for each $\alpha \in F$, and the remaining uncovered portion of K_F is a closed subset of a finite union of copies of compact X . Then K'_F is also compact as it is a closed subset of K_F , and L_m is compact as it is a finite union of copies of compact X .

Let D be compact. Consider the open cover

$$\{B_{\alpha, 0} : \alpha < \omega_1\} \cup \{X \times \{s\} : s \in 2^{<\omega}\}$$

and note that the finite subcover for D contains subsets of some $K'_F \cup L_m$. \square

Theorem 3.4. $\mathcal{K} \uparrow \text{Gru}_{K,P}(\mathbf{X})$.

Proof. Since $\{U_\alpha : \alpha < \omega_1\}$ is a point-countable cover, for each $x \in X$ let $\alpha_{x,n} < \omega_1$ yield ordinals such that $x \in U_{\alpha_{x,n}}$ for $n < \omega$.

Let $M : \mathbf{X} \times \omega \rightarrow \mathcal{K}(\mathbf{X})$ as follows:

$$M(\mathbf{x}, n) = \begin{cases} K_{\{\alpha_{x,m} : m \leq n\}} & : \mathbf{x} = \langle x, s \rangle \in X \times 2^{<\omega} \\ K_{\{\alpha\}} & : \mathbf{x} = c_\alpha \in C \end{cases}$$

and use M to define the strategy σ for each $\mathbf{a} \in \mathbf{X}^{<\omega}$:

$$\sigma(\mathbf{a}) = L_{|\mathbf{a}|} \cup \bigcup_{i < |\mathbf{a}|} M(\mathbf{a}(i), |\mathbf{a}|)$$

Let $\mathbf{p} : \omega \rightarrow \mathbf{X}$ be a legal attack against σ . Then as $\mathbf{p}(n) \notin L_n$, for each $\mathbf{x} = \langle x, s \rangle \in X \times 2^{<\omega}$, $X \times \{s\}$ is an open neighborhood of \mathbf{x} which contains finitely many $\mathbf{p}(n)$.

Now consider $\mathbf{x} = c_\alpha$ for some $\alpha < \omega_1$, and let $n < \omega$. Then if $\mathbf{p}(n) = \langle x, s \rangle$ with $\alpha = \alpha_{x,N}$ for some $N < \omega$, then $\mathbf{p}(m) \notin B_{\alpha,0}$ for $\max(n, N) < m < \omega$. Or, if $\mathbf{p}(n) = c_\alpha$, then $\mathbf{p}(m) \notin B_{\alpha,0}$ for $n < m < \omega$. Otherwise, $\mathbf{p}(m) \notin B_{\alpha,0}$ for any $m < \omega$. In any case, $B_{\alpha,0}$ is a neighborhood of \mathbf{x} which contains finitely many $\mathbf{p}(n)$. Therefore, σ is a winning strategy. \square

One might hope then that $\mathcal{K} \not\uparrow_{\text{tact}} \text{Gru}_{K,P}(\mathbf{X})$ but $\mathcal{K} \uparrow_{2\text{-tact}} \text{Gru}_{K,P}(\mathbf{X})$, giving us our counterexample. However, we will see that in fact, any winning k -mark for \mathcal{K} may be improved to a winning tactic by exploiting the structure of the Cantor tree.

In particular, knowledge of round number does not assist \mathcal{K} , since she may force \mathcal{P} to either stay within C , or to seed a growing integer which could be used in place of the round number by forcing her to play outside $L_{|s|+1}$ in response to $\langle x, s \rangle \in X \times 2^{<\omega}$.

Lemma 3.5. *If $\mathcal{K} \uparrow_{(k+1)\text{-mark}} \text{Gru}_{K,P}(\mathbf{X})$, then $\mathcal{K} \uparrow_{(k+1)\text{-tact}} \text{Gru}_{K,P}(\mathbf{X})$.*

Proof. Let σ be a winning $(k+1)$ -mark for \mathcal{K} such that $m \leq n$ and $\text{range}(r) \subseteq \text{range}(s)$ implies $\sigma(r, m) \subseteq \sigma(s, n)$. For a sequence p , let $p \upharpoonright^k n = \nu_k(p \upharpoonright n)$ give the last k terms of $p \upharpoonright n$.

Define $r : \mathbf{X} \rightarrow \omega$ by

$$r(\mathbf{x}) = \begin{cases} |s| & : \mathbf{x} = \langle x, s \rangle \in X \times 2^{<\omega} \\ 0 & : \mathbf{x} \in C \end{cases}$$

and use r to define the $(k+1)$ -tactic τ by

$$\tau(\emptyset) = \sigma(\emptyset, 0)$$

$$\tau(\mathbf{t} \smallfrown \langle \mathbf{x} \rangle) = L_{r(\mathbf{x})+1} \cup \{\mathbf{x}\} \cup \sigma(\mathbf{t} \smallfrown \langle \mathbf{x} \rangle, r(\mathbf{x}) + 1)$$

Let $\mathbf{p} : \omega \rightarrow \mathbf{X}$ be a legal attack by \mathcal{P} against τ . If $\mathbf{p}(n) \in C$ for $N < n < \omega$, then since no $\mathbf{p}(n)$ may be legally repeated, $\{\{\mathbf{p}(n)\} : N < n < \omega\}$ is a discrete collection, making the points $\mathbf{p}(n)$ locally finite.

Otherwise, let $f \in \omega^\omega$ be increasing and define $\mathbf{q} : \omega \rightarrow X \times 2^{<\omega}$ such that $\mathbf{q}(i) = \mathbf{p}(f(i))$, and $\mathbf{p}(j) \in X \times 2^{<\omega}$ implies there is some i with $j = f(i)$. It follows that

$$\mathbf{q}(0) = \mathbf{p}(f(0)) \notin \bigcup_{m \leq f(0)} \tau(\mathbf{p} \upharpoonright^{k+1} m) \supseteq \tau(\emptyset) = \sigma(\emptyset, 0)$$

Denoting $\mathbf{q}(n) = \langle x_n, s_n \rangle$, it's trivial to note that $|s_0| \geq 0$. Assuming that $|s_m| \geq m$ for $m \leq n$, it then follows that

$$\begin{aligned} \mathbf{q}(n+1) = \mathbf{p}(f(n+1)) &\notin \bigcup_{m \leq f(n+1)} \tau(\mathbf{p} \upharpoonright^{k+1} m) \\ &\supseteq \bigcup_{m \leq n} \tau(\mathbf{q} \upharpoonright^{k+1} m) \supseteq \sigma(\emptyset, 0) \cup \bigcup_{m < n} \sigma(\mathbf{q} \upharpoonright^{k+1} (m+1), |s_m| + 1) \\ &\supseteq \sigma(\emptyset, 0) \cup \bigcup_{m < n} \sigma(\mathbf{q} \upharpoonright^{k+1} (m+1), m+1) \end{aligned}$$

and

$$\mathbf{q}(n+1) \notin \tau(\mathbf{q} \upharpoonright^{k+1} (n+1)) \supseteq L_{r(\mathbf{q}(n))} = L_{|s_n|+1}$$

gives $|s_{n+1}| \geq |s_n| + 1 \geq n + 1$. Thus \mathbf{q} is a legal attack on the winning $(k+1)$ -Markov strategy σ , so the points $\mathbf{q}(n)$ are locally finite, and it follows that the points $\mathbf{p}(n)$ are also locally finite. \square

Corollary 3.6. *\mathbf{X} is σ -metacompact if and only if \mathbf{X} is metacompact.*

On the other hand, recalling a maximum of $k+1$ moves is only as good as recalling the most recent move for \mathcal{H} , since \mathcal{P} may always choose to burn k out of every $k+1$ moves by moving down an antichain of the Cantor tree. Such movement would certainly be locally finite (and therefore not directly benefit \mathcal{P}), but would at least succeed in overloading \mathcal{H} 's limited memory.

Lemma 3.7. *If $\mathcal{H} \uparrow_{(k+1)\text{-tact}} \text{Gru}_{K,P}(\mathbf{X})$, then $\mathcal{H} \uparrow_{\text{tact}} \text{Gru}_{K,P}(\mathbf{X})$.*

Proof. Let σ be a winning $(k+1)$ -tactical strategy, and without loss of generality assume it ignores order.

Define $F(x_0, \dots, x_k, n) \in [C]^{<\omega}$ and $m(x_0, \dots, x_k, n) \in \omega \setminus (n+1)$, both increasing on n , such that for each $\langle x_0, \dots, x_k \rangle \in X^{k+1}$,

$$\bigcup_{s_0, \dots, s_k \in 2^{\leq n}} \sigma(\langle x_0, s_0 \rangle, \dots, \langle x_k, s_k \rangle) \subseteq K'_{F(x_0, \dots, x_k, n)} \cup L_{m(x_0, \dots, x_k, n)}$$

Select an arbitrary point $y \in X$. Let

$$M^0(x, n) = n$$

$$M^{i+1}(x, n) = m(x, y, \dots, y, M^i(x, n) + 1)$$

and define the tactical strategy τ as follows:

$$\tau(\emptyset) = \sigma(\emptyset)$$

$$\tau(\langle c_\alpha \rangle) = \{c_\alpha\}$$

$$\tau(\langle \langle x, s \rangle \rangle) = K'_{F(x, y, \dots, y, M^k(x, |s|) + 1)} \cup L_{m(x, y, \dots, y, M^k(x, |s|) + 1)}$$

Let $\mathbf{p} : \omega \rightarrow \mathbf{X}$ be a legal attack against τ , and assume $\mathbf{p}(n) = \langle x_n, s_n \rangle \in X \times 2^{<\omega}$. Then consider the attack $\mathbf{q} : \omega \rightarrow X \times 2^{<\omega}$ against σ defined by, for $n < \omega$ and $i < k$,

$$\begin{aligned} \mathbf{q}((k+1)n) &= \mathbf{p}(n) = \langle x_n, s_n \rangle \\ \mathbf{q}((k+1)n + (i+1)) &= \langle y, a_{M^{i+1}(x_n, |s_n|)} \rangle \end{aligned}$$

Since

$$\langle x_{n+1}, s_{n+1} \rangle = \mathbf{p}(n+1) \notin \tau(\langle \mathbf{p}(n) \rangle) \supseteq L_{M^{k+1}(x_n, |s_n|) + 1}$$

it follows that $|s_{n+1}| \geq M^{i+1}(x_n, |s_n|) + 1$ for $i < k$; furthermore,

$$|s_n| \leq M^i(x_n, |s_n|) < M^i(x_n, |s_n|) + 1 \leq M^{i+1}(x_n, |s_n|) < M^{i+1}(x_n, |s_n|) + 1 \leq |s_{n+1}|$$

so the second coordinate of $\mathbf{q}(n)$ is always strictly increasing.

By the definition of τ ,

$$\mathbf{q}((k+1)n) = \mathbf{p}(n) \notin \bigcup_{m \leq n} \tau(\mathbf{p} \upharpoonright^1 m) \supseteq \bigcup_{m \leq (k+1)n} \sigma(\mathbf{q} \upharpoonright^{k+1} m)$$

Since

$$\mathbf{q}((k+1)n + (i+1)) = \langle y, a_{M^{i+1}(x_n, |s_n|)} \rangle \in X \times A$$

it follows that $\mathbf{q}((k+1)n + (i+1)) \notin K'_F$ for any $F \in [\omega_1]^{<\omega}$.

Then it's sufficient to note that

$$|a_{M^{i+1}(x_n, |s_n|)}| = M^{i+1}(x_n, |s_n|) + 1 > m(x_n, y, \dots, y, M^i(x_n, |s_n|) + 1)$$

to show that

$$\mathbf{q}((k+1)n + (i+1)) = \langle y, a_{M^{i+1}(x_n, |s_n|)} \rangle \notin L_{m(x_n, y, \dots, y, M^i(x_n, |s_n|) + 1)}$$

and therefore $\mathbf{q}((k+1)n + (i+1))$ is a legal move.

As a result, \mathbf{q} is a legal attack against σ , and $\{\{\mathbf{q}(n)\} : n < \omega\} \supseteq \{\{\mathbf{p}(n)\} : n < \omega\}$ are both locally finite.

Finally, if the range of \mathbf{p} intersects C , those moves may be safely ignored as they cannot be repeated and lay in a closed discrete set, so the proof is complete. \square

So our hopes for a counterexample to our main question in \mathbf{X} are thusly defeated. Our consolation prize will be to show that for a certain choice of X and $\{U_\alpha : \alpha < \omega_1\}$, \mathbf{X} is at least an example of a space which allows a winning strategy for $\text{Gru}_{K,P}(\mathbf{X})$ but no winning k -tactics or k -marks.

To this end, we will show that $\{U_\alpha : \alpha < \omega_1\}$ may be chosen such that \mathbf{X} is not metacompact, and therefore \mathcal{K} lacks a k -Markov strategy for any $k < \omega$.

Theorem 3.8. *\mathbf{X} is metacompact if and only if $\{U_\alpha : \alpha < \omega_1\}$ is σ -point-finite.*

Proof. Let $\omega_1 = \bigcup_{n < \omega} A_n$ such that $\{U_\alpha : \alpha \in A_n\}$ is point-finite for each $n < \omega$.

Let \mathcal{U} be a cover of \mathbf{X} , and for each $s \in 2^\omega$ let \mathcal{V}_s be a finite open refinement of \mathcal{C} covering the compact set $X \times \{s\}$. Then let $\mathcal{W}_n = \{B_{\alpha, n_\alpha} : \alpha \in A_n\}$ be an open refinement of \mathcal{C} for each $n < \omega$, and note that it is point-finite. It follows that $\mathcal{U}' = \bigcup_{s \in 2^{<\omega}} \mathcal{V}_s \cup \bigcup_{n < \omega} \mathcal{W}_n$ is an open σ -point-finite refinement of \mathcal{U} , so \mathbf{X} is σ -metacompact, and therefore it is metacompact.

For the other direction, consider the open cover $\mathcal{U} = \{B_{\alpha, 0} : \alpha < \omega_1\}$ of the closed subset C of metacompact \mathbf{X} , and let $\{B_{\alpha, n_\alpha} : \alpha < \omega_1\}$ be a point-finite refinement. Then $\mathcal{U}_s = \{U_\alpha : c_\alpha \upharpoonright n_\alpha = s\}$ is point-finite for $s \in 2^{<\omega}$ and $U_\alpha \in \mathcal{U}_{c_\alpha \upharpoonright n_\alpha}$ for each $\alpha < \omega_1$. Therefore $\mathcal{U} = \bigcup_{s \in 2^{<\omega}} \mathcal{U}_s$ is σ -point-finite. \square

Theorem 3.9. *There exists a compact, zero-dimensional topological space X with a clopen cover $\{U_\alpha : \alpha < \omega_1\}$ of distinct sets which is not σ -point-finite.*

Proof. Let Y be a zero-dimensional Corson compact space which is not Eberlein compact; one such space was constructed in [4]. Let $X = Y^2$. Then by characterizations of Corson and Eberlein compacts found in [2], $Y^2 \setminus \Delta$ is meta-Lindelöf but not σ -metacompact, so there exists a point-countable clopen cover \mathcal{U} of $Y^2 \setminus \Delta$ which is not σ -point-finite. Then $\mathcal{U} \cup \{X\}$ is a point-countable clopen cover of X which is not σ -point-finite. \square

Corollary 3.10. *There exists a locally compact space X such that $\mathcal{K} \uparrow \text{Gru}_{K,P}(X)$ but $\mathcal{K} \not\uparrow_{k\text{-mark}} \text{Gru}_{K,P}(X)$ for all $k < \omega$.*

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