ON k-TACTICS IN GRUENHAGE'S COMPACT-POINT GAME

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Abstract. Gary Gruenhage showed in [3] that metacompactness and σ-metacompactness may be characterized for locally compact spaces by way of a certain topological game using limited information strategies which consider only the most recent move of the opponent (tactical/stationary strategies) and possibly the round number (Markov strategies). This paper demonstrates a non-trivial example of a space for which a winning strategy exists but every limited information strategy considering a maximum of k moves of the opponent and the round number may be defeated. The question follows: are metacompactness and σ-metacompactness characterized by winning k-tactical and k-Markov strategies?

1. Introduction

Consider the following topological game.

Game 1.1. Let $Gru_{K,P}(X)$ denote the *Gruenhage compact/point game* with players $\mathscr{K}$, $\mathscr{P}$. During round n, $\mathscr{K}$ chooses a compact subset K_n of X, followed by $\mathscr{P}$ choosing a point $p_n \in X$ such that $p_n \notin \bigcup_{m \leq n} K_m$.

$\mathscr{K}$ wins the game if the points p_n are locally finite in the space, and $\mathscr{P}$ wins otherwise.

Definition 1.2. A *strategy* for a game G with moveset M is a function $\sigma : M^{<\omega} \to M$; intuitively, this is a fixed rule for one player's choices in consideration of her opponent's moves. If using such a strategy always results in a win for the player $\mathscr{P}$ using it, then it is called a *winning strategy* and we write $\mathscr{P} \uparrow G$.

When $G(X)$ is a topological game played with space X, then $\mathscr{P} \uparrow G(X)$ and $\mathscr{P} \not\uparrow G(X)$ are topological properties of the space X.

The above game was used by the author's adviser Gary Gruenhage in [3] to characterize metacompactness (every open cover of the space has a point-finite open refinement covering the space) and σ-metacompactness (every open cover of the space has a σ-point-finite open refinement covering the space) in locally compact spaces. However, the characterization considers so-called *limited information strategies* which do not use full information of the history of the game.

Definition 1.3. A k-tactical strategy for a game G with moveset M is a function $\sigma : M^{\leq k} \to M$; intuitively, it is a strategy which only considers the previous k moves of the opponent. If using such a strategy always results in a win for the player $\mathscr{P}$ using it, then we write $\mathscr{P} \underset{k\text{-tact}}{\uparrow} G$.

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Definition 1.4. A k-Markov strategy for a game G with moveset M is a function $\sigma : M^{\leq k} \times \omega \to M$; intuitively, it is a strategy which only considers the previous k moves of the opponent and the round number. If using such a strategy always results in a win for the player $\mathscr{P}$ using it, then we write $\mathscr{P} \underset{k\text{-mark}}{\uparrow} G$.

We will call k-tactical strategies "k-tactics" and k-Markov strategies "k-marks". If the k is omitted then it is assumed that $k = 1$. In addition, note that some authors refer to tactics as *stationary strategies*.

In this language, one may write Gruenhage's results as follows:

Theorem 1.5. *If X is a locally compact space, then*

- *X is metacompact if and only if $\mathscr{K} \underset{tact}{\uparrow} Gru_{K,P}(X)$, and*
- *X is σ-metacompact if and only if $\mathscr{K} \underset{mark}{\uparrow} Gru_{K,P}(X)$*

The main question essentially asks whether (at least for locally compact spaces) a winning $(k+2)$-tactic or $(k+2)$-mark may always be improved to a 1-tactic or 1-mark.

Question 1.6. Let X be a locally compact space and $k < \omega$. Does $\mathscr{K} \underset{(k+2)\text{-tact}}{\uparrow} Gru_{K,P}(X)$ imply X is metacompact? Does $\mathscr{K} \underset{(k+2)\text{-mark}}{\uparrow} Gru_{K,P}(X)$ imply X is σ-metacompact?

This question is very similar to the open question on the well-known Banach-Mazur (also known as Choquet) game $BM_{E,N}(X)$ where $\mathscr{N}$ is the player wishing for a nonempty intersection: does there exist a space for which $\mathscr{N} \underset{3\text{-tact}}{\uparrow} BM_{E,N}(X)$ but $\mathscr{N} \underset{2\text{-tact}}{\not\uparrow} BM_{E,N}(X)$? (The analogous question for 2- and 1-tactics was answered in the affirmative by Gabriel Debs in [1].) For fans of such infinite combinatorial game theory puzzles, we may restate the main question as follows.

Question 1.7. Does there exist a locally compact space X such that $\mathscr{K} \underset{2\text{-tact}}{\uparrow} Gru_{K,P}(X)$ but $\mathscr{K} \underset{\text{tact}}{\not\uparrow} Gru_{K,P}(X)$? What about for Markov strategies?

2. Related results

The game $Gru_{K,P}(X)$ has its roots in another topological game due to Gruenhage.

Game 2.1. Let $Gru_{\vec{O},P}(X, x)$ denote *Gruenhage's W-convergence game* with players $\mathscr{O}$, $\mathscr{P}$, for a topological space X and point $x \in X$. In round n, $\mathscr{O}$ chooses an open neighborhood O_n of x, followed by $\mathscr{P}$ choosing a point $p_n \in \bigcap_{m \leq n} O_m$.

$\mathscr{O}$ wins the game if the points p_n converge to x, and $\mathscr{P}$ wins otherwise.

Let $X^\star = X \cup \{\infty\}$ be the *one-point compactification* of a noncompact locally compact space X, where points in X have their usual neighborhoods, and neighborhoods of ∞ are complements of compact sets in X. Then convergence to ∞ in $X^\star$ corresponds to local finiteness in the subspace X. One may then assume that $Gru_{K,P}(X)$ and $Gru_{\vec{O},P}(X^\star, \infty)$ are equivalent games when X is locally compact. Considering perfect information, 1-tactical, and 1-Markov strategies, this is essentially true.

Theorem 2.2. *If X is locally compact, then*

- $\mathscr{K} \uparrow Gru_{K,P}(X)$ *if and only if* $\mathscr{O} \uparrow Gru_{\vec{O},P}(X^\star, \infty)$.
- $\mathscr{K} \underset{mark}{\uparrow} Gru_{K,P}(X)$ *if and only if* $\mathscr{O} \underset{mark}{\uparrow} Gru_{\vec{O},P}(X^\star, \infty)$.
- $\mathscr{K} \underset{tact}{\uparrow} Gru_{K,P}(X)$ *if and only if* $\mathscr{O} \underset{tact}{\uparrow} Gru_{\vec{O},P}(X^\star, \infty)$.

Proof. Let σ be a winning mark for $\mathscr{K}$ in $Gru_{K,P}(X)$. Define the tactic τ for $\mathscr{O}$ in $Gru_{\vec{O},P}(X^\star, \infty)$ as follows:

$$\tau(\emptyset, 0) = X^\star \setminus \sigma(\emptyset, 0)$$

$$\tau(\langle x\rangle, n) = \begin{cases} X^\star & : \quad x = \infty \\ X^\star \setminus \bigcup_{m \leq n} \sigma(\langle x\rangle, m) & : \quad x \neq \infty \end{cases}$$

Then for any legal attack p against τ, consider its subsequence p' which removes all instances of ∞. If p' is a finite sequence, then p contains ∞ co-finitely and therefore converges to ∞.

Note that for each $n < \omega$, $p'(n) = p(f(n))$ for some $f(n) \geq n$. Since p is a legal attack against τ,

$$p'(n) = p(f(n)) \in \tau(\emptyset, 0) \cap \bigcap_{m < f(n)} \tau(\langle p(m)\rangle, m+1)$$

$$= \tau(\emptyset, 0) \cap \bigcap_{m<n} \tau(\langle p'(m)\rangle, f(m)+1) = X^\star \setminus \left(\sigma(\emptyset, 0) \cup \bigcup_{m<n, i \leq f(m)+1} \sigma(\langle p'(m)\rangle, i)\right)$$

$$\subseteq X^\star \setminus \left(\sigma(\emptyset, 0) \cup \bigcup_{m<n} \sigma(\langle p'(m)\rangle, m+1)\right)$$

so p' is a legal attack against σ. Since σ is a winning strategy, the points $p'(n)$ are locally finite in X, so p' and therefore p converge to ∞.

If σ is a winning mark for $\mathscr{O}$ in $Gru_{\vec{O},P}(X^\star, \infty)$, let τ be a mark for $\mathscr{K}$ in $Gru_{K,P}(X)$ such that

$$\tau(s, n) = X \setminus \sigma(s, n)$$

Then for any legal attack p against τ, p is a legal attack against σ. Since σ is a winning strategy, p converges to ∞, and therefore the points $p(n)$ are locally finite in X.

The proofs of the first and third bullets are similar and are left to the reader. $\square$

The reason why the games are not entirely equivalent is related to the extra point ∞ in $X^\star$, which gives $\mathscr{P}$ an extra choice in $Gru_{\vec{O},P}(X^\star, \infty)$ unavailable in $Gru_{K,P}(X)$. In fact, a generalization of the above proof for a $(k+2)$-mark would not hold.

For instance, suppose $\mathscr{P}$ wants to use a winning 2-tactic σ from $Gru_{K,P}(X)$ to create a winning 2-tactic for $Gru_{\vec{O},P}(X^\star, \infty)$. If $\mathscr{P}$ is attacked by $\langle x_0, \infty, x_2, \infty, \ldots\rangle$, then $\mathscr{P}$ must ensure that the x_{2n} converge without using the point $\infty \notin X$. It's difficult to see how, as $\mathscr{P}$ may not take advantage of $\sigma(\langle x_{2n}, x_{2n+2}\rangle)$; x_{2n}, x_{2n+2} are not consecutive moves in the original game.

So we cannot (at least easily) find a result for $Gru_{K,P}(X)$ comparable to the following result for $Gru_{\vec{O},P}(X, x)$:

Proposition 2.3. *For any $x \in X$ and $k < \omega$,*

- $\mathscr{O} \underset{(k+1)\text{-}tact}{\uparrow} Gru_{\vec{O},P}(X,x) \Leftrightarrow \mathscr{O} \underset{tact}{\uparrow} Gru_{\vec{O},P}(X,x)$
- $\mathscr{O} \underset{(k+1)\text{-}mark}{\uparrow} Gru_{\vec{O},P}(X,x) \Leftrightarrow \mathscr{O} \underset{mark}{\uparrow} Gru_{\vec{O},P}(X,x)$

Proof. If σ witnesses $\mathscr{O} \underset{(k+1)\text{-tact}}{\uparrow} Gru_{\vec{O},P}(X,x)$, let $\tau(\emptyset) = \sigma(\emptyset)$ and

$$\tau(\langle p \rangle) = \bigcap_{i<k} \sigma(\langle \underbrace{x, \dots, x}_{k-i}, p, \underbrace{x, \dots, x}_{i+1} \rangle)$$

Then τ is easily verified to be a winning tactic, and the proof for the second part is analogous. □

Note that this proof ironically relies on $\mathscr{P}$ playing the point x she wishes to avoid convergence to; a luxury not allowed to $\mathscr{P}$ in $Gru_{K,P}(X)$ as that point (∞) is fictional.

So we do have this corollary at least.

Corollary 2.4. *If X is a locally compact space, then*

- *X is metacompact if and only if $\mathscr{K} \underset{tact}{\uparrow} Gru_{\vec{O},P}(X^\star, \infty)$ if and only if $\mathscr{K} \underset{(k+1)\text{-}tact}{\uparrow} Gru_{\vec{O},P}(X^\star, \infty)$ for some $k < \omega$, and*
- *X is σ-metacompact if and only if $\mathscr{K} \underset{mark}{\uparrow} Gru_{\vec{O},P}(X^\star, \infty)$ if and only if $\mathscr{K} \underset{(k+1)\text{-}mark}{\uparrow} Gru_{\vec{O},P}(X^\star, \infty)$ for some $k < \omega$.*

3. A non-trivial example

So if the analogous result does not hold for $Gru_{K,P}(X)$, then we should be able to find a counterexample. Gruenhage suggested the following class of spaces to the author:

Definition 3.1. Let $\boldsymbol{X} = (X \times 2^{<\omega}) \cup C$ denote a Cantor tree of copies of a zero-dimensional, compact space X with a point-countable cover $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ of distinct clopen sets, along with an uncountable subset of the Cantor set $C = \{c_\alpha : \alpha < \omega_1\} \in [2^\omega]^{\omega_1}$. The topology on $\boldsymbol{X}$ is given by decaring $U \times \{s\}$ to be a open neighborhood of $\langle x, s \rangle \in X \times 2^{<\omega}$ for each open neighborhood U of x in X, and declaring $B_{\alpha,m} = (U_\alpha \times \{c_\alpha \restriction n : m \leq n < \omega\}) \cup \{c_\alpha\}$ to be a clopen neighborhood of $c_\alpha \in C$ for each $\alpha < \omega_1$, $m < \omega$.

Definition 3.2. Let $F \in \omega_1^{<\omega}$ and $m, n < \omega$.

$$K_F = \bigcup_{\alpha \in F} B_{\alpha,0}$$

$$a_n = \{\langle i, 0 \rangle : i < n\} \cup \{\langle n, 1 \rangle\}$$

$$A = \{a_n : n < \omega\}$$

$$K'_F = K_F \setminus (X \times A)$$

$$L_m = X \times 2^{<m}$$

Figure 1 provides a rough illustration of $\boldsymbol{X}$, L_m (the triangle at the base), and K'_F (the branches of open sets descending from C).

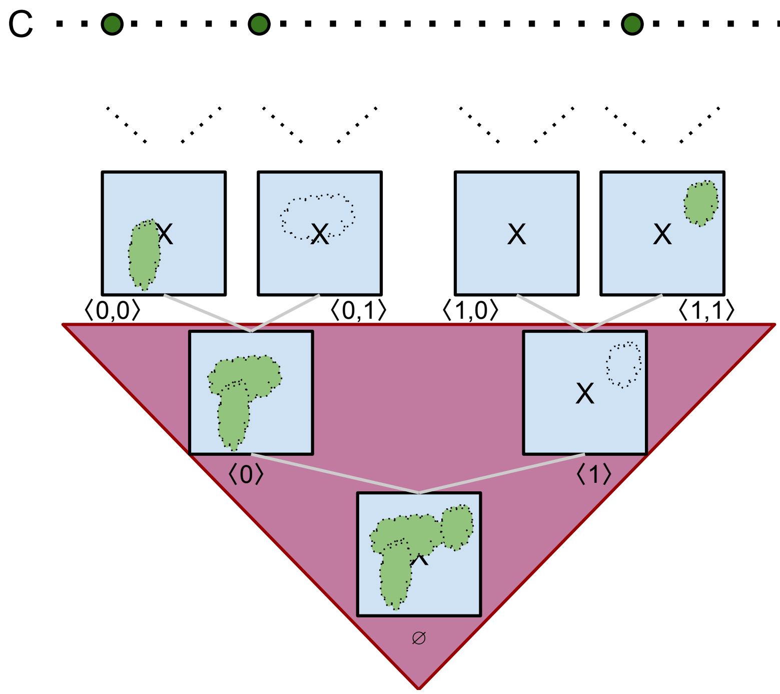

FIGURE 1. $\boldsymbol{X}$, with L_m and K'_F

Lemma 3.3. *K_F, K'_F, and L_m are compact in $\boldsymbol{X}$. Furthermore, every compact set is contained in a union of K'_F, L_m for some $F \in C^{<\omega}$ and $m < \omega$.*

Proof. K_F contains $C_F = \{c_\alpha : \alpha \in F\} \subseteq C$, so any cover of basic open sets must include B_{α,n_α} for each $\alpha \in F$, and the remaining uncovered portion of K_F is a closed subset of a finite union of copies of compact X. Then K'_F is also compact as it is a closed subset of K_F, and L_m is compact as it is a finite union of copies of compact X.

Let D be compact. Consider the open cover

$$\{B_{\alpha,0} : \alpha < \omega_1\} \cup \{X \times \{s\} : s \in 2^{<\omega}\}$$

and note that the finite subcover for D contains subsets of some $K'_F \cup L_m$. □

Theorem 3.4. $\mathscr{K} \uparrow Gru_{K,P}(\boldsymbol{X})$.

Proof. Since $\{U_\alpha : \alpha < \omega_1\}$ is a point-countable cover, for each $x \in X$ let $\alpha_{x,n} < \omega_1$ yield ordinals such that $x \in U_{\alpha_{x,n}}$ for $n < \omega$.

Let $M : \boldsymbol{X} \times \omega \to \mathcal{K}(\boldsymbol{X})$ as follows:

$$M(\boldsymbol{x}, n) = \begin{cases} K_{\{\alpha_{x,m} : m \leq n\}} & : \quad \boldsymbol{x} = \langle x, s\rangle \in X \times 2^{<\omega} \\ K_{\{\alpha\}} & : \quad \boldsymbol{x} = c_\alpha \in C \end{cases}$$

and use M to define the strategy σ for each $\boldsymbol{a} \in \boldsymbol{X}^{<\omega}$:

$$\sigma(\boldsymbol{a}) = L_{|\boldsymbol{a}|} \cup \bigcup_{i < |\boldsymbol{a}|} M(\boldsymbol{a}(i), |\boldsymbol{a}|)$$

Let $\boldsymbol{p} : \omega \to \boldsymbol{X}$ be a legal attack against σ. Then as $\boldsymbol{p}(n) \notin L_n$, for each $\boldsymbol{x} = \langle x, s\rangle \in X \times 2^{<\omega}$, $X \times \{s\}$ is an open neighborhood of $\boldsymbol{x}$ which contains finitely many $\boldsymbol{p}(n)$.

Now consider $\boldsymbol{x} = c_\alpha$ for some $\alpha < \omega_1$, and let $n < \omega$. Then if $\boldsymbol{p}(n) = \langle x, s\rangle$ with $\alpha = \alpha_{x,N}$ for some $N < \omega$, then $\boldsymbol{p}(m) \notin B_{\alpha,0}$ for $\max(n, N) < m < \omega$. Or, if $\boldsymbol{p}(n) = c_\alpha$, then $\boldsymbol{p}(m) \notin B_{\alpha,0}$ for $n < m < \omega$. Otherwise, $\boldsymbol{p}(m) \notin B_{\alpha,0}$ for any $m < \omega$. In any case, $B_{\alpha,0}$ is a neighborhood of $\boldsymbol{x}$ which contains finitely many $\boldsymbol{p}(n)$. Therefore, σ is a winning strategy. $\square$

One might hope then that $\mathscr{K} \underset{\text{tact}}{\not\uparrow} Gru_{K,P}(\boldsymbol{X})$ but $\mathscr{K} \underset{\text{2-tact}}{\uparrow} Gru_{K,P}(\boldsymbol{X})$, giving us our counterexample. However, we will see that in fact, any winning k-mark for $\mathscr{K}$ may be improved to a winning tactic by exploiting the structure of the Cantor tree.

In particular, knowledge of round number does not assist $\mathscr{K}$, since she may force $\mathscr{P}$ to either stay within C, or to seed a growing integer which could be used in place of the round number by forcing her to play outside $L_{|s|+1}$ in response to $\langle x, s\rangle \in X \times 2^{<\omega}$.

Lemma 3.5. *If* $\mathscr{K} \underset{(k+1)\text{-}mark}{\uparrow} Gru_{K,P}(\boldsymbol{X})$*, then* $\mathscr{K} \underset{(k+1)\text{-}tact}{\uparrow} Gru_{K,P}(\boldsymbol{X})$.

Proof. Let σ be a winning $(k+1)$-mark for $\mathscr{K}$ such that $m \leq n$ and $\mathrm{range}(r) \subseteq \mathrm{range}(s)$ implies $\sigma(r, m) \subseteq \sigma(s, n)$. For a sequence p, let $p \restriction^k n = \nu_k(p \restriction n)$ give the last k terms of $p \restriction n$.

Define $r : \boldsymbol{X} \to \omega$ by

$$r(\boldsymbol{x}) = \begin{cases} |s| & : \boldsymbol{x} = \langle x, s\rangle \in X \times 2^{<\omega} \\ 0 & : \boldsymbol{x} \in C \end{cases}$$

and use r to define the $(k+1)$-tactic τ by

$$\tau(\emptyset) = \sigma(\emptyset, 0)$$

$$\tau(\boldsymbol{t}^\frown\langle \boldsymbol{x}\rangle) = L_{r(\boldsymbol{x})+1} \cup \{\boldsymbol{x}\} \cup \sigma(\boldsymbol{t}^\frown\langle \boldsymbol{x}\rangle, r(\boldsymbol{x}) + 1)$$

Let $\boldsymbol{p} : \omega \to \boldsymbol{X}$ be a legal attack by $\mathscr{P}$ against τ. If $\boldsymbol{p}(n) \in C$ for $N < n < \omega$, then since no $\boldsymbol{p}(n)$ may be legally repeated, $\{\{\boldsymbol{p}(n)\} : N < n < \omega\}$ is a discrete collection, making the points $p(n)$ locally finite.

Otherwise, let $f \in \omega^\omega$ be increasing and define $\boldsymbol{q} : \omega \to X \times 2^{<\omega}$ such that $\boldsymbol{q}(i) = \boldsymbol{p}(f(i))$, and $\boldsymbol{p}(j) \in X \times 2^{<\omega}$ implies there is some i with $j = f(i)$. It follows that

$$\boldsymbol{q}(0) = \boldsymbol{p}(f(0)) \notin \bigcup_{m \leq f(0)} \tau(\boldsymbol{p} \restriction^{k+1} m) \supseteq \tau(\emptyset) = \sigma(\emptyset, 0)$$

Denoting $\boldsymbol{q}(n) = \langle x_n, s_n\rangle$, it's trivial to note that $|s_0| \geq 0$. Assuming that $|s_m| \geq m$ for $m \leq n$, it then follows that

$$\boldsymbol{q}(n+1) = \boldsymbol{p}(f(n+1)) \notin \bigcup_{m \leq f(n+1)} \tau(\boldsymbol{p} \restriction^{k+1} m)$$

$$\supseteq \bigcup_{m \leq n} \tau(\boldsymbol{q} \restriction^{k+1} m) \supseteq \sigma(\emptyset, 0) \cup \bigcup_{m < n} \sigma(\boldsymbol{q} \restriction^{k+1} (m+1), |s_m| + 1)$$

$$\supseteq \sigma(\emptyset, 0) \cup \bigcup_{m < n} \sigma(\boldsymbol{q} \restriction^{k+1} (m+1), m + 1)$$

and

$$\boldsymbol{q}(n+1) \notin \tau(\boldsymbol{q} \restriction^{k+1} (n+1)) \supseteq L_{r(\boldsymbol{q}(n))} = L_{|s_n|+1}$$

gives $|s_{n+1}| \geq |s_n| + 1 \geq n + 1$. Thus $\boldsymbol{q}$ is a legal attack on the winning $(k+1)$-Markov strategy σ, so the points $q(n)$ are locally finite, and it follows that the points $p(n)$ are also locally finite. $\square$

Corollary 3.6. $\boldsymbol{X}$ *is* σ*-metacompact if and only if* $\boldsymbol{X}$ *is metacompact.*

On the other hand, recalling a maximum of $k+1$ moves is only as good as recalling the most recent move for $\mathscr{K}$, since $\mathscr{P}$ may always choose to burn k out of every $k+1$ moves by moving down an antichain of the Cantor tree. Such movement would certainly be locally finite (and therefore not directly benefit $\mathscr{P}$), but would at least succeed in overloading $\mathscr{K}$'s limited memory.

Lemma 3.7. *If* $\mathscr{K} \underset{(k+1)\text{-}tact}{\uparrow} Gru_{K,P}(\boldsymbol{X})$, *then* $\mathscr{K} \underset{tact}{\uparrow} Gru_{K,P}(\boldsymbol{X})$.

Proof. Let σ be a winning $(k+1)$-tactical strategy, and without loss of generality assume it ignores order.

Define $F(x_0, \dots, x_k, n) \in [C]^{<\omega}$ and $m(x_0, \dots, x_k, n) \in \omega \setminus (n+1)$, both increasing on n, such that for each $\langle x_0, \dots, x_k \rangle \in X^{k+1}$,

$$\bigcup_{s_0,\dots,s_k \in 2^{\leq n}} \sigma(\langle x_0, s_0\rangle, \dots, \langle x_k, s_k\rangle) \subseteq K'_{F(x_0,\dots,x_k,n)} \cup L_{m(x_0,\dots,x_k,n)}$$

Select an arbitrary point $y \in X$. Let

$$M^0(x,n) = n$$

$$M^{i+1}(x,n) = m(x, y, \dots, y, M^i(x,n)+1)$$

and define the tactical strategy τ as follows:

$$\tau(\emptyset) = \sigma(\emptyset)$$

$$\tau(\langle c_\alpha \rangle) = \{c_\alpha\}$$

$$\tau(\langle\langle x, s\rangle\rangle) = K'_{F(x,y,\dots,y,M^k(x,|s|)+1)} \cup L_{m(x,y,\dots,y,M^k(x,|s|)+1)}$$

Let $\boldsymbol{p} : \omega \to \boldsymbol{X}$ be a legal attack against τ, and assume $\boldsymbol{p}(n) = \langle x_n, s_n \rangle \in X \times 2^{<\omega}$. Then consider the attack $\boldsymbol{q} : \omega \to X \times 2^{<\omega}$ against σ defined by, for $n < \omega$ and $i < k$,

$$\boldsymbol{q}((k+1)n) = \boldsymbol{p}(n) = \langle x_n, s_n \rangle$$

$$\boldsymbol{q}((k+1)n + (i+1)) = \langle y, a_{M^{i+1}(x_n,|s_n|)} \rangle$$

Since

$$\langle x_{n+1}, s_{n+1} \rangle = \boldsymbol{p}(n+1) \notin \tau(\langle \boldsymbol{p}(n) \rangle) \supseteq L_{M^{k+1}(x_n,|s_n|)+1}$$

it follows that $|s_{n+1}| \geq M^{i+1}(x_n, |s_n|) + 1$ for $i < k$; furthermore,

$$|s_n| \leq M^i(x_n,|s_n|) < M^i(x_n,|s_n|)+1 \leq M^{i+1}(x_n,|s_n|) < M^{i+1}(x_n,|s_n|)+1 \leq |s_{n+1}|$$

so the second coordinate of $\boldsymbol{q}(n)$ is always strictly increasing.

By the definition of τ,

$$\boldsymbol{q}((k+1)n) = \boldsymbol{p}(n) \notin \bigcup_{m \leq n} \tau(\boldsymbol{p} \restriction^1 m) \supseteq \bigcup_{m \leq (k+1)n} \sigma(\boldsymbol{q} \restriction^{k+1} m)$$

Since

$$\boldsymbol{q}((k+1)n + (i+1)) = \langle y, a_{M^{i+1}(x_n,|s_n|)} \rangle \in X \times A$$

it follows that $\boldsymbol{q}((k+1)n+(i+1)) \notin K'_F$ for any $F \in [\omega_1]^{<\omega}$.

Then it's sufficient to note that

$$|a_{M^{i+1}(x_n,|s_n|)}| = M^{i+1}(x_n,|s_n|)+1 > m(x_n,y,\dots,y,M^i(x_n,|s_n|)+1)$$

to show that

$$\boldsymbol{q}((k+1)n+(i+1)) = \langle y, a_{M^{i+1}(x_n,|s_n|)}\rangle \notin L_{m(x_n,y,\dots,y,M^i(x_n,|s_n|)+1)}$$

and therefore $\boldsymbol{q}((k+1)n+(i+1))$ is a legal move.

As a result, $\boldsymbol{q}$ is a legal attack against σ, and $\{\{\boldsymbol{q}(n)\} : n < \omega\} \supseteq \{\{\boldsymbol{p}(n)\} : n < \omega\}$ are both locally finite.

Finally, if the range of $\boldsymbol{p}$ intersects C, those moves may be safely ignored as they cannot be repeated and lay in a closed discrete set, so the proof is complete. □

So our hopes for a counterexample to our main question in $\boldsymbol{X}$ are thusly defeated. Our consolation prize will be to show that for a certain choice of X and $\{U_\alpha : \alpha < \omega_1\}$, $\boldsymbol{X}$ is at least an example of a space which allows a winning strategy for $Gru_{K,P}(\boldsymbol{X})$ but no winning k-tactics or k-marks.

To this end, we will show that $\{U_\alpha : \alpha < \omega_1\}$ may be chosen such that $\boldsymbol{X}$ is not metacompact, and therefore $\mathscr{K}$ lacks a k-Markov strategy for any $k < \omega$.

Theorem 3.8. *$\boldsymbol{X}$ is metacompact if and only if $\{U_\alpha : \alpha < \omega_1\}$ is σ-point-finite.*

Proof. Let $\omega_1 = \bigcup_{n<\omega} A_n$ such that $\{U_\alpha : \alpha \in A_n\}$ is point-finite for each $n < \omega$.

Let $\mathcal{U}$ be a cover of $\boldsymbol{X}$, and for each $s \in 2^\omega$ let $\mathcal{V}_s$ be a finite open refinement of $\mathcal{C}$ covering the compact set $X \times \{s\}$. Then let $\mathcal{W}_n = \{B_{\alpha,n_\alpha} : \alpha \in A_n\}$ be an open refinement of $\mathcal{C}$ for each $n < \omega$, and note that it is point-finite. It follows that $\mathcal{U}' = \bigcup_{s\in 2^{<\omega}} \mathcal{V}_s \cup \bigcup_{n<\omega} \mathcal{W}_n$ is an open σ-point-finite refinement of $\mathcal{U}$, so $\boldsymbol{X}$ is σ-metacompact, and therefore it is metacompact.

For the other direction, consider the open cover $\mathcal{U} = \{B_{\alpha,0} : \alpha < \omega_1\}$ of the closed subset C of metacompact $\boldsymbol{X}$, and let $\{B_{\alpha,n_\alpha} : \alpha < \omega_1\}$ be a point-finite refinement. Then $\mathcal{U}_s = \{U_\alpha : c_\alpha \restriction n_\alpha = s\}$ is point-finite for $s \in 2^{<\omega}$ and $U_\alpha \in \mathcal{U}_{c_\alpha \restriction n_\alpha}$ for each $\alpha < \omega_1$. Therefore $\mathcal{U} = \bigcup_{s\in 2^{<\omega}} \mathcal{U}_s$ is σ-point-finite. □

Theorem 3.9. *There exists a compact, zero-dimensional topological space X with a clopen cover $\{U_\alpha : \alpha < \omega_1\}$ of distinct sets which is not σ-point-finite.*

Proof. Let Y be a zero-dimensional Corson compact space which is not Eberlein compact; one such space was constructed in [4]. Let $X = Y^2$. Then by characterizations of Corson and Eberlien compacts found in [2], $Y^2 \setminus \Delta$ is meta-Lindelöf but not σ-metacompact, so there exists a point-countable clopen cover $\mathcal{U}$ of $Y^2 \setminus \Delta$ which is not σ-point-finite. Then $\mathcal{U} \cup \{X\}$ is a point-countable clopen cover of X which is not σ-point-finite. □

Corollary 3.10. *There exists a locally compact space X such that $\mathscr{K} \uparrow Gru_{K,P}(X)$ but $\mathscr{K} \underset{k\text{-}mark}{\not\uparrow} Gru_{K,P}(X)$ for all $k < \omega$.*

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