Definition 1. A space X is *strong Eberlein compact* if it embeds in $\sigma 2^\kappa = \{x \in 2^\kappa : |\{\alpha : x(\alpha) = 1\}| < \omega\}$.

Theorem 2 (Gruenhage)**.** *For compact spaces X, X is strong Eberlein compact if and only if X is scattered and X is a W-space ($\mathscr{O} \uparrow Gru^{\rightarrow}_{O,P}(X, x)$ for all $x \in X$).*

Theorem 3. $\mathscr{D} \underset{tact}{\uparrow} Bell^{\rightarrow}_{D,P}(\sigma 2^\kappa)$.

Proof. Let $\mathrm{supp}(x) = \{\alpha : x(\alpha) = 1\} \in [\kappa]^{<\omega}$.

Define the tactic σ for $\mathscr{D}$ such that

$$\sigma(\langle x \rangle) = \bigcap \{P_\alpha(\Delta) : \alpha \in \mathrm{supp}(x)\}$$

Fix a legal attack $p : \omega \to \sigma 2^\kappa$, and let $\alpha < \kappa$. If $p_\alpha : \omega \to \sigma 2^\kappa$ defined by $p_\alpha(n) = p(n)(\alpha)$ converges for each $\alpha < \kappa$, then σ is a winning tactic. So assume $p_\alpha(n) = 1$ for some n, and as $\alpha \in \mathrm{supp}(p(n))$, $\sigma(p(n)) \subseteq P_\alpha(\Delta)$. As p is a legal attack, it follows that $p_\alpha(m) = p_\alpha(m + 1)$ for all $m > n$, so p_α converges. Otherwise $p_\alpha(n) = 0$ for all n so p_α converges to 0. $\square$

Corollary 4. *If X is strong Eberlein compact, then $\mathscr{D} \underset{tact}{\uparrow} Bell^{\rightarrow}_{D,P}(X)$.*

Theorem 5. *If X contains a copy of the Cantor set, then $\mathscr{D} \underset{tact}{\not\uparrow} Bell^{\rightarrow}_{D,P}(X)$.*

Proof. The result follows from showing that $\mathscr{D} \underset{\text{tact}}{\not\uparrow} Bell^{\rightarrow}_{D,P}(2^\omega)$ (any copy of the Cantor set within a Hausdorff space is a compact and thus closed subspace). Let σ be a tactic for $\mathscr{D}$ in $Bell^{\rightarrow}_{D,P}(2^\omega)$ and let $D_k = \{\langle f, g \rangle : f \upharpoonright k = g \upharpoonright k\}$. Since $\{D_k : k < \omega\}$ is a base for the uniformity on 2^ω, we may fix $k(f) < \omega$ for each $f \in 2^\omega$ such that $D_{k(f)} \subseteq \sigma(\langle f \rangle)$.

Then there exists $k < \omega$ such that $\{f : k = k(f)\}$ is uncountable, and therefore there exist distinct f, g such that $k = k(f) = k(g)$ and $f \upharpoonright k = g \upharpoonright k$. Then $p : \omega \to 2^\omega$ defined by $p(2n) = f$ and $p(2n + 1) = g$ is an attack against σ which obviously doesn't converge. This attack is legal since $f \in D_k[g] \subseteq \sigma(\langle g \rangle)[g]$ and $g \in D_k[f] \subseteq \sigma(\langle f \rangle)[f]$. $\square$

Lemma 6. *Every non-scattered Corson compact space contains a homeomorphic copy of the Cantor set.*

Proof. Every non-scattered space contains a closed subspace without isolated points. Let X be such a subspace, and assume that this Corson compact is embedded in $\Sigma\mathbb{R}^\kappa$. Let $B_{\alpha,\epsilon}(x) = \{y : d(x(\alpha), y(\alpha)) < \epsilon\}$. For each $x \in X$ and $n < \omega$, let $\beta(x, n) < \kappa$ be defined such that $\{\alpha : x(\alpha) \neq 0\} = \{\beta(x, n) : n < \omega\}$.

Choose an arbitrary $x_\emptyset \in X$ and $\epsilon_0 > 0$, and and let $A_0 = \emptyset$.

Suppose then that for some $n < \omega$, $x_s \in X$ is defined for all $s \in 2^n$, and $\epsilon_n > 0$ and $A_n \in [\kappa]^{<\omega}$ are defined. Since each x_s is not isolated in X, let U_s be the open set

$$U_s = X \cap \bigcap_{\alpha \in A_{|s|}} B_{\alpha, \epsilon_{|s|}}(x_s)$$

and choose $x_{s^\frown\langle 0 \rangle}, x_{s^\frown\langle 1 \rangle} \in U_s$ distinct. Then let $\alpha_s < \kappa$ such that $x_{s^\frown\langle 0 \rangle}(\alpha_s) \neq x_{s^\frown\langle 1 \rangle}(\alpha_s)$. Let

$$A_{n+1} = \{\alpha_s : s \in 2^{\leq n}\} \cup \{\beta(x_s, i) : s \in 2^{\leq n}, i \leq n\}$$

Then choose $0 < \epsilon_{n+1} < \frac{1}{2}\epsilon_n$ such that

$$B_{\alpha_s, \epsilon_{n+1}}(x_s{}^\frown\langle 0 \rangle) \cap B_{\alpha_s, \epsilon_{n+1}}(x_s{}^\frown\langle 1 \rangle) = \emptyset$$

and

$$\overline{\bigcap_{\alpha \in A_{n+1}} B_{\alpha, \epsilon_{n+1}}(x_s{}^\frown\langle 0 \rangle)} \cup \overline{\bigcap_{\alpha \in A_{n+1}} B_{\alpha, \epsilon_{n+1}}(x_s{}^\frown\langle 1 \rangle)} \subseteq \bigcap_{\alpha \in A_n} B_{\alpha, \epsilon_n}(x_s)$$

for all $s \in 2^n$.

Let $x_f = \lim_{n<\omega} x_{f\restriction n} \in X$ for each $f \in 2^\omega$. We claim $C = \{x_f : f \in 2^\omega\}$ is a copy of the Cantor set. This will follow if we can show that $\{U_s : s \in 2^{<\omega}\}$ is a base for C, since it has the structure of the Cantor tree.

Consider x_f for some $f \in 2^\omega$, and a subbasic open ball $B_{\alpha,\epsilon}(x_f)$. Observe that $x_f \in \bigcap_{n<\omega} U_{f\restriction n}$ since $x_{f\restriction n} \in U_{f\restriction m}$ for all $m < n < \omega$.

If $\alpha \in \{\beta(x_s, n) : s \in 2^{<\omega}, n < \omega\}$, choose $k < \omega$ with $\alpha \in A_k$. Then choose $l < \omega$ such that $\epsilon_l < \epsilon$. Then $U_{f\restriction(l+k)} \subseteq B_{\alpha,\epsilon}(x_f)$.

Otherwise, $x_s(\alpha) = 0$ for all $s \in 2^{<\omega}$, so $x_g(\alpha) = 0$ for all $g \in 2^\omega$ and therefore $C \subseteq B_{\alpha,\epsilon}(x_f)$. □

Corollary 7. *For compact spaces X, X is strong Eberlein compact if and only if $\mathscr{D} \underset{tact}{\uparrow} Bell^{\rightarrow}_{D,P}(X)$.*

Proof. Suppose X is not strong Eberlien compact; then X is either not a W-space or not scattered. If $\mathscr{D} \not\uparrow Bell^{\rightarrow}_{D,P}(X)$, then the result follows immediately, which only leaves non-scattered proximal compact spaces to be considered. But non-scattered proximal compacts are non-scattered Corson compacts, and thus contain a copy of the Cantor set, so the result follows from Theorem 5. □

Definition 8. A space X is *Eberlein compact* if it embeds in $\Sigma^\star\mathbb{R}^\kappa = \{x \in 2^\kappa : |\{\alpha : |x(\alpha)| \geq \epsilon\}| < \omega$ for all $\epsilon > 0\}$.

Theorem 9. $\mathscr{D} \underset{mark}{\uparrow} Bell_{D,P}^{\rightarrow}(\Sigma^\star\mathbb{R}^\kappa)$.

Proof. Let $\mathrm{supp}_\epsilon(x) = \{\alpha : |x(\alpha)| \geq \epsilon\} \in [\kappa]^{<\omega}$. Let D_ϵ be the entourage of the diagonal formed by balls of radius ϵ. Finally, for $z \in \mathbb{R}$ and $n < \omega$, let

$$\epsilon(z, n) = \min\left(\frac{1}{2^n}, \frac{|z|}{2}\right)$$

noting in particular that if $|z| \geq \frac{1}{2^n}$, then for any z' with $|z' - z| < \epsilon(z, n)$, it follows that $|z'| \geq \frac{1}{2^{n+1}}$.

Define the mark σ for $\mathscr{D}$ such that

$$\sigma(\langle x \rangle, n) = \bigcap\{P_\alpha(D_{\epsilon(x(\alpha),n)}) : \alpha \in \mathrm{supp}_{2^{-n}}(x)\}$$

Fix a legal attack $p : \omega \to \Sigma^\star\mathbb{R}^\kappa$, and let $\alpha < \kappa$. If $p_\alpha : \omega \to \Sigma^\star\mathbb{R}^\kappa$ defined by $p_\alpha(n) = p(n)(\alpha)$ converges for each $\alpha < \kappa$, then σ is a winning mark. So assume $|p_\alpha(n)| \geq \frac{1}{2^n}$ for some n, and as $\alpha \in \mathrm{supp}(p(n))$, $\sigma(p(n)) \subseteq P_\alpha(D_{\epsilon(p_\alpha(n),n)})$. As p is a legal attack, it follows that $|p_\alpha(n+1) - p_\alpha(n)| < \epsilon(p_\alpha(n), n)$, and therefore $|p_\alpha(n+1)| \geq \frac{1}{2^{n+1}}$.

Thus $|p_\alpha(m)| \geq \frac{1}{2^m}$ for all $m \geq n$, and as $\alpha \in \mathrm{supp}(p(m))$, $\sigma(p(m)) \subseteq P_\alpha(D_{\epsilon(p_\alpha(m),m)})$. As p is a legal attack, it follows that $|p_\alpha(m+1) - p_\alpha(m)| < \epsilon(p_\alpha(m), m) \leq \frac{1}{2^m}$, and therefore p_α is a Cauchy sequence and converges.

Otherwise $p_\alpha(n) < 2^{-n}$ for all n, so p_α converges to 0. □

Theorem 10. *If X is compact and $\mathscr{D} \underset{mark}{\uparrow} Bell_{D,P}^{\rightarrow}(X)$, then $X^2 \setminus \Delta$ is σ-metacompact.*

Proof. Recall that uniformizable spaces have a base of cozero (in particular, F_σ) sets. Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be an open cover of $X^2 \setminus \Delta$; without loss of generality assume it contains F_σ sets with compact closures. Let σ be a Markov strategy for $\mathscr{D}$ in $Bell_{D,P}^{\rightarrow}(X)$; without loss of generality assume that $\sigma(\langle x \rangle, n) \supseteq \overline{\sigma(\langle x \rangle, n+1)}$. Let $\tau(x, y, n) = \sigma(\langle x \rangle, n) \cap \sigma(\langle y \rangle, n)$ for each $x, y \in X$.

Note that if there is $\langle x, y \rangle \in X^2 \setminus \Delta$ such that $\langle x, y \rangle \in \tau(x, y, n)$ for all n, then $\langle x, y, x, y, \ldots \rangle$ is a legal counterattack to σ which does not converge. Otherwise, for each $\langle x, y \rangle \in X^2 \setminus \Delta$, let $n_{x,y}$ satisfy $\langle x, y \rangle \notin \tau(x, y, n_{x,y})$, and let $\upsilon(x, y) = \tau(x, y, n_{x,y})$.

Let $K(\langle x, y \rangle) = X^2 \setminus \upsilon(x, y)$.

We wish to find $\mathcal{U}_\alpha \subseteq \mathcal{U}$ such that

- $|\mathcal{U}_\alpha| \leq \omega$
- $\{U_\beta : \beta < \alpha\} \subseteq \bigcup_{\beta<\alpha} \mathcal{U}_\beta$
- Let $N_\alpha = (\bigcup \mathcal{U}_\alpha) \setminus (\bigcup \bigcup_{\beta<\alpha} \mathcal{U}_\beta)$. There exists $S_\alpha \in [N_\alpha]^{\leq\omega}$ such that

$$N_\alpha \subseteq \bigcup_{\langle x,y\rangle \in S_\alpha} \upsilon(x,y) \subseteq \bigcup \mathcal{U}_\alpha$$

Let $\mathcal{U}_0 = \emptyset$ and $\mathcal{U}_\alpha = \emptyset$ for limit ordinals α.

Let $S_{\alpha+1,0} = \emptyset$ and $\mathcal{U}_{\alpha+1,0} = \{U_\alpha\}$. □