(joint work with Alan Dow)

Definition 1. Two functions f, g are almost compatible if $\{a \in \text{dom} f \cap \text{dom} g : f(a) \neq g(a)\}$ is finite.

Definition 2. $S'(\theta)$ states that there exists a cofinal family $\mathcal{S} \subseteq [\theta]^\omega$ and a collection of pairwise almost compatible finite-to-one functions $\{f_S \in \omega^S : S \in \mathcal{S}\}$

Definition 3. $S(\theta)$ strengthens $S'(\theta)$ by requiring the collection to contain one-to-one functions.

We wish to show that Scheeper's original $S(\theta)$ is strictly stronger than $S'(\theta)$.

Definition 4. A topological space is said to be ω-bounded if each countable subset of the space has compact closure.

Theorem 5. *For each $n \in \omega$, there is a locally countable, ω-bounded topology on ω_n. Note that this means that the closure of any set has the same cardinality and weight as the set.*

To prove the theorem, we must actually prove a stronger lemma.

Lemma 6. *Assume that X has cardinality at most ω_n (for any $n \in \omega$), and is locally countable, locally compact, and the closure of each set has the same cardinality as the set. Then X has an ω-bounded extension with the same properties.*

Proof. We prove this by induction on n. In fact, we make our inductive statement that if $\tilde{X}$ is the extension of X, then $\tilde{X} \setminus X$ also has cardinality ω_n. If $n = 0$, then we can just take the free union of two copies of X and then the one-point compactification. So suppose $n > 0$ and that X is such a topology on the ordinal ω_n. For each $\alpha < \omega_n$, the closure of the initial segment α is bounded by some γ_α. Also, because X is locally countable, γ_α can be chosen so that α is contained in the interior of γ_α. There is a cub $C \subset \omega_n$ with the property that for each $\delta \in C$ and $\alpha < \delta$, γ_α is also less than δ. This implies that for each $\delta \in C$, the initial segment δ is open, and if δ has uncountable cofinality, then δ is clopen.

The proof will be easier to visualize if we now identify the points of X with the point set $\omega_n \times \{0\}$ and we will add the points $\omega_n \times \{1\}$ to create the extension. By induction on $\lambda \in C$ we define a topology on $\omega_n \times \{0\} \cup \lambda \times \{1\}$ so that $\omega_n \times \{0\}$ is an open subset. We also ensure, by induction, for each $\alpha < \lambda$, the closure of $\alpha \times 2$ is an ω-bounded subset of $\lambda \times 2$.

In the case that $n = 1$, then choose any sequence $\lambda_n : n \in \omega$ increasing cofinal in λ. If λ is a limit in C, then we simply take the topology we have constructed so far on $\lambda \times 2$ and there's nothing more needs to be done. Otherwise we may assume that λ_0 is the predecessor of $\lambda \in C$ and we set Y_λ to equal the countable set $\overline{\lambda} \setminus \lambda$. For convenience, and with no loss, we assume that λ itself is a limit of limits. And we have a topology on

$$\lambda_0 \times 2 \cup (\lambda \cup Y_\lambda) \times \{0\} \ .$$

Recursively choose clopen sets U_n in this topology so that $\lambda_0 \times 2 \subset U_0$, $U_n \cup \lambda_{n+1} \times \{0\}$ is contained U_{n+1} while U_{n+1} is disjoint from Y_λ. It is easy to see that we can have all the points in $(\lambda \setminus \{\lambda_n : n \in \omega\}) \times \{1\}$ be isolated, and arrange that $(\lambda_n, 1)$ is the point at infinity in the one-point compactification $U_n \cup (\lambda_n \times \{1\})$.

Now we handle the case $n > 1$ and we can shrink C and now assume that C is the closure of $\{\lambda \in C : \mathrm{cf}(\lambda) > \omega\}$. We again proceed by induction on $\lambda \in C$. If λ is a limit in C, then there is nothing to do: we simply have defined an appropriate topology on $\omega_n \times \{0\} \cup \lambda \times \{1\}$ so that for each $\mu \in C \cap \lambda$ with $\mathrm{cf}(\mu) > \omega$, $\mu \times 2$ is a clopen ω-bounded subspace. In case λ is not a limit of C, then λ has uncountable cofinality and a predecessor $\mu \in C$. We therefore have that $\lambda \times \{0\}$ is clopen in $\omega_n \times \{0\}$. We apply the induction hypothesis to the space $\lambda \times \{0\} \cup \mu \times 2$ to choose the topology on $\lambda \times 2$.

□

Definition 7. A Kurepa family $\mathcal{K} \subseteq [\theta]^\omega$ on θ satisfies that $\mathcal{K} \restriction A = \{K \cap A : K \in \mathcal{K}\}$ is countable for each $A \in [\theta]^\omega$.

Corollary 8. *There exists a Kurepa family cofinal in $[\omega_k]^\omega$ for each $k < \omega$.*

Proof. This is actually a corollary of an observation of Todorcevic communicated by Dow in [TODO cite Gen Prog in Top I]: if every Kurepa family of size at most θ extends to a cofinal Kurepa family, then the same is true of θ^+. So the result follows as every Kurepa family $\mathcal{K}$ of size ω extends to the cofinal Kurepa family $[\bigcup \mathcal{K}]^\omega$.

We may alternatively obtain the result from the previous topological argument by using the family $\mathcal{K}$ of compact sets in the constructed topology on ω_k as our witness. Of course, every Lindelöf set in a locally countable space is countable. Thus $\mathcal{K}$ is cofinal in $[\omega_k]^\omega$ since for every countable set A, $\overline{A}$ is compact and countable. It is Kurepa since for every countable set A, let (TODO) □

Theorem 9. *$S'(\theta)$ holds whenever there exists a cofinal Kurepa family on θ.*

Proof. Let $k < \omega$, and $\mathcal{K} = \{K_\alpha : \alpha < \kappa\}$ be a cofinal Kurepa family on θ. We should define $f_\alpha : K_\alpha \to \omega$ for each $\alpha < \kappa$.

Suppose we've defined pairwise almost compatible $\{f_\beta : \beta < \alpha\}$. To define f_α, we first recall that $\mathcal{K} \restriction K_\alpha$ is countable, so we may choose $\beta_n < \alpha$ for $n < \omega$ such that $\{K_\beta : \beta < \alpha\} \restriction K_\alpha \setminus \{\emptyset\} = \{K_\alpha \cap K_{\beta_n} : n < \omega\}$. Let $K_\alpha = \{\delta_{i,j} : i \leq \omega, j < w_i\}$ where $w_i \leq \omega$ for each $i \leq \omega$, $K_\alpha \cap \left(K_{\beta_n} \setminus \bigcup_{m<n} K_{\beta_m}\right) = \{\delta_{n,j} : j < w_n\}$, and $K_\alpha \setminus \bigcup_{n<\omega} K_{\beta_n} = \{\delta_{\omega,j} : j < w_\omega\}$. Then let $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ for $n < \omega$ and $f_\alpha(\delta_{\omega,j}) = j$ otherwise.

We should show that f_α is finite-to-one. Let $n < \omega$. We need only worry about $\delta_{m,j}$ for $m \leq n$ since $f_\alpha(\delta_{m,j}) \geq m$. Since each f_{β_m} is finite-to-one, $f_{\beta_m}(\delta_{m,j}) \leq n$ for only finitely many j. Thus f_α maps to n only finitely often.

We now want to demonstrate that $f_\alpha \sim f_{\beta_n}$ for all $n < \omega$. We again need only concern ourselves with $\delta_{m,j}$ for $m \leq n$ since otherwise $\delta_{m,j} \notin K_{\beta_n}$. For $m = n$, we have $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ which differs from $f_{\beta_n}(\delta_{n,j})$ for only the finitely many j which are mapped below n by f_{β_n}. For $m < n$ and $\delta_{m,j} \in K_{\beta_n}$,we have $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$ which can only differ from $f_{\beta_n}(\delta_{m,j})$ for only the finitely many j which are mapped below m by f_{β_m} or the finitely many j for which the almost compatible $f_{\beta_n} \sim f_{\beta_m}$ differ. □

Corollary 10. *$S'(\omega_k)$ holds for all $k < \omega$.*

As noted in [TODO cite Dow], Jensen's one-gap two-cardinal theorem under $V = L$ [TODO cite] can be used to show that there exist cofinal Kurepa families on every cardinal.

Corollary 11 ($V = L$)**.** *$S'(\theta)$ holds for all cardinals.*

In particular, $S(\omega_2)$ fails under CH, showing the two are unique. Actually, CH is not required to have $S(\omega_2)$ fail.

Theorem 12. *Adding ω_2 Cohen reals to a model of CH forces $\mathfrak{c} = \omega_2$ and $\neg S(\omega_2)$.*

Proof. TODO add Alan's proof □