LEXICOGRAPHIC PRODUCTS AND IDEMPOTENT INVERSE LIMITS

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Abstract. The inverse limit $\varprojlim\{X, \gamma, L\}$ may be characterized as the quotient of a generalized lexicogrphic product of the compactification $\hat{L}$ and X. A special case of this fact is used to show that when f satisfies condition Γ and L is uncountable, the inverse limit $\varprojlim\{X, f, L\}$ cannot be Corson compact, and therefore cannot be metrizable.

1. Introduction

All topological spaces are assumed to be T_1.

Definition 1. Let L, M be total orders. Define their *lexicographic product* $L \times_{\text{lex}} M$ by the total ordering $\langle l, m\rangle < \langle l', m'\rangle$ if and only if $l < l'$ or both $l = l'$ and $m < m'$.

Definition 2. Let L be a total order, and let X be a topological space with two distinguished points $0, 1$. Define their *generalized lexicographic product* $L \times_{\text{gl}} X$ to have the topology generated by the following neighborhood bases:

- For $l \in L$ and $t \in X \setminus \{0, 1\}$, $\langle l, t\rangle$ has neighborhoods of the form $\{l\} \times U \setminus \{0, 1\}$ for all neighborhoods U of t in X.
- For non-minimal $l \in L$, $\langle l, 0\rangle$ has neighborhoods of the form $(\{l^-\} \times V \setminus \{0\}) \cup ((l^-, l) \times X) \cup (\{l\} \times U \setminus \{1\})$ such that $l^- < l$, U is a neighborhood of 0 in X, and V is either empty or a neighborhood of 1 in X.
- For $l = \min L$, $\langle l, 0\rangle$ has neighborhoods of the form $(\{l\} \times U \setminus \{1\})$ such that U is a neighborhood of 0 in X.
- For non-maximal $l \in L$, $\langle l, 1\rangle$ has neighborhoods of the form $(\{l\} \times V \setminus \{0\}) \cup ((l, l^+) \times X) \cup (\{l^+\} \times U \setminus \{1\})$ such that $l < l^+$, U is either empty or a neighborhood of 0 in X, and V is a neighborhood of 1 in X.
- For $l = \max L$, $\langle l, 1\rangle$ has neighborhoods of the form $(\{l\} \times V \setminus \{0\})$ such that V is a neighborhood of 1 in X

Proposition 3. *Let L, M be LOTS such that M has distinguished points $0 = \min M$ and $1 = \max M$. Then $L \times_{\text{gl}} M \cong L \times_{\text{lex}} M$.*

Definition 4. Let X be a topological space with two distinguished points $0, 1$. Define $\gamma \subseteq X^2$ to be the idempotent bonding relation defined by $\gamma(0) = X$ and $\gamma(t) = \{1\}$ for all $t \in X \setminus \{0\}$.

Theorem 5. *Let L be a total order, and let X be a topological space with two distinguished points $0, 1$. Then $\varprojlim\langle X, \gamma, L\rangle = (\hat{L} \times_{\text{gl}} X)/\sim$, where the equivalence relation $\sim$ is given by*

- *$\langle A, t\rangle \sim \langle A, 1\rangle$ for all $A \in \hat{L} \setminus \dot{L}$.*

- $\langle A,1\rangle \sim \langle B,0\rangle$ *whenever* $|B\setminus A|=1$.

Proof. We first observe that for $\vec{x}\in \varprojlim\langle X,\gamma,L\rangle$ there exists a maximal left-closed set $A_{\vec{x}}\in\hat{L}$ such that $\vec{x}[\operatorname{int}A_{\vec{x}}]=\{1\}$ and $\vec{x}[\operatorname{ext}A_{\vec{x}}]=\{0\}$. Note that if $A_{\vec{x}}$ has a maximum element, i.e. $A_{\vec{x}}\in\dot{L}$, then $\vec{x}(\max A_{\vec{x}})$ may be valued at almost any point of X. The lone exception occurs when $\max A_{\vec{x}}$ has a successor in L; $\vec{x}(\max A_{\vec{x}})=1$ would then violate the maximality of $A_{\vec{x}}$.

Define $\theta:\varprojlim\langle X,\gamma,L\rangle\to(\hat{L}\times_{\mathrm{gl}}X)/\sim$ by $\theta(\vec{x})=\langle A_{\vec{x}},1\rangle$ whenever $A_{\vec{x}}\in\hat{L}\setminus\dot{L}$, and $\theta(\vec{x})=\langle A_{\vec{x}},\vec{x}(\max A_{\vec{x}})\rangle$ otherwise. It's then evident from our construction of $A_{\vec{x}}$ that θ is a bijection.

Let $B=(\leftarrow,l]\in\dot{L}$. If $t\in X\setminus\{0,1\}$, note $\langle B,t\rangle$ has a base of neighborhoods of the form $\{B\}\times U\setminus\{0,1\}$ which is equal to $\theta[\pi_l^{-1}[U\setminus\{0,1\}]]$.

Suppose $|B\setminus A|=1$ for $A=(\leftarrow,k]\in\dot{L}$. Then $\langle B,0\rangle$ has a base of neighborhoods of the form $(\{A\}\times V\setminus\{0\})\cup(\{B\}\times U\setminus\{1\})$, which is equal to $\theta[\pi_k^{-1}[V\setminus\{0\}]]\cap\theta[\pi_l^{-1}[U\setminus\{1\}]]$.

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References

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