

**Definition 1.** Let a V-map be a u.s.c. idempotent surjection.

**Definition 2.** For any LOS  $\langle L, \leq \rangle$ , let  $\check{L}$  be the collection of leftward subsets of  $L$  (subsets for which  $b \in L, a \leq b \Rightarrow a \in L$ ) linearly ordered by  $\subseteq$ , and let  $\hat{L}$  be the collection of left-open subsets of  $L$  (leftward subsets which are open) linearly ordered by  $\subseteq$ .

**Proposition 3.**  $\check{L}, \hat{L}$  are compact.

*Proof.* Each subset  $S$  has a supremum  $\cup S$  and infimum  $\cap S$  (or  $\text{int}(\cap S)$ ). □

Note that  $\check{L}$  this is not a “compactification” as  $L$  does not necessarily embed as a dense subspace of  $\check{L}$ : if  $L = I$ , we might attempt to embed  $t \mapsto [0, t)$ , but then note that the subspace topology induces the reverse Sorgenfrey interval as  $([0, s), [0, t]) = ([0, s), [0, t])$  is open. However  $\hat{L}$  is basically the typical way of compactifying a linearly ordered space  $L$  (actually, left-closed is more typical, but this works similarly and fits our applications later), provided  $L$  lacks a maximum element (otherwise the whole space is an [easily removable] isolated point in  $\hat{L}$ ).

**Definition 4.** For any compact LOTS  $K$  with minimum 0 and maximum 1, let  $\gamma$  be the V-map on  $K$  where  $\gamma(0) = K$  and  $\gamma(t) = \{1\}$  for  $t > 0$ .

**Definition 5.** For any LOTS  $M$  with minimum element 0, let  $\nu$  be the V-map on  $M$  where  $\nu(0) = K$  and  $\nu(t) = \{t\}$  for  $t > 0$ .

Note for  $K = M = 2$  that  $\gamma = \nu$ .

**Theorem 6.**  $X = \varprojlim \{2, \nu, L\} \cong \check{L}$

*Proof.* We start by placing an order on  $X$ . Let  $\vec{x} < \vec{y}$  if there exists  $a \in L$  with  $\vec{x}(a) = 0, \vec{y}(a) = 1$ . We claim this is a total order inducing the topology on  $X$ .

We first observe that if  $\vec{x}(b) = 1$ , then for all  $a \leq b$ ,  $\vec{x}(a) \in \nu(1) = \{1\}$ . If  $\vec{x} \neq \vec{y}$ , then assume without loss of generality that  $\vec{x}(a) = 0, \vec{y}(a) = 1$ , so  $\vec{x} < \vec{y}$ . Also, whenever  $\vec{x}(b) = 1$ , we have that  $b < a$ , so  $\vec{y}(b) = 1$ , preventing  $\vec{y} < \vec{x}$ . Finally if  $\vec{x} < \vec{y}$  and  $\vec{y} < \vec{z}$ , take  $a, b$  with  $\vec{x}(a) = 0, \vec{y}(a) = 1, \vec{y}(b) = 0, \vec{z}(b) = 1$ . It follows that  $a < b$  so  $\vec{z}(a) = 1$  and  $\vec{x} < \vec{z}$ .

Consider the basic open set  $B(\vec{x}, F)$  for a finite set  $F \in [L]^{<\omega}$  about the sequence  $\vec{x} \in X$  which contains all sequences  $\vec{y}$  agreeing with  $\vec{x}$  on  $F$ . If  $\vec{x}(a) = 1$  for all  $a \in F$ , then let  $\vec{w} \in X$  be 0 on the maximum of  $F$ , and 1 for anything less. It follows that  $B(\vec{x}, F) = (\vec{w}, \rightarrow)$ . If  $\vec{x}(a) = 0$  for all  $a \in F$ , then let  $\vec{y} \in X$  be 1 on the minimum of  $F$ , and 0 for anything greater. It follows that  $B(\vec{x}, F) = (\leftarrow, \vec{y})$ . Finally if  $\vec{x}(a) = 1$  and  $\vec{x}(b) = 0$  for  $a < b$  in  $F$  and nothing between  $a, b$  is in  $F$ , then let  $\vec{w} \in X$  be 0 on  $a$  and 1 for anything less, and let  $\vec{y} \in X$  be 1 on  $b$  and 0 for anything greater. It follows that  $B(\vec{x}, F) = (\vec{w}, \vec{y})$ .

Let  $\phi$  evaluate each  $\vec{x} \in X \subseteq 2^L$  as the characteristic function for a subset of  $L$ . It's easy to see that  $\phi$  is an order isomorphism between  $\langle X, \leq \rangle$  and  $\langle \check{L}, \subseteq \rangle$ .  $\square$

We introduce an alternate definition of an arbitrarily indexed inverse limit.

**Definition 7.** Let  $\varprojlim^* \{X, f, L\} \subseteq \varprojlim \{X, f, L\}$  satisfy that  $\vec{x}(a) = \lim_{t \rightarrow a} \vec{x}(t)$  for all  $a \in L$  (for any open neighborhood  $U$  of  $\vec{x}(a)$  there is  $b < a$  where  $\vec{x}(t) \in U$  for all  $t \in (b, a]$ ).

**Theorem 8.**  $Y = \varprojlim^* \{2, \nu, L\} \cong \hat{L}$ .

*Proof.* Consider  $Y$  as a subspace of  $X = \varprojlim \{2, \nu, L\}$  with the linear order described above. We claim that if  $\phi$  is the characteristic function for a subset of  $L$ , then  $\text{int} \circ \phi$  is an order isomorphism between  $\langle Y, \leq \rangle$  and  $\langle \hat{L}, \subseteq \rangle$ .

Let  $A$  be a left-open subset of  $L$ . Then let  $\vec{x}(a) = 1$  when  $a \in \text{cl}A$  and 0 otherwise. Then  $\vec{x} \in Y$ ,  $\phi(\vec{x}) = \text{cl}A$ , and  $\text{int} \circ \phi(\vec{x}) = \text{intcl}A = A$ .

Let  $\vec{x}, \vec{y} \in Y$ . If  $\text{int} \circ \phi(\vec{x}) = \text{int} \circ \phi(\vec{y}) = A$ , then  $A$  is a left-open set where  $\vec{x}(a) = \vec{y}(a) = 1$  for  $a \in A$ . It follows that if  $b \in \text{cl}A$ , then as  $\vec{x}, \vec{y} \in Y$ ,  $\vec{x}(b) = \vec{y}(b) = 1$  as well. And if  $c \notin \text{cl}A$ , then as  $c \notin \phi(x)$ ,  $\vec{x}(c) = \vec{y}(c) = 0$ , so  $\vec{x} = \vec{y}$ .

Finally let  $\vec{x} < \vec{y}$ , so there exists  $a \in L$  with  $\vec{x}(a) = 0, \vec{y}(a) = 1$ . Then  $\text{int} \circ \phi(\vec{x}) \subseteq \phi(\vec{x}) \subseteq (\leftarrow, a) \subseteq \text{int} \circ \phi(\vec{y})$ . Thus  $\text{int} \circ \phi$  preserves order.  $\square$

**Corollary 9.**  $\varprojlim \{2, \nu, \kappa\} \cong \varprojlim^* \{2, \nu, \kappa\} \cong \kappa + 1$

*Proof.* Since  $\hat{\kappa} = \kappa + 1$  (actually equals, not just homeomorphic!), we get  $\varprojlim^* \{2, \nu, \kappa\} \cong \kappa + 1$  for free. Then observe that all leftward sets in  $\kappa$  are open:  $A = [0, \min(\kappa + 1 \setminus A))$ . Thus we have  $\varprojlim \{2, \nu, \kappa\} = \check{\kappa} = \hat{\kappa} = \kappa + 1$ .  $\square$

## References