DUAL SELECTION GAMES

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ABSTRACT. Often, a given selection game studied in the literature has a known dual game. In dual games, a winning strategy for a player in either game may be used to create a winning strategy for the opponent in the dual. For example, the Rothberger selection game involving open covers is dual to the point-open game. This extends to a general theorem: if $\{\text{range}(f) : f \in \mathbf{C}(\mathcal{R})\}$ is coinitial in $\mathcal{A}$ with respect to $\subseteq$, where $\mathbf{C}(\mathcal{R}) = \{f \in (\bigcup \mathcal{R})^{\mathcal{R}} : R \in \mathcal{R} \Rightarrow f(R) \in R\}$ collects the choice functions on the set $\mathcal{R}$, then $G_1(\mathcal{A}, \mathcal{B})$ and $G_1(\mathcal{R}, \neg\mathcal{B})$ are dual selection games.

1. Introduction

Definition 1. An *ω-length game* is a pair $G = \langle M, W\rangle$ such that $W \subseteq M^\omega$. The set M is the *moveset* of the game, and the set W is the *payoff set* for the second player.

In such a game G, players I and II alternate making choices $a_n \in M$ and $b_n \in M$ during each round $n < \omega$, and II wins the game if and only if $\langle a_0, b_0, a_1, b_1, \ldots\rangle \in W$.

Often when defining games, I and II are restricted to choosing from different movesets A, B. Of course, this can be modeled with $\langle M, W\rangle$ by simply letting $M = A \cup B$ and adding/removing sequences from W whenever player I/II makes the first "illegal" move.

A class of such games heavily studied in the literature, particularly topology (see [9] and its many sequels), are selection games.

Definition 2. The *selection game* $G_1(\mathcal{A}, \mathcal{B})$ is an ω-length game involving Players I and II. During round n, I chooses $A_n \in \mathcal{A}$, followed by II choosing $B_n \in A_n$. Player II wins in the case that $\{B_n : n < \omega\} \in \mathcal{B}$, and Player I wins otherwise.

Let $\mathcal{P}(Z) = \{z : z \subseteq Z\}$ denote the power set of Z, so $\{B_n : n < \omega\} \in \mathcal{P}(\bigcup \mathcal{A})$ for any choices $B_n \in A_n \in \mathcal{A}$ made by the players. Then for brevity, let

$$G_1(\mathcal{A}, \neg\mathcal{B}) = G_1\left(\mathcal{A}, \mathcal{P}\left(\bigcup \mathcal{A}\right) \setminus \mathcal{B}\right).$$

That is, this is the analogous game where II wins provided $\{B_n : n < \omega\} \notin \mathcal{B}$, and I wins otherwise.

Notation 3. For a set X, let $\mathbf{C}(X) = \{f \in (\bigcup X)^X : x \in X \Rightarrow f(x) \in x\}$ be the collection of all choice functions on X.

Definition 4. Write $X \preceq Y$ if X is coinitial in Y with respect to $\subseteq$; that is, $X \subseteq Y$, and for all $y \in Y$, there exists $x \in X$ such that $x \subseteq y$.

In the context of selection games, we will say $\mathcal{A}'$ is a *selection basis* for $\mathcal{A}$ when $\mathcal{A}' \preceq \mathcal{A}$.

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Definition 5. The set $\mathcal{R}$ is said to be a *reflection* of the set $\mathcal{A}$ if

$$\{\mathrm{range}(f) : f \in \mathbf{C}(\mathcal{R})\}$$

is a selection basis for $\mathcal{A}$.

Put another way, $\mathcal{R}$ is a reflection of $\mathcal{A}$ if $\mathrm{range}(f) \in \mathcal{A}$ for all $f \in \mathbf{C}(\mathcal{R})$, and for each $A \in \mathcal{A}$ there exists $f_A \in \mathbf{C}(\mathcal{R})$ such that $\mathrm{range}(f_A) \subseteq A$.

As we will see, reflections of selection sets are used frequently (but implicitly) throughout the literature to define dual selection games.

We use the following conventions to describe strategies for playing games.

Notation 6. For $f \in B^A$ and $X \subseteq A$, let $f \restriction X$ be the restriction of f to X. In particular, for $f \in B^\omega$ and $n < \omega$, $f \restriction n$ describes the first n terms of the sequence f.

Definition 7. A *strategy* for the first player I (resp. second player II) in a game G with moveset M is a function $\sigma : M^{<\omega} \to M$. This strategy is said to be *winning* if for all possible *attacks* $\alpha \in M^\omega$ by their opponent, where $\alpha(n)$ is played by the opponent during round n, the player wins the game by playing $\sigma(\alpha \restriction n)$ (resp. $\sigma(\alpha \restriction n+1)$) during round n.

That is, a strategy is a rule that determines the moves of a player based upon all previous moves of the opponent. (It could also rely on all previous moves of the player using the strategy, since these can be reconstructed from the previous moves of the opponent and the strategy itself.)

Definition 8. A *predetermined strategy* for the first player I in a game G with moveset M is a function $\sigma : \omega \to M$. This strategy is said to be winning if for all possible attacks $\alpha \in M^\omega$ by their opponent, the first player wins the game by playing $\sigma(n)$ during round n.

So a predetermined strategy ignores all moves of the opponent during the game (all moves were decided before the game began). Such strategies are also known as 0-Markov strategies or 0-Markov tactics. The following definition is similarly also known as a 1-Markov strategy.

Definition 9. A *Markov strategy* for the second player II in a game G with moveset M is a function $\sigma : M \times \omega \to M$. This strategy is said to be winning if for all possible attacks $\alpha \in M^\omega$ by their opponent, the second player wins the game by playing $\sigma(\alpha(n), n)$ during round n.

So a Markov strategy may only consider the most recent move of the opponent, and the current round number. Note that unlike perfect-information or predetermined strategies, a Markov strategy cannot use knowledge of moves used previously by the player (since they depend on previous moves of the opponent that have been "forgotten").

We also consider similar strategies that ignore the round number.

Definition 10. A *constant strategy* for the first player I in a game G with moveset M is simply a choice $m \in M$. This strategy is said to be winning if for all possible attacks $\alpha \in M^\omega$ by their opponent, the first player wins the game by playing m during every round.

Definition 11. A *tactical strategy* for the second player II in a game G with moveset M is a function $\sigma : M \to M$. This strategy is said to be winning if for all possible attacks $\alpha \in M^\omega$ by their opponent, the second player wins the game by playing $\sigma(\alpha(n))$ during round n.

Notation 12. Write $\mathrm{I} \uparrow G$ (resp. $\mathrm{I} \underset{\text{pre}}{\uparrow} G$, $\mathrm{I} \underset{\text{con}}{\uparrow} G$) if player I has a winning strategy (resp. winning predetermined/constant strategy) for the game G. Similarly, write $\mathrm{II} \uparrow G$ (resp. $\mathrm{II} \underset{\text{mark}}{\uparrow} G$, $\mathrm{II} \underset{\text{tact}}{\uparrow} G$) if player II has a winning strategy (resp. winning Markov/tactical strategy) for the game G.

Of course:

$$\mathrm{II} \underset{\text{tact}}{\uparrow} G \Rightarrow \mathrm{II} \underset{\text{mark}}{\uparrow} G \Rightarrow \mathrm{II} \uparrow G \Rightarrow \mathrm{I} \not\uparrow G \Rightarrow \mathrm{I} \underset{\text{pre}}{\not\uparrow} G \Rightarrow \mathrm{I} \underset{\text{con}}{\not\uparrow} G.$$

In general, none of these implications (not even the middle [6]) can be reversed.

While predetermined and constant strategies are rarely explicitly studied in the literature, they are implicitly considered when studying the following well-known principles.

Definition 13 ([9])**.** The selection principle $S_1(\mathcal{A}, \mathcal{B})$ asserts that for each sequence $\{A_n : n < \omega\} \in [\mathcal{A}]^\omega$, there exist $B_n \in A_n$ such that $\{B_n : n < \omega\} \in \mathcal{B}$.

For example, if $\mathcal{O}_X$ denotes the open covers of a space X, then $S_1(\mathcal{O}_X, \mathcal{O}_X)$ is the Rothberger covering property.

Definition 14 ([13])**.** The choice principle $\binom{\mathcal{A}}{\mathcal{B}}^\kappa$ asserts that for each $A \in \mathcal{A}$, there exists a subset $B \subseteq A$ where $|B| = \kappa$ and $B \in \mathcal{B}$. (When κ is omitted, the criterion $|B| = \kappa$ is also omitted.)

For example, $\binom{\mathcal{O}_X}{\mathcal{O}_X}^\omega$ is the Lindelöf covering property.

Proposition 15. *$S_1(\mathcal{A}, \mathcal{B})$ is equivalent to $\mathrm{I} \underset{pre}{\not\uparrow} G_1(\mathcal{A}, \mathcal{B})$, and $\binom{\mathcal{A}}{\mathcal{B}}^\omega$ is equivalent to $\mathrm{I} \underset{con}{\not\uparrow} G_1(\mathcal{A}, \mathcal{B})$.*

Proof. The first equivalence is direct from their defintions.

To see the second, assume $\binom{\mathcal{A}}{\mathcal{B}}^\omega$. Then given a constant strategy $A \in \mathcal{A}$ for I, II uses $\binom{\mathcal{A}}{\mathcal{B}}^\omega$ to choose $B = \{b_n : n < \omega\} \subseteq A$ where $B \in \mathcal{B}$; thus playing b_n in round n defeats I's constant strategy.

Likewise, assuming $\mathrm{I} \underset{\text{con}}{\not\uparrow} G_1(\mathcal{A}, \mathcal{B})$, for each constant strategy $A \in \mathcal{A}$ for I there must be a counterattack playing $b_n \in A$ during each round n such that $\{b_n : n < \omega\} \in \mathcal{B}$; this witnesses $\binom{\mathcal{A}}{\mathcal{B}}^\omega$. □

The goal of this paper is to characterize when two games are "dual" in the following senses.

Definition 16. A pair of games $G(X), H(X)$ defined for a topological space X are *tactical information dual* if both of the following hold.

- $I \underset{\text{con}}{\uparrow} G(X)$ if and only if $II \underset{\text{tact}}{\uparrow} H(X)$.
- $II \underset{\text{tact}}{\uparrow} G(X)$ if and only if $I \underset{\text{con}}{\uparrow} H(X)$.

Definition 17. A pair of games $G(X), H(X)$ defined for a topological space X are *Markov information dual* if both of the following hold.

- $I \underset{\text{pre}}{\uparrow} G(X)$ if and only if $II \underset{\text{mark}}{\uparrow} H(X)$.
- $II \underset{\text{mark}}{\uparrow} G(X)$ if and only if $I \underset{\text{pre}}{\uparrow} H(X)$.

Definition 18. A pair of games $G(X), H(X)$ defeind for a topological space X are *perfect information dual* if both of the following hold.

- $I \uparrow G(X)$ if and only if $II \uparrow H(X)$.
- $II \uparrow G(X)$ if and only if $I \uparrow H(X)$.

Definition 19. A pair of games $G(X), H(X)$ defined for a topological space X are *dual* if they are tactial, Markov, and perfect information dual.

2. Main Results

The following six theorems demonstrate that reflections characterize dual selection games for perfect information strategies and certain limited information strategies.

For example, the duality of the Rothberger game $G_1(\mathcal{O}_X, \mathcal{O}_X)$ and the point-open game on X for perfect information strategies was first noted by Galvin in [7], and for Markov-information strategies by Clontz and Holshouser in [5]. These proofs may be generalized as follows.

Theorem 20. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*
Then $\text{I} \underset{\text{con}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$ *if and only if* $\text{II} \underset{\text{tact}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let A witness $\text{I} \underset{\text{con}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. Since $A \in \mathcal{A}$, $\text{range}(\tau) \subseteq A$ for some $\tau \in \mathbf{C}(\mathcal{R})$. Suppose $R_n \in \mathcal{R}$ for all $n < \omega$. Note that since A is winning and $\tau(R_n) \in \text{range}(\tau) \subseteq A$, $\{\tau(R_n) : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\text{II} \underset{\text{tact}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Now let σ witness $\text{II} \underset{\text{tact}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$. Then $\sigma \in \mathbf{C}(\mathcal{R})$ and we may let $A = \text{range}(\sigma) \in \mathcal{A}$. Suppose that $B_n \in A = \text{range}(\sigma)$ for all $n < \omega$. Choose $R_n \in \mathcal{R}$ such that $B_n = \sigma(R_n)$. Since σ is winning, $\{B_n : n < \omega\} \notin \mathcal{B}$. Thus A witnesses $\text{I} \underset{\text{con}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. □

Theorem 21. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*
Then $\text{II} \underset{\text{tact}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$ *if and only if* $\text{I} \underset{\text{con}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let σ witness $\text{II} \underset{\text{tact}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. Suppose that for each $R \in \mathcal{R}$, there was $g(R) \in R$ such that for all $A \in \mathcal{A}$, $\sigma(A) \neq g(R)$. Then $g \in \mathbf{C}(\mathcal{R})$ and $\text{range}(g) \in \mathcal{A}$, thus $\sigma(\text{range}(g)) \neq g(R)$ for all $R \in \mathcal{R}$, a contradiction.

So choose $R \in \mathcal{R}$ such that for all $r \in R$ there exists $A_r \in \mathcal{A}$ such that $\sigma(A_r) = r$. It follows that when $r_n \in R$ for $n < \omega$, $\{r_n : n < \omega\} = \{\sigma(A_{r_n}) : n < \omega\} \in B$, so R witnesses $\text{I} \underset{\text{con}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Now let R witness $\text{I} \underset{\text{con}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$. Then $R \in \mathcal{R}$, so for $A \in \mathcal{A}$, let $f_A \in \mathbf{C}(\mathcal{R})$ satisfy $\text{range}(f_A) \subseteq A$, and let $\tau(A) = f_A(R) \in A \cap R$. Then if $A_n \in \mathcal{A}$ for $n < \omega$, $\tau(A_n) \in R$, so $\{\tau(A_n) : n < \omega\} \in \mathcal{B}$. Thus τ witnesses $\text{II} \underset{\text{tact}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. □

Theorem 22. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*

Then $\mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$ *if and only if* $\mathrm{II} \underset{mark}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let σ witness $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. Since $\sigma(n) \in \mathcal{A}$, $\mathrm{range}(f_n) \subseteq \sigma(n)$ for some $f_n \in \mathbf{C}(\mathcal{R})$. So let $\tau(R, n) = f_n(R)$ for all $R \in \mathcal{R}$ and $n < \omega$. Suppose $R_n \in \mathcal{R}$ for all $n < \omega$. Note that since σ is winning and $\tau(R_n, n) = f_n(R_n) \in \mathrm{range}(f_n) \subseteq \sigma(n)$, $\{\tau(R_n, n) : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Now let σ witness $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$. Let $f_n \in \mathbf{C}(\mathcal{R})$ be defined by $f_n(R) = \sigma(R, n)$, and let $\tau(n) = \mathrm{range}(f_n) \in \mathcal{A}$. Suppose that $B_n \in \tau(n) = \mathrm{range}(f_n)$ for all $n < \omega$. Choose $R_n \in \mathcal{R}$ such that $B_n = f_n(R_n) = \sigma(R_n, n)$. Since σ is winning, $\{B_n : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. $\square$

Theorem 23. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*

Then $\mathrm{II} \underset{mark}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$ *if and only if* $\mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let σ witness $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. Let $n < \omega$. Suppose that for each $R \in \mathcal{R}$, there was $g(R) \in R$ such that for all $A \in \mathcal{A}$, $\sigma(A, n) \neq g(R)$. Then $g \in \mathbf{C}(\mathcal{R})$ and $\mathrm{range}(g) \in \mathcal{A}$, thus $\sigma(\mathrm{range}(g), n) \neq g(R)$ for all $R \in \mathcal{R}$, a contradiction.

So choose $\tau(n) \in \mathcal{R}$ such that for all $r \in \tau(n)$ there exists $A_{r,n} \in \mathcal{A}$ such that $\sigma(A_{r,n}, n) = r$. It follows that when $r_n \in \tau(n)$ for $n < \omega$, $\{r_n : n < \omega\} = \{\sigma(A_{r_n,n}) : n < \omega\} \in B$, so τ witnesses $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$.

Now let σ witness $\mathrm{I} \underset{\mathrm{pre}}{\uparrow} G_1(\mathcal{R}, \neg\mathcal{B})$. Then $\sigma(n) \in \mathcal{R}$, so for $A \in \mathcal{A}$, let $f_A \in \mathbf{C}(\mathcal{R})$ satisfy $\mathrm{range}(f_A) \subseteq A$, and let $\tau(A, n) = f_A(\sigma(n)) \in A \cap \sigma(n)$. Then if $A_n \in \mathcal{A}$ for $n < \omega$, $\tau(A_n, n) \in \sigma(n)$, so $\{\tau(A_n, n) : n < \omega\} \in \mathcal{B}$. Thus τ witnesses $\mathrm{II} \underset{\mathrm{mark}}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$. $\square$

Theorem 24. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*

Then $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$ *if and only if* $\mathrm{II} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$.

Proof. Let σ witness $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$. Let $c(\emptyset) = \emptyset$. Suppose $c(s) \in (\bigcup A)^{<\omega}$ is defined for $s \in \mathcal{R}^{<\omega}$. Since $\sigma(c(s)) \in \mathcal{A}$, let $f_s \in \mathbf{C}(\mathcal{R})$ satisfy $\mathrm{range}(f_s) \subseteq \sigma(c(s))$, and let $c(s^\frown\langle R\rangle) = c(s)^\frown\langle f_s(R)\rangle$. Then let $c(\alpha) = \bigcup\{c(\alpha \restriction n) : n < \omega\}$ for $\alpha \in \mathcal{R}^\omega$, so

$$c(\alpha)(n) = f_{\alpha\restriction n}(\alpha(n)) \in \mathrm{range}(f_{\alpha\restriction n}) \subseteq \sigma(c(\alpha \restriction n))$$

demonstrating that $c(\alpha)$ is a legal attack against σ.

Let $\tau(s^\frown\langle R\rangle) = f_s(R)$. Consider the attack $\alpha \in \mathcal{R}^\omega$ against τ. Then since σ is winning and $\tau(\alpha \restriction n+1) = f_{\alpha\restriction n}(\alpha(n)) \in \mathrm{range}(f_{\alpha\restriction n}) \subseteq \sigma(c(\alpha \restriction n))$, it follows that $\{\tau(\alpha \restriction n+1) : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\mathrm{II} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$.

Now let σ witness $\mathrm{II} \uparrow G_1(\mathcal{R}, \neg\mathcal{B})$. For $s \in \mathcal{R}^{<\omega}$, define $f_s \in \mathbf{C}(\mathcal{R})$ by $f_s(R) = \sigma(s^\frown\langle R\rangle)$. Let $\tau(\emptyset) = \mathrm{range}(f_\emptyset) \in \mathcal{A}$, and for $x \in \tau(\emptyset)$, choose $R_{\langle x\rangle} \in \mathcal{R}$ such that $x = f_\emptyset(R_{\langle x\rangle})$ (for other $x \in \bigcup A$, choose $R_{\langle x\rangle}$ arbitrarily as it won't be used). Now let $s \in (\bigcup A)^{<\omega}$, and suppose $R_{s\restriction n^\frown\langle x\rangle} \in \mathcal{R}$ has been defined for $n \leq |s|$ and $x \in \bigcup A$. Then let $\tau(s^\frown\langle x\rangle) = \mathrm{range}(f_{\langle R_{s\restriction 0}, \dots, R_s, R_{s^\frown\langle x\rangle}\rangle})$ and for $y \in \tau(s)$ choose $R_{s^\frown\langle x,y\rangle}$ such that $x = f_{\langle R_{s\restriction 0}, \dots, R_s, R_{s^\frown\langle x\rangle}\rangle}(R_{s^\frown\langle x,y\rangle})$ (and again, choose $R_{s^\frown\langle x,y\rangle}$ arbitrarily for other $y \in \bigcup\mathcal{A}$ as it won't be used).

Then let α attack τ, so $\alpha(n) \in \tau(\alpha \restriction n)$ and thus $\alpha(n) = f_{\langle R_{\alpha\restriction 0},\dots,R_{\alpha\restriction n}\rangle}(R_{\alpha\restriction n+1}) = \sigma(\langle R_{\alpha\restriction 0},\dots,R_{\alpha\restriction n+1}\rangle)$. Since σ is winning, $\{\sigma(\langle R_{\alpha\restriction 0},\dots,R_{\alpha\restriction n+1}\rangle) : n < \omega\} = \{\alpha(n) : n < \omega\} \notin \mathcal{B}$. Thus τ witnesses $\mathrm{I} \uparrow G_1(\mathcal{A},\mathcal{B})$. □

Theorem 25. *Let $\mathcal{R}$ be a reflection of $\mathcal{A}$.*

Then $\mathrm{II} \uparrow G_1(\mathcal{A},\mathcal{B})$ if and only if $\mathrm{I} \uparrow G_1(\mathcal{R},\neg\mathcal{B})$.

Proof. Let σ witness $\mathrm{II} \uparrow G_1(\mathcal{A},\mathcal{B})$. Let $s \in (\bigcup A)^{<\omega}$ and assume $a(s) \in \mathcal{A}^{|s|}$ is defined (of course, $a(\emptyset) = \emptyset$). Suppose for all $R \in \mathcal{R}$ there existed $f(R) \in R$ such that for all $A \in \mathcal{A}$, $\sigma(a(s)^\frown\langle A\rangle) \neq f(R)$. Then $f \in \mathbf{C}(\mathcal{R})$ and $\mathrm{range}(f) \in \mathcal{A}$, and thus $\sigma(a(s)^\frown\langle \mathrm{range}(f)\rangle) \neq f(R)$ for all $R \in \mathcal{R}$, a contradiction. So let $\tau(s) \in \mathcal{R}$ satisfy for all $x \in \tau(s)$ there exists $a(s^\frown\langle x\rangle) \in \mathcal{A}^{|s|+1}$ extending $a(s)$ such that $x = \sigma(a(s^\frown\langle x\rangle))$.

If τ is attacked by $\alpha \in (\bigcup R)^\omega$, then $\alpha(n) \in \tau(\alpha \restriction n)$. So $\alpha(n) = \sigma(a(\alpha \restriction n+1))$, and since σ is winning, $\{\sigma(a(\alpha \restriction n+1)) : n < \omega\} = \{\alpha(n) : n < \omega\} \in \mathcal{B}$. Therefore τ witnesses $\mathrm{I} \uparrow G_1(\mathcal{R},\neg\mathcal{B})$.

Now let σ witness $\mathrm{I} \uparrow G_1(\mathcal{R},\neg\mathcal{B})$. Let $s \in \mathcal{A}^{<\omega}$, and suppose $r(s) \in (\bigcup\mathcal{R})^{|s|}$ is defined (again, $r(\emptyset) = \emptyset$). For $A \in \mathcal{A}$ choose $f_A \in \mathbf{C}(\mathcal{R})$ where $\mathrm{range}(f_A) \subseteq A$, and let $\tau(s^\frown\langle A\rangle) = f_A(\sigma(r(s)))$, and let $r(s^\frown\langle A\rangle)$ extend $r(s)$ by letting $r(s^\frown\langle A\rangle)(|s|) = \tau(s^\frown\langle A\rangle)$.

If τ is attacked by $\alpha \in \mathcal{A}^\omega$, then since $\tau(\alpha \restriction n+1) = f_{\alpha(n)}(\sigma(r(\alpha \restriction n)) \in \alpha(n) \cap \sigma(r(\alpha \restriction n))$ and σ is winning, we conclude that τ is a legal strategy and $\{\tau(\alpha \restriction n+1) : n < \omega\} \in \mathcal{B}$. Therefore τ witnesses $\mathrm{II} \uparrow G_1(\mathcal{A},\mathcal{B})$. □

Corollary 26. *If $\mathcal{R}$ is a reflection of $\mathcal{A}$, then $G_1(\mathcal{A},\mathcal{B})$ and $G_1(\mathcal{R},\neg\mathcal{B})$ are dual.*

3. Applications of Reflections

Definition 27. Let X be a topological space and $\mathcal{T}_X$ be a chosen basis of nonempty sets for its topology.

- Let $\mathcal{T}_{X,x} = \{U \in \mathcal{T}_X : x \in U\}$ be the local point-base at $x \in X$.
- Let $\Omega_{X,x} = \{Y \subseteq X : \forall U \in \mathcal{T}_{X,x}(U \cap Y \neq \emptyset)\}$ be the fan at $x \in X$.
- Let $\mathcal{T}_{X,F} = \{U \in \mathcal{T}_X : F \subseteq U\}$ be the local finite-base at $F \in [X]^{<\aleph_0}$.
- Let $\mathcal{O}_X = \{\mathcal{U} \subseteq \mathcal{T}_X : \bigcup\mathcal{U} = X\}$ be the collection of basic open covers of X.
- Let $\mathcal{P}_X = \{\mathcal{T}_{X,x} : x \in X\}$ be the collection of local point-bases of X.
- Let $\Omega_X = \{\mathcal{U} \subseteq \mathcal{T}_X : \forall F \in [X]^{<\aleph_0} \exists U \in \mathcal{U}(F \subseteq U)\}$ be the collection of basic ω-covers of X.
- Let $\mathcal{F}_X = \{\mathcal{T}_{X,F} : F \in [X]^{<\aleph_0}\}$ be the collection of local finite-bases of X.
- Let $\mathcal{D}_X = \{Y \subseteq X : \forall U \in \mathcal{T}_X(U \cap Y \neq \emptyset)\}$ be the collection of dense subsets of X.
- Let $\Gamma_{X,x} = \{Y \subseteq X : \forall U \in \mathcal{T}_{X,x}(Y \setminus U \in [X]^{<\aleph_0})\}$ be the collection of converging fans at $x \in X$. (When intersected with $[X]^{\aleph_0}$, these are the non-trivial sequences of X converging to x.)

We may now establish the following dual games.

Proposition 28. *$\mathcal{P}_X$ is a reflection of $\mathcal{O}_X$.*

Proof. For every basic open cover $\mathcal{U}$, the corresponding choice function $f_\mathcal{U} \in \mathbf{C}(\mathcal{P}_X)$ is simply the witness that for each $\mathcal{T}_{X,x} \in \mathcal{P}_X$, there exists $f_\mathcal{U}(\mathcal{T}_{X,x}) \in \mathcal{U}$ such that $x \in f_\mathcal{U}(\mathcal{T}_{X,x})$. □

Corollary 29. *$G_1(\mathcal{O}_X, \mathcal{B})$ and $G_1(\mathcal{P}_X, \neg\mathcal{B})$ are dual.*

In the case that $\mathcal{B} = \mathcal{O}_X$, $G_1(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known Rothberger game, and $G_1(\mathcal{P}_X, \neg\mathcal{O}_X)$ is isomorphic to the point-open game $PO(X)$: I chooses points of X, II chooses an open neighborhood of each chosen point, and I wins if II's choices are a cover. So this encapsulates the classic result that the Rothberger game and point-open game are perfect-information dual [7], the more recent result that these games are Markov-information dual [5], and the quickly verified fact that a Lindelöf space X may be characterized as follows: for each neighborhood assignment (i.e. tactic for II in $PO(X)$) there exists a countable subset of X such that its neighborhoods cover the space.

Proposition 30. *$\mathcal{F}_X$ is a reflection of Ω_X.*

Proof. For every basic open ω-cover $\mathcal{U}$, the corresponding choice function $f_{\mathcal{U}} \in \mathbf{C}(\mathcal{F}_X)$ is simply the witness that for each $\mathcal{T}_{X,F} \in \mathcal{F}_X$, there exists $f_{\mathcal{U}}(\mathcal{T}_{X,F}) \in \mathcal{U}$ such that $F \subseteq f_{\mathcal{U}}(\mathcal{T}_{X,F})$. □

Corollary 31. *$G_1(\Omega_X, \mathcal{B})$ and $G_1(\mathcal{F}_X, \neg\mathcal{B})$ are dual.*

Note that in the case that $\mathcal{B} = \Omega_X$, $G_1(\Omega_X, \Omega_X)$ is the Rothberger game played with ω-covers, and $G_1(\mathcal{F}_X, \neg\Omega_X)$ is isomorphic to the Ω-finite-open game $\Omega FO(X)$: I chooses finite subsets of X, II chooses an open neighborhood of each chosen finite set, and I wins if II's choices are an ω-cover. These games were directly shown to be Markov and perfect-information dual in [5].

Proposition 32. *$\mathcal{T}_X$ is a reflection of $\mathcal{D}_X$.*

Proof. For every dense D, the corresponding choice function $f_D \in \mathbf{C}(\mathcal{T}_X)$ is simply the witness that for each $U \in \mathcal{T}_X$, there exists $f_D(U) \in U \cap D$. □

Corollary 33. *$G_1(\mathcal{D}_X, \mathcal{B})$ and $G_1(\mathcal{T}_X, \neg\mathcal{B})$ are perfect-information and Markov-information dual.*

In the case that $\mathcal{B} = \Omega_{X,x}$ for some $x \in X$, $G_1(\mathcal{D}_X, \Omega_{X,x})$ is the strong countable dense fan-tightness game at x, see e.g. [1]. $G_1(\mathcal{T}_X, \neg\Omega_{X,x})$ is the game $CL(X, x)$ first studied by Tkachuk in [12]. Tkachuk showed in that paper that these games are perfect-information dual; Clontz and Holshouser previously showed these were Markov-information dual in the case that $X = C_p(Y)$ [5].

In the case that $\mathcal{B} = D_X$, then $G_1(\mathcal{D}_X, \mathcal{D}_X)$ is the strong selective separability game introduced by Scheepers in [10], and $G_1(\mathcal{T}_X, \neg\mathcal{D}_X)$ is the point-picking game of Berner and Juhász defined in [2]. Scheepers showed that these were perfect-information dual in [10].

Proposition 34. *$\mathcal{T}_{X,x}$ is a reflection of $\Omega_{X,x}$.*

Proof. For every set Y with limit point x, the corresponding choice function $f_Y \in \mathbf{C}(\mathcal{T}_{X,x})$ is simply the witness that for each $U \in \mathcal{T}_{X,x}$, there exists $f_Y(U) \in U \cap Y$. □

Corollary 35. *$G_1(\Omega_{X,x}, \mathcal{B})$ and $G_1(\mathcal{T}_{X,x}, \neg\mathcal{B})$ are dual.*

In the case that $\mathcal{B} = \Gamma_{X,x}$ for some $x \in X$, $G_1(\mathcal{T}_{X,x}, \neg\Gamma_{X,x})$ is Gruenhage's W game [8]. Its dual $G_1(\Omega_{X,x}, \Gamma_{X,x})$ characterizes the strong Fréchet-Urysohn property I $\underset{\text{pre}}{\not\uparrow}$ $G_1(\Omega_{X,x}, \Gamma_{X,x})$ at x, which now seen to be equivalent to II $\underset{\text{mark}}{\not\uparrow}$ $G_1(\mathcal{T}_{X,x}, \neg\Gamma_{X,x})$. This allows us to obtain the following result.

Corollary 36. $\mathrm{I} \underset{pre}{\not\uparrow} G_1(\Omega_{X,x}, \Gamma_{X,x})$ *if and only if* $\mathrm{I} \not\uparrow G_1(\Omega_{X,x}, \Gamma_{X,x})$.

Proof. As shown in [11], $\mathrm{II} \not\uparrow G_1(\mathcal{T}_{X,x}, \neg\Gamma_{X,x})$ (i.e. X is w in the terminology of that paper) if and only if $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_1(\Omega_{X,x}, \Gamma_{X,x})$. The result follows as $G_1(\mathcal{T}_{X,x}, \neg\Gamma_{X,x})$ and $G_1(\Omega_{X,x}, \Gamma_{X,x})$ are dual. □

For $\mathcal{B} = \Omega_{X,x}$, $G_1(\mathcal{T}_{X,x}, \neg\Omega_{X,x})$ is the variant of Gruenhage's W game for clustering. This game is now seen to be dual to the strong countable fan tightness game $G_1(\Omega_{X,x}, \Omega_{X,x})$ at x.

We conclude by noting how Corollary 26 was used by the authors of [4] to easily strengthen Propositions 29 and 31 while this paper was still in preparation.

Definition 37. Let $\mathcal{Q}$ be a collection of subsets of a topological space X. Then $\mathcal{O}_{X,\mathcal{Q}}$ is the collection of basic open covers $\mathcal{U}$ such that for each $Q \in \mathcal{Q}$, there is $U \in \mathcal{U}$ with $Q \subseteq U$. Likewise, $\mathcal{N}_{X,\mathcal{Q}} = \{\mathcal{T}_{X,Q} : Q \in \mathcal{Q}\}$ where each $\mathcal{T}_{X,Q}$ collects all basic open sets U such that $Q \subseteq U$.

In particular, $\mathcal{O}_{X,[X]^1} = \mathcal{O}_X$ and $\mathcal{O}_{X,[X]^{<\omega}} = \Omega_X$. Likewise $\mathcal{N}_{X,[X]^1} = \mathcal{P}_X$ and $\mathcal{N}_{X,[X]^{<\omega}} = \mathcal{F}_X$.

Theorem 38 ([4])**.** *The games* $G_1(\mathcal{O}_{X,\mathcal{Q}}, \mathcal{B})$ *and* $G_1(\mathcal{N}_{X,\mathcal{Q}}, \neg\mathcal{B})$ *are dual.*

Proof. This is immediate as $\mathcal{N}_{X,\mathcal{Q}}$ reflects $\mathcal{O}_{X,\mathcal{Q}}$. To see this, for each $\mathcal{U} \in \mathcal{O}_{X,\mathcal{Q}}$, choose $f_{\mathcal{U}} \in \mathbf{C}(\mathcal{N}_{X,\mathcal{Q}})$ satisfying that for each $\mathcal{T}_{X,Q} \in \mathcal{N}_{X,\mathcal{Q}}$, there exists $f_{\mathcal{U}}(\mathcal{T}_{X,Q}) \in \mathcal{U}$ such that $Q \subseteq f_{\mathcal{U}}(\mathcal{T}_{X,Q})$. □

4. Open Questions

Let $\Gamma_X = \{\mathcal{U} \subseteq \mathcal{T}_X : \forall x \in X(\mathcal{U} \setminus \mathcal{T}_{X,x} \in [\mathcal{T}_X]^{<\aleph_0})\}$. Such γ-covers are related to the convergent sequences of $C_p(X)$ (that is, $\Gamma_{C_p(X),\mathbf{0}}$ as defined in Definition 27), see e.g. [3].

Question 39. *Does there exist a natural reflection for* $\Gamma_{X,x}$ *or* Γ_X*?*

The game $G_{fin}(\mathcal{A}, \mathcal{B})$ is defined analogously to $G_1(\mathcal{A}, \mathcal{B})$, except II may choose a finite subset each round rather than a single set.

Question 40. *Do there exist any duality results for* $G_{fin}(\mathcal{A}, \mathcal{B})$ *similar to the technique of reflections?*

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