Limited Information Strategies for Topological Games

by

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Abstract

I talk a lot about topological games.

TODO: Write this.

Acknowledgments

TODO: Thank people.

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3.1 Basic game

Definition 3.1. Gruenhage's open-point convergence game $Con_{O,P}(X, x)$ has O choosing nested open sets and P choosing a point within the last chosen open set by O. O wins if the points chosen by P converge to x.

TODO: This version of the game (where $\mathscr{P}$ need not play within all previously chosen open sets) is more difficult for $\mathscr{O}$. Need to differientiate these two versions somehow...

Definition 3.2. The one-point compactification of a space X is X^*, where neighborhoods of points in X are the same as they were originally, and neighborhoods of ∞ are sets $X^* \setminus K$ for compact K. If X is discrete then neighborhoods of ∞ are cofinite sets containing ∞, and the game is equivalent to O choosing finite "forbidden" sets and P choosing points not forbidden by O.

Proposition 3.3. *$O \underset{code}{\uparrow} Con_{O,P}(\kappa^*, \infty)$ for all cardinals κ.*

Proof. Use $F(N, p) = N \cup \{p\}$. □

Proposition 3.4. *$O \underset{pre}{\uparrow} Con_{O,P}(\omega^*, \infty)$.*

Proof. Use $F(n) = n$. □

Proposition 3.5. *$O \underset{pre}{\not\uparrow} Clus_{O,P}(\kappa^*, \infty)$ for $\kappa \geq \omega_1$.*

Proof. Let $F(n)$ be O's predetermined forbidding strategy, let $\alpha \in \kappa \setminus \bigcup_{n<\omega} F(n)$, and have P counter with $\langle \alpha, \alpha, \dots \rangle$. $\square$

Proposition 3.6. $O \underset{tact}{\uparrow} Con_{O,P}(\omega^*, \infty)$.

Proof. Use $F(n) = n + 1$. $\square$

Theorem 3.7. *If κ is a regular uncountable cardinal, for every function $f : [\kappa]^{<\omega} \to [\kappa]^{<\omega}$ the set $C_f = \{\alpha < \kappa : S \in [\alpha]^{<\omega} \Rightarrow f(S) \in [\alpha]^{<\omega}\}$ is club.*

Proof. First assume $\alpha_0 < \alpha_1 < \dots \in C_f$. It is easily seen that $\sup(\alpha_n) \in C_f$, showing C_f is closed.

Now assume $\gamma_0 \in C_f$. Let $\gamma_{n+1} > \gamma_n$ be the least ordinal such that if $S \in [\gamma_n + 1]^{<\omega}$ then $f(S) \in [\gamma_{n+1}]^{<\omega}$. We claim $\gamma_\omega = \sup(\gamma_n) \in C_f$. Let $S \in [\gamma_\omega]^{<\omega}$. Then $S \in [\gamma_n + 1]^{<\omega}$ for some n, and thus $f(S) \in [\gamma_{n+1}]^{<\omega} \subset [\gamma_\omega]^{<\omega}$. Therefore C_f is unbounded. $\square$

We may thus assume, for the purposes of countering a tactical or Markov strategy, that the strategy is **downward** on some regular uncountable cardinal.

Theorem 3.8. $O \underset{k\text{-}tact}{\not\uparrow} Clus_{O,P}(\kappa^*, \infty)$ *for* $\kappa \geq \omega_1$.

Proof. Let $F : [\kappa]^{\leq k} \to [\kappa]^{<\omega}$ be a forbidding strategy by O against P which is downward on ω_1. We define n_i for $0 \leq i < k$ to be a natural number such that

$$n_i \in \omega \setminus (F(n_0, \dots, n_{i-1}) \cup F(n_0, \dots, n_{i-1}, \omega + i, \dots, \omega + k - 1))$$

and note that

$$\langle n_0, n_1, \dots, n_{k-1}, \omega, \omega + 1, \dots, \omega + k - 1, n_0, n_1, \dots, n_{k-1}, \omega, \omega + 1, \dots, \omega + k - 1, \dots \rangle$$

counters F. $\square$

Theorem 3.9. $O \underset{mark}{\uparrow} Clus_{O,P}(\omega_1^*, \infty)$.

Proof. For $\alpha < \omega_1$ let $A_{\alpha,n}$ be a sequence of finite sets such that $A_{\alpha,n} \subset A_{\alpha,n+1}$ and $\bigcup_{n<\omega} A_{\alpha,n} = \alpha + 1$.

Give O the Markov forbidding strategy $F(n, \alpha) = A_{\alpha,n}$. To observe that any legal play by P against the strategy F has infinite range, we observe that for any $\alpha_0 < \cdots < \alpha_{k-1}$, there is some round n such that $\{\alpha_0, \ldots, \alpha_{k-1}\} \subseteq \bigcup_{0\leq i<k} F(n, \alpha_i)$, and thus P cannot legally play any of these ordinals until another ordinal is played. □

Peter J. Nyikos has shown the following:

Theorem 3.10. $O \underset{mark}{\not\uparrow} Con_{O,P}(\omega_1 \cup \{\infty\}, \infty)$.

which improves to:

Theorem 3.11. $O \underset{k\text{-}mark}{\not\uparrow} Clus_{O,P}(\kappa \cup \{\infty\}, \infty)$ *for* $\kappa > \omega_1$.

Proof. Let $F : \omega \times [\kappa]^{\leq k} \to [\kappa]^{<\omega}$ be a forbidding strategy by O against P which is downward on ω_2. We define α_i for $0 \leq i < k$ to be a countable ordinal such that

$$\alpha_i \in \omega_1 \setminus \bigcup_{n<\omega} (F(n, \{\alpha_0, \ldots, \alpha_{i-1}\}) \cup F(n, \{\alpha_0, \ldots, \alpha_{i-1}, \omega_1 + i, \ldots, \omega_1 + k - 1\}))$$

and note that

$$\langle \alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \omega_1, \omega_1 + 1, \ldots, \omega_1 + k - 1, \alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \omega_1, \omega_1 + 1, \ldots, \omega_1 + k - 1, \ldots \rangle$$

counters F. □

3.2 $Con_{O,P}(X, x)$ for Sigma Product

Proposition 3.12. $O \uparrow Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$.

Proof. For $s \in \Sigma\mathbb{R}^\kappa$ let $C(s) = \{\alpha < \kappa : s(\alpha) \neq 0\}$ denote the countable nonzero coordinates of s. Let $\Phi : [\kappa]^{\leq\omega} \times \omega \to [\kappa]^{<\omega}$ be such that $\Phi(C, n) \subseteq \Phi(C, n + 1)$ and $\bigcup_{n<\omega} \Phi(C, n) = C$.

If τ is the usual topology on $\mathbb{R}$, let $\sigma_\alpha : (\Sigma\mathbb{R}^\kappa)^{<\omega} \to \tau$ be such that

$$U_\alpha(s_0, \ldots, s_{n-1}) = \begin{cases} (-\frac{1}{n}, \frac{1}{n}) \text{ if } \alpha \in \bigcup_{i<n} \Phi(C(s_i), n) \\ \mathbb{R} \text{ otherwise} \end{cases}$$

Finally, give O the winning strategy $\sigma(s_0, \ldots, s_{n-1}) = \Sigma\mathbb{R}^\kappa \cap \prod_{\alpha<\kappa} U_\alpha(s_0, \ldots, s_{n-1})$. □

Theorem 3.13. *For all cardinals $\kappa \leq 2^\omega$, $O \underset{code}{\uparrow} Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$.*

Proof. Note that $|\Sigma\mathbb{R}^\kappa| \leq 2^\omega = |\mathbb{R}|$. Define the following:

- Encode every $S \in (\Sigma\mathbb{R}^\kappa)^{<\omega}$ as a real number $0 < r(S) < 1$.
- Let $\gamma(U)$ be the function which, for basic open sets $U = \Sigma\mathbb{R}^\kappa \cap \prod_{\alpha<\kappa} U_\alpha$ where for all $\alpha < \kappa$ either $U_\alpha = \mathbb{R}$ or $(-\frac{1}{r}, \frac{1}{r})$, returns $\lfloor r \rfloor$.
- Let $n(U)$ be the number of non-$\mathbb{R}$ components of a basic open set U.
- Let $\psi(U, s) = r^{-1}(\gamma(U))^\frown \langle y \rangle$.
- For $s \in \Sigma\mathbb{R}^\kappa$ let $C(s) = \{\alpha < \kappa : s(\alpha) \neq 0\}$ denote the countable nonzero coordinates of s.
- Let $\Phi : [\kappa]^{\leq\omega} \times \omega \to [\kappa]^{<\omega}$ be such that $\Phi(C, n) \subseteq \Phi(C, n+1)$ and $\bigcup_{n<\omega} \Phi(C, n) = C$.
- For each $\alpha < \kappa$, define the interval $\sigma_\alpha(U, s)$ about 0 as follows:
  - If $\alpha \leq n(U)$ or $\alpha \in \bigcup_{s\in\psi(U,s)} \Phi(C(s), n(U))$ then $\sigma_\alpha(U, s) = (-\frac{1}{n(U)+r(\psi(U,s))}, -\frac{1}{n(U)+r(\psi(U,s))})$.
  - Otherwise, $\sigma_\alpha(U, s) = \mathbb{R}$.

It follows that $\sigma(U, s) = \Sigma\mathbb{R}^\kappa \cap \prod_{\alpha<\kappa} \sigma_\alpha(U, s)$ is a winning coding strategy. □

Theorem 3.14. *Let κ be a cardinal such that there exists a function $f : \kappa \to [\kappa]^{\leq\omega}$ where for every $W \in [\kappa]^{\leq\omega}$ there exists $\alpha_W < \kappa$ with $W \subseteq f(\alpha_W)$. (That is, $cf([\kappa]^{\leq\omega}) = \kappa$.) Then $F \underset{code}{\uparrow} PF_{F,C}(\kappa)$ and $O \underset{code}{\uparrow} Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$.*

Proof. Let $W \upharpoonright n \in [\kappa]^n$ be a subset of $W \in [\kappa]^\omega$ such that $W \upharpoonright n \subset W \upharpoonright (n+1)$ and $\bigcup_{n<\omega} W \upharpoonright n = W$.

Define

$$\sigma(N, W) = N \cup (|N|+1) \cup \{\alpha_W\} \cup \bigcup_{\alpha \in N} f(\alpha) \upharpoonright |N|$$

Consider the play $\langle \emptyset, W_0, N_1, W_1, N_2, W_2, \dots \rangle$ with F following the strategy σ. Let $\gamma \in W_i$, and note $\gamma \in f(\alpha_{W_i})$ (and $\gamma \in f(\alpha_{W_i}) \upharpoonright |N_n|$ for sufficiently large n).

$$N_{i+1} = \sigma(N_i, W_i) \supseteq \{\alpha_{W_i}\}$$

and thus

$$N_{n+1} = \sigma(N_n, W_n) \supseteq \bigcup_{\alpha \in N_n} f(\alpha) \upharpoonright |N_n| \supseteq \bigcup_{\alpha \in N_{i+1}} f(\alpha) \upharpoonright |N_n| \supseteq f(\alpha_{W_i}) \upharpoonright |N_n|$$

showing $\gamma \in N_{n+1}$. Since γ is forbidden in round $n+1$, γ appears in finitely many sets chosen by C.

We turn our attention to $Con_{O,P}(\Sigma\mathbb{R}^\kappa)$. We define the winning strategy $\tau(U, p)$ for O as follows: let $N(U)$ be the non-$\mathbb{R}$ coordinates in the basic open set U and $W(p)$ be the non-0 coordinates in p. Then $\tau(U, p) = \left(\prod_{\alpha<\kappa} U_\alpha\right) \cap \Sigma\mathbb{R}^\kappa$ where if $\alpha \in \sigma(N(U), W(p))$ then $U_\alpha = (-\frac{1}{|N(U)|}, \frac{1}{|N(U)|})$ and $U_\alpha = \mathbb{R}$ otherwise.

Consider the play $\langle \emptyset, p_0, U_1, p_1, U_2, p_2, \dots \rangle$ with O following the strategy τ. Observe that $N(\tau(U, p)) = \sigma(N(U), W(p))$. Thus $p_i(\gamma) \neq 0$ is equivalent to $\gamma \in W(p_i)$, and by the above argument, for sufficiently large n, $\gamma \in \sigma(N(U_n), W(p_n))$. Therefore from round n onward the γ-coordinates of points chosen by P must lay in $(-\frac{1}{|N(U)|}, \frac{1}{|N(U)|})$ and converge to 0. $\square$

Theorem 3.15. *Let κ be the limit of cardinals κ_n such that cf$([\kappa_n]^{\leq\omega}, \subseteq) = \kappa_n$. Then $F \underset{code}{\uparrow} PF_{F,C}(\kappa)$ and $O \underset{code}{\uparrow} Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$.*

Proof. Let $f_n : \kappa_n \to [\kappa_n]^{\leq\omega}$ be such that for every $W \in [\kappa_n]^{\leq\omega}$ there exists $\alpha_{W,n} < \kappa_n$ such that $f_n(\alpha_{W,n}) \supseteq W$.

Define

$$\sigma(N,W) = N \cup (|N|+1) \cup \{\alpha_{W\cap\kappa_{|N|},|N|}\} \cup \bigcup_{n\leq |N|} \bigcup_{\alpha\in N} f_n(\alpha) \restriction |N|$$

We claim that σ is a winning coding strategy.

Consider the play $\langle N_0, W_0, N_1, W_1, \dots \rangle$ where O follows the strategy σ. For σ to be a winning strategy for $F \underset{\text{code}}{\uparrow} PF_{F,C}(\kappa)$, it must follow that for each $\gamma \in \bigcup_{i<\omega} W_i$, γ is forbidden by some $\sigma(N_j, W_j)$.

Let $\gamma \in W_i \cap \kappa_{|N_i|}$. For all $j > i$, $\alpha_{W\cap\kappa_{N_i},|N_i|} \in N_j$. Also, $\gamma \in f_{N_i}(\alpha_{W\cap\kappa_{N_i},|N_i|}) \restriction |N_j|$ for some sufficiently large j. So we observe that $\gamma \in \bigcup_{n\leq |N_j|} \bigcup_{\alpha\in N_j} f_n(\alpha) \restriction |N_j| \subseteq \sigma(N_j, W_j)$.

We turn our attention to $Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$. We define the winning strategy $\tau(U,p)$ for O as follows: let $N(U)$ be the non-$\mathbb{R}$ coordinates in the basic open set U and $W(p)$ be the non-0 coordinates in p. Then $\tau(U,p) = \Sigma\mathbb{R}^\kappa \cap \prod_{\alpha<\kappa} U_\alpha$ where if $\alpha \in \sigma(N(U), W(p))$ then $U_\alpha = (-\frac{1}{|N(U)|}, \frac{1}{|N(U)|})$ and $U_\alpha = \mathbb{R}$ otherwise.

Consider the play $\langle \Sigma\mathbb{R}^\kappa, p_0, U_1, p_1, U_2, p_2, \dots \rangle$ with O following the strategy τ. Observe that $N(\tau(U,p)) = \sigma(N(U), W(p))$. Thus $p_i(\gamma) \neq 0$ is equivalent to $\gamma \in W(p_i)$, and by the above argument, for sufficiently large n, $\gamma \in \sigma(N(U_n), W(p_n))$. Therefore from round n onward the γ-coordinates of points chosen by P must lay in $(-\frac{1}{|N(U)|}, \frac{1}{|N(U)|})$ and converge to 0. □

Theorem 3.16. *$F \underset{code}{\uparrow} PF_{F,C}(\kappa)$ for all cardinals κ.*

Proof. Let κ be the limit of cardinals κ_n such that $F \underset{\text{code}}{\uparrow} PF_{F,C}(\kappa_n)$ using the strategy $\sigma_n(N,W)$ such that for $M \subseteq N$, $\sigma_n(M,W) \subseteq \sigma_n(N,W)$. Define

$$\sigma(N,W) = (|N|+1) \cup \bigcup_{n<|N|} \sigma_n(N\cap\kappa_n, W\cap\kappa_n)$$

Let $\langle N_0, W_0, N_1, W_1, \dots \rangle$ be a legal play of the game with $N_{i+1} = \sigma(N_i, W_i)$. Suppose $\gamma \in W_i$ for infinitely-many i. $\gamma \in \kappa_n$ for some n, so observe the play $\langle M_0, W_0 \cap \kappa_n, M_1, W_1 \cap \kappa_n, \dots \rangle$

with $M_0 = N_0 \cap \kappa_n$ and $M_{i+1} = \sigma_n(M_i, W_i \cap \kappa_n) \subseteq \sigma_n(N_i \cap \kappa_n, W_i \cap \kappa_n)$ which is $\subseteq \sigma(N_i, W_i) = N_{i+1}$ for sufficiently large i.

Since σ_n is a winning strategy, $\gamma \in M_{m+1} \subseteq N_{m+1}$ for sufficiently large i, making $\langle N_0, W_0, N_1, W_1, \ldots \rangle$ illegal, contradiction.

Now suppose $F \underset{\text{code}}{\uparrow} PF_{F,C}(\kappa)$. For each $\alpha < \kappa^+$, let $\sigma_\alpha(N, W)$ be a winning coding strategy for $PF_{F,C}(\alpha)$ such that for $M \subseteq N$, $\sigma_\alpha(M, W) \subseteq \sigma_\alpha(N, W)$. We define the following strategy for F in $PF_{F,C}(\kappa^+)$:

$$\sigma(N, W) = N \cup \bigcup_{\alpha \in N} \sigma_{\alpha+1}(N \cap (\alpha+1), W \cap (\alpha+1))$$

Let $\langle N_0, W_0, N_1, W_1, \ldots \rangle$ be a legal play of the game with $N_{i+1} = \sigma(N_i, W_i)$. Suppose $\gamma \in W_i$ for infinitely-many i. Observe the play $\langle M_0, W_0 \cap (\gamma+1), M_1, W_1 \cap (\gamma+1), \ldots \rangle$ with $M_0 = N_0 \cap (\gamma+1)$ and $M_{i+1} = \sigma_{\gamma+1}(M_i, W_i \cap (\gamma+1)) \subseteq \sigma_{\gamma+1}(N_i \cap (\gamma+1), W_i \cap (\gamma+1))$.

Since $\sigma_{\gamma+1}$ is a winning strategy, $\gamma \in M_{m+1} \subseteq \sigma_{\gamma+1}(N_i \cap (\gamma+1), W_i \cap (\gamma+1)) \subseteq \sigma(N_i, W_i) = N_{i+1}$ for some sufficiently large m, making $\langle N_0, W_0, N_1, W_1, \ldots \rangle$ illegal, contradiction. □

Corollary 3.17. $O \underset{\textit{code}}{\uparrow} Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$ *for all cardinals* κ.

Proof. Let $\tau(N, W)$ be the winning coding strategy for F in $PF_{F,C}(\kappa)$, $N(U) \in [\kappa]^{<\omega}$ represent the non-$\mathbb{R}$ coordinates of a basic open set U of $\Sigma\mathbb{R}^\kappa$, and $W(p) \in [\kappa]^{\leq\omega}$ represent the non-0 coordinates of a point p in $\Sigma\mathbb{R}^\kappa$. For each $\alpha < \kappa$, let

$$\sigma_\alpha(U, p) = \begin{cases} \left(-\frac{1}{|N(U)|}, \frac{1}{|N(U)|}\right) & \text{if } \alpha \in \tau(N(U), W(p)) \\ \mathbb{R} & \text{otherwise} \end{cases}$$

and $\sigma(U, p) = \Sigma\mathbb{R}^\kappa \cap \prod_{\alpha<\kappa} \sigma_\alpha(U, p)$. □

Chapter 4

Locally Finite Games

Results pertaining to the Locally Finite game related to the W games.

4.1 basic results

Theorem 4.1. *The following are equivalent for a locally compact space X:*

- *X is paracompact*
- *$K \uparrow G_{K,L}(X)$.*

However, often it is the presence of "limited information" strategies which can characterize interesting properties of a space.

Definition 4.2. A **limited information strategy** for a game is a function whose domain is restricted to less information than all previous moves by the opposing player.

In the above mentioned paper, Gruenhage used the following limited information strategies to prove some interesting characterizations based on the game $G_{K,P}(X)$.

Definition 4.3. A **tactical strategy** considers only the most recent move by the opposing player. If Player Z has a winning tactical strategy for a game G, this may be denoted $Z \uparrow_{\text{tact}} G$.

Definition 4.4. A **Markov strategy** considers only the most recent move by the opposing player and the turn number. If Player Z has a winning Markov strategy for a game G, this may be denoted $Z \uparrow_{\text{mark}} G$.

Theorem 4.5. *The following are equivalent for a locally compact space X:*

- *X is metacompact*

- $K \uparrow_{tact} G_{K,P}(X)$.

Theorem 4.6. *The following are equivalent for a locally compact space X:*

- *X is σ-metacompact*

- $K \uparrow_{mark} G_{K,P}(X)$.

Upon learning these results, one might wonder the consequences of the existence of this type of limited information strategy:

Definition 4.7. A **predetermined strategy** considers only the turn number. If Player Z has a winning predetermined strategy for a game G, this may be denoted $Z \underset{\text{pre}}{\uparrow} G$.

Intuitively, if a player is using a predetermined strategy, then that player decides every move he or she will make before the game even begins, ignoring the other player's moves.

Consider the following trivial result:

Definition 4.8. A **button-mashing strategy** is a constant function. If Player Z has a winning button-mashing strategy for a game G, this may be denoted $Z \uparrow_{\text{mash}} G$.

Proposition 4.9. *The following are equivalent for any space X:*

- *X is compact*

- $K \uparrow_{mash} G_{K,P}(X)$.

Observing that giving K the added information of turn number to a tactical strategy (making it Markov) changed the characterization of a metacompact space into a σ-metacompact space, it would be very convenient if adding that same information to a button-mashing strategy (making it predetermined) would similarly change the characterization of a compact space into σ-compact.

Proposition 4.10. *If* $K \underset{pre}{\uparrow} G_{K,P}(X)$*, then* X *is* σ*-compact.*

Proof. Let K_n be the sets given by the winning predetermined strategy. If they did not union to X, then the counter play $p_n = p$ for some $p \in X \setminus \bigcup_n K_n$ would defeat the "winning" strategy. $\square$

Theorem 4.11. *If* Y *is a locally compact, Lindelöf space, then* $K \underset{pre}{\uparrow} G_{K,P}(X)$*.*

Proof. Let $\mathcal{K}$ be a collection of compact neighborhoods whose interiors cover X. By Lindelöf, let $\{K_n : n < \omega\}$ be a countable subcollection whose interiors cover X. We then define the predetermined strategy $\sigma(n) = \bigcup_{m \leq n} K_n$.

Let p_n give a play by P. If p is a cluster point of the p_n, then every open set about p contains infinitely many p_n. Let K_N be some compact neighborhood in $\{K_n : n < \omega\}$ which covers p. Then K_N contains infinitely many p_n, which means sometime after round N, P played in a set already covered by the strategy σ, which is an illegal move. Thus σ is a winning predetermined strategy. $\square$

Corollary 4.12. *The following are equivalent for a locally compact space* X*:*

- X *is* σ*-compact*
- X *is Lindelöf*
- $K \underset{pre}{\uparrow} G_{K,P}(X)$.

We now turn our attention to an example of a σ-compact space for which no predetermined strategy exists (which must, of course, not be locally compact). In fact, P will instead have a winning tactical strategy.

Definition 4.13. Let $M = \omega^2 \cup \{\infty\}$ denote the **metric fan space** with the topology generated by the singletons in ω^2 and sets of the form $((\omega \setminus n) \times \omega) \cup \{\infty\}$ for $n < \omega$.

Proposition 4.14. *For each compact set* C *in* M*, there exists a minimal dominating function* f_C *such that for each* $(x, y) \in C \setminus \{\infty\}$*,* $f(x) > y$*.*

Lemma 4.15. *$P \underset{mark}{\uparrow} G_{K,P}(M)$ where M is the metric fan space. (This implies $K \not\uparrow G_{K,P}(M)$.)*

Proof. Let P respond to the move $C \in K[X]$ by K on round n with the point $p = (n, s_C)$ such that $s_C = \min(\{y < \omega : f_C(n) < y\}$. It is easy to see that either $p_n \to \infty$, so P has a winning tactical strategy. $\square$

Furthermore, by a theorem due to Eric van Douwen...

Theorem 4.16. *Every first-countable non-locally countably compact space has the metric fan space M as a closed subspace.*

... we have the following corollary:

Corollary 4.17. *$P \underset{mark}{\uparrow} G_{K,P}(X)$ where X is a first-countable non-locally countably compact space. (This implies $K \not\uparrow G_{K,P}(X)$.)*

(Open question: does $P \underset{\text{tact}}{\uparrow} G_{K,P}(X)$?)

Theorem 4.18. *$P \underset{tact}{\not\uparrow} D_{K,P}(M)$ where M is the metric fan space.*

Proof. Give P the tactic σ. Suppose that for all $n < \omega$, there is an upper bound $m < \omega$ so that for each $C \in K[M]$, if $\pi_1(\sigma(C)) = n$, then $\pi_2(\sigma(C)) < m$. We may then define $f(n) = m$, and let $C_f = \{(x, y) : f(x) < y\} \in K[M]$. $\sigma(C_f)$ must show a contradiction if σ is legal.

So it follows that there is some $n < \omega$ such that there are compact sets $C_i \in K[M]$ with $\pi_1(\sigma(C_i)) = n$ and $\pi_2(\sigma(C_i)) < \pi_2(\sigma(C_{i+1}))$. The play $\langle C_0, \sigma(C_0), C_1, \sigma(C_1), \dots \rangle$ is a counter to σ. $\square$

While $K \underset{\text{pre}}{\uparrow} G_{K,P}(X)$ implies X is σ-compact, it in fact implies something stronger.

Definition 4.19. A space X is **hemicompact** if there exists a chain of increasing compact sets $K_0 \subseteq K_1 \subseteq \dots$ such that every compact set in X is a subset of some K_n.

Lemma 4.20. *If $K \underset{pre}{\uparrow} G_{K,P}(X)$, then X is hemicompact. Furthermore, any predetermined winning strategy for K witnesses hemicompactness.*

Proof. Let σ be a predetermined strategy for K in the game $G_{K,P}(Y)$ such that there exists a compact set C with $C \not\subseteq \sigma(n)$ for all n. On each turn, have P play some $y_n \in C \setminus \sigma(n)$. Then the y_n are an infinite subset of the compact set C and must have a cluster point in C, showing σ is not a winning strategy.

Thus if K has a winning predetermined strategy, it witnesses that Y is hemicompact. □

In fact, for locally compact spaces, finding winning predetermined strategies for $G_{K,P}(X)$ and $G_{K,L}(X)$ are equivalent problems.

Theorem 4.21. *The following are equivalent for any locally compact space X:*

- *X is hemicompact.*
- *$K \underset{pre}{\uparrow} G_{K,L}(X)$.*
- *$K \underset{pre}{\uparrow} G_{K,P}(X)$.*

Proof. Let Y be hemicompact, witnessed by $K_n = \sigma(n)$. Let $L_0, L_1, \ldots$ be a play by L in $G_{K,L}(X)$. Suppose that this play defeats σ. Then let $x \in X$ be the point such that for all neighborhoods U of x, U hits infinite L_n. Let C be a compact neighborhood of x, which must hit infinite L_n. As K_n witnesses hemicompactness, $C \subseteq K_N = \sigma(N)$ for some N. But then $C \subset K_N$ intersects infinitely many L_n, which shows that the play $L_0, L_1, \ldots$ was illegal. Thus σ defeats every legal play by L and is thus a winning predetermined strategy for K in $G_{K,L}(X)$.

We conclude by noting that any winning strategy for $G_{K,L}(X)$ is a winning strategy for $G_{K,P}(X)$, and the existence of a winning predetermined strategy for $G_{K,P}(X)$ implies hemicompact by the previous lemma. □

Corollary 4.22. *The following are equivalent for any locally compact space X:*

- *X is Lindelöf.*

- *X is σ-compact.*

- *X is hemicompact.*

- $K \underset{pre}{\uparrow} G_{K,L}(X)$.

- $K \underset{pre}{\uparrow} G_{K,P}(X)$.

The compact-point and compact-compact games are also useful in inspecting compactly generated "k"-spaces.

Definition 4.23. A topological space is called a ***k*-space** if the following condition is satisfied:

$$C \subseteq X \text{ is closed in } X \Leftrightarrow C \cap K \text{ is closed in } K \text{ for all compact sets } K \in K[X]$$

Definition 4.24. A topological space is called a **k_ω-space** if there exist $K_0, K_1, \cdots \in K[X]$ that satisfy the following condition:

$$C \subseteq X \text{ is closed in } X \Leftrightarrow C \cap K_n \text{ is closed in } K_n \text{ for all } n$$

Theorem 4.25. *The following are equivalent for any Hausdorff k-space X:*

- *X is hemicompact.*

- *X is k_ω.*

- $K \underset{pre}{\uparrow} G_{K,P}(X)$.

Furthermore, all predetermined strategies for K witness hemicompact and k_ω, and any witness to hemicompact/k_ω witnesses the other and serves as a predetermined strategy for K.

Proof. If X is hemicompact, then let it be witnessed by K_n. We claim K_n also witnesses k_ω. Note that the forward implication of k_ω always holds for T_1 spaces as $C \cap K_n$ is closed in X, and thus in every K_n. So assume $C \cap K_n$ is closed in K_n for all n. Let H be any compact set. As X is hemicompact, $H \subseteq K_n$ for some n. Note $C \cap H = (C \cap K_n) \cap H$. As both $C \cap K_n$ and H are closed in K_n, $C \cap H$ is closed in K_n, and thus $C \cap H$ is closed in H. As Y is k and $C \cap H$ is closed in H for all compact H, C is closed, showing the backwards implication.

Now if Y is k_ω, let it be witnessed by K_n. Give K the predeterined strategy $\sigma(n) = K_n$ for the game $G_{K,P}(X)$, and let p_n be the result of a legal counter by P. Suppose by way of contradiction that p is a cluster point of the p_n. Note $p \in \sigma(N)$ for some N. p is a cluster point of $\{p_n : n \geq N\}$ but $p \notin \{p_n : n \geq N\}$. Also, $\{p_n : n \geq N\} \cap \sigma(m)$ is finite for all m, and thus closed, so as $\sigma(n)$ witnesses k_ω, $\{p_n : n \geq N\}$ is closed and must contain its cluster point p, which is a contradiction. Thus σ is a winning predetermined strategy for K in $G_{K,P}(Y)$.

Finally, if $K \underset{\text{pre}}{\uparrow} G_{K,P}(X)$, X is hemicompact by the previous lemma. $\square$

For k-spaces, it turns out that finding winning predetermined strategies for $G_{K,P}(X)$ and $G_{K,L}(X)$ are also equivalent problems.

Theorem 4.26. *For any hemicompact Hausdorff k-space X, $K \underset{pre}{\uparrow} G_{K,L}(X)$.*

Proof. Let X's hemicompactness be witnessed by $K_n = \sigma(n)$. Note that this also witnesses k_ω by the proof of the previous theorem. Let $H_0, H_1, \ldots$ be a counter by H for the game $G_{K,L}(X)$ in response to σ. Suppose by way of contradiction the counter was legal and defeats σ. Then there is a point x such that every neighborhood of x hits infinitely many of the H_n.

Now, $x \in \sigma(N)$ for some N, and since the play $H_0, H_1, \ldots$ is legal, $x \notin H_n$ for all $n \geq N$. Consider the set $H_\omega = \bigcup_{n \geq N} H_n$. Note that as the K_n witness k_ω, H_ω is closed if and only if $H_\omega \cap \sigma(m)$ is closed in $\sigma(m)$ for all m. In fact, since every H_n is a subset of some $\sigma(m)$ (by hemicompactness), $H_\omega \cap \sigma(m)$ is a finite union of some H_n, and is thus closed in Y.

We thus have that H_ω is a closed set not containing x. But since every neighborhood of x intersects H_ω, x is a limit point of the closed set H_ω and should be included, demonstrating our contradiction. Thus σ is a winning predetermined strategy for K in the game $G_{K,L}(X)$.

$\square$

Corollary 4.27. *The following are equivalent for any Hausdorff k-space X:*

- *X is hemicompact.*
- *X is k_ω.*
- *K has a winning predetermined strategy in $G_{K,L}(X)$.*
- *K has a winning predetermined strategy in $G_{K,P}(X)$.*

It's natural to question whethere there is ever any difference between finding winning predetermined strategies for $G_{K,P}(X)$ and $G_{K,L}(X)$. We now look to a (non-locally compact, non-k) Hausdorff space where the distinction arises:

Definition 4.28. Given a set X, an ultrafilter on X is a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ such that

1. $\emptyset \notin \mathcal{F}$
2. $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
3. $A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$
4. $\forall A \subseteq X (A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F})$

As a result, ultrafilters which contain a finite set contain only one singleton (and are called **principal**). Otherwise, ultrafilters which contain no finite sets are called **free**.

Definition 4.29. The **Stone-Cech compactification** $\beta\omega$ of ω is the collection of ultrafilters on ω. The principal ultrafilters containing a singleton $\{n\}$ are each identified with n itself and are isolated. Free ultrafilters $\mathcal{F}$ are given neighborhoods of the form

$$\{\mathcal{G} : \mathcal{G} \text{ is an ultrafilter on } \omega \text{ and } A \in \mathcal{G}\} = A \cup \{\mathcal{G} : \mathcal{G} \text{ is a free ultrafilter on } \omega \text{ and } A \in \mathcal{G}\}$$

for each $A \in \mathcal{F}$.

Alternately $\beta\omega = \omega \cup \{\mathcal{F} : \mathcal{F}$ is a free ultrafilter on $\omega\}$ where ω is discrete and the free ultrafilters have the local base described above.

Definition 4.30. A **single-ultrafilter space** is a subset of $\beta\omega$ containing all elements of ω and a single ultrafilter $\mathcal{F}$.

Proposition 4.31. *The compact sets of a single-ultrafilter space are exactly the finite subsets of the space. Thus a single-ultrafilter space is neither locally compact nor* k*.*

Regardless of the ultrafilter chosen, we can see that K has no hope of having a winning predetermined strategy for $G_{K,L}$ played on a single-ultrafilter space.

Proposition 4.32. *If* X *is any single-ultrafilter space with the ultrafilter* $\mathcal{F}$*, then* $K \underset{pre}{\not\uparrow} G_{K,L}(X)$*.*

Proof. Compact sets are exactly finite sets in this space. Therefore, the difference of any two compact sets is compact.

Give K the predetermined strategy $\sigma(n)$. H counters with

$$H_n = (n \cup \sigma(n+1)) \setminus \sigma(n)$$

on turn n. Since any free ultrafilter contains only unbounded sets, every neighborhood $A \cup \{\mathcal{F}\}$ of $\mathcal{F}$ must intersect infinitely many H_n, defeating σ. □

However, while it is consistant that there is an ultrafilter which defies the existance of a predetermined winning strategy for K in $G_{K,P}$...

Proposition 4.33. *If a selective ultrafilter* $\mathcal{F}$ *exists (this is independent of ZFC), then* K *has no winning predetermined strategy in the compact-point game* $G_{K,P}(Y)$ *for the single selective ultrafilter space* $Y = \omega \cup \{\mathcal{F}\}$*.*

Proof. Let σ be a predetermined strategy for K. By the definition of a selective ultrafilter, for every partition $\{B_n : n < \omega\}$ of subsets of ω such that $B_n \notin \mathcal{F}$ for all n, there exists $A \in \mathcal{F}$ such that $|A \cap B_n| = 1$ for all n. So then let

$$B_n = \omega \cap \sigma(n) \setminus \sigma(n-1)$$

Note that B_n is finite and thus $B_n \notin \mathcal{F}$, so there exists $A \in \mathcal{F}$ such that $|A \cap B_n| = 1$. Let p_n be the singleton in $A \cap B_{n+1}$, so $\{p_n : n < \omega\}$ is cofinite in A, and thus is also a member of $\mathcal{F}$. Thus p_n converges to $\mathcal{F}$, and counters the strategy σ. $\square$

... in general we can find many ultrafilters for which $K \underset{\text{pre}}{\uparrow} G_{K,P}$.

Theorem 4.34. *Let a_n be a sequence such that the sequence $\frac{a_n}{n}$ is unbounded above. Then there is an ultrafilter $\mathcal{F}$ such that $\sigma(n) = (\sum_{m\leq n} a_m) \cup \{\mathcal{F}\}$ is a winning predetermined strategy for K in $G_{K,P}(\omega \cup \{\mathcal{F}\})$.*

Proof. Let $\mathcal{P}$ be the collection of all legal plays by P against the strategy σ. Consider a finite collection of plays $P_0, \ldots, P_{n-1} \in \mathcal{P}$. As $\frac{a_m}{m}$ is unbounded above, we may find infinitely many m such that $\frac{a_m}{m} > n \Rightarrow mn < a_m$. As the a_m partition ω such that P may only play at most m points in each part, there are infinitely many parts which are not filled, and thus $\bigcup_{m<n} P_m$ is not cofinite.

It then follows that the closure of $\mathcal{P}$ under finite unions and subsets, along with all finite sets, is an ideal. Its dual filter may then be extended to an ultrafilter $\mathcal{F}$ such that every possible play by P is the complement of some member of $\mathcal{F}$. $\square$

So we can see that there are non-k spaces X for which $K \underset{\text{pre}}{\uparrow} G_{K,P}(X)$. However, we have found no such spaces for the game $G_{K,L}(X)$. So we conclude with this open question:

Question 4.35. $K \underset{pre}{\uparrow} G_{K,L}(X) \Rightarrow X$ *is a k-space?*

4.2 Cantor space example

Example 4.36. Let X be a zero-dimensional, compact L-space (hereditarally Lindeloff and non-separable). It is a fact that there exists a point-countable collection $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ of clopen sets in X, and it is also true that any point-finite subcollection of $\mathcal{U}$ is countable.

Let $C = \{c_\alpha : \alpha < \omega_1\}$ be any uncountable subset of the Cantor space 2^ω. Let $X_s = X \times \{s\}$ for each $s \in 2^{<\omega}$, and $U_{\alpha,s} = U_\alpha \times \{s\}$.

Finally, let

$$\mathbb{X} = C \cup \bigcup_{s \in 2^{<\omega}} X_s$$

be a tree of $2^{<\omega}$ copies of X, and where

$$c_\alpha \cup \bigcup_{n<\omega} U_{\alpha, x_\alpha \restriction n}$$

is an open set about each c_α.

Definition 4.37. Let $S \in [\omega_1]^{<\omega}$ and $m < \omega$. Define

$$K_S = \bigcup_{\alpha \in S} \left(c_\alpha \cup \left(\bigcup_{s < c_\alpha} U_{\alpha,s} \right) \right)$$

$$A = \{z^\frown\langle 1 \rangle : z \in 1^{<\omega}\}$$

$$K_S^* = K_S \setminus \bigcup_{s \in A} X_s$$

and

$$L_m = \bigcup_{s \in 2^{<m}} X_s$$

and observe that every compact set is dominated by $K_S^* \cup L_m$ for some S, m. Intuitively, K_S^* collects the branches of U_α converging up to c_α for each $\alpha \in S$ while avoiding copies X_s of X for each s in an antichain A, and L_m collects the copies X_s of X with $|s| < m$ at the base of the tree.

Proposition 4.38. *Without loss of generality, P always plays points in $\bigcup_{s\in 2^{<\omega}} X_s$.*

Proposition 4.39. $K \uparrow LF_{K,P}(\mathbb{X})$.

Proof. In response to a point $\langle x, s\rangle$, K observes that there are only countably many α such that $U_\alpha \times \{s\}$ contains $\langle x, s\rangle$ (by point-countability of X). Enumerate these as α_n. K makes a promise that during round m, K will forbid some superset of $K_{\{\alpha_n : n\leq m\}}$. Finally, K also always forbids a superset of $L_{|s|+1}$.

Suppose P's moves clustered at some point. Since K forbade $L_{|s|+1}$ during each round, that point must be c_α for some α. P's play then must have included a subsequence of points $\langle x_0, s_0\rangle, \langle x_1, s_1\rangle, \langle x_2, s_2\rangle \ldots$ such that $x_n \in U_\alpha$ and $s_n \leq s_{n+1} \leq c_\alpha$. However, in response to $\langle x_0, s_0\rangle$, K made a promise to eventually forbid a superset of $K_{\{\alpha\}}$, making every $\langle x_n, t_n\rangle$ illegal after that round. □

Theorem 4.40. $K \underset{tact}{\not\uparrow} LF_{K,P}(\mathbb{X})$.

Proof. This is actually a corollary of G's theorem in [?]. The following is a direct game-theoretic proof.

Suppose that $\sigma(\langle x, s\rangle)$ was a winning strategy for K and assume

$$\sigma(\langle x, s\rangle) = \bigcup_{|t|\leq|s|} \sigma(\langle x, t\rangle) = \sigma'(x, |s|)$$

Thus there exists some $f : \omega_1 \to \omega$ such that $\sigma'(x, f(\alpha))$ covers every neighborhood of c_α for all $x \in U_\alpha$. (If not, P wins by taking the α for which f is not defined, and may always play $\langle x, s\rangle$ in a neighborhood of c_α for which $\sigma'(x, |s|)$ doesn't cover a neighborhood of c_α.) Fix n for which $f(\alpha) = n$ for α in an uncountable set A.

Since the collection $\{U_\alpha : \alpha \in A\}$ is uncountable, it is not point-finite. Fix x so that x belongs to U_α for all α in an infinite $B \subseteq A$. Finally, consider $\sigma'(x, n)$. For each $\alpha \in B$, $\sigma'(x, f(\alpha)) = \sigma'(x, n)$ covers c_α. Since $\{c_\alpha : \alpha \in B\}$ is a closed infinite discrete set, we have a contradiction to the compactness of $\sigma'(x, n)$. □

Theorem 4.41. $K \underset{2\text{-}tact}{\not\uparrow} LF_{K,P}(\mathbb{X})$.

Proof. Suppose $\sigma(\langle x, s\rangle, \langle y, t\rangle)$ was a winning 2-tactical strategy. We may define $S(x, y, n) \in [\omega_1]^{<\omega}$ (increasing on n) and $n < m(x, y, n) < \omega$ such that for each (x, y),

$$\bigcup_{s,t\in 2^{\leq n}} \sigma(\langle x, s\rangle, \langle y, t\rangle) \subseteq K^*_{S(x,y,n)} \cup L_{m(x,y,n)}$$

and so we assume

$$\sigma(\langle x, s\rangle, \langle y, t\rangle) = K^*_{S(x,y,\max(|s|,|t|))} \cup L_{m(x,y,\max(|s|,|t|))}$$

Select an arbitrary point $x' \in X$. We define a tactical strategy

$$\tau(x, s) = K^*_{S(x,x',m(x,x',|s|)+1)} \cup L_{m(x,x',m(x,x',|s|)+1)}$$

We complete the proof by showing τ is a winning tactical strategy (a contradiction).

Suppose

$$\langle x_0, s_0\rangle, \langle x_1, s_1\rangle, \langle x_2, s_2\rangle, \ldots$$

successfully countered τ by clustering at $c \in C$ (the strategy trivially prevents clustering elsewhere). Let $z_n = \langle 0, \ldots, 0\rangle$ with n zeros. We claim

$$\langle x_0, s_0\rangle, \langle x', z_{m(x_0,x',|s_0|)}{}^\frown\langle 1\rangle\rangle, \langle x_1, s_1\rangle, \langle x', z_{m(x_1,x',|s_1|)}{}^\frown\langle 1\rangle\rangle, \langle x_2, s_2\rangle, \langle x', z_{m(x_2,x',|s_2|)}{}^\frown\langle 1\rangle\rangle, \ldots$$

is a successful counter to σ.

We will need the fact that, as $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ:

$$|s_i| < m(x_i, x', |s_i|) + 1 = |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|$$

$$< m(x_i, x', m(x_i, x', |s_i|) + 1) = m(x_i, x', |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|) \leq |s_{i+1}|$$

Note that $m(x, y, \max(|s|, |t|))$ is increasing throughout this play of the game versus σ:

$$m(x_i, x', \max(|s_i|, |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|))$$

$$= m(x_i, x', |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|)$$

$$\leq |s_{i+1}|$$

$$< m(x_{i+1}, x', |s_{i+1}|)$$

$$= m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|))$$

$$= |z_{m(x_{i+1},x',|s_{i+1}|)}|$$

$$< |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle|$$

$$< m(x_{i+1}, x', |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle|)$$

$$= m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle|))$$

We turn to showing that $\langle x', z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle\rangle$ is always a legal move. Since $z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle$ is on the antichain avoided by any K^*, the problem is reduced to showing that this move isn't forbidden by

$$L_{m(x_{i+1},x',\max(|s_{i+1}|,|z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|))}$$

which we can see here:

$$m(x_{i+1}, x', \max(|s_{i+1}|, |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|)) = m(x_{i+1}, x', |s_{i+1}|) < |z_{m(x_{i+1},x',|s_{i+1}|)}{}^\frown\langle 1\rangle|$$

We can conclude by showing that $\langle x_{i+1}, s_{i+1}\rangle$ is always a legal move. We can see it avoids

$$L_{m(x_i,x',\max(|s_i|,|z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|))}$$

since

$$m(x_i, x', \max(|s_i|, |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|)) = m(x_i, x', |z_{m(x_i,x',|s_i|)}{}^\frown\langle 1\rangle|) \leq |s_{i+1}|$$

Since $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ, it avoided

$$K^*_{S(x_h,x',m(x_h,x',|s_h|)+1)} = K^*_{S(x_h,x',\max(|s_h|,|z_{m(x_h,x',|s_h|)}{}^\frown\langle 1\rangle|))}$$

for $h \leq i$. And when $h < i$, we see it avoids:

$$K^*_{S(x_{h+1},x',\max(|s_{h+1}|,|z_{m(x_h,x',|s_h|)}{}^\frown\langle 1\rangle|))} = K^*_{S(x_{h+1},x',|s_{h+1}|)}$$

$$\subseteq K^*_{S(x_{h+1},x',m(x_{h+1},x',|s_{h+1}|)+1)}$$

This concludes the proof. $\square$

Theorem 4.42. *$K \underset{k\text{-tact}}{\not\uparrow} LF_{K,P}(\mathbb{X})$.*

Proof. The proof proceeds in parallel to the proof of $K \underset{\text{2-tact}}{\not\uparrow} LF_{K,P}(\mathbb{X})$.

Suppose $\sigma(\langle x_0, s_0\rangle, \ldots, \langle x_k, s_k\rangle)$ was a winning $(k+1)$-tactical strategy. We may define $S(x_0, \ldots, x_k, n) \in [\omega_1]^{<\omega}$ (increasing on n) and $n < m(x_0, \ldots, x_k, n) < \omega$ such that for each $(x_0, \ldots, x_k)$,

$$\bigcup_{s_0,\ldots,s_k \in 2^{\leq n}} \sigma(\langle x_0, s_0\rangle, \ldots, \langle x_k, s_k\rangle) \subseteq K^*_{S(x_0,\ldots,x_k,n)} \cup L_{m(x_0,\ldots,x_k,n)}$$

and so we assume

$$\sigma(\langle x_0, s_0\rangle, \ldots, \langle x_k, s_k\rangle) = K^*_{S(x_0,\ldots,x_k,\max(|s_0|,\ldots,|s_k|))} \cup L_{m(x_0,\ldots,x_k,\max(|s_0|,\ldots,|s_k|))}$$

Select an arbitrary point $x' \in X$. Let $M^0(x,n) = m(x, x', \ldots, x', n)$ and $M^{i+1}(x,n) = M^0(x, M^i(x,n)+1)$. We define a tactical strategy

$$\tau(x,s) = K^*_{S(x,x',\ldots,x',M^{k-1}(x,|s|)+1)} \cup L_{m(x,x',\ldots,x',M^{k-1}(x,|s|)+1)}$$

We complete the proof by showing τ is a winning tactical strategy (a contradiction).

Suppose

$$\langle x_0, s_0\rangle, \langle x_1, s_1\rangle, \langle x_2, s_2\rangle, \ldots$$

successfully countered τ by clustering at $c \in C$ (the strategy trivially prevents clustering elsewhere). Let $z_n = \langle 0, \ldots, 0\rangle$ with n zeros. We claim

$$\langle x_0, s_0\rangle, \langle x', z_{M^0(x_0,|s_0|)}{}^\frown\langle 1\rangle\rangle, \langle x', z_{M^1(x_0,|s_0|)}{}^\frown\langle 1\rangle\rangle, \ldots, \langle x', z_{M^{k-1}(x_0,|s_0|)}{}^\frown\langle 1\rangle\rangle,$$

$$\langle x_1, s_1\rangle, \langle x', z_{M^0(x_1,|s_1|)}{}^\frown\langle 1\rangle\rangle, \langle x', z_{M^1(x_1,|s_1|)}{}^\frown\langle 1\rangle\rangle, \ldots, \langle x', z_{M^{k-1}(x_1,|s_1|)}{}^\frown\langle 1\rangle\rangle, \ldots$$

is a successful counter to σ.

We will need the fact that, as $\langle x_{i+1}, s_{i+1}\rangle$ was legal against τ:

$$|s_i| < M^0(x_i,|s_i|)+1 = |z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle| < M^0(x_i, M^0(x_i,|s_i|)+1)+1$$

$$= M^1(x_i,|s_i|)+1 = |z_{M^1(x_i,|s_i|)}{}^\frown\langle 1\rangle| < \cdots < |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|$$

$$= M^{k-1}(x_i,|s_i|)+1 < m(x_i, x', \ldots, x', M^{k-1}(x_i,|s_i|)+1) \le |s_{i+1}|$$

Note that $m(x_0, \ldots, x_k, \max(|s_0|, \ldots, |s_k|))$ is increasing throughout this play of the game versus σ:

$$m(x_i, x', \ldots, x', \max(|s_i|, |z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|))$$

$$= m(x_i, x', \ldots, x', |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|)$$

$$= m(x_i, x', \ldots, x', M^{k-1}(x_i, |s_i|) + 1)$$

$$\leq |s_{i+1}|$$

$$< M^0(x_{i+1}, |s_{i+1}|)$$

$$= m(x_{i+1}, x', \ldots, x', |s_{i+1}|)$$

$$= m(x_{i+1}, x', \ldots, x', \max(|s_{i+1}|, |z_{M^0(x_i,|s_i|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|))$$

$$= |z_{m(x_{i+1},x',\ldots,x',|s_{i+1}|)}|$$

$$= |z_{M^0(x_{i+1},|s_{i+1}|)}|$$

$$< |z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|$$

$$< m(x_{i+1}, x', \ldots, x', |z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|)$$

$$= m(x_{i+1}, x', \ldots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|, |z_{M^1(x_i,|s_i|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_i,|s_i|)}{}^\frown\langle 1\rangle|))$$

$$\vdots$$

$$< m(x_{i+1}, x', \ldots, x', \max(|s_{i+1}|, |z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|, \ldots, |z_{M^{k-1}(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|))$$

We turn to showing that $\langle x', z_{M^j(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle\rangle$ is always a legal move. Since $z_{M^j(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle$ is on the antichain avoided by any K^*, the problem is reduced to showing that this move isn't forbidden by

$$L_{m(x_{i+1},x',\ldots,x',\max(|s_{i+1}|,|z_{M^0(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|,\ldots,|z_{M^{j-1}(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|,|z_{M^j(x_i,|s_i|)}{}^\frown\langle 1\rangle|,\ldots,|z_{M^k(x_i,|s_i|)}{}^\frown\langle 1\rangle|))}$$

$$= L_{m(x_{i+1},x',\ldots,x',|z_{M^{j-1}(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|)}$$

which we can see here:

$$m(x_{i+1}, x', \ldots, x', |z_{M^{j-1}(x_{i+1},|s_{i+1}|)}{}^\frown\langle 1\rangle|)$$

$$= m(x_{i+1}, x', \ldots, x', M^{j-1}(x_{i+1}, |s_{i+1}|) + 1)$$

$$= M^0(x_{i+1}, M^{j-1}(x_{i+1}, |s_{i+1}|) + 1)$$

$$= M^j(x_{i+1}, s_{i+1})$$

$$< |z_{M^j(x_{i+1}, |s_{i+1}|)} {}^\frown \langle 1 \rangle|$$

We can conclude by showing that $\langle x_{i+1}, s_{i+1} \rangle$ is always a legal move. We can see it avoids

$$L_{m(x_i, x', \ldots, x', \max(|s_i|, |z_{M^0(x_i,|s_i|)} {}^\frown \langle 1 \rangle|, \ldots, |z_{M^{k-1}(x_i,|s_i|)} {}^\frown \langle 1 \rangle|))}$$

since

$$m(x_i, x', \ldots, x', \max(|s_i|, |z_{M^0(x_i,|s_i|)} {}^\frown \langle 1 \rangle|, \ldots, |z_{M^{k-1}(x_i,|s_i|)} {}^\frown \langle 1 \rangle|))$$

$$= m(x_i, x', \ldots, x', |z_{M^{k-1}(x_i,|s_i|)} {}^\frown \langle 1 \rangle|)$$

$$= m(x_i, x', \ldots, x', M^{k-1}(x_i, |s_i|) + 1)$$

$$\leq |s_{i+1}|$$

Since $\langle x_{i+1}, s_{i+1} \rangle$ was legal against τ, it avoided

$$K^*_{S(x_h, x', \ldots, x', M^{k-1}(x_h, |s_h|)+1)}$$

$$= K^*_{S(x_h, x', \ldots, x', \max(|s_h|, |z_{M^0(x_h,|s_h|)} {}^\frown \langle 1 \rangle|, \ldots, |z_{M^{k-1}(x_h,|s_h|)} {}^\frown \langle 1 \rangle|))}$$

for $h \leq i$. And when $h < i$, we see it avoids both:

$$K^*_{S(x_{h+1}, x', \ldots, x', \max(|s_{h+1}|, |z_{M^0(x_{h+1},|s_{h+1}|)} {}^\frown \langle 1 \rangle|, \ldots, |z_{M^{j-1}(x_{h+1},|s_{h+1}|)} {}^\frown \langle 1 \rangle|, |z_{M^j(x_h,|s_h|)} {}^\frown \langle 1 \rangle|, \ldots, |z_{M^k(x_h,|s_h|)} {}^\frown \langle 1 \rangle|))}$$

$$= K^*_{S(x_{h+1}, x', \ldots, x', |z_{M^{j-1}(x_{h+1},|s_{h+1}|)} {}^\frown \langle 1 \rangle|)}$$

$$= K^*_{S(x_{h+1}, x', \ldots, x', M^{j-1}(x_{h+1},|s_{h+1}|)+1)}$$

$$\subseteq K^*_{S(x_{h+1},x',\ldots,x',M^{k-1}(x_{h+1},|s_{h+1}|)+1)}$$

and:

$$K^*_{S(x_{h+1},x',\ldots,x',\max(|s_{h+1}|,|z_{M^0(x_h,|s_h|)}{}^\frown\langle 1\rangle|,\ldots,|z_{M^k(x_h,|s_h|)}{}^\frown\langle 1\rangle|))}$$

$$= K^*_{S(x_{h+1},x',\ldots,x',|s_{k+1}|)}$$

$$\subseteq K^*_{S(x_{h+1},x',\ldots,x',M^{k-1}(x_{h+1},|s_{h+1}|)+1)}$$

This concludes the proof. □

4.3 various examples

Example 4.43. If $\mathcal{F}$ is a free ultrafilter on ω, let $L(\mathcal{F}) = \omega \cup \{\mathcal{F}\}$ as a subspace of the Stone-Cech compactification $\beta\omega$ be the **single ultrafilter line**. There is some ultrafilter $\mathcal{F}$ such that $K \underset{\text{pre}}{\uparrow} LF_{K,P}(L(\mathcal{F}))$ and $K \underset{\text{tact}}{\uparrow} LF_{K,P}(L(\mathcal{F}))$.

($L(\mathcal{F})$ is not compactly generated, and thus not locally compact.)

Proof. Let a_n be a sequence such that the sequence $\frac{a_n}{n}$ is unbounded above. Then there is an ultrafilter $\mathcal{F}$ such that $\sigma(n) = (\sum_{m\leq n} a_m) \cup \{\mathcal{F}\}$ is a winning predetermined strategy for K in $LF_{K,P}(L(\mathcal{F}))$.

Let $\mathcal{P}$ be the collection of all legal plays by P against the strategy σ. Consider a finite collection of plays $P_0, \ldots, P_{n-1} \in \mathcal{P}$. As $\frac{a_m}{m}$ is unbounded above, we may find infinitely many m such that $\frac{a_m}{m} > n \Rightarrow mn < a_m$. As the a_m partition ω such that P may only play at most m points in each part, there are infinitely many parts which are not filled, and thus $\bigcup_{m<n} P_m$ is not cofinite.

It then follows that the closure of $\mathcal{P}$ under finite unions and subsets, along with all finite sets, is an ideal. Its dual filter may then be extended to an ultrafilter $\mathcal{F}$ such that every possible play by P is the complement of some member of $\mathcal{F}$, making σ a winning predetermined strategy.

A winning tactic can then be easily constructed by using the moves by P as the round number in the predetermined strategy. $\square$

Example 4.44. Let $T(\mathcal{F}) = 2^{\leq\omega}$ where $2^{<\omega}$ is discrete and for each $c \in 2^\omega$, $\{c \upharpoonright \alpha : \alpha \leq \omega\}$ is homeomorphic to $L(\mathcal{F})$. This is called the **single ultrafilter tree**. There is some ultrafilter $\mathcal{F}$ such that $K \underset{\text{pre}}{\uparrow} LF_{K,P}(L(\mathcal{F}))$ and $K \underset{\text{tact}}{\uparrow} LF_{K,P}(L(\mathcal{F}))$.

($T(\mathcal{F})$ is not compactly generated, and thus not locally compact.)

Proof. Assume without loss of generality that P does not play points in 2^ω.

We use a winning predetermined strategy $\sigma^*(n)$ for $L(\mathcal{F})$ and let $\sigma(n) = \bigcup_{m \in \sigma^*(n)} 2^m$. Note that if P has a counter which converges to some $c \in {}^\omega 2$, then P would have a counter within a single branch. Since each branch is homeomorphic to $L(\mathcal{F})$; this is impossible.

A winning tactic can then be easily constructed by using the moves by P, taking the level of the tree played upon as the round number in the predetermined strategy. $\square$

Example 4.45. Let $M = \omega^2 \cup \{\infty\}$ be the **metric fan** where ω^2 is discrete and ∞ has neighborhoods of the form $M \setminus (n \times \omega)$ for any $n < \omega$. Then $K \not\uparrow LF_{K,P}(M)$. (In fact, $P \underset{\text{mark}}{\uparrow} LF_{K,P}(M)$.)

(M is not locally compact, but is compactly generated.)

Proof. For each compact set C in M, there exists a minimal dominating function f_C such that for each $(x, y) \in C \setminus \{\infty\}$, $f(x) > y$.

So let P respond to the move $C \in K[X]$ by K on round n with the point $p = (n, s_C)$ such that $s_C = \min(\{y < \omega : f_C(n) < y\}$. It is easy to see that $p_n \to \infty$, so P has a winning Markov strategy. $\square$

Example 4.46. Let $S = \omega^2 \cup \{\infty\}$ be the **sequential fan** where ω^2 is discrete and ∞ has neighborhoods of the form $M \setminus \{(x, y) : x < f(y)\}$ for any $f : \omega \to \omega$. Then $K \underset{\text{pre}}{\uparrow} LF_{K,P}(S)$ and $K \underset{\text{tact}}{\uparrow} LF_{K,P}(S)$.

(S is not locally compact, but is compactly generated.)

Proof. Let $\sigma(n) = \omega \times (n+1) \cup \{\infty\}$. By defining $f(y)$ to be greater than the x-coordinate of all P's plays through round y, we see that $M \setminus \{(x, y) : x < f(y)\}$ misses every move by P, so P cannot converge to ∞.

A winning tactic can be easily constructed by using the y-coordinate of P's moves as the round number in the predetermined strategy. $\square$

Chapter 5

Menger Game

Results pertaining to the Menger game characterizing the Menger property.

5.1 cut-and-paste

Definition 5.1. X is **Menger** if for all open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$ there exist finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Proposition 5.2. *σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelof*

Definition 5.3. In the two-player game $Cov_{C,F}(X)$ player C chooses open covers $\mathcal{U}_n$ of X, followed by player F choosing a finite subcollection $\mathcal{F}_n \subseteq \mathcal{U}_n$. F wins if $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Theorem 5.4. *X is Menger if and only if $C \not\uparrow Cov_{C,F}(X)$.*

Proof. Result due to Hurewicz.

First, suppose X wasn't Menger. Then there would exist open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of X such that for any choice of finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$, $\bigcup_{n<\omega} \mathcal{F}_n$ isn't a cover of X. Thus $C \underset{\text{pre}}{\uparrow} Cov_{C,F}(X) \Rightarrow S \not\uparrow Cov_{C,F}(X)$.

Now, assume X is Menger, and consider a strategy for C in $Cov_{C,F}(X)$.

Since X is Lindelof, we can assume C plays only countable covers of X. Then, since F is choosing finite subsets, we may assume F chooses some initial segement of the countable cover. In turn, we can assume C plays an increasing open cover $\{U_0, U_1, \ldots\}$ where $U_n \subseteq U_{n+1}$. And in that case, it's suffient to assume F simply chooses a singleton subset of each cover. And finally, since choices made by F are already covered, we can assume that every open set in a cover played by C covers the sets chosen by F previously.

As a result, we have the following figure of a tree of plays which I need to draw:

(Insert figure here.)

Note that for $a, b \in \omega^{<\omega}$ and $m \leq n$, we know:

(a) $U_{a^\frown m} \subseteq U_{a^\frown n}$

(for example, $U_{1627} \subseteq U_{1629}$ - increasing the final digit yields supersets)

(b) $U_a \subseteq U_{a^\frown b}$

(for example, $U_{1627} \subseteq U_{162789}$ - appending any sequence to the end yields supersets)

(c) $U_{a^\frown m} \subseteq U_{a^\frown n} \subseteq U_{a^\frown n^\frown b} \subseteq U_{a^\frown n^\frown b^\frown m}$

(for example: $U_{1627} \subseteq U_{1629283287}$ - injecting a subseqence with initial number larger than the original's final number, prior to the final number, yields supersets)

We may observe that if F can find an $f : \omega \to \omega$ such that $\bigcup_{n<\omega} U_{f\restriction(n+1)} = X$, she can use $\{U_{f\restriction 0}\}, \{U_{f\restriction 1}\}, \ldots$ to counter C's strategy.

Let $V_k^n = \bigcap\limits_{a\in\omega^{\leq n}} U_{a^\frown k}$. We claim that (1) V_k^n is open, (2) $\mathcal{V}^n = \{V_0^n, V_1^n, \ldots\}$ is increasing, and (3) $\mathcal{V}^n$ is a cover. Proofs:

1. Since due to (c) for each $b \in \omega^{\leq n} \setminus k^{\leq n}$, there is an $a \in k^{\leq n}$ with $U_{a^\frown k} \subseteq U_{b^\frown k}$:

$$V_k^n = \bigcap_{a\in\omega^{\leq n}} U_{a^\frown k} = \bigcap_{a\in k^{\leq n}} U_{a^\frown k} \cap \bigcap_{b\in\omega^{\leq n}\setminus k^{\leq n}} U_{b^\frown k} = \bigcap_{a\in k^{\leq n}} U_{a^\frown k}$$

   making V_k^n a finite intersection of open sets.

2. We show $V_k^0 \subseteq V_{k+1}^0$:

$$V_k^0 = U_k \subseteq U_{k+1} = V_{k+1}^0$$

   and then assume $V_k^n \subseteq V_{k+1}^n$:

$$V_k^{n+1} = \bigcap_{a\in\omega^{\leq n+1}} U_{a^\frown k} = V_k^n \cap \bigcap_{a\in\omega^{n+1}} U_{a^\frown k} \subseteq V_{k+1}^n \cap \bigcap_{a\in\omega^{n+1}} U_{a^\frown(k+1)} = V_{k+1}^{n+1}$$

3. We easily see that $\mathcal{V}^0 = \{U_0, U_1, \dots\}$ is a cover, and then assume $\mathcal{V}^n$ is a cover.

   Let $x \in X$ and pick $l < \omega$ such that $x \in V_l^n$. For $a \in l^{n+1}$ choose l_a such that $x \in U_{a^\frown l_a}$, giving

$$x \in \bigcap_{a \in l^{n+1}} U_{a^\frown l_a}$$

   We will assume $k > l, l_a$ for all $a \in l^{\leq n+1}$.

   For any $a \in k^{n+1} \setminus l^{n+1}$ note that $a = b^\frown c$ where $b \in l^{\leq n}$ and c begins with a number l or greater:

$$V_l^n \subseteq U_{b^\frown l} \subseteq U_{b^\frown c} \subseteq U_{b^\frown c^\frown l_a} = U_{a^\frown l_a}$$

   Thus:

$$\begin{aligned} x &\in V_l^n \cap \left(\bigcap_{a \in l^{n+1}} U_{a^\frown l_a} \right) \\ &= V_l^n \cap \left(\bigcap_{a \in k^{n+1} \setminus l^{n+1}} U_{a^\frown l_a} \right) \cap \left(\bigcap_{a \in l^{n+1}} U_{a^\frown l_a} \right) \\ &= V_l^n \cap \left(\bigcap_{a \in k^{n+1}} U_{a^\frown l_a} \right) \\ &\subseteq V_k^n \cap \left(\bigcap_{a \in k^{n+1}} U_{a^\frown k} \right) \\ &= V_k^{n+1} \end{aligned}$$

Finally, apply Menger to $\mathcal{V}^n$, resulting in the cover $\{V^0_{f(0)}, V^1_{f(1)}, \dots\}$, noting

$$X = \bigcup_{n<\omega} V^n_{f(n)} \subseteq \bigcup_{n<\omega} U_{(f\restriction n)^\frown f(n)} = \bigcup_{n<\omega} U_{f\restriction(n+1)}$$

$\square$

Proposition 5.5. *X is compact if and only if $F \underset{tact}{\uparrow} Cov_{C,F}(X)$ if and only if $F \underset{k\text{-}tact}{\uparrow} Cov_{C,F}(X)$*

Proof. Assume X is compact. For each open cover played by C, pick a finite subcover, and this yields a winning tactical strategy.

Assume F has a winning k-tactical strategy. For any open cover, have C play only it during the entire game. F's only choice must be a finite subcover. $\square$

Proposition 5.6. *If X is σ-compact then $F \underset{mark}{\uparrow} Cov_{C,F}(X)$*

Proof. Let $X = \bigcup_{n<\omega} X_n$ for compact X_n. On round n, F picks the finite subcover of C's open cover of X_n. $\square$

For Menger's game, there is no useful distinction between a k-Markov strategy for F, and a 2-Markov strategy.

Theorem 5.7. *For any topological space X and all $k \geq 2$, $F \underset{k\text{-}mark}{\uparrow} Cov_{C,F}(X)$ if and only if $F \underset{2\text{-}mark}{\uparrow} Cov_{C,F}(X)$.*

Proof. Assume $\sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{k-1}, n)$ is a winning k-Markov strategy. Define the 2-Markov strategy $\tau(\mathcal{U}, \mathcal{V}, n)$ so that it contains $\sigma(\mathcal{W}_0, \ldots, \mathcal{W}_{k-2}, \mathcal{V}, m)$ for the following conditions on $\mathcal{W}_0, \ldots, \mathcal{W}_{k-2}, m$:

- Each $\mathcal{W}_i \in \{\mathcal{U}, \mathcal{V}\}$
- $m \leq (n+1)k$; in particular, for $i < k$,

$$\sigma(\mathcal{W}_0, \ldots, \mathcal{W}_{k-2}, \mathcal{V}, (n+1)k + i) \subseteq \tau(\mathcal{U}, \mathcal{V}, n+1)$$

Considering an arbitrary play $\mathcal{U}_0, \mathcal{U}_1, \ldots$ by C versus τ, we note that σ defeats the play

$$\underbrace{\mathcal{U}_0, \mathcal{U}_0, \ldots, \mathcal{U}_0}_{k}, \underbrace{\mathcal{U}_1, \mathcal{U}_1, \ldots, \mathcal{U}_1}_{k} \ldots$$

So we have that

$$\bigcup_{i<k, n<\omega} \sigma(\underbrace{\mathcal{U}_n, \ldots, \mathcal{U}_n}_{k-i-1}, \underbrace{\mathcal{U}_{n+1}, \ldots, \mathcal{U}_{n+1}}_{i+1}, (n+1)k + i)$$

is a cover for X, and as

$$\sigma(\underbrace{\mathcal{U}_n,\dots,\mathcal{U}_n}_{k-i-1},\underbrace{\mathcal{U}_{n+1},\dots,\mathcal{U}_{n+1}}_{i+1},(n+1)k+i)\subseteq\tau(\mathcal{U}_n,\mathcal{U}_{n+1},n+1)$$

τ defeats the play $\mathcal{U}_0,\mathcal{U}_1,\dots$. □

But there are spaces for which there is no Markov strategy, but there is a 2-Markov strategy.

In a question I posed to G, he answered:

Lemma 5.8. *For all functions $\tau:\omega_1\times\omega\to[\omega_1]^{<\omega}$, there exists a sequence $\alpha_0,\alpha_1,\dots<\omega_1$ such that $\{\tau(\alpha_n,n):n<\omega\}$ is not a cover for $\{\beta:\forall n<\omega(\beta<\alpha_n)\}$.*

Proof. Let $P_n=\{\beta:\beta<\alpha\Rightarrow\beta\in\tau(\alpha,n)\}$. Observe that each P_n is finite; else there is some α larger than every member of some countably infinite $P_n^*\subseteq P_n$ such that $P_n^*\subseteq\tau(\alpha,n)$.

Choose $\beta\notin\bigcup_{n<\omega}P_n$. Then for each $n<\omega$, pick $\alpha_n>\beta$ such that $\beta\notin\tau(\alpha_n,n)$. □

Note that the one-point Lindelöfication of discrete ω_1, $\omega_1^\dagger$, is not σ-compact. With the above lemma, we may see that:

Example 5.9. $F\uparrow Cov_{C,F}(\omega_1^\dagger)$ but $F\underset{\text{mark}}{\not\uparrow}Cov_{C,F}(\omega_1^\dagger)$.

Proof. First, we see F has a simple perfect information strategy: in response to the initial cover of $\omega_1^\dagger$, F chooses a co-countable neighborhood of ∞. On successive turns she may pick a single set from C's covers to cover the countable remainder.

Now, suppose that $\sigma(\mathcal{U},n)$ was a winning Markov strategy and aim for a contradiction. Consider the covers

$$\mathcal{U}(\alpha)=\{[\alpha,\omega_1)\cup\{\infty\}\}\cup\{\{\beta\}:\beta<\alpha\}$$

and define $\tau(\alpha,n)$ to be the union of singletons chosen by $\sigma(\mathcal{U}(\alpha),n)$.

Using the sequence $\alpha_0,\alpha_1,\dots<\omega_1$ from the previous lemma, we consider the play $\mathcal{U}(\alpha_0),\mathcal{U}(\alpha_1),\dots$.

As σ was a winning strategy, $\{\sigma(\mathcal{U}(\alpha_n), n) : n < \omega\}$ must cover $\omega_1^\dagger$, and thus $\{\tau(\alpha_n, n) : n < \omega\}$ must cover $\{\beta : \forall n < \omega(\beta < \alpha_n)\}$, contradiction. □

Telgarski showed in "On Games of Topsoe" that a metrizable space is σ-compact if and only if there exists a winning strategy for F in the Menger game, and Scheepers gave a more direct proof later. We generalize Scheeper's proof to handle a number of cases.

Definition 5.10. A set $R \subseteq X$ is relatively compact to the topological space X if for every open cover of the entire space X, there is a finite subcover of the set R.

Proposition 5.11. *If X is regular, then Y is relatively compact if and only if $\overline{Y}$ is compact.*

Proof. The reverse implication is trivial.

Assume Y is relatively compact, let $\mathcal{U}$ be an open cover of $\overline{Y}$, and define $x \in V_x \subseteq \overline{V_x} \subseteq U_x \in \mathcal{U}$ for each $x \in X$. Then if we take a cover $\mathcal{F} = \{V_{x_i} : i < n\}$ of Y by relative compactness, then $\{U_{x_i} : i < n\}$ is a finite cover of $\overline{Y}$, showing compactness. □

Lemma 5.12. *Let $\sigma(\mathcal{U}, n)$ be a winning Markov strategy for F in $Cov_{C,F}(X)$, and $\mathfrak{C}$ collect all open covers of X. Then for*

$$R_n = \bigcap_{\mathcal{U}\in\mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

it follows that R_n is relatively compact to X, and $\bigcup_{n<\omega} R_n = X$.

Proof. First, we see that $\sigma(\mathcal{U}, n)$ witnesses the relative compactness of R_n. Suppose that $x \notin R_n = \bigcap_{\mathcal{U}\in\mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$ for any $n < \omega$. Then for each n, pick $\mathcal{U}_n \in \mathfrak{C}$ such that $x \notin \bigcup \sigma(\mathcal{U}_n, n)$. Then σ does not defeat the play $\mathcal{U}_0, \mathcal{U}_1, \ldots$ □

Corollary 5.13. *A space X is σ-(relatively compact) if and only if $F \underset{mark}{\uparrow} Cov_{C,F}(X)$.*

Corollary 5.14. *For regular spaces X, the following are equivalent:*

(a) X is σ-compact

(b) X is σ-(relatively compact)

(c) $F \underset{mark}{\uparrow} Cov_{C,F}(X)$

Theorem 5.15. *For second-countable X, the following are equivalent:*

(a) X *is* σ*-(relatively compact)*

(b) $F \uparrow Cov_{C,F}(X)$

(c) $F \underset{mark}{\uparrow} Cov_{C,F}(X)$

Proof. . We need only show $(b) \Rightarrow (a)$, so let $\sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{n-1})$ be a winning strategy for F, and observe that since X is second-countable, we may assume all covers are countable. Let $\mathfrak{C}$ be the collection of all countable covers of X. We define $R_s, \mathcal{U}_s$ for $s \in \omega^{<\omega}$ as follows:

- $R_\emptyset = \bigcap_{\mathcal{U} \in \mathfrak{C}} \left(\bigcup \sigma(\mathcal{U}) \right)$
- Note that there are only countably many distinct finite subsets $\sigma(\mathcal{U})$ of the countable collection $\mathcal{U}$. Thus define each $\mathcal{U}_{\langle n \rangle}$ so that

$$R_\emptyset = \bigcap_{n<\omega} \left(\bigcup \sigma(\mathcal{U}_{\langle n \rangle}) \right)$$

- $R_s = \bigcap_{\mathcal{U} \in \mathfrak{C}} \left(\bigcup \sigma(\mathcal{U}_{s\restriction 1}, \mathcal{U}_{s\restriction 2}, \ldots, \mathcal{U}_s, \mathcal{U}) \right)$
- Again, note that there are only countably many distinct finite subsets $\sigma(\mathcal{U}_{s\restriction 1}, \mathcal{U}_{s\restriction 2}, \ldots, \mathcal{U}_s, \mathcal{U})$ of the countable collection $\mathcal{U}$. Thus define each $\mathcal{U}_{s^\frown\langle n \rangle}$ so that

$$R_s = \bigcap_{n<\omega} \left(\bigcup \sigma(\mathcal{U}_{s\restriction 1}, \mathcal{U}_{s\restriction 2}, \ldots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle n \rangle}) \right)$$

We quickly confirm that each R_s is relatively compact as for each open cover $\mathcal{U}$ of X we have the finite subcover $\sigma(\mathcal{U}_{s\restriction 1}, \mathcal{U}_{s\restriction 2}, \ldots, \mathcal{U}_s, \mathcal{U})$ of R_s.

Finally, we claim that $X = \bigcup_{s \in \omega^{<\omega}} R_s$. If not, let x be missed by every R_s, and define $f \in \omega^\omega$ such that $x \notin \bigcup \sigma(\mathcal{U}_{f\restriction 1}, \ldots, \mathcal{U}_{f\restriction n})$ for any n. Then $\mathcal{U}_{f\restriction 1}, \mathcal{U}_{f\restriction 2}, \ldots$ is a counter to the winning strategy σ, a contradiction. $\square$

Corollary 5.16. *For metric spaces X, the following are equivalent:*

(a) X is σ-compact

(b) X is σ-(relatively compact)

(c) $F \uparrow Cov_{C,F}(X)$

(d) $F \underset{mark}{\uparrow} Cov_{C,F}(X)$

Example 5.17. Let R be given the topology from example 63 from Counterexamples in Topology, the topology generated by open intervals with countable sets removed. This space is a non-regular example where $F \uparrow Cov_{C,F}(R)$, but $F \underset{\text{mark}}{\not\uparrow} Cov_{C,F}(R)$, that is, R is not σ-(relatively compact).

Proof. From Counterexamples: The irrationals are open, but contain no closed neighborhood, showing non-regular.

Take open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$. Define $\sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{2n})$ to be a finite subcover of $[-n, n] \setminus C_n$ for some countable $C_n = \{c_{n,0}, c_{n,1}, \ldots\}$. For $\sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{2n+1})$, use any subcover of $\{c_{i,j} : i, j < n\}$. It is easily seen that σ is a winning perfect information strategy.

For any $A = \{x_n : n < \omega\} \in [R]^\omega$, we may choose the open cover $\mathcal{U} = \{R \setminus \{x_i : i \neq n\} : n < \omega\}$ of R with no finite subcover. Thus all relatively compact sets are finite, and the countable union of finite sets cannot contain R, making R not σ-(relatively compact). $\square$

Example 5.18. Let R be given the topology from example 67 from Counterexamples in Topology, the topology generated by open intervals with or without the rationals removed. This space is non-regular, and non-σ-compact, but is second-countable and σ-(relatively compact).

Proof. From Counterexamples: The rationals are closed, but the closure of any open neighborhood is the whole real line, so they cannot be separated from any irrational point. Compact sets in this topology are nowhere dense in the Euclidean topology, so there cannot be countably many which union to the whole space. $\{(a,b)\setminus D : a,b\in\mathbb{Q}, D\in\{\emptyset,\mathbb{Q}\}\}$ is a countable base for the space.

To see that R is σ-relatively compact, it suffices to show that $[a,b]\setminus\mathbb{Q}$ is relatively compact. Let $\mathcal{U}$ be a cover of R, and let $\mathcal{U}'$ fill in the missing rationals for any open set in $\mathcal{U}$. There is a finite subcover $\mathcal{V}'\subseteq\mathcal{U}'$ for $[a,b]$ since $\mathcal{U}'$ contains open sets from the Euclidean topology. Let $\mathcal{V}=\{V\setminus\mathbb{Q} : V\in V'\}$: this is a finite refinment of $\mathcal{U}$ covering $[a,b]\setminus\mathbb{Q}$, so $[a,b]\setminus\mathbb{Q}$ is relatively compact. $\square$

We define a new property "almost-σ-(relatively compact)" to describe a sufficient condition for $F \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(X)$.

Definition 5.19. Let $\mathcal{U}$ be a cover of X. We say $C\subseteq X$ is $\mathcal{U}$-compact if there exists a finite subcover of $\mathcal{U}$ which covers C.

We say X is almost-σ-(relatively compact) if there exist functions $r_{\mathcal{V}}: X\to\omega$ for each open cover $\mathcal{V}$ of X such that both of the following sets are $\mathcal{V}$-compact for all open covers $\mathcal{U}$, $\mathcal{V}$ and $n<\omega$:

$$c(\mathcal{V},n)=\{x\in X : r_{\mathcal{V}}(x)\le n\}$$

$$p(\mathcal{U},\mathcal{V})=\{x\in X : 0<r_{\mathcal{U}}(x)<r_{\mathcal{V}}(x)\}$$

Definition 5.20. For two functions f,g we say f is μ**-almost compatible** with g ($f\|^*_\mu g$) if $|\{x\in\operatorname{dom}(f)\cap\operatorname{dom}(g) : f(x)\neq g(x)\}|<\mu$. If $\mu=\omega$ then we say f,g are **almost compatible** ($f\|^*g$).

Lemma 5.21. *For each $\alpha<\omega_1$, there exist injective functions $f_\alpha:\alpha\to\omega$ such that if $\alpha<\beta$, then*

$$f_\alpha\|^*f_\beta$$

that is, f_α and $f_\beta\restriction\alpha$ agree on all but finitely many ordinals. In addition, the range of each f_α is co-infinite.

Proof. Taken from Kunen (used for the construction of an ω_1-Aronszajn tree).

We begin with the empty function $f_0 : 0 \to \omega_1$ which satisfies the hypothesis, and assume f_α is defined by induction. Let $f_{\alpha+1} = f_\alpha \cup \{\langle \alpha, n\rangle\}$ where n is not defined for f_α, and this satisfies the hypothesis.

Finally, suppose γ is the limit of $\alpha_0, \alpha_1, \ldots$, and f_α is defined for $\alpha < \gamma$. Let $g_0 = f_{\alpha_0}$, and assuming $g_n \|^* f_{\alpha_n}$ with coinfinite range, define $g_{n+1} : \alpha_{n+1} \to \omega$ so that $g_{n+1} \restriction \alpha_n = g_n$ and $g_{n+1} \restriction (\alpha_{n+1} \setminus \alpha_n) \|^* f_{\alpha_{n+1}}$ with coinfinite range. Then $g = \bigcup_{n<\omega} g_n$ is an injective function from $\gamma \to \omega$ and $g \|^* f_\alpha$ for $\alpha < \gamma$, but the range need not be coinfinite. So let

$$f_\gamma(\beta) = \begin{cases} g(\alpha_{2n}) & \beta = \alpha_n \\ g(\beta) & \text{otherwise} \end{cases}$$

which frees up $\{g(\alpha_{2n+1}) : n < \omega\}$ from the range of f_γ, and allows $f_\gamma \|^* f_\alpha$. □

Theorem 5.22. *The one-point Lindelöfication of the uncountable discrete space, $\omega_1^\dagger$, is almost-σ-(relatively compact).*

Proof. Take the injective funcions f_α from Kunen's lemma such that $f_\alpha \|^* f_\beta$. For each open cover $\mathcal{V}$ of $\omega_1^\dagger$ let $\gamma(\mathcal{V})$ identify the least ordinal such that $\omega_1^\dagger \setminus \gamma(\mathcal{U})$ is in a refinement of $\mathcal{V}$. Then $r_\mathcal{V}$ defined by

$$r_\mathcal{V}(x) = \begin{cases} 0 & x \in \omega_1^\dagger \setminus \gamma(\mathcal{V}) \\ f_{\gamma(\mathcal{V})}(x) + 1 & x \in \gamma(\mathcal{V}) \end{cases}$$

witnesses the property as $c(\mathcal{V}, 0)$ is contained in a single open set in $\mathcal{V}$, $c(\mathcal{V}, n+1)$ is a singleton or empty set, and

$$p(\mathcal{U}, \mathcal{V}) = \{\alpha < \min(\gamma(\mathcal{U}), \gamma(\mathcal{V})) : f_{\gamma(\mathcal{U})}(\alpha) < f_{\gamma(\mathcal{V})}(\alpha)\}$$

is finite. □

Theorem 5.23. *If X is almost-σ-(relatively compact), then $F \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(X)$.*

Proof. Let $\sigma(\mathcal{U}_0, 0)$ cover $c(\mathcal{U}_0, 0)$, and let $\sigma(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1)$ cover both $c(\mathcal{U}_{n+1}, n+1)$ and $p(\mathcal{U}_n, \mathcal{U}_{n+1})$. If $\mathcal{U}_0, \mathcal{U}_1, \dots$ is any play by C, then for each $x \in X$, we note that one of the following must occur:

- $r_{\mathcal{U}_0}(x) = 0$ and thus $x \in c(\mathcal{U}_0, 0) \subseteq \bigcup \sigma(\mathcal{U}_0, 0)$.
- $r_{\mathcal{U}_0}(x) = N + 1$ for some $N \geq 0$ and:
  - For all $n \leq N$,
    $$r_{\mathcal{U}_{n+1}}(x) \leq N + 1$$
    and thus $x \in c(\mathcal{U}_{N+1}, N+1)$.
  - For some $n \leq N$,
    $$r_{\mathcal{U}_n}(x) \leq N + 1 < r_{\mathcal{U}_{n+1}}(x)$$
    and thus $x \in p(\mathcal{U}_n, \mathcal{U}_{n+1})$

□

Corollary 5.24. $F \underset{2\text{-}mark}{\uparrow} Cov_{C,F}(\omega_1^\dagger)$

Definition 5.25. The statement $S(\kappa, \mu, \lambda)$ due to Scheepers is shorthand for the following: there exist injective functions $f_A : A \to \lambda$ for each $A \in [\kappa]^\mu$ such that $f_A \|_\mu^* f_B$ for all $A, B \in [\kappa]^\mu$.

Proposition 5.26. *$\neg S(\kappa, \omega, \omega)$ for $\kappa > 2^\omega$*

Proof. Let $A_\alpha = \{\alpha \cdot \omega + n : n < \omega\} \in [\kappa]^\omega$ and $f_{A_\alpha} : A_\alpha \to \omega$ be injective for $\alpha < \kappa$. Since there are $\kappa > |[\omega]^\omega|$ different A_α, there must be α, β where $\text{ran}(f_{A_\alpha}) = \text{ran}(f_{A_\beta})$. Then there is no way to define $f_{A_\alpha \cup A_\beta}$ so that it is almost compatible with both f_{A_α} and f_{A_β}. □

Theorem 5.27. *$S(\kappa, \omega, \omega)$ implies $\kappa^\dagger$ is almost-σ-(relatively compact).*

Proof. Take the injective funcions $f_A : A \to \omega$ witnessing $S(\kappa, \omega, \omega)$. For each cover $\mathcal{V}$ of $\kappa^\dagger$ let $A(\mathcal{V})$ define a set such that $\kappa^\dagger \setminus A(\mathcal{V})$ is in a refinement of $\mathcal{V}$. Then $r_{\mathcal{V}}$ defined by

$$r_{\mathcal{V}}(x) = \begin{cases} 0 & x \in \kappa^\dagger \setminus A(\mathcal{V}) \\ f_{A(\mathcal{V})}(x) + 1 & x \in A(\mathcal{V}) \end{cases}$$

witnesses the property as $c(\mathcal{V}, 0)$ is contained in a single open set in $\mathcal{V}$, $c(\mathcal{V}, n+1)$ is a singleton or empty set, and

$$p(\mathcal{U}, \mathcal{V}) = \{\alpha \in A(\mathcal{U}) \cap A(\mathcal{V}) : f_{A(\mathcal{U})}(\alpha) < f_{A(\mathcal{V})}(\alpha)\}$$

is finite. $\square$

Corollary 5.28. *$S(\kappa, \omega, \omega)$ implies $F \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(\kappa^\dagger)$.*

Definition 5.29. A finite partial function p from A to B has a domain which is a finite subset of A and a range which is a finite subset of B. Let the set of all finite partial functions from A to B be denoted by $Fn(A, B)$.

Then let $Fn^2(\mathcal{A}, B) \subset Fn(\mathcal{A}, Fn(\bigcup \mathcal{A}, B))$ such that for each $p \in Fn^2(\mathcal{A}, B)$, $p(A) = p_A \in Fn(A, B)$.

Definition 5.30. Let $\mathbb{P}_\kappa \subset Fn^2([\kappa]^\omega, \omega)$ be such that each p_A is injective, and give it the partial order $\leq$ defined by $q \leq p$ if and only if:

- $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$
- For each $A \in \mathrm{dom}(p)$, $q_A \supseteq p_A$
- For each $A, B \in \mathrm{dom}(p)$, if p_A and p_B are not defined for some $\alpha \in A \cap B$, but q_A is, then q_B is also defined for α and $q_A(\alpha) = q_B(x)$. That is, for $\alpha \in A \cap B$

$$\alpha \in \mathrm{dom}(q_A) \setminus (\mathrm{dom}(p_A) \cup \mathrm{dom}(p_B)) \Rightarrow \alpha \in \mathrm{dom}(q_B) \text{ and } q_A(x) = q_B(x)$$

Lemma 5.31. *$\mathbb{P}_\kappa$ has property K (and thus is c.c.c.). That is, let $P \subseteq \mathbb{P}_\kappa$ be uncountable: there is an uncountable $Q \subseteq P$ such that points in Q are pairwise compatible.*

Proof. If $|\{\text{dom}(p) : p \in P\}| > \omega$, we will use the Δ-system lemma to find an uncountable $P' \subseteq P$ such that for $p, q \in P'$, $\text{dom}(p) \cap \text{dom}(q) = \mathcal{R}$. Otherwise, we may fix an uncountable $P' \subseteq P$ such that for $p, q \in P'$, $\text{dom}(p) = \text{dom}(q) = \mathcal{R}$.

Similarly, for each $A \in \mathcal{R}$ we may find that $|\{\text{dom}(p_A) : p \in P'\}| > \omega$, and we can use the Δ-system lemma to find an uncountable $P'' \subseteq P'$ where $\text{dom}(p_A) \cap \text{dom}(q_A) = A'$ for all $p, q \in P''$, or otherwise we may find $P'' \subseteq P'$ where $\text{dom}(p_A) = \text{dom}(q_A) = A'$ for all $p, q \in P''$.

Finally, for each $A \in \mathcal{R}$ and $\alpha \in A'$, we may find $n_{A,\alpha}$ such that there are uncountable $p \in P''$ with $p_A(\alpha) = n_{A,\alpha}$, and thus we may choose $Q \subseteq P''$ to be an uncountable collection such that for $p, q \in Q$, $p_A = q_A$ for $A \in \mathcal{R}$.

Then it is easily verified that $p \cup q \in \mathbb{P}_\kappa$ and $p \cup q \leq p, q$ for all $p, q \in Q$. □

Proposition 5.32. *For $A \in [\kappa]^\omega$ and $\alpha \in A$, the sets*

$$D_A = \{p \in \mathbb{P}_\kappa : A \in dom(p)\}$$

$$D_{A,\alpha} = \{p \in \mathbb{P}_\kappa : A \in dom(p), \alpha \in dom(p_A)\}$$

are dense in $\mathbb{P}_\kappa$.

Proof. Let $A \in [\kappa]^\omega$ and $p \in \mathbb{P}_\kappa$. Either $p' = p \in D_A$, or $p' = p \cup \{\langle A, \emptyset\rangle\} \in D_A$ with $p' \leq p$.

Let $\alpha \in A$. Either $p'' = p' \in D_{A,\alpha}$, or we may find $n < \omega$ not in the range of p'_A, and then $p'' = p' \setminus \{\langle A, p'_A\rangle\} \cup \{\langle A, p'_A \cup \{\langle \alpha, n\rangle\}\rangle\} \in D_{A,\alpha}$ with $p'' \leq p' \leq p$. □

Theorem 5.33. *$S(\kappa, \omega, \omega) + (\kappa = 2^\omega)$ is consistent with ZFC for any cardinal κ with $cf(\kappa) > \omega$.*

Proof. We adapt a forcing argument due to Scheepers. Let M be a countable transitive submodel of ZFC. Consider the c.c.c. poset $\mathbb{P}_\kappa$ realized in the model M. Let G be a $\mathbb{P}_\kappa$-generic filter over M.

We now work in the smallest model $M[G]$ extending M and containing G. Observe that by (Kunen), $M[G]$ preserves cofinalities and cardinals.

For each $A \in [\kappa]^\omega$, note $[\kappa]^\omega \cap M$ is cofinal in $[\kappa]^\omega$, so let $A' \supseteq A$ be in $[\kappa]^\omega \cap M$ and let $f_A = \bigcup_{p \in G \cap D_{A'}} p_{A'} \restriction A$. Since G is a $\mathbb{P}_\kappa$-generic filter over M, it is easily verified (considering the dense sets $D_{A,\alpha}$) that f_A is an injective function from A into ω.

In addition, for $A, B \in [\kappa]^\omega \cap M$, let $p \in G \cap D_A \cap D_B$. For all $q \leq p$ it follows that $\{\alpha \in \mathrm{dom}(q_A) \cap \mathrm{dom}(q_B) : q_A(\alpha) \neq q_B(\alpha)\} \subseteq \mathrm{dom}(p_A) \cup \mathrm{dom}(p_B)$. Thus $|\{\alpha \in A \cap B : f_A(\alpha) \neq f_B(\alpha)\}| < \omega$ and $f_A \|^* f_B$ for $A, B \in [\kappa]^\omega \cap M$, and it's immediate that $f_A \|^* f_B$ for $A, B \in [\kappa]^\omega$ as well.

The f_A witness $S(\kappa, \omega, \omega)$. Since $\kappa \geq 2^\omega$ by $\mathbb{P}_\kappa$ c.c.c., and $S(\kappa, \omega, \omega)$ is a contradiction for $\kappa > 2^\omega$, we know $\kappa = 2^\omega$. $\square$

Corollary 5.34. *For each κ, $F \underset{2\text{-}mark}{\uparrow} Cov_{C,F}(\kappa^\dagger)$ is consistent with ZFC.*

Besides various limited information characterizations of $Cov_{C,F}(X)$, there are other interesting covering properties between σ-(relatively compact) and Menger.

Definition 5.35. A colleciton of subsets of a space X is (**really**) **ample** if has a finite subcover for each (relatively) compact subset of X.

Proposition 5.36. *Every ample cover of a regular space X is really ample.*

Proof. Every relatively compact set R is a subset of the compact set $\overline{R}$. □

Definition 5.37. A space X is (**relatively**) **Alster** if there exists a countable subcover for each (really) ample cover of X by G_δ sets.

Proposition 5.38. *Every regular relatively Alster space is Alster.*

Theorem 5.39. *X σ-compact $\Rightarrow$ X Alster $\Rightarrow$ X Menger*

Proof. Due to Leandro F. Aurichi and Franklin D. Tall in "Lindelöf spaces which are indestructible, productive, or D". □

Proposition 5.40. *X σ-(relatively compact) $\Rightarrow$ X relatively Alster $\Rightarrow$ X Menger*

Proof. If X is σ-(relatively compact), then there exists a countable subcover for every really ample cover of X by arbitrary sets.

If X is relatively Alster, we adapt Aurichi and Tall's argument for the previous theorem.

Since relatively Alster implies Lindelöf (all open covers of X are relatively ample by definition of relatively compact), it is sufficient to consider open covers $\mathcal{U}_n = \{U_n^m : m < \omega\}$ such that $U_n^m \subseteq U_n^{m+1}$. Then $\mathcal{G} = \{\bigcap_{n<\omega} U_n^{f(n)} : f \in \omega^\omega\}$ is a G_δ cover. To see that $\mathcal{G}$ is relatively ample, note for each relatively compact K in X, K is covered by something in each $\mathcal{U}_n$, and thus there is $f_K(n) \in \omega^\omega$ such that $K \subseteq \bigcap_{n<\omega} U_n^{f_K(n)}$.

By relatively Alster, let $f_m \in \omega^\omega$ for $m < \omega$ so that $\mathcal{G}' = \{\bigcap_{n<\omega} U_n^{f_m(n)} : m < \omega\}$ covers X.

Then let $\mathcal{F}_n = \{U_n^{f_m(n)} : m \leq n\} \in [\mathcal{U}_n]^{<\omega}$. Note that if $x \in X$, then $x \in \bigcap_{n<\omega} U_n^{f_{m_x}(n)}$ for some $m_x < \omega$ as $\mathcal{G}'$ is a cover. Thus $x \in U_{m_x}^{f_{m_x}(m_x)}$ and is covered by $\mathcal{F}_{m_x}$. $\square$

Example 5.41. Let the real numbers R be given the topology generated by open intervals with countable sets removed. R is not relatively Alster and $F \uparrow Cov_{C,F}(R)$. If $S(2^\omega, \omega, \omega)$ holds, then $F \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(R)$.

Proof. Since points are G_δ and its relatively compact sets are all finite, a cover by singletons is really ample and has no countable subcover, showing R is not relatively Alster.

It was proven earlier that $F \uparrow Cov_{C,F}(R)$. Assuming $S(2^\omega, \omega, \omega)$, we construct a 2-Marköv strategy by first defining the following:

- Let z_n enumerate the integers for $n < \omega$.
- For each open cover $\mathcal{V}$ of R, let $A(\mathcal{V})$ be a countable set such that $[z_n, z_n + 1) \setminus A(V)$ is $\mathcal{V}$-compact for each $n < \omega$.
- Let $f_A : A \to \omega$ for $A \subseteq R$ witness $S(2^\omega, \omega, \omega)$.

Then we may define $\sigma(\mathcal{U}_0, 0) = \emptyset$ and define $\sigma(\mathcal{U}_n, \mathcal{U}_{n+1}, n + 1)$ such that it covers each of the following:

$$\left(\bigcup_{m\leq n} [z_m, z_m + 1)\right) \setminus A(\mathcal{U}_{n+1})$$

$$\{f^{-1}_{A(\mathcal{U}_{n+1})}(m) : m \leq n\}$$

$$\{x \in A(\mathcal{U}_n) \cap A(\mathcal{U}_{n+1}) : f_{A(\mathcal{U}_n)}(x) \neq f_{A(\mathcal{U}_{n+1})}(x)\}$$

For any attack $\mathcal{U}_0, \mathcal{U}_1, \ldots$ on σ, let $x \in [z_n, z_n + 1)$.

- If $x \notin A(\mathcal{U}_{n+1})$, then x is covered in round $n + 1$.
- If $x \in A(\mathcal{U}_{n+1})$, let $N = f_{A(\mathcal{U}_{n+1})}(x)$.
  - If $N \leq n$, then x is covered in round $n + 1$.

- If $N > n$ and $N = f_{A(\mathcal{U}_{p+1})}(x)$ for all $p \geq n$, then x is covered in round N.
- If $N > n$ and $N \neq f_{A(\mathcal{U}_{p+1})}(x)$ for some $p > n$, then x is covered in round $p + 1$.

$\square$

Definition 5.42. A space X is **Hurewicz** if for each sequence of open covers $\mathcal{U}_n$, there are $\mathcal{F}_n \in [\mathcal{U}_n]^{<\omega}$ such that $X = \bigcup_{m<\omega} \bigcap_{m \leq n < \omega} \cup\mathcal{F}_n$.

Definition 5.43. The **filling game** $Fill^{\subseteq}_{M,N}(J)$ on an ideal J proceeds as follows: player M chooses $M_0 \in \langle J \rangle$, the σ-completion of J, in the initial round, followed by N choosing $N_0 \in J$. In round $n+1$, player M chooses M_{n+1} where $M_n \subseteq M_{n+1} \in \langle J \rangle$, and player N replies with $N_{n+1} \in J$. Player N wins the game if $\bigcup_{n<\omega} N_n \supseteq \bigcup_{n<\omega} M_n$. (The sets in J and $\langle J \rangle$ are thought of as nowhere-dense and meager sets, respectively.)

The **strict filling game** $Fill^{\subsetneq}_{M,N}(J)$ proceeds analogously, with the added requirement that $M_n \subsetneq M_{n+1}$. This game has been studied by Scheepers.

Theorem 5.44. $N \underset{2\text{-}tact}{\uparrow} Fill^{\subsetneq}_{M,N}(J) \Rightarrow N \underset{2\text{-}mark}{\uparrow} Fill^{\subseteq}_{M,N}(J)$

Proof. Enumerate the sets in J as A_α for $\alpha < |J|$. For $M \in \langle J \rangle$ and $n < \omega$, let $M + 0 = M$ and $M + n + 1$ be the union of $M + n$ and the least A_α not contained in $M + n$.

Let σ be a winning 2-tactical strategy for N in $Fill^{\subsetneq}_{M,N}(\kappa)$, and assume $\sigma(M) \cup \sigma(M') \subseteq \sigma(M, M')$.

We define a 2-Markov strategy τ for F in $Fill^{\subseteq}_{M,N}(\kappa)$ as follows:

$$\tau(M_0, 0) = \sigma(M_0)$$

$$\tau(M_n, M_{n+1}, n+1) = \begin{cases} \sigma(M_n, M_{n+1}) & \text{if } M_n \subsetneq M_{n+1} \\ \bigcup_{m \leq n} \sigma(M_n + m, M_{n+1} + m + 1) & \text{otherwise} \end{cases}$$

Let $M_0 \subseteq M_1 \subseteq \dots$ be an attack by C against τ. There are two possible cases:

- Assume $M_n = M_N$ for all $n \geq N$.

  The collection produced by σ versus the attack

  $$M_N + 0 \subsetneq M_N + 1 \subsetneq \dots$$

  must cover M_N as σ is a winning strategy.

Let $x \in M_N$. If $x \in \sigma(M_N + 0)$, then x will be covered in round $N + 1$ by

$$\tau(M_N, M_N, N+1) \supseteq \sigma(M_N + 0, M_N + 1) \supseteq \sigma(M_N + 0)$$

Otherwise, $x \in \sigma(M_N + n, M_N + n + 1)$, and x will be covered in round $N + n + 1$ by

$$\tau(M_N, M_N, N+n+1) \supseteq \sigma(M_N + n, M_N + n + 1)$$

- Otherwise we may find $0 < f(0) < f(1) < \dots$ such that $M_{f(n)} \subsetneq M_{f(n)+1} = M_{f(n+1)}$.

  Then the collection produced by σ versus the attack

$$M_{f(0)} \subsetneq M_{f(1)} \subsetneq M_{f(2)} \dots$$

  must cover $\bigcup_{n<\omega} M_n$ as σ is a winning strategy.

  Let $x \in \bigcup_{n<\omega} M_n$. If $x \in \sigma(M_{f(0)})$, then x will be covered by τ in round $f(0) + 1$ by

$$\tau(M_{f(0)}, M_{f(0)+1}, f(0)+1) = \sigma(M_{f(0)}, M_{f(0)+1}) \supseteq \sigma(M_{f(0)})$$

  Otherwise, $x \in \sigma(M_{f(n)}, M_{f(n+1)})$, and x will be covered by τ in round $f(n) + 1$ by

$$\tau(M_{f(n)}, M_{f(n)+1}, f(n)+1) = \sigma(M_{f(n)}, M_{f(n)+1}) = \sigma(M_{f(n)}, M_{f(n+1)})$$

Thus τ is a winning strategy. □

Example 5.45. There is a free ideal J such that $N \underset{\text{2-tact}}{\not\uparrow} Fill^{\subseteq}_{M,N}(J)$ but $N \underset{\text{2-mark}}{\uparrow} Fill^{\subseteq}_{M,N}(J)$.

Proof. Result based on "Meager nowhere dense games" Prop 9 by Scheepers. Assume $\mathbb{R}$ has the usual Euclidean topology.

Choose $A \subseteq \mathbb{R}$ such that $|A| = \omega$ and A is meager but not nowhere dense. Then choose $V \subseteq \mathbb{R}$ such that $|V| = 2^\omega$, V is meager, and V is disjoint from A. Assume $A = \{a_n : n < \omega\}$.

Certainly, if J is the collection of nowhere dense subsets of $A \cup V$, then $F \underset{\text{2-mark}}{\uparrow} Fill^{\subseteq}_{M,N}(J)$. In fact, since $A \cup V$ is meager, $F \underset{\text{pre}}{\uparrow} Fill^{\subseteq}_{M,N}(J)$.

By Prop 9 in Scheeper's paper, $F \underset{\text{2-tact}}{\not\uparrow} Fill^{\subsetneq}_{M,N}(J)$ immediately. A proof follows: let σ be a 2-tactical strategy such that $\sigma(M) \subseteq \sigma(M, M')$.

We may define K_n to be the collection of pairs of comparable sets $\{B, C\}$ such that $B \subsetneq C$ and n is the least integer where $a_n \in A \setminus \sigma(A \cup B, A \cup C)$.

By Cor 28 of Scheeper's "A partition relation for partially ordered sets", for every partition $\{K_n : n < \omega\}$ of the comparable pairs in $[\mathcal{P}(V)]^2$ there is some $n' < \omega$ and branch $C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq V$ where $\{C_m, C_{m+1}\} \in K_{n'}$ for all $m < \omega$.

Then σ may be countered by the attack $A \cup C_0, A \cup C_1, \ldots$, since $a_{n'} \in A \setminus \sigma(A \cup C_m, A \cup C_{m+1})$ for all $m < \omega$ and thus is never covered. $\square$

Definition 5.46. X is **Rothberger** if for all open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$ there exist open sets $U_n \in \mathcal{U}_n$ such that $\{U_n : n < \omega\}$ is a cover of X.

Proposition 5.47. *Rothberger $\Rightarrow$ Menger*

Definition 5.48. In the two-player game $Cov_{C,S}(X)$ player C chooses open covers $\mathcal{U}_n$ of X, followed by player S choosing an open set $U_n \in \mathcal{U}_n$. S wins if $\{U_n : n < \omega\}$ is a cover of X.

Theorem 5.49. *X is Rothberger if and only if $C \not\uparrow Cov_{C,S}(X)$.*

Proof. Due to Pawlikowski ☐

Definition 5.50. A space X is scattered if every subspace contains an isolated point. By convention, $X = \bigcup_{\alpha < \mathrm{rank}(X)} X^\alpha$ where X^α is the set of isolated points of $X \setminus \bigcup_{\beta<\alpha} X^\beta$.

Proposition 5.51. *A space X is scattered if and only if every closed subspace contains an isolated point.*

Proposition 5.52. *The rank of a compact scattered T_1 space is a successor ordinal, and $X^{rank(X)-1}$ is finite.*

Proof. Suppose that the rank of X was a limit ordinal β. Then by choosing $\beta_n \to \beta$, we may pick a point $x_n \in X^{\beta_n}$, and $\{x_n : n < \omega\}$ may be shown to be a closed discrete subspace of X.

It's easily seen that $X^{\mathrm{rank}(X)-1}$ must be finite - it is a closed discrete subspace of compact X. ☐

Theorem 5.53. *The following are equivalent for compact T_2 X:*

(a) X is Rothberger

(b) X is scattered

(c) $S \uparrow Cov_{C,S}(X)$

(d) $C \not\uparrow Cov_{C,S}(X)$

Proof. To show $(a) \Rightarrow (b)$, we use Aurichi's proof in *D-Spaces*: suppose X has a closed subspace without isolated points. Then it is compact and contains a closed copy of the Cantor set, which is not Rothberger, contradiction.

To show $(b) \Rightarrow (c)$, if X is scattered, suppose during a particular round n, player S observes that the uncovered subspace $Y \subseteq X$ is nonempty. Then as Y is also compact scattered, select one of the finite points in $Y^{\text{rank}(Y)-1}$, label it x_n, and choose an open set containing x_n from the given cover.

We claim that if S follows this strategy, player S will observe that the uncovered subspace $Y \subseteq X$ is empty during some round. If not, consider the x_n chosen by Y by the end of the game - the rank of each point within X is nonincreasing, and does not contain a constant final sequence, contradiction.

Of course, $(c) \Rightarrow (a)$ is trivial, and $(a) \Leftrightarrow (d)$. $\square$

Definition 5.54. In the two-player game $Cov_{P,O}(X)$ player P chooses points $x_n \in X$, followed by player O choosing an open neighborhood U_n of x_n. P wins if $\{U_n : n < \omega\}$ is a cover of X.

Theorem 5.55. *$Cov_{P,O}(X)$ is "perfect information equivalent" to $Cov_{C,S}(X)$. That is:*

- *$P \uparrow Cov_{P,O}(X)$ if and only if $S \uparrow Cov_{C,S}(X)$*
- *$O \uparrow Cov_{P,O}(X)$ if and only if $C \uparrow Cov_{C,S}(X)$.*

Proof. Due to Galvin.

- Let σ be a strategy for S in $Cov_{C,S}(X)$.

  Let $n < \omega$, and consider open covers $\mathcal{U}_m$ for each $m < n$. Suppose that for each $x \in X$, there was an open neighborhood U_x of x where for every open cover $\mathcal{U}$,

$\sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{n-1}, \mathcal{U}) \neq U_x$. The open cover $\{U_x : x \in X\}$ demonstrates the contradiction.

We define a strategy for P in $Cov_{P,O}(X)$ as follows: during round n, P chooses a point x_n for which every open neighborhood is of the form $U_n = \sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{n-1}, \mathcal{U}_n)$ for some open cover $\mathcal{U}_0$.

If σ was a winning strategy for S in $Cov_{C,S}(X)$, then the open sets chosen by O in response to P's strategy for $Cov_{P,O}(X)$ are of the form $\{\sigma(\mathcal{U}_0), \sigma(\mathcal{U}_0, \mathcal{U}_1), \ldots\}$ and are an open cover of X.

- Let σ be a strategy for P in $Cov_{P,O}(X)$.

  We define a strategy for S in $Cov_{C,S}(X)$ as follows: during round n, if S has chosen $U_0, \ldots, U_{n-1}$ in previous rounds, S chooses any open set in C's latest cover containing the point $\sigma(U_0, \ldots, U_{n-1})$. If σ was a winning strategy for P, then for any open sets $U_0, U_1, \ldots$ containing $\sigma(\cdot), \sigma(U_0), \ldots$, the collection $\{U_0, U_1, \ldots\}$ is a cover for X.

- Let σ be a strategy for C in $Cov_{C,S}(X)$.

  We define a strategy for O in $Cov_{P,O}(X)$ as follows: during round n, if O has chosen $U_0, \ldots, U_{n-1}$ in previous rounds, O chooses an open set from the cover $\sigma(U_0, \ldots, U_{n-1})$ containing the point chosen by P that round. If σ was a winning strategy for C, then for any open sets $U_0, U_1, \ldots$ from the covers $\sigma(\cdot), \sigma(U_0), \ldots$, the collection $\{U_0, U_1, \ldots\}$ is not a cover for X.

- Let σ be a strategy for O in $Cov_{P,O}(X)$.

  We define a strategy for C in $Cov_{C,S}(X)$ as follows: during round 0, C chooses $\mathcal{U}_0 = \{\sigma(x) : x \in X\}$. In response, S chooses some $\sigma(x_0)$. During round $n+1$, if S has chosen $\sigma(x_0), \ldots, \sigma(x_0, \ldots, x_n)$ in previous rounds, C chooses $\mathcal{U}_{n+1} = \{\sigma(x_0, \ldots, x_n, x) : x \in X\}$. If σ was a winning strategy for O, then $\{\sigma(x_0), \sigma(x_0, x_1), \ldots\}$ is not a cover for X.

□

A similar theorem exists for limited information strategies.

Theorem 5.56.
- *$P \underset{pre}{\uparrow} Cov_{P,O}(X)$ if and only if $S \underset{mark}{\uparrow} Cov_{C,S}(X)$*
- *$O \underset{mark}{\uparrow} Cov_{P,O}(X)$ if and only if $C \underset{pre}{\uparrow} Cov_{C,S}(X)$.*

Proof.
- Let $\sigma(\mathcal{U}_n, n)$ be a Markov strategy for S in $Cov_{C,S}(X)$.

  Let $n < \omega$. Suppose that for each $x \in X$, there was an open neighborhood U_x of x where for every open cover $\mathcal{U}$, $\sigma(\mathcal{U}, n) \neq U_x$. The open cover $\{U_x : x \in X\}$ demonstrates the contradiction.

  We define a predetermined strategy for P in $Cov_{P,O}(X)$ as follows: during round n, P chooses a point x_n for which every open neighborhood is of the form $U_n = \sigma(\mathcal{U}, n)$ for some open cover $\mathcal{U}$.

  If σ was a winning strategy for S in $Cov_{C,S}(X)$, then the open sets chosen by O in response to P's strategy for $Cov_{P,O}(X)$ are of the form $\{\sigma(\mathcal{U}_0, 0), \sigma(\mathcal{U}_1, 1), \dots\}$ and are an open cover of X.

- Let $\sigma(n)$ be a predetermined strategy for P in $Cov_{P,O}(X)$.

  We define a Markov strategy for S in $Cov_{C,S}(X)$ as follows: during round n, S chooses any open set in C's cover containing the point $\sigma(n)$. If σ was a winning strategy for P, then for any open sets $U_0, U_1, \dots$ containing $\sigma(0), \sigma(1), \dots$, the collection $\{U_0, U_1, \dots\}$ is a cover for X.

- Let $\sigma(n)$ be a predetermined strategy for C in $Cov_{C,S}(X)$.

  We define a Markov strategy for O in $Cov_{P,O}(X)$ as follows: during round n, O chooses an open set from the cover $\sigma(n)$ containing the point chosen by P that round. If σ was a winning strategy for C, then for any open sets $U_0, U_1, \dots$ from the covers $\sigma(0), \sigma(1), \dots$, the collection $\{U_0, U_1, \dots\}$ is not a cover for X.

- Let $\sigma(x_n, n)$ be a Markov strategy for O in $Cov_{P,O}(X)$.

  We define a predetermined strategy for C in $Cov_{C,S}(X)$ as follows: during round n, C chooses $\mathcal{U}_n = \{\sigma(x, n) : x \in X\}$. If σ was a winning strategy for O, we observe any play $\{\sigma(x_0, 0), \sigma(x_1, 1), \dots\}$ by S is not a cover for X.

$\square$

Definition 5.57. Let $\mathcal{N}(x)$ be the **neighborhood network** of x, that is, the collection of all neighborhoods of x.

Definition 5.58. A topological space X is **almost countable** if there exist $x_n \in X$ for each $n < \omega$ such that $X = \bigcup_{n<\omega} \bigcap \mathcal{N}(x_n)$.

Theorem 5.59. *For any space X, the following are equivalent:*

- $S \underset{mark}{\uparrow} Cov_{C,S}(X)$
- $P \underset{pre}{\uparrow} Cov_{P,O}(X)$
- *X is almost countable*

Proof. If there exist $x_n \in X$ for each $n < \omega$ such that $X = \bigcup_{n<\omega} \bigcap \mathcal{N}(x_n)$, then let $\sigma(n) = x_n$ be a predetermined strategy for P in $Cov_{P,O}(X)$. For any neighborhoods O_n of x_n chosen by O to attack σ, note that $\bigcup_{n<\omega} O_n \supseteq \bigcup_{n<\omega} \bigcap \mathcal{N}(x_n) = X$ results in a win for P.

Likewise, if for each sequence $x_n \in X$ there is $x \in X$ with $x \notin \bigcup_{n<\omega} \bigcap \mathcal{N}(x_n)$, then for a fixed strategy $\sigma(n)$ for P, O may counter σ by choosing $O_n \in \mathcal{N}(x_n)$ which misses x during each round, causing σ to lose. $\square$

Theorem 5.60. *For any T_1 space X, the following are equivalent:*

- $S \underset{mark}{\uparrow} Cov_{C,S}(X)$
- $P \underset{pre}{\uparrow} Cov_{P,O}(X)$
- *X is almost countable*

- $|X| \leq \omega$

Proof. If $|X| = \omega$ then the winning Markov strategy is obvious, so let $\sigma(\mathcal{U}, n)$ be a Markov strategy.

Let $n < \omega$. Suppose that for each $x \in X$, there was an open neighborhood U_x of x where for every open cover $\mathcal{U}$, $\sigma(\mathcal{U}, n) \neq U_x$. The open cover $\{U_x : x \in X\}$ demonstrates the contradiction.

So let $x_n \in X$ be chosen such that for each open neighborhood U of x_n, there is an open cover $\mathcal{U}$ such that $\sigma(\mathcal{U}, n) = U$. Then if $x \neq \{x_n : n < \omega\}$, C may counter σ as follows: during round n, choose U_n which contains x_n but not x, and then choose $\mathcal{U}_n$ such that $\sigma(\mathcal{U}_n, n) = U_n$. $\square$

Example 5.61. Let $X = \omega_1 \cup \{\infty\}$ be a "weak Lindelöfication" of discrete ω_1 such that open neighborhoods of ∞ contain $\omega_1 \setminus \omega$. This space is T_0 but not T_1, and note that $S \underset{\text{mark}}{\uparrow} Cov_{C,S}(X)$ and $|X| > \omega$.

Theorem 5.62. *The following are equivalent for points-G_δ X:*

(a) $S \uparrow Cov_{C,S}(X)$

(b) $P \uparrow Cov_{P,O}(X)$

(c) $S \underset{k\text{-}mark}{\uparrow} Cov_{C,S}(X)$ *for some* $k \geq 1$

(d) $P \underset{k\text{-}mark}{\uparrow} Cov_{P,O}(X)$ *for some* $k \geq 1$

(e) $S \underset{mark}{\uparrow} Cov_{C,S}(X)$

(f) $P \underset{pre}{\uparrow} Cov_{P,O}(X)$

(g) *X is almost countable*

(h) $|X| \leq \omega$

Proof. Due to Galvin. Let σ be a strategy for S in $Cov_{C,S}(X)$.

Let $G_{x,m}$ designate open sets such that $\{x\} = \bigcap_{m<\omega} G_{x,m}$ for all $x \in X$.

Let $n < \omega$, $s \in \omega^n$, and consider open covers $\mathcal{U}_t$ for each $t \leq s$. Suppose that for each $x \in X$, there was an open neighborhood U_x of x where for every open cover $\mathcal{U}$, $\sigma(\mathcal{U}_{s\restriction 1}, \ldots, \mathcal{U}_s, \mathcal{U}) \neq U_x$. The open cover $\{U_x : x \in X\}$ demonstrates the contradiction.

Thus C may find points x_s such that for each $m < \omega$, there exists an open cover $\mathcal{U}_{s^\frown\langle m\rangle}$ where $\sigma(\mathcal{U}_{s\restriction 1}, \ldots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle m\rangle}) = G_{x_s,m}$. Then if $x \neq \{x_s : s \in \omega^{<\omega}\}$, C may counter σ as follows: during round n, choose $f(n)$ so that $x \notin G_{x_{f\restriction n},f(n)}$, and then choose $\mathcal{U}_{f\restriction n^\frown\langle f(n)\rangle}$ such that $\sigma(\mathcal{U}_{f\restriction 1}, \ldots, \mathcal{U}_{f\restriction n}, \mathcal{U}_{f\restriction n^\frown\langle f(n)\rangle}) = G_{x_{f\restriction n},f(n)}$. $\square$

Corollary 5.63. *The following are equivalent for compact points-G_δ X:*

(a) $S \uparrow Cov_{C,S}(X)$

(b) $P \uparrow Cov_{P,O}(X)$

(c) $S \underset{k\text{-}mark}{\uparrow} Cov_{C,S}(X)$ *for some* $k \geq 1$

(d) $P \underset{k\text{-}mark}{\uparrow} Cov_{P,O}(X)$ *for some* $k \geq 1$

(e) $S \underset{mark}{\uparrow} Cov_{C,S}(X)$

(f) $P \underset{pre}{\uparrow} Cov_{P,O}(X)$

(g) X *is almost countable*

(h) $|X| \leq \omega$

(i) $C \not\uparrow Cov_{C,S}(X)$

(j) $O \not\uparrow Cov_{P,O}(X)$

(k) X *is Rothberger*

(l) X *is scattered*

Definition 5.64. The game $Rec^m_{F,S}(\kappa)$ proceeds as follows: during round 0, player F chooses $F_0 \in [\kappa]^m$, followed by player S choosing $x_0 \in F_0 \cup \{\infty\}$. During round $n+1$, F chooses $F_{n+1} \in [\kappa]^{m^{n+2}}$ such that $F_{n+1} \supset F_n$, followed by S choosing $x_{n+1} \in F_{n+1} \cup \{\infty\}$.

S wins the game if $\{x_n : n < \omega\} \supseteq F_0 \cup \{\infty\}$, and F wins otherwise.

Proposition 5.65. $S \underset{limit}{\uparrow} Cov_{C,S}(\kappa^\dagger) \Rightarrow S \underset{limit}{\uparrow} Rec^m_{F,S}(\kappa)$

Proof. Let σ be a limited information strategy for S in $Cov_{C,S}(\kappa^\dagger)$.

Suppose $C(\cdot)$ converts any finite set G played by F in $Rec^m_{F,S}(\kappa)$ into the open cover $\mathcal{U}_G = [G]^1 \cup \{\kappa^* \setminus G\}$. Then we may define a strategy τ using the same type of information as σ by setting $\tau(\cdot) \in \sigma(C(\cdot))$.

Suppose that the attack $F_0, F_1, \ldots$ countered τ. Let x_n be the point given by τ during round n, and choose $\alpha \in (F_0 \cup \{\infty\}) \setminus \{x_n : n < \omega\}$.

If $\alpha = \infty$, then σ may be countered by the attack $\mathcal{U}_{F_0}, \mathcal{U}_{F_1}, \ldots$ since no neighborhood of ∞ is ever chosen.

Similarly, if $\alpha \in F_0$, then σ may also be countered by the attack $\mathcal{U}_{F_0}, \mathcal{U}_{F_1}, \ldots$ since the singleton $\{\alpha\}$ is never chosen. $\square$

Proposition 5.66. $S \underset{k\text{-}mark}{\uparrow} Rec^m_{F,S}(\kappa) \Leftrightarrow S \underset{k\text{-}tact}{\uparrow} Rec^m_{F,S}(\kappa)$

Proof. The round number is deterimined by the size of the sets played by C. $\square$

Chapter 6

Proximal Game

Results pertaining to Bell's proximal game for uniform spaces.

6.1 cut-and-paste

Definition 6.1. A **uniform space** $\langle X, \mathcal{D} \rangle$ is a set X paired with a filter $\mathcal{D}$ (called its **uniformity**) of relations (called **entourages**) on X such that for each entourage $D \in \mathcal{D}$:

- D is reflexive, i.e., the diagonal $\Delta \subseteq D$.
- Its inverse $D^{-1} = \{\langle y, x \rangle : \langle x, y \rangle \in D\} \in \mathcal{D}$.
- There exists $\frac{1}{2}D \in \mathcal{D}$ such that

$$2(\frac{1}{2}D) = \frac{1}{2}D \circ \frac{1}{2}D = \{\langle x, z \rangle : \exists y(\langle x, y \rangle, \langle y, z \rangle \in \frac{1}{2}D)\} \subseteq D$$

Note that since $\mathcal{D}$ is a filter, for each $D \in \mathcal{D}$, the symmetric relation $D \cap D^{-1} \in \mathcal{D}$.

Proposition 6.2. *For each $D \in \mathcal{D}$ and $n < \omega$ there exists $\frac{1}{2^{n+1}}D \in \mathcal{D}$ such that*

$$2(\frac{1}{2^{n+1}}D) = \frac{1}{2^{n+1}}D \circ \frac{1}{2^{n+1}}D \subseteq \frac{1}{2^n}D$$

and if $2E \subseteq \frac{1}{2^n}D$, then $E \subseteq \frac{1}{2^{n+1}}D$.

Definition 6.3. For an entourage $D \in \mathcal{D}$, let $D[x] = \{y : (x, y) \in D\}$ be the D-**neighborhood** of x. The uniform topology for a uniform space $\langle X, \mathcal{D} \rangle$ is generated by the base $\{D[x] : x \in X, D \in \mathcal{D}\}$.

Theorem 6.4. *A space X is uniformizable (its topology is the uniform topology for some uniformity) if and only if X is completely regular ($T_{3\frac{1}{2}}$).*

Proposition 6.5. *If X is a uniform space, then for all $x \in X$ and symmetric entourages D:*

$$x \in \frac{1}{2}D[y] \text{ and } y \in \frac{1}{2}D[z] \Rightarrow x \in D[z]$$

and

$$\frac{1}{2}D[x] \subseteq \overline{\frac{1}{2}D[x]} \subseteq D[x]$$

Proof. The first is by definition of $\frac{1}{2}D$.

If $z \in \overline{\frac{1}{2}D[x]}$, it follows that there is $y \in \frac{1}{2}D[x] \cap \frac{1}{2}D[z]$ since $\frac{1}{2}D[z]$ is an open neighborhood of z. Thus $(x, z) \in D \Rightarrow z \in D[x] \Rightarrow \overline{\frac{1}{2}D[x]} \subseteq D[x]$. $\square$

Definition 6.6. For a uniform space X, Bell's proximity game proceeds as follows.

In round 0, $\mathscr{D}$ chooses an entourage D_0, followed by $\mathscr{P}$ choosing a point $p_0 \in X$.

In round $n+1$, $\mathscr{D}$ chooses an entourage $D_{n+1} \subseteq D_n$, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in 4D_n[p_n]$.

Player $\mathscr{D}$ wins if either $\bigcap_{n<\omega} 4D_n[p_n] = \emptyset$ or $\langle p_0, p_1, \ldots \rangle$ converges.

Definition 6.7. For a uniform space X, the simplified proximal game $Prox_{D,P}(X)$ can be defined as follows:

In round 0, $\mathscr{D}$ chooses a symmetric entourage D_0, followed by $\mathscr{P}$ choosing a point $p_0 \in X$.

In round $n+1$, $\mathscr{D}$ chooses a symmetric entourage D_{n+1}, followed by $\mathscr{P}$ choosing a point $p_{n+1} \in \left(\bigcap_{m \leq n} D_m\right)[p_n]$.

Player $\mathscr{D}$ wins if either $\bigcap_{n<\omega} \left(\bigcap_{m \leq n} D_m\right)[p_n] = \emptyset$ or $\langle p_0, p_1, \ldots \rangle$ converges.

Theorem 6.8. *$\mathscr{D}$ has a winning perfect-information strategy in Bell's game if and only if $\mathscr{D} \uparrow Prox_{D,P}(X)$.*

Proof. Let σ be a winning perfect information strategy for $\mathscr{D}$ in Bell's game. We define a perfect information strategy τ in the simplified game to yield symmetric entourages $\tau(p \upharpoonright n) = \sigma(p \upharpoonright n) \cap (\sigma(p \upharpoonright n))^{-1}$ for all partial attacks $p \upharpoonright n$. Note that $\tau(p \upharpoonright n) = \bigcap_{m \leq n} \tau(p \upharpoonright m)$.

If p attacks τ in the simplified game, $p(n+1) \in \left(\bigcap_{m \leq n} \tau(p \upharpoonright m)\right)[p(n)] = \tau(p \upharpoonright n)[p(n)] \subseteq \sigma(p \upharpoonright n)[p(n)] \subseteq 4\sigma(p \upharpoonright n)[p(n)]$, so p attacks σ in Bell's game. Thus either p converges, or

$$\emptyset = \bigcap_{n<\omega} 4\sigma(p \upharpoonright n)[p(n)] \supseteq \bigcap_{n<\omega} \tau(p \upharpoonright n)[p(n)] = \bigcap_{n<\omega} \left(\bigcap_{m \leq n} \tau(p \upharpoonright n) \right) [p(n)]$$

For the other direction, let σ be a winning perfect information strategy for $\mathscr{D}$ in the simplified game such that $\sigma(p \upharpoonright n) = \bigcap_{m \leq n} \sigma(p \upharpoonright m)$. Define the perfect information strategy τ in Bell's Game such that $4\tau(p \upharpoonright n) \subseteq \sigma(p \upharpoonright n)$ and $\tau(p \upharpoonright n) = \bigcap_{m \leq n} \tau(p \upharpoonright m)$ for all partial attacks $p \upharpoonright n$.

If p attacks τ in Bell's game, $p(n) \in 4\tau(p \upharpoonright n) \subseteq \sigma(p \upharpoonright n) = \bigcap_{m \leq n} \sigma(p \upharpoonright m)$, so p attacks σ in the simplified game. Thus either p converges, or

$$\emptyset = \bigcap_{n<\omega} \left(\bigcap_{m \leq n} \sigma(p \upharpoonright n) \right) [p(n)] = \bigcap_{n<\omega} \sigma(p \upharpoonright n)[p(n)] \supseteq \bigcap_{n<\omega} 4\tau(p \upharpoonright n)[p(n)] \supseteq \bigcap_{n<\omega} \tau(p \upharpoonright n)[p(n)]$$

□

Proposition 6.9. *$\mathscr{P}$ has a winning perfect-information strategy in Bell's game if and only if $\mathscr{P} \uparrow Prox_{D,P}(X)$.*

Proof. Similar to the previous. □

Definition 6.10. A uniform space is **proximal** if $\mathscr{D} \uparrow Prox_{D,P}(X)$.

Definition 6.11. For a space X and a point $x \in X$, the **W-convergence-game** $Con_{O,P}(X, x)$ proceeds as follows.

In round 0, $\mathscr{O}$ chooses a neighborhood U_n of x, followed by $\mathscr{P}$ choosing a point $p_n \in \bigcap_{m \leq n} U_m$.

Player $\mathscr{O}$ wins if $\langle p_0, p_1, \ldots \rangle$ converges.

Definition 6.12. A space is W if $\mathscr{O} \uparrow Con_{O,P}(X,x)$ for all $x \in X$.

Definition 6.13. For each finite tuple $(m_0, \ldots, m_{n-1})$, we define the k**-tactical fog-of-war**

$$T_k(\langle m_0, \ldots, m_{n-1} \rangle) = \langle m_{n-k}, \ldots, m_{n-1} \rangle$$

and the k**-Marköv fog-of-war**

$$M_k(\langle m_0, \ldots, m_{n-1} \rangle) = \langle \langle m_{n-k}, \ldots, m_{n-1} \rangle, n \rangle$$

So $P \underset{k\text{-tact}}{\uparrow} G$ if and only if there exists a winning strategy for P of the form $\sigma \circ T_k$, and $P \underset{k\text{-mark}}{\uparrow} G$ if and only if there exists a winning strategy of the form $\sigma \circ M_k$.

Theorem 6.14. *For all $x \in X$:*

- $\mathscr{D} \uparrow Prox_{D,P}(X) \Rightarrow \mathscr{O} \uparrow Con_{O,P}(X,x)$
- $\mathscr{D} \underset{2k\text{-}tact}{\uparrow} Prox_{D,P}(X) \Rightarrow \mathscr{O} \underset{k\text{-}tact}{\uparrow} Con_{O,P}(X,x)$
- $\mathscr{D} \underset{2k\text{-}mark}{\uparrow} Prox_{D,P}(X) \Rightarrow \mathscr{O} \underset{k\text{-}mark}{\uparrow} Con_{O,P}(X,x)$

Proof. Let σ witness $\mathscr{D} \underset{2k\text{-tact}}{\uparrow} Prox_{D,P}(X)$ (resp. $\mathscr{D} \underset{2k\text{-mark}}{\uparrow} Prox_{D,P}(X)$, $\mathscr{D} \uparrow Prox_{D,P}(X)$). We define the k-tactical (resp. k-Marköv, perfect info) strategy τ such that

$$\tau \circ L_k(p) = \sigma \circ L_{2k}(\langle x, p(0), \ldots, x, p(|p|-1) \rangle)[x] \cap \sigma \circ L_{2k}(\langle x, p(0), \ldots, x, p(|p|-1), x \rangle)[x]$$

where L_{2k} is the $2k$-tactical fog-of-war (resp. $2k$-Marköv fog-of-war, identity) and L_k is the k-tactical fog-of-war (resp. k-Marköv fog-of-war, identity).

Let p attack τ. Consider the attack q against the winning strategy σ such that $q(2n) = x$ and $q(2n+1) = p(n)$, and let $D_n = \sigma \circ L_{2k}(q)$ and $E_n = \bigcap_{m \le n} D_n$.

Certainly, $x \in E_{2n}[x] = E_{2n}[q(2n)]$ for any $n < \omega$. Note also for any $n < \omega$ that

$$p(n) \in \bigcap_{m \leq n} \tau \circ L_k(p \restriction n)$$

$$= \bigcap_{m \leq n} (\sigma \circ L_{2k}(\langle x, p(0), \ldots, x, p(m-1)\rangle)[x] \cap \sigma \circ L_{2k}(\langle x, p(0), \ldots, x, p(m-1), x\rangle)[x])$$

$$= \bigcap_{m \leq n} (D_{2m}[x] \cap D_{2m+1}[x]) = \bigcap_{m \leq 2n+1} D_m[x] = E_{2n+1}[x]$$

so by the symmetry of E_{2n+1}, $x \in E_{2n+1}[p(n)] = E_{2n+1}[q(2n+1)]$. Thus $x \in \bigcap_{n<\omega} E_n[q(n)] \neq \emptyset$, and since σ is a winning strategy, the attack q converges. Since $q(2n) = x$, q must converge to x. Thus its subsequence p converges to x, and τ is a winning strategy in $Con_{O,P}(X, x)$. $\square$

Corollary 6.15. *For all $x \in X$:*

- $\mathscr{D} \underset{k\text{-}tact}{\uparrow} Prox_{D,P}(X) \Rightarrow \mathscr{O} \underset{k\text{-}tact}{\uparrow} Con_{O,P}(X, x)$
- $\mathscr{D} \underset{k\text{-}mark}{\uparrow} Prox_{D,P}(X) \Rightarrow \mathscr{O} \underset{k\text{-}mark}{\uparrow} Con_{O,P}(X, x)$

Corollary 6.16. *All proximal spaces are W-spaces.*

Theorem 6.17. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete. Then*

- $\mathscr{O} \uparrow Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathscr{D} \uparrow Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \underset{k\text{-}tact}{\uparrow} Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathscr{D} \underset{k\text{-}tact}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \underset{k\text{-}mark}{\uparrow} Con_{O,P}(X \cup \{\infty\}, \infty) \Rightarrow \mathscr{D} \underset{k\text{-}mark}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$

Proof. Note that the topology on $X \cup \{\infty\}$ is induced by the uniformity with equivalence relation entourages $D(U) = \Delta \cup U^2$ for each open neighborhood U of ∞.

Let σ witness $\mathscr{D} \underset{k\text{-tact}}{\uparrow} Con_{O,P}(X \cap \{\infty\}, \infty)$ (resp. $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Con_{O,P}(X \cap \{\infty\}, \infty)$, $\mathscr{D} \uparrow Con_{O,P}(X \cap \{\infty\}, \infty)$). We define the k-tactical (resp. k-Marköv, perfect info) strategy τ such that

$$\tau \circ L(p) = D(\sigma \circ L(p))$$

where L is the k-tactical fog-of-war (resp. k-Markōv fog-of-war, identity).

Let $p \in (X \cup \{\infty\})^\omega$ attack τ such that $\bigcap_{n<\omega} \tau(p \restriction n)[p(n)] \neq \emptyset$.

If $\infty \in \bigcap_{n<\omega} \tau(p \restriction n)[p(n)]$, it follows that p is an attack on σ. Since σ is a winning strategy, it follows that q and its subsequence p must coverge to ∞.

Otherwise, $\infty \notin \tau(p \restriction N)[p(N)]$ for some $N < \omega$, and then $\tau(p \restriction N)[p(N)] = \{p(N)\}$ implies $p \to p(N)$.

Thus $\tau \circ L$ is a winning strategy. □

Corollary 6.18. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete. Then*

- $\mathscr{O} \uparrow Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \uparrow Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \underset{k\text{-}tact}{\uparrow} Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \underset{k\text{-}tact}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$
- $\mathscr{O} \underset{k\text{-}mark}{\uparrow} Con_{O,P}(X \cup \{\infty\}, \infty) \Leftrightarrow \mathscr{D} \underset{k\text{-}mark}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$

Proposition 6.19. *For any $x \in X$ and $k \geq 1$,*

- $\mathscr{O} \underset{k\text{-}tact}{\uparrow} Con_{O,P}(X, x) \Leftrightarrow \mathscr{O} \underset{tact}{\uparrow} Con_{O,P}(X, x)$
- $\mathscr{O} \underset{k\text{-}mark}{\uparrow} Con_{O,P}(X, x) \Leftrightarrow \mathscr{O} \underset{mark}{\uparrow} Con_{O,P}(X, x)$

Proof. If σ witnesses $\mathscr{O} \underset{k\text{-tact}}{\uparrow} Con_{O,P}(X, x)$, let $\tau(\emptyset) = \sigma(\emptyset)$ and

$$\tau(\langle q \rangle) = \bigcap_{i<k} \sigma(\langle \underbrace{x, \dots, x}_{k-i-1}, q, \underbrace{x, \dots, x}_{i} \rangle)$$

This is easily verified to be a winning strategy. The proof for $\mathscr{O} \underset{k\text{-mark}}{\uparrow} Con_{O,P}(X, x)$ is analogous. □

Corollary 6.20. *Let $X \cup \{\infty\}$ be a uniformizable space such that X is discrete, and $k \geq 1$. Then*

- $\mathscr{D} \underset{k\text{-}tact}{\uparrow} Prox_{D,P}(X \cup \{\infty\}) \Leftrightarrow O \underset{tact}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$

- $\mathscr{D} \underset{k\text{-}mark}{\uparrow} Prox_{D,P}(X \cup \{\infty\}) \Leftrightarrow O \underset{mark}{\uparrow} Prox_{D,P}(X \cup \{\infty\})$

Proposition 6.21. *For any uniform space X,*

- $\mathscr{O} \underset{k\text{-}tact}{\uparrow} Prox_{D,P}(X) \Leftrightarrow \mathscr{O} \underset{2\text{-}tact}{\uparrow} Prox_{D,P}(X)$
- $\mathscr{O} \underset{k\text{-}mark}{\uparrow} Prox_{D,P}(X) \Leftrightarrow \mathscr{O} \underset{2\text{-}mark}{\uparrow} Prox_{D,P}(X)$

Proof. If σ witnesses $\mathscr{O} \underset{k\text{-tact}}{\uparrow} Con_{O,P}(X,x)$, let $\tau(\emptyset) = \sigma(\emptyset)$ and

$$\tau(\langle q \rangle) = \bigcap_{i<k} \sigma(\langle \underbrace{q, \dots, q}_{i} \rangle)$$

$$\tau(\langle q, q' \rangle) = \bigcap_{i<k} \sigma(\langle \underbrace{q, \dots, q}_{k-i}, \underbrace{q', \dots, q'}_{i} \rangle)$$

This is easily verified to be a winning strategy. The proof for $\mathscr{O} \underset{k\text{-mark}}{\uparrow} Con_{O,P}(X,x)$ is analogous. $\square$

Definition 6.22. The absolute proximal game $aProx_{D,P}(X)$ is analogous to $Prox_{D,P}(X)$, except $\mathscr{D}$ may only win if p converges.

Definition 6.23. A **uniformly locally compact** space is a uniformizable space with a **uniformly compact entourage** M where $\overline{M[x]}$ is compact for all x.

Theorem 6.24. *For any uniformly locally compact space X, $\mathscr{D} \uparrow Prox_{D,P}(X) \Leftrightarrow \mathscr{D} \uparrow aProx_{D,P}(X)$*

Proof. Let M be a uniformly locally compact entourage. Let σ witness $\mathscr{D} \uparrow Prox_{D,P}(X)$ such that $\sigma(a) \subseteq M$ always (so $\overline{\sigma(a)[x]} \subseteq \overline{M[x]}$ is compact), and $a \supseteq b$ implies $\sigma(a) \subseteq \frac{1}{4}\sigma(b)$.

Let $\tau(p \upharpoonright n) = \frac{1}{2}\sigma(p \upharpoonright n)$. If p attacks τ in $aProx_{D,P}(X)$, then

$$p(n+1) \in \tau(p \upharpoonright n)[p(n)] = \frac{1}{2}\sigma(p \upharpoonright n)[p(n)]$$

and for

$$x \in \overline{\sigma(p \upharpoonright (n+1))[p(n+1)]} \subseteq \overline{\frac{1}{4}\sigma(p \upharpoonright n)[p(n+1)]} \subseteq \frac{1}{2}\sigma(p \upharpoonright n)[p(n+1)]$$

we can conclude $x \in \sigma(p \upharpoonright n)[p(n)]$. Thus

$$\sigma(p \upharpoonright (n+1))[p(n+1)] \subseteq \overline{\sigma(p \upharpoonright (n+1))[p(n+1)]} \subseteq \sigma(p \upharpoonright n)[p(n)]$$

Finally, note that p attacks the winning strategy σ in $Prox_{D,P}(X)$, but since the intersection of a chain of nonempty compact sets is nonempty:

$$\bigcap_{n<\omega} \sigma(p \upharpoonright n)[p(n)] = \bigcap_{n<\omega} \overline{\sigma(p \upharpoonright n)[p(n)]} \neq \emptyset$$

We conclude that p converges. □

Corollary 6.25. *A uniformaly locally compact space X is proximal if and only if $\mathscr{D} \uparrow aProx_{D,P}(X)$.*

Theorem 6.26. *For any uniformly locally compact proximal space X, $\mathscr{O} \uparrow Clus_{O,P}(X, H)$ for all compact $H \subseteq X$.*

Proof. Let σ witness $\mathscr{D} \uparrow aProx_{D,P}(X)$ such that $p \supseteq q$ implies $\sigma(p) \subseteq \frac{1}{4}\sigma(q)$.

Let $o(t)$ be the subsequence of t consisting of its odd-indexed terms.

We define $T(\emptyset)$, etc. as follows:

- Let $\emptyset \in T(\emptyset)$.
- Choose $m_\emptyset < \omega$, $h_{\emptyset,i} \in H$ for $i < m_\emptyset$, and $h_{\emptyset,i,j} \in H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$ for $i, j < m_\emptyset$ such that

  $$\{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}] : i < m_\emptyset\}$$

  is a cover for H and such that for each $i < m_\emptyset$

  $$\{\frac{1}{4}\sigma(\langle h_{\emptyset,i}\rangle)[h_{\emptyset,i,j}] : j < m_\emptyset\}$$

  is a cover for $H \cap \overline{\frac{1}{4}\sigma(\emptyset)[h_{\emptyset,i}]}$.
- Let $\langle i\rangle \in T(\emptyset)$, $\langle i, h_{\emptyset,i}\rangle \in T(\emptyset)$, and $\langle i, h_{\emptyset,i}, j\rangle \in T(\emptyset)$ for $i, j < m_\emptyset$.

Suppose $T(a)$, etc. are defined. We then define $T(a^\frown\langle x\rangle)$, etc. for

$$x \in \bigcup_{s^\frown\langle i,h_{s,i},j\rangle \in \max(T(a))} \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

as follows:

- Let $T(a) \subseteq T(a^\frown\langle x\rangle)$.
- Choose $t = s^\frown\langle i, h_{s,i}, j, x\rangle$ such that $s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(a))$ and $x \in \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$.
- Note that, assuming $o(s)^\frown\langle h_{s,i}\rangle$ is a legal partial attack against σ, then

  $$x \in \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}] \subseteq \frac{1}{4}\sigma(o(s))[h_{s,i,j}]$$

and

$$h_{s,i,j} \in \overline{\frac{1}{4}\sigma(o(s))[h_{s,i}]} \subseteq \frac{1}{2}\sigma(o(s))[h_{s,i}]$$

implies

$$x \in \sigma(o(s))[h_{s,i}]$$

and thus $o(s)^\frown\langle h_{s,i}, x\rangle = o(t)$ is a legal partial attack against σ.

- Choose $m_t < \omega$, $h_{t,k} \in H \cap \overline{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]}$ for $k < m_t$, and $h_{t,k,l} \in H \cap \overline{\frac{1}{4}\sigma(t)[h_{t,k}]}$ for $k, l < m_t$ such that

$$\{\frac{1}{4}\sigma(o(t))[h_{t,k}] : k < m_t\}$$

is a cover for $H \cap \overline{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]}$ and such that for each $k < m_t$

$$\{\frac{1}{4}\sigma(o(t)^\frown\langle h_{t,k}\rangle)[h_{t,i,j}] : l < m_t\}$$

is a cover for $H \cap \overline{\frac{1}{4}\sigma(o(t))[h_{t,k}]}$.

- Note that, assuming $o(t)$ is a legal partial attack against σ, then

$$h_{t,k} \in \overline{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]} \subseteq \frac{1}{2}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

and

$$x \in \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

implies

$$h_{t,k} \in \sigma(o(s)^\frown\langle h_{s,i}\rangle)[x]$$

and thus $o(t)^\frown\langle h_{t,k}\rangle$ is a legal partial attack against σ.

- Let $t \in T(a^\frown\langle x\rangle)$, $t^\frown\langle k\rangle \in T(a^\frown\langle x\rangle)$, $t^\frown\langle k, h_{t,k}\rangle \in T(a^\frown\langle x\rangle)$, and $t^\frown\langle k, h_{t,k}, l\rangle \in T(a^\frown\langle x\rangle)$ for $k, l < m_t$.

- Note that assuming

$$\{\frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}] : s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(a))\}$$

covers H, then since

$$\{\frac{1}{4}\sigma(o(t)^\frown\langle h_{t,k}\rangle)[h_{t,k,l}] : s^\frown\langle i, h_{s,i}, j, x, k, h_{t,k}, l\rangle \in \max(T(a^\frown\langle x\rangle)) \setminus \max(T(a))\}$$

covers $H \cap \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$, we have that

$$\{\frac{1}{4}\sigma(o(t)^\frown\langle h_{t,k}\rangle)[h_{t,k,l}] : t^\frown\langle k, h_{t,k}, l\rangle \in \max(T(a^\frown\langle x\rangle))\}$$

covers H.

With this we may define the perfect information strategy τ for $\mathscr{O}$ in $Con_{O,P}(X, H)$ such that:

$$\tau(p \restriction n) = \bigcup_{s^\frown\langle i, h_{s,i}, j\rangle \in \max(T(p\restriction n))} \frac{1}{4}\sigma(o(s)^\frown\langle h_{s,i}\rangle)[h_{s,i,j}]$$

If p attacks τ, then it follows that $T(p \restriction n)$ is defined for all $n < \omega$, so let $T(p) = \bigcup_{n<\omega} T(p \restriction n)$. We note $T(p)$ is an infinite tree with finite levels:

- $\emptyset$ has exactly $m_\emptyset$ successors $\langle i\rangle$.
- $s^\frown\langle i\rangle$ has exactly one successor $t^\frown\langle i, h_{s,i}\rangle$
- $s^\frown\langle i, h_{s,i}\rangle$ has exactly m_s successors $t^\frown\langle i, h_{s,i}, j\rangle$
- $s^\frown\langle i, h_{s,i}, j\rangle$ has either no successors or exactly one successor $t^\frown\langle i, h_{s,i}, j, x\rangle$
- $t = s^\frown\langle i, h_{s,i}, j, x\rangle$ has exactly m_t successors $t^\frown\langle k\rangle$

Let $q' = \langle i_0, h_0, j_0, x_0, i_1, h_1, j_1, x_1, \ldots\rangle$ correspond to this infinite branch in $T(p)$, and let $q = o(q') = \langle h_0, x_0, h_1, x_1, \ldots\rangle$. Note that by the construction of $T(p)$, q is an attack on the winning strategy σ in $aProx_{D,P}(X)$, so it must converge. Since every other term of q is in H, it must converge to H. Then since q is a subsequence of p, p must cluster at H. $\square$

Corollary 6.27. *For any uniformly locally compact proximal space, $\mathscr{O} \uparrow Con_{O,P}(X, H)$ for all compact $H \subseteq X$.*

Proof. $\mathscr{O} \uparrow Con_{O,P}(X, H)$ if and only if $\mathscr{O} \uparrow Clus_{O,P}(X, H)$. $\square$

Corollary 6.28. *A compact uniform space X is Corson compact if and only if it is proximal.*

Proof. A characterization of Corson compact is having a W-set diagonal. If X is proximal compact, then X^2 is proximal compact, and its compact diagonal is a W-set. $\square$

Theorem 6.29. $\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X, H)$ *if and only if there exists a countable base around* H.

Proof. Let $\{U_n : n < \omega\}$ be a countable base around H. We define the predetermined strategy $\sigma(n) = \bigcap_{m \leq n} U_m$. Let p attack $\sigma(n)$ - then if U is any neighborhood of H, we may choose $H \subseteq U_m \subseteq U$, and note that $\sigma(n) \subseteq U_m$ for $n \geq m$, and thus $p(n) \in U_m \subseteq U$ for all $n \geq m$. Thus σ is a winning strategy.

For the other direction, suppose there does not exist a countable base around H, and let $\sigma(n)$ be an arbitrary predetermined strategy. Since $\{\bigcap_{m \leq n} \sigma(m) : n < \omega\}$ is not a countable base around H, we may choose an open set U around H such that $\bigcap_{m \leq n} \sigma(m) \not\subseteq U$ for all $n < \omega$. We may easily verify that if $p(n) \in \bigcap_{m \leq n} \sigma(m) \setminus U$ for all $n < \omega$, then p is a successful counterattack to σ. $\square$

Corollary 6.30. X *is first countable if and only if* $\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X, x)$ *for all* $x \in X$

Corollary 6.31. $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$ *implies* X *is first countable.*

Definition 6.32. Scattered Eberlein compact spaces are known as **strong Eberlein compact** spaces.

Theorem 6.33 (folklore)**.** *Scattered compact first-countable spaces are metrizable.*

Corollary 6.34. *If* X *is scattered compact and* $\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X, x)$ *for all* $x \in X$ *(or* $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$*), then* X *is metrizable.*

Example 6.35. $\mathscr{D} \underset{\text{pre}}{\not\uparrow} Prox_{D,P}(\omega_1^*)$

Proof. There does not exist a countable base around ∞, so $\mathscr{O} \underset{\text{pre}}{\not\uparrow} Con_{O,P}(X, \omega_1)$. $\square$

Example 6.36. $\mathscr{O} \underset{\text{tact}}{\uparrow} Con_{O,P}(\kappa^*, \infty)$ and $\mathscr{D} \underset{\text{tact}}{\uparrow} Prox_{D,P}(\kappa^*)$ for all cardinals κ

Proof. For $Con_{O,P}(\kappa^*, \infty)$, let $\sigma() = \sigma(\infty) = \kappa^*$ and $\sigma(x) = \kappa^* \setminus \{x\}$ otherwise. $\square$

Theorem 6.37. *If H is a closed subset of X, then $\mathscr{D} \underset{limit}{\uparrow} Prox_{D,P}(X) \Rightarrow \mathscr{D} \underset{limit}{\uparrow} Prox_{D,P}(H)$ where $\underset{limit}{\uparrow}$ is any of $\uparrow$, $\underset{k\text{-}tact}{\uparrow}$, or $\underset{k\text{-}mark}{\uparrow}$.*

Proof. Let $\sigma \circ L$ witness $\mathscr{D} \underset{\text{limit}}{\uparrow} Prox_{D,P}(X)$. We define $\tau \circ L$ for $\mathscr{D}$ in $Prox_{D,P}(H)$ as follows:

$$\tau \circ L(p \restriction n) = \sigma \circ L(p \restriction n) \cap H^2$$

Let p attack $\tau \circ L$. p also attacks the winning strategy $\sigma \circ L$, so either

$$\bigcap_{n<\omega} \left(\bigcap_{m \leq n} \tau \circ L(p \restriction n) \right) [p(n)] \subseteq \bigcap_{n<\omega} \left(\bigcap_{m \leq n} \sigma \circ L(p \restriction n) \right) [p(n)] = \emptyset$$

or p converges in X, and thus converges in H. □

Theorem 6.38. *If $\mathscr{D} \underset{limit}{\uparrow} Prox_{D,P}(X_i)$ for $i < \omega$, then $\mathscr{D} \underset{limit}{\uparrow} Prox_{D,P}(\prod_{i<\omega} X_i)$, where $\underset{limit}{\uparrow}$ is either $\uparrow$ or $\underset{k\text{-}mark}{\uparrow}$.*

Proof. A subbase for $\prod_{i<\omega} X_i$ is

$$\{\pi_i^{-1}(D) : i < \omega, D \in \mathcal{D}_i\}$$

where π_i is the natural projection from $\left(\prod_{i<\omega} X_i\right)^2$ onto X_i^2. (See Bell.)

For $p \in \left(\prod_{i<\omega} X_i\right)^\omega$, let $p_i \in X_i^\omega$ such that $p_i(n) = p(n)(i)$.

Let $\sigma_i \circ L$ witness $\mathscr{D} \underset{\text{limit}}{\uparrow} Prox_{D,P}(X_i)$ for $i < \omega$, and assume without loss of generality that $\sigma_i \circ L$ always yields X_i^2 before round i.

Then we define the strategy $\tau \circ L$ for $\mathscr{D}$ in $Prox_{D,P}(\prod_{i<\omega} X_i)$ as follows:

$$\tau \circ L(p \restriction n) = \bigcap_{i \leq n} \pi_i^{-1}(\sigma_i \circ L(p_i \restriction n))$$

Let p attack $\tau \circ L$. If $\bigcap_{n<\omega} \left(\bigcap_{m \leq n} \sigma_i(p_i \restriction n)\right) [p_i(n)] = \emptyset$ for any $i < \omega$, it easily follows that $\bigcap_{n<\omega} \left(\bigcap_{m \leq n} \tau(p \restriction n)\right) [p(n)] = \emptyset$.

Otherwise, we assume that for each $i < \omega$, p_i converges to some $x_i \in X_i$. Thus p converges to $x = \langle x_0, x_1, \ldots \rangle$. $\square$

Note: I expect I should be able to do some clever things assuming $S(\kappa, \omega, \omega)$ to get a similar result for sigma products of dimension κ.

Example 6.39. $\mathscr{D} \underset{\text{mark}}{\uparrow} Prox_{D,P}((\kappa^*)^\omega)$

Proof. $\mathscr{D} \underset{\text{tact}}{\uparrow} Prox_{D,P}(\kappa^*)$ + previous result $\square$

Lemma 6.40. *$\mathscr{O} \underset{pre}{\uparrow} Clus_{O,P}(X,S)$ if and only if $\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X,S)$.*

Proof. Suppose that σ is a predetermined winning strategy for $Clus_{O,P}(X,S)$. Let p attack σ, and q be a subsequence of p. It follows that q also attacks σ, so q clusters at S. Thus p conveges to S, and σ is a predetermined winning strategy for $Con_{O,P}(X,S)$. $\square$

Theorem 6.41. *For any predetermined absolutely proximal space X, $\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X,H)$ for all compact $H \subseteq X$.*

Proof. Let $\sigma(n)$ be a winning predetermined strategy for $\mathscr{D}$ in the absolutely proximal game such that $\sigma(n+1) \subseteq \sigma(n)$. For a given tree T, let $\max(T)$ denote its maximal nodes.

First we define $T(0) \subseteq \omega^{\leq 2}$.

- Let $\emptyset \in T(0)$.
- Choose

  $$m_\emptyset < \omega$$

  and for $i < m_\emptyset$ choose

  $$h_{\langle i \rangle} \in H$$

  and for $i, j < m_\emptyset$ choose

  $$h_{\langle i,j \rangle} \in H \cap \overline{\frac{1}{4}\sigma(0)[h_{\langle i \rangle}]}$$

  such that

  $$\left\{\frac{1}{4}\sigma(0)[h_{\langle i \rangle}] : i < m_\emptyset\right\}$$

  is a cover for H and such that for each $i < m_\emptyset$

  $$\left\{\frac{1}{4}\sigma(1)[h_{\langle i,j \rangle}] : j < m_\emptyset\right\}$$

  is a cover for $H \cap \overline{\frac{1}{4}\sigma(0)[h_{\langle i \rangle}]}$.

- Let $\langle i\rangle$ and $\langle i,j\rangle$ be in $T(0)$ for $i,j<m_\emptyset$.

Now suppose $T(n)\subseteq\omega^{\leq 2n+2}$ is defined. We then define $T(n+1)\subseteq\omega^{\leq 2n+4}$ as follows:

- Let $T(n)\subseteq T(n+1)$.
- For each $t^\frown\langle i,j\rangle\in\max(T(n))$, choose

$$m_{t^\frown\langle i,j\rangle}<\omega$$

  and for $k<m_{t^\frown\langle i,j\rangle}$ choose

$$h_{t^\frown\langle i,j,k\rangle}\in H\cap\overline{\frac{1}{4}\sigma(2n+2)[h_{t^\frown\langle i,j\rangle}]}$$

  and for $k,l<m_{t^\frown\langle i,j\rangle}$ choose

$$h_{t^\frown\langle i,j,k,l\rangle}\in H\cap\overline{\frac{1}{4}\sigma(2n+3)[h_{t^\frown\langle i,j,k\rangle}]}$$

  such that

$$\left\{\frac{1}{4}\sigma(2n+2)[h_{t^\frown\langle i,j,k\rangle}]:k<m_{t^\frown\langle i,j\rangle}\right\}$$

  is a cover for $H\cap\overline{\frac{1}{4}\sigma(2n+1)[h_{t^\frown\langle i,j\rangle}]}$, and such that for each $k<m_{t^\frown\langle i,j\rangle}$

$$\left\{\frac{1}{4}\sigma(2n+3)[h_{t^\frown\langle i,j,k\rangle}]:l<m_t\right\}$$

  is a cover for $H\cap\overline{\frac{1}{4}\sigma(2n+2)[h_{t^\frown\langle i,j,k\rangle}]}$.

- For each $t\in\max(T(n))$ and each $i,j<m_t$, put $t^\frown\langle i\rangle$ and $t^\frown\langle i,j\rangle$ in $T(n+1)$.

We now define the predetermined strategy τ for $\mathscr{O}$ in $Clus_{O,P}(X,H)$ such that:

$$\tau(n)=\bigcup_{t\in\max(T(n))}\bigcap_{m<2n+2}\frac{1}{4}\sigma(m)[h_{t\restriction m+1}]$$

noting that $\tau(n) = \bigcap_{m \leq n} \tau(m)$ by definition.

Since $\left\{\frac{1}{4}\sigma(2n+2)[h_{t^\frown\langle i\rangle}] : i < m_t\right\}$ is a cover for $H \cap \frac{1}{4}\sigma(2n+1)[h_t]$, since $\left\{\frac{1}{4}\sigma(2n+1)[h_{t^\frown\langle i,j\rangle}] : j < m_t\right\}$ is a cover for $H \cap \frac{1}{4}\sigma(2n)[h_{t^\frown\langle i\rangle}]$, and since $\left\{\frac{1}{4}\sigma(0)[h_{\langle i\rangle}] : i < m_\emptyset\right\}$ is a cover for H, it follows that $\tau(n)$ contains H and is τ is a legal strategy.

Let p be an attack against τ such that $p(n) \in \tau(n)$. If we can construct an attack q against σ which shares a subsequence of p, then p must cluster since q must converge. To find such a q, we construct a subtree $T' \subseteq T$.

We begin by setting $T'(0) = T(0)$.

For $n < \omega$, suppose $T'(n) \subseteq T(n)$ is defined such that:

- If $t'^\frown\langle i, j\rangle \in T'(n)$, then $|t'| \leq 2n$
- If $s' \leq t' \in T'(n)$, then $s \in T'(n)$.
- If $t' \in T'(n) \setminus \max(T'(n))$ and $|t'|$ is even, then $t'^\frown\langle i, j\rangle \in T'(n)$ for $i, j < m_{t'}$.

Since $p(n) \in \tau(n)$, there exists some $t_n \in \max(T(n))$ such that

$$p(n) \in \bigcap_{m<2n+2} \frac{1}{4}\sigma(m)[h_{t_n\restriction m+1}]$$

and in turn, there exists $t_n'^\frown\langle i, j\rangle \in \max(T'(n))$ with $t_n'^\frown\langle i, j\rangle \leq t_n$ and

$$p(n) \in \bigcap_{m<|t_n'|+2} \frac{1}{4}\sigma(m)[h_{t_n'^\frown\langle i,j\rangle\restriction m+1}]$$

since $|t_n'| \leq 2n$.

Let $p_{t_n'^\frown\langle i,j\rangle} = p(n)$

Take note that, in particular,

$$p_{t_n'^\frown\langle i,j\rangle} \in \sigma(|t_n'|)[h_{t_n'^\frown\langle i\rangle}]$$

and

$$p_{t'_n \frown \langle i,j \rangle} \in \frac{1}{4}\sigma(|t'_n|+1)[h_{t'_n \frown \langle i,j \rangle}]$$

We then define

$$T'(n+1) = T'(n) \cup \{t'_n{}^\frown \langle i,j,k \rangle : k \leq m_{t'_n \frown \langle i,j \rangle}\} \cup \{t'_n{}^\frown \langle i,j,k,l \rangle : k,l \leq m_{t'_n \frown \langle i,j \rangle}\}$$

while noting that for all $k \leq m_{t'_n \frown \langle i,j \rangle}$,

$$h_{t'_n \frown \langle i,j,k \rangle} \in H \cap \overline{\frac{1}{4}\sigma(|t'_n|+2)[h_{t'_n \frown \langle i,j \rangle}]} \subseteq \frac{1}{2}\sigma(|t'_n|+1)[h_{t'_n \frown \langle i,j \rangle}]$$

and thus

$$h_{t'_n \frown \langle i,j,k \rangle} \in \sigma(|t'_n|+1)[p_{t'_n \frown \langle i,j \rangle}]$$

Finally, we let $T' = \bigcup_{n<\omega} T'(n)$. Since T' is an infinite tree with finite levels, we may pick an infinite branch b. From b, we construct the sequence

$$q = \langle h_{b\restriction 1}, p_{b\restriction 2}, h_{b\restriction 3}, p_{b\restriction 4}, \dots \rangle$$

and claim it attacks σ and thus must converge. If so, since $\langle p_{b\restriction 2}, p_{b\restriction 4}, \dots \rangle$ is a subsequence of p, p must cluster. To see this, recall that for some t'_n:

$$p_{b\restriction 2n+2} = p_{t'_n \frown \langle i,j \rangle} \in \sigma(|t'_n|)[h_{t'_n \frown \langle i \rangle}]$$

and

$$h_{b\restriction 2n+3} = h_{t'_n \frown \langle i,j,k \rangle} \in \sigma(|t'_n|+1)[p_{t'_n \frown \langle i,j \rangle}]$$

We have thus proven $\mathscr{O} \underset{\text{pre}}{\uparrow} Clus_{O,P}(X,H)$, and thus $\mathscr{O} \underset{\text{pre}}{\uparrow} Con_{O,P}(X,H)$. $\square$

Example 6.42. Let $X = I \times 2$ be the Alexandrov double interval. Then $\mathscr{D} \underset{\text{pre}}{\not\uparrow} Prox_{D,P}(X)$, but $\mathscr{D} \underset{\text{mark}}{\uparrow} Prox_{D,P}(X)$.

Proof. We assume that the uniformity on X is given by entourages

$$D(\epsilon, F) = \{\langle x, 0\rangle, \langle y, 0\rangle : |x - y| < \epsilon\} \cup \{\langle x, 1\rangle, \langle y, 0\rangle : |x - y| < \epsilon \vee x \notin F\}$$

$$\cup \{\langle x, 0\rangle, \langle y, 1\rangle : |x - y| < \epsilon \vee y \notin F\} \cup \{\langle x, 1\rangle, \langle y, 1\rangle : x = y\}$$

That is, points are $D(\epsilon, F)$-close if they are the same point, or the first coordinates are within ϵ of each other while neither second coordinate is in F.

Suppose $\mathscr{D}$ had a predetermined winning strategy $\sigma(n) = D(\epsilon_n, F_n)$. Then $\mathscr{P}$ can choose $x \notin \bigcup_{n<\omega} F_n$, and play $\langle x, 1\rangle$ during even rounds, and $\langle x_{2n+1}, 0\rangle$ where $|x - x_{2n+1}| < \epsilon_{2n}$ during odd rounds, preventing convergence.

However, assume $\mathscr{D}$ uses the Marköv strategy $\sigma(x, n) = D(2^{-n}, \{x\})$. If $\mathscr{P}$ repeats a point of the form $\langle x, 1\rangle$, then since $D(2^{-n}, \{x\})[\langle x, 1\rangle] = \{\langle x, 1\rangle\}$, $\mathscr{P}$ must repeat $\langle x, 1\rangle$ for the rest of the game, and $\mathscr{D}$ wins. Otherwise, $\mathscr{P}$ cannot repeat points played in $I \times \{1\}$, and as the first coordinates form a Cauchy sequence and converge to some z, any open set about $\langle z, 0\rangle$ contains all but finitely many points of $\mathscr{P}$'s sequence, and $\mathscr{D}$ wins. $\square$

Theorem 6.43. *For any uniformly locally compact space X, $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X) \Leftrightarrow \mathscr{D} \underset{pre}{\uparrow} aProx_{D,P}(X)$*

Proof. Let M be a uniformly locally compact entourage. Let σ witness $\mathscr{D} \underset{\text{pre}}{\uparrow} Prox_{D,P}(X)$ such that $\sigma(n) \subseteq M$ always (so $\overline{\sigma(a)[x]} \subseteq \overline{M[x]}$ is compact), $\sigma(n+1) \subseteq \frac{1}{4}\sigma(n)$.

Let $\tau(n) = \frac{1}{2}\sigma(n)$. If p attacks τ in $aProx_{D,P}(X)$, then

$$p(n+1) \in \tau(n)[p(n)] = \frac{1}{2}\sigma(n)[p(n)]$$

and for

$$x \in \overline{\sigma(n+1)[p(n+1)]} \subseteq \overline{\frac{1}{4}\sigma(n)[p(n+1)]} \subseteq \frac{1}{2}\sigma(n)[p(n+1)]$$

we can conclude $x \in \sigma(n)[p(n)]$. Thus

$$\sigma(n+1)[p(n+1)] \subseteq \overline{\sigma(n+1)[p(n+1)]} \subseteq \sigma(n)[p(n)]$$

Finally, note that p attacks the winning strategy σ in $Prox_{D,P}(X)$, but since the intersection of a chain of nonempty compact sets is nonempty:

$$\bigcap_{n<\omega} \sigma(n)[p(n)] = \bigcap_{n<\omega} \overline{\sigma(n)[p(n)]} \neq \emptyset$$

We conclude that p converges. $\square$

Proposition 6.44. *If $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$, then X has a G_δ diagonal.*

Proof. If $\mathscr{D} \underset{\text{pre}}{\uparrow} Prox_{D,P}(X)$ with strategy σ, then consider $\langle x, y\rangle \in \bigcap_{n<\omega} \sigma(n)$. It follows that $\langle x, y, x, y, \ldots\rangle$ attacks σ, and $\{x, y\} \subseteq \bigcap_{n<\omega} \sigma(n)[x] \cap \bigcap_{n<\omega} \sigma(n)[y] \neq 0$ so it must converge, and $x = y$. Thus $\bigcap_{n<\omega} \sigma(n) = \Delta$ is G_δ. $\square$

Example 6.45. The Sorgenfrey line S has a G_δ diagonal but $\mathscr{P} \uparrow Prox_{D,P}(S)$.

Corollary 6.46. *For X with uniformity $\mathbb{D}$ inducing the compact Hausdorff topology τ, the following are equivalent:*

(a) $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$

(b) $\mathscr{D} \underset{pre}{\uparrow} aProx_{D,P}(X)$

(c) X has a G_δ diagonal

(d) $\mathbb{D}$ is metrizable

(e) τ is metrizable

Proof. For compact Hausdorff spaces, it is well known that there is exactly one uniformity inducing the topology. Thus $(d) \Leftrightarrow (e)$. Since X is uniformly locally compact, $(a) \Leftrightarrow (b)$. Also, compact spaces with a G_δ diagonal are metrizable, so $(c) \Rightarrow (e)$. Bell noted $(d) \Rightarrow (a)$ for arbitrary uniform spaces, and the previous proposition shows $(a) \Rightarrow (c)$. □

Theorem 6.47. *A uniformly locally compact space with a G_δ diagonal is metrizable.*

Proof. Based on several folklore results.

Uniformly locally compact implies the topological sum of σ-compact spaces implies paracompact. Locally compact plus G_δ diagonal implies locally metrizable. Locally metrizable plus paracompact characterizes metrizable. □

Corollary 6.48. *If X is uniformly locally compact, then $\mathscr{D} \underset{pre}{\uparrow} Prox_{D,P}(X)$ implies X's topology is metrizable.*

Example 6.49. Let R be the Michael Line. Then $\mathscr{P} \uparrow Prox_{D,P}(X)$.

Proof. During round 0, $\mathscr{P}$ may choose $m(0) = 0$ and $p(0) = 1$, and during round $n+1$, $\mathscr{P}$ may choose $m(n+1) > m(n)$ and $p(n+1) = p(n) + \frac{1}{10^{m(n+1)}}$ such that p is a legal attack.

It follows that p "converges" to $x = \sum_{n<\omega} \frac{1}{10^{m(n)}}$, except x is an irrational number composed of 1s separated by strings of 0s of strictly increasing size. $\square$

Example 6.50. Let ω_1 be given a ladder topology:

- All successor ordinals are isolated.
- Strictly increasing sequences (ladders) $L_\alpha : \omega \to \alpha$ are defined for each limit ordinal α such that L_α converges to α in the order topology, and each limit α is given neighborhoods of the form $L(\alpha, m) = \{\alpha\} \cup \{L_\alpha(n) : n \geq m\}$.
- $\omega_1 = \bigcup_{\alpha \in \omega_1^L} L(\alpha, 0)$

Let

$$A(\alpha, n) = [L(\alpha, 0) \setminus L(\alpha, n)]^1 \cup \{\omega_1^* \setminus (L(\alpha, 0) \setminus L(\alpha, n))\}$$

$$B(\alpha) = \{L(\alpha, 0), \omega_1^* \setminus L(\alpha, 0)\}$$

Finite refinements of $A(\alpha, n)$ and $B(\alpha)$ give partitions witnessing a uniformization of the ladder topology.

Then $Prox_{D,P}(\omega_1^*)$ is indetermined.

Proof. $\mathscr{D} \not\uparrow Prox_{D,P}(\omega_1^*)$ since ω_1^* isn't Corson compact.

Let σ be a perfect information strategy for $\mathscr{P}$.

Let

$$C(\sigma) = \{\delta \in \omega_1^L : \omega_1 \setminus D[\infty] \subseteq \delta \Rightarrow \sigma(D) < \delta\}$$

and note that by the closing up lemma, $C(\sigma)$ is always club.

We define a counterattack for $\mathscr{D}$ with $D_0 = D(F_0, 0)$ and $F_0 = \emptyset$ as her initial move, and let $p_0 = \sigma(D_0)$ be the initial move of $\mathscr{P}$. We fix $p_0 < \delta_0 < \delta_1 < \cdots \in C(\sigma)$ such that

$\lim_{n\to\infty}\delta_n = \delta_\infty \in C$. Enumerate all the limit ordinals $< \delta_n$ as $\alpha_{n,m}$ for $m < \omega$. Let $i_n < \omega$ be the least integer such that $L_{\delta_\infty}(i_n) > \delta_n$.

If $D_0, \ldots, D_n$ are the first n moves of $\mathscr{D}$, and $p_0, \ldots, p_n$ are the first n moves of $\mathscr{P}$ determined by σ, then we define

$$D_{n+1} = \bigwedge_{j\le n} (A(\delta_j, n) \wedge B(\delta_j)) \wedge \bigwedge_{j,k\le n} (A(\alpha_{j,k}, n) \wedge B(\alpha_{j,k})) \wedge A(\delta_\infty, i_n)$$

(TODO: show this is a winning counterattack) $\square$