Limited information strategies for topological games
PhD Defense

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Topological Games
Gruenhage's Convergence/Clustering Games
Bell's Proximal Game
Gruenhage's Locally Finite Games
Menger's Game

Defintion
Example
Motivation

A *topological game* is a two-player game $G(X)$ of length $\omega = \{0, 1, 2, \dots\}$ defined for certain topological spaces X.

During each round n, the first and second player take turns choosing certain topological objects from X (e.g. point, open set, open cover, etc.).
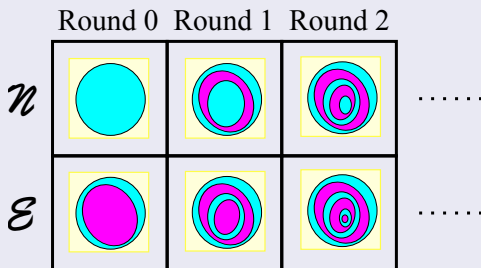
At the "end" of the game, a winner is declared by inspecting the sequences of choices made throughout the game.

The study of such games involves finding when a player has a *winning strategy* which defeats every possible counterattack by the opponent.

See Telgarsky's excellent survey on topological games for more details: [11]

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The *Banach-Mazur Game* $BM_{N,E}(X)$ (1935) [5]

The first player $\mathcal{N}$ wins the game if the intersection of all the chosen open sets is nonempty.

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X is Baire if and only if $\mathcal{N}$ lacks a winning strategy in the Banach Mazur game ($\mathcal{N} \not\uparrow BM_{N,E}(X)$).

Thus the topological property of being a Baire space has a game-theoretic characterization using $BM_{N,E}(X)$.

By considering *limited information strategies*, we may characterize more properties.

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Consider the following:

Theorem

X is α-favorable $\Rightarrow$ X is weakly α-favorable $\Rightarrow$ X is Baire

α-favorability is characterized by $\mathscr{E} \underset{\text{tact}}{\uparrow} BM_{N,E}(X)$: player $\mathscr{E}$ has a *tactical* winning strategy which only considers the most recent move of the opponent.

This is stronger than weak α-favorability [2], characterized by $\mathscr{E} \uparrow BM_{N,E}(X)$. In this case $\mathscr{E}$ still has a winning strategy, but it may rely on perfect information of the history of the game.

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By characterizing topological properties using the theory of topological games, we introduce new proof techniques for demonstrating the structure of given topological spaces.

The aim of my dissertation was to investigate four topological games from the literature with unknown limited information implications.

In doing so I uncovered several new results in general topology, advancing research done by G. Gruenhage, P. Nyikos, R. Telgársky, J. Bell, M. Scheepers, and others.

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Gruenhage's convergence game $Gru_{O,P}^{\rightarrow}(X, x)$ and clustering game $Gru_{O,P}^{\rightsquigarrow}(X, x)$ proceed as follows:
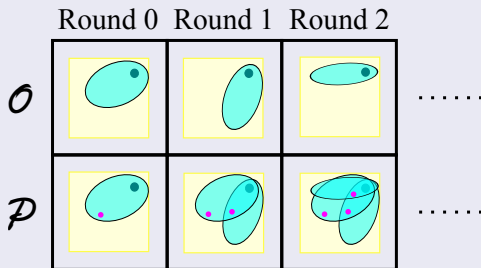
$\mathscr{O}$ wins the game if the points chosen by $\mathscr{P}$ converge/cluster to the given point $x \in X$. Otherwise, $\mathscr{P}$ wins.

Note that $\mathscr{O}$ need not know anything about the history of the game to play each round.

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If $\mathscr{O} \uparrow Gru_{O,P}^{\rightarrow}(X, x)$, then x is called a W-point in X. Obviously, all points of first-countablity are W-points, but $\mathscr{O} \uparrow Gru_{O,P}^{\rightarrow}(\kappa^*, \infty)$ also, where ∞ is the added point in the one-point compactification κ^* of uncountable discrete κ.

Points of first-countability may in fact be characterized by this game as well:

Theorem

x has a countable local base in X if and only if
$\mathscr{O} \underset{pre}{\uparrow} Gru_{O,P}^{\rightarrow}(X, x)$ *(*$\mathscr{O}$ *has a winning predetermined strategy using only the round number).*
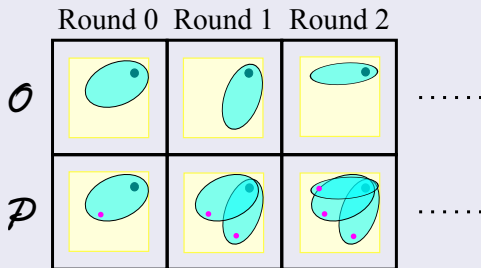
If every point in a space X is a W-point, then X is a W-space.

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A variation of this game which is harder for $\mathcal{O}$ yields some difficult infinite combinatorial questions:

Game

Gruenhage's hard convergence game $Gru_{O,P}^{\rightarrow,\star}(X, x)$ and hard clustering game $Gru_{O,P}^{\rightsquigarrow,\star}(X, x)$ proceed as follows:

Round 0   Round 1   Round 2

$\mathcal{O}$

$\mathcal{P}$

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Nyikos observed in [6] that:

Theorem

$\mathscr{O} \underset{mark}{\not\uparrow} Gru_{O,P}^{\to,\star} (\omega_1^*, \infty).$

($\mathscr{O}$ cannot guarantee a win using a Markov strategy which considers only the round number and most recent move.)

Some more work shows that in fact

Theorem

$\mathscr{O} \underset{k\text{-mark}}{\not\uparrow} Gru_{O,P}^{\to,\star} (\omega_1^*, \infty).$

($\mathscr{O}$ cannot guarantee a win using a k-Markov strategy which considers only the round number and k most recent moves.)

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Interestingly, the strategy which prevents convergence won't prevent clustering as well unless the cardinality of the space is sufficiently large.

Theorem

$$\mathscr{O} \underset{mark}{\uparrow} Gru_{O,P}^{\leadsto,\star}(\omega_1^*, \infty), \text{ but } \mathscr{O} \underset{k\text{-}mark}{\not\uparrow} Gru_{O,P}^{\leadsto,\star}(\omega_2^*, \infty).$$

But knowledge of the round number is used non-trivially in doing so.

Theorem

$$\mathscr{O} \underset{k\text{-}tact}{\not\uparrow} Gru_{O,P}^{\leadsto,\star}(\omega_1^*, \infty).$$

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Proof that $\mathscr{O} \underset{mark}{\uparrow} Gru_{O,P}^{\leadsto,\star}(\omega_1^*, \infty)$

For each $\alpha < \omega_1$, let $f_\alpha : \alpha \to \omega$ be injective.

Note that for each $F \in [\omega_1]^{<\omega}$, there is some $n_F < \omega$ such that $f_{\alpha+1}(\beta) < n_F$ for all $\beta, \alpha \in F$.

Let σ be a Markov strategy for $\mathscr{O}$ such that

$$\sigma(\langle \alpha \rangle, n) \subseteq \omega_1^* \setminus \{\beta < \omega_1 : f_{\alpha+1}(\beta) < n\}$$

for $\alpha < \omega_1$.

Thus as any legal counterattack cannot have finite range in ω_1 without repeating ∞ co-finitely, σ is a winning Markov strategy. $\quad\square$

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Proof that $\mathscr{O} \not\uparrow_{\text{tact}} Gru_{O,P}^{\leadsto,\star}(\omega_1^*, \infty)$

Let σ be a tactic for $\mathscr{O}$ in $Gru_{O,P}^{\leadsto,\star}(\omega_1^*, \infty)$.

Then this set is closed and unbounded in ω_1:
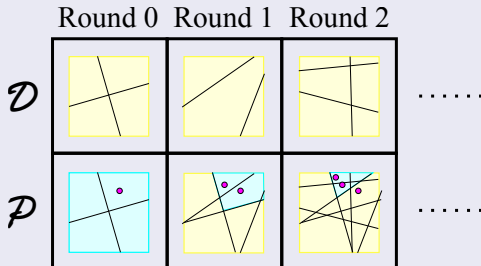
$$C_\sigma = \{\alpha < \omega_1 : \beta < \alpha \Rightarrow \omega_1^* \setminus \sigma(\langle\beta\rangle) \subseteq \alpha\}$$

If $a_\sigma : \omega_1 \to C_\sigma$ is an order isomorphism, then there is $n < \omega$ such that $a_\sigma(n) \in \sigma(\langle a_\sigma(\omega)\rangle)$.

Then since $a_\sigma(\omega) > a_\sigma(n)$ implies $a_\sigma(\omega) \in \sigma(\langle a_\sigma(n)\rangle)$ (by the definition of C_σ), it follows that $\langle a_\sigma(n), a_\sigma(\omega), a_\sigma(n), a_\sigma(\omega), \ldots\rangle$ is a legal counterattack against σ which does not cluster at ∞. $\square$

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Bell's proximal game $Bell_{D,P}^{\rightarrow}(X)$ for compact zero-dimensional X:

Round 0 Round 1 Round 2

$\mathscr{D}$

$\mathscr{P}$

$\mathscr{D}$ wins the game if the points chosen by $\mathscr{P}$ converge. Otherwise, $\mathscr{P}$ wins.

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If $\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(X)$, then X is called a proximal compact. This game was brought to my attention due to this result: [1]

Theorem

Every proximal space is a W-space. So
$\mathscr{D} \uparrow Bell_{D,P}^{\rightarrow}(X) \Rightarrow \mathscr{O} \uparrow Gru_{O,P}^{\rightarrow}(X,x)$ *for all* $x \in X$.

Proximal spaces have strong preservation properties, as any closed subset or Σ-product of proximal spaces is proximal. Since any proximal space is collectionwise normal, Bell's game gives an elegant proof of the classic result:

Theorem

A Σ-product of metrizable spaces is collection-wise normal.

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Nyikos [7] observed that

Proposition

Corson compact spaces are proximal.

and asked if the converse holds as well.

With Gruenhage, I showed that the answer is yes:

Theorem

A compact space is Corson compact if and only if it is proximal.

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A winning limited information strategy may always be passed down to win in a closed subspace, but Bell's result that winning strategies are preserved for Σ-products does not quite generalize as well:

Theorem

For $k < \omega$, if $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell_{D,P}^{\rightarrow}(X_i)$ for all $i < \omega$, then
$\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell_{D,P}^{\rightarrow}(\prod_{i<\omega} X_i)$.

Other limited information lemmas proved in my dissertation allowed me to prove a game-theoretic characterization of another compactness property (paper in preparation):

Theorem

A compact space is strong Eberlein compact if and only if
$\mathscr{D} \underset{tact}{\uparrow} Bell_{D,P}^{\rightarrow}(X)$.

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Gruenhage's locally finite games $Gru_{K,P}(X)$ and $Gru_{K,L}(X)$ proceed as follows:

Round 0 Round 1 Round 2
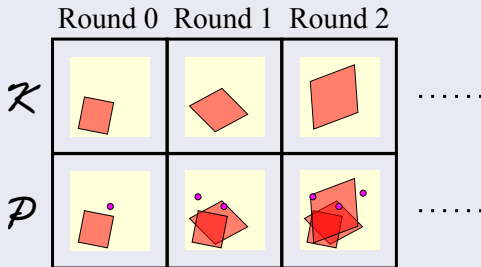
$\mathcal{K}$ wins the game if the points/sets chosen by $\mathcal{P}/\mathcal{L}$ are locally finite in the space. Otherwise, $\mathcal{P}/\mathcal{L}$ wins.

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Gruenhage used these games in [3] to characterize metacompactness and σ-metacompactness amongst locally compact spaces:

Theorem

For locally compact spaces, $\mathscr{K} \underset{tact}{\uparrow} Gru_{K,P}(X)$ if and only if X is metacompact.

Theorem

For locally compact spaces, $\mathscr{K} \underset{mark}{\uparrow} Gru_{K,P}(X)$ if and only if X is σ-metacompact.

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By removing knowledge of the round number, an analogous result is unsurfaced:

Theorem

For locally compact spaces, $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(X)$ if and only if X is σ-compact.

Actually, for locally compact or even compactly-generated spaces, $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(X)$ if and only if $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,L}(X)$.

However, there is a non-compactly-generated counterexample:

Theorem

There exists a free ultrafilter $\mathcal{F}$ such that $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,P}(\omega \cup \{\mathcal{F}\})$, but $\mathscr{K} \underset{pre}{\not\uparrow} Gru_{K,L}(\omega \cup \{\mathcal{F}\})$ for any free ultrafilter $\mathcal{F}$.

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For the related game $Gru^{\rightarrow}_{O,P}(X, x)$, a $(k + 1)$-tactic/mark may always be improved to only use the most recent move of the opponent. If this also holds for $Gru_{K,P}(X)$, then metacompactness and σ-metacompactness may would be characterized by the existence of any winning $(k + 1)$-tactic/mark.
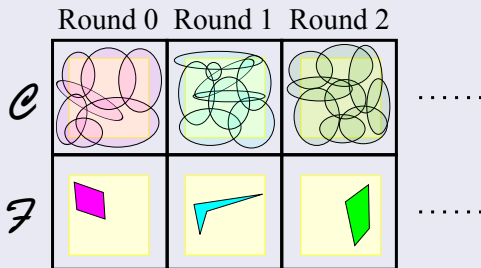
Due to a technical difference in the games, it's unclear if this is true. However, for a tricky non-σ-metacompact example $\boldsymbol{X}$:

Theorem

$\mathscr{K} \uparrow Gru_{K,P}(\boldsymbol{X})$ but $\mathscr{K} \underset{k\text{-mark}}{\not\uparrow} Gru_{K,P}(\boldsymbol{X})$.

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Game

Menger's game $Men_{C,F}(X)$ proceeds as follows:

Round 0 Round 1 Round 2

$\mathscr{C}$

$\mathscr{F}$

$\mathscr{F}$ wins the game if her finitely coverable subsets union to the space. Otherwise, $\mathscr{C}$ wins.

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A covering property generalizing σ-compactness is characterized by this game, demonstrated by Hurewicz in the 1920's. [4]

Theorem

A space is Menger if and only if $\mathscr{C} \not\uparrow Men_{C,F}(X)$.

It was originally suspected that Menger subspaces of the real line were exactly the σ-compact subspaces, but as was shown by Telgarsky and Scheepers, σ-compact spaces have slightly more structure. [10] [9]

Theorem

A metrizable space X is σ-compact if and only if $\mathscr{F} \uparrow Men_{C,F}(X)$.

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By considering Markov strategies, the previous theorem may be factored into two subresults.

Theorem

A regular space X is σ-compact if and only if
$\underset{mark}{\mathscr{F} \uparrow} Men_{C,F}(X)$.

Theorem

For a second-countable space X, $\mathscr{F} \uparrow Men_{C,F}(X)$ if and only if
$\underset{mark}{\mathscr{F} \uparrow} Men_{C,F}(X)$.

Note that since the spaces we are considering are all Lindelöf, metrizability is characterized by regularity and second-countability.

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Limited information strategies in topological games such as $Men_{C,F}(X)$ often have set-theoretic consequences. The statement $S(\kappa)$ due to M. Scheepers [8] says that there exist almost-compatible functions $f_A : A \to \omega$ for each $A \in [\kappa]^\omega$.
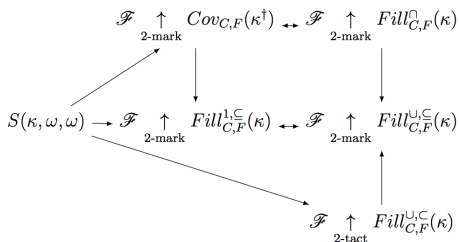
Theorem

$S(\omega_1)$ and $\neg S((2^\omega)^+)$ are theorems of ZFC, but $S(\kappa)$ is independent of ZFC for $\omega_1 < \kappa \leq 2^\omega$.

Theorem

If $S(\kappa)$ holds, then $\mathscr{F} \underset{\text{2-tact}}{\uparrow} Fill_{C,F}^{\cup,\subseteq}(\kappa)$.

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Let $\kappa^\dagger$ be the one-point "Lindelöf-ication" of discrete κ.

Theorem

For most of the games in the previous chart, including the Menger game $Men_{C,F}(X)$, there's no need to consider larger amounts of limited information.

Theorem

For each $k < \omega$, $\mathscr{F} \underset{k+2\text{-mark}}{\uparrow} Men_{C,F}(X)$ if and only if $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men_{C,F}(X)$

The topological property $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men_{C,F}(X)$ seems to depend on the set-theoretic axioms at play.

Theorem

If $S(2^\omega)$, then $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Men_{C,F}(R_\omega)$.

Any questions?

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