Definition 1. Let a V-map be a u.s.c. idempotent surjection.

Definition 2. For any LOS $\langle L, \leq \rangle$, let $\check{L}$ be the collection of leftward subsets of L (subsets for which $b \in L, a \leq b \Rightarrow a \in L$) linearly ordered by $\subseteq$, and let $\hat{L}$ be the collection of left-closed subsets of L (leftward subsets which are closed) linearly ordered by $\subseteq$.

Proposition 3. *$\check{L}$, $\hat{L}$ are compact.*

Proof. Each subset S has an infimum $\cap S$ and a supremum $\cup S$ (or $\mathrm{cl}(\cap S)$). □

Note that $\check{L}$ is not a "compactification" as L does not necessarily embed as a dense subspace of $\check{L}$: if $L = I$, we might attempt to embed $t \mapsto [0,t)$, but then note that the subspace topology induces the reverse Sorgenfrey interval as $([0,s),[0,t]) = ([0,s),[0,t)]$ is open. However $\hat{L}$ is the typical way of compactifying a linearly ordered space L, provided L lacks a least element (otherwise the empty set is an [easily removable] isolated point in $\hat{L}$). Note that we **always** assume that $\emptyset \in \hat{L}$:

Example 4. $\hat{I} \cong \{-\infty\} \cup I$ where $\emptyset \mapsto -\infty$ and $[0,t] \mapsto t$.

Example 5. For limit ordinals α, $\hat{\alpha} \cong \alpha + 1$, and for all other infinite ordinals, $\hat{\alpha} \cong \alpha$. (The addition of a new least isolated point is of course irrelevant).

Definition 6. For any compact LOTS K with minimum 0 and maximum 1, let γ be the V-map on K where $\gamma(0) = K$ and $\gamma(t) = \{1\}$ for $t > 0$.

Definition 7. For any LOTS M with minimum element 0, let ν be the V-map on M where $\nu(0) = K$ and $\nu(t) = \{t\}$ for $t > 0$.

Note for $K = M = 2$ that $\gamma = \nu$.

Theorem 8. $X = \varprojlim\{2, \gamma, L\} \cong \check{L}$

Proof. We start by placing an order on X. Let $\vec{x} < \vec{y}$ if there exists $a \in L$ with $\vec{x}(a) = 0, \vec{y}(a) = 1$. We claim this is a total order inducing the topology on X.

We first observe that if $\vec{x}(b) = 1$, then for all $a \leq b$, $\vec{x}(a) \in \gamma(1) = \{1\}$. If $\vec{x} \neq \vec{y}$, then assume without loss of generality that $\vec{x}(a) = 0, \vec{y}(a) = 1$, so $\vec{x} < \vec{y}$. Also, whenever $\vec{x}(b) = 1$, we have that $b < a$, so $\vec{y}(b) = 1$, preventing $\vec{y} < \vec{x}$. Finally if $\vec{x} < \vec{y}$ and $\vec{y} < \vec{z}$, take a, b with $\vec{x}(a) = 0$, $\vec{y}(a) = 1$,$\vec{y}(b) = 0$,$\vec{z}(b) = 1$. It follows that $a < b$ so $\vec{z}(a) = 1$ and $\vec{x} < \vec{z}$.

Consider the basic open set $B(\vec{x}, F)$ for a finite set $F \in [L]^{<\omega}$ about the sequence $\vec{x} \in X$ which contains all sequences $\vec{y}$ agreeing with $\vec{x}$ on F. If $\vec{x}(a) = 1$ for all $a \in F$, then let $\vec{w} \in X$ be 0 on the maximum of F, and 1 for anything less. It follows that $B(\vec{x}, F) = (\vec{w}, \rightarrow)$. If $\vec{x}(a) = 0$ for all $a \in F$, then let $\vec{y} \in X$ be 1 on the minimum of F, and 0 for anything greater. It follows that $B(\vec{x},F)=(\leftarrow,\vec{y})$. Finally if $\vec{x}(a)=1$ and $\vec{x}(b)=0$ for $a<b$ in F and nothing between a,b is in F, then let $\vec{w}\in X$ be 0 on a and 1 for anything less, and let $\vec{y}\in X$ be 1 on b and 0 for anything greater. It follows that $B(\vec{x},F)=(\vec{w},\vec{y})$.

Let ϕ evaluate each $\vec{x}\in X\subseteq 2^L$ as the characteristic function for a subset of L. It's easy to see that ϕ is an order isomorphism between $\langle X,\leq\rangle$ and $\langle \check{L},\subseteq\rangle$. □

Corollary 9. $\varprojlim\{2,\gamma,\alpha\}\cong\alpha+1$ *for every ordinal* α.

Proof. Since $\breve{\alpha}=\alpha+1$ (actually equals, not just homeomorphic!), we get $\varprojlim^\star\{2,\gamma,\alpha\}\cong\breve{\alpha}=\alpha+1$ for free. Note that C and Varagona used this in (TODO create citataion) to break metrizability in uncountable-ordinal-indexed inverse limits (for any V-map there exists a two-point set 2 such that $f\restriction 2\supseteq\gamma$, that is, "$f$ has condition Γ"). □

We may generalize theorem 8 as follows:

Theorem 10. *If M is a LOTS with minimum 0 and maximum 1, then* $\varprojlim\{M,\gamma,L\}\cong\hat{L}\times_{lex}M/\sim$*, where* $\langle(\leftarrow,l_0],1\rangle\sim\langle(\leftarrow,l_1],0\rangle$ *if* $l_0<l_1$ *and* $(l_0,l_1)=\emptyset$*, and where* $\langle A,m\rangle\sim\langle A,m'\rangle$ *if* $A\in\hat{L}\setminus L$*.*

Proof. Let $\rho(\vec{x})=\mathrm{cl}\{l\in L:\vec{x}(l)>0\}$, $v(\vec{0})=0$, and $v(\vec{x})=\min\{\vec{x}(l):l\in\rho(\vec{x})\}$ otherwise. Say $\vec{x}<\vec{y}$ if $\rho(\vec{x})\subsetneq\rho(\vec{y})$ or both $\rho(\vec{x})=\rho(\vec{y})$ and $v(\vec{x})<v(\vec{y})$. The reader may verify that this is a linear order on $\varprojlim\{M,\gamma,L\}$, and $\theta(\vec{x})=\langle\rho(\vec{x}),v(\vec{x})\rangle\in\hat{L}\times_{\mathrm{lex}}M/\sim$ preserves order. For each left-closed set A and $m\in M$, let $\vec{x}_{A,m}(l)=1$ for $l\in A$ unless l is the supremum element of A, $\vec{x}_{A,m}(l)=m$ if l is the supremum of A, and $\vec{x}_{A,m}(l)=0$ for $l\notin A$. To complete the proof, we should demonstrate that the linear order we defined induces the topology of the inverse limit, and that θ is a surjection.

A basic open set in $\varprojlim\{M,\gamma,L\}\subseteq L^m$ is of the form $[U,F]$ where $U(l)$ is an open interval in M for each $l\in F\in[L]^{<\omega}$, and $[U,F]=\{\vec{x}:l\in F\Rightarrow\vec{x}(l)\in U(l)\}$. If we assume that $[U,F]$ is non-empty, one of the following must hold:

- $U[l_0]=(a,b)$ for some $l_0\in F$. Then $[U,F]=[U,\{l_0\}]$, and note that $[U,\{l_0\}]=(\vec{x}_{(\leftarrow,l_0],a},\vec{x}_{(\leftarrow,l_0],b})$.
- $U(l_0)=(a,1]$ and $U(l_1)=[0,b)$ for some $l_0<l_1\in L$ and $[U,F]=[U,\{l_0,l_1\}]$. Then $[U,\{l_0,l_1\}]=(\vec{x}_{l_0,a},\vec{x}_{l_1,b})$.

In the other direction, consider $\vec{y}\in(\vec{x},\vec{z})$.

- In the case that $l_0\in\rho(\vec{y})\setminus\rho(\vec{x})$ and $l_1\in\rho(\vec{z})\setminus\rho(\vec{y})$, let $U(l_0)=(0,1]$, $U(l_1)=[0,v(\vec{z}))$ and note $\vec{y}\in[U,\{l_0,l_1\}]\subseteq(\vec{x},\vec{z})$.

- In the case that $l_0 \in \rho(\vec{y}) \setminus \rho(\vec{x})$, $\rho(\vec{y}) = \rho(\vec{z})$, and $v(\vec{y}) < v(\vec{z})$, it follows that $\rho(\vec{y}) = \rho(\vec{z}) = (\leftarrow, l_1]$, so let $U(l_0) = (0, 1]$, $U(l_1) = [0, v(\vec{z}))$ and note $\vec{y} \in [U, \{l_0, l_1\}] \subseteq (\vec{x}, \vec{z})$.

- In the case that $\rho(\vec{x}) = \rho(\vec{y})$, $v(\vec{x}) < v(\vec{y})$, and $l_1 \in \rho(\vec{z}) \setminus \rho(\vec{y})$, it follows that $\rho(\vec{x}) = \rho(\vec{y}) = (\leftarrow, l_0]$, so let $U(l_0) = (v(\vec{x}), 1]$, $U(l_1) = [0, v(\vec{z}))$ and note $\vec{y} \in [U, \{l_0, l_1\}] \subseteq (\vec{x}, \vec{z})$.

- In the case that $\rho(\vec{x}) = \rho(\vec{y}) = \rho(\vec{z})$ and $v(\vec{x}) < v(\vec{y}) < v(\vec{z})$, it follows that $\rho(\vec{x}) = \rho(\vec{y}) = \rho(\vec{z}) = (\leftarrow, l_0]$, so let $U(l_0) = (v(\vec{x}), v(\vec{z}))$ and note $\vec{y} \in [U, \{l_0\}] = (\vec{x}, \vec{z})$.

We conclude by showing that θ is a surjection. If $B \in \hat{L} \setminus L$ and $m \in M$, consider $\langle B, m\rangle$. B lacks a supremum in L, so $\vec{x}_{B,0}(l) = 1$ for $l \in B$ and $\vec{x}_{B,0}(l) = 0$ otherwise. So $\theta(\vec{x}_{B,0}) = \langle \mathrm{cl}B, 1\rangle = \langle B, 1\rangle \sim \langle B, m\rangle$ for all $m \in M$. Otherwise, $B = (\leftarrow, l_1]$ for some $l_1 \in L$. Let $m > 0$. Then $\theta(\vec{x}_{(\leftarrow,l_1],m}) = \langle \mathrm{cl}(\leftarrow, l_1], v(\vec{x}_{(\leftarrow,l_1],m})\rangle = \langle(\leftarrow, l_1], m\rangle$. Finally, we want to map onto $\langle(\leftarrow, l_1], 0\rangle$. If there exists $l_0 < l_1$ with $(l_0, l_1) = \emptyset$, then $\theta(\vec{x}_{(\leftarrow,l_1],0}) = \theta(\vec{x}_{(\leftarrow,l_0],1}) = \langle(\leftarrow, l_0], 1\rangle$ will suffice. Otherwise, $\theta(\vec{x}_{(\leftarrow,l_1],0}) = \langle \mathrm{cl}(\leftarrow, l_1), v(\vec{x}_{(\leftarrow,l_1),0})\rangle = \langle(\leftarrow, l_1], 0\rangle$. □

Here are some applications:

Example 11. $\varprojlim\{2, \gamma, I\} \cong (\hat{I} \setminus \emptyset) \times_{\text{lex}} 2 \cong I \times_{\text{lex}} 2 \cong \check{I}$ (of course, this could be found quicker with theorem 8).

Example 12. $\varprojlim\{I, \gamma, I\} \cong (\hat{I} \setminus \emptyset) \times_{\text{lex}} I \cong I \times_{\text{lex}} I$.

Example 13. For infinite ordinals α, $\varprojlim\{I, \gamma, \alpha\} \cong (\alpha \times_{\text{lex}} [0, 1)) \cup \{\infty\}$. In particular, $\alpha = \kappa$ for an infinite cardinal κ gives the closed long ray of length κ.

We introduce an alternate definition of an arbitrarily indexed inverse limit.

Definition 14. Let $\varprojlim^\star\{X, f, L\} \subseteq \varprojlim\{X, f, L\}$ satisfy that $\vec{x}(a) = \lim_{t\to a} \vec{x}(t)$ for all $a \in L$ (for any open neighborhood $U$of $\vec{x}(a)$ there is $b < a$ where $\vec{x}(t) \in U$ for all $t \in (b, a]$).

Theorem 15. $Y = \varprojlim^\star\{2, \gamma, L\} \cong \hat{L}$.

Proof. Consider Y as a subspace of $X = \varprojlim\{2, \gamma, L\}$ with the linear order described above. We claim that if ϕ is the characteristic function for a subset of L, then ϕ is an order isomorphsim between $\langle Y, \leq\rangle$ and $\langle \hat{L}, \subseteq\rangle$.

Let A be a left-closed subset of L. Let $\vec{x}(a) = 1$ when $a \in A$ and $\vec{x}(a) = 0$ otherwise. Then $\vec{x} \in Y$ and $\phi(\vec{x}) = A$.

Let $\vec{x}, \vec{y} \in Y$. If $\phi(\vec{x}) = \phi(\vec{y}) = A$, then A is a left-closed set where $\vec{x}(a) = \vec{y}(a) = 1$ for $a \in A$ and $\vec{x}(a) = \vec{y}(a) = 0$ otherwise, so $\vec{x} = \vec{y}$.

Finally let $\vec{x} < \vec{y}$, so there exists $a \in L$ with $\vec{x}(a) = 0$, $\vec{y}(a) = 1$. Then $\phi(\vec{x}) \subseteq (\leftarrow, a) \subseteq \phi(\vec{y})$. Thus ϕ preserves order. □

Corollary 16. $\varprojlim^\star\{2,\gamma,\alpha\} \cong \alpha+1$ *for every infinite limit or finite ordinal* α.

Proof. If α is finite, then of course all (leftward) sets are closed and we get $\hat{\alpha} = \check{\alpha} = \alpha + 1$ for free. Otherwise, as observed previously $\hat{\alpha}$ is homeomorphic to its usual compactification $\alpha + 1$ for limit ordinals. □

In fact, $\hat{\alpha} = \alpha + 1 \setminus L(\alpha)$ where $L(\alpha)$ is the colleciton of all limit ordinals less than α, which also shows $\hat{\alpha} \cong \alpha$ for infinite successor ordinals α.

References