ARHANGELSKII'S α-PRINCIPLES AND SELECTION GAMES

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Abstract. Arhangelskii's convergence properties α_2 and α_4 may be characterized in terms of Scheeper's selection games. We generalize these folklore results to hold for more general collections.

The following characterizations were given as Definition 1 by Kocinac in [cite Kocinac selection principles related].

Definition 1. *Arhangelskii's α-principles* $\alpha_i(\mathcal{A},\mathcal{B})$ are defined as follows for $i \in \{1,2,3,4\}$. Let $A_n \in \mathcal{A}$ for all $n < \omega$; then there exists $B \in \mathcal{B}$ such that:

α_1: $A_n \cap B$ is cofinite in A_n for all $n < \omega$.
α_2: $A_n \cap B$ is infinite for all $n < \omega$.
α_3: $A_n \cap B$ is infinite for infinitely-many $n < \omega$.
α_4: $A_n \cap B$ is non-empty for infinitely-many $n < \omega$.

When $(\mathcal{A},\mathcal{B})$ is omitted, it is assumed that $\mathcal{A} = \mathcal{B}$ is the collection $\Gamma_{X,x}$ of sequences converging to some point $x \in X$, as introduced by Arhangelskii in [cite Arhangelskii frequency spectrum]. Provided $\mathcal{A}$ only contains infinite sets, it's easy to see that $\alpha_n(\mathcal{A},\mathcal{B})$ implies $\alpha_{n+1}(\mathcal{A},\mathcal{B})$.

We aim to relate these to the following games.

Definition 2. The *selection game* $G_1(\mathcal{A},\mathcal{B})$ (resp. $G_{fin}(\mathcal{A},\mathcal{B})$) is an ω-length game involving Players I and II. During round n, I chooses $A_n \in \mathcal{A}$, followed by II choosing $a_n \in A_n$ (resp. $F_n \in [A_n]^{<\aleph_0}$). Player II wins in the case that $\{a_n : n < \omega\} \in \mathcal{B}$ (resp. $\bigcup\{F_n : n < \omega\} \in \mathcal{B}$), and Player I wins otherwise.

Such games are well-represented in the literature; see [cite Scheepers combinatorics ramsey] for example. We will also consider the similarly-defined games $G_{<2}(\mathcal{A},\mathcal{B})$ (II chooses 0 or 1 points from each choice by I) and $G_{cf}(\mathcal{A},\mathcal{B})$ (II chooses cofinitely-many points).

Definition 3. Let P be a player in a game G. P has a *winning strategy* for G, denoted $P \uparrow G$, if P has a strategy that defeats every possible counterplay by their opponent. If a strategy only relies on the round number and ignores the moves of the opponent, the strategy is said to be *predetermined*; the existence of a predetermined winning strategy is denoted $P \underset{\text{pre}}{\uparrow} G$.

We briefly note that the statement I $\underset{\text{pre}}{\not\uparrow} G_\star(\mathcal{A},\mathcal{B})$ is often denoted as the *selection principle* $S_\star(\mathcal{A},\mathcal{B})$.

The equivalence of $\alpha_2(\Gamma_{X,x}\Gamma_{X,x})$ and I $\underset{\text{pre}}{\not\uparrow} G_1(\Gamma_{X,x},\Gamma_{X,x})$ was briefly asserted by Sakai in the introduction of [cite Sakai sequence selection properties]; the similar

Key words and phrases. Selection principle, selection game, α_i property, convergence.

equivalence of $\alpha_4(\Gamma_{X,x}\Gamma_{X,x})$ and $\mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} G_{fin}(\Gamma_{X,x}, \Gamma_{X,x})$ seems to be folklore. In fact, these relationships hold in more generality.

1. Main Results

Definition 4. Let $\Gamma_{X,x}$ be the collection of non-trivial sequences $S \subseteq X$ converging to x, that is, infinite subsets of $X \setminus \{x\}$ such that for each neighborhood U of x, $S \cap U$ is cofinite in S.

Definition 5. Let Γ_X be the collection of open γ-*covers* $\mathcal{U}$ of X, that is, infinite open covers of X such that $X \notin \mathcal{U}$ and for each $x \in X$, $\{U \in \mathcal{U} : x \in U\}$ is cofinite in $\mathcal{U}$.

The similarity in nomenclature follows from the observation that every non-trivial sequence in $C_p(X)$ converging to the zero function $\mathbf{0}$ naturally defines a corresponding γ-cover in X, see e.g. Theorem 4 of [Scheepers a sequential property and covering property].

Note that by these definitions, convergent sequences (resp. γ-covers) may be uncountable, but any infinite subset of either would remain a convergent sequence (resp. γ-cover), in particular, countably infinite subsets. We capture this idea as follows.

Definition 6. Say a collection $\mathcal{A}$ is Γ-*like* if it satisfies the following for each $A \in \mathcal{A}$.

- $|A| \geq \aleph_0$.
- If $A' \subseteq A$ and $|A'| \geq \aleph_0$, then $A' \in \mathcal{A}$.

We also require the following.

Definition 7. Say a collection $\mathcal{A}$ is *almost*-Γ-*like* if for each $A \in \mathcal{A}$, there is $A' \subseteq A$ such that:

- $|A'| = \aleph_0$.
- If A'' is a cofinite subset of A', then $A'' \in \mathcal{A}$.

So all Γ-like sets are almost-Γ-like.

We are now able to prove a few general equivalences between α-princples and selection games.

Theorem 8. *Let $\mathcal{A}$ be almost-Γ-like and $\mathcal{B}$ be Γ-like. Then $\alpha_2(\mathcal{A}, \mathcal{B})$ holds if and only if* $\mathrm{I} \underset{pre}{\not\uparrow} G_1(\mathcal{A}, \mathcal{B})$.

Proof. We first assume $\alpha_2(\mathcal{A}, \mathcal{B})$ and let $A_n \in \mathcal{A}$ for $n < \omega$ define a predetermined strategy for I. We may apply $\alpha_2(\mathcal{A}, \mathcal{B})$ to choose $B \in \mathcal{B}$ such that $|A_n \cap B| \geq \aleph_0$. We may then choose $a_n \in (A_n \cap B) \setminus \{a_i : i < n\}$ for each $n < \omega$. It follows that $B' = \{a_n : n < \omega\} \in \mathcal{B}$ since B' is an infinite subset of $B \in \mathcal{B}$; therefore A_n does not define a winning predetermined strategy for I.

Now suppose $\mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} G_1(\mathcal{A}, \mathcal{B})$. Given $A_n \in \mathcal{A}$ for $n < \omega$, first choose $A'_n \in \mathcal{A}$ such that $A'_n = \{a_{n,j} : j < \omega\} \subseteq A_n$, $j < k$ implies $a_{n,j} \neq a_{n,k}$, and $A_{n,m} = \{a_{n,j} : m \leq j < \omega\} \in \mathcal{A}$. Finally choose some $\theta : \omega \to \omega$ such that $|\theta^{\leftarrow}(n)| = \aleph_0$ for each $n < \omega$.

Since playing $A_{\theta(m),m}$ during round m does not define a winning strategy for I in $G_1(\mathcal{A}, \mathcal{B})$, II may choose $x_m \in A_{\theta(m),m}$ such that $B = \{x_m : m < \omega\} \in \mathcal{B}$. Choose $i_m < \omega$ for each $m < \omega$ such that $x_m = a_{\theta(m),i_m}$, noting $i_m \geq m$. It follows that $A_n \cap B \supseteq \{a_{\theta(m),i_m} : m \in \theta^{\leftarrow}(n)\}$. Since for each $m \in \theta^{\leftarrow}(n)$ there exists $M \in$

$\theta^{\leftarrow}(n)$ such that $m \leq i_m < M \leq i_M$, and therefore $a_{\theta(m),i_m} \neq a_{\theta(m),i_M} = a_{\theta(M),i_M}$, we have shown that $A_n \cap B$ is infinite. Thus B witnesses $\alpha_2(\mathcal{A}, \mathcal{B})$. $\square$

Theorem 9. *Let $\mathcal{A}$ be almost-Γ-like and $\mathcal{B}$ be Γ-like. Then $\alpha_4(\mathcal{A}, \mathcal{B})$ holds if and only if $\mathrm{I} \underset{pre}{\not\uparrow} G_{<2}(\mathcal{A}, \mathcal{B})$ if and only if $\mathrm{I} \underset{pre}{\not\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$.*

Proof. We first assume $\alpha_4(\mathcal{A}, \mathcal{B})$ and let $A_n \in \mathcal{A}$ for $n < \omega$ define a predetermined strategy for I in $G_{<2}(\mathcal{A}, \mathcal{B})$. We then may choose $A'_n \in \mathcal{A}$ where $A'_n = \{a_{n,j} : j < \omega\} \subseteq A_n$, $j < k$ implies $a_{n,j} \neq a_{n,k}$, and $A''_n = A'_n \setminus \{a_{i,j} : i, j < n\} \in \mathcal{A}$.

By applying $\alpha_4(\mathcal{A}, \mathcal{B})$ to A''_n, we obtain $B \in \mathcal{B}$ such that $A''_n \cap B \neq \emptyset$ for infintely-many $n < \omega$. We then let $F_n = \emptyset$ when $A''_n \cap B = \emptyset$, and $F_n = \{x_n\}$ for some $x_n \in A''_n \cap B$ otherwise. Then we will have that $B' = \bigcup\{F_n : n < \omega\} \subseteq B$ belongs to $\mathcal{B}$ once we show that B' is infinite. To see this, for $m \leq n < \omega$ note that either F_m is empty (and we let $j_m = 0$) or $F_m = \{a_{m,j_m}\}$ for some $j_m \geq m$; choose $N < \omega$ such that $j_m < N$ for all $m \leq n$ and $F_N = \{x_N\}$. Thus $F_m \neq F_N$ for all $m \leq n$ since $x_N \notin \{a_{i,j} : i, j < N\}$. Thus II may defeat the predetermined strategy A_n by playing F_n each round.

Since $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_{<2}(\mathcal{A}, \mathcal{B})$ immediately implies $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$, we assume the latter. Given $A_n \in \mathcal{A}$ for $n < \omega$, we note this defines a (non-winning) predetermined strategy for I, so II may choose $F_n \in [A_n]^{<\aleph_0}$ such that $B = \bigcup\{F_n : n < \omega\} \in \mathcal{B}$. Since B is infinite, we note $F_n \neq \emptyset$ for infinitely-many $n < \omega$. Thus B witnesses $\alpha_4(\mathcal{A}, \mathcal{B})$ since $A_n \cap B \supseteq F_n \neq \emptyset$ for infinitely-many $n < \omega$. $\square$

This shows that II gains no advantage from picking more than one point per round. This in fact only depends on $\mathcal{B}$ being Γ-like, which we formalize in the following results.

Theorem 10. *Let $\mathcal{B}$ be Γ-like. Then $\mathrm{I} \underset{pre}{\uparrow} G_{<2}(\mathcal{A}, \mathcal{B})$ if and only if $\mathrm{I} \underset{pre}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$.*

Proof. Assume $\bigcup \mathcal{A}$ is well-ordered. Given a winning predetermined strategy A_n for I in $G_{<2}(\mathcal{A}, \mathcal{B})$, consider $F_n \in [A_n]^{<\aleph_0}$. We set

$$F_n^* = \begin{cases} \emptyset & \text{if } F_n \setminus \bigcup\{F_m : m < n\} = \emptyset \\ \{\min(F_n \setminus \bigcup\{F_m : m < n\})\} & \text{otherwise} \end{cases}$$

Since $|F_n^*| < 2$, we have that $\bigcup\{F_n^* : n < \omega\} \notin \mathcal{B}$. In the case that $\bigcup\{F_n^* : n < \omega\}$ is finite, we immediately see that $\bigcup\{F_n : n < \omega\}$ is also finite and therefore not in $\mathcal{B}$. Otherwise $\bigcup\{F_n^* : n < \omega\} \notin \mathcal{B}$ is an infinite subset of $\bigcup\{F_n : n < \omega\}$, and thus $\bigcup\{F_n : n < \omega\} \notin \mathcal{B}$ too. Therefore A_n is a winning predetermined strategy for I in $G_{fin}(\mathcal{A}, \mathcal{B})$ as well. $\square$

Theorem 11. *Let $\mathcal{B}$ be Γ-like. Then $\mathrm{I} \uparrow G_{<2}(\mathcal{A}, \mathcal{B})$ if and only if $\mathrm{I} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$.*

Proof. Assume $\bigcup \mathcal{A}$ is well-ordered. Suppose $\mathrm{I} \uparrow G_{<2}(\mathcal{A}, \mathcal{B})$ is witnessed by the strategy σ. Let $\langle\rangle^\star = \langle\rangle$, and for $s^\frown \langle F \rangle \in ([\bigcup \mathcal{A}]^{<\aleph_0})^{<\omega} \setminus \{\langle\rangle\}$ let

$$(s^\frown \langle F \rangle)^\star = \begin{cases} s^{\star\frown} \langle \emptyset \rangle & \text{if } F \setminus \bigcup \mathrm{range}(s) = \emptyset \\ s^{\star\frown} \langle \{\min(F \setminus \bigcup \mathrm{range}(s))\} \rangle & \text{otherwise} \end{cases}$$

We then define the strategy τ for I in $G_{fin}(\mathcal{A}, \mathcal{B})$ by $\tau(s) = \sigma(s^\star)$. Then given any counterattack $\alpha \in ([\bigcup \mathcal{A}]^{<\aleph_0})^\omega$ by II played against τ, we note that $\alpha^* =$

$\bigcup\{(\alpha \restriction n)^* : n < \omega\}$ is a counterattack to σ, and thus loses. This means $B = \bigcup \mathrm{range}(\alpha^*) \notin \mathcal{B}$.

We consider two cases. The first is the case that $\bigcup \mathrm{range}(\alpha^*)$ is finite. Noting that $\alpha^*(m) \cap \alpha^*(n) = \emptyset$ whenever $m \neq n$, there exists $N < \omega$ such that $\alpha^*(n) = \emptyset$ for all $n > N$. As a result, $\bigcup \mathrm{range}(\alpha) = \bigcup \mathrm{range}(\alpha \restriction n)$, and thus $\bigcup \mathrm{range}(\alpha)$ is finite, and therefore not in $\mathcal{B}$.

In the other case, $\bigcup \mathrm{range}(\alpha^*) \notin \mathcal{B}$ is an infinite subset of $\bigcup \mathrm{range}(\alpha)$, and therefore $\bigcup \mathrm{range}(\alpha) \notin \mathcal{B}$ as well. Thus we have shown that τ is a winning strategy for I in $G_{fin}(\mathcal{A}, \mathcal{B})$. $\square$

We note that the above proof technique could be used to establish that perfect-information and limited-information strategies for II in $G_{fin}(\mathcal{A}, \mathcal{B})$ may be improved to be valid in $G_{<2}(\mathcal{A}, \mathcal{B})$, provided $\mathcal{B}$ is Γ-like. As such, $G_{<2}(\mathcal{A}, \mathcal{B})$ and $G_{fin}(\mathcal{A}, \mathcal{B})$ are effectively equivalent games under this hypothesis, so we will no longer consider $G_{<2}(\mathcal{A}, \mathcal{B})$.

We now demonstrate the following, in the spirit of Pawlikowskii's celebrated result that a winning strategy for the first player in the Rothberger game may always be improved to a winning predetermined strategy [cite pawlikowskii].

Theorem 12. *Let $\mathcal{A}$ be almost-Γ-like and $\mathcal{B}$ be Γ-like. Then*

- $\mathrm{I} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ *if and only if* $\mathrm{I} \underset{pre}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$, *and*
- $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$ *if and only if* $\mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$.

Proof. We assume $\mathrm{I} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ and let the symbol $\dagger$ mean $< \aleph_0$ (respectively, $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$ and $\dagger = 1$, and for convenience we assume II plays singleton subsets of $\mathcal{A}$ rather than elements). As $\mathcal{A}$ is almost-Γ-like, there is a winning strategy σ where $|\sigma(s)| = \aleph_0$ and $\sigma(s) \cap \bigcup \mathrm{range}(s) = \emptyset$ (that is, σ never replays the choices of II) for all partial plays s by II.

For each $s \in \omega^{<\omega}$, suppose $F_{s \restriction m} \in [\bigcup A]^\dagger$ is defined for each $0 < m \leq |s|$. Then let $s^\star : |s| \to [\bigcup \mathcal{A}]^\dagger$ be defined by $s^\star(m) = F_{s \restriction m+1}$, and define $\tau' : \omega^{<\omega} \to \mathcal{A}$ by $\tau'(s) = \sigma(s^\star)$. Finally, set $[\sigma(s^\star)]^\dagger = \{F_{s^\frown \langle n \rangle} : n < \omega\}$, and for some bijection $b : \omega^{<\omega} \to \omega$ let $\tau(n) = \tau'(b(n))$ be a predetermined strategy for I in $G_{fin}(\mathcal{A}, \mathcal{B})$ (resp. $G_1(\mathcal{A}, \mathcal{B})$).

Suppose α is a counterattack by II against τ, so

$$\alpha(n) \in [\tau(n)]^\dagger = [\tau'(b(n))]^\dagger = [\sigma(b(n)^\star)]^\dagger$$

It follows that $\alpha(n) = F_{b(n)^\frown \langle m \rangle}$ for some $m < \omega$. In particular, there is some infinite subset $W \subseteq \omega$ and $f \in \omega^\omega$ such that $\{\alpha(n) : n \in W\} = \{F_{f \restriction n+1} : n < \omega\}$. Note here that $(f \restriction n+1)^\star = (f \restriction n)^{\star\frown} \langle F_{f \restriction n+1} \rangle$. This shows that $F_{f \restriction n+1} \in [\sigma((f \restriction n)^\star)]^\dagger$ is an attempt by II to defeat σ, which fails. Thus $\bigcup\{F_{f \restriction n+1} : n < \omega\} = \bigcup\{\alpha(n) : n \in W\} \notin \mathcal{B}$, and since this set is infinite (as σ prevents II from repeating choices) we have $\bigcup\{\alpha(n) : n < \omega\} \notin \mathcal{B}$ too. Therefore τ is winning. $\square$

Note that the assumption in Theorem 12 that $\mathcal{A}$ be almost-Γ-like cannot be omitted. In [todo cite Clontz k-tactics in Gruenhage game] an example of a space X^* and point $\infty \in X^*$ where $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$ but $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_1(\mathcal{A}, \mathcal{B})$ is given, where $\mathcal{A}$ is the set of open neighborhoods of ∞ (which are all uncountable), and $\mathcal{B}$ is the set $\Gamma_{X^*,\infty}$ of sequences converging to that point. (Note that $G_1(\mathcal{A}, \mathcal{B})$ is called $Gru_{O,P}(X^*, \infty)$ in that paper, and an equivalent game $Gru_{K,P}(X)$ is what is directly studied. In fact, more is shown: I has a winning perfect-information strategy, but for any natural number k, any strategy that only uses the most recent k moves of II and the round number can be defeated.)

While $\mathcal{A}$ is often not almost-Γ-like in general, it may satisfy that property in combination with the selection principles being considered.

Proposition 13. *Let $\mathcal{B}$ be Γ-like, $\mathcal{B} \subseteq \mathcal{A}$, and* $\mathrm{I} \underset{pre}{\not\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$*. Then $\mathcal{A}$ is almost-Γ-like.*

Proof. Let $A \in \mathcal{A}$, and for all $n < \omega$ let $A_n = A$. Then A_n is not a winning predetermined strategy for I, so II may choose finite sets $B_n \subseteq A_n = A$ such that $A' = \bigcup\{B_n : n < \omega\} \in \mathcal{B} \subseteq \mathcal{A}$.

It follows that $A' \subseteq A$ and $|A'| = \aleph_0$, and for any infinite subset $A'' \subseteq A'$ (in particular, any cofinite subset), $A'' \in \mathcal{B} \subseteq \mathcal{A}$. Thus $\mathcal{A}$ is almost-Γ-like. $\square$

Note that in the previous result, $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$ could be weakened to the choice principle $\binom{\mathcal{A}}{\mathcal{B}}$: for every member of $\mathcal{A}$, there is some countable subset belonging to $\mathcal{B}$.

Corollary 14. *Let $\mathcal{B}$ be Γ-like and $\mathcal{B} \subseteq \mathcal{A}$. Then*

- $\mathrm{I} \uparrow G_{fin}(\mathcal{A},\mathcal{B})$ *if and only if* $\mathrm{I} \underset{pre}{\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$*, and*
- $\mathrm{I} \uparrow G_1(\mathcal{A},\mathcal{B})$ *if and only if* $\mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{A},\mathcal{B})$*.*

Proof. Assuming $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$, we have $\mathrm{I} \not\uparrow G_{fin}(\mathcal{A},\mathcal{B})$ by Proposition 13 and Theorem 12.

Similarly, assuming $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_1(\mathcal{A},\mathcal{B}) \Rightarrow \mathrm{I} \underset{\text{pre}}{\not\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$, we have $\mathrm{I} \not\uparrow G_1(\mathcal{A},\mathcal{B})$ by Proposition 13 and Theorem 12. $\square$

This corollary generalizes e.g. Theorems 26 and 30 of [cite Scheepers 1996 Ramsey], Theorem 5 of [cite MR2119791], and Corollary 36 of [cite Clontz dual games].

In summary, using the selection principle notation $S_\star(\mathcal{A},\mathcal{B})$:

Corollary 15. *Let $\mathcal{B}$ be Γ-like and $\mathcal{B} \subseteq \mathcal{A}$. Then*

- $\mathrm{I} \not\uparrow G_{fin}(\mathcal{A},\mathcal{B})$ *if and only if* $S_{fin}(\mathcal{A},\mathcal{B})$ *if and only if* $\alpha_2(\mathcal{A},\mathcal{B})$*, and*
- $\mathrm{I} \not\uparrow G_1(\mathcal{A},\mathcal{B})$ *if and only if* $S_1(\mathcal{A},\mathcal{B})$ *if and only if* $\alpha_4(\mathcal{A},\mathcal{B})$*.*

2. Conclusion

We conclude with the following easy result, and a couple questions.

Proposition 16. *Let $\mathcal{B}$ be Γ-like. Then $\alpha_1(\mathcal{A},\mathcal{B})$ holds if and only if* $\mathrm{I} \underset{pre}{\not\uparrow} G_{cf}(\mathcal{A},\mathcal{B})$*.*

Proof. We first assume $\alpha_1(\mathcal{A},\mathcal{B})$ and let $A_n \in \mathcal{A}$ for $n < \omega$ define a predetermined strategy for I. By $\alpha_1(\mathcal{A},\mathcal{B})$, we immediately obtain $B \in \mathcal{B}$ such that $|A_n \setminus B| < \aleph_0$. Thus $B_n = A_n \cap B$ is a cofinite choice from A_n, and $B' = \bigcup\{B_n : n < \omega\}$ is an infinite subset of B, so $B' \in \mathcal{B}$. Thus II may defeat I by choosing $B_n \subseteq A_n$ each round, witnessing $\mathrm{I} \underset{\text{pre}}{\not\uparrow} G_{cf}(\mathcal{A},\mathcal{B})$.

On the other hand, let $\mathrm{I} \underset{\mathrm{pre}}{\not\uparrow} G_{cf}(\mathcal{A}, \mathcal{B})$. Given $A_n \in \mathcal{A}$ for $n < \omega$, we note that II may choose a cofinite subset $B_n \subseteq A_n$ such that $B = \bigcup\{B_n : n < \omega\} \in \mathcal{B}$. Then B witnesses $\alpha_1(\mathcal{A}, \mathcal{B})$ since $|A_n \setminus B| \leq |A_n \setminus B_n| \leq \aleph_0$. $\square$

Question 17. *Is there a game-theoretic characterization of* $\alpha_3(\mathcal{A}, \mathcal{B})$*?*

Noting that $\mathrm{I} \uparrow G_1(\Gamma_X, \Gamma_X)$ if and only if $\mathrm{I} \uparrow G_{fin}(\Gamma_X, \Gamma_X)$ [cite Kocinac], but the same is not true of $G_\star(\Gamma_{X,x}, \Gamma_{X,x})$ (i.e. there are α_4 spaces that are not α_2 [cite Arhangelskii]), we also ask the following.

Question 18. *Is there an elegant condition on* $\mathcal{A}, \mathcal{B}$ *guaranteeing* $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B}) \Rightarrow \mathrm{I} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$*?*

References

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