# Limited Information Strategies for Topological Games

by

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Abstract

I talk a lot about topological games.

TODO: Write this.

Acknowledgments

TODO: Thank people.

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# Chapter 1

# Introduction

Basic overview of combinatorial games, topological games, limited info strategies, and applications in topology.

Chapter 2

Toplogical Games and Strategies

of Perfect and Limited Information

The goal of this paper is to explore the applications of limited information strategies in existing topological games. There are a variety of frameworks for modeling such games, so we establish one within this chapter which we will use for this manuscript.

## 2.1 Games

Intuitively, the games studied in this paper are two-player games for which each player takes turns making a choice from a set of possible moves. At the conclusion of the game, the choices made by both players are examined, and one of the players is declared the winner of that playthrough.

Games may be modeled mathematically in various ways, but we will find it convenient to think of them in terms defined by Gale and Stewart. [3]

**Definition 2.1.1.** A *game* is a tuple $\langle M, W \rangle$ such that $W \subseteq M^\omega$. $M$ is set of *moves* for the game, and $M^\omega$ is the set of all possible *playthroughs* of the game.

$W$ is the set of *winning playthroughs* or *victories* for the first player, and $M^\omega \setminus W$ is the set of victories for the second player. ($W$ is often called the *payoff set* for the first player.) $\Diamond$

Within this model, we may imagine two players $\mathscr{A}$ and $\mathscr{B}$ playing a game which consists of *rounds* enumerated for each $n < \omega$. During round $n$, $\mathscr{A}$ chooses $a_n \in M$, followed by $\mathscr{B}$ choosing $b_n \in M$. The playthrough corresponding to those choices would be the sequence $p = \langle a_0, b_0, a_1, b_1, \dots \rangle$. If $p \in W$, then $\mathscr{A}$ is the winner of that playthrough, and if $p \notin W$, then $\mathscr{B}$ is the winner. Note that no ties are allowed.

Rather than explicitly defining $W$, we typically define games by declaring the *rules* that each player must follow and the *winning condition* for the first player. Then a playthrough is in $W$ if either the first player made only *legal moves* which observed the game's rules and the playthrough satisified the winning condition, or the second player made an *illegal move* which contradicted the game's rules. Often, we will consider *legal playthoughs* where both players only made legal moves, in which case only the winning condition need be considered.

As an illustration, we could model a game of chess (ignoring stalemates) by letting

$$M = \{\langle p, s\rangle : p \text{ is a chess piece and } s \text{ is a space on the board }\}$$

representing moving a piece $p$ to the space $s$ on the board. Then the rules of chess restict White from moving pieces which belong to Black, or moving a piece to an illegal space on the board.[1] The winning condition could then "inspect" the resulting positions of pieces on the board after each move to see if White attained a checkmate. This winning condition along with the rules implicitly define the set $W$ of winning playthroughs for White.

[1] In practice, $M$ is often defined as the union of two sets, such as white pieces and black pieces in chess. For example, the first player may choose open sets in a topology, while the second player chooses points within the topological space.

### 2.1.1 Infinite and Topological Games

Games never technically end within this model, since playthroughs of the game are infinite sequences. However, for all practical purposes many games end after a finite number of turns.

**Definition 2.1.2.** A game is said to be an *finite game* if for every playthrough $p \in M^\omega$ there exists a round $n < \omega$ such that $[p \upharpoonright n] = \{q \in M^\omega : q \supseteq p \upharpoonright n\}$ is a subset of either $W$ or $M^\omega \setminus W$. $\lozenge$

Put another way, a finite game is decided after a finite number of rounds, after which the game's winner could not change even if further rounds were played. Games which are not finite are called *infinite games*.

As an illustration of an infinite game, we may consider a simple example due to Baker [1].

**Game 2.1.3.** Let $Lim_{A,B}(X)$ denote a game with players $\mathscr{A}$ and $\mathscr{B}$, defined for each subset $A \subset \mathbb{R}$. In round 0, $\mathscr{A}$ chooses a number $a_0$, followed by $\mathscr{B}$ choosing a number $b_0$ such that $a_0 < b_0$. In round $n+1$, $\mathscr{A}$ chooses a number $a_{n+1}$ such that $a_n < a_{n+1} < b_n$, followed by $\mathscr{B}$ choosing a number $b_{n+1}$ such that $a_{n+1} < b_{n+1} < b_n$.

$\mathscr{A}$ wins the game if the sequence $\langle a_n : n < \omega \rangle$ limits to a point in $X$, and $\mathscr{B}$ wins otherwise. $\lozenge$

Certainly, $\mathscr{A}$ and $\mathscr{B}$ will never be in a position without (infinitely many) legal moves available, and provided that $A$ is non-trivial, there is a playthrough such that for all $n < \omega$, the segment $(a_n, b_n)$ intersects both $A$ and $\mathbb{R} \setminus A$. Such a playthrough could never be decided in a finite number of moves, so the winning condition considers the infinite sequence of moves made by the players and declares a victor at the "end" of the game.

**Definition 2.1.4.** A *topological game* is a game defined in terms of an arbitrary topological space. $\Diamond$

Topological games are usually infinite games, ignoring trivial examples. One of the earliest examples of a topological game is the Banach-Mazur game, proposed by Stanislaw Mazur as Problem 43 in Stefan Banach's Scottish Book (1935). A more comprehensive history of the Banacy-Mazur and other topological games may be found in Telgarsky's survey on the subject [18].

The original game was defined for subsets of the real line; however, we give a more general definition here.

**Game 2.1.5.** Let $Empty_{E,N}(X)$ denote the *Banach-Mazur game* with players $\mathscr{E}$, $\mathscr{N}$ defined for each topological space $X$. In round 0, $\mathscr{E}$ chooses a nonempty open set $E_0 \subseteq X$, followed by $\mathscr{N}$ choosing a nonempty open subset $N_0 \subseteq E_0$. In round $n+1$, $\mathscr{E}$ chooses a nonempty open subset $E_{n+1} \subseteq N_n$, followed by $\mathscr{N}$ choosing a nonempty open subset $N_{n+1} \subseteq E_{n+1}$.

$\mathscr{E}$ wins the game if $\bigcap_{n<\omega} E_n = \emptyset$, and $\mathscr{N}$ wins otherwise. $\Diamond$

For example, if $X$ is a locally compact Hausdorff space, $\mathscr{N}$ can "force" a win by choosing $N_0$ such that $\overline{N_0}$ is compact, and choosing $N_{n+1}$ such that $N_{n+1} \subseteq \overline{N_{n+1}} \subseteq O_{n+1} \subseteq N_n$ (possible since $N_n$ is a compact Hausdorff $\Rightarrow$ normal space). Since $\bigcap_{n<\omega} E_n = \bigcap_{n<\omega} N_n$ is the decreasing intersection of compact sets, it cannot be empty.

This concept of when (and how) a player can "force" a win in certain topological games is the focus of this manuscript.

## 2.2 Strategies

We shall make the notion of forcing a win in a game rigorous by introducing "strategies" and "attacks" for games.

**Definition 2.2.1.** A *strategy* for a game $G = \langle M, W\rangle$ is a function from $M^{<\omega}$ to $M$. $\Diamond$

**Definition 2.2.2.** An *attack* for a game $G = \langle M, W\rangle$ is a function from $\omega$ to $M$. $\Diamond$

Intuitively, a strategy is a rule for one of the players on how to play the game based upon the previous (finite) moves of her opponent, while an attack is a fixed strike by an opponent indexed by round number.

**Definition 2.2.3.** The *result* of a game given a strategy $\sigma$ for the first player and an attack $\langle a_0, a_1, \ldots\rangle$ by the second player is the playthrough

$$\langle \sigma(\emptyset), a_0, \sigma(\langle a_0\rangle), a_1, \sigma(\langle a_0, a_1\rangle), \ldots\rangle$$

Likewise, if $\sigma$ is a strategy for the second player, and $\langle a_0, a_1, \ldots\rangle$ is an attack by the first player, then the result is the playthrough

$$\langle a_0, \sigma(\langle a_0\rangle), a_1, \sigma(\langle a_0, a_1\rangle), \ldots\rangle$$

$\Diamond$

We now may rigorously define the notion of "forcing" a win in a game.

**Definition 2.2.4.** A strategy $\sigma$ is a *winning strategy* for a player if for every attack by the opponent, the result of the game is a victory for that player.

If a winning strategy exists for a player $\mathscr{A}$ in the game $G$, then we write $\mathscr{A} \uparrow G$. Otherwise, we write $\mathscr{A} \not\uparrow G$. $\Diamond$

To show that a winning strategy exists for a player (i.e. $\mathscr{A} \uparrow G$), we typically begin by defining it and showing that it is *legal*: it only yields moves which are legal according to the rules of the game. Then, we consider an arbitrary legal attack, and prove that the result of the game is a victory for that player.

If we wish to show that a winning strategy does not exist for a player (i.e. $\mathscr{A} \not\uparrow G$), we often consider an arbitrary legal strategy, and use it to define a legal *counter-attack* for the opponent. If we can prove that the result of the game for that strategy and counter-attack is a victory for the opponent, then a winning strategy does not exist.

Unlike finite games, is not the case that a winning strategy must exist for one of the players in an infinite game.

**Definition 2.2.5.** A game $G$ with players $\mathscr{A}$, $\mathscr{B}$ is said to be *determined* if either $\mathscr{A} \uparrow G$ or $\mathscr{B} \uparrow G$. Otherwise, the game is *undetermined.* $\Diamond$

The Borel Determinacy Theorem states that $G = \langle M, W \rangle$ is determined whenever $W$ is a Borel subset of $M^\omega$ [9]. It's an easy corollary that all finite games are determined; $W$ must be clopen.

However, as stated earlier, most topological games are infinite, and many are undetermined for certain spaces constructed using the Axiom of Choice.[2]

Often, we will build new strategies based on others.

**Definition 2.2.6.** A strategy $\tau$ is a *strengthening* of another strategy $\sigma$ for a player if whenever the result of the game for $\sigma$ and an attack $a$ by the opponent is a victory for the player, then the result of the game for $\tau$ and $a$ is also a victory for the player. $\Diamond$

**Proposition 2.2.7.** *If $\sigma$ is a winning strategy, and $\tau$ strengthens $\sigma$, then $\tau$ is also a winning strategy.* $\Diamond$

[2]These spaces cannot be constructed just only the axioms of ZF. In fact, mathematicians have studied an Axiom of Determinacy which declares that that all Gale-Stewart games are determined (and implies that the Axiom of Choice is false). [11]

### 2.2.1 Applications of Strategies

The power of studying these infinite-length games can be illustrated by considering the following proposition.

**Proposition 2.2.8.** *If $X$ is countable, then $\mathscr{B} \uparrow Lim_{A,B}(X)$.* $\Diamond$

*Proof.* Adapted from [1]. Let $X = \{x_0, x_1, \dots\}$. Let $i(a, b)$ be the least integer such that $a < x_{i(x,y)} < b$, if it exists. We define a strategy $\sigma$ for $\mathscr{B}$ such that:

- $\sigma(\langle a_0 \rangle) = x_{i(a_0,\infty)}$. If $i(a_0, \infty)$ does not exist, then the choice of $\sigma(\langle a_0 \rangle)$ is arbitary, say, $a_0 + 1$.

- $\sigma(\langle a_0, ..., a_{n+1} \rangle) = x_{i(a_{n+1},b_n)}$, where $b_n = \sigma(\langle a_0, \dots, a_n \rangle)$. If $i(a_{n+1}, b_n)$ does not exist, then the choice of $\sigma(\langle a_0, ..., a_{n+1} \rangle)$ is arbitrary, say, $\sigma(\langle a_0, ..., a_{n+1} \rangle) = \frac{a_{n+1}+b_n}{2}$.

Observe that $\sigma$ is a legal strategy according to the rules of the game since $a_0 < \sigma(\langle a_0 \rangle)$ and $a_{n+1} < \sigma(\langle a_0, \dots, a_{n+1} \rangle) < b_n$. We claim this is a winning strategy for $\mathscr{B}$. Let $a = \langle a_0, a_1, \dots \rangle$ be a legal attack by $\mathscr{A}$ against $\sigma$: we will show that the resulting playthrough is a victory for $\mathscr{B}$, that is, $\lim_{n\to\infty} a_n \notin X$. Let $b_n = \sigma(\langle a_0, \dots, a_n \rangle)$. Note that

$$a_0 < a_1 < \cdots < \lim_{n\to\infty} a_n < \cdots < b_1 < b_0$$

If $i(a_0, \infty)$ does not exist, then $a_0$ is greater than every element of $X$, and thus $\lim_{n\to\infty} a_n \notin X$. A similar argument follows if some $i(a_{n+1}, b_n)$ does not exist.

Otherwise,

$$i(a_0, \infty) < i(a_1, b_0) < i(a_2, b_1) < \dots$$

and for each $i < \omega$, one of the following must hold.

- $i < i(a_0, \infty)$. Then $x_i \leq a_0 < \lim_{n\to\infty} a_n$.

- $i = i(a_0, \infty)$. Then $x_i = b_0 > \lim_{n\to\infty} a_n$.

- $i(a_0, \infty) < i < i(a_1, b_0)$. Then $x_i \leq a_1 < \lim_{n\to\infty} a_n$ or $x_i \geq b_0 > \lim_{n\to\infty} a_n$.

- $i = i(a_{n+1}, b_n)$ for some $n < \omega$. Then $x_i = b_{n+1} > \lim_{n\to\infty} a_n$.

- $i(a_{n+1}, b_n) < i < i(a_{n+2}, b_{n+1})$ for some $n < \omega$. Then $x_i \leq a_{n+2} < \lim_{n\to\infty} a_n$ or $x_i \geq b_{n+1} > \lim_{n\to\infty} a_n$.

In any case, $x_i \neq \lim_{n\to\infty} a_n$, and thus $\lim_{n\to\infty} a_n \notin X$. $\square$

More informally, $\mathscr{B}$ can force a win by enumerating the countable set $X$ and playing every legal choice by the end of the game. This yields a classical result.

**Corollary 2.2.9.** $\mathbb{R}$ *is uncountable.* $\Diamond$

*Proof.* $\mathscr{A} \uparrow Lim_{A,B}(\mathbb{R})$, since $a_n$ must converge to some real number. This implies $\mathscr{B} \not\uparrow Lim_{A,B}(\mathbb{R})$, and thus $\mathbb{R}$ is not countable. $\square$

Infinite games thus provide a rich framework for considering questions in set theory and topology. In general, the presence or absence of a winning strategy for a player in a topological game characterizes a property of the topological space in question.

**Theorem 2.2.10.** $\mathscr{E} \not\uparrow Empty_{E,N}(X)$ *if and only if* $X$ *is a Baire space. [6]* $\Diamond$

### 2.2.2 Limited Information Strategies

So far we have assumed both players enjoy *perfect information*, and may develop strategies which use all of the previous moves of the opponent as input.

**Definition 2.2.11.** For a game $G = \langle M, W \rangle$, the *$k$-tactical fog-of-war* is the function $\nu_k : M^{<\omega} \to M^{\leq k}$ defined by

$$\nu_k(\langle m_0, \ldots, m_{n-1} \rangle) = \langle m_{n-k}, \ldots, m_{n-1} \rangle$$

and the *$k$-Markov fog-of-war* is the function $\mu_k : M^{<\omega} \to (M^{\leq k} \times \omega)$ defined by

$$\mu_k(\langle m_0, \ldots, m_{n-1} \rangle) = \langle \langle m_{n-k}, \ldots, m_{n-1} \rangle, n \rangle$$

$\Diamond$

Essentially, these fogs-of-war represent a limited memory: $\nu_k$ filters out all but the last $k$ moves of the opponent, and $\mu_k$ filters out all but the last $k$ moves of the opponent and the round number.

We call strategies which do not require full recollection of the opponent's moves *limited information strategies*.

**Definition 2.2.12.** A *$k$-tactical strategy* or *$k$-tactic* is a function $\sigma : M^{\leq k} \to M$ yielding a corresponding strategy $\sigma \circ \nu_k : M^{<\omega} \to M$.

A *$k$-Markov strategy* or *$k$-mark* is a function $\sigma : M^{\leq k} \times \omega \to M$ yielding a corresponding strategy $\sigma \circ \mu_k : M^{<\omega} \to M$. $\Diamond$

$k$-tactics and $k$-marks may then only use the last $k$ moves of the opponent, and in the latter case, also the round number.

The $k$ is usually omitted when $k = 1$. A (1-)tactic is called a *stationary strategy* by some authors. 0-tactics are not usually interesting (such strategies would be constant functions); however, we will discuss 0-Markov strategies, called *predetermined strategies* since such a strategy only uses the round number and does not rely on knowing which moves the opponent will make. Of course, a limited information strategy $\sigma$ is *winning* for a player if its corresponding strategy $\sigma \circ \nu_k$ or $\sigma \circ \mu_k$ is winning for that player.

**Definition 2.2.13.** If a winning $k$-tactical strategy exists for a player $\mathscr{A}$ in the game $G$, then we write $\mathscr{A} \underset{k\text{-tact}}{\uparrow} G$. If $k = 1$, then $\mathscr{A} \underset{\text{tact}}{\uparrow} G$.

If a winning $k$-Markov strategy exists for a player $\mathscr{A}$ in the game $G$, then we write $\mathscr{A} \underset{k\text{-mark}}{\uparrow} G$. If $k = 1$, then $\mathscr{A} \underset{\text{mark}}{\uparrow} G$, and if $k = 0$, then $\mathscr{A} \underset{\text{pre}}{\uparrow} G$. $\Diamond$

The existence of a winning limited information strategy can characterize a stronger property than the property characterized by a perfect information strategy.

**Definition 2.2.14.** $X$ is an *$\alpha$-favorable space* when $\mathscr{N} \underset{\text{tact}}{\uparrow} \mathit{Empty}_{E,N}(X)$. $X$ is a *weakly $\alpha$-favorable space* when $\mathscr{N} \uparrow \mathit{Empty}_{E,N}(X)$. $\Diamond$

**Proposition 2.2.15.** *$X$ is $\alpha$-favorable $\Rightarrow$ $X$ is weakly $\alpha$-favorable $\Rightarrow$ $X$ is Baire* $\Diamond$

Those arrows may not be reversed. A Bernstein subset of the real line is an example of a Baire space which is not weakly $\alpha$-favorable, and Gabriel Debs constructed an example of a completely regular space for which $\mathscr{N}$ has a winning 2-tactic, but lacks a winning 1-tactic. [2]

## Chapter 3

## $W$ Convergence and Clustering Games

We begin by investigating a game due to Gary Gruenhage.

**Game 3.0.16.** Let $Con_{O,P}(X, S)$ denote the *$W$-convergence game* with players $\mathscr{O}$, $\mathscr{P}$, for a topological space $X$ and $S \subseteq X$.

In round $n$, $\mathscr{O}$ chooses an open neighborhood $O_n \supseteq S$, followed by $\mathscr{P}$ choosing a point $x_n \in \bigcap_{m \leq n} O_m$.

$\mathscr{O}$ wins the game if the points $x_n$ converge to the set $S$; that is, for every open neighborhood $U \supseteq S$, $x_n \in U$ for all but finite $n < \omega$.

If $S = \{x\}$ then we write $Con_{O,P}(X, x)$ for short. ◊

The "W" in the name merely refers to $\mathscr{O}$'s goal: to "win" the game. Gruenhage defined this game in his doctoral dissertation to define a class of spaces generalizing first-countability. [5]

**Definition 3.0.17.** The spaces $X$ for which $\mathscr{O} \uparrow Con_{O,P}(X, x)$ for all $x \in X$ are called *$W$-spaces.* ◊

In fact, using limited information strategies, one may characterize the first-countable spaces using this game.

**Proposition 3.0.18.** *$X$ is first countable if and only if $\mathscr{O} \underset{pre}{\uparrow} Con_{O,P}(X, x)$ for all $x \in X$.* ◊

*Proof.* The forward implication shows that all first-countable spaces are $W$ spaces, and was proven in [5]: if $\{U_n : n < \omega\}$ is a countable base at $x$, let $\sigma(n) = \bigcap_{m \leq n} U_m$. $\sigma$ is easily seen to be a winning predetermined strategy.

If $X$ is not first countable at some $x$, let $\sigma$ be a predetermined strategy for $\mathscr{O}$ in $Con_{O,P}(X,x)$. There exists an open neighborhood $U$ of $x$ which does not contain any $\bigcap_{m\leq n}\sigma(m)$ (otherwise $\{\bigcap_{m\leq n}\sigma(m) : n<\omega\}$ would be a countable base at $x$). Let $x_n$ be an element of $\bigcap_{m\leq n}\sigma(m)\setminus U$ for all $n<\omega$. Then $\langle x_0,x_1,\ldots\rangle$ is a winning counter-attack to $\sigma$ for $\mathscr{P}$, so $\mathscr{O}$ lacks a winning predetermined strategy. □

At first glance, the difficulty of $Con_{O,P}(X,S)$ could be increased for $\mathscr{O}$ by only restricting the choices for $\mathscr{P}$ to be within the most recent open set played by $\mathscr{O}$, rather than all the previously played open sets.

**Definition 3.0.19.** Let $Con^{\star}_{O,P}(X,S)$ denote the *hard $W$-convergence game* which proceeds as $Con_{O,P}(X,S)$, except that $\mathscr{P}$ need only choose $x_n\in O_n$ rather than $x_n\in\bigcap_{m\leq n}O_m$ during each round. ◊

This seemingly more difficult game for $\mathscr{O}$ is Gruenhage's original formulation. But with perfect information, there is no real difference for $\mathscr{O}$.

**Proposition 3.0.20.** *$\mathscr{O} \underset{limit}{\uparrow} Con_{O,P}(X,S)$ if and only if $\mathscr{O} \underset{limit}{\uparrow} Con^{\star}_{O,P}(X,S)$, where $\underset{limit}{\uparrow}$ is either $\uparrow$ or $\underset{pre}{\uparrow}$.* ◊

*Proof.* The backwards implication is immediate.

For the forward implication, let $\sigma$ be a winning predetermined (perfect information) strategy, and $\lambda$ be the 0-Markov fog-of-war $\mu_0$ (the identity).

We define a new predetermined (perfect information) strategy $\tau$ by

$$\tau\circ\lambda(\langle x_0,\ldots,x_{n-1}\rangle)=\bigcap_{m\leq n}\sigma\circ\lambda(\langle x_0,\ldots,x_{m-1}\rangle)$$

so that each move by $\mathscr{O}$ according to $\tau\circ\lambda$ is the intersection of $\mathscr{O}$'s previous moves. Then any attack against $\tau\circ\lambda$ is an attack against $\sigma\circ\lambda$, and since $\sigma\circ\lambda$ is a winning strategy, so is $\tau\circ\lambda$. □

Put more simply, $\tau(n) = \bigcap_{m \leq n} \sigma(m)$ in the predetermined case, and $\tau(\langle x_0, \ldots, x_{n-1}\rangle) = \bigcap_{m \leq n} \sigma(\langle x_0, \ldots, x_{m-1}\rangle)$ in the perfect information case. The original proof would have been invalid if $\lambda$ was required to be, say, the tactical fog-of-war $\nu_1$, since the value of $\mathscr{O}$'s own round 1 move $\sigma \circ \nu_1(\langle x_0\rangle) = \sigma(\langle x_0\rangle)$ could not be determined from the information she has during round 2: $\nu_1(\langle x_0, x_1\rangle) = \langle x_1\rangle$.

Due to the equivalency of the "hard" and "normal" variations of the convergence game in the perfect information case, many authors use them interchangibly. However, it is possible to find spaces for which the games are not equivalent when considering $k+1$-tactics and $k+1$-marks, as we will soon see.

In addition to the $W$-convergence games, we will also investigate "clustering" analogs to both variations.

**Game 3.0.21.** Let $Clus_{O,P}(X, S)$ ($Clus^{\star}_{O,P}(X, S)$) be a variation of $Con_{O,P}(X, S)$ ($Con^{\star}_{O,P}(X, S)$) such that $x_n$ need only cluster at $S$, that is, for every open neighborhood $U$ of $S$, $x_n \in U$ for infinitely many $n < \omega$. $\Diamond$

This variation seems to make $\mathscr{O}$'s job easier, but Gruenhage noted that the clustering game is perfect-information equivalent to the convergence game for $\mathscr{O}$. This can easily be extended for some limited information cases as well.

**Proposition 3.0.22.** *$\mathscr{O} \underset{limit}{\uparrow} Con_{O,P}(X, S)$ if and only if $\mathscr{O} \underset{limit}{\uparrow} Clus_{O,P}(X, S)$ where $\underset{limit}{\uparrow}$ is any of $\uparrow$, $\underset{pre}{\uparrow}$, $\underset{tact}{\uparrow}$, or $\underset{mark}{\uparrow}$.* $\Diamond$

*Proof.* For the perfect information case we refer to [5].

In the predetermined (resp. tactical) case, suppose that $\sigma$ is a winning predetermined (resp. tactical) strategy for $\mathscr{O}$ in $Clus_{O,P}(X, S)$. Let $p$ be a legal attack against $\sigma$, and $q$ be a subsequence of $p$. It's easily seen that $q$ is also a legal attack against $\sigma$, so $q$ clusters at $S$. Since every subsequence of $p$ clusters at $S$, $p$ converges to $S$, and $\sigma$ is a winning predetermined (resp. tactical) strategy for $\mathscr{O}$ in $Con_{O,P}(X, S)$ as well.

In the final case, note that any Markov strategy $\sigma'$ for $\mathscr{O}$ may be strengthened to $\sigma$ defined by $\sigma(x,n)=\bigcap_{m\leq n}\sigma'(x,m)$. So, suppose that $\sigma$ is a winning Markov strategy for $\mathscr{O}$ in $Clus_{O,P}(X,S)$ such that $\sigma(x,m)\supseteq\sigma(x,n)$ for all $m\leq n$.

Let $p$ be a legal attack against $\sigma$, and $q$ be a subsequence of $p$. For $m<\omega$, there exists $f(m)\geq m$ such that $q(m)=p(f(m))$. It follows that $q(0)=p(f(0))\in\sigma(\emptyset,0)\cap\bigcap_{m\leq f(0)}\sigma(\langle p(m)\rangle,m)\subseteq\sigma(\emptyset,0)$ and

$$\begin{aligned}
q(n+1)=p(f(n+1)) &\in \sigma(\emptyset,0)\cap\bigcap_{m<f(n+1)}\sigma(\langle p(m)\rangle,m+1)\\
&\subseteq \sigma(\emptyset,0)\cap\bigcap_{m<n+1}\sigma(\langle p(f(m))\rangle,f(m)+1)\\
&= \sigma(\emptyset,0)\cap\bigcap_{m<n+1}\sigma(\langle q(m)\rangle,f(m)+1)\\
&\subseteq \sigma(\emptyset,0)\cap\bigcap_{m<n+1}\sigma(\langle q(m)\rangle,m+1)
\end{aligned}$$

so $q$ is also a legal attack against $\sigma$. Since $\sigma$ is a winning strategy, $q$ clusters at $S$, and since every subsequence of $p$ clusters at $S$, $p$ must converge to $S$. Thus $\sigma$ is also a winning Markov strategy for $\mathscr{O}$ in $Con_{O,P}(X,S)$ as well. $\square$

Two types of questions emerge from these results.

**Question 3.0.23.** Does $\mathscr{O}\underset{\text{2-tact}}{\uparrow}Clus_{O,P}(X,S)$ imply $\mathscr{O}\underset{\text{2-tact}}{\uparrow}Con_{O,P}(X,S)$? What about for $\underset{\text{2-mark}}{\uparrow}$? $\Diamond$

**Question 3.0.24.** Could $\mathscr{O}\underset{k+1\text{-tact}}{\uparrow}Con_{O,P}(X,S)$ actually imply $\mathscr{O}\underset{\text{tact}}{\uparrow}Con_{O,P}(X,S)$? What about for $Clus_{O,P}(X,S)$? $\Diamond$

## 3.1 Fort spaces

In his original paper, Gruenhage suggested the one-point-compactification of a discrete space as an example of a $W$-space which is not first-countable.

**Definition 3.1.1.** A *Fort space* $\kappa^* = \kappa \cup \{\infty\}$ is defined for each cardinal $\kappa$. Its subspace $\kappa$ is discrete, and the neighborhoods of $\infty$ are of the form $\kappa^* \setminus F$ for each $F \in [\kappa]^{<\omega}$. $\Diamond$

**Proposition 3.1.2.** $\mathscr{O} \underset{tact}{\uparrow} Con_{O,P}(\kappa^*, \infty)$ *for all cardinals* $\kappa$ $\Diamond$

*Proof.* Let $\sigma(\emptyset) = \sigma(\langle\infty\rangle) = \kappa^*$ and $\sigma(\langle\alpha\rangle) = \kappa^* \setminus \{\alpha\}$. Any legal attack against the tactic $\sigma$ could not repeat non-$\infty$ points, so it must converge to $\infty$. $\square$

**Corollary 3.1.3.** $\mathscr{O} \uparrow Con^{\star}_{O,P}(\kappa^*, \infty)$ *for all cardinals* $\kappa$ $\Diamond$

*Proof.* Propositions 3.0.20 and 3.1.2. $\square$

Since it's trivial to show that $\mathscr{O} \underset{\text{pre}}{\uparrow} Con_{O,P}(\kappa^*,)$ if and only if $\kappa \leq \omega$, this closes the question on limited information strategies for $Con_{O,P}(\kappa^*, \infty)$. However, limited information analysis of the harder $Con^{\star}_{O,P}(\kappa^*, \infty)$ is more interesting.

Peter Nyikos noted Proposition 3.1.2 and the following in [12].

**Theorem 3.1.4.** $\mathscr{O} \underset{mark}{\not\uparrow} Con^{\star}_{O,P}(\omega_1^*, \infty)$. $\Diamond$

This actually can be generalized to any $k$-Markov strategy with just a little more book-keeping.

**Theorem 3.1.5.** $\mathscr{O} \underset{k\text{-}mark}{\not\uparrow} Con^{\star}_{O,P}(\omega_1^*, \infty)$. $\Diamond$

*Proof.* Let $\sigma$ be a $k$-mark for $\mathscr{O}$. Since the set

$$D_\sigma = \bigcap_{n<\omega, s\in\omega^{\leq k}} \sigma(s, n)$$

is co-countable, we may choose $\alpha_\sigma \in D_\sigma \cap \omega_1$. Thus, we may choose $n_0 < n_1 < \cdots < \omega$ such that

$$\langle n_0, \ldots, n_{k-1}, \alpha_\sigma, n_k, \ldots, n_{2k-1}, \alpha_\sigma, \ldots \rangle$$

is a legal counterattack, which fails to converge to $\infty$ since $\alpha_\sigma$ is repeated infinitely often. $\square$

However, while the clustering and convergence variants are equivalent for Markov strategies in the "normal" version of the $W$ game, they are *not* equivalent in the "hard" version.

**Theorem 3.1.6.** $\mathscr{O} \underset{mark}{\uparrow} Clus^\star_{O,P}(\omega_1^*, \infty)$. $\Diamond$

*Proof.* For each $\alpha < \omega_1$ let $A_\alpha = \langle A_\alpha(0), A_\alpha(1), \ldots \rangle$ be a countable sequence of finite sets such that $A_\alpha(n) \subset A_\alpha(n+1)$ and $\bigcup_{n<\omega} A_\alpha(n) = \alpha + 1$.

We define the Markov strategy $\sigma$ by setting

$$\sigma(\emptyset, 0) = \sigma(\langle \infty \rangle, n) = \omega_1^*$$

and for all $\alpha < \omega_1$ setting

$$\sigma(\langle \alpha \rangle, n) = \omega_1^* \setminus A_\alpha(n)$$

Note that for any $\alpha_0 < \cdots < \alpha_{k-1}$, there is some $n < \omega$ such that $\{\alpha_0, \ldots, \alpha_{k-1}\} \subseteq A_{\alpha_i}(n)$ for all $i < k$. Thus for any legal attack $p$ against $\sigma$, the range of $p$ cannot be finite. Since the range of $p$ is infinite, every open neighborhood of $\infty$ contains infinitely many points of $p$, so $p$ clusters at $\infty$. $\square$

However, knowledge of the round number is critical.

**Theorem 3.1.7.** $O \underset{k\text{-}tact}{\not\uparrow} Clus^\star_{O,P}(\omega_1^*, \infty)$. $\Diamond$

*Proof.* Let $\sigma$ be a $k$-tactic for $\mathscr{O}$ in $Clus^\star_{O,P}(\omega_1^*, \infty)$. By the closing-up lemma, the set

$$C_\sigma = \{\alpha < \omega_1 : s \in \alpha^{\leq k} \Rightarrow \omega_1^* \setminus \sigma(s) \subset \alpha\}$$

is closed and unbounded. Let $a_\sigma : \omega_1 \to C_\sigma$ be an order isomorphism.

Choose $n_0 < \cdots < n_{k-1} < \omega$ such that for each $i < k$:

$$a_\sigma(n_i) \in \sigma(\langle a_\sigma(n_0), \ldots, a_\sigma(n_{i-1}), a_\sigma(\omega + i), \ldots, a_\sigma(\omega + k - 1)\rangle)$$

Finally, observe that the legal counterattack

$$\langle a_\sigma(n_0), \ldots, a_\sigma(n_{k-1}), a_\sigma(\omega), \ldots, a_\sigma(\omega+k-1), a_\sigma(n_0), \ldots, a_\sigma(n_{k-1}), a_\sigma(\omega), \ldots, a_\sigma(\omega+k-1), \ldots\rangle$$

has a range outside the open neighborhood

$$\omega_1^* \setminus \{a_\sigma(n_0), \ldots, a_\sigma(n_{k-1}), a_\sigma(\omega), \ldots, a_\sigma(\omega + k - 1)\}$$

of $\infty$. Thus $\sigma$ is not a winning $k$-tactic. $\square$

Once the discrete space is larger than $\omega_1$, knowing the round number is not sufficient to construct a limited information strategy, due to a similar argument.

**Theorem 3.1.8.** $O \underset{k\text{-}mark}{\not\uparrow} Clus^\star_{O,P}(\omega_2^*, \infty)$. $\Diamond$

*Proof.* Let $\sigma$ be a $k$-mark for $\mathscr{O}$ in $Clus_{O,P}(\omega_2^*, \infty)$. By the closing-up lemma, the set

$$C_\sigma = \{\alpha < \omega_2 : s \in \alpha^{<\omega} \Rightarrow \omega_2^* \setminus \sigma \circ \mu_k(s) \subset \alpha\}$$

is closed and unbounded. (Recall that $\mu_k$ is the $k$-Markov fog-of-war which turns perfect information into the last $k$ moves and the round number.) Let $a_\sigma : \omega_2 \to C_\sigma$ be an order isomorphism.

Choose $\beta_0 < \cdots < \beta_{k-1} < \omega_1$ such that for each $i < k$:

$$a_\sigma(\beta_i) \in \bigcap_{n<\omega} \sigma(\langle a_\sigma(\beta_0), \ldots, a_\sigma(\beta_{i-1}), a_\sigma(\omega_1 + i), \ldots, a_\sigma(\omega_1 + k - 1)\rangle, n)$$

Finally, observe that the legal counterattack

$$\langle a_\sigma(\beta_0),\ldots,a_\sigma(\beta_{k-1}),a_\sigma(\omega_1),\ldots,a_\sigma(\omega_1{+}k{-}1),a_\sigma(\beta_0),\ldots,a_\sigma(\beta_{k-1}),a_\sigma(\omega_1),\ldots,a_\sigma(\omega_1{+}k{-}1),\ldots\rangle$$

has a range outside the open neighborhood

$$\omega_2^* \setminus \{a_\sigma(\beta_0),\ldots,a_\sigma(\beta_{k-1}),a_\sigma(\omega_1),\ldots,a_\sigma(\omega_1+k-1)\}$$

of $\infty$. Thus $\sigma$ is not a winning $k$-mark. □

## 3.2 Sigma-products

Knowing the status of $W$-games in simpler spaces yields insight to larger spaces.

**Proposition 3.2.1.** *Suppose $S \subseteq Y \subseteq X$, $\underset{limit}{\uparrow}$ is any of $\uparrow$, $\underset{k\text{-}tact}{\uparrow}$, or $\underset{k\text{-}mark}{\uparrow}$, and $G(X,S)$ is any of $Con_{O,P}(X,S)$, $Con^\star_{O,P}(X,S)$, $Clus_{O,P}(X,S)$, or $Clus^\star_{O,P}(X,S)$.*

*Then $\mathscr{O} \underset{limit}{\uparrow} G(X,S)$ implies $\mathscr{O} \underset{limit}{\uparrow} G(Y,S)$.* ◊

*Proof.* Simply intersect the output of the winning strategy in $G(X,S)$ with $Y$. □

A natural superspace of a Fort space is the sigma-product of a discrete cardinal.

**Definition 3.2.2.** Let $\Sigma_y X^\kappa$ be a *sigma product* of $X$ with dimension $\kappa$ for each $y \in X^\kappa$, the subset of the usual Tychonoff product space $X^\kappa$ such that $x \in \Sigma_y X^\kappa$ if and only if $\{\alpha < \kappa : x(\alpha) \neq y(\alpha)\}$ is countable.

For homogeneous spaces $X$ containing 0, $y$ is usually assumed to be the zero vector $\vec{0}$ and the sigma product is written $\Sigma X^\kappa$. ◊

**Proposition 3.2.3.** *$\kappa^*$ is homeomorphic to the space*

$$\{x \in \Sigma 2^\kappa : x(\alpha) = 0 \textit{ for all but one } \alpha < \kappa\}$$

◊

*Proof.* Map $\alpha < \kappa$ to $x_\alpha$ such that

$$x_\alpha(\beta) = \begin{cases} 0 & \beta \neq \alpha \\ 1 & \beta = \alpha \end{cases}$$

and map $\infty$ to the zero vector $\vec{0}$. $\square$

**Corollary 3.2.4.** *$\mathscr{O} \underset{k\text{-tact}}{\not\uparrow} Clus^{\star}_{O,P}(\Sigma\mathbb{R}^{\omega_1}, \vec{0})$, $\mathscr{O} \underset{k\text{-mark}}{\not\uparrow} Con^{\star}_{O,P}(\Sigma\mathbb{R}^{\omega_1}, \vec{0})$, and $\mathscr{O} \underset{k\text{-mark}}{\not\uparrow} Clus^{\star}_{O,P}(\Sigma\mathbb{R}^{\omega_2}, \vec{0})$.* $\Diamond$

While this closes the question on tactics and marks for high dimensional sigma- (and Tychonoff-) products of the real line, there is another type of limited information strategy to investigate.

**Definition 3.2.5.** For a game $G = \langle M, W\rangle$ and *coding strategy* or *code* $\sigma : M^2 \to M$, the *$\sigma$-coding fog-of-war* $\gamma_\sigma : M^{<\omega} \to M^{\leq 2}$ is the function defined such that

$$\gamma_\sigma(\emptyset) = \emptyset$$

and

$$\gamma_\sigma(s^\frown\langle x\rangle) = \langle \sigma \circ \gamma_\sigma(s), x\rangle$$

For a coding strategy $\sigma$, its coresponding strategy is $\sigma \circ \gamma_\sigma$. For a game $G$, if $\sigma \circ \gamma_\sigma$ is a winning strategy for $\mathscr{A}$, then $\sigma$ is a winning coding strategy and we write $\mathscr{A} \underset{\text{code}}{\uparrow} G$. $\Diamond$

Intuitively, a $\sigma$-coding fog-of-war converts perfect information of the game into the last moves of both the player and her opponent, so a player has a winning coding strategy when she only needs to know the move of her opponent and her own last move. The term "coding" comes from the fact that a player may encode information about the history of the game into her own moves, and use this encoded information in later rounds.

As an example, the existence of a winning coding strategy is necessary for the second player to force a win the Banach-Mazur game.

**Theorem 3.2.6.** *$\mathscr{N} \uparrow Empty_{E,N}(X)$ if and only if $\mathscr{N} \underset{code}{\uparrow} Empty_{E,N}(X)$ [2] [4].* $\Diamond$

We are interested in whether the same holds for $W$ games.

The hard and normal versions of the $W$ games are all equivalent with regards to coding strategies since $\mathscr{O}$ may always ensure her new move is a subset of her previous move. For Fort spaces, the question is immediately closed.

**Proposition 3.2.7.** *$\mathscr{O} \underset{code}{\uparrow} Con_{O,P}(\kappa^*, \infty)$.* $\Diamond$

*Proof.* Let $\sigma(\emptyset) = \kappa^*$, $\sigma(\langle U, \alpha\rangle) = U \setminus \{\alpha\}$ for $\alpha < \kappa$, and $\sigma(\langle U, \infty\rangle) = U$. $\mathscr{P}$ cannot legally repeat non-$\infty$ points of the set, so her points converge to $\infty$. $\square$

This trick does not simply extend to the $\Sigma\mathbb{R}^\kappa$ case, however. An open set may only restrict finitely many coordinates of the product, and a point in $\Sigma\mathbb{R}^\kappa$ may have countably infinite non-zero coordinates. Thus, information about the previous non-zero coordinates cannot be directly encoded into the open set.

Circumventing this takes a bit of extra machinery. We proceed by defining a simpler infinite game for each cardinal $\kappa$.

**Game 3.2.8.** Let $PF_{F,C}(\kappa)$ denote the *point-finite game* with players $\mathscr{F}$, $\mathscr{C}$ for each cardinal $\kappa$.

In round $n$, $\mathscr{F}$ chooses $F_n \in [\kappa]^{<\omega}$, followed by $\mathscr{C}$ choosing $C_n \in [\kappa \setminus \bigcup_{m \leq n} F_m]^{\leq\omega}$.

$\mathscr{F}$ wins the game if the collection $\{C_n : n < \omega\}$ is a point-finite cover of its union $\bigcup_{n<\omega} C_n$, that is, each point in $\bigcup_{n<\omega} C_n$ is in $C_n$ only for finitely many $n < \omega$. $\Diamond$

This game has a strong resemblance to a game defined by Scheepers in [14] in relationship to the Banach-Mazur game and studied specifically with finite and countable sets in [15]. Scheeper's game and the results pertaining to it aren't of use here; however, they will be referenced in a later chapter in studying a different topological game.

This game of finite and countable sets is directly applicable to the $W$ games played upon the sigma-product of real lines.

**Lemma 3.2.9.** $\mathscr{F} \underset{code}{\uparrow} PF_{F,C}(\kappa)$ *implies* $\mathscr{O} \underset{code}{\uparrow} Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$. $\Diamond$

*Proof.* Let $\sigma$ be a winning coding strategy for $\mathscr{F}$ in $PF_{F,C}(\kappa)$ such that $\sigma(\emptyset) \supset \emptyset$ and $\sigma(F, C) \supset F$.

For $F \in [\kappa]^{<\omega}$ and $\epsilon > 0$ let $U(F, \epsilon)$ be the basic open set in $\mathbb{R}^\kappa$ such that each projection is of the form

$$\pi_\alpha(U(F,\epsilon)) = \begin{cases} (-\epsilon, \epsilon) & \alpha \in F \\ \mathbb{R} & \alpha \notin F \end{cases}$$

Note that $F \supset \emptyset$ and $\epsilon$ are uniquely identifible given $U(F, \epsilon) \cap \Sigma\mathbb{R}^\kappa$.

For each point $x \in \Sigma\mathbb{R}^\kappa$ and $\epsilon > 0$, let $C_\epsilon(x) \in [\kappa]^{\leq\omega}$ such that $\alpha \in C_\epsilon(\alpha)$ if and only if $|x(\alpha)| \geq \epsilon$.

We define the coding strategy $\tau$ for $\mathscr{O}$ in $Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$ as follows:

$$\tau(\emptyset) = U(\sigma(\emptyset), 1) \cap \Sigma\mathbb{R}^\kappa$$

$$\tau(\langle U(F,\epsilon) \cap \Sigma\mathbb{R}^\kappa, x\rangle) = U\left(\sigma(\langle F, C_\epsilon(x)\rangle), \frac{\epsilon}{2}\right) \cap \Sigma\mathbb{R}^\kappa$$

Let $\langle a_0, a_1, a_2, \ldots\rangle$ be a legal attack by $\mathscr{P}$ against $\tau$. It then follows that

$$b = \langle C_1(a_0), C_{1/2}(a_1), C_{1/4}(a_2), \ldots\rangle$$

is a legal attack by $\mathscr{C}$ against $\sigma$. Since $\sigma$ is a winning strategy, each ordinal in $\bigcup_{n<\omega} C_{2^{-n}}(a_n)$ is in $C_{2^{-n}}(a_n)$ only for finitely many $n < \omega$. Thus for every coordinate $\alpha < \kappa$ it follows that there exists some $n_\alpha < \omega$ such that $a_n(\alpha) \leq 2^{-n}$ for $n \geq n_\alpha$. We conclude $a_n \to \vec{0}$, showing that $\tau$ is a winning strategy. $\square$

This lemma simplifies our notation in proving the main result. Intuitively, we aim to show that when $\kappa$ has cofinality $\omega$, $\mathscr{F}$ can split up the game among $\omega$-many smaller cardinals converging to $\kappa$, and when $\kappa$ has a larger cofinality, $\mathscr{F}$ may exploit the fact that $\mathscr{C}$ may only play within some ordinal smaller than $\kappa$.

**Theorem 3.2.10.** *$\mathscr{F} \underset{code}{\uparrow} PF_{F,C}(\kappa)$ for all cardinals $\kappa$.* $\Diamond$

*Proof.* For each cardinal $\kappa$ and $\lambda < \kappa$, assume $\sigma_\lambda$ is a winning strategy for $\mathscr{F}$ in $PF_{F,C}(\lambda)$ such that $\sigma_\lambda(\emptyset) \supset \emptyset$ and $\sigma_\lambda(\langle F, C\rangle) \supset F$.

In the case that $\mathrm{cf}(\kappa) = \omega$, let $\langle \kappa_0, \kappa_1, \ldots\rangle$ be an increasing sequence of cardinals limiting to $\kappa$. Then we define the coding strategy $\sigma$ for $\mathscr{F}$ as follows:

$$\sigma(\emptyset) = \sigma_{\kappa_0}(\emptyset)$$

$$\sigma(\langle F, C\rangle) = \bigcup_{n \leq |F|} \sigma_{\kappa_n}(\langle F \cap \kappa_n, C \cap \kappa_n\rangle)$$

Then for each legal attack $a = \langle a(0), a(1), \ldots\rangle$ by $\mathscr{C}$ against $\sigma$ and each $n < \omega$, the sequence $b_n = \langle a(n) \cap \kappa_n, a(n+1) \cap \kappa_n, \ldots\rangle$ is a legal attack by $\mathscr{C}$ against the winning coding strategy $\sigma_{\kappa_n}$. It follows then that $\{a(i+n) \cap \kappa_n : i < \omega\}$ is a point-finite cover of $\bigcup_{i<\omega} a(i+n) \cap \kappa_n$. We conclude that $\{a(i) : i < \omega\}$ is a point-finite cover of $\bigcup_{i<\omega} a(i)$ and $\sigma$ is a winning strategy.

It remains to consider the case where $\mathrm{cf}(\kappa) > \omega$. Note that now, for each $C \in [\kappa]^{\leq\omega}$, $C$ is bounded above in $\kappa$. So we define the coding strategy $\sigma$ for $\mathscr{F}$ as follows:

$$\sigma(\emptyset) = \emptyset$$

$$\sigma(\langle F, C\rangle) = \{\sup(C)\} \cup \bigcup_{\alpha \in F} \sigma_{\alpha+1}(\langle F \cap (\alpha+1), C \cap (\alpha+1)\rangle)$$

Then for each legal attack $a = \langle a(0), a(1), \ldots\rangle$ by $\mathscr{C}$ against $\sigma$ and each $n < \omega$, the sequence $b_n = \langle a(n) \cap (\sup(a(n)) + 1), a(n+1) \cap (\sup(a(n)) + 1), \ldots\rangle$ is a legal attack by $\mathscr{C}$ against the winning coding strategy $\sigma_{\sup(a(n))+1}$. It follows then that $\{a(i+n) \cap (\sup(a(n)) +$

1) : $i < \omega\}$ is a point-finite cover of $\bigcup_{i<\omega} a(i+n) \cap (\sup(a(n))+1)$. We conclude that $\{a(i) : i < \omega\}$ is a point-finite cover of $\bigcup_{i<\omega} a(i)$ and $\sigma$ is a winning strategy. $\square$

**Corollary 3.2.11.** $\mathscr{O} \underset{code}{\uparrow} Con_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$ *for all cardinals* $\kappa$. $\Diamond$

This leaves open the question analogous to Theorem 3.2.6.

**Question 3.2.12.** Does $\mathscr{O} \uparrow Con_{O,P}(X,x)$ imply $\mathscr{O} \underset{\text{code}}{\uparrow} Con_{O,P}(X,x)$? $\Diamond$

# Chapter 4

# The Proximal Game

placeholder

# Chapter 5

# Locally Finite Games

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## Chapter 6

## The Menger Game

In 1924 Karl Menger introduced a covering property generalizing $\sigma$-compactness [10].

**Definition 6.0.13.** A space $X$ is Menger if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ of open covers of $X$ there exists a sequence $\langle \mathcal{F}_0, \mathcal{F}_1, \ldots \rangle$ such that $\mathcal{F}_n \subseteq \mathcal{U}_n$, $|\mathcal{F}_n| < \omega$, and $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of $X$. $\Diamond$

**Proposition 6.0.14.** *$X$ is $\sigma$-compact $\Rightarrow$ $X$ is Menger $\Rightarrow$ $X$ is Lindelöf.* $\Diamond$

None of these implications may be reversed; the irrationals are a simple example of a Lindelöf space which is not Menger, and we'll see several examples of Menger spaces which are not $\sigma$-compact.

It can be shown via a non-trivial proof that the following game can be used to characterize the Menger property.

**Definition 6.0.15.** For each cover $\mathcal{U}$ of $X$, $S \subseteq X$ is *$\mathcal{U}$-finite* if there exists a finite subcollection of $\mathcal{U}$ which covers $C$. $\Diamond$

Of course, a compact space is $\mathcal{U}$-finite for all open covers $\mathcal{U}$.

**Game 6.0.16.** Let $Cov_{C,F}(X)$ denote the *Menger game* with players $\mathscr{C}$, $\mathscr{F}$. In round $n$, $\mathscr{C}$ chooses an open cover $\mathcal{U}_n$, followed by $\mathscr{F}$ choosing a $\mathcal{U}_n$-finite subset $F_n$ of $X$.

$\mathscr{F}$ wins the game if $X = \bigcup_{n<\omega} F_n$, and $\mathscr{C}$ wins otherwise. $\Diamond$

**Theorem 6.0.17.** *A space $X$ is Menger if and only if $\mathscr{C} \not\uparrow Cov_{C,F}(X)$ [7].* $\Diamond$

The typical characterization of the Menger game involves $\mathscr{F}$ choosing a finite subcollection $\mathcal{F}_n$ of $\mathcal{U}_n$, but it is easy to see that the characterization given above is equivalent, and will be convenient for use in our proofs.

## 6.1 Markov strategies

To the author's knowledge, no other direct work has been done on limited information strategies pertaining to the Menger game, although as we'll see there are results which can be sharpened when considering them. However, we immediately see that tactics are not of any real interest.

**Proposition 6.1.1.** *$X$ is compact if and only if $\mathscr{F} \underset{tact}{\uparrow} Cov_{C,F}(X)$ if and only if $\mathscr{F} \underset{k\text{-}tact}{\uparrow} Cov_{C,F}(X)$.* $\Diamond$

*Proof.* If $\sigma$ is a winning $k$-tactic, then for each open cover $\mathcal{U}$, $\sigma$ defeats the attack $\langle \mathcal{U}, \mathcal{U}, \ldots \rangle$. Then

$$\bigcup_{i \leq k} \sigma(\langle \underbrace{\mathcal{U}, \ldots, \mathcal{U}}_{i} \rangle) = X$$

and $X$ is $\mathcal{U}$-finite. $\square$

Essentially, because $\mathscr{C}$ may repeat the same finite sequence of open covers, $\mathscr{F}$ needs to be seeded with knowledge of the round number to prevent being trapped in a loop.

If $\mathscr{F}$'s memory of $\mathscr{C}$'s past moves is bounded, then there is no need to consider more than the two most recent moves. The intuitive reason is that $\mathscr{C}$ could simply play the same cover repeatedly until $\mathscr{F}$'s memory is exhausted, in which case $\mathscr{F}$ would only ever see the change from one cover to another.

**Theorem 6.1.2.** *There exists $k < \omega$ such that $F \underset{(k+2)\text{-}mark}{\uparrow} Cov_{C,F}(X)$ if and only if $F \underset{2\text{-}mark}{\uparrow} Cov_{C,F}(X)$.* $\Diamond$

*Proof.* Let $\sigma$ be a winning $(k+2)$-mark. We define the 2-mark $\tau$ as follows:

$$\tau(\langle \mathcal{U} \rangle, 0) = \bigcup_{m < k+2} \sigma(\langle \underbrace{\mathcal{U}, \ldots, \mathcal{U}}_{m+1} \rangle, m)$$

$$\tau(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) = \bigcup_{m < k+2} \sigma(\langle \underbrace{\mathcal{U}, \ldots, \mathcal{U}}_{k+1-m}, \underbrace{\mathcal{V}, \ldots, \mathcal{V}}_{m+1} \rangle, (n+1)(k+2) + m)$$

Let $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ be an attack by $\mathscr{C}$ against $\tau$. Then consider the attack

$$\langle \underbrace{\mathcal{U}_0, \dots, \mathcal{U}_0}_{k+2}, \underbrace{\mathcal{U}_1, \dots, \mathcal{U}_1}_{k+2}, \dots \rangle$$

by $\mathscr{C}$ against $\sigma$. Since $\sigma$ is a winning $(k+2)$-mark,

$$X = \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}_0, \dots, \mathcal{U}_0}_{m+1} \rangle, m) \cup \bigcup_{n<\omega} \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}_n, \dots, \mathcal{U}_n}_{k+1-m}, \underbrace{\mathcal{U}_{n+1}, \dots, \mathcal{U}_{n+1}}_{m+1} \rangle, (n+1)(k+2)+m)$$

$$= \tau(\langle \mathcal{U}_0 \rangle, 0) \cup \bigcup_{n<\omega} \tau(\langle \mathcal{U}_n, \mathcal{U}_{n+1} \rangle, n+1)$$

Thus $\tau$ is a winning 2-mark. $\square$

A natural question arises: is there an example of a space $X$ for which $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(X)$ but $\mathscr{F} \underset{\text{mark}}{\not\uparrow} Cov_{C,F}(X)$? We quickly see that perhaps the simplest example of a Lindelöf non-$\sigma$-compact space has this property.

**Definition 6.1.3.** For any cardinal $\kappa$, let $\kappa^\dagger = \kappa \cup \{\infty\}$ denote the *one-point Lindelöfication* of discrete $\kappa$, where points in $\kappa$ are isolated, and the neighborhoods of $\infty$ are the co-countable sets containing it. $\Diamond$

**Theorem 6.1.4.** *$F \underset{mark}{\not\uparrow} Cov_{C,F}(\omega_1^\dagger)$.* $\Diamond$

*Proof.* This result will later follow from the fact that $\omega_1^\dagger$ is not a $\sigma$-compact space (all its compact subsets are finite).

For now, let $\sigma$ be a Markov strategy for $\mathscr{F}$. For each $\alpha < \omega_1$, let $\mathcal{U}_\alpha$ be the open cover $\{\{\beta\} : \beta < \alpha\} \cup \{\omega_1^\dagger \setminus \alpha\}$ of $\omega_1^\dagger$, and set $F(\alpha, n)$ to be the finite set $\alpha \cap \sigma(\langle \mathcal{U}_\alpha \rangle, n)$.

If $P_n = \{\beta : \beta < \alpha < \omega_1 \Rightarrow \beta \in F(\alpha, n)\}$, then $P_n \subseteq F(\sup(P_n)+1, n)$. Thus $P_n$ is finite for $n < \omega$. Choose $\beta \in \omega_1 \setminus \bigcup_{n<\omega} P_n$ and $\alpha_n > \beta$ such that $\beta \notin F(\alpha_n, n)$. Then $\mathscr{C}$ may attack $\sigma$ with $\langle \mathcal{U}_{\alpha_0}, \mathcal{U}_{\alpha_1}, \dots \rangle$, and it follows that $\beta \notin \bigcup_{n<\omega} F(\alpha_n, n)$ and $\beta \notin \bigcup_{n<\omega} \sigma(\langle \mathcal{U}_{\alpha_n} \rangle, n)$. $\square$

The greatest advantage of a strategy which has knowledge of two or more previous moves of the opponent, versus only knowledge of the most recent move, is the ability to react to changes from one round to the next. It's this ability to react that will give $\mathscr{F}$ her winning 2-Markov strategy in the Menger game on $\omega_1^\dagger$.

For inspiration, we turn to a game whose $n$-tactics were studied by Marion Scheepers in [14] which has similar goals to the Menger game when played upon $\kappa^\dagger$.

**Game 6.1.5.** Let $Fill_{C,F}^{\cup,\subset}(\kappa)$ denote the *strict union filling game* with two players $\mathscr{C}$, $\mathscr{F}$. In round 0, $\mathscr{C}$ chooses $C_0 \in [\kappa]^{\leq\omega}$, followed by $\mathscr{F}$ choosing $F_0 \in [\kappa]^{<\omega}$. In round $n+1$, $\mathscr{C}$ chooses $C_{n+1} \in [\kappa]^{\leq\omega}$ such that $C_{n+1} \supset C_n$, followed by $\mathscr{F}$ choosing $F_{n+1} \in [\kappa]^{<\omega}$.

$\mathscr{F}$ wins the game if $\bigcup_{n<\omega} F_n \supseteq \bigcup_{n<\omega} C_n$; otherwise, $\mathscr{C}$ wins. $\Diamond$

In $Cov_{C,F}(\kappa^\dagger)$, $\mathscr{C}$ essentially chooses a countable set to not include in her neighborhood of $\infty$, followed by $\mathscr{F}$ choosing a finite subset of this complement to cover during each round. Thus, $\mathscr{F}$ need only be concerned with the *intersection* of the countable sets chosen by $\mathscr{C}$ in $Cov_{C,F}(\kappa^\dagger)$, rather than the union as in $Fill_{C,F}^{\cup,\subset}(\kappa)$.

Another difference: Scheepers required that $\mathscr{C}$ always choose strictly growing countable sets. The reasoning is clear: if the goal is to study tactics, then $\mathscr{C}$ cannot be allowed to trap $\mathscr{F}$ in a loop by repeating the same moves. But by eliminating this requirement, the study can then turn to Markov srategies, bringing the game further in line with the Menger game played upon $\kappa^\dagger$.

We introduce a few games to make the relationship between Scheeper's $Fill_{C,F}^{\cup,\subset}(\kappa)$ and $Cov_{C,F}(\kappa^\dagger)$ more precise.

**Game 6.1.6.** Let $Fill_{C,F}^{\cup,\subseteq}(\kappa)$ denote the *union filling game* which proceeds analogously to $Fill_{C,F}^{\cup,\subset}(\kappa)$, except that $\mathscr{C}$'s restriction in round $n+1$ is reduced to $C_{n+1} \supseteq C_n$. $\Diamond$

**Game 6.1.7.** Let $Fill_{C,F}^{1,\subseteq}(\kappa)$ denote the *initial filling game* which proceeds analogously to $Fill_{C,F}^{\cup,\subseteq}(\kappa)$, except that $\mathscr{F}$ wins whenever $\bigcup_{n<\omega} F_n \supseteq C_0$. $\Diamond$

**Game 6.1.8.** Let $Fill_{C,F}^{\cap}(\kappa)$ denote the *intersection filling game* which proceeds analogously to $Fill_{C,F}^{1,\subseteq}(\kappa)$, except that $\mathscr{C}$ may choose any $C_n \in [\kappa]^{\leq\omega}$ each round, and $\mathscr{F}$ wins whenever $\bigcup_{n<\omega} F_n \supseteq \bigcap_{n<\omega} C_n$. $\Diamond$

$$
\begin{tikzcd}
\mathscr{F} \underset{k\text{-mark}}{\uparrow} Cov_{C,F}(\kappa^\dagger) \arrow[r, leftrightarrow] \arrow[d] & \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa) \arrow[d] \\
\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{1,\subseteq}_{C,F}(\kappa) \arrow[r, leftrightarrow] & \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cup,\subseteq}_{C,F}(\kappa) \\
 & \mathscr{F} \underset{k\text{-tact}}{\uparrow} Fill^{\cup,\subset}_{C,F}(\kappa) \arrow[u]
\end{tikzcd}
$$

Figure 6.1: Diagram of Filling/Menger game implications

**Theorem 6.1.9.** *For any cardinal $\kappa > \omega$ and integer $k < \omega$, Figure 6.1 holds.* $\diamondsuit$

*Proof.* $\mathscr{F} \underset{k\text{-mark}}{\uparrow} Cov_{C,F}(\kappa^\dagger) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa)$: Let $\sigma$ be a winning $k$-mark for $\mathscr{F}$ in $Cov_{C,F}(\kappa^\dagger)$. Let $\mathcal{U}(C)$ (resp. $\mathcal{U}(s)$) convert each countable subset $C$ of $\kappa$ (resp. finite sequence $s$ of such subsets) into the open cover $[C]^1 \cup \{\kappa^\dagger \setminus C\}$ (resp. finite sequence of such open covers). Then $\tau$ defined by

$$\tau(s^\frown\langle C\rangle, n) = C \cap \sigma(\mathcal{U}(s^\frown\langle C\rangle), n)$$

is a winning $k$-mark for $\mathscr{F}$ in $Fill^{\cap}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Cov_{C,F}(\kappa^\dagger)$: Let $\sigma$ be a winning $k$-mark for $\mathscr{F}$ in $Fill^{\cap}_{C,F}(\kappa)$. Let $C(\mathcal{U})$ (resp. $C(s)$) convert each open cover $\mathcal{U}$ of $\kappa^\dagger$ (resp. finite sequence $s$ of such covers) into a countable set $C$ which is the complement of some neighborhood of $\infty$ in $\mathcal{U}$ (resp. finite sequence of such countable sets). Then $\tau$ defined by

$$\tau(s^\frown\langle\mathcal{U}\rangle, n) = (\kappa^\dagger \setminus C(\mathcal{U})) \cup \sigma(C(s^\frown\langle\mathcal{U}\rangle\rangle), n)$$

is a winning $k$-mark for $\mathscr{F}$ in $Cov_{C,F}(\kappa^\dagger)$.

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{1,\subsetneq}_{C,F}(\kappa)$: Let $\sigma$ be a winning $k$-mark for $\mathscr{F}$ in $Fill^{\cap}_{C,F}(\kappa)$. $\sigma$ is also a winning $k$-mark for $\mathscr{F}$ in $Fill^{1,\subsetneq}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{1,\subsetneq}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cup,\subsetneq}_{C,F}(\kappa)$: Let $\sigma$ be a winning $k$-mark for $\mathscr{F}$ in $Fill^{1,\subsetneq}_{C,F}(\kappa)$. For each finite sequence $s$, let $t \preceq s$ mean $t$ is a final subsequence of $s$. Then $\tau$ defined by

$$\tau(s^\frown\langle C\rangle, n) = \bigcup_{t \preceq s, m \leq n} \sigma(t^\frown\langle C\rangle, m)$$

is a winning $k$-mark for $\mathscr{F}$ in $Fill^{\cup,\subsetneq}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cup,\subsetneq}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{1,\subsetneq}_{C,F}(\kappa)$: Let $\sigma$ be a winning $k$-mark for $\mathscr{F}$ in $Fill^{\cup,\subsetneq}_{C,F}(\kappa)$. $\sigma$ is also a winning $k$-mark for $\mathscr{F}$ in $Fill^{1,\subsetneq}_{C,F}(\kappa)$.

$\mathscr{F} \underset{k\text{-tact}}{\uparrow} Fill^{\cup,\subset}_{C,F}(\kappa) \Rightarrow \mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cup,\subsetneq}_{C,F}(\kappa)$: Let $\sigma$ be a winning $k$-tactic for $\mathscr{F}$ in $Fill^{\cup,\subset}_{C,F}(\kappa)$. For each countable subset $C$ of $\kappa$, let $C+n$ be the union of $C$ with the $n$ least ordinals in $\kappa \setminus C$. Then $\tau$ defined by

$$\tau(\langle C_0, \ldots, C_i\rangle, n) = \sigma(\langle C_0 + (n-i), \ldots, C_i + n\rangle)$$

is a winning $k$-mark for $\mathscr{F}$ in $Fill^{\cup,\subsetneq}_{C,F}(\kappa)$. $\square$

While we have not proven a direct implication between the Menger game and Scheeper's original filling game, Scheepers introduced the statement $S(\kappa, \omega, \omega)$ relating to the almost-compatability of functions from countable subsets of $\kappa$ into $\omega$ which may be applied to both.

**Definition 6.1.10.** For two functions $f, g$ we say $f$ is **$\mu$-almost compatible** with $g$ ($f\|_\mu^* g$) if $|\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(x) \neq g(x)\}| < \mu$. If $\mu = \omega$ then we say $f, g$ are **almost compatible** ($f\|^* g$). $\Diamond$

**Definition 6.1.11.** $S(\kappa, \omega, \omega)$ states that there exist functions $f_A : A \to \omega$ for each $A \in [\kappa]^{\leq\omega}$ such that $|\{\alpha \in A : f_A(\alpha) \leq n\}| < \omega$ for all $n < \omega$ and $f_A\|^* f_B$ for all $A, B \in [\kappa]^\omega$. [1] $\Diamond$

Scheepers went on to show that $S(\kappa, \omega, \omega)$ implies $\mathscr{F} \underset{\text{2-tact}}{\uparrow} Fill_{C,F}^{\cup,\subset}(\kappa)$. This proof, along with the following facts, give us inspiration for finding a winning 2-Markov strategy in the Menger game played on $\kappa^\dagger$.

**Theorem 6.1.12.** *$S(\omega_1, \omega, \omega)$ and $\kappa > 2^\omega \Rightarrow \neg S(\kappa, \omega, \omega)$ are theorems of $ZFC$. $S(2^\omega, \omega, \omega)$ is a theorem of $ZFC + CH$ and consistent with $ZFC + \neg CH$.* $\Diamond$

*Proof.* For $S(\omega_1, \omega, \omega)$, look at pg. 70 of [8]; this of course implies $S(2^\omega, \omega, \omega)$ under $CH$. $\neg S((2^\omega)^+, \omega, \omega)$ is shown by a cardinality argument in [14]. The consistency result under $ZFC + \neg CH$ is a lemma for the main theorem in [14]. $\square$

**Theorem 6.1.13.** *$S(\kappa, \omega, \omega)$ implies the game-theoretic results in Figure 6.2.* $\Diamond$

*Proof.* Since $S(\kappa, \omega, \omega) \Rightarrow \mathscr{F} \underset{\text{2-tact}}{\uparrow} Fill_{C,F}^{\cup,\subset}(\kappa)$ was a main result of [14], it is sufficient to show that $S(\kappa, \omega, \omega) \Rightarrow \mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill_{C,F}^{\cap}(\kappa)$.

Let $f_A$ for $A \in [\kappa]^{\leq\omega}$ witness $S(\kappa, \omega, \omega)$. We define the 2-mark $\sigma$ as follows:

$$\sigma(\langle A \rangle, 0) = \{\alpha \in A : f_A(\alpha) \leq 0\}$$

$$\sigma(\langle A, B \rangle, n+1) = \{\alpha \in A \cap B : f_B(\alpha) \leq n+1 \text{ or } f_A(\alpha) \neq f_B(\alpha)\}$$

[1] This is equivalent to the original characterization given in [14]: there exist injections $g_A : A \to \omega$ such that $g_A\|^* g_B$ for all $A, B \in [\kappa]^\omega$ and $A \subset B$.

$$
\begin{tikzcd}
 & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(\kappa^\dagger) \arrow[r, leftrightarrow] \arrow[d] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa) \arrow[d] \\
S(\kappa,\omega,\omega) \arrow[ur] \arrow[r] \arrow[drr] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill^{1,\subseteq}_{C,F}(\kappa) \arrow[r, leftrightarrow] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill^{\cup,\subseteq}_{C,F}(\kappa) \\
 & & \mathscr{F} \underset{\text{2-tact}}{\uparrow} Fill^{\cup,\subseteq}_{C,F}(\kappa) \arrow[u]
\end{tikzcd}
$$

Figure 6.2: Diagram of Filling/Menger game implications with $S(\kappa,\omega,\omega)$

For any attack $\langle A_0, A_1, \ldots \rangle$ by $\mathscr{C}$ and $\alpha \in \bigcap_{n<\omega} A_n$, either $f_{A_n}(\alpha)$ is constant for all $n$, or $f_{A_n}(\alpha) \neq f_{A_{n+1}}(\alpha)$ for some $n$; either way, $\alpha$ is covered. $\square$

**Corollary 6.1.14.** $\mathscr{F} \underset{\textit{2-mark}}{\uparrow} Cov_{C,F}(\omega_1^\dagger)$. $\lozenge$

## 6.2 Menger game derived covering properties

$$
\begin{tikzcd}
\mathscr{F} \underset{\text{mark}}{\uparrow} Cov_{C,F}(X) \arrow[r] \arrow[d, leftrightarrow] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(X) \arrow[r] \arrow[d, leftrightarrow] \arrow[l, dotted, bend right, "/" description] & \mathscr{F} \uparrow Cov_{C,F}(X) \arrow[r] \arrow[d, leftrightarrow] \arrow[l, dotted, bend right, "?" description] & \mathscr{C} \not\uparrow Cov_{C,F}(X) \arrow[d, leftrightarrow] \arrow[l, dotted, bend right, "/" description] \\
\text{Markov Menger} \arrow[r] & \text{2-Markov Menger} \arrow[r] \arrow[l, dotted, bend right, "/" description] & \text{Perfect Menger} \arrow[r] \arrow[l, dotted, bend right, "?" description] & \text{Menger} \arrow[l, dotted, bend right, "/" description]
\end{tikzcd}
$$

Figure 6.3: Diagram of covering properties related to the Menger game

Limited information strategies for the Menger game naturally define a spectrum of covering properties, see Figure 6.3. However, we do not know if the middle two properties are actually distinct.

**Question 6.2.1.** Does there exist a space $X$ such that $\mathscr{F} \uparrow Cov_{C,F}(X)$ but $\mathscr{F} \underset{\text{2-mark}}{\not\uparrow} Cov_{C,F}(X)$? ◊

We are also interested in non-game-theoretic characterizations of these covering properties. It has been known for some time that for metrizable spaces, winning Menger spaces are exactly the $\sigma$-compact spaces, shown first by Telgarksy in [17] and later directly by Scheepers in [16].

In the interest of generality, we will first characterize the Markov Menger spaces without any separation axioms.

**Definition 6.2.2.** A subset $Y$ of $X$ is *relatively compact* to $X$ if for every open cover of $X$, there exists a finite subcollection which covers $Y$. ◊

For example, any bounded subset of Euclidean space is relatively compact whether it is closed or not. Actually, relative compactness can be thought of as an analogue of boundedness for regular spaces.

**Proposition 6.2.3.** *For regular spaces, $Y$ is relatively compact to $X$ if and only if $\overline{Y}$ is compact in $X$.* [2] ◊

*Proof.* For any space, any subset of a compact set is relatively compact.

Assume $Y$ is relatively compact, let $\mathcal{U}$ be an open cover of $\overline{Y}$, and define $x \in V_x \subseteq \overline{V_x} \subseteq U_x \in \mathcal{U}$ for each $x \in X$. Then if we take a subcollection $\mathcal{F} = \{V_{x_i} : i < n\}$ covering $Y$ by relative compactness, then $\{U_{x_i} : i < n\}$ is a finite subcollection of $\mathcal{U}$ covering $\overline{Y}$, showing compactness. □

[2] It should be noted that some authors define relative compactness in this way, but such a definition creates pathological implications for non-regular spaces. For example, the singleton containing the particular point of an infinite space with the particular point topology would not be relatively compact since its closure is not compact, even though it is finite.

We now begin the process of factoring out Scheeper's proof to reveal the limited information implications at work.

**Lemma 6.2.4.** *Let $\sigma(\mathcal{U}, n)$ be a Markov strategy for $F$ in $Cov_{C,F}(X)$, and $\mathfrak{C}$ collect all open covers of $X$. Then the set*

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \sigma(\mathcal{U}, n)$$

*is relatively compact to $X$. If $\sigma$ is a winning Markov startegy, then $\bigcup_{n<\omega} R_n = X$.* $\Diamond$

*Proof.* First, for every open cover $\mathcal{U} \in \mathfrak{C}$, $R_n \subseteq \sigma(\mathcal{U}, n)$ is covered by a finite subcollection of $\mathcal{U}$.

Suppose that $x \notin R_n$ for any $n < \omega$. Then for each $n$, pick $\mathcal{U}_n \in \mathfrak{C}$ such that $x \notin \sigma(\mathcal{U}_n, n)$. Then $\mathscr{C}$ may counter $\sigma$ with the attack $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$. $\square$

**Definition 6.2.5.** A *$\sigma$-relatively-compact* space is the countable union of relatively compact subsets. $\Diamond$

**Corollary 6.2.6.** *The following are equivalent:*

- *$X$ is $\sigma$-relatively-compact*
- $F \underset{pre}{\uparrow} Cov_{C,F}(X)$
- $F \underset{mark}{\uparrow} Cov_{C,F}(X)$

$\Diamond$

*Proof.* If $X = \bigcup_{n<\omega} R_n$ for $R_n$ relatively compact, then $\sigma(n) = R_n$ is a winning predetermined strategy, which yields a winning Markov strategy. The previous lemma finishes the proof. $\square$

**Corollary 6.2.7.** *Let $X$ be a regular space. The following are equivalent:*

- *$X$ is $\sigma$-compact*
- *$X$ is $\sigma$-relatively-compact*
- *$F \underset{pre}{\uparrow} Cov_{C,F}(X)$*
- *$F \underset{mark}{\uparrow} Cov_{C,F}(X)$*

$\diamondsuit$

For Lindelöf spaces, metrizability is characterized by regularity and second-countability, the latter of which was essentially used by Scheepers in this way:

**Lemma 6.2.8.** *Let $X$ be a second-countable space. $\mathscr{F} \uparrow Cov_{C,F}(X)$ if and only if $\mathscr{F} \underset{mark}{\uparrow} Cov_{C,F}(X)$.* $\diamondsuit$

*Proof.* Let $\sigma$ be a strategy for $\mathscr{F}$, and note that it's sufficient to consider playthroughs with only basic open covers.

So if $\mathcal{U}_t$ is a basic open cover for $t < s \in \omega^{<\omega}$, and $\mathcal{V}$ is any basic open cover, we may choose a finite subcollection $\mathcal{F}(s, \mathcal{V})$ of $\mathcal{V}$ such that

$$\sigma(\langle \mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{V}\rangle) \subseteq \bigcup \mathcal{F}(s, \mathcal{V})$$

Note that there are only countably-many finite collections of basic open sets. Thus we may choose basic open covers $\mathcal{U}_{s^\frown\langle n\rangle}$ for $n < \omega$ such that for any basic open cover $\mathcal{V}$, there exists $n < \omega$ where $\mathcal{F}(s, \mathcal{V}) = \mathcal{F}(s, \mathcal{U}_{s^\frown\langle n\rangle})$.

Let $t : \omega \to \omega^{<\omega}$ be a bijection. We define the Marköv strategy $\tau$ as follows:

$$\tau(\langle \mathcal{V}\rangle, n) = \bigcup \mathcal{F}(t(n), \mathcal{V})$$

Suppose there exists a counter-attack $\langle \mathcal{V}_0, \mathcal{V}_1, \ldots \rangle$ of basic open covers which defeats $\tau$. Then there exists $f : \omega \to \omega$ such that, letting $t(m_n) = f \restriction n$:

$$\begin{aligned} x &\notin \tau(\langle \mathcal{V}_{m_n} \rangle, m_n) \\ &= \bigcup \mathcal{F}(f \restriction n, \mathcal{V}_{m_n}) \\ &= \bigcup \mathcal{F}(f \restriction n, \mathcal{U}_{f \restriction (n+1)}) \\ &\supseteq \sigma(\langle \mathcal{U}_{f \restriction 1}, \ldots, \mathcal{U}_{f \restriction (n+1)} \rangle) \end{aligned}$$

Thus $\langle \mathcal{U}_{f \restriction 1}, \mathcal{U}_{f \restriction 2}, \ldots \rangle$ is a successful counter-attack by $\mathscr{C}$ against the perfect information strategy $\sigma$. □

**Corollary 6.2.9.** *Let $X$ be a second-countable space. The following are equivalent:*

- *$X$ is $\sigma$-relatively-compact*
- $F \underset{pre}{\uparrow} Cov_{C,F}(X)$
- $F \underset{mark}{\uparrow} Cov_{C,F}(X)$
- $F \uparrow Cov_{C,F}(X)$

◊

**Corollary 6.2.10.** *Let $X$ be a metrizable space. The following are equivalent:*

- *$X$ is $\sigma$-compact*
- *$X$ is $\sigma$-relatively-compact*
- $F \underset{pre}{\uparrow} Cov_{C,F}(X)$
- $F \underset{mark}{\uparrow} Cov_{C,F}(X)$
- $F \uparrow Cov_{C,F}(X)$

◊

*Proof.* Each property implies Lindelöf, so $X$ may be assumed to be regular and second-countable. □

### 6.3 Robustly Lindelöf

To help describe $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(X)$ topologically, we introduce a subset variant of the Menger game and a related covering property.

**Game 6.3.1.** Let $Cov_{C,F}(X, Y)$ denote the *Menger subspace game* which proceeds analogously to the Menger game, except that $\mathscr{F}$ wins whenever $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover for $Y \subseteq X$. ◊

Note of course that $Cov_{C,F}(X, X) = Cov_{C,F}(X)$.

**Definition 6.3.2.** A subset $Y$ of $X$ is *relatively robustly Menger* if there exist functions $r_{\mathcal{V}} : Y \to \omega$ for each open cover $\mathcal{V}$ of $X$ such that for all open covers $\mathcal{U}, \mathcal{V}$ and numbers $n < \omega$, the following sets are $\mathcal{V}$-finite:

$$c(\mathcal{V}, n) = \{x \in Y : r_{\mathcal{V}}(x) \leq n\}$$

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{x \in Y : n < r_{\mathcal{U}}(x) < r_{\mathcal{V}}(x)\}$$

◊

**Definition 6.3.3.** A space $X$ is *robustly Menger* if it is relatively robustly Menger to itself. ◊

**Proposition 6.3.4.** *All $\sigma$-relatively-compact spaces are robustly Menger.* ◊

*Proof.* If $X = \bigcup_{n<\omega} R_n$, then for all $\mathcal{U}$, let $r_{\mathcal{U}}(x)$ be the least $n$ such that $x \in R_n$. Then $c(\mathcal{V}, n) = \bigcup_{m \leq n} R_m$ and $p(\mathcal{U}, \mathcal{V}) = \emptyset$. □

**Theorem 6.3.5.** *If $Y \subseteq X$ is relatively robustly Menger, then $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(X, Y)$.* ◊

*Proof.* We define the Markov strategy $\sigma$ as follows. Let $\sigma(\langle\mathcal{U}\rangle, 0) = c(\mathcal{U}, 0)$, and let $\sigma(\langle\mathcal{U}, \mathcal{V}\rangle, n+1) = c(\mathcal{V}, n+1) \cup p(\mathcal{U}, \mathcal{V}, n+1)$.

For any attack $\langle\mathcal{U}_0, \mathcal{U}_1, \ldots\rangle$ by $\mathscr{C}$ and $x \in Y$, one of the following must occur:

- $r_{\mathcal{U}_0}(x) = 0$ and thus $x \in c(\mathcal{U}_0, 0) \subseteq \sigma(\langle\mathcal{U}_0\rangle, 0)$.
- $r_{\mathcal{U}_0}(x) = N + 1$ for some $N \geq 0$ and:
  - For all $n \leq N$,
  $$r_{\mathcal{U}_{n+1}}(x) \leq N + 1$$
  and thus $x \in c(\mathcal{U}_{N+1}, N+1) \subseteq \sigma(\langle U_N, U_{N+1}\rangle, N+1)$.
  - For some $n \leq N$,
  $$r_{\mathcal{U}_n}(x) \leq n$$
  and thus $x \in c(\mathcal{U}_{n+1}, n+1) \subseteq \sigma(\langle U_n, U_{n+1}\rangle, n+1)$.
  - For some $n \leq N$,
  $$n < r_{\mathcal{U}_n}(x) \leq N + 1 < r_{\mathcal{U}_{n+1}}(x)$$
  and thus $x \in p(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1) \subseteq \sigma(\langle U_n, U_{n+1}\rangle, n+1)$

□

**Theorem 6.3.6.** *$S(\kappa, \omega, \omega)$ implies $\kappa^\dagger$ is robustly Menger, and thus $\mathscr{F} \underset{\textit{2-mark}}{\uparrow} Cov_{C,F}(\kappa^\dagger)$.* ◊

*Proof.* Let $f_A$ for $A \in [\kappa]^{\leq\omega}$ witness $S(\kappa, \omega, \omega)$ and fix $A(\mathcal{U}) \in [\kappa]^{\leq\omega}$ for each open cover $\mathcal{U}$ such that $\kappa^\dagger \setminus A(\mathcal{U})$ is contained in some element of $\mathcal{U}$. Then let $r_{\mathcal{U}}(x) = 0$ for $x \in \kappa^\dagger \setminus A(\mathcal{U})$, and $r_{\mathcal{U}}(\alpha) = f_{A(\mathcal{U})}(\alpha)$ for $\alpha \in A(\mathcal{U})$.

It follows that

$$c(\mathcal{U}, n) = \left(\kappa^\dagger \setminus A(\mathcal{U})\right) \cup \{\alpha \in A(\mathcal{U}) : f_{A(\mathcal{U})}(\alpha) \leq n\}$$

is $\mathcal{U}$-finite, $\bigcup_{n<\omega} c(\mathcal{U}, n) = X$, and

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{\alpha \in A(\mathcal{U}) \cap A(\mathcal{V}) : n < f_{A(\mathcal{U})}(\alpha) < f_{A(\mathcal{V})}(\alpha)\}$$

is finite. $\square$

We may also consider common (non-regular) counterexamples which are finer than the usual Euclidean line.

**Definition 6.3.7.** Let $R_{\mathbb{Q}}$ be the real line with the topology generated by open intervals with or without the rationals removed. $\Diamond$

**Theorem 6.3.8.** *$R_{\mathbb{Q}}$ is non-regular and non-$\sigma$-compact, but is second-countable and $\sigma$-relatively-compact.* $\Diamond$

*Proof.* Compact sets in $R_{\mathbb{Q}}$ can be shown to not contain open intervals, and thus are nowhere dense in nonmeager $\mathbb{R}$, so $R_{\mathbb{Q}}$ is not $\sigma$-compact. The usual base of intervals with rational endpoints (with or without rationals removed) witnesses second-countability.

To see that $R_{\mathbb{Q}}$ is $\sigma$-relatively compact, consider $[a, b] \setminus \mathbb{Q}$. Let $\mathcal{U}$ be a cover of $R_{\mathbb{Q}}$, and let $\mathcal{U}'$ fill in the missing rationals for any open set in $\mathcal{U}$. There is a finite subcover $\mathcal{V}' \subseteq \mathcal{U}'$ for $[a, b]$ since $\mathcal{U}'$ contains open sets from the Euclidean topology. Let $\mathcal{V} = \{V \setminus \mathbb{Q} : V \in V'\}$: this is a finite refinment of $\mathcal{U}$ covering $[a, b] \setminus \mathbb{Q}$, so $[a, b] \setminus \mathbb{Q}$ is relatively compact. It follows then that $R_{\mathbb{Q}} \setminus \mathbb{Q}$ is $\sigma$-relatively-compact, and since $\mathbb{Q}$ is countable, $R_{\mathbb{Q}}$ is $\sigma$-relatively-compact. Non-regularity follows since regular and $\sigma$-relatively-compact implies $\sigma$-compact. $\square$

**Definition 6.3.9.** Let $R_\omega$ be the real line with the topology generated by open intervals with countably many points removed. $\Diamond$

**Theorem 6.3.10.** *$R_\omega$ is non-regular, non-second-countable, and non-$\sigma$-relatively-compact, but $\mathscr{F} \uparrow Cov_{C,F}(R_\omega)$.* $\Diamond$

*Proof.* The closure of any open set is its closure in the usual Euclidean topology, so $R_\omega$ is not regular. If $S \supseteq \{s_n : n < \omega\}$ for $s_n$ discrete, then $U_m = R_\omega \setminus \{s_n : m < n < \omega\}$ yields an infinite cover $\{U_m : m < \omega\}$ with no finite subcollection covering $S$, showing that all relatively compact sets are finite, and $R_\omega$ is not $\sigma$-relatively-compact.

Define the winning strategy $\sigma$ for $\mathscr{F}$ in $Cov_{C,F}(R_\omega)$ as follows: let $\sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{2n}) = [-n, n] \setminus C_n$ for some countable $C_n = \{c_{n,m} : m < \omega\}$, and let $\sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{2n+1}) = \{c_{i,j} : i, j < n\}$. Non-second-countable follows since second-countable and $\mathscr{F} \uparrow Cov_{C,F}(X)$ implies $\sigma$-relatively-compact. □

We will soon see that, assuming $S(2^\omega, \omega, \omega)$, $\mathscr{F}$ has a winning 2-Marköv strategy for $Cov_{C,F}(R_\omega)$ as well.

**Proposition 6.3.11.** *Let* $\underset{limit}{\uparrow}$ *be either* $\underset{k\text{-}mark}{\uparrow}$ *or* $\uparrow$. *If* $X = \bigcup_{i<\omega} X_i$ *and* $\mathscr{F} \underset{limit}{\uparrow} Cov_{C,F}(X, X_i)$ *for* $i < \omega$, *then* $\mathscr{F} \underset{limit}{\uparrow} Cov_{C,F}(X)$ ◊

*Proof.* Let $L$ be the $k$-Markov fog-of-war $\mu_k$ (resp. the identity), and let $\sigma_i$ be a $k$-Markov strategy (resp. perfect information strategy) for $\mathscr{F}$ in $Cov_{C,F}(X, X_i)$.

We define the $k$-Markov strategy (resp. perfect information strategy) $\sigma$ for $Cov_{C,F}(X)$ as follows:

$$\sigma \circ L(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle) = \bigcup_{i \leq n} \sigma_i \circ L(\langle \mathcal{U}_i, \ldots, \mathcal{U}_n \rangle)$$

Let $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ be a successful counter-attack by $\mathscr{C}$ against $\sigma$. Then there exists $x \in X_i$ for some $i < \omega$ such that $x$ is not covered by $\bigcup_{n<\omega} \sigma \circ L(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$. It follows that $x$ is not covered by $\bigcup_{n<\omega} \sigma_i \circ L(\langle \mathcal{U}_i, \ldots, \mathcal{U}_{i+n} \rangle)$, and $\langle \mathcal{U}_i, \mathcal{U}_{i+1}, \ldots \rangle$ is a successful counter-attack by $\mathscr{C}$ against $\sigma_i$. □

**Theorem 6.3.12.** *If* $S(2^\omega, \omega, \omega)$, *then* $\mathscr{F} \underset{2\text{-}mark}{\uparrow} Cov_{C,F}(R_\omega)$. ◊

*Proof.* It's sufficient to show that $[0, 1] \subseteq R_\omega$ is relatively robustly Menger. Let $f_A$ witness $S(2^\omega, \omega, \omega)$ for $A \in [[a, b]]^{\leq \omega}$. For each open cover $\mathcal{U}$, let $A_\mathcal{U}$ be such that $[0, 1] \setminus A_\mathcal{U}$ is $\mathcal{U}$-finite. Let $r_\mathcal{U}(x) = 0$ if $x \in [0, 1] \setminus A_\mathcal{U}$ and $r_\mathcal{U}(x) = f_{A_\mathcal{U}}(x)$ otherwise.

It follows then that

$$c(\mathcal{U}, n) = [0, 1] \setminus \{x \in A_\mathcal{U} : f_{A_\mathcal{U}}(x) > n\}$$

is $\mathcal{U}$-finite and

$$p(\mathcal{U}, \mathcal{V}, n+1) = \{x \in A_{\mathcal{U}} \cap A_{\mathcal{V}} : n < f_{\mathcal{A}_{\mathcal{U}}}(x) < f_{\mathcal{A}_{\mathcal{V}}}(x)\}$$

is finite. $\square$

## 6.4 Rothberger game

We conclude this chapter with some results on another variation of the Menger property and game.

**Definition 6.4.1.** A space $X$ is Rothberger if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ of open covers of $X$ there exists a sequence $\langle S_0, S_1, \dots \rangle$ such that $S_n \in \mathcal{U}_n$ and $\{S_n : n < \omega\}$ is a cover of $X$. $\Diamond$

**Game 6.4.2.** Let $Cov_{C,S}(X)$ denote the *Rothberger game* with players $\mathscr{C}$, $\mathscr{S}$. In round $n$, $\mathscr{C}$ chooses an open cover $\mathcal{U}_n$, followed by $\mathscr{S}$ choosing a subset $S_n$ of $X$ contained in some open set in $\mathcal{U}_n$. $\mathscr{S}$ wins the game if $X = \bigcup_{n<\omega} S_n$, and $\mathscr{C}$ wins otherwise. $\Diamond$

**Theorem 6.4.3.** *$X$ is Rothberger if and only if $\mathscr{C} \not\uparrow Cov_{C,S}(X)$. [13]* $\Diamond$

(TODO: cite)

**Theorem 6.4.4.** *Let $X$ be a compact $T_2$ space. The following are equivalent:*

- *$X$ is Rothberger*
- *$X$ is scattered*
- *$\mathscr{S} \uparrow Cov_{C,S}(X)$*
- *$\mathscr{C} \not\uparrow Cov_{C,S}(X)$*

$\Diamond$

Another game exists which is "perfect information equivalent" to the Rothberger game.

**Game 6.4.5.** Let $Cov_{P,O}(X)$ denote the *alternate Rothberger game* with players $\mathscr{P}$, $\mathscr{O}$. In round $n$, $\mathscr{P}$ chooses a point $p_n \in X$, followed by $\mathscr{O}$ choosing a neighborhood $O_n$ of $p_n$. $\mathscr{P}$ wins if $X = \bigcup_{n<\omega} O_n$; $\mathscr{O}$ wins otherwise. $\Diamond$

(TODO: cite Galvin (Indeterminacy of Point Open Games?) )

**Theorem 6.4.6.** *$\mathscr{P} \uparrow Cov_{P,O}(X)$ if and only if $\mathscr{S} \uparrow Cov_{C,S}(X)$; $\mathscr{O} \uparrow Cov_{P,O}(X)$ if and only if $\mathscr{C} \uparrow Cov_{C,S}(X)$.* $\Diamond$

An analogous relationship exists for the winning limited information strategies.

**Theorem 6.4.7.** *$P \underset{pre}{\uparrow} Cov_{P,O}(X)$ if and only if $S \underset{mark}{\uparrow} Cov_{C,S}(X)$; $O \underset{mark}{\uparrow} Cov_{P,O}(X)$ if and only if $C \underset{pre}{\uparrow} Cov_{C,S}(X)$.* ◊

*Proof.* TODO □

As it turns out, $\mathscr{S}$ requires a lot of structure on $X$ in order to guarantee victory with a Markov strategy in the Rothberger game.

**Definition 6.4.8.** TODO: almost countable ◊

**Theorem 6.4.9.** *For any space $X$, the following are equivalent:*

- $S \underset{mark}{\uparrow} Cov_{C,S}(X)$
- $P \underset{pre}{\uparrow} Cov_{P,O}(X)$
- *$X$ is almost countable*

◊

*Proof.* TODO □

**Theorem 6.4.10.** *For any $T_1$ space $X$, the following are equivalent:*

- $S \underset{mark}{\uparrow} Cov_{C,S}(X)$
- $P \underset{pre}{\uparrow} Cov_{P,O}(X)$
- *$X$ is almost countable*
- *$X$ is countable*

$\Diamond$

Let $\kappa \cup \{\infty\}$ be a "weak Lindelöf-ication" of discrete $\kappa > \omega$ such that open neighborhoods of $\infty$ contain $\kappa \setminus \omega$. This space is an example of an almost countable space which is not countable.

**Theorem 6.4.11.** *The following are equivalent for points-$G_\delta$ $X$:*

- $S \uparrow Cov_{C,S}(X)$
- $P \uparrow Cov_{P,O}(X)$
- $S \underset{mark}{\uparrow} Cov_{C,S}(X)$
- $P \underset{pre}{\uparrow} Cov_{P,O}(X)$
- *$X$ is Rothberger*
- *$X$ is almost countable*
- *$X$ is countable*

$\Diamond$

*Proof.* TODO $\square$

**Corollary 6.4.12.** *The following are equivalent for compact points-$G_\delta$ $X$:*

- $S \uparrow Cov_{C,S}(X)$
- $P \uparrow Cov_{P,O}(X)$
- $S \underset{mark}{\uparrow} Cov_{C,S}(X)$
- $P \underset{pre}{\uparrow} Cov_{P,O}(X)$
- $C \not\uparrow Cov_{C,S}(X)$
- $O \not\uparrow Cov_{P,O}(X)$
- *$X$ is scattered*
- *$X$ is Rothberger*
- *$X$ is almost countable*
- *$X$ is countable*

$\Diamond$

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