# RELATING GAMES OF MENGER, COUNTABLE FAN TIGHTNESS, AND SELECTIVE SEPARABILITY

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ABSTRACT. By adapting techniques of Arhangel'skii, Barman, and Dow, we may equate the existence of perfect-information, Markov, and tactical strategies between two interesting selection games. These results shed some light on Gruenhage's question asking whether all strategically selectively separable spaces are Markov selectively separable.

## 1. INTRODUCTION

**Definition 1.** The *selection principle* $S_{fin}(\mathcal{A}, \mathcal{B})$ states that given $A_n \in \mathcal{A}$ for $n < \omega$, there exist $B_n \in [A_n]^{<\omega}$ such that $\bigcup_{n<\omega} B_n \in \mathcal{B}$.

**Definition 2.** The *selection game* $G_{fin}(\mathcal{A}, \mathcal{B})$ is the analogous game to $S_{fin}(\mathcal{A}, \mathcal{B})$, where during each round $n < \omega$, Player I first chooses $A_n \in \mathcal{A}$, and then Player II chooses $B_n \in [A_n]^{<\omega}$. Player II wins in the case that $\bigcup_{n<\omega} B_n \in \mathcal{B}$, and Player I wins otherwise.

This game and property were first formally investigated by Scheepers in "Combinatorics of open covers" [13], which inspired a series of ten sequels with several co-authors. We may quickly observe that if II has a winning strategy for the game $G_{fin}(\mathcal{A}, \mathcal{B})$, then $S_{fin}(\mathcal{A}, \mathcal{B})$ will hold, but the converse need not follow.

The power of this selection principle and game comes from their ability to characterize several properties and games from the literature. Of interest to us are the following.

**Definition 3.** Let $\mathcal{O}_X$ be the collection of open covers for a topological space $X$. Then $S_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger property* for $X$ (*M* for short), and $G_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger game*.

**Definition 4.** An $\omega$-*cover* $\mathcal{U}$ for a topological space $X$ is an open cover such that for every $F \in [X]^{<\omega}$, there exists some $U \in \mathcal{U}$ such that $F \subseteq U$.

**Definition 5.** Let $\Omega_X$ be the collection of $\omega$-covers for a topological space $X$. Then $S_{fin}(\Omega_X, \Omega_X)$ is the $\Omega$-*Menger property* for $X$ ($\Omega M$ for short), and $G_{fin}(\Omega_X, \Omega_X)$ is the $\Omega$-*Menger game*.

In [7, Theorem 3.9] it was shown that $X$ is $\Omega$-Menger if and only if $X^n$ is Menger for all $n < \omega$.

**Definition 6.** Let $\Omega_{X,x}$ be the collection of subsets $A \subset X$ where $x \in \mathrm{cl}\, A$. (Call $A$ a *blade* of $x$.) Then $S_{fin}(\Omega_{X,x}, \Omega_{X,x})$ is the *countable fan tightness property* for

$X$ at $x$ ($CFT_x$ for short), and $G_{fin}(\Omega_{X,x}, \Omega_{X,x})$ is the *countable fan tightness game* for $X$ at $x$.

The similar notations for $\omega$-covers $\Omega_X$ and blades $\Omega_{X,x}$ is well-established, see e.g. [10, Topological games and Ramsey theory] (wherein the notation is more succinctly given as $\Omega, \Omega_x$). The relationship between such sets becomes apparent in the study of $C_p$-theory; for this reason we choose to always denote the space explicitly to help distinguish the ambient spaces when discussing the relationship between $\Omega_X$ and $\Omega_{C_p(X),\mathbf{0}}$.

**Definition 7.** A space $X$ has *countable fan tightness* ($CFT$ for short) if it has countable fan tightness at each point $x \in X$.

**Definition 8.** Let $\mathcal{D}_X$ be the collection of dense subsets of a topological space $X$. Then $S_{fin}(\mathcal{D}_X, \Omega_{X,x})$ is the *countable dense fan tightness property* for $X$ at $x$ ($CDFT_x$ for short), and $G_{fin}(\mathcal{D}_X, \Omega_{X,x})$ is the *countable dense fan tightness game* for $X$ at $x$.

**Definition 9.** A space $X$ has *countable dense fan tightness* ($CDFT$ for short) if it has countable dense fan tightness at each point $x \in X$.

Note that $CFT \Rightarrow CDFT$ for any space $X$ as $\mathcal{D}_X \subseteq \Omega_{X,x}$ for all $x \in X$.

The notion of countable fan tightness was first studied by by Arhangel'skii in [1]. A result of that paper showed that for $T_{3\frac{1}{2}}$ spaces $X$, the countable fan tightness of the space of real-vaued continuous functions with pointwise convergence $C_p(X)$ is characterized by the $\Omega$-Menger property of $X$.

**Definition 10.** $S_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability property* for $X$ ($SS$ for short), and $G_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability game* for $X$.

Of course, one may easily observe that a selectively separable space is separable, and $SS \Rightarrow CDFT$. In [2] Barman and Dow demonstrated that all separable Frechet spaces are selectively separable. They were also able to produce a space which is selectively separable, but does not allow II a winning strategy in the selective separability game.

The object of this paper is to investigate the game-theoretic properties characterized by the presence of winning *limited information* strategies in these selection games.

**Definition 11.** A *strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function $\sigma$ satisfying $\sigma(\langle A_0, \dots, A_n \rangle) \in [A_n]^{<\omega}$ for $\langle A_0 \dots, A_n \rangle \in \mathcal{A}^{n+1}$. We say this strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(\langle A_0, \dots, A_n \rangle)$ during each round $n < \omega$. If a winning strategy exists, then we write II $\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$.

**Definition 12.** A *Markov strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function $\sigma$ satisfying $\sigma(A, n) \in [A_n]^{<\omega}$ for $A \in \mathcal{A}$ and $n < \omega$. We say this Markov strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(A_n, n)$ during each round $n < \omega$. If a winning Markov strategy exists, then we write II $\underset{\mathrm{mark}}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$.

**Notation 13.** If $S_{fin}(\mathcal{A}, \mathcal{B})$ characterizes the property $P$, then we say II $\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes $P^+$ ("strategically $P$"), and II $\underset{\mathrm{mark}}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes $P^{+\mathrm{mark}}$ ("Markov $P$"). Of course, $P^{+\mathrm{mark}} \Rightarrow P^+ \Rightarrow P$.

In this notation, Barman and Dow showed that $SS$ does not imply $SS^+$. Our goal is to make progress on the following question attributed to Gary Gruenhage:

**Question 14.** *Does $SS^+$ imply $SS^{+\mathrm{mark}}$?*

The solution is already known to be "yes" in the context of countable spaces [2]. However in general, winning strategies in selection games cannot be improved to be winning Markov strategies. In [4] the author showed that $M^+$ implies $M^{+\mathrm{mark}}$ for second-countable spaces; however, the one-point Lindelöf-ication $L(\omega_1)$ of a discrete space of size $\omega_1$ is a simple example of a regular non-second-countable space which is $M^+$ but not $M^{+\mathrm{mark}}$ (equivalently, not $\sigma$-compact). It will later be shown that it is no coincidence that countable $SS^+$ implies $SS^{+\mathrm{mark}}$ and second-countable $M^+$ implies $M^{+\mathrm{mark}}$.

## 2. $CFT$, $CDFT$ AND $SS$

We begin by generalizing the following result:

**Theorem 15** (Lemma 2.7 of [2])**.** *The following are equivalent for any topological space $X$.*

- *$X$ is $SS$.*
- *$X$ is separable and $CDFT$.*
- *$X$ has a countable dense subset $D$ where $CDFT_x$ holds for all $x \in D$.*

**Theorem 16.** *The following are equivalent for any topological space $X$.*

- *$X$ is $SS$ (resp. $SS^+$, $SS^{+\mathrm{mark}}$).*
- *$X$ is separable and $CDFT$ (resp. $CDFT^+$, $CDFT^{+\mathrm{mark}}$).*
- *$X$ has a countable dense subset $D$ where $CDFT_x$ (resp. $CDFT_x^+$, $CDFT_x^{+\mathrm{mark}}$) holds for all $x \in D$.*

*Proof.* We need only show that the final condition implies the first. Let $D = \{d_i : i < \omega\}$.

Let $\sigma_i$ be a witness for $CDFT_{d_i}^+$ for each $i < \omega$. We define the strategy $\tau$ for the $SS$ game by
$$\tau(\langle D_0, \ldots, D_n \rangle) = \bigcup_{i \leq n} \sigma_i(\langle D_i, \ldots, D_n \rangle).$$

Let $\langle D_0, D_1, \ldots \rangle \in \mathcal{D}_X^\omega$. By $CDFT_{d_i}^+$, we have
$$d_i \in \overline{\bigcup_{i \leq n < \omega} \sigma_i(\langle D_i, \ldots, D_n \rangle)} \subseteq \overline{\bigcup_{i \leq n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)} \subseteq \overline{\bigcup_{n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)}$$

and as $D \subseteq \overline{\bigcup_{n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)}$ it follows that
$$X \subseteq \overline{D} \subseteq \overline{\overline{\bigcup_{n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)}} = \overline{\bigcup_{n < \omega} \tau(\langle D_0, \ldots, D_n \rangle)}.$$

Therefore $\tau$ witnesses $SS^+$.

Now let $\sigma_i$ be a witness for $CDFT_{d_i}^{+\mathrm{mark}}$ for each $i < \omega$. We define the Markov strategy $\tau$ for the $SS$ game by
$$\tau(D, n) = \bigcup_{i \leq n} \sigma_i(D, n - i).$$

Let $\langle D_0, D_1, \ldots \rangle \in \mathcal{D}_X^\omega$. By $CDFT_{d_i}^{+\mathrm{mark}}$, we have

$$d_i \in \overline{\bigcup_{i \leq n < \omega} \sigma_i(D_n, n-i)} \subseteq \overline{\bigcup_{i \leq n < \omega} \tau(D_n, n)} \subseteq \overline{\bigcup_{n < \omega} \tau(D_n, n)}$$

and as $D \subseteq \overline{\bigcup_{n < \omega} \tau(D_n, n)}$ it follows that

$$X \subseteq \overline{D} \subseteq \overline{\overline{\bigcup_{n < \omega} \tau(D, n)}} = \overline{\bigcup_{n < \omega} \tau(D, n)}.$$

Therefore $\tau$ witnesses $SS^{+\mathrm{mark}}$. $\hfill\square$

So amongst separable spaces, we see that $SS$ (resp. $SS^+$, $SS^{+\mathrm{mark}}$) and $CDFT$ (resp. $CDFT^+$, $CDFT^{+\mathrm{mark}}$) are equivalent. We now further bridge the gap between $CDFT$ and $CFT$ in the context of function spaces. Consider the following result of Arhangel'skii.

**Theorem 17** ([1])**.** *The following are equivalent for any $T_{3\frac{1}{2}}$ topological space $X$.*

- *$X$ is $\Omega M$.*
- *$C_p(X)$ is $CFT$.*

This result may similarly be generalized in a game-theoretic sense. In addition, this proof will demonstrate the equivalence of $CFT$ and $CDFT$ in $C_p(X)$. It is unknown to the author whether Arhangel'skii used a strategy similar to the following proof in [1], but Sakai employed a similar technique in [12] to relate the $\Omega$-Rothberger and countable strong fan tightness properties (and essentially, the countable strong dense fan tightness property). Due to the difficulty in obtaining an English translation of [1], we reprove Arhangel'skii's theorem above in our more general context below.

**Proposition 18.** *Let $\mathcal{A}$, $\mathcal{B}$ be families of sets such that $B \in \mathcal{B}$ and $B \subseteq B' \subseteq \bigcup \mathcal{A}$ implies $B' \in \mathcal{B}$. Then any witness for $S_{fin}(\mathcal{A}, \mathcal{B})$ may be improved so that any final sequence of the sets chosen by II belongs to $\mathcal{B}$.*

*Furthermore, any witness for $\mathrm{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ or $\mathrm{II} \underset{mark}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$ may be improved so that any subsequence of the sets chosen by $\mathrm{II}$ belongs to $\mathcal{B}$.*

*Proof.* Let $A_n \in \mathcal{A}$ for $n < \omega$, and suppose $\langle B_{n,n}, B_{n+1,n}, \ldots \rangle$ witnesses $S_{fin}(\mathcal{A}, \mathcal{B})$ for $\langle A_n, A_{n+1}, \ldots \rangle$. Let $B_n = \bigcup_{i \leq n} B_{n,i}$; it follows that $\langle B_0, B_1, \ldots \rangle$ satisfies the proposition.

Let $s \preceq t$ indicate that $s$ is a subsequence of $t$. Then for $\mathrm{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$, the winning strategy $\sigma$ may be improved to $\tau$ where $\tau(t^\frown \langle A \rangle) = \bigcup_{s \preceq t} \sigma(s^\frown \langle A \rangle)$. For $\mathrm{II} \underset{mark}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$, the winning Markov strategy $\sigma$ may be improved to $\tau$ where $\tau(A, n) = \bigcup_{i \leq n} \sigma(A, i)$. Both improved strategies are easily seen to satisfy the proposition. $\hfill\square$

It's worth noting that Gruenhage applied a similar technique used above for strategies in $G_{fin}(\mathcal{A}, \mathcal{B})$ to obtain strengthened strategies in his game played with open neighborhoods and points in [6].

**Definition 19.** Let $X$ be a $T_{3\frac{1}{2}}$ topological space. For $\mathbf{x} \in C_p(X)$, $F \in [X]^{<\omega}$, and $\epsilon > 0$, let

$$[\mathbf{x}, F, \epsilon] = \{\mathbf{y} \in C_p(x) : |\mathbf{y}(t) - \mathbf{x}(t)| < \epsilon \text{ for all } t \in F\}$$

give a basic open neighborhood of $\mathbf{x}$.

**Lemma 20.** *If $X$ is $\Omega M$ (resp. $\Omega M^+$, $\Omega M^{+\mathrm{mark}}$), then $C_p(X)$ is $CFT_{\mathbf{0}}$ (resp. $CFT_{\mathbf{0}}^+$, $CFT_{\mathbf{0}}^{+\mathrm{mark}}$).*

*Proof.* For each $B \in \Omega_{C_p(X),\mathbf{0}}$ define

$$\mathcal{U}_n(B) = \left\{ \mathbf{x}^{\leftarrow}\left[\left(-\frac{1}{2^n}, \frac{1}{2^n}\right)\right] : \mathbf{x} \in B \right\}.$$

Consider the finite set $F \in [X]^{<\omega}$. Since $\mathbf{0} \in \overline{B}$, choose $\mathbf{x} \in B \cap [\mathbf{0}, F, 1/2^n]$. It follows that $F \subseteq \mathbf{x}^{\leftarrow}[(-1/2^n, 1/2^n)] \in \mathcal{U}_n(B)$, so $\mathcal{U}_n(B)$ is an $\omega$-cover of $X$.

Consider the sequence of blades $\langle B_0, B_1, \ldots \rangle \in \Omega_{C_p(X),\mathbf{0}}^{\omega}$, and the corresponding sequence of $\omega$-covers $\langle \mathcal{U}_0(B_0), \mathcal{U}_1(B_1), \ldots \rangle \in \Omega_X^{\omega}$. For $U \in \mathcal{U}_n(B_n)$, choose $\mathbf{x}_U \in B_n$ such that $U = \mathbf{x}_U^{\leftarrow}\left[\left(-\frac{1}{2^n}, \frac{1}{2^n}\right)\right]$.

Assuming $X$ is $\Omega M$, choose a witness $\langle \mathcal{F}_0, \mathcal{F}_1, \ldots \rangle$ such that $\mathcal{F}_n \in [\mathcal{U}_n(B_n)]^{<\omega}$ and

$$\bigcup_{i \leq n < \omega} \mathcal{F}_n$$

is an $\omega$-cover of $X$ for all $i < \omega$. Let

$$F_n = \{\mathbf{x}_U : U \in \mathcal{F}_n\}.$$

We claim that $\mathbf{0} \in \overline{\bigcup_{n<\omega} F_n}$. Let $G \in [X]^{<\omega}, \epsilon > 0$. Choose $i < \omega$ such that $\frac{1}{2^i} < \epsilon$. Since $\bigcup_{i \leq n < \omega} \mathcal{F}_n$ is an $\omega$-cover, we may choose $i \leq n < \omega$ and $U \in \mathcal{F}_n$ such that

$$G \subseteq U = \mathbf{x}_U^{\leftarrow}\left[\left(-\frac{1}{2^n}, \frac{1}{2^n}\right)\right] \subseteq \mathbf{x}_U^{\leftarrow}[(-\epsilon, \epsilon)].$$

It follows that $\mathbf{x}_U \in F_n \cap [\mathbf{0}, G, \epsilon]$, and thus every basic open neighborhood of $\mathbf{0}$ intersects $\bigcup_{n<\omega} F_n$.

Assuming $X$ is $\Omega M^+$, choose a witness $\sigma$ such that

$$\bigcup_{i \leq n < \omega} \sigma(\langle \mathcal{U}_0(B_0), \ldots, \mathcal{U}_n(B_n) \rangle)$$

is an $\omega$-cover of $X$ for all $i < \omega$. Let

$$\tau(\langle B_0, \ldots, B_n \rangle) = \{\mathbf{x}_U : U \in \sigma(\langle \mathcal{U}_0(B_0), \ldots, \mathcal{U}_n(B_n) \rangle)\}.$$

We claim that $\mathbf{0} \in \overline{\bigcup_{n<\omega} \tau(\langle B_0, \ldots, B_n \rangle)}$. Let $G \in [X]^{<\omega}, \epsilon > 0$. Choose $i < \omega$ such that $\frac{1}{2^i} < \epsilon$. Since $\bigcup_{i \leq n < \omega} \sigma(\langle \mathcal{U}_0(B_0), \ldots, \mathcal{U}_n(B_n) \rangle)$ is an $\omega$-cover, we may choose $i \leq n < \omega$ and $U \in \sigma(\langle \mathcal{U}_0(B_0), \ldots, \mathcal{U}_n(B_n) \rangle)$ such that

$$G \subseteq U = \mathbf{x}_U^{\leftarrow}\left[\left(-\frac{1}{2^n}, \frac{1}{2^n}\right)\right] \subseteq \mathbf{x}_U^{\leftarrow}[(-\epsilon, \epsilon)].$$

It follows that $\mathbf{x}_U \in \tau(\langle B_0, \ldots, B_n \rangle) \cap [\mathbf{0}, G, \epsilon]$, and thus every basic open neighborhood of $\mathbf{0}$ intersects $\bigcup_{n<\omega} \tau(\langle B_0, \ldots, B_n \rangle)$.

Assuming $X$ is $\Omega M^{+\mathrm{mark}}$, choose a witness $\sigma$ such that

$$\bigcup_{i \leq n < \omega} \sigma(\mathcal{U}_n(B_n), n)$$

is an $\omega$-cover of $X$ for all $i < \omega$. Let

$$\tau(B_n, n) = \{\mathbf{x}_U : U \in \sigma(\mathcal{U}_n(B_n), n)\}.$$

We claim that $\mathbf{0} \in \overline{\bigcup_{n<\omega} \tau(B_n, n)}$. Let $G \in [X]^{<\omega}, \epsilon > 0$. Choose $i < \omega$ such that $\frac{1}{2^i} < \epsilon$. Since $\bigcup_{i \leq n < \omega} \sigma(\mathcal{U}_n(B_n), n)$ is an $\omega$-cover, we may choose $i \leq n < \omega$ and $U \in \sigma(\mathcal{U}_n(B_n), n)$ such that

$$G \subseteq U = \mathbf{x}_U^{\leftarrow}\left[\left(-\frac{1}{2^n}, \frac{1}{2^n}\right)\right] \subseteq \mathbf{x}_U^{\leftarrow}[(-\epsilon, \epsilon)].$$

It follows that $\mathbf{x}_U \in \tau(B_n, n) \cap [\mathbf{0}, G, \epsilon]$, and thus every basic open neighborhood of $\mathbf{0}$ intersects $\bigcup_{n<\omega} \tau(B_n, n)$.                                                  $\square$

**Lemma 21.** *Let $X$ be a $T_{3\frac{1}{2}}$ topological space. If $C_p(X)$ is $CDFT_{\mathbf{0}}$ (resp. $CDFT_{\mathbf{0}}^+$, $CDFT_{\mathbf{0}}^{+\mathrm{mark}}$), then $X$ is $\Omega M$ (resp. $\Omega M^+$, $\Omega M^{+\mathrm{mark}}$).*

*Proof.* For each $\mathcal{U} \in \Omega_X$ define

$$D(\mathcal{U}) = \{\mathbf{y} \in C_p(X) : \mathbf{y}[X \setminus U_{\mathbf{y}, \mathcal{U}}] = \{1\} \text{ for some } U_{\mathbf{y}, \mathcal{U}} \in \mathcal{U}\}.$$

Consider the point $\mathbf{x} \in C_p(X)$ and its basic open neighborhood $[\mathbf{x}, G, \epsilon]$. If $\mathcal{U}$ is an $\omega$-cover of $X$, $G \subseteq U$ for some $U \in \mathcal{U}$. Since $X$ is $T_{3\frac{1}{2}}$, $X \setminus U$ is closed, and $G$ is finite and disjoint from $X \setminus U$, we may choose some function $\mathbf{y} \in C_p(X)$ where $\mathbf{y}[X \setminus U] = \{1\}$ and $\mathbf{x}(t) = \mathbf{y}(t)$ for each $t \in G$. It follows $\mathbf{y} \in [\mathbf{x}, G, \epsilon] \cap D(\mathcal{U})$, so $D(\mathcal{U})$ is dense in $C_p(X)$.

Consider the sequence of $\omega$-covers $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle \in \Omega_X^\omega$, and the corresponding sequence of dense subsets $\langle D(\mathcal{U}_0), D(\mathcal{U}_1), \ldots \rangle \in \mathcal{D}_{C_p(X)}^\omega$.

Assuming $C_p(X)$ is $CDFT_{\mathbf{0}}$, choose a witness $\langle F_0, F_1, \ldots \rangle$ such that

$$\mathbf{0} \in \overline{\bigcup_{n<\omega} F_n}.$$

Now let

$$\mathcal{F}_n = \{U_{\mathbf{y}, \mathcal{U}_n} : \mathbf{y} \in F_n\} \in [\mathcal{U}_n]^{<\omega}.$$

We claim that $\bigcup_{n<\omega} \mathcal{F}_n$ is an $\omega$-cover. Let $G \in [X]^{<\omega}$. The neighborhood $[\mathbf{0}, G, \frac{1}{2}]$ contains some point $\mathbf{y} \in F_n$ for some $n < \omega$. It follows that $U_{\mathbf{y}, \mathcal{U}_n} \in \mathcal{U}_n$ and $\mathbf{y}[X \setminus U_{\mathbf{y}, \mathcal{U}_n}] = \{1\}$. It follows that $G \cap (X \setminus U_{\mathbf{y}, \mathcal{U}_n}) = \emptyset$, and therefore $G \subseteq U_{\mathbf{y}, \mathcal{U}_n} \in \mathcal{F}_n$.

Assuming $C_p(X)$ is $CDFT_{\mathbf{0}}^+$, choose a witness $\sigma$ such that

$$\mathbf{0} \in \overline{\bigcup_{n<\omega} \sigma(\langle D(\mathcal{U}_0), \ldots, D(\mathcal{U}_n)\rangle)}.$$

Now let

$$\tau(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n\rangle) = \{U_{\mathbf{y}, \mathcal{U}_n} : \mathbf{y} \in \sigma(\langle D(\mathcal{U}_0), \ldots, D(\mathcal{U}_n)\rangle)\} \in [\mathcal{U}_n]^{<\omega}.$$

We claim that $\bigcup_{n<\omega} \tau(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n\rangle)$ is an $\omega$-cover. Let $G \in [X]^{<\omega}$. The neighborhood $[\mathbf{0}, G, \frac{1}{2}]$ contains some point $\mathbf{y} \in \sigma(\langle D(\mathcal{U}_0), \ldots, D(\mathcal{U}_n)\rangle)$ for some $n < \omega$. It follows that $U_{\mathbf{y}, \mathcal{U}_n} \in \mathcal{U}_n$ and $\mathbf{y}[X \setminus U_{\mathbf{y}, \mathcal{U}_n}] = \{1\}$. As a result $G \cap (X \setminus U_{\mathbf{y}, \mathcal{U}_n}) = \emptyset$, and therefore $G \subseteq U_{\mathbf{y}, \mathcal{U}_n} \in \tau(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n\rangle)$.

Assuming $C_p(X)$ is $CDFT_{\mathbf{0}}^{+\mathrm{mark}}$, choose a witness $\sigma$ such that

$$\mathbf{0} \in \overline{\bigcup_{n<\omega} \sigma(D(\mathcal{U}_n), n)}.$$

Now let

$$\tau(\mathcal{U}_n, n) = \{U_{\mathbf{y}, \mathcal{U}_n} : \mathbf{y} \in \sigma(D(\mathcal{U}_n), n)\} \in [\mathcal{U}_n]^{<\omega}.$$

We claim that $\bigcup_{n<\omega} \tau(\mathcal{U}_n, n)$ is an $\omega$-cover. Let $G \in [X]^{<\omega}$. The neighborhood $[\mathbf{0}, G, \frac{1}{2}]$ contains some point $\mathbf{y} \in \sigma(D(\mathcal{U}_n), n)$ for some $n < \omega$. It follows that $U_{\mathbf{y}, \mathcal{U}_n} \in \mathcal{U}_n$ and $\mathbf{y}[X \setminus U_{\mathbf{y}, \mathcal{U}_n}] = \{1\}$. As a result $G \cap (X \setminus U_{\mathbf{y}, \mathcal{U}_n}) = \emptyset$, and therefore $G \subseteq U_{\mathbf{y}, \mathcal{U}_n} \in \tau(\mathcal{U}_n, n)$. $\qquad\square$

**Theorem 22.** *The following are equivalent for any $T_{3\frac{1}{2}}$ topological space $X$.*

- *$X$ is $\Omega M$ (resp. $\Omega M^+$, $\Omega M^{+\mathrm{mark}}$).*
- *$C_p(X)$ is $CFT$ (resp. $CFT^+$, $CFT^{+\mathrm{mark}}$).*
- *$C_p(X)$ is $CDFT$ (resp. $CDFT^+$, $CDFT^{+\mathrm{mark}}$).*

*Proof.* Since $\mathcal{D}_X \subseteq \Omega_{X,x}$, the second condition trivially implies the third. As $C_p(X)$ is homogeneous, the $C(D)FT$ properties follow from $C(D)FT_{\mathbf{0}}$. So the result follows from the previous lemmas. $\qquad\square$

At this point, it's worth noting the following open question asked by Angelo Bella in personal correspondance with the author. The following term is defined in [8].

**Definition 23.** A space is *dense-separable* if every dense subspace is separable.

In dense-separable spaces, it is safe to assume the first player only chooses countable dense subsets in the $SS$ and $CDFT$ principles/games.

**Question 24.** *Suppose $X$ is a dense-separable space such that every countable dense subspace $D \subseteq X$ is $SS^+$. Is $X$ itself $SS^+$?*

Bella noted that this is certainly true for non-strategic $SS$, as verified below. Note the use of knowledge of all $D_n$ to define $E_n$ in this proof, demonstrating why it cannot be trivially modified to the $SS^+$ case.

**Proposition 25.** *Let $X$ be a dense-separable space such that every countable dense subspace $D \subseteq X$ is $SS$. Then $X$ is $SS$.*

*Proof.* Take $\{D_n : n < \omega\}$ with $D_n \subseteq X$ countable dense, so $E_n = \bigcup\{D_m : n \leq m < \omega\}$ is countable dense in the $SS$ subspace $D = \bigcup\{D_n : n < \omega\}$. So choose $F_n \in [E_n]^{<\omega}$ with $cl_D(\bigcup\{F_n : n < \omega\}) = D$ and therefore $cl(\bigcup\{F_n : n < \omega\}) = X$.

Define $G_n \in [D_n]^{<\omega}$ by $G_n = D_n \cap \bigcup\{F_m : m \leq n\}$. The result follows as $\bigcup\{G_n : n < \omega\} = \bigcup\{F_n : n < \omega\}$. $\qquad\square$

## 3. A space which is $CFT^+$ but not $CDFT^{+\mathrm{mark}}$

Our goal is to now demonstrate a space which is $CFT^+$, but not $CFT^{+\mathrm{mark}}$ or even $CDFT^{+\mathrm{mark}}$. To do so, we will find a space $X$ which is $\Omega M^+$ but not $\Omega M^{+\mathrm{mark}}$, yielding $C_p(X)$ as our example.

**Proposition 26.** *A space $X$ is compact if and only if for each $\omega$-cover $\mathcal{U}$ of $X$ and $n < \omega$, there exists a finite subcollection $\mathcal{F} \in [\mathcal{U}]^{<\omega}$ such that for each $F \in [X]^{\leq n}$ there exists some $U \in \mathcal{F}$ where $F \subseteq U$.*

*Proof.* Let $X$ and therefore $X^n$ be compact. Let $\mathcal{F}_n$ be the finite subcover of the open cover $\mathcal{U}_n = \{U^n : U \in \mathcal{U}\}$. Then $\mathcal{F} = \{U : U^n \in \mathcal{F}_n\}$ satisfies that for each $F \in [X]^{\leq n}$ there exists some $U \in \mathcal{F}$ where $F \subseteq U$.

Conversely, assume that for each $\omega$-cover $\mathcal{U}$ of $X$ and $n < \omega$, there exists a finite subcollection $\mathcal{F} \in [\mathcal{U}]^{<\omega}$ such that for each $F \in [X]^{\leq n}$ there exists some $U \in \mathcal{F}$

where $F \subseteq U$. Given an arbitrary open cover $\mathcal{V}$, close it under finite unions to obtain an $\omega$-cover $\mathcal{U}$. Then for $n = 1$, the finite subcollection $\mathcal{F}$ of $\mathcal{U}$ satisfying $\{x\} \subseteq U$ for all $x \in X$ and some $U \in \mathcal{U}$ is of course a finite subcover. Since each member of $\mathcal{F}$ is a finite union of open sets in $\mathcal{V}$, there exists a finite subcover $\mathcal{G}$ of $\mathcal{V}$. $\qquad\square$

**Lemma 27.** *The following are equivalent for a regular topological space $X$:*

- *$X$ is $\sigma$-compact*
- *$X$ is $\Omega M^{+\mathrm{mark}}$*
- *$X$ is $M^{+\mathrm{mark}}$*

*Proof.* The equivalence of $\sigma$-compact and $M^{+\mathrm{mark}}$ in regular spaces was shown in [4]. As $\Omega M^{+\mathrm{mark}}$ trivially implies $M^{+\mathrm{mark}}$, we need only demonstrate that if $X = \bigcup_{n<\omega} K_n$ for $K_n$ compact and increasing, then $X$ is $\Omega M^{+\mathrm{mark}}$.

We define $\sigma(\mathcal{U}, n)$ for each $\omega$-cover $\mathcal{U}$ and $n < \omega$ to witness Proposition 26 for $\mathcal{U}$, $K_n$, and $n$. It follows that for every sequence of $\omega$-covers $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ and $F \in [X]^{<\omega}$, $F \in [K_n]^{\leq n}$ for some $n \geq |F|$, and thus there exists some $U \in \sigma(\mathcal{U}_n, n)$ where $F \subseteq U$. Therefore $\bigcup_{n<\omega} \sigma(\mathcal{U}, n)$ is an $\omega$-cover of $X$. $\qquad\square$

The reader may note that with this lemma, we may view Theorem 22 as a generalization of [3, Proposition 2.6].

**Definition 28.** Let $X$ be a topological space such that all countable sets are closed. Then $L(X) = X \cup \{\infty\}$ is the *one-point Lindelöf-ication* of $X$, with a basis given by the usual open sets of $X$ and the co-countable sets containing $\infty$.

**Theorem 29.** *Let $\omega_1$ have the discrete topology. Then $L(\omega_1)$ is $\Omega M^+$ but not $\Omega M^{+\mathrm{mark}}$.*

*Proof.* The proof that $X = L(\omega_1)$ is not $\sigma$-compact (and therefore not $\Omega M^{+\mathrm{mark}}$) follows from the fact that its countably infinite subsets are closed and discrete, so all of its compact subsets are finite.

To show $\Omega M^+$, we define the strategy $\sigma$ for II in $G_{fin}(\Omega_X, \Omega_X)$ as follows. For a given partial play $\langle \mathcal{U}_0, \dots, \mathcal{U}_n \rangle$, suppose $\{\alpha_{i,m} : i < n, m < \omega\} \subseteq \omega_1$ has already been defined. Then as $\mathcal{U}_n$ is an $\omega$-cover, we may choose $U(\langle \mathcal{U}_0, \dots, \mathcal{U}_n \rangle) \in \mathcal{U}_n$ that contains $\{\infty\} \cup \{\alpha_{i,j} : i, j < n\}$. Since $U(\langle \mathcal{U}_0, \dots, \mathcal{U}_n \rangle)$ is co-countable, define $\{\alpha_{n,m} : m < \omega\}$ such that $U(\langle \mathcal{U}_0, \dots, \mathcal{U}_n \rangle) = L(\omega_1) \setminus \{\alpha_{n,m} : m < \omega\}$. Then let $\sigma(\langle \mathcal{U}_0, \dots, \mathcal{U}_n \rangle) = \{U(\langle \mathcal{U}_0, \dots, \mathcal{U}_n \rangle)\}$.

Consider the arbitrary sequence of moves $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ by I. For $F \in [L(\omega_1)]^{<\omega}$, choose $n < \omega$ such that

$$F \cap \{\alpha_{i,j} : i, j < \omega\} = F \cap \{\alpha_{i,j} : i, j < n\} \subseteq U(\langle \mathcal{U}_0, \dots, \mathcal{U}_n \rangle)$$

It follows that as

$$F \setminus \{\alpha_{i,j} : i, j < \omega\} \subseteq F \setminus \{\alpha_{n,m} : m < \omega\} \subseteq U(\langle \mathcal{U}_0, \dots, \mathcal{U}_n \rangle)$$

$F$ is a subset of $U(\langle \mathcal{U}_0, \dots, \mathcal{U}_n \rangle)$, making $\bigcup_{n<\omega} \sigma(\langle \mathcal{U}_0, \dots, \mathcal{U}_n \rangle)$ an $\omega$-cover. $\qquad\square$

Note that the preceding result could be worded more strongly: $L(\omega_1)$ is strategically $\Omega$-Rothberger, since the winning strategy only uses a single set of the $\omega$-cover during each round. This topic will be discussed further in a later section.

**Corollary 30.** $C_p(L(\omega_1))$ *is $CFT^+$ but not $CDFT^{+\mathrm{mark}}$.*

If $C_p(L(\omega_1))$ were separable, it would be a negative solution to Question 14. However, it is not.

**Lemma 31** ([9])**.** *For a $T_{3\frac{1}{2}}$ topological space $X$, $C_p(X)$ is separable if and only if $X$ has a coarser separable metrizable topology.*

**Corollary 32.** *$C_p(L(\omega_1))$ is not separable.*

*Proof.* Consider a topology coarser than the topology of $L(\omega_1)$. Since open neighborhoods of $\infty$ in this topology are co-countable, a countable intersection of $\infty$ neighborhoods is not $\{\infty\}$. Since every metrizable topological space has points $G_\delta$, this topology cannot be metrizable. $\square$

An affirmative answer to either of these questions would answer Question 14 negatively.

**Question 33.** *Does there exist a separable subspace of $C_p(L(\omega_1))$ which is not $CDFT^{+\mathrm{mark}}$?*

**Question 34.** *Does there exist a $T_{3\frac{1}{2}}$, non-$\sigma$-compact, and $\Omega M^+$ space with a coarser separable metrizable topology?*

We conclude this section by covering a related result observed by the anonymous referee in their initial review of this manuscript.

**Theorem 35.** *A topological space is $M^+$ if and only if it is $\Omega M^+$.*

*Proof.* Assume $X$ is $M^+$. In [5, Proposition 5.1] it is shown that that $X^m$ must be $M^+$ for all powers $m < \omega$, so we shall proceed in the spirit of Proposition 26 and its following lemma. Given an $\omega$-cover $\mathcal{U}$, it follows that $(\mathcal{U})^m = \{U^m : U \in \mathcal{U}\}$ is an open cover of $X^m$. Given a finite subset $\mathcal{F} \in [(\mathcal{U})^m]^{<\omega}$, let $\pi_m \mathcal{F} = \{U : U^m \in \mathcal{F}\} \in [\mathcal{U}]^{<\omega}$.

Let $\sigma_m$ witness $M^+$ for $X^m$, and define the following strategy for II in $G_{fin}(\Omega, \Omega)$:

$$\tau(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle) = \bigcup_{m \leq n} \pi_m \sigma_m(\langle (\mathcal{U}_m)^m, \ldots, (\mathcal{U}_n)^m \rangle).$$

Since

$$\bigcup_{m \leq n < \omega} \sigma_m(\langle (\mathcal{U}_m)^m, \ldots, (\mathcal{U}_n)^m \rangle)$$

is an open cover of $X^m$ for $m < \omega$, it follows that

$$\bigcup_{n < \omega} \tau(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle) \supseteq \bigcup_{m \leq n < \omega} \pi_m \sigma_m(\langle (\mathcal{U}_m)^m, \ldots, (\mathcal{U}_n)^m \rangle)$$

covers all finite sets of size $m < \omega$, that is, $\bigcup_{n<\omega} \tau(\langle \mathcal{U}_0, \ldots, \mathcal{U}_n \rangle)$ is an $\omega$-cover, making $\tau$ a winning strategy. $\square$

**Corollary 36.** *The following are equivalent for any $T_{3\frac{1}{2}}$ topological space $X$.*
- *$X$ is $M^+$*
- *$X$ is $\Omega M^+$*
- *$C_p(X)$ is $CFT^+$*
- *$C_p(X)$ is $CDFT^+$*

## 4. Equivalence of certain strategic and Markov selection properties

Barman and Dow previously demonstrated that $SS^+$ is equivalent to $SS^{+\mathrm{mark}}$ amongst countable spaces. A similar result by the author showed that $M^+$ is equivalent to $M^{+\mathrm{mark}}$ in second-countable spaces. The following result generalizes both.

**Lemma 37.** *Let $\mathcal{A}$ be a family of sets where $|\bigcup \mathcal{A}| \leq \omega$ and let $\mathcal{B}$ be a family of sets such that $B \in \mathcal{B}$ and $B \subseteq B' \subseteq \bigcup \mathcal{A}$ implies $B' \in \mathcal{B}$. Then $\mathrm{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ if and only if $\mathrm{II} \underset{mark}{\uparrow} G_{fin}(\mathcal{A}, \mathcal{B})$.*

*Proof.* Let $\sigma$ witness $\mathrm{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$.

For $t \in \omega^{<\omega}$, suppose $Z_s \in \mathcal{A}$ has been defined for all nonempty $s \leq t$. We use the convention $\langle Z_{\emptyset \upharpoonright 1}, Z_{\emptyset \upharpoonright 2}, \ldots, Z_\emptyset, A \rangle = \langle A \rangle$. Note then that

$$\{\sigma(\langle Z_{t \upharpoonright 1}, Z_{t \upharpoonright 2}, \ldots, Z_t, A \rangle) : A \in \mathcal{A}\} \subseteq \left[\bigcup \mathcal{A}\right]^{<\omega}$$

and therefore is countable. So choose $Z_{t^\frown \langle n \rangle} \in \mathcal{A}$ for $n < \omega$ such that

$$\{\sigma(\langle Z_{t \upharpoonright 1}, Z_{t \upharpoonright 2}, \ldots, Z_t, A \rangle) : A \in \mathcal{A}\} = \{\sigma(\langle Z_{t \upharpoonright 1}, Z_{t \upharpoonright 2}, \ldots, Z_t, Z_{t^\frown \langle n \rangle} \rangle) : n < \omega\}.$$

Let $b : \omega \to \omega^{<\omega}$ be a bijection, and define $\tau(A, n) = \sigma(\langle Z_{b(n) \upharpoonright 1}, Z_{b(n) \upharpoonright 2}, \ldots, Z_{b(n)}, A \rangle)$. Consider $\langle A_0, A_1, \ldots \rangle \in \mathcal{A}^\omega$.

Define $f \in \omega^\omega$ as follows. Let $n < \omega$ and suppose that $f \upharpoonright n$ has already been defined. Then choose $f(n) < \omega$ such that $\sigma(\langle Z_{f \upharpoonright 1}, Z_{f \upharpoonright 2}, \ldots, Z_{f \upharpoonright n}, A_{b \leftarrow (f \upharpoonright n)} \rangle) = \sigma(\langle Z_{f \upharpoonright 1}, Z_{f \upharpoonright 2}, \ldots, Z_{f \upharpoonright n}, Z_{f \upharpoonright (n+1)} \rangle)$.

Since $\langle Z_{f \upharpoonright 1}, Z_{f \upharpoonright 2}, \ldots \rangle \in \mathcal{A}^\omega$, it follows that $\bigcup_{n < \omega} \sigma(\langle Z_{f \upharpoonright 1}, Z_{f \upharpoonright 2}, \ldots, Z_{f \upharpoonright n} \rangle) \in \mathcal{B}$. Let $b(n) \prec f$ denote that $b(n) \in \omega^{<\omega}$ is a finite restriction of $f \in \omega^\omega$. The result then follows from

$$\begin{aligned}
\bigcup_{n < \omega} \tau(A_n, n) &= \bigcup_{n < \omega} \sigma(\langle Z_{b(n) \upharpoonright 1}, Z_{b(n) \upharpoonright 2}, \ldots, Z_{b(n)}, A_n \rangle) \\
&\supseteq \bigcup_{n < \omega, b(n) \prec f} \sigma(\langle Z_{b(n) \upharpoonright 1}, Z_{b(n) \upharpoonright 2}, \ldots, Z_{b(n)}, A_n \rangle) \\
&= \bigcup_{n < \omega, b(n) \prec f} \sigma(\langle Z_{f \upharpoonright 1}, Z_{f \upharpoonright 2}, \ldots, Z_{f \upharpoonright |b(n)|}, A_{b \leftarrow (f \upharpoonright |b(n)|)} \rangle) \\
&= \bigcup_{n < \omega, b(n) \prec f} \sigma(\langle Z_{f \upharpoonright 1}, Z_{f \upharpoonright 2}, \ldots, Z_{f \upharpoonright |b(n)|}, Z_{f \upharpoonright (|b(n)|+1)} \rangle) \\
&= \bigcup_{n < \omega} \sigma(\langle Z_{f \upharpoonright 1}, Z_{f \upharpoonright 2}, \ldots, Z_{f \upharpoonright n} \rangle) \\
&\in \mathcal{B}
\end{aligned}$$

as $\mathcal{B}$ is closed under supersets. $\qquad \square$

**Corollary 38** ([3])**.** *A countable space is $SS^+$ if and only if it is $SS^{+\mathrm{mark}}$.*

*Proof.* $\bigcup \mathcal{D}_X = X$ is countable, and any set containing a dense set is dense. $\qquad \square$

**Corollary 39.** *A second-countable space is $\Omega M^+$ if and only if it is $\Omega M^{+\mathrm{mark}}$.*

*Proof.* Fix a countable base $\mathcal{B}$ for $X$ closed under finite unions, and let $\Omega_X^*$ be the collection of $\omega$-covers of $X$ which only use basic open sets. One may quickly verify that $\Omega M^+$ implies $\mathrm{II} \uparrow G_{fin}(\Omega_X^*, \Omega_X^*)$.

Note now that $\bigcup \Omega_X^* = \mathcal{B}$ is countable, and any set containing an $\omega$-cover is an $\omega$-cover; therefore II $\uparrow G_{fin}(\Omega_X^*, \Omega_X^*)$ if and only if II $\underset{\text{mark}}{\uparrow} G_{fin}(\Omega_X^*, \Omega_X^*)$.

So let $\sigma$ witness II $\underset{\text{mark}}{\uparrow} G_{fin}(\Omega_X^*, \Omega_X^*)$. For each $\omega$-cover $\mathcal{U}$ and finite set $F \in [X]^{<\omega}$, choose $U_{\mathcal{U},F} \in \mathcal{U}$ and a basic open set $B_{\mathcal{U},F}$ such that $F \subseteq B_{\mathcal{U},F} \subseteq U_{\mathcal{U},F}$. Finally, for each $\omega$-cover $\mathcal{U}$, let

$$\mathcal{U}^* = \{B_{\mathcal{U},F} : F \in [X]^{<\omega}\} \in \Omega_X^*$$

be an $\omega$-cover of basic open sets refining $\mathcal{U}$.

Given $\mathcal{U}$ and $V \in \mathcal{U}^*$, choose $F(V) \in [X]^{<\omega}$ such that $V = B_{\mathcal{U},F(V)}$. Define the strategy $\tau$ such that $\tau(\mathcal{U}, n) = \{U_{\mathcal{U},F(V)} : V \in \sigma(\mathcal{U}^*, n)\}$. For any sequence of $\omega$-covers $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$, it follows that $\bigcup_{n<\omega} \sigma(\mathcal{U}_n^*, n)$ is an $\omega$-cover refining $\bigcup_{n<\omega} \tau(\mathcal{U}_n, n)$, and therefore $\bigcup_{n<\omega} \tau(\mathcal{U}_n, n)$ is an $\omega$-cover also, demonstrating that $X$ is $\Omega M^{+\text{mark}}$. □

**Corollary 40** ([4])**.** *A second-countable space is $M^+$ if and only if it is $M^{+\text{mark}}$.*

*Proof.* An easy adaptation of the preceding proof, replacing $\omega$-covers with open covers, and replacing $F \in [X]^{<\omega}$ with $x \in X$. □

## 5. Strong variants of selection principles and games

Let $S_1(\mathcal{A}, \mathcal{B}), G_1(\mathcal{A}, \mathcal{B})$ be the natural variants of $S_{fin}(\mathcal{A}, \mathcal{B}), G_{fin}(\mathcal{A}, \mathcal{B})$ where each choice by II must either be a single element or singleton (whichever is more convenient for the proof at hand), rather than a finite set. Convention calls for denoting these as *strong* versions of the corresponding selection principles and games, although the "strong Menger" property is commonly known as "Rothberger" who first investigated the principle in [11]. We will thus call "strong $\Omega$-Menger" "$\Omega$-Rothberger" and shorten it with $\Omega R$, and otherwise attach the prefix "s" when abbreviating to all other strong variants.

**Theorem 41.** *The following are equivalent for any topological space $X$.*
- *$X$ is sSS (resp. $sSS^+$, $sSS^{+\text{mark}}$).*
- *$X$ is separable and sCDFT (resp. $sCDFT^+$, $sCDFT^{+\text{mark}}$).*
- *$X$ has a countable dense subset $D$ where $sCDFT_x$ (resp. $sCDFT_x^+$, $sCDFT_x^{+\text{mark}}$) holds for all $x \in D$.*

*Proof.* We need only show that the final condition implies the first. Let $D = \{d_i : i < \omega\}$.

Let $\{D_{m,n} \in \mathcal{D}_X : m, n < \omega\}$, and let $x_{i,n} \in D_{i,n}$ witness $sCDFT_{d_i}$, so

$$d_i \in \overline{\{x_{i,n} : n < \omega\}} \subseteq \overline{\{x_{m,n} : m, n < \omega\}}$$

and as $D \subseteq \overline{\{x_{m,n} : m, n < \omega\}}$ it follows that

$$X \subseteq \overline{D} \subseteq \overline{\overline{\{x_{m,n} : m, n < \omega\}}} = \overline{\{x_{m,n} : m, n < \omega\}}.$$

Therefore $x_{m,n} \in D_{m,n}$ witnesses $sSS$.

Let $\sigma_i$ be a witness for $sCDFT_{d_i}^+$ for each $i < \omega$. Define $p : \omega \to \omega$ partitioning $\omega$ such that $|p^{\leftarrow}(i)| = \omega$ for all $i < \omega$. Essentially, we will partition each attack by I according to $p$ into $\omega$-many subattacks, and defeat each subattack to produce a blade of $d_n$.

For a nonempty finite sequence $t$, let $t'$ be its subsequence deleting all terms of index $n$ where $p(n) \neq p(|t| - 1)$. Note that the final term of $t$ is the final term of $t'$. Thus the function $\tau$ defined by

$$\tau(t) = \sigma_{p(|t|-1)}(t')$$

is a legal strategy.

Let $\alpha \in \mathcal{D}_X^\omega$ be an attack by I, and let $\alpha_i$ be its subsequence deleting all terms of index $n$ where $p(n) \neq i$. By $sCDFT_{d_i}^+$, we have

$$d_i \in \overline{\{\sigma_i(\alpha_i \upharpoonright (n+1)) : n < \omega\}} = \overline{\{\tau(\alpha \upharpoonright (n+1)) : n \in p^\leftarrow(i)\}} \subseteq \overline{\{\tau(\alpha \upharpoonright (n+1)) : n < \omega\}}$$

and as $D \subseteq \overline{\{\tau(\alpha \upharpoonright (n+1)) : n < \omega\}}$ it follows that

$$X \subseteq \overline{D} \subseteq \overline{\overline{\{\tau(\alpha \upharpoonright (n+1)) : n < \omega\}}} = \overline{\{\tau(\alpha \upharpoonright (n+1)) : n < \omega\}}.$$

Therefore $\tau$ witnesses $SS^+$.

Let $\sigma_i$ be a witness for $sCDFT_{d_i}^{+\text{mark}}$ for each $i < \omega$. Again, we will partition each attack by I according to $p$ into $\omega$-many subattacks, and defeat each subattack to produce a blade of $d_n$.

We define the Markov strategy $\tau$ by

$$\tau(D, n) = \sigma_{p(n)}(D, |\{m < n : p(m) = p(n)\}|).$$

Let $\alpha \in \mathcal{D}_X^\omega$, and let $\alpha_i$ be its subsequence removing all terms of index $n$ such that $p(n) \neq i$. By $sCDFT_{d_i}^{+\text{mark}}$, we have

$$d_i \in \overline{\{\sigma_i(\alpha_i(n), n) : n < \omega\}} = \overline{\{\tau(\alpha(n), n) : n \in p^\leftarrow(i)\}} \subseteq \overline{\{\tau(\alpha(n), n) : n < \omega\}}$$

and as $D \subseteq \overline{\{\tau(\alpha(n), n) : n < \omega\}}$ it follows that

$$X \subseteq \overline{D} \subseteq \overline{\overline{\{\tau(\alpha(n), n) : n < \omega\}}} = \overline{\{\tau(\alpha(n), n) : n < \omega\}}.$$

Therefore $\tau$ witnesses $SS^{+\text{mark}}$. $\qquad\square$

As mentioned earlier, the following is a result of Sakai:

**Theorem 42** ([12])**.** *The following are equivalent for any $T_{3\frac{1}{2}}$ topological space $X$.*

- *$X$ is $\Omega R$.*
- *$C_p(X)$ is $sCFT$.*
- *$C_p(X)$ is $sCDFT$.*

The corresponding game-theoretic results also hold.

**Theorem 43.** *The following are equivalent for any $T_{3\frac{1}{2}}$ topological space $X$.*

- *$X$ is $\Omega R$ (resp. $\Omega R^+$, $\Omega R^{+\text{mark}}$).*
- *$C_p(X)$ is $sCFT$ (resp. $sCFT^+$, $sCFT^{+\text{mark}}$).*
- *$C_p(X)$ is $sCDFT$ (resp. $sCDFT^+$, $sCDFT^{+\text{mark}}$).*

*Proof.* In the proof of Lemmas 20 and 21, if the strategy $\sigma$ produces singletons, so does the strategy $\tau$. $\qquad\square$

**Theorem 44.** *Let $\omega_1$ have the discrete topology. Then $L(\omega_1)$ is $\Omega R^+$ but not $\Omega M^{+\text{mark}}$.*

*Proof.* The strategy constructed in Theorem 29 is a witness. $\qquad\square$

**Corollary 45.** *$C_p(L(\omega_1))$ is $sCFT^+$ but not $CDFT^{+\text{mark}}$.*

**Lemma 46.** *Let $\mathcal{A}$ be a family of sets where $|\bigcup \mathcal{A}| \leq \omega$ and let $\mathcal{B}$ be a family of sets such that $B \in \mathcal{B}$ and $B \subseteq B' \subseteq \bigcup \mathcal{A}$ implies $B' \in \mathcal{B}$. Then $\mathrm{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$ if and only if $\mathrm{II} \underset{mark}{\uparrow} G_1(\mathcal{A}, \mathcal{B})$.*

*Proof.* In the proof of Lemma 37, if the perfect information strategy $\sigma$ always produces singletons, so does the Markov strategy $\tau$. $\qquad\square$

**Corollary 47.** *A countable space is $sSS^+$ if and only if it is $sSS^{+\mathrm{mark}}$.*

**Corollary 48.** *A second-countable space is $\Omega R^+$ if and only if it is $\Omega R^{+\mathrm{mark}}$.*

**Corollary 49.** *A second-countable space is $R^+$ if and only if it is $R^{+\mathrm{mark}}$.*

## 6. Acknowledgements

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