Definition 1. X is **Menger** if for all open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ there exist finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Proposition 2. *σ-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelof*

Definition 3. In the two-player game $Cov_{C,F}(X)$ player C chooses open covers $\mathcal{U}_n$ of X, followed by player F choosing a finite subcollection $\mathcal{F}_n \subseteq \mathcal{U}_n$. F wins if $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Theorem 4. *X is Menger if and only if $C \not\uparrow Cov_{C,F}(X)$.*

Proof. Result due to (???)

First, suppose X wasn't Menger. Then there would exist open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$ of X such that for any choice of finite subcollections $\mathcal{F}_n \subseteq \mathcal{U}_n$, $\bigcup_{n<\omega} \mathcal{F}_n$ isn't a cover of X. Thus $C \uparrow_{\text{pre}} Cov_{C,F}(X) \Rightarrow S \not\uparrow Cov_{C,F}(X)$.

The other direction is based upon Gruenhage's topological game presentation. Assume X is Menger, and consider a strategy for C in $Cov_{C,F}(X)$.

Since X is Lindelof, we can assume C plays only countable covers of X. Then, since F is choosing finite subsets, we may assume F chooses some initial segement of the countable cover. In turn, we can assume C plays an increasing open cover $\{U_0, U_1, \dots\}$ where $U_n \subseteq U_{n+1}$. And in that case, it's suffient to assume F simply chooses a singleton subset of each cover. And finally, since choices made by F are already covered, we can assume that every open set in a cover played by C covers the sets chosen by F previously.

As a result, we have the following figure of a tree of plays which I need to draw:

(Insert figure here.)

Note that for $a, b \in \omega^{<\omega}$ and $m \leq n$, we know:

(a) $U_{a^\frown m} \subseteq U_{a^\frown n}$
(for example, $U_{1627} \subseteq U_{1629}$ - increasing the final digit yields supersets)

(b) $U_a \subseteq U_{a^\frown b}$
(for example, $U_{1627} \subseteq U_{162789}$ - appending any sequence to the end yields supersets)

(c) $U_{a^\frown m} \subseteq U_{a^\frown n} \subseteq U_{a^\frown n^\frown b} \subseteq U_{a^\frown n^\frown b^\frown m}$
(for example: $U_{1627} \subseteq U_{1629283287}$ - injecting a subseqence with initial number larger than the original's final number, prior to the final number, yields supersets)

We may observe that if F can find an $f : \omega \to \omega$ such that $\bigcup_{n<\omega} U_{f\restriction(n+1)} = X$, she can use $\{U_{f\restriction 0}\}, \{U_{f\restriction 1}\}, \dots$ to counter C's strategy.

Let $V_k^n = \bigcap_{a \in \omega^{\leq n}} U_{a^\frown k}$. We claim that (1) V_k^n is open, (2) $\mathcal{V}^n = \{V_0^n, V_1^n, \dots\}$ is increasing, and (3) $\mathcal{V}^n$ is a cover. Proofs:

1. Since due to (c) for each $b \in \omega^{\leq n} \setminus k^{\leq n}$, there is an $a \in k^{\leq n}$ with $U_{a^\frown k} \subseteq U_{b^\frown k}$:

$$V_k^n = \bigcap_{a \in \omega^{\leq n}} U_{a^\frown k} = \bigcap_{a \in k^{\leq n}} U_{a^\frown k} \cap \bigcap_{b \in \omega^{\leq n} \setminus k^{\leq n}} U_{b^\frown k} = \bigcap_{a \in k^{\leq n}} U_{a^\frown k}$$

making V_k^n a finite intersection of open sets.

2. We show $V_k^0 \subseteq V_{k+1}^0$:

$$V_k^0 = U_k \subseteq U_{k+1} = V_{k+1}^0$$

and then assume $V_k^n \subseteq V_{k+1}^n$:

$$V_k^{n+1} = \bigcap_{a \in \omega^{\leq n+1}} U_{a^\frown k} = V_k^n \cap \bigcap_{a \in \omega^{n+1}} U_{a^\frown k} \subseteq V_{k+1}^n \cap \bigcap_{a \in \omega^{n+1}} U_{a^\frown (k+1)} = V_{k+1}^{n+1}$$

3. We easily see that $\mathcal{V}^0 = \{U_0, U_1, \dots\}$ is a cover, and then assume $\mathcal{V}^n$ is a cover.

   Let $x \in X$ and pick $l < \omega$ such that $x \in V_l^n$. For $a \in l^{n+1}$ choose l_a such that $x \in U_{a^\frown l_a}$, giving

$$x \in \bigcap_{a \in l^{n+1}} U_{a^\frown l_a}$$

   We will assume $k > l, l_a$ for all $a \in l^{\leq n+1}$.

   For any $a \in k^{n+1} \setminus l^{n+1}$ note that $a = b^\frown c$ where $b \in l^{\leq n}$ and c begins with a number l or greater:

$$V_l^n \subseteq U_{b^\frown l} \subseteq U_{b^\frown c} \subseteq U_{b^\frown c^\frown l_a} = U_{a^\frown l_a}$$

   Thus:

$$x \in V_l^n \cap \left(\bigcap_{a \in l^{n+1}} U_{a^\frown l_a} \right)$$

$$= V_l^n \cap \left(\bigcap_{a \in k^{n+1} \setminus l^{n+1}} U_{a^\frown l_a} \right) \cap \left(\bigcap_{a \in l^{n+1}} U_{a^\frown l_a} \right)$$

$$= V_l^n \cap \left(\bigcap_{a \in k^{n+1}} U_{a^\frown l_a} \right)$$

$$\subseteq V_k^n \cap \left(\bigcap_{a \in k^{n+1}} U_{a^\frown k} \right)$$

$$= V_k^{n+1}$$

Finally, apply Menger to $\mathcal{V}^n$, resulting in the cover $\{V^0_{f(0)}, V^1_{f(1)}, \dots\}$, noting

$$X = \bigcup_{n<\omega} V^n_{f(n)} \subseteq \bigcup_{n<\omega} U_{(f\restriction n)^\frown f(n)} = \bigcup_{n<\omega} U_{f\restriction(n+1)}$$

$\square$

Proposition 5. *X is compact if and only if $F \uparrow_{tact} Cov_{C,F}(X)$ if and only if $F \uparrow_{k\text{-}tact} Cov_{C,F}(X)$*

Proof. Assume X is compact. For each open cover played by C, pick a finite subcover, and this yields a winning tactical strategy.

Assume F has a winning k-tactical strategy. For any open cover, have C play only it during the entire game. F's only choice must be a finite subcover. $\square$

Proposition 6. *If X is σ-compact then $F \uparrow_{mark} Cov_{C,F}(X)$*

Proof. Let $X = \bigcup_{n<\omega} X_n$ for compact X_n. On round n, F picks the finite subcover of C's open cover of X_n. $\square$

For Menger's game, there is no useful distinction between a k-Markov strategy for F, and a 2-Markov strategy.

Theorem 7. *For any topological space X and all $k \geq 2$, $F \uparrow_{k\text{-}mark} Cov_{C,F}(X)$ if and only if $F \uparrow_{2\text{-}mark} Cov_{C,F}(X)$.*

Proof. Assume $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{k-1}, n)$ is a winning k-Markov strategy. Define the 2-Markov strategy $\tau(\mathcal{U}, \mathcal{V}, n)$ so that it contains $\sigma(\mathcal{W}_0, \dots, \mathcal{W}_{k-2}, \mathcal{V}, m)$ for the following conditions on $\mathcal{W}_0, \dots, \mathcal{W}_{k-2}, m$:

- Each $\mathcal{W}_i \in \{\mathcal{U}, \mathcal{V}\}$
- $m \leq (n+1)k$; in particular, for $i < k$,

$$\sigma(\mathcal{W}_0, \dots, \mathcal{W}_{k-2}, \mathcal{V}, (n+1)k + i) \subseteq \tau(\mathcal{U}, \mathcal{V}, n+1)$$

Considering an arbitrary play $\mathcal{U}_0, \mathcal{U}_1, \dots$ by C versus τ, we note that σ defeats the play

$$\underbrace{\mathcal{U}_0, \mathcal{U}_0, \dots, \mathcal{U}_0}_{k}, \underbrace{\mathcal{U}_1, \mathcal{U}_1, \dots, \mathcal{U}_1}_{k} \dots$$

So we have that

$$\bigcup_{i<k, n<\omega} \sigma(\underbrace{\mathcal{U}_n, \dots, \mathcal{U}_n}_{k-i-1}, \underbrace{\mathcal{U}_{n+1}, \dots, \mathcal{U}_{n+1}}_{i+1}, (n+1)k + i)$$

is a cover for X, and as

$$\sigma(\underbrace{\mathcal{U}_n, \dots, \mathcal{U}_n}_{k-i-1}, \underbrace{\mathcal{U}_{n+1}, \dots, \mathcal{U}_{n+1}}_{i+1}, (n+1)k+i) \subseteq \tau(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1)$$

τ defeats the play $\mathcal{U}_0, \mathcal{U}_1, \dots$. □

But there are spaces for which there is no Markov strategy, but there is a 2-Markov strategy.

In a question I posed to G, he answered:

Lemma 8. *For all functions $\tau : \omega_1 \times \omega \to [\omega_1]^{<\omega}$, there exists a sequence $\alpha_0, \alpha_1, \dots < \omega_1$ such that $\{\tau(\alpha_n, n) : n < \omega\}$ is not a cover for $\{\beta : \forall n < \omega(\beta < \alpha_n)\}$.*

Proof. Let $P_n = \{\beta : \beta < \alpha \Rightarrow \beta \in \tau(\alpha, n)\}$. Observe that each P_n is finite; else there is some α larger than every member of some countably infinite $P_n^* \subseteq P_n$ such that $P_n^* \subseteq \tau(\alpha, n)$.

Choose $\beta \notin \bigcup_{n<\omega} P_n$. Then for each $n < \omega$, pick $\alpha_n > \beta$ such that $\beta \notin \tau(\alpha_n, n)$. □

Note that the one-point Lindelöfication of discrete ω_1, $\omega_1^\dagger$, is not σ-compact. With the above lemma, we may see that:

Example 9. *$F \uparrow Cov_{C,F}(\omega_1^\dagger)$ but $F \not\uparrow_{mark} Cov_{C,F}(\omega_1^\dagger)$.*

Proof. First, we see F has a simple perfect information strategy: in response to the initial cover of $\omega_1^\dagger$, F chooses a co-countable neighborhood of ∞. On successive turns she may pick a single set from C's covers to cover the countable remainder.

Now, suppose that $\sigma(\mathcal{U}, n)$ was a winning Markov strategy and aim for a contradiction. Consider the covers

$$\mathcal{U}(\alpha) = \{[\alpha, \omega_1) \cup \{\infty\}\} \cup \{\{\beta\} : \beta < \alpha\}$$

and define $\tau(\alpha, n)$ to be the union of singletons chosen by $\sigma(\mathcal{U}(\alpha), n)$.

Using the sequence $\alpha_0, \alpha_1, \dots < \omega_1$ from the previous lemma, we consider the play $\mathcal{U}(\alpha_0), \mathcal{U}(\alpha_1), \dots$.

As σ was a winning strategy, $\{\sigma(\mathcal{U}(\alpha_n), n) : n < \omega\}$ must cover $\omega_1^\dagger$, and thus $\{\tau(\alpha_n, n) : n < \omega\}$ must cover $\{\beta : \forall n < \omega(\beta < \alpha_n)\}$, contradiction. □

Telgarski showed in "On Games of Topsoe" that a metrizable space is σ-compact if and only if there exists a winning strategy for F in the Menger game, and Scheepers gave a more direct proof later. We generalize Scheeper's proof to handle a number of cases.

Definition 10. A set $R \subseteq X$ is relatively compact to the topological space X if for every open cover of the entire space X, there is a finite subcover of the set R.

Proposition 11. *If X is regular, then Y is relatively compact if and only if $\overline{Y}$ is compact.*

Proof. The reverse implication is trivial.

Assume Y is relatively compact, let $\mathcal{U}$ be an open cover of $\overline{Y}$, and define $x \in V_x \subseteq \overline{V_x} \subseteq U_x \in \mathcal{U}$ for each $x \in X$. Then if we take a cover $\mathcal{F} = \{V_{x_i} : i < n\}$ of Y by relative compactness, then $\{U_{x_i} : i < n\}$ is a finite cover of $\overline{Y}$, showing compactness. □

Lemma 12. *Let $\sigma(\mathcal{U}, n)$ be a winning Markov strategy for F in $Cov_{C,F}(X)$, and $\mathfrak{C}$ collect all open covers of X. Then for*

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

it follows that R_n is relatively compact to X, and $\bigcup_{n<\omega} R_n = X$.

Proof. First, we see that $\sigma(\mathcal{U}, n)$ witnesses the relative compactness of R_n. Suppose that $x \notin R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$ for any $n < \omega$. Then for each n, pick $\mathcal{U}_n \in \mathfrak{C}$ such that $x \notin \bigcup \sigma(\mathcal{U}_n, n)$. Then σ does not defeat the play $\mathcal{U}_0, \mathcal{U}_1, \ldots$ □

Theorem 13. *A space X is σ-(relatively compact) if and only if $F \uparrow_{mark} Cov_{C,F}(X)$.*

Proof. For the forward implication, let $X = \bigcup_{n<\omega} R_n$ for R_n relatively compact, and define $\sigma(\mathcal{U}, n)$ to be a finite subcover of R_n. The previous lemma proves the other direction. □

Corollary 14. *For regular spaces X, the following are equivalent:*

(a) X is σ-compact

(b) X is σ-(relatively compact)

(c) $F \uparrow_{mark} Cov_{C,F}(X)$

Theorem 15. *For second-countable X, the following are equivalent:*

(a) X is σ-(relatively compact)

(b) $F \uparrow Cov_{C,F}(X)$

(c) $F \uparrow_{mark} Cov_{C,F}(X)$

Proof. . We need only show $(b) \Rightarrow (a)$, so let $\sigma(\mathcal{U}_0, \ldots, \mathcal{U}_{n-1})$ be a winning strategy for F, and observe that since X is second-countable, we may assume all covers are countable. Let $\mathfrak{C}$ be the collection of all countable covers of X. We define $R_s, \mathcal{U}_s$ for $s \in \omega^{<\omega}$ as follows:

- $R_\emptyset = \bigcap_{\mathcal{U}\in\mathfrak{C}} \left(\bigcup \sigma(\mathcal{U})\right)$
- Note that there are only countably many distinct finite subsets $\sigma(\mathcal{U})$ of the countable collection $\mathcal{U}$. Thus define each $\mathcal{U}_{\langle n\rangle}$ so that

$$R_\emptyset = \bigcap_{n<\omega} \left(\bigcup \sigma(\mathcal{U}_{\langle n\rangle})\right)$$

- $R_s = \bigcap_{\mathcal{U}\in\mathfrak{C}} \left(\bigcup \sigma(\mathcal{U}_{s\restriction 1}, \mathcal{U}_{s\restriction 2}, \dots, \mathcal{U}_s, \mathcal{U})\right)$
- Again, note that there are only countably many distinct finite subsets $\sigma(\mathcal{U}_{s\restriction 1}, \mathcal{U}_{s\restriction 2}, \dots, \mathcal{U}_s, \mathcal{U})$ of the countable collection $\mathcal{U}$. Thus define each $\mathcal{U}_{s^\frown\langle n\rangle}$ so that

$$R_s = \bigcap_{n<\omega} \left(\bigcup \sigma(\mathcal{U}_{s\restriction 1}, \mathcal{U}_{s\restriction 2}, \dots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle n\rangle})\right)$$

We quickly confirm that each R_s is relatively compact as for each open cover $\mathcal{U}$ of X we have the finite subcover $\sigma(\mathcal{U}_{s\restriction 1}, \mathcal{U}_{s\restriction 2}, \dots, \mathcal{U}_s, \mathcal{U})$ of R_s.

Finally, we claim that $X = \bigcup_{s\in\omega^{<\omega}} R_s$. If not, let x be missed by every R_s, and define $f \in \omega^\omega$ such that $x \notin \bigcup \sigma(\mathcal{U}_{f\restriction 1}, \dots, \mathcal{U}_{f\restriction n})$ for any n. Then $\mathcal{U}_{f\restriction 1}, \mathcal{U}_{f\restriction 2}, \dots$ is a counter to the winning strategy σ, a contradiction. $\square$

Corollary 16. *For metric spaces X, the following are equivalent:*

(a) X *is* σ*-compact*

(b) X *is* σ*-(relatively compact)*

(c) $F \uparrow Cov_{C,F}(X)$

(d) $F \uparrow_{mark} Cov_{C,F}(X)$

Example 17. *Let R be given the topology from example 63 from Counterexamples in Topology, the topology generated by open intervals with countable sets removed. This space is non-regular, non-σ-compact, and Lindelöf. It is also Menger as $F \uparrow Cov_{C,F}(R)$, but $F \not\uparrow_{mark} Cov_{C,F}(R)$.*

Proof. From Counterexamples: The irrationals are open, but contain no closed neighborhood, showing non-regular. Compact subsets are exactly finite subsets, showing non-σ-compact.

Take open covers $\mathcal{U}_0, \mathcal{U}_1, \dots$. Define $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{2n})$ to be a finite subcover of $[-n, n] \setminus C_n$ for some countable $C_n = \{c_{n,0}, c_{n,1}, \dots\}$. For $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_{2n+1})$, use any subcover of $\{c_{i,j} : i, j < n\}$. It is easily seen that σ is a winning perfect information strategy.

There cannot be a winning Markov strategy $\sigma(\mathcal{U}, n)$, however. Define

$$R_n = \bigcap_{\mathcal{U}\in\mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

where $\mathfrak{C}$ is the collection of open covers of R. For any $x_0, x_1, \dots \in R$, we may define the open cover $\mathcal{U} = \{R \setminus \{x_i : i \neq n\} : n < \omega\}$, and observe that $\bigcup \sigma(\mathcal{U}, n) \supseteq R_n$ contains only finitely many x_i. Thus R_n is finite, but since the previous lemma requires $\bigcup_{n<\omega} R_n = R$ if σ is a winning strategy, there exists a counter to σ. $\square$

Example 18. *Let R be given the topology from example 67 from Counterexamples in Topology, the topology generated by open intervals with or without the rationals removed. This space is non-regular, non-σ-compact, and Lindelöf.*

This space is an example of non-σ-compact but $F \uparrow_{mark} Cov_{C,F}(R)$ (and is thus also Menger).

Proof. From Counterexamples: The rationals are closed, but the closure of any open neighborhood is the whole real line, so they cannot be separated from any irrational point. Compact sets in this topology are nowhere dense in the Euclidean topology, so there cannot be countably many which union to the whole space. $\{(a, b) \setminus D : a, b \in \mathbb{Q}, D \in \{\emptyset, \mathbb{Q}\}\}$ is a countable base for the space, and second-countability implies Lindelöf.

To see $F \uparrow_{\text{mark}} Cov_{C,F}(R)$, we define $\sigma(\mathcal{U}_{2n}, 2n)$ to be a finite cover of $[-n, n] \setminus \mathbb{Q}$, and $\sigma(\mathcal{U}_{2n+1}, 2n + 1)$ to be a finite cover of $\{q_n\}$ for each $q_n \in \mathbb{Q}$. $\square$

We define a new property "σ-compactish " to describe a sufficient condition for $F \uparrow_{\text{2-mark}} Cov_{C,F}(X)$.

Definition 19. Let $\mathcal{U}$ be a cover of X. We say $C \subseteq X$ is $\mathcal{U}$-compact if there exists a finite subcover of $\mathcal{U}$ which covers C.

Let $\mathfrak{C}$ collect all the open covers of X. We say X is σ-compactish if there exists a function $f : \mathfrak{C} \times \omega \to \mathcal{P}(X)$ such that:

- $f(\mathcal{V}, n)$ is $\mathcal{V}$-compact
- $f(\mathcal{V}, n) \subseteq f(\mathcal{V}, n + 1)$
- $\bigcup_{n<\omega} f(\mathcal{V}, n) = X$
- The set
$$g(\mathcal{U}, \mathcal{V}, n) = \bigcup_{m\geq n} f(\mathcal{U}, m) \setminus (f(\mathcal{U}, m - 1) \cup f(\mathcal{V}, m))$$
is $\mathcal{V}$-compact

Obviously σ-compact implies σ-compactish implies Lindelöf. We shall see that the non-σ-compact space $\omega_1^\dagger$ is σ-compactish .

Lemma 20. *There exist injective functions $f_\alpha : \alpha \to \omega$ such that if $\alpha < \beta$, then*

$$f_\beta \restriction \alpha =^* f_\alpha$$

that is, $f_\beta \restriction \alpha$ and f_α agree on all but finitely many ordinals. (In addition, the range of each f_α is co-infinite.)

Proof. Taken from Kunen (used for the construction of an ω_1-Aronszajn tree).

We begin with the empty function $f_0 : 0 \to \omega_1$ which satisfies the hypothesis, and assume f_α is defined by induction. Let $f_{\alpha+1} = f_\alpha \cup \{\langle \alpha, n\rangle\}$ where n is not defined for f_α, and this satisfies the hypothesis.

Finally, suppose γ is the limit of $\alpha_0, \alpha_1, \dots$, and f_α is defined for $\alpha < \gamma$. Let $g_0 = f_{\alpha_0}$, and define $g_n : \alpha_n \to \omega$ to be injective, $g_n =^* f_{\alpha_n}$, and $g_{n+1} \restriction \alpha_n = g_n$. $g = \bigcup_{n<\omega} g_n$ is an injective function from $\gamma \to \omega$ and $g =^* f_\alpha$ for $\alpha < \gamma$, but the range need not be coinfinte. So let

$$f_\gamma(\beta) = \begin{cases} g(\alpha_{2n}) & \beta = \alpha_n \\ g(\beta) & \text{otherwise} \end{cases}$$

which frees up $\{g(\alpha_{2n+1}) : n < \omega\}$ from the range of f_γ, but still agrees with all but finitely many points compared to previous f's. □

Theorem 21. *The one-point Lindelöfication of the uncountable discrete space, $\omega_1^\dagger$, is σ-compactish .*

Proof. Take the injective funcions f_α from Kunen's lemma such that $f_\alpha \restriction \beta =^* f_\beta$. Let $\gamma(\mathcal{U})$ identify the least ordinal such that $[\gamma(\mathcal{U}), \omega_1) \cup \{\infty\}$ is in a refinement of $\mathcal{U}$. Then $f(\mathcal{U}, n) = f^{-1}_{\gamma(\mathcal{U})}([0, n]) \cup [\gamma(\mathcal{U}), \omega_1) \cup \{\infty\}$ is easily seen to witness the property. □

Theorem 22. *If X is σ-compactish , then $F \uparrow_{\textit{2-mark}} Cov_{C,F}(X)$.*

Proof. Let $\sigma(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1)$ cover $f(\mathcal{U}_{n+1}, n+1)$ and $g(\mathcal{U}_n, \mathcal{U}_{n+1}, n+1)$. If $\mathcal{U}_0, \mathcal{U}_1, \dots$ is any play by C, then for each $x \in X$, we note that $x \in f(\mathcal{U}_0, N)$ for some N. So either $x \in \bigcap_{m \le N} f(\mathcal{U}_m, N)$ and is covered by the strategy during round N, or for some $m < N$, $x \in f(\mathcal{U}_m, N) \setminus (f(\mathcal{U}_m, N-1) \cup f(\mathcal{U}_{m+1}, N))$ and is covered by the strategy during round $m + 1$. □

Corollary 23. $F \uparrow_{\textit{2-mark}} Cov_{C,F}(\omega_1^\dagger)$

Definition 24. X is **Rothberger** if for all open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$ there exist open sets $U_n \in \mathcal{U}_n$ such that $\{U_n : n < \omega\}$ is a cover of X.

Proposition 25. *Rothberger* $\Rightarrow$ *Menger*

Definition 26. In the two-player game $Cov_{C,S}(X)$ player C chooses open covers $\mathcal{U}_n$ of X, followed by player S choosing an open set $U_n \in \mathcal{U}_n$. S wins if $\{U_n : n < \omega\}$ is a cover of X.

Theorem 27. *X is Rothberger if and only if $C \not\uparrow Cov_{C,S}(X)$.*

Proof. Due to Pawlikowski $\square$

Definition 28. A space X is scattered if every subspace contains an isolated point. By convention, $X = \bigcup_{\alpha < \text{rank}(X)} X^\alpha$ where X^α is the set of isolated points of $X \setminus \bigcup_{\beta<\alpha} X^\beta$.

Proposition 29. *A space X is scattered if and only if every closed subspace contains an isolated point.*

Proposition 30. *The rank of a compact scattered T_1 space is a successor ordinal, and $X^{rank(X)-1}$ is finite.*

Proof. Suppose that the rank of X was a limit ordinal β. Then by choosing $\beta_n \to \beta$, we may pick a point $x_n \in X^{\beta_n}$, and $\{x_n : n < \omega\}$ may be shown to be a closed discrete subspace of X.

It's easily seen that $X^{\text{rank}(X)-1}$ must be finite - it is a closed discrete subspace of compact X. $\square$

Theorem 31. *The following are equivalent for compact T_2 X:*

(a) *X is Rothberger*

(b) *X is scattered*

(c) *$O \uparrow Cov_{C,S}(X)$*

Proof. To show $(a) \Rightarrow (b)$, we use Aurichi's proof in *D-Spaces*: suppose X has a closed subspace without isolated points. Then it is compact and contains a closed copy of the Cantor set, which is not Rothberger, contradiction.

To show $(b) \Rightarrow (c)$, if X is scattered, suppose during a particular round n, player S observes that the uncovered subspace $Y \subseteq X$ is nonempty. Then as Y is also compact scattered, select one of the finite points in $Y^{\text{rank}(Y)-1}$, label it x_n, and choose an open set containing x_n from the given cover.

We claim that if S follows this strategy, player S will observe that the uncovered subspace $Y \subseteq X$ is empty during some round. If not, consider the x_n chosen by Y by the end of the game - the rank of each point within X is nonincreasing, and does not contain a constant final sequence, contradiction.

Of course, $(c) \Rightarrow (a)$ is trivial. $\square$

Definition 32. In the two-player game $Cov_{P,O}(X)$ player P chooses points $x_n \in X$, followed by player O choosing an open neighborhood U_n of x_n. P wins if $\{U_n : n < \omega\}$ is a cover of X.

Theorem 33. *$Cov_{P,O}(X)$ is equivalent to $Cov_{C,S}(X)$. That is:*

- *$P \uparrow Cov_{P,O}(X)$ if and only if $S \uparrow Cov_{C,S}(X)$*
- *$O \uparrow Cov_{P,O}(X)$ if and only if $C \uparrow Cov_{C,S}(X)$.*

Proof. Due to Galvin.

- Let σ be a strategy for S in $Cov_{C,S}(X)$.

  We define a strategy for P in $Cov_{P,O}(X)$ as follows: during round 0, P chooses a point x_0 for which every open neighborhood is of the form $U_0 = \sigma(\mathcal{U}_0)$ for some open cover $\mathcal{U}_0$. (If not, let U_x witness for every $x \in X$, and note that $\sigma(\{U_x : x \in X\})$ is a contradiction.)

  During round $n+1$, P chooses point x_{n+1} for which every open neighborhood is of the form $U_{n+1} = \sigma(\mathcal{U}_0, \dots, \mathcal{U}_n, \mathcal{U}_{n+1})$ for some open cover $\mathcal{U}_{n+1}$. (If not, let U_x witness for every $x \in X$, and note that $\sigma(\mathcal{U}_0, \dots, \mathcal{U}_n, \{U_x : x \in X\})$ is a contradiction.)

  If σ was a winning strategy for S, then the open sets chosen by P are of the form $\{\sigma(\mathcal{U}_0), \sigma(\mathcal{U}_0, \mathcal{U}_1), \dots\}$ and are an open cover of X.

- Let σ be a strategy for P in $Cov_{P,O}(X)$.

  We define a strategy for S in $Cov_{C,S}(X)$ as follows: during round n, if S has chosen $U_0, \dots, U_{n-1}$ in previous rounds, S chooses an open set covering the point $\sigma(U_0, \dots, U_{n-1})$. If σ was a winning strategy for P, then for any open sets $U_0, U_1, \dots$ containing $\sigma(\cdot), \sigma(U_0), \dots$, the collection $\{U_0, U_1, \dots\}$ is a cover for X.

- Let σ be a strategy for C in $Cov_{C,S}(X)$.

  We define a strategy for O in $Cov_{P,O}(X)$ as follows: during round n, if O has chosen $U_0, \dots, U_{n-1}$ in previous rounds, O chooses an open set from the cover $\sigma(U_0, \dots, U_{n-1})$ containing the point chosen by P that round. If σ was a winning strategy for C, then for any open sets $U_0, U_1, \dots$ from the covers $\sigma(\cdot), \sigma(U_0), \dots$, the collection $\{U_0, U_1, \dots\}$ is not a cover for X.

- Let σ be a strategy for O in $Cov_{P,O}(X)$.

  We define a strategy for C in $Cov_{C,S}(X)$ as follows: during round 0, C chooses $\mathcal{U}_0 = \{\sigma(x) : x \in X\}$. In response, S chooses some $\sigma(x_0)$. During round $n+1$, if S has chosen $\sigma(x_0), \dots, \sigma(x_0, \dots, x_n)$ in previous rounds, C chooses $\mathcal{U}_{n+1} = \{\sigma(x_0, \dots, x_n, x) : x \in X\}$. If σ was a winning strategy for O, then $\{\sigma(x_0), \sigma(x_0, x_1), \dots\}$ is not a cover for X.

$\square$

Theorem 34. *The following are equivalent for points-G_δ X:*

(a) $S \uparrow Cov_{C,S}(X)$

(b) $S \uparrow_{mark} Cov_{C,S}(X)$

(c) $|X| = \omega$

Proof. $(c) \Rightarrow (b) \Rightarrow (a)$ are all trivial. Galvin has shown $(a) \Rightarrow (c)$ in *Indeterminacy*, but a direct proof follows. Let σ be a strategy for S in $Cov_{C,S}(X)$.

Let $G_{x,m}$ designate open sets such that $\{x\} = \bigcap_{m<\omega} G_{x,m}$ for all $x \in X$.

Suppose for $s \in \omega^{<\omega}$, $\mathcal{U}_t$ is defined for each $t \leq s$. Then C may find a point x_s such that for each $m < \omega$, there exists an open cover $\mathcal{U}_{s^\frown\langle m\rangle}$ where $\sigma(\mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle m\rangle}) = G_{x_s,m}$. (If not, let $m(x)$ witness the contrary for each x, and consider $\sigma(\mathcal{U}_{s\restriction 0}, \dots, \mathcal{U}_s, \{G_{x,m(x)} : x \in X\})$ to see the contradiction.)

If $x \notin \{x_s : s \in \omega^{<\omega}\}$, then C may counter the strategy σ by choosing $f \in \omega^\omega$ where $x \notin G_{x_{f\restriction n}, f(n)}$, and playing $\mathcal{U}_{f\restriction 1}, \mathcal{U}_{f\restriction 2}, \dots$, for which

$$x \notin \bigcup_{n<\omega} G_{x_{f\restriction n}, f(n)} = \bigcup_{n<\omega} \sigma(\mathcal{U}_{f\restriction 1}, \dots, \mathcal{U}_{f\restriction n+1})$$

Thus if σ is a winning strategy, then $X = \{x_s : s \in \omega^{<\omega}\}$ is countable. $\square$