

*Remark 1.* Scheeper's  $S(\kappa)$  requiring injections is stronger than my  $S'(\kappa)$  requiring finite-to-one maps. Dow suggests that  $S'(\omega_\omega)$  holds in ZFC by the following.

**Definition 2.** A topological space is said to be  $\omega$ -bounded if each countable subset of the space has compact closure.

**Theorem 3.** *There exists a topology on  $\omega_2$  which is  $\omega$ -bounded, locally compact, and locally countable.*

*Proof.* Note that the natural linear order on  $\omega_1$  induces such a topology for it.

Let  $\alpha < \omega_2$ , and suppose we've defined compatible topologies on  $\omega_1 \cdot (\beta + 1)$  for all  $0 \leq \beta < \alpha$ . If  $\alpha = \beta + 1$ , then let  $\omega_1 \cdot (\alpha + 1) = \omega_1 \cdot (\beta + 2)$  be the topological sum of the previously defined  $\omega_1 \cdot (\beta + 1)$  and the linear order on  $\omega_1 \cdot (\beta + 2) \setminus \omega_1 \cdot (\beta + 1) \cong \omega_1$ . Similarly, if  $cf(\alpha) > \omega$ , then let  $\omega_1 \cdot (\alpha + 1)$  be the topological sum of  $\bigcup_{\beta < \alpha} \omega_1 \cdot (\beta + 1)$  and the linear order on  $\omega_1 \cdot (\alpha + 1) \setminus \omega_1 \cdot \alpha \cong \omega_1$ .

The remaining case is where  $\alpha$  is the limit of increasing  $\alpha_n$  for  $n < \omega$ . Fix a bijection  $f_\alpha : \omega_1 \cdot (\alpha + 1) \setminus \omega_1 \cdot \alpha \rightarrow \omega_1 \cdot \alpha$ . Points in  $\omega_1 \cdot (\alpha_n + 1)$  for some  $n < \omega$  have their usual base induced by that previously defined topology. So let  $\gamma \in \omega_1 \cdot (\alpha + 1) \setminus \omega_1 \cdot \alpha$ . Basic open neighborhoods of  $\gamma$  are of the form  $[\gamma', \gamma] \cup f_\alpha[[\omega_1 \cdot \alpha, \gamma]] \setminus \omega_1 \cdot (\alpha_n + 1)$ , where  $\omega_1 \cdot \alpha \leq \gamma' < \gamma$  and  $n < \omega$ .

We wish to show that  $\omega_2$  with the topology induced by  $\bigcup_{\alpha < \omega_2} \omega_1 \cdot (\alpha + 1)$  is  $\omega$ -bounded, locally compact, and locally countable. If  $\gamma \in \omega_1 \cdot (\alpha + 1) \setminus \omega_1 \cdot \alpha$  where  $cf(\alpha) \neq \omega$ , then we immediately see that it is in a clopen copy of  $\omega_1$  giving us local compactness and local countability immediately. Otherwise,  $\gamma$  has a basic open neighborhood of the form  $[\gamma', \gamma] \cup f_\alpha[[\omega_1 \cdot \alpha, \gamma]] \setminus \omega_1 \cdot (\alpha_n + 1)$ , which is obviously countable. Likewise, its closure is compact: the closure operation will never add any greater ordinals, and any open cover contains another basic open neighborhood of  $\gamma$  such as  $[\gamma'', \gamma] \cup f_\alpha[[\omega_1 \cdot \alpha, \gamma]] \setminus \omega_1 \cdot (\alpha_m + 1)$  which misses only the compact set  $[\gamma', \gamma'']$  and the closure of the countable set  $f_\alpha[[\omega_1 \cdot \alpha, \gamma]] \cap \omega_1 \cdot (\alpha_{\min(m,n)} + 1)$  for which we may assume by induction is compact.

□

**Theorem.**  $S'(\omega_\omega)$ .