Games and Mathematics

Huntingdon College

Steven Clontz
http://stevenclontz.com

Department of Mathematics and Statistics
Auburn University

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Abstract

Two player games of perfect information such as chess and checkers have been played for centuries. Such games can be analyzed using tools from the field of game theory.

We will define two simple games: Takeaway and Nim. Using game theory, we will show how to create a **winning strategy** for one of the players in each game.

Also, we will show how *any* game can be analyzed in order to produce a winning strategy (given enough computational power).

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About game theory

Game theory is a broad subject, but in any context, it typically involves the mathematics of decision making.

Economists use game theory to study **two-player simultaneous** games. Suppose two companies have to set a price on competiting products. The demand for each product depends on the decisions made by both companies. The **Prisoner's Delimma** is another example of this type of game.

Another example would be **games of chance** like Solitaire or Yahtzee. One or more players use dice rolls or a shuffled deck of cards to play such games, and have to use probability to make decisions based upon future rolls of the dice or face-down cards.

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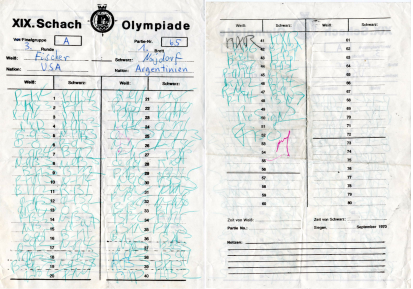
Two-player Sequential Games

We'll be talking about **two-player sequential** games today. These are games of **perfect information**: the players take turns with full knowledge of the history of their opponent's moves.



Coin games

The two games we'll talk about are **coin games**, because they can be played with whatever loose change you have in your pocket.

Takeaway

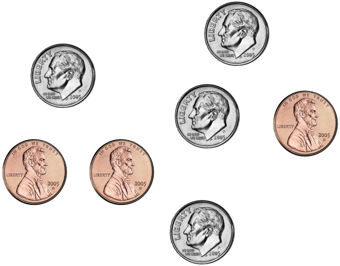
In **Takeaway**, the Players $\mathscr{A}$, $\mathscr{B}$ take turns removing 1, 2, or 3 coins from the table. The player who removes the last coin wins.

Round 1a: Player $\mathscr{A}$ takes away 3 coins, leaving 12.

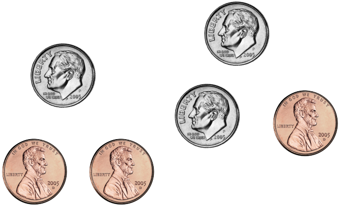
Round 1b: Player $\mathscr{B}$ takes away 2 coins, leaving 10.

Round 2a: Player $\mathscr{A}$ takes away 1 coin, leaving 9.

Round 2b: Player $\mathscr{B}$ takes away 2 coins, leaving 7.



Round 3a: Player $\mathscr{A}$ takes away 1 coin, leaving 6.



Round 3b: Player $\mathscr{B}$ takes away 2 coins, leaving 4.

Round 4a: Player $\mathscr{A}$ takes away 1 coin, leaving 3.

Round 4b: Player $\mathscr{B}$ takes away the last 3 coins, and wins!

A winning strategy

When studying sequential games, we often want to find what's called a **winning strategy**. Such a strategy should guarantee that the player following it cannot lose the game.

I claim that when Takeaway starts with 12 coins, then Player $\mathscr{B}$ has a winning strategy.

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I claim that when Takeaway starts with 12 coins, then Player $\mathscr{B}$ has a winning strategy.

Proof: Player $\mathscr{B}$ can always end her round so that there's 8, then 4, then 0 coins. For example:

$12 - 1 = 11$

$11 - 3 = 8$

$12 - 2 = 10$

$10 - 2 = 8$

$12 - 3 = 9$

$9 - 1 = 8$

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Two puzzles to try out.

Puzzle 1: Show that Player $\mathscr{A}$ had a winning strategy in Takeaway played with 15 coins (which she obviously didn't follow in the example).

Puzzle 2: Make a general rule about the number of starting coins which tells whether Player $\mathscr{A}$ or $\mathscr{B}$ has a winning strategy in Takeaway.

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Nim

In **Nim**, the Players $\mathscr{A}$, $\mathscr{B}$ take turns removing 1 or more coins from the table, but cannot take more than one type of coin at a time. The player who removes the last coin wins.

Round 1a: Player $\mathscr{A}$ takes away 3 pennies.

Round 1b: Player $\mathscr{B}$ takes away 1 dime.

Round 2a: Player $\mathscr{A}$ takes away 2 dimes.

Round 2b: Player $\mathscr{B}$ takes away 2 pennies.

Round 3a: Player $\mathscr{A}$ takes away the last penny.

Round 3b: Player $\mathscr{B}$ takes away the last 2 dimes, and wins!

A winning strategy

Just like in Takeaway, we want to know when each player has a winning strategy which guarantees that they cannot lose the game.

I claim that when Takeaway starts with 4 pennies and 4 dimes, then Player $\mathscr{B}$ has a winning strategy.

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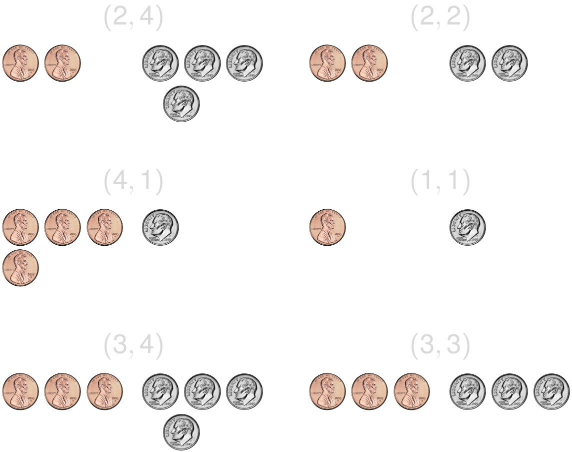
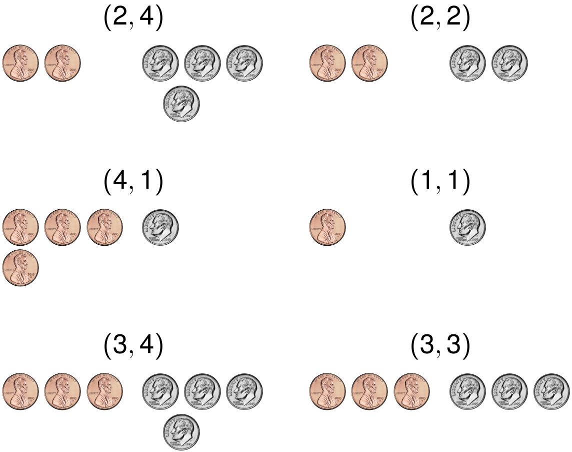
I claim that when Takeaway starts with 4 pennies and 4 dimes, then Player $\mathscr{B}$ has a winning strategy.

Proof: Player $\mathscr{B}$ can always end her round so that there's the same number of coins in both piles.



Proof: Player $\mathscr{B}$ can always end her round so that there's the same number of coins in both piles.



Three more puzzles!

Puzzle 3: Show that Player $\mathscr{A}$ had a winning strategy in Takeaway played with 6 pennies and 5 dimes (which she obviously didn't follow in the example).

Puzzle 4: Make a general rule about the number of starting pennies and dimes which tells whether Player $\mathscr{A}$ or $\mathscr{B}$ has a winning strategy in Nim.

Puzzle 5: Investigate how the game changes if more types of coins are allowed.

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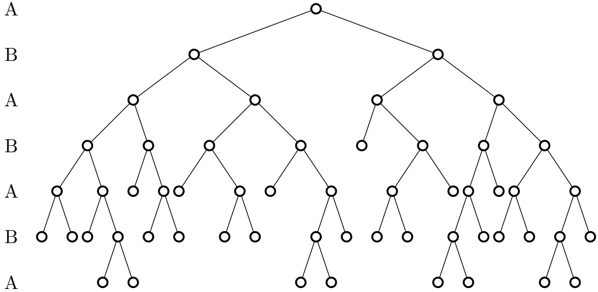
Decision Trees

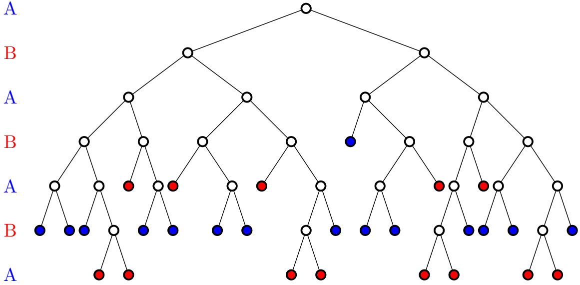
Since our games involve two players making decisions, we can model them by drawing a tree which maps all possible sequences of decisions (known as **extensive form**).



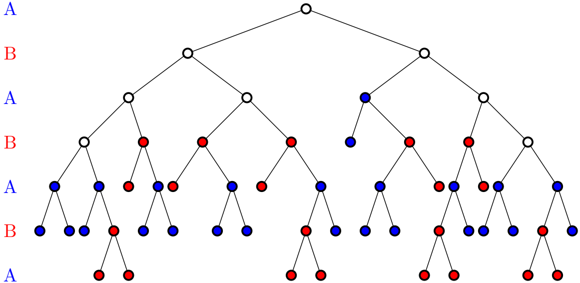
The player which starts their turn in a bottom node loses the game. (We assume no ties.)

Zermelo's Theorem

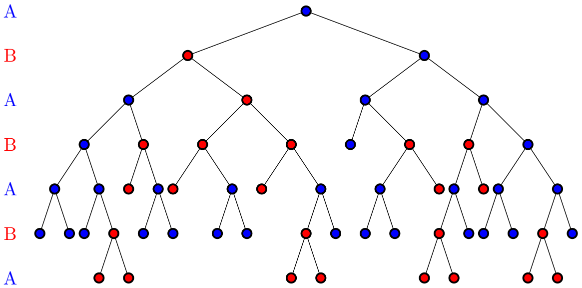
Zermelo's Theorem states that for every two-player sequential game of finite length, one of the players has a winning strategy which can be discovered by "backwards induction" on its decision tree.



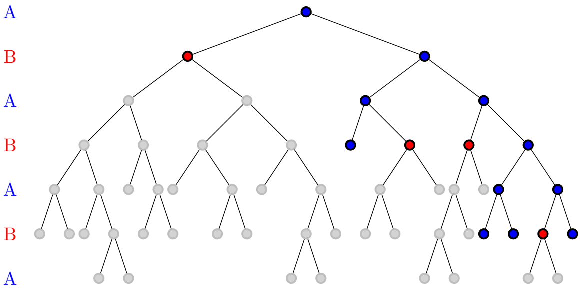
Label the tree by first showing the states where $\mathscr{A}$ (blue) and $\mathscr{B}$ (red) have already won the game.



Then move back and label the spaces where the active player is able to move to a vertex of their color.



Continue labeling the entire tree based on when the active player has the option to move into their color or not.



The color of the top vertex determines which player has the winning strategy. This strategy is to always make the decision which moves into the appropriate color.

So what about chess?

Due to Zermelo's theorem, we know that either White or Black has a winning strategy in chess: a guaranteed way to prevent losing the game.

But the problem is, we don't have a way to know exactly what it is! The decision tree for chess has millions upon millions of nodes, and our computers aren't powerful enough to figure it out (yet). And also, it would be so complicated that no human player could memorize it, so there's no reason to stop playing now. :-)

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Questions? Thanks for having me!