Finite and Infinite Games

Auburn REU Presentation

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June 24, 2014

Abstract

Two player games of perfect information such as chess and checkers have been played for centuries.

Such games may be mathematically modeled as a tuple $G = \langle M, W \rangle$, where M represents the moves of the game, and W represents the playthroughs (sequences of choices in M) which result in a victory for the first player.

We will investigate the classic result that all finite length games are determined: that is, exactly one player has a strategy which guarantees victory in the game regardless of the moves of her opponent.

In addition, we will learn how infinite length games are used in fields such as set theory and topology by using a game to prove that the real numbers are uncountable.

Heads up

This talk is about **sequential** or **combinatorial** games of perfect information.

Game theory is a broad subject, including classic games like the **Prisoner's Delimma** where two players make a **simultaneous** choice, or **Yahtzee** where the players face randomness from dice rolls.

However, we're going to look at games in the family of **Tic Tac Toe** or **Chess**, where two players take turns making moves with full knowledge of their options and the history of their previous moves.

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Some Definitions

Definition

Let $\omega = \{0, 1, 2, \ldots\}$, and let B^A be the set of all functions with domain A and range B.
Then X^ω contains all functions from $\{0, 1, 2, \ldots\}$ to X, or equivalently, all sequences of the form $\langle x_0, x_1, x_2, \ldots \rangle$ with $x_i \in X$.

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A **game** G is a tuple $\langle M, W \rangle$ where $W \subseteq M^\omega$. M represents the set of possible moves of the game, and W contains certain sequences of moves $\langle a_0, b_0, a_1, b_1, \ldots \rangle$ called **victories** (for the first player).

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In this context, all games have two players, and there are no ties.

If the players of the game are $\mathscr{A}$ and $\mathscr{B}$, then a **playthrough** of the game is some sequence in M^ω:

$$p = \langle a_0, b_0, a_1, b_1, a_2, b_2, \dots \rangle$$

If p is in W, then the first player $\mathscr{A}$ has won the game; otherwise, the second player $\mathscr{B}$ has won the game.

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Example

As an example, let $\langle M, W\rangle$ be the game **Sylver Coinage** with players $\mathscr{A}$, $\mathscr{B}$ where $M = \{2, 3, 4, \ldots\}$.

Defining W directly as a set is usually obnoxious, so we'll define it implicitly by setting this rule: no player can choose a number which is the sum of previously chosen numbers, perhaps with repetition.

Thus, if 4 and 7 have been chosen previously, then $25 = 4 + 7 + 7 + 7$ is not a legal move.

Thus a sequence $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ will be in W if it shows the second player $\mathscr{B}$ breaking the rules before the first player $\mathscr{A}$.

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For example, consider the playthrough beginning with

$$\langle 4, 11, 6, 5, 7, 3, 2, \ldots \rangle$$

- $\mathscr{A}$ chose 4
- $\mathscr{B}$ chose 11
- $\mathscr{A}$ chose 6 (legal moves remaining: $\{2, 3, 5, 7, 9, 13\}$)
- $\mathscr{B}$ chose 5 (legal moves remaining: $\{2, 3, 7\}$)
- $\mathscr{A}$ chose 7 (legal moves remaining: $\{2, 3\}$)
- $\mathscr{B}$ chose 3 (legal moves remaining: $\{2\}$)
- $\mathscr{A}$ chose 2 (legal moves remaining: $\emptyset$)
- $\mathscr{B}$ chose something illegal and lost.

This game, invented by John Conway, is an example of a **finite game**, since eventually one of the players are forced to break the given rules. (A puzzle I'll leave for you to work on!)

We've just seen that every sequence of the form $\langle 4, 11, 6, 5, 7, 3, 2, \dots \rangle$ is in W, since $\mathscr{A}$ wins those playthroughs of the game.

An artifact of this game model is that all playthroughs are infinite sequences. After $\mathscr{B}$ makes an illegal move, there's no point to keep playing in reality, but the sequences in W stretch on in every possible combination...

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Strategies

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A **strategy** is a function σ which turns a finite sequence of moves in M into a new move in M.

Put another way, a strategy is a fixed rule which tells a player what move to make during each round *in response* to all the previous moves of the game.

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Winning Strategies

Definition

The **result** of a game for which $\mathscr{A}$ uses the strategy σ $\mathscr{B}$ uses the attack $\langle b_0, b_1, \ldots \rangle$ is the playthrough of the game $\langle \sigma(\emptyset), b_0, \sigma(b_0), b_1, \sigma(b_0, b_1), \ldots \rangle$

(Or the similar definition when $\mathscr{B}$ has a strategy and $\mathscr{A}$ has an attack.)

Definition

If σ is a strategy for $\mathscr{A}$ such that the result of the game for every possible attack by $\mathscr{B}$ is in W, then σ is a **winning strategy**.

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Definition

If one of the players has a winning strategy for a game, then that game is said to be **determined**.

Obviously, both players can't have a guaranteed way to win the same game, but is it possible that neither player can guarantee a way to win? That is, for every fixed strategy by either player, could the opponent always have some chance of getting lucky and beating it?

Borel Determinacy Theorem

We could use a very strong topological and set-theoretic result to prove that finite games are determined.

Theorem

If M is given the discrete topology, and M^ω is given the usual product topology, then the game $G = \langle M, W \rangle$ is determined whenever W is a Borel subset of the space M^ω.

With a little topology, you can show that if G is finite, then W is an open set, which implies it's Borel. Thus finite games *are* determined: one of the players has a winning strategy.

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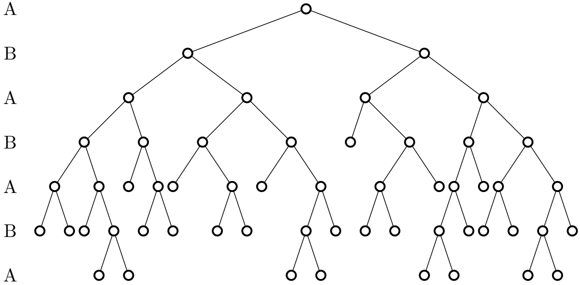
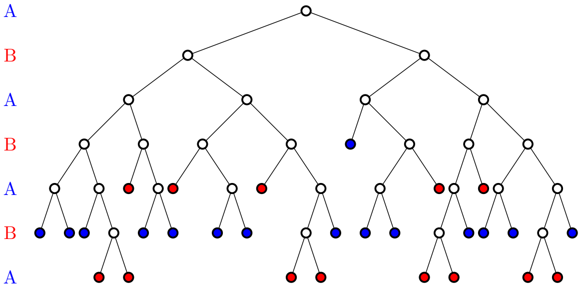
Decision Trees

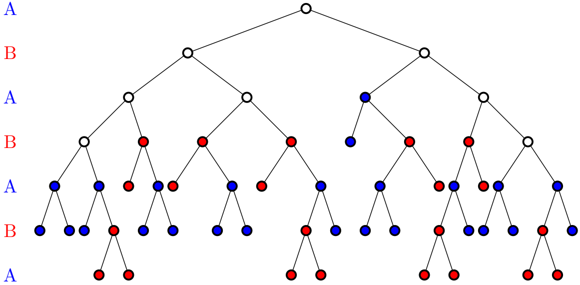
Finite games can be modeled as a decision tree.



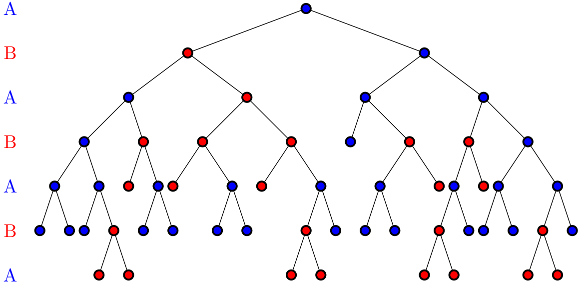
The above tree models a game where $\mathscr{A}$ and $\mathscr{B}$ alternate choosing "left" or "right" moves to descend the tree. A player wins if they move into a terminal node of the tree, since the opponent cannot move farther.



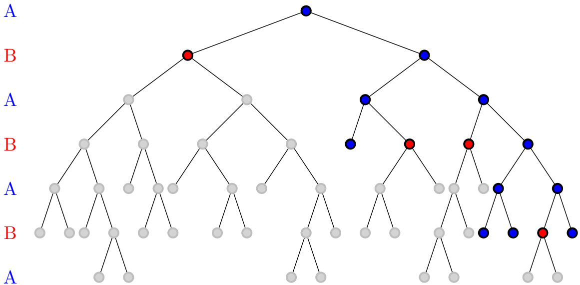
We can label the tree by first showing the states where $\mathscr{A}$ (blue) and $\mathscr{B}$ (red) have already won the game.



Then, we can move back and label the spaces where the active player is able to move to a vertex of their color.



Eventually, we label the entire tree based on when the active player has the option to move into their color or not.



Since the top vertex is blue, $\mathscr{A}$ has a winning strategy: always make the choice which leads to another blue vertex.

Infinite Games

Why do we care so much about determinacy? Why did we define game playthroughs to be infinite sequences?

These topics come into play when considering infinite games.

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A game G is **infinite** if there exists a playthrough such that it is still possible for either player to win during every round of the game.

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How are they played?

Even though a game could never actually be played, we can still construct strategies (functions) and attacks (sequences), and we can compute the result of a game given a strategy and attack.

Put another way, for every infinite game, there is a finite analog of the game which lasts exactly one round: one player chooses a strategy, followed by the opponent choosing an attack based upon it. The result of the infinite game is computed, and that determines the result of the single-round finite game.

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Determinacy

Like many theorems about infinite mathematical objects, whether infinite games are determined depend on your set-theoretic axioms. Mathematicians who work in foundations often use the Zermelo-Fraenkel (ZF) set theory, which isn't powerful enough to write proofs on the subject.

The **Axiom of Determinacy** states that all games which involve (countably) infinite moves are determined: one of the players always can construct a winning strategy.

But the more commonly used **Axiom of Choice** can be used to construct a game where if either player fixes a strategy, the other player can always create a counter-attack which defeats it. (See the Banach-Mazur game and Bernstein subsets of the real numbers.)

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Example Game

Let $\langle M, W\rangle$ be **Convergence Game** A where M is the set of real numbers $\mathbb{R}$, and A is a subset of the real numbers.

Players $\mathscr{A}$, $\mathscr{B}$ must follow the rule that every real number chosen is strictly between the latest numbers chosen by $\mathscr{A}$ and $\mathscr{B}$.

The start of a playthrough could be

$$\left\langle 5, \underset{(5<12)}{12}, \underset{(5<2\pi<12)}{2\pi}, \underset{(2\pi<7<12)}{7}, \underset{(2\pi<6.5<7)}{6.5}, \dots \right\rangle$$

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A Winning Condition

Since there's always infinitely many numbers between every two real numbers, $\mathscr{A}$ and $\mathscr{B}$ always have legal moves to choose from.

That's why we must add a winning condition: if both players always make legal moves, then $\mathscr{A}$ wins if the numbers she chose form a sequence converging to a number in the set A, and $\mathscr{B}$ wins otherwise.

For example, $\mathscr{A}$ won the earlier playthrough if the sequence $\langle 5, 2\pi, 6.5, \ldots \rangle$ converges to a number in A.

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Who wins?

Theorem

$\mathscr{B}$ has a winning strategy in Convergence Game when A can be indexed by the non-negative integers: $A = \{a_0, a_1, a_2, \dots\}$ (that is, when A is a ***countable*** *set).*

Proof.

$\mathscr{B}$'s strategy is to take the list of numbers $\{a_0, a_1, \ldots\}$, and every turn, $\mathscr{B}$ chooses the number furthest to the left of the list which is legal to play.

Then, at the "end" of the game, assuming that $\mathscr{A}$ also followed the rules, every number in the list $\{a_0, a_1, \ldots\}$ is either to the left of one of $\mathscr{A}$'s points, or to the right of one of $\mathscr{B}$'s points.

Thus, $\mathscr{A}$'s points cannot converge to any of the numbers in $\{a_0, a_1, \ldots\}$ no matter what attack she attempts. $\square$

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The Application

Using this game, we get a classic set theory result due to Cantor:

Theorem

The real numbers $\mathbb{R}$ cannot be exhaustively indexed by the non-negative integers (they are ***uncountable****).*

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Every increasing bounded above sequence converges to a real number (see Cal II). Thus $\mathscr{A}$ has a winning strategy for the Convergence Game $A = \mathbb{R}$.
But since $\mathscr{B}$ has a winning strategy when A is a countable set, we've proven that $\mathbb{R}$ cannot be countable. □

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Questions? Thanks for having me!