Game-theoretic strengthenings of Menger's property

29th Summer Conference on Topology and its Applications

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July 23, 2014

The Menger property

Definition

A space X is Menger if for every sequence $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots \rangle$ of open covers of X there exists a sequence $\langle \mathcal{F}_0, \mathcal{F}_1, \ldots \rangle$ such that $\mathcal{F}_n \subseteq \mathcal{U}_n$, $|\mathcal{F}_n| < \omega$, and $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover of X.

Proposition

X is σ-compact $\Rightarrow$ X is Menger $\Rightarrow$ X is Lindelöf.

The Menger game

Game

Let $Cov_{C,F}(X)$ denote the *Menger game* with players $\mathscr{C}$, $\mathscr{F}$. In round n, $\mathscr{C}$ chooses an open cover $\mathcal{C}_n$, followed by $\mathscr{F}$ choosing a finite subcollection $\mathcal{F}_n \subseteq \mathcal{C}_n$.
$\mathscr{F}$ wins the game, that is, $\mathscr{F} \uparrow Cov_{C,F}(X)$ if $\bigcup_{n<\omega} \mathcal{F}_n$ is a cover for the space X, and $\mathscr{C}$ wins otherwise.

Theorem

X is Menger if and only if $\mathscr{C} \not\uparrow Cov_{C,F}(X)$. (Hurewicz 1926, effectively)

Menger suspected that the subsets of the real line with his property were exactly the σ-compact spaces; however:

Theorem

There are ZFC examples of non-σ-compact subsets of the real line which are Menger. (Fremlin, Miller 1988)

But metrizable non-σ-compact Menger spaces will be *undetermined* for the Menger game.

Theorem

Let X be metrizable. $\mathscr{F} \uparrow Cov_{C,F}(X)$ if and only if X is σ-compact. (Telgarsky 1984, Scheepers 1995)

Note that for Lindelöf spaces, metrizability is characterized by regularity and secound countability.

Sketch of Scheeper's proof:

- Using second-countability and the winning strategy for $\mathscr{F}$, construct certain subsets K_s for $s \in \omega^{<\omega}$ such that $X = \bigcup_{s\in\omega^{<\omega}} K_s$.
- Using regularity, show that each K_s is compact.
- The result follows since $|\omega^{<\omega}| = \omega$.

By considering winning *limited-information strategies*, we'll be able to factor out this proof a bit.

Limited information strategies

Definition

A *(perfect information) strategy* has knowledge of all the past moves of the opponent.

Definition

A *k-tactical strategy* has knowledge of only the past k moves of the opponent.

Definition

A *k-Markov strategy* has knowledge of only the past k moves of the opponent and the round number.

Obviously,

$$\mathscr{A} \underset{k\text{-tact}}{\uparrow} G \Rightarrow \mathscr{A} \underset{k\text{-mark}}{\uparrow} G \Rightarrow \mathscr{A} \underset{\text{(perfect)}}{\uparrow} G$$

But tactical strategies aren't interesting for the Menger game.

Proposition

For any $k < \omega$, $\mathscr{F} \underset{k\text{-tact}}{\uparrow} Cov_{C,F}(X)$ if and only if X is compact.

Effectively, $\mathscr{F}$ needs some sort of seed to prevent from being stuck in a loop: there's nothing stopping $\mathscr{C}$ from playing the same open cover during every round of the game.

Comparitively, Marköv strategies are very powerful.

Proposition

If X is σ-compact, then $\mathscr{F} \underset{1\text{-mark}}{\uparrow} Cov_{C,F}(X)$.

Proof.

Let $X = \bigcup_{n<\omega} K_n$. During round n, $\mathscr{F}$ picks a finite subcollection of the last open cover played by $\mathscr{C}$ (the only one $\mathscr{F}$ remembers) which covers K_n. $\square$

Without assuming regularity, we can't quite reverse the implication, but we can get close.

Definition

A subset Y of X is *relatively compact* if for every open cover for X, there exists a finite subcollection which covers Y.

Proposition

If X is σ-relatively-compact, then $\mathscr{F} \underset{1\text{-mark}}{\uparrow} Cov_{C,F}(X)$.

Proposition

For regular spaces, $Y \subseteq X$ is relatively compact if and only if $\overline{Y}$ is compact. So σ-relatively-compact regular spaces are exactly the σ-compact regular spaces.

Theorem

$\mathscr{F} \underset{1\text{-mark}}{\uparrow} Cov_{C,F}(X)$ if and only if X is σ-relatively-compact.

Proof.

Let $\sigma(\mathcal{U}, n)$ represent a 1-Marköv strategy. For every open cover $\mathcal{U} \in \mathfrak{C}$, $\sigma(\mathcal{U}, n)$ witnesses relative compactness for the set

$$R_n = \bigcap_{\mathcal{U} \in \mathfrak{C}} \bigcup \sigma(\mathcal{U}, n)$$

If X is not σ-relatively compact, fix $x \notin R_n$ for any $n < \omega$. Then $\mathscr{C}$ can beat σ by choosing $\mathcal{U}_n \in \mathfrak{C}$ during each round such that $x \notin \bigcup \sigma(\mathcal{U}_n, n)$. $\square$

So for regular spaces, a winning strategy for $\mathscr{F}$ in the Menger game isn't sufficient to characterize σ-compactness, but a winning 1-Markōv strategy does the trick.

We can complete Telgarsky's/Scheeper's result by showing the following:

Theorem

For second countable spaces X, $\mathscr{F} \uparrow Cov_{C,F}(X)$ if and only if $\mathscr{F} \underset{1\text{-}mark}{\uparrow} Cov_{C,F}(X)$.

Proof

Let σ be a perfect information strategy. Since X is a second-countable space, we may pretend that there are only countably many finite collections of open sets. Thus for $s \in \omega^{<\omega}$, we may define open covers $\mathcal{U}_{s^\frown\langle n\rangle}$ such that for each open cover $\mathcal{U}$, there is some $n < \omega$ where

$$\sigma(\mathcal{U}_{s\restriction 1}, \ldots, \mathcal{U}_s, \mathcal{U}) = \sigma(\mathcal{U}_{s\restriction 1}, \ldots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle n\rangle})$$

Let $t : \omega \to \omega^{<\omega}$ be a bijection. During round n and seeing only the latest open cover $\mathcal{U}$, $\mathscr{F}$ may play the finite subcollection

$$\tau(\mathcal{U}, n) = \sigma(\mathcal{U}_{t(n)\restriction 1}, \ldots, \mathcal{U}_{t(n)}, \mathcal{U})$$

Proof (cont.)

Suppose there exists a counter-attack $\langle \mathcal{V}_0, \mathcal{V}_1, \dots \rangle$ which defeats the 1-Markröv strategy τ. Then there exists $f : \omega \to \omega$ such that, if $\mathcal{V}^n = \mathcal{V}_{t^{-1}(f \restriction n)}$

$$\begin{aligned} x &\notin \bigcup \tau(\mathcal{V}^n, t^{-1}(f \restriction n)) \\ &= \bigcup \sigma(\mathcal{U}_{f\restriction 1}, \dots, \mathcal{U}_{f\restriction n}, \mathcal{V}^n) \\ &= \bigcup \sigma(\mathcal{U}_{f\restriction 1}, \dots, \mathcal{U}_{f\restriction n}, \mathcal{U}_{f\restriction (n+1)}) \end{aligned}$$

Thus $\langle \mathcal{U}_{f\restriction 1}, \mathcal{U}_{f\restriction 2}, \dots \rangle$ is a successful counter-attack by $\mathscr{C}$ against the perfect information strategy σ. $\square$

Unlike the Banach-Mazur game, we can immediately see that knowledge of more than two previous moves of $\mathscr{F}$'s opponent must be infinite to be of any use.

Theorem

If $\mathscr{F} \underset{k\text{-mark}}{\uparrow} Cov_{C,F}(X)$, *then* $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Cov_{C,F}(X)$.

Proof.

$$\tau(\langle \mathcal{U}, \mathcal{V} \rangle, n+1) = \bigcup_{m<k+2} \sigma(\langle \underbrace{\mathcal{U}, \ldots, \mathcal{U}}_{k+1-m}, \underbrace{\mathcal{V}, \ldots, \mathcal{V}}_{m+1} \rangle, (n+1)(k+2)+m)$$

$\square$

Knowledge of two previous moves versus one is an important distinction: in the former case, the player is able to react to change by the opponent.

Definition

Let $\kappa^\dagger = \kappa \cup \{\infty\}$ be the *one point Lindelöf-ication* of discrete κ: neighborhoods of ∞ are exactly the co-countable sets containing it.

$\kappa^\dagger$ is a simple space which is a regular and Lindelöf, but not second-countable space or σ-compact. Thus $\mathscr{F} \underset{\text{1-mark}}{\not\uparrow} Cov_{C,F}(\kappa^\dagger)$, but it's easy to see that $\mathscr{F} \uparrow Cov_{C,F}(\kappa^\dagger)$.
What about 2-Markōv strategies?

In 1991, Scheepers introduced the statement $S(\kappa, \omega, \omega)$ to study an infinite game involving the countable and finite subsets of κ.

Game

Let $Fill_{C,F}^{\cup,\subset}(\kappa)$ denote the *strict union filling game* with two players $\mathscr{C}$, $\mathscr{F}$. In round 0, $\mathscr{C}$ chooses $C_0 \in [\kappa]^{\leq\omega}$, followed by $\mathscr{F}$ choosing $F_0 \in [\kappa]^{<\omega}$. In round $n+1$, $\mathscr{C}$ chooses $C_{n+1} \in [\kappa]^{\leq\omega}$ such that $C_{n+1} \supset C_n$, followed by $\mathscr{F}$ choosing $F_{n+1} \in [\kappa]^{<\omega}$. $\mathscr{F}$ wins the game if $\bigcup_{n<\omega} F_n \supseteq \bigcup_{n<\omega} C_n$; otherwise, $\mathscr{C}$ wins.

Definition

For two functions f, g we say f is *almost compatible* with g ($f \wr g$) if $|\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(x) \neq g(x)\}| < \omega$.

Definition

$S(\kappa, \omega, \omega)$ states that there exist functions $f_A : A \to \omega$ for each $A \in [\kappa]^{\leq\omega}$ such that $|f_A^{-1}(n)| < \omega$ for all $n < \omega$ and $f_A \wr f_B$ for all $A, B \in [\kappa]^{\omega}$.

Theorem

$S(\omega_1, \omega, \omega)$; $Con(S(2^\omega, \omega, \omega) + \neg CH)$; $\neg S(\kappa, \omega, \omega)$ for $\kappa > 2^\omega$.

The round number is unnecessary in Scheeper's game, since $\mathscr{C}$ must choose strictly increasing sets.

Theorem

If $S(\kappa, \omega, \omega)$, then $\mathscr{F} \underset{2\text{-tact}}{\uparrow} Fill^{\cup,\subset}_{C,F}(\kappa)$.

As it turns out, a related game characterizes $Cov_{C,F}(\kappa^\dagger)$.

Definition

Let $Fill^{\cap}_{C,F}(\kappa)$ denote the *intersection filling game* analogous to $Fill^{\cup,\subset}_{C,F}(\kappa)$, except that $\mathscr{C}$ has no restriction on the countable sets she chooses, but $\mathscr{F}$ need only ensure that $\bigcup_{n<\omega} F_n \supseteq \bigcap_{n<\omega} C_n$ to win the game.

Theorem

$\mathscr{F} \underset{k\text{-mark}}{\uparrow} Cov_{C,F}(\kappa^\dagger)$ if and only if $\mathscr{F} \underset{k\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa)$.

Theorem

If $S(\kappa, \omega, \omega)$, then $\mathscr{F} \underset{2\text{-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa)$.

Proof.

Let $f_A : A \to \omega$ witness $S(\kappa, \omega, \omega)$. Then we define the winning 2-Markëv strategy σ as follows:

$$\sigma(\langle A \rangle, 0) = \{\alpha \in A : f_A(\alpha) = 0\}$$

$$\sigma(\langle A, B \rangle, n+1) = \{\alpha \in A \cap B : f_B(\alpha) \leq n+1 \text{ or } f_A(\alpha) \neq f_B(\alpha)\}$$

$\square$

Corollary

$\mathscr{F} \underset{2\text{-}mark}{\uparrow} Cov_{C,F}(\omega_1^\dagger)$, but $\mathscr{F} \underset{1\text{-}mark}{\not\uparrow} Cov_{C,F}(\omega_1^\dagger)$.

$$
\begin{tikzcd}
 & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(\kappa^\dagger) \arrow[r, leftrightarrow] \arrow[d] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill^{\cap}_{C,F}(\kappa) \arrow[d] \\
S(\kappa,\omega,\omega) \arrow[r] \arrow[ur] \arrow[drr] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill^{1,\subseteq}_{C,F}(\kappa) \arrow[r, leftrightarrow] & \mathscr{F} \underset{\text{2-mark}}{\uparrow} Fill^{\cup,\subseteq}_{C,F}(\kappa) \\
 & & \mathscr{F} \underset{\text{2-tact}}{\uparrow} Fill^{\cup,\subseteq}_{C,F}(\kappa) \arrow[u]
\end{tikzcd}
$$

Question

Does $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(\kappa^\dagger)$ imply $S(\kappa, \omega, \omega)$?

Question

Are $\mathscr{F} \uparrow Cov_{C,F}(X)$ and $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(X)$ distinct?

An affirmative answer to the first question answers this since $\neg S(\kappa, \omega, \omega)$ for $\kappa > 2^\omega$.

Question

Where does $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(X)$ fit in with other properties between σ-(relatively-)compact and Menger?

$\mathscr{F} \underset{\text{2-mark}}{\uparrow} Cov_{C,F}(X)$ seems to characterize an "almost-σ-(relative-)compactness".

Any sufficient property would imply $\mathscr{F} \uparrow Cov_{C,F}(X)$, and any (interesting) necessary property shouldn't be implied by $\mathscr{F} \uparrow Cov_{C,F}(X)$. Assuming T_3, properties which come to mind from the literature fit the latter: e.g. Alster (Aurichi, Tall 2013), and thus productively Lindelöf (Alster 1988) and Hurewicz (Tall 2009).

Questions? Thanks for listening!