

Proximal compact spaces are Corson compact

2015 Joint Mathematics Meetings at San Antonio

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January 11, 2015

A *topological game* is a two-player game $G(X)$ of length $\omega = \{0, 1, 2, \dots\}$ defined for certain topological spaces X .

During each round n , the first and second player take turns choosing certain topological objects from X (e.g. point, open set, open cover, etc.).

At the “end” of the game, a winner is declared by inspecting the sequences of choices made throughout the game.

The study of such games involves finding when a player has a *winning strategy* which defeats every possible counterattack by the opponent.

See Telgarsky's excellent survey on topological games for more details: [7]

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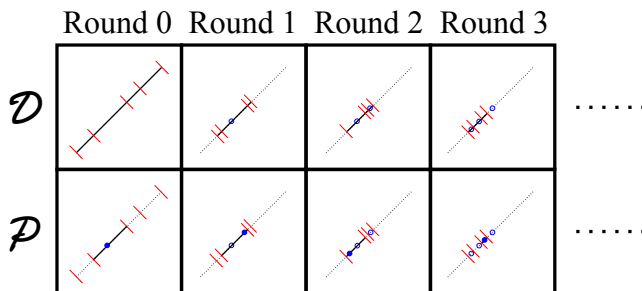
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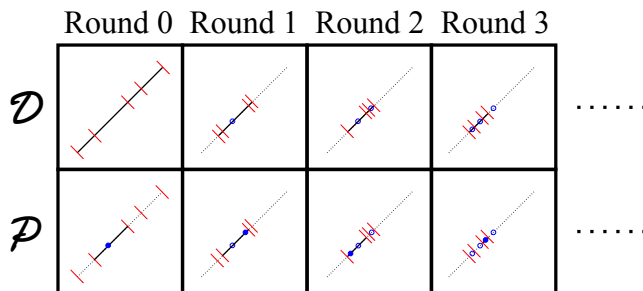
Proximal Game (2014) [1]



for compact T_1 0-dim spaces

The first player \mathcal{D} wins the game if the points chosen by the second player \mathcal{P} converge. If \mathcal{D} has a winning strategy for this game, call X *proximal*.

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Some results related to the Proximal Game due to Jocelyn Bell:

Proposition

If X is metrizable, then X is proximal.

Theorem

If X is proximal, then X is collectionwise normal.

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Σ -products and closed subspaces of proximal spaces are proximal.

Corollary

*The Σ -product of metrizable spaces is collectionwise normal.
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A *Corson compact* space is a space homeomorphic to a compact subset of the Σ -product of real lines.

Peter Nyikos observed:

Proposition

Every Corson compact space is proximal compact. [5]

C. and Gruenhage showed in [2] that any compact proximal space must be Corson compact, using another game-theoretic characterization of Corson compact due to Gruenhage:

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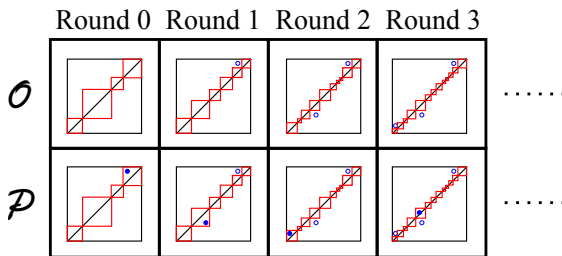
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Diagonal Game (1984) [3]:



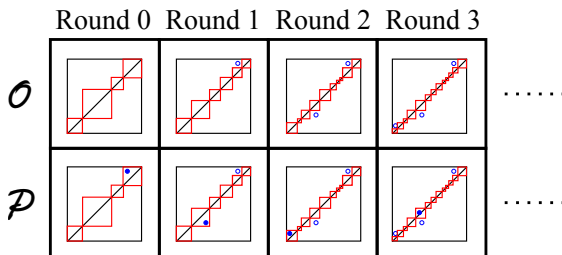
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The first player \mathcal{O} wins the game if any open set containing the diagonal also contains infinitely many of \mathcal{P} 's chosen points.

Theorem

For X compact, \mathcal{O} has a winning strategy for the diagonal game if and only if X is Corson compact.

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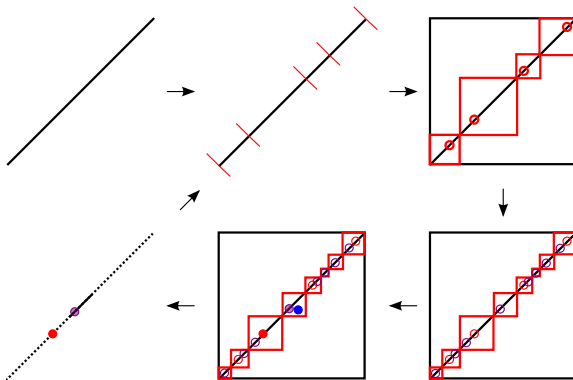
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The first player O wins the game if any open set containing the diagonal also contains infinitely many of P 's chosen points.

Theorem

For X compact, O has a winning strategy for the diagonal game if and only if X is Corson compact.

One may use a winning strategy σ for \mathcal{D} in the proximal game to construct a strategy τ for \mathcal{O} in the diagonal game.



In general:

$$\tau(a) = \bigcup_{s \frown \langle i, h_{s,i}, j \rangle \in \max(T(a))} \frac{1}{4} \sigma(o_s \frown \langle h_{s,i} \rangle) [h_{s,i}, j]$$

Using the strategy τ defined for every proximal compact space, \mathcal{O} cannot be defeated in the diagonal game, and therefore all proximal compacts are Corson compact. □

Open questions:

- If compactness is dropped, does the proximal game characterize all copies of *closed* subspaces of a Σ -product of reals? (Nyikos)
- If the winning strategy for the proximal game is *Markov* (relies on only the latest move and round number) for a compact space, does that imply that the space is *Eberlein* compact? (This holds for the diagonal game.)



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Steven Clontz and Gary Gruenhage.

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Gary Gruenhage.

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Any questions?