Properties of Convergent Sequences

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Abstract

Convergence is a difficult definition to use directly. We investigate how other, more easily verifiable properties of sequences of real numbers are related to convergence.

This is a short paragraph to introduce the article (but it is not the abstract). It is optional, in case it would be preferable to have the first section be titled an "Introduction."

1 Bounded Sequences

Convergence is a very strong property for a sequence to have, since it requires the tails of the sequence to all grow arbitrarily close to a specified real number (its limit). Let's look at some simpler properties, each of which is weaker than convergence, and their relationships to convergence.

Definition 1.1 (Bounded sequence). s_n *bounded* $M \in \mathbb{R} n \in \mathbb{N}$

$$|s_n| \leq M.$$

Intuitively, we might say that *all* terms of a bounded sequence lie between a constant "ceiling" and a constant "floor:" another way to write the inequality at the end of the definition is $-M \leq |s_n| \leq M$.

Boundedness is indeed a weaker condition than convergence; while it is not true that every bounded sequence is convergent, is *is* true that every convergent sequence is bounded:

Theorem 1.2 (Convergent implies bounded). $s_n s_n$

Proof. Intuitively, convergence is a strong condition. Given any $\epsilon > 0$, it produces an $N \in \mathbb{N}$ which divides the sequence into a (finite) head and an (infinite) tail. We imagine that each will have a ceiling of its own: images/video-1.jpg www.youtube.com/watch?v=uWjC7e

Now, let us define $\epsilon = 1$. Since s_n is a convergent sequence, let us denote $L = \lim_{n\to\infty} s_n$.

Then by definition of convergence there exists an $N \in \mathbb{N}$ such that, for all $n \geq N$, we have

$$|s_n - L| < \epsilon.$$

This defines for us a head of the sequence, $\{s_1, s_2, \ldots, s_N\}$, and a tail of the sequence, $\{s_N, s_{N+1}, s_{N+2}, \ldots\}$, and all of the terms in the *tail* are within a distance of ϵ of the limit L.

Using an add-subtract trick can shift the inequality $|s_n - L| < \epsilon$ from a measurement of the sequence's distance from L into a measurement of its distance from zero (i.e., its absolute value):

$$|s_n| = |s_n - L + L| \leq |s_n - L| + |L| < \epsilon + |L| \qquad \text{for all } n \geq N. \tag{1.1}$$

In other words, $\epsilon + |L|$ is an upper bound for the tail of the sequence.

Meanwhile, since the head of the sequence is a *finite* set, it will in particular have a largest element that can be used as an upper bound for that set. So we define $m = \max\{|s_1|, |s_2|, \ldots, |s_N|\}$.

Now define $M = \max\{m, \epsilon + |L|\}$.

Let $n \in \mathbb{N}$ be arbitrarily chosen. Then there are two cases, depending on whether the nth term belongs to the head of the sequence or the tail:

1. *If* $n \leq N$, then s_n belongs to the head of the sequence and $|s_n|$ is one of the values in the finite list which was used to define the maximum of the head, m. Hence $|s_n| \leq m \leq M$.

2. *If* $n > N$, then s_n belongs to the tail of the sequence and $|s_n|$ is governed by the tail inequality (1.1). Hence $|s_n| < \epsilon + |L| \leq M$.

This covers all cases, so we conclude that for all $n \in \mathbb{N}$ we have $|s_n| \leq M$. images/video-2.jpg www.youtube.com/watch?v=DpfmgXilu_8 □

2 Monotonic Sequences

Another class of sequences whose behavior is well regulated is the class of sequences which "do not change direction." These are the monotonic sequences.

Definition 2.1 (Monotonic sequence)**.** s_n *monotonic*

- For all $n \in \mathbb{N}$, we have $s_n \leq s_{n+1}$.
- For all $n \in \mathbb{N}$, we have $s_n \geq s_{n+1}$.

In the first case, we say the sequence is *increasing.* In the second case, we say the sequence is *decreasing.* If either inequality is a strict inequality ($<$ or $>$), then we say the sequence is "strictly" increasing or decreasing respectively.

3 Cauchy Sequences

Hi there.