# BUSINESS CALCULUS

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If you find any errors, I would very much appreciate it if you could email me at scalaway@shoreline.edu and tell me about them.

# Introduction

## A Preview of Calculus

Calculus was first developed more than three hundred years ago by Sir Isaac Newton and Gottfried Leibniz to help them describe and understand the rules governing the motion of planets and moons. Since then, thousands of other men and women have refined the basic ideas of calculus, developed new techniques to make the calculations easier, and found ways to apply calculus to problems besides planetary motion. Perhaps most importantly, they have used calculus to help understand a wide variety of physical, biological, economic and social phenomena and to describe and solve problems in those areas.

Part of the beauty of calculus is that it is based on a few very simple ideas. Part of the power of calculus is that these simple ideas can help us understand, describe, and solve problems in a variety of fields.

## About this book

*Chapter 0 Introduction and Preliminaries* gives a brief introduction to calculus in general and this course in particular.

*Chapter 1 Review* contains review material that you should recall before we begin calculus.

*Chapter 2 The Derivative* builds on the precalculus idea of the slope of a line to let us find and use rates of change in many situations.

*Chapter 3 The Integral* builds on the precalculus idea of the area of a rectangle to let us find accumulated change in more complicated and interesting settings.

*Chapter 4 Functions of Two Variables* extends the calculus ideas of chapter 2 to functions of more than one variable.

*Chapter 5 Optional Topics* contains a few topics that are not part of most Business Calculus courses. You might need them for your course, or you might simply find them interesting.

# Two Powerful Precalculus Ideas

Calculus is a way to extend some powerful precalculus ideas to more complicated functions. In this course, we will extend two powerful ideas – slope and area.

## Slope

The slope of a line measures how fast a line rises or falls as we move from left to right along the line. It measures the rate of change of the y-coordinate with respect to changes in the x-coordinate. If the line represents the distance traveled over time, for example, then its slope represents the velocity. In Figure 1, you can remind yourself of how we calculate slope using two points on the line:

$$m = \{\text{slope from P to Q}\} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

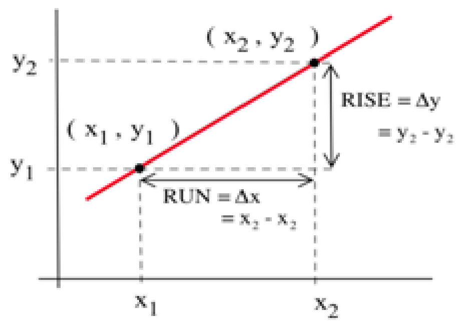


**Figure 1**

We would like to be able to get that same sort of information (how fast the curve rises or falls, velocity from distance) even if the graph is not a straight line. But what happens if we try to find the slope of a curve, as in Figure 1? We need two points in order to determine the slope of a line. How can we find a slope of a curve, at just one point?

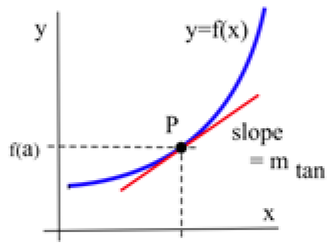


Figure 2

In Chapter 2, we'll extend the idea of slope to curves, and we'll be able to use it to answer many different kinds of questions.

Here is an example of a question that can be answered with slope (in Chapter 2):

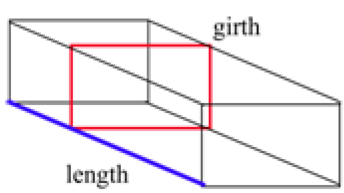


Figure 3

**Ex:** The US Post Office requires that the length plus the girth (Figure 3) of a package not exceed 84 inches. What are the dimensions of the rectangular box with the largest volume that the Post Office will accept?

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## Area

It's easy to compute the area of a rectangle, or something made of rectangles. But what if the object we want to find the area of is irregular and curvy? We don't have formulas (like length x width) for every shape, and we don't want to. We want a way to find the area of any shape.

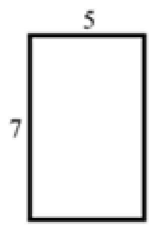



Figure 4

Figure 5

It turns out that area tells us much more than simply how much carpet we will need to buy to cover a region. And in Chapter 3, we'll extend the idea of area and use it to answer many different kinds of questions.

Here is an example of a question that can be answered with area (in Chapter 3):

**Ex:** For a certain product, the demand function is given by $p = 300 - 4q^2$ and the supply function is given by $p = 3 + q^2$, where *p* is the price in dollars for the product and *q* is the quantity. At the equilibrium point, find the total gains from trade (consumer surplus + producer surplus).

## How is Business Calculus Different?

Students who plan to go into science, engineering, or mathematics take a year-long sequence of classes that cover many of the same topics as we do in our one-quarter or one-semester course. Here are some of the differences:

### No trigonometry ☺

We will not be using trigonometry at all in this course. The scientists and engineers need trigonometry frequently, and so a great deal of the engineering calculus course is devoted to trigonometric functions and the situations they can model.

### The applications are different

The scientists and engineers learn how to apply calculus to physics problems, such as work. They do a lot of geometric applications, like finding minimum distances, volumes of revolution, or arclengths. In this class, we will do only a few of these (distance/velocity problems, areas between curves). On the other hand, we will learn to apply calculus in some economic and business settings, like maximizing profit or minimizing average cost, finding elasticity of demand, or finding the present value of a continuous income stream. These are applications that are seldom seen in a course for engineers.

### Fewer theorems, no proofs

The focus of this course is applications rather than theory. In this course, we will use the results of some theorems, but we won't prove any of them. When you finish this course, you should be able to solve many kinds of problems using calculus. But you won't be prepared to go on to higher mathematics.

### Less algebra

In this class, you will not need clever algebra. If you need to solve an equation, it will either be relatively simple, or you can use technology to solve it. In most cases, you won't need "exact answers," calculator numbers will be good enough.

### Simplification and Calculator Numbers

When you were in tenth grade, your math teacher may have impressed you with the need to simplify your answers. I'm here to tell you – she was wrong. The form your answer should be in depends entirely on what you will do with it next. In addition, the process of "simplifying," often messy algebra, can ruin perfectly correct answers. From the teacher's point of view, "simplifying" obscures how a student arrived at his answer, and makes problems harder to grade. Moral: don't spend a lot of extra time simplifying your answer. Leave it as close to how you arrived at it as possible.

## When should you simplify?

1. Simplify when it actually makes your life easier. For example, in Chapter 2 it's easier to find a second derivative if you simplify the first derivative.

2. Simplify your answer when you need to match it to an answer in the book. You may need to do some algebra to be sure your answer and the book answer are the same.

## When you use your calculator

A calculator is required for this course, and it can be a wonderful tool. However, you should be careful not to rely too strongly on your calculator. Follow these rules of thumb:

1. Estimate your answers. If you expect an answer of about 4, and your calculator says 2500, you've made an error somewhere.

2. Don't round until the very end. Every time you make a calculation with a rounded number, your answer gets a little bit worse.

3. When you answer an applied problem, find a calculator number. It doesn't mean much to suggest that the company should produce $\frac{\sqrt{12100}(2.4)}{2.5}$ items; it's much more meaningful to report that they should produce about 106 items.

4. When you present your final answer, round it to something that makes sense. If you've found an amount of US money, round it to the nearest cent. If you've computed the number of people, round to the nearest person. If there's no obvious context, show your teacher at least two digits after the decimal place.

5. Occasionally in this course, you will need to find the "exact answer." That means – not a calculator approximation. (You can still use your calculator to check your answer.)

# Chapter 1: Review

## Section 1: Algebra Review

The following is a list of some algebra skills you are expected to have. The example problems here are only a brief review. This is not enough to teach you these skills. If you need more review, you can look in any book called Intermediate Algebra.

### *Laws of Exponents*

The Laws of Exponents let you rewrite algebraic expressions that involve exponents. The last three listed here are really definitions rather than rules.

**Laws of Exponents:**

All variables here represent real numbers and all variables in denominators are nonzero.

$x^a \cdot x^b = x^{a+b}$ $\qquad$ $\dfrac{x^a}{x^b} = x^{a-b}$ $\qquad$ $\left(x^a\right)^b = x^{ab}$

$(xy)^a = x^a y^a$ $\qquad$ $\left(\dfrac{x}{y}\right)^b = \dfrac{x^b}{y^b}$ $\qquad$ $x^0 = 1$, provided $x \neq 0$

$x^{-n} = \dfrac{1}{x^n}$, provided $x \neq 0$ $\qquad$ $x^{1/n} = \sqrt[n]{x}$, provided this is a real number

**Example:** Simplify as much as possible and write your answer using only positive exponents:

$$\left(\frac{x^{-2}}{y^{-3}}\right)^2$$

**Solution:**

$$\left(\frac{x^{-2}}{y^{-3}}\right)^2 = \frac{\left(x^{-2}\right)^2}{\left(y^{-3}\right)^2} = \frac{x^{-4}}{y^{-6}} = \frac{y^6}{x^4}$$

**Example:** Rewrite using only positive exponents:

$$\left(\sqrt{p^5}\right)^{-1/3}$$

**Solution:**

$$\left(\sqrt{p^5}\right)^{-1/3} = \left(\left(p^5\right)^{1/2}\right)^{-1/3} = p^{-5/6} = \frac{1}{p^{5/6}}$$

## *Writing Equations of Lines*

**Identify the independent and dependent variables.** The independent variable is the one that you think explains the relationship, and the dependent variable is the one that you think responds. If you are counting cricket chirps per minute at various temperatures, the temperature could affect how the crickets chirp, but the cricket chirps are unlikely to affect the temperature. In this example, the independent variable will be temperature and the dependent variable will be the number of chirps per minute.

**Slope** is a number that tells you which direction the line points. If the slope is positive, the line points uphill as you read from left to right. If the slope is negative, the line points downhill. Horizontal lines have a slope of zero. The closer the slope is to zero, the closer the line is to horizontal. The further the slope is from zero, either positive or negative, the steeper the line is. Vertical lines have undefined slope – because the “run” in the rise over run calculation is zero. One way to define a straight line is as a curve with a constant slope. You can calculate slope using any two points on the line. Parallel lines have the same slope. Perpendicular lines have negative reciprocal slopes (that is, their slopes multiply to make –1).

Slope is a rate of change. The units of slope are fractional, *y*-units over *x*-units, like miles per hour or dollars per day. In an application problem, look for the fractional units to help you find the slope. The identifying feature of a linear equation is that the slope is constant.

**Equations of lines:** There are several different forms of an equation of a line that you might encounter. (Here I'm assuming *x* is the independent variable and *y* is the dependent variable.)

> **Slope-Intercept form, y *= mx + b***: This is the favorite of most students. The form is easy to remember. You can read the slope *m* and *y*-intercept *b* right off the equation. If you don't have the *y*-intercept, you will have to do some algebra to use this form.
>
> **Point-Slope form,** $y - y_1 = m(x - x_1)$**:** This is my favorite form. The slope *m* is visible, and $(x_1, y_1)$ is some known point on the line. I like this form the best because there is no algebra required – just plop the slope and one point into place and you're done.
>
> **Standard form, *Ax + By = C*:** This form is useful for comparing different types of equations. But it's not a very helpful form for graphing or writing the equation of a line. You have to do algebra to find either the slope or any point on the line.

All you need in order to write the equation of a line is the slope and one point. The slope might be given to you (look for fractional units!), or you might compute it from two points, or perhaps get it from another line that is parallel or perpendicular to it. The one point is usually given to you, or you could need to find the intersection of some curves to get the point.

**Find and interpret the rate of change (slope).** If you have two points, whether they are given to you numerically or if you read them off a graph, you can compute the slope using that familiar rise-over-run formula (the difference in the *y*'s over the difference in the *x*'s). If you have an equation, you can algebraically maneuver it into slope-intercept form and read the slope right off. If the situation is described in English, then the constant rate of change is the slope.

Remember the units (*y*-units over *x*-units)! Simply writing down a sentence like "the rate of change is 15 dollars per year" is the biggest step in interpreting the slope.

If the slope is positive, then the function is increasing. If the slope is negative, then the function is decreasing. If the slope is zero, the function is neither increasing or decreasing (staying constant).

You can compare the rates of change for two functions by comparing their slopes. The function whose graph is steeper, whose slope is further from zero, is changing more rapidly.

**Find and interpret various points of the linear function (for example, the *y*-intercept).** The points that are on the line, the points that satisfy the algebraic equation, are individual examples that fit your situation. The *x*- and *y*-intercepts are usually important, but they are not the only important points.

The *y*-intercept is the place where the line crosses the *y*-axis. This is the *y*-value when $x = 0$.

The *x*-intercept is the place where the line crosses the *x*-axis. This is the *x*-value that makes $y = 0$.

There may be other important points that arise because of the applied setting for your problem.

The units can help you decide what the important points mean in each particular situation.

> **Example:** The cost of a diet program is $299 to join, plus $70 per week for the food. Describe this function and tell how much it would cost to join for ten weeks.
>
> **Solution:** You can tell this is a linear function, because the rate of change (look for those fractional units – dollars per week) is a constant. The slope of this function is 70 dollars a week – this means that each week I will spend another 70 dollars on the food. The independent variable ($x$) is weeks and the dependent variable ($y$) is dollars. The *y*-intercept is the $299 fee to join – I will pay this no matter how many weeks I belong. The *x*-intercept for the line doesn't make sense in this situation. The line itself has an *x*-intercept, with $x$ some negative number of weeks. But it's not possible to join the diet program for a negative number of weeks and pay zero dollars. To stay with the program for ten weeks I would pay $299 + 10($70) = $999. This is the point $(x, y) = (10, 999)$, which lies on the line.

If you know the function is linear, two points are enough to write the formula. Use the two points to find the slope, and then you can solve to find the *y*-intercept.

**Example:** A faucet is dripping water at a constant rate into a bowl. At 1:00, there was ½ cup of water in the bowl. At 1:45, there was ¾ cup of water in the bowl. How much water will be in the bowl at 3:30?

**Solution:** This is a linear function, because the faucet is dripping at a constant rate. The domain is the set of times (hours past noon). The range is the set of volumes in cups (numbers $\geq 0$). Let $t$ be the time, measured in hours past noon, and let $W$ be the amount of water in the bowl, measured in cups. There are two points given: when $t = 1$, $W = 0.5$, and when $t = 1.75$, $W = 0.75$. The slope is rise/run, $\Delta W/\Delta t = (0.75 - 0.5) / (1.75 - 1) = .25 / .75 = 1/3$ cups per hour. So the equation will be

$$W = \frac{1}{3}t + b.$$

To find the $W$-intercept, just plug in one of the points you know and solve for $b$:

$$\frac{1}{2} = \frac{1}{3} \cdot 1 + b, \text{ or } b = \frac{1}{6}.$$

The function that tells us how much water is in the bowl after $t$ hours is given by

$$W = \frac{1}{3}t + \frac{1}{6}.$$

As a check, let's make sure this gives us the right answer at the other known point – if I plug in $t = 1.75$, I get $W = 0.75$, which is right. At 3:30, $t = 3.5$, and $W = 4/3$ cup.

## *Factoring and the Quadratic Formula*

By this time, you should have seen how to solve quadratic equations in several different ways. In this class, you will only need a couple: you will be expected to factor easy things, use the quadratic formula, and to approximate the solutions using technology (such as your calculator).

**Factor out common monomial factors.**

Example: $70b^3 + 49b^6 = 7b^3(10 + 7b^3)$

**Recognize and be able to factor a sum or difference of squares and perfect square trinomials.** These "special products" come up all the time, and you should be able to handle them automatically.

**Factor quadratic trinomials with leading coefficient of 1** – if they're easy! This is a guess-and-check process; look for two numbers whose sum is the coefficient of the linear term (exponent = 1) and whose product is the constant term.

**Example:** Factor $x^2 - 11x + 30$

**Solution:** I'm looking for two numbers who add to –11 and multiply to 30; –5 and –6 work. So the factorization is $x^2 - 11x + 30 = (x-5)(x-6)$.

Unfortunately, factoring this way can take a long time, and it's hard to know if you should stop. If you can't find a factorization, is it because you didn't try the right factors yet, or maybe the factors involve square roots, or maybe the quadratic is already fully factored?

I usually don't spend very long searching for a factorization of a quadratic trinomial. If I can't see a factorization quickly, I turn to the quadratic formula (see below).

**Quadratic formula:** The solutions of the quadratic equation $ax^2 + bx + c = 0$ (where *a*, *b*, and *c* are real numbers and $a \neq 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

**Understand and frequently use that deep connection between roots and factors.** This is often called the Zero Product Property. This is the primary reason we factor – to find the roots, solutions, of an equation. But remember that it goes the other way, also. If you know the solutions of a polynomial equation, you can use them to construct the factors.

**Use the quadratic formula to give you the roots, and use them to construct the factors.** If you can't easily factor a quadratic, you can always exploit that deep connection between roots and factors. This takes a little bit of time, too, but it will always give you an answer. This is also the easiest way to find factors that involve square roots.

**Example:** Factor $10x^2 + 14x + 4$.

**Solution:**

I don't immediately see a factorization, but I can use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-14 \pm \sqrt{(14)^2 - 4(10)(4)}}{2(10)} = \frac{-14 \pm \sqrt{196 - 160}}{20} = \frac{-14 \pm 6}{20}.$$

The two roots are $-1$ and $-\frac{2}{5}$, so the factors are $(x+1)$ and $\left(x+\frac{2}{5}\right)$. I'll need to multiply by 10 so that the leading coefficient is right: $10x^2+14x+4=10(x+1)\left(x+\frac{2}{5}\right)$.

## *Solving Exponential Equations*

You will need to remember logarithms, but you won't have to do a lot of algebra with them. You won't have to simplify expressions involving logarithms, so you won't need many of the laws of logarithms. Here they are, just in case you want to look at them – the only ones you are likely to need are Law 3 and Law 4.

**Laws of Exponents:**

1: $\log_a(xy)=\log_a(x)+\log_a(y)$. In English: The log of a product is the sum of the logs.

2: $\log_a\left(\frac{x}{y}\right)=\log_a(x)-\log_a(y)$. In English: The log of a quotient is the difference of the logs.

3: $\log_a(x^n)=n\log_a(x)$. In English: When you take the log of a power, the exponent comes down in front.

4: $\log_a(a) = 1$ and $\log_a(1) = 0$

5: Change of Base Formula: $\log_a(x)=\frac{\log_b(x)}{\log_b(a)}$

An exponential equation is any equation that involves an exponential function. This is the technique you can use to solve for the exponent.

1. Do as much ordinary algebra (adding, subtracting, multiplying or dividing – always to both sides of the equation) as you can in order to isolate the exponent.
2. Take a logarithm of both sides. You can use any base you want here. If you intend to get a calculator approximation, your life will be easier if you use common log or natural log.
3. Use the 3rd Law of Logarithms to bring the exponent down in front. This is the whole point of using logarithms – it gets the exponent on ground level where you can do ordinary algebra to it.
4. Use ordinary algebra to solve for the exponent.

**Example:** A bacteria colony doubles every 20 minutes. It starts with 3 million bacteria at noon. When will there be 8 million bacteria in the colony?

**Solution**: If *t* is in hours and *A*(*t*) is in millions of bacteria, the function that tells how many bacteria in the colony is $A(t) = 3(2)^{3t}$. (Review on your own if you don't remember how to find this function.) So the equation we want to solve is

$$8 = 3(2)^{3t}$$

First, we do as much ordinary algebra as possible to isolate the exponent. For this example, that means dividing both sides of the equation by 3:

$$\frac{8}{3} = 2^{3t}$$

That's as much as we can do without logarithms. Now it's time to take the log of both sides. I want a calculator approximation when I'm done here (so I can write down a time), so I'll use natural log. You can use any log you like, as long as you do the same thing to both sides.

$$\ln\left(\frac{8}{3}\right) = \ln\left(2^{3t}\right) = 3t\ln(2)$$

Taking the log brings the exponent down in front (third Law of Logarithms), which is just what we want. Now we have an equation of the form number = number times *t*; it's time to do ordinary algebra again to solve for *t*. Divide both sides of the equation by $3\ln(2)$ to get

$$t = \frac{\ln(8/3)}{3\ln(2)}$$

This is the exact answer. I can't just look at this answer and see how big it is, though, so I want a calculator answer.

$$t = \frac{\ln(8/3)}{3\ln(2)} \cong 0.4717$$

This tells me that the colony will have 8 million bacteria about 0.47 hours, or a bit more than 28 minutes, past noon. Does this make sense? By counting up we can see that the colony would have 6 million bacteria at 12:20 and 12 million at 12:40, so this is reasonable. There will be 8 million bacteria at about 12:28.

## Section 2: What is a Function?

### Functions

The notion of a function is one of the most powerful in mathematics. It's a surprisingly simple idea, though. The reason students are so often confused when they encounter functions for the first time in an algebra class is the notation. Before we get to the notation, we'll concentrate on the core idea.

Our lives are full of relationships and correspondences between sets, although we don't always think of them in these terms. For example, we know that the number of plates we take out of the cupboard corresponds to the number of people we're expecting at the table. We know that each telephone number we know corresponds to one person that we want to reach. We know that the size of our electric bill corresponds to the amount of electricity we use. A function is just a special type of correspondence.

**Definition:** A *function* is a correspondence between two sets that assigns to each element of the first set **exactly one** element of the second set. The first set, the set of inputs, is called the *domain*. The second set, the set of outputs, is called the *range*.

Functions do not have to have anything to do with numbers. The key point is those words "**exactly one**." That makes them predictable, and that's the reason they're so important.

**Example:** Every person has a birthday. This is an example of a function. Notice that each person gets exactly one birthday. Notice also that lots of people can have the same birthday – that doesn't affect whether this relationship is a function or not. The "exactly one" only needs to work the one direction. In this example, the domain is the set of all people, and the range is the set of all possible birthdays (the days of the year).

**Example**: Every number has a square. This is also an example of a function. Again, notice that every number has exactly one square – if you give me a number, I can give you its square (a function is predictable). In this case, the domain is the set of all numbers, and the range is the set of all possible squares.

The point of a function is to be predictable, so it's nicest if we can write down a rule. There are several different ways to write a function:

- A function could come as a table. The income tax tables in the back of the tax booklet are examples of this kind of function. There's one such function every year for each type of taxpayer: single, married filing jointly, etc. Within each of these tables, the assignment of a tax amount to a taxable income amount is the function, and the information comes from a table. In this example, the domain is the set of possible taxable income amounts and the range is the set of possible tax amounts.

  To tell if a table represents a function, you need to check whether any input has two outputs. Remember, a function associates **exactly one** output to each input. In our income tax example, you can tell it's a function because no matter how many times you look it up, the amount you owe the government doesn't change. Notice that it doesn't matter that several taxable income amounts yield the same tax amount – it's OK for many different inputs to give the same output.

- A function could come as a graph. For example, the graph that shows the Dow Jones average in the newspaper represents a function. The domain is along the horizontal axis (in my newspaper, that represents the set of the last five business days), and the range is represented vertically (the Dow Jones average for that day). The information about this function comes from the graph. In order to find the Dow Jones average for last Friday, say, you read the graph. Every time you read this week's graph for last Friday, you'll see the same Dow Jones average – the graph is predictable.

  To tell if a graph represents a function, you need to check whether any input (along the horizontal axis) has two outputs (values above or below it on the graph). An easy way to tell is to use the vertical line test. If any vertical line hits the graph more than once, then the graph does not represent a function.

  **Example:** The graph of a circle is not a function, because there are lots of vertical lines that cross the circle more than once . This graph fails the vertical line test. The graph of the top half of a circle is a function.

- A function could come as an algebraic rule. This is the way most students think about functions (which may be why so many people become confused about functions). This is a great shorthand way to write a function that has to do with numbers. For example, our square number example from above could be written this way:

  $$f(x) = x^2$$

  This is read "*f* of *x* equals *x* squared."

The *f* here is the name of the function. You'll often see *f* used for function, because *f* is the first letter in the word "function." But any letter or combination of letters would be fine. In fact, it's a good idea to pick a letter that will remind you of what you're doing.

The parentheses here do not denote multiplication. They're read aloud as "of." The fact that they're right next to the name of the function tells you that this is a function, and you should look inside them to see what the variable will be.

The *x* here is the variable name. Again, *x* is very commonly used, but there's nothing magic about it. You could use any letter or symbol that you like. The point is to look within the parentheses to see what letter is there, because that's what will stand for the input in the rule.

The algebraic stuff on the right hand side of the equals sign is the rule. This is the part that tells you what to do with your input. Your input goes exactly in place of the variable (which you identified right above). This rule says "take the input and square it."

**Example:** In the function $C = \frac{5}{9}(F - 32)$, the function name is *C*, the variable name is *F*, and the rule says "first subtract 32 from your input, then multiply the result by 5/9." This is the algebraic representation of the function that associates degrees Celsius to degrees Fahrenheit. The domain here is the set of all possible temperatures, measured in degrees Fahrenheit, and the range is the set of all possible temperatures, measured in degrees Celsius. This is a function, because there is exactly one Celsius measurement corresponding to each Fahrenheit measurement.

One convenient thing about having an algebraic representation for a function is that you don't have to check whether the "exactly one" condition is satisfied. Algebra has that property built in – you always get the same answer when you plug in the same input.

## The Rule of Four

There are four ways that mathematical information can be communicated to you.

- Numerically- as a list of numbers in a table, for example.
- Algebraically or analytically - as a formula.
- Graphically or geometrically, as a graph or a picture.
- In English - the story or word problem.

Each of these ways has distinct advantages and disadvantages. Depending on what kind of mathematical information you need to communicate, you might choose just one of these ways, or some combination of these ways.

Many students are most familiar with algebra and formulas. And many math textbooks seem to focus on formulas. But all of these ways of looking at mathematical information are important. We'll be communicating mathematics in all four ways during this course.

## Numerically

### Advantages

You get precise information - actual numbers. This is often how real-world information comes to you, as numerical data that's been collected.

**Disadvantages**

There's no information about anything that isn't already on your list. Patterns and trends are difficult or impossible to find.

## Algebraically or Analytically (with formulas)

**Advantages**

You get precise information - you can solve for an actual number. You can use a formula to predict information about any number you're interested in. Patterns in the situation may be revealed by what we know about the formula.

**Disadvantages**

Trends may be difficult or impossible to find. Formulas are mathematical models only -- the real world is usually not as neat and tidy as the formula suggests.

## Communicating Graphically or Geometrically

**Advantages**

You get big picture information - you can easily see trends, change, and growth. You can easily approximate the interesting points on the curve.  It's a quick way to see what's really going on.

**Disadvantages**

You can only approximate numbers, except for certain known and labeled points.

## Communicating in English

**Advantages**

This is how real-world problems come. Nobody outside of a math class will ever ask you to solve a quadratic equation. Instead, they'll ask you how many pounds of salmon you'll need to feed a dinner party of eight, or how much is their share of the phone bill, or what's the most efficient speed for running the machinery on the factory floor.

**Disadvantages**

You usually need to use one of the other ways to solve such a problem. Translation can be difficult. English is a fluid language with many meanings. Sometimes there are legitimate but contradictory interpretations of the same English statements.

## Section 3: Library of Functions

There are a few functions that you should be completely familiar with. By this time, you should have seen linear, quadratic, and exponential functions many times.

### Linear Functions

Linear functions are the simplest kind of functions to work with. Many relationships are truly linear, and many more can be approximated well enough with a linear function. Linear functions have many helpful features – their graphs are straight lines, which we know a lot about. Their rates of change are simply slopes, which we know how to find.

**Recognize linear growth, no matter how the information is given to you.** Remember that there are four ways quantitative information can be presented.

**Numerically**: Linear functions have a constant change in $y$ for every constant change in $x$. This reflects the graphical idea of a linear function – the change in $y$ over the change in $x$, or $\Delta y/\Delta x$, is the constant slope of the line. One way to recognize a line is if you see a constant slope in a table of numbers.

**Algebraically**: The formula for a linear function can always be algebraically maneuvered into one of the common forms given above. You can recognize that a function is linear if it has only one independent variable, which is raised to the first power only (no squares, no one-overs, no roots), and some constants.

**Graphically**: Linear functions are the ones whose graphs are straight lines.

**In English**: Linear functions have a constant rate of change. You can often recognize the slope by its units; look for fractional units, rise/run units, like miles per hour, or dollars per pound, or people per year. The $y$-intercept is like the fixed cost or the overhead – how much $y$ you have when $x$ is zero.

### Quadratic Functions

Quadratic functions have lots of applications (for example, the height of a baseball can be modeled with a quadratic function). We already discussed how to solve quadratic equations.

**Numerically**: The best way to tell if a table displays a quadratic function is to graph it.

**Algebraically**: Quadratic functions can always be algebraically maneuvered to look like $f(x) = ax^2 + bx + c$. You can recognize that a function is quadratic if there is only one independent variable, and the only powers you see are 1 and 2.

**Graphically**: The graph of a quadratic function is a parabola, a sort of curvy V-shape. The formula can tell you a lot of information about the graph:

The sign of *a* tells you if the graph opens up (*a* > 0) or down (*a* < 0)

The *y*-intercept (where *x* = 0) is at *y* = *c*.

The solutions of the equation $f(x) = ax^2 + bx + c = 0$ are the zeros, the roots of the function – these are the *x*-intercepts of the parabola.

The vertex formula tells you where the high or low point of the parabola is:

$$x = \frac{-b}{2a}; y = \ldots \text{ plug it in.}$$

You can use the same kind of information to go from the graph to a formula – use the zeros of the function to find the factors, adjust the leading coefficient using the *y*-intercept.

**In English**: If the function is quadratic, they'll need to say so specifically. One of the most common applications is the height of a falling body.

## Exponential Functions

Exponential functions are very common. For example, the compound interest formula is an exponential function. And many natural things grow (or decay) exponentially.

**Numerically**: Exponential functions show a constant *ratio* in *y* for a constant change in *x*. That is, if you increase *x* by 1, you multiply *y* by a constant multiplier. The multiplier is the easiest base to use for the exponential function.

**Algebraically**: An exponential function is of the form $f(x) = A_0 b^x$, where *b* > 0 and *b* ≠ 1. The domain of the exponential function is the set of all real numbers – we can use any real number as the exponent. The range of the exponential function is the set of all positive real numbers. (If we raise a positive number to any power, we get a positive number back.)

Note – some books are totally *e*-happy. That is, they want every exponential function to have base *e*. Now, *e* is a lovely number, and it's the perfect base for some applications – for example, continuously compounded interest. But it's a better idea to let *b* be the multiplier. That is, if you have a quantity that doubles every hour, you'll be much happier if you use *b* = 2.

**Example:** Suppose a bacteria colony is growing in such a way that it doubles in size every 20 minutes. There are 3 million bacteria at noon.

a. How many will there be at 1:00 pm?
b. How many will there be at 1:30 pm?

**Solution:** Because the doubling time is constant, we know the bacteria are growing exponentially.

a. This part is easy to figure out without writing a formula, by just counting up. If they double every 20 minutes, then there are 6 million at 12:20, there are 12 million at 12:40, and there are 24 million at 1:00.
b. This part is not so easy – 1:30 isn't a whole number of 20-minute chunks after noon. So we will build the formula. Our units will be millions of bacteria and hours. The initial amount, the principal, $A_0$ is the 3 million bacteria we started with at noon. Our population is doubling every twenty minutes, so it's being multiplied by 2 every 1/3 hour. Over one hour, then, it will be multiplied by $2^3$.
   The formula that tells how many million bacteria there are in this colony *t* hours after noon is

$$A(t) = 3(2)^{3t}.$$

   1:30 is *t* = 1.5 hours past noon, so there should be $A(1.5) = 3(2)^{3\times 1.5} \cong 67.89$ million bacteria. Does this make sense? Yes, by counting up we find that there should be 48 million at 1:20 and 96 million at 1:40, so this seems right.

**Note:** You can pick whatever units are convenient for you. Your formula may end up looking different, but your answers will be correct. In the bacteria example, you could have used the units of (single) bacterium and twenty-minute-intervals. Then the formula would look different:

$A(t) = 3{,}000{,}000(2)^t$ , but you'd use *t* = 4.5 (because 1:30 is 4.5 twenty-minute-intervals past noon) and you'd get the same answer – about 67.89 million bacteria.

**Graphically**: If *a* > 1, $f(x) = a^x$ represents exponential growth, and the graph of the function will be incredibly flat on the left, incredibly steep on the right. If 0 < *a* < 1, $f(x) = a^x$ represents exponential decay, and the graph of the function will be the mirror image, left to right, of an exponential growth graph. It will be incredibly steep on the left and incredibly flat on the right.

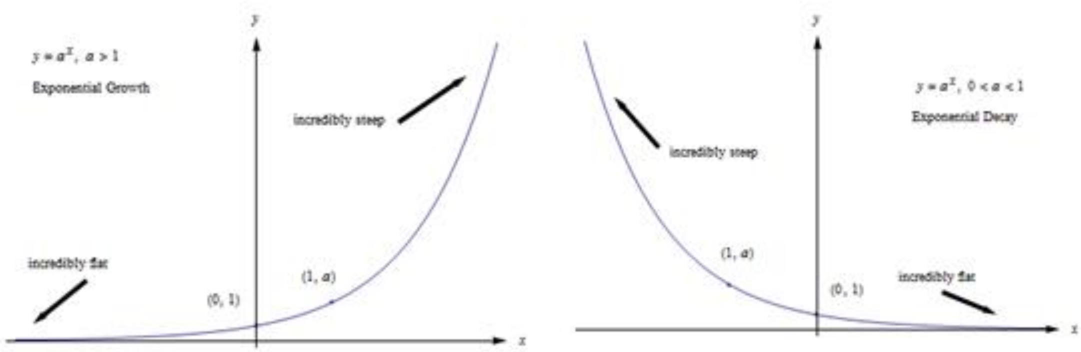



**Figure 6**

We always get two points for free on any simple exponential graph: (0, 1) and (1, *a*).

**In English**: Exponential functions show up when the increase depends on how much is already there. For example, compound interest (the additional interest depends on how much is in the account), or simple population growth (the number of additional babies depends on how many people are in the account).

### *Other functions*

There are several other functions that you should know something about – you should recognize their formulas and their graphs. You should know

- the absolute value functions
- polynomial graphs in general, cubics (3rd degree) in particular
- rational functions (remember vertical asymptotes?)
- power functions ( $f(x) = x^n$ for some *n*), including the square root function
- logarithmic functions – with base 10 and the natural log, with base *e*

## Section 4: New Functions from Old

### *Transformations*

Changing the constants that appear in an algebraic formula changes the graph in some predictable ways.

Here are the principles:

- Changing the *x* (the input) changes the horizontal.
- Changing the *y* (the output) changes the vertical.
- Multiplying by a constant stretches (or squashes) the graph.
- Multiplying by –1 reflects the graph.
- Adding a constant shifts the graph.

Here are the details:

Start with the graph of $y = f(x)$. The graph of each of the following will have the same basic shape as $y = f(x)$, altered as noted. For all of these, *a* is a constant

- $y = af(x)$ is *a* times as tall.
- $y = -f(x)$ is reflected vertically across the *x*-axis (upside down).
- $y = f(x) + a$ is shifted up *a* units.

- $y = f(ax)$ is $\frac{1}{a}$ times as wide.
- $y = f(-x)$ is reflected horizontally across the *y*-axis.
- $y = f(x + a)$ is shifted to the left *a* units.
- Notice that for horizontal stretches or shifts, the effect is sort of backwards from what you might expect at first. These can be confusing – check your answers by plotting a couple of points.

You can handle these all at once. You can do horizontal and vertical changes independently. For each of these, follow the order of operations – stretch and reflect first, then shift. Use the origin as your anchor point, even if it's not on your graph.

**Example:** The graph of $y = -4|x - 2| + 5$ has the same basic shape as $y = |x|$. It's been shifted to the right 2 units. It's upside down, 4 times as tall, and has been shifted up 5 units. So the new graph has its vertex at (2, 5), opens down, and looks stretched vertically compared to the original. You can plug in a point or two to confirm your answer. The new graph goes through the point (0, –3), which makes sense. If you move 2 units to the left of the vertex, the unchanged graph goes up 2 units. Here, it goes down (because the graph is upside down) 8 units (because the graph is stretched by a factor of 4.)

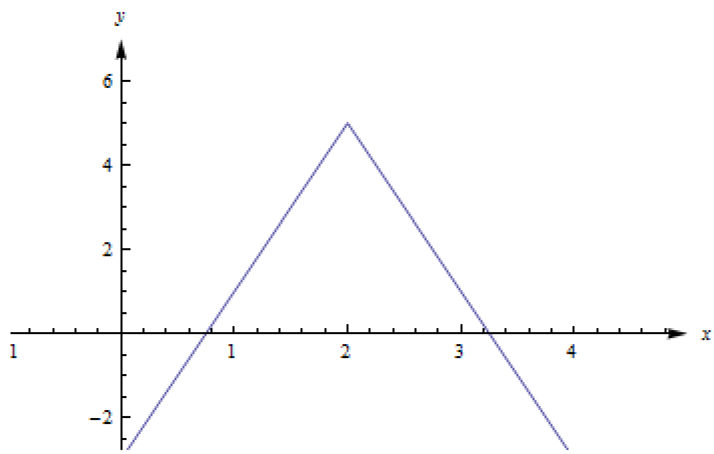



Figure 7

## Composition

One of the most important ways to combine functions is to chain them together, using the output from one as the input into another. A simple example of this is unit conversion – we have one function that tells us how many meters high the ball is after $t$ seconds, and another that tells how many feet are in a certain number of meters. We can use the output of the first function (meters) as the input to the second function to find how many feet high the ball is after $t$ seconds. The chaining together of functions in this way is called composition:

The composition of *f* with *g*, written $f \circ g(x)$, is the function that takes *x*, first does *g* to it, and then does *f* to the output. That is, $f \circ g(x) = f(g(x))$.

**Example:** Let $f(x) = 4x^2 - 13$ and $g(x) = \sqrt{x-2}$. Then their composition $f \circ g(x) = f(g(x)) = f\left(\sqrt{x-2}\right) = 4\left(\sqrt{x-2}\right)^2 - 13.$

## Decomposition

Sometimes you will be given the composition and be asked to identify the component pieces. This is called "decomposition." It turns out to be a very useful skill in calculus. There are often several correct decompositions for a function, but usually only one of them is useful. It may take some practice before you can see which composition is the useful one.

**Example:** The function $G(x) = \dfrac{1}{2x+3}$ is a composition $f \circ g$. Identify the component functions *f* and *g*.

**The most useful solution**: In many cases, there is an "obvious" choice, which you can find by thinking about the inside and the outside. In $G(x) = \dfrac{1}{2x+3}$, the "inside function" is the denominator and the "outside function" is the reciprocal function (that is, "one over"). In the composition $f \circ g$, *g* is the inside function and *f* is the outside function. So this decomposition would be $f(x) = \dfrac{1}{x}$ and $g(x) = 2x+3$. Then $F(x) = f(g(x)) = f(2x+3) = \dfrac{1}{2x+3}$. This is the most useful solution. This is the solution you would see in the answer pages of the textbook. This is the type of decomposition you should look for.

There are usually lots of correct solutions, some of which involve some creativity to find. In this class, you don't have to ever find any of these clever decompositions. If you do find one, it will be correct. But your teacher may suggest that you stick to the more useful decomposition.

## Inverse Functions

The word "inverse" means backwards, and that's what inverse functions are about – going backwards. There are a few different and useful ways to think about inverse functions.

### Swapping the roles of input and output

One important reason we care about inverse functions is that, in many cases, the same relationship can give two different functions, depending on what questions you're interested in answering. Which function you use depends on which quantity you want to use as your input.

**Example:** A private investigator charges a \$500 fee per case, plus \$80 per hour that she works on the case. There is a functional relationship between the hours she works and the amount she bills. But which is the input and which is the output?

If the number of hours she works is the input, then the number of dollars she bills is the output. And it's a function, because each possible number of hours is associated with exactly one billing amount. This might be the function you'd think of first. If we let $h$ be the number of hours the detective works and $b$ be the number of dollars she bills, then this function might be written as

$$b = f(h) = 500 + 80h\,.$$

You'd use this function if you knew how many hours she worked on your case and you wanted to know how much she would charge you.

But the very same relationship yields a different function, whose input is the billing amount and whose output is the number of hours she works. This is also a function, because each possible bill is associated with exactly one amount of time. Again, letting $h$ be the number of hours and $b$ be the amount she bills in dollars, we can write this function:

$$h = f^{-1}(b) = \frac{b-500}{80}$$

This would be a helpful function if you had a certain amount of money to spend and you wanted to know how many hours she would work on your case.

The two functions here are inverse functions. They model the same relationship, but the roles of input and output have been exchanged. That little –1 that looks like an exponent for the *f* in the second formula indicates it is the inverse function for *f*. (It is not an exponent.)

## Undoing

The most important reason we want to study inverse functions is that they undo each other. Remember the algebraic definition of inverse functions:

$f(x)$ and $f^{-1}(x)$ are inverse functions means that their composition in either order is the identity function. That is, both

$$f\left(f^{-1}(x)\right) = x \text{ and } f^{-1}\left(f(x)\right) = x$$

The arrow diagram may be clearer:

$$x \xrightarrow{f^{-1}} f^{-1}(x) \xrightarrow{f} x \text{ and } x \xrightarrow{f} f(x) \xrightarrow{f^{-1}} x$$

## Graphically

If you graph a function and its inverse on the same axes, the inverse will be a reflection of the original across the line $y = x$. That's because the inverse function swaps the roles of input and output. On a graph, that means interchanging the order of the coordinates for every point. That is, if $(x, y)$ is on the graph of $y = f(x)$, then $(y, x)$ will be on the graph of $y = f^{-1}(x)$.

## Chapter 1 Exercises

1. Use the rules of exponents to simplify the following. Write your answer using only positive exponents. Assume all variables represent non-zero numbers.

a. $\dfrac{4y^3}{12y^7}$

b. $\left(2xy^{-3}z^0\right)^3$

c. $\left(m^{-2}\right)^8$

d. $(xyz)^0$

e. $\left(5x^3\right)\left(-7x^5\right)$

f. $\sqrt[4]{\left(4ab^2\right)^4\left(b^{-3}\right)^{-2}}$

2. Write the equation of the line that is parallel to $y = 1.5x - 11$ and has *y*-intercept at *y* = 3.

3. Write the equation of the line that passes through the points (1, 5) and (3, 11).

4. A coffee supplier finds that it costs $850 to roast and package 100 pounds of coffee in a day and $2850 to produce 500 pounds of coffee in a day. Assume that the cost function is linear. Express the cost as a function of the number of pounds of coffee they produce each day.

5. Factor $y^5 - 6y^4 + 9y^3$.

6. Factor $a^4 - 5a^2 - 36$.

7. The two points $\left(-3, \dfrac{5}{8}\right)$ and $(2,20)$ both lie on the graph of a function $y = g(x)$.

   a. Find the formula for $y = g(x)$ if it is a linear function.
   b. Find the formula for $y = g(x)$ if it is of the form $g(x) = Ca^x$.

8. An object is thrown into the air. Its height in feet above the ground *t* seconds later is given by $h(t) = -16t^2 + 30t + 25$.

   a. Find the time when the object reaches its maximum height.
   b. How high does it get?
   c. When does it land?
   d. The graph of *y* = *h*(*t*) is a parabola. What is the physical meaning of the *y*-intercept of this parabola? (That is, what does the *y*-intercept of the parabola tell us about the object?)

9. Suppose $u \circ v(x) = \frac{1}{x^2} + 1$ and $v \circ u(x) = \frac{1}{(x+1)^2}$. Find possible formulas for $u(x)$ and $v(x)$.

10. Suppose $f(x) = x^2 - 2x$.

a. Compute $\frac{f(x) - f(3)}{x-3}$. Simplify your answer.
b. In English, what is the meaning of your answer to part a?

c. Graphically, what is the meaning of your answer to part a?

11. Here is the graph of $y = f(x)$. Use this graph to sketch the graph ***on graph paper*** of $y = -3f(x-2)$. Be careful and neat, and remember to label your graph. Briefly explain what you did to find the graph.

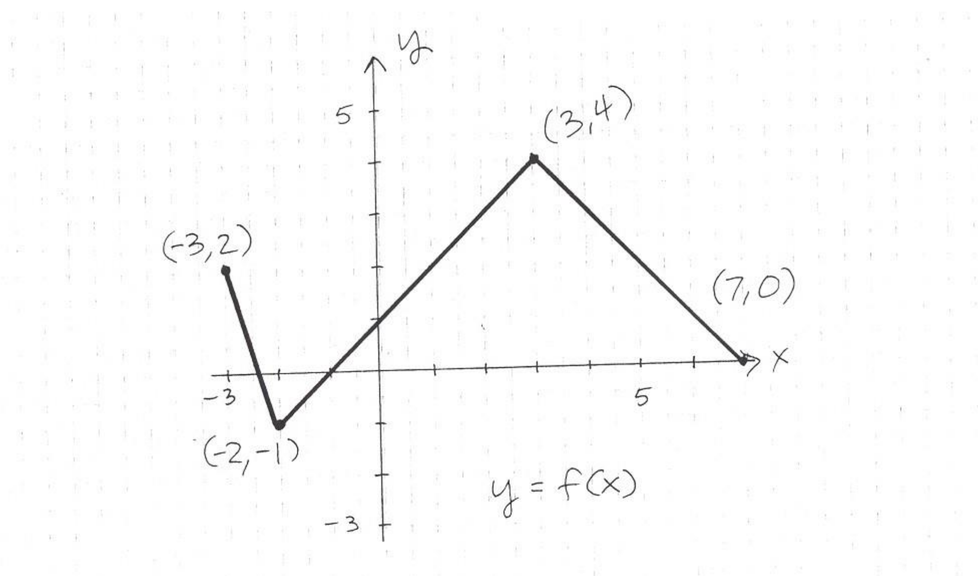



Figure 8

12. In 2000, the number of people infected by a virus was $P_0$. Due to a new vaccine, the number of infected people has decreased by 14% each year since 2000.

a. Find a formula for $P = f(n)$, the number of infected people *n* years after 2000.
b. When will there be (or when were there) just half as many people infected as there were in 2000?

13. Here is the graph of an exponential or logarithmic function.

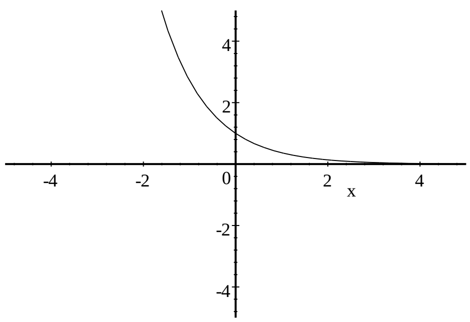



**Figure 9**

a. Is this an exponential function with base $b > 1$, an exponential function with $0 < b < 1$, or a logarithmic function with base $b > 1$?

b. What is the value of *b* (the base) for this graph? How do you know?

# Chapter 2: The Derivative

## Precalculus Idea: Slope and Rate of Change

The slope of a line measures how fast a line rises or falls as we move from left to right along the line. It measures the rate of change of the y-coordinate with respect to changes in the x-coordinate. If the line represents the distance traveled over time, for example, then its slope represents the velocity. In Figure 1, you can remind yourself of how we calculate slope using two points on the line:

$$m = \{\text{ slope from P to Q }\} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

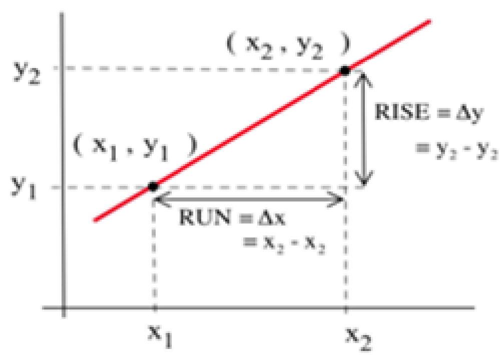



Figure 10

We would like to be able to get that same sort of information (how fast the curve rises or falls, velocity from distance) even if the graph is not a straight line. But what happens if we try to find the slope of a curve, as in Figure 2? We need two points in order to determine the slope of a line. How can we find a slope of a curve, at just one point?

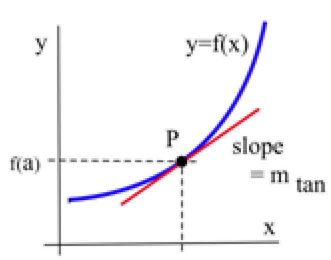



Figure 11

The answer, as suggested in Figure 2 is to find the slope of the tangent line to the curve at that point. Most of us have an intuitive idea of what a tangent line is. Unfortunately, “tangent line” is hard to define precisely.

**Definition:** A **secant line** is a line between two points on a curve.

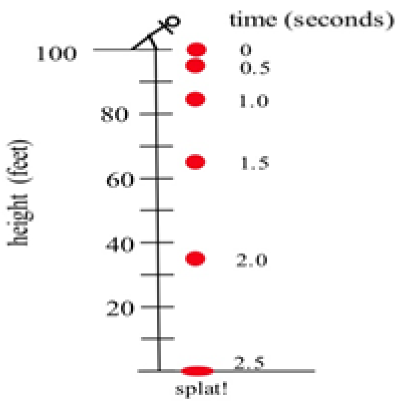
**Can’t-quite-do-it-yet Definition:** A **tangent line** is a line at one point on a curve …. that does its best to be the curve at that point?

It turns out that the easiest way to define the tangent line is to define its slope.

## Section 1: Instantaneous Rate of Change and Tangent Lines

### Instantaneous Velocity

Suppose we drop a tomato from the top of a 100 foot building and time its fall .




| time (s) | height (ft) |
|---|---|
| 0 | 100 |
| 0.5 | 96 |
| 1.0 | 84 |
| 1.5 | 64 |
| 2.0 | 36 |
| 2.5 | 0 |

Figure 12

Some questions are easy to answer directly from the table:

(a) How long did it take for the tomato n to drop 100 feet? (2.5 seconds)

(b) How far did the tomato fall during the first second? (100 – 84 = 16 feet)

(c) How far did the tomato fall during the last second? (64 – 0 = 64 feet)

(d) How far did the tomato fall between t =.5 and t = 1? (96 – 84 = 12 feet)

Some other questions require a little calculation:

(e) What was the average velocity of the tomato during its fall?

$$\textbf{Average velocity} = \frac{\textbf{distance fallen}}{\textbf{total time}} = \frac{\boldsymbol{\Delta}\textbf{ position}}{\boldsymbol{\Delta}\textbf{ time}} = \frac{-100 \text{ ft}}{2.5 \text{ s}} = -40 \text{ ft/s} .$$

(f) What was the average velocity between t=1 and t=2 seconds?

$$\text{Average velocity} = \frac{\Delta \text{ position}}{\Delta \text{ time}} = \frac{36 \text{ ft} - 84 \text{ ft}}{2 \text{ s} - 1 \text{ s}} = \frac{-48 \text{ ft}}{1 \text{ s}} = -48 \text{ ft/s} .$$

Some questions are more difficult.

(g) How fast was the tomato falling 1 second after it was dropped?

This question is significantly different from the previous two questions about average velocity. Here we want the **instantaneous velocity,** the velocity at an instant in time. Unfortunately the tomato is not equipped with a speedometer so we will have to give an approximate answer.

One crude approximation of the instantaneous velocity after 1 second is simply the average velocity during the entire fall, –40 ft/s . But the tomato fell slowly at the beginning and rapidly near the end so the "–40 ft/s" estimate may or may not be a good answer.

We can get a better approximation of the instantaneous velocity at t=1 by calculating the average velocities over a short time interval near t = 1 . The average velocity between t = 0.5 and t = 1 is $\frac{-12 \text{ feet}}{0.5 \text{ s}} = -24$ ft/s, and the average velocity between t = 1 and t = 1.5 is $\frac{-20 \text{ feet}}{.5 \text{ s}} = -40$ ft/s so we can be reasonably sure that the instantaneous velocity is between –24 ft/s and –40 ft/s.

In general, the shorter the time interval over which we calculate the average velocity, the better the average velocity will approximate the instantaneous velocity. The average velocity over a time interval is $\frac{\Delta \text{ position}}{\Delta \text{ time}}$, which is the slope of the **secant line** through two points on the graph of height versus time (Fig. 4). The instantaneous velocity at a particular time and height is the slope of the **tangent line** to the graph at the point given by that time and height.

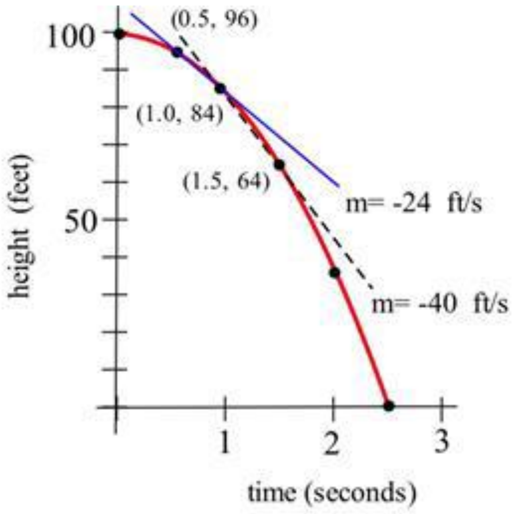



Figure 13

**Average velocity** $= \dfrac{\Delta \text{ position}}{\Delta \text{ time}}$ **= slope of the secant line through 2 points.**

**Instantaneous velocity = slope of the line tangent to the graph.**

## Tangent Lines

Do this!

The graph below is the graph of $y = f(x)$. We want to find the slope of the tangent line at the point (1, 2).

First, draw the secant line between (1, 2) and (2, −1) and compute its slope.

Now draw the secant line between (1, 2) and (1.5, 1) and compute its slope.

Compare the two lines you have drawn. Which would be a better approximation of the tangent line to the curve at

(1, 2)?

Now draw the secant line between (1, 2) and (1.3, 1.5) and compute its slope. Is this line an even better approximation of the tangent line?

Now draw your best guess for the tangent line and measure its slope. Do you see a pattern in the slopes?

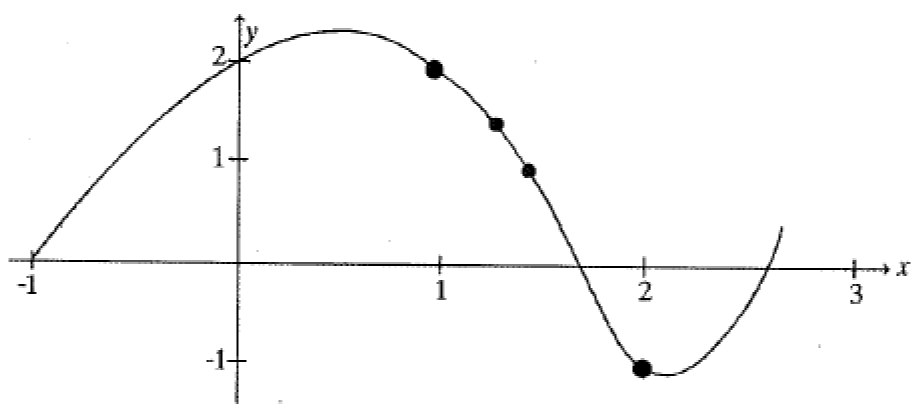



**Figure 14**

You should have noticed that as the interval got smaller and smaller, the secant line got closer to the tangent line and its slope got closer to the slope of the tangent line. That's good news – we know how to find the slope of a secant line.

**Example:** Now let's look at the problem of finding the slope of the line L (Figure 6) which is tangent to $f(x) = x^2$ at the point (2,4).

We could estimate the slope of L from the graph, but we won't. Instead, we will use the idea that secant lines over tiny intervals approximate the tangent line.

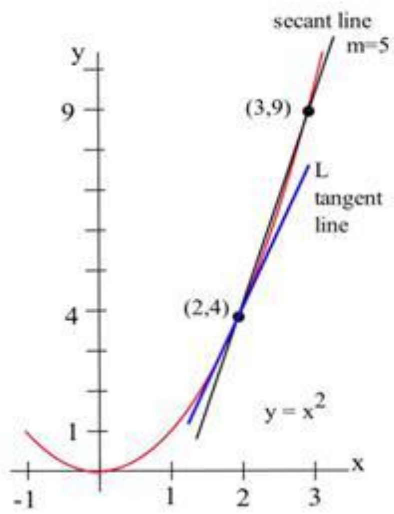



Figure 15

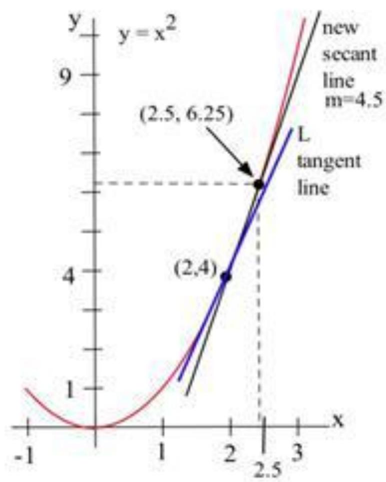



Figure 16

We can see that the line through (2,4) and (3,9) on the graph of f is an approximation of the slope of the tangent line, and we can calculate that slope exactly: m = Δy/Δx = (9–4)/(3–2) = 5. But m = 5 is only an estimate of the slope of the tangent line and not a very good estimate. It's too big. We can get a better estimate by picking a second point on the graph of f which is closer to (2,4) — the point (2,4) is fixed and it must be one of the points we use. From Figure 7, we can see that the slope of the line through the points (2,4) and (2.5,6.25) is a better approximation of the slope of the tangent line at (2,4): m = Δy/Δx = (6.25 – 4)/(2.5 – 2) = 2.25/.5 = 4.5 , a better estimate, but still

an approximation. We can continue picking points closer and closer to (2,4) on the graph of f, and then calculating the slopes of the lines through each of these points and the point (2,4):

Points to the left of (2,4)

| x | $y = x^2$ | slope of line through (x,y) and (2,4) |
|---|---|---|
| 1.5 | 2.25 | 3.5 |
| 1.9 | 3.61 | 3.9 |
| 1.99 | 3.9601 | 3.99 |

Points to the right of (2,4)

| x | $y = x^2$ | slope of line through (x,y) and (2,4) |
|---|---|---|
| 3 | 9 | 5 |
| 2.5 | 6.25 | 4.5 |
| 2.01 | 4.0401 | 4.01 |

The only thing special about the x–values we picked is that they are numbers which are close, and very close, to

x = 2. Someone else might have picked other nearby values for x. As the points we pick get closer and closer to

the point (2,4) on the graph of $y = x^2$ , the slopes of the lines through the points and (2,4) are better approximations of the slope of the tangent line, and these slopes are getting closer and closer to 4.

We can bypass much of the calculating by not picking the points one at a time: let's look at a general point near (2,4). Define $x = 2 + h$ so h is the increment from 2 to x (Fig. 8). If h is small, then $x = 2 + h$ is close to 2 and the point $(2+h, f(2+h)) = (2+h, (2+h)^2)$ is close to (2,4). The slope m of the line through the points (2,4) and $(2+h, (2+h)^2)$ is a good approximation of the slope of the tangent line at the point (2,4):

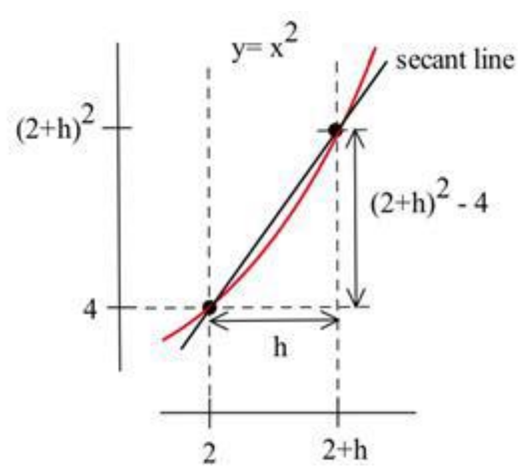



Figure 17

$$m = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 4}{(2+h) - 2} = \frac{\{4 + 4h + h^2\} - 4}{h} = \frac{4h + h^2}{h} = \frac{h(4 + h)}{h} = 4 + h\,.$$

If $h$ is very small, then $m = 4 + h$ is a very good approximation to the slope of the tangent line, and $m = 4 + h$ is very close to the value 4.

The value $m = 4 + h$ is the slope of the secant line through the two points $(2,4)$ and $(2+h, (2+h)^2)$. As $h$ gets smaller and smaller, this slope approaches the slope of the tangent line to the graph of $f$ at $(2,4)$.

In some applications, we need to know where the graph of a function $f(x)$ has horizontal tangent lines (slopes = 0). In Fig. 3, the slopes of the tangent lines to graph of $y = f(x)$ are 0 when $x = 2$ or $x \approx 4.5$ .

**Example:** At right is the graph of $y = g(x)$. At what values of $x$ does the graph of $y = g(x)$ in Fig. 9 have horizontal tangent lines?

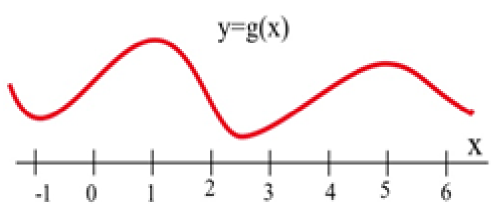



Figure 18

**Solution:** The tangent lines to the graph of $g$ are horizontal (slope = 0) when $x \approx -1, 1, 2.5$, and $5$.

# Section 2: The Derivative

## Definition of the Derivative

The tangent line problem and the instantaneous velocity problem are the same problem. In each problem we wanted to know how rapidly something was **changing at an instant in time**, and the answer turned out to be finding the **slope of a tangent line,** which we approximated with the **slope of a secant line**. This idea is the key to defining the slope of a curve.

## The Derivative:

The **derivative** of a function f at a point (x, f(x)) is the instantaneous rate of change.

The **derivative** is the slope of the tangent line to the graph of f at the point (x, f(x)).

The **derivative** is the slope of the curve f(x) at the point (x, f(x)).

A function is called **differentiable** at (x, f(x)) if its derivative exists at (x, f(x)).

**Notation for the Derivative:**

The **derivative of y = f(x) with respect to x** is written as

$f\,'(x)$ (read aloud as "f prime of x"), or $y'$ ("y prime")

or $\frac{dy}{dx}$ (read aloud as "dee why dee ex"), or $\frac{df}{dx}$

The notation that resembles a fraction is called **Leibniz notation**. It displays not only the name of the function (f or y), but also the name of the variable (in this case, x). It looks like a fraction because the derivative is a slope. In fact, this is simply $\frac{\Delta y}{\Delta x}$ written in Roman letters instead of Greek letters.

**Verb forms:**

We **find the derivative** of a function, or **take the derivative** of a function, or **differentiate** a function.

We use an adaptation of the $\frac{dy}{dx}$ notation to mean "find the derivative of f(x):"

$$\frac{d}{dx}\left(f(x)\right) = \frac{df}{dx}$$

**Formal Algebraic Definition:**

$$f\,'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad (*)$$

**Practical Definition:**

The derivative can be approximated by looking at an average rate of change, or the slope of a secant line,
over a very tiny interval. The tinier the interval, the closer this is to the true instantaneous rate of change, slope of the tangent line, or slope of the curve.

**Looking Ahead:**

We will have methods for computing exact values of derivatives from formulas soon. If the function is
given to you as a table or graph, you will still need to approximate this way.

(* information about "lim" is in Chapter 5: Optional Topics)

This is the foundation for the rest of this chapter. It's remarkable that such a simple idea (the slope of a tangent line) and such a simple definition (for the derivative f ' ) will lead to so many important ideas and applications.

## The Derivative as a Function

We now know how to find (or at least approximate) the derivative of a function for any x-value; this means we can think of the derivative as a function, too. The inputs are the same x's; the output is the value of the derivative at

**Example:** Fig. 10 is the graph of a function $y = f(x)$. We can use the information in the graph to fill in a table showing values of $f'(x)$:

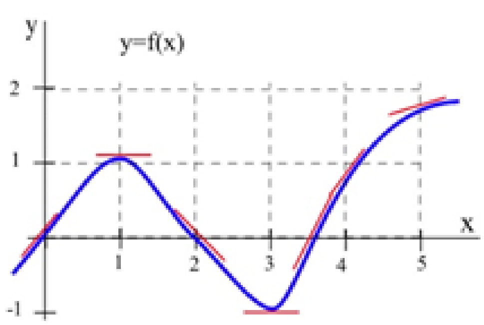



Figure 19

At various values of *x*, draw your best guess at the tangent line and measure its slope. You might have to extend your lines so you can read some points. In general, your estimate of the slope will be better if you choose points that are easy to read and far away from each other. Here are my estimates for a few values of *x* (parts of the tangent lines I used are shown):

| $x$ | $y = f(x)$ | $f'(x)$ = the estimated SLOPE of the tangent line to the curve at the point $(x, y)$. |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |
| 2 | 0 | −1 |
| 3 | −1 | 0 |
| 4 | 1 | 1 |
| 5 | 2 | 0.5 |

We can estimate the values of f’(x) at some non-integer values of x, too: $f'(.5) \approx 0.5$ and $f'(1.3) \approx -0.3$.

We can even think about entire intervals. For example, if $0 < x < 1$, then f(x) is increasing, all the slopes are positive, and so f’(x) is positive.

The values of f’(x) definitely depend on the values of x , and f’(x) is a function of x. We can use the results in the table to help sketch the graph of f’(x) in Fig. 11.

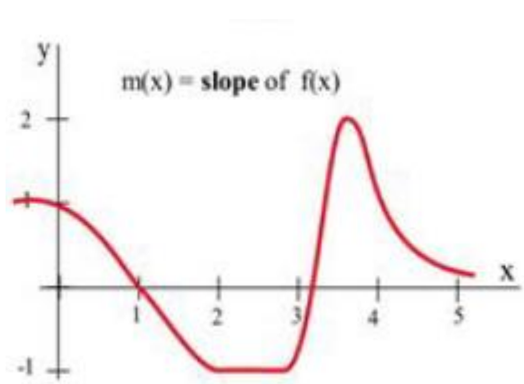



Figure 20

**Example:** Fig. 12 is the graph of the height h(t) of a rocket at time t. Sketch the graph of the **velocity** of the rocket at time t. (Velocity is the **derivative** of the height function, so it is the **slope of the tangent** to the graph of position or height.)

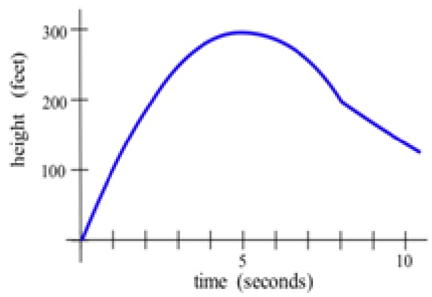



Figure 21

**Solution:** The lower graph in Fig. 13 shows the velocity of the rocket. This is v(t) = h'(t).

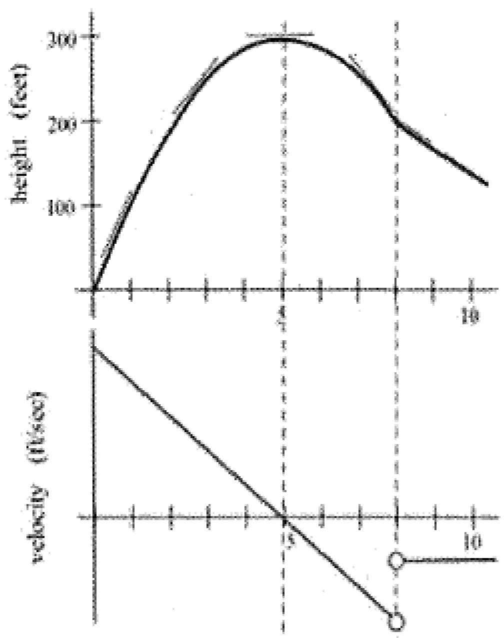



Figure 22

## Section 3: Rates in Real Life

So far we have emphasized the derivative as the slope of the line tangent to a graph . That interpretation is very visual and useful when examining the graph of a function, and we will continue to use it. Derivatives, however, are used in a wide variety of fields and applications, and some of these fields use other interpretations. The following are a few interpretations of the derivative that are commonly used.

**General**

Rate of Change: f '(x) is the **rate of change** of the function at x. If the units for x are years and the units for f(x) are people, then the units for $\frac{df}{dx}$ are $\frac{\text{people}}{\text{year}}$ , a rate of change in population.

**Graphical**

Slope: f '(x) is the **slope of the line tangent to the graph of f at the point ( x, f(x) ).**

**Physical**

Velocity: If f(x) is the position of an object at time x, then f '(x) is the **velocity** of the object at time x. If

the units for x are hours and f(x) is distance measured in miles, then the units for **f** **'**(x) = $\frac{df}{dx}$ are $\frac{\text{miles}}{\text{hour}}$ ,

miles per hour, which is a measure of velocity.

Acceleration If f(x) is the velocity of an object at time x, then f '(x) is the **acceleration** of the object at time

x. If the units are for x are hours and f(x) has the units $\frac{\text{miles}}{\text{hour}}$ , then the units for the acceleration **f '(x)** =

$\frac{df}{dx}$ are $\frac{\text{miles/hour}}{\text{hour}} = \frac{\text{miles}}{\text{hour}^2}$ , miles per hour per hour.

**<u>Business</u>**

Marginal Cost If $f(x)$ is the total cost of $x$ objects, then **$f'(x)$** is the **marginal cost**, at a production level

of $x$. This marginal cost is approximately the <u>additional</u> cost of making one more object once we have

already made $x$ objects. If the units for $x$ are bicycles and the units for $f(x)$ are dollars, then the units for

$f'(x) = \frac{df}{dx}$ are $\frac{\text{dollars}}{\text{bicycle}}$, the cost per bicycle.

Marginal Profit If $f(x)$ is the total profit from producing and selling $x$ objects, then **$f'(x)$** is the **marginal**

**profit**, the profit to be made from producing and selling one more object. If the units for $x$ are bicycles and

the units for $f(x)$ are dollars, then the units for $f'(x) = \frac{df}{dx}$ are $\frac{\text{dollars}}{\text{bicycle}}$, dollars per bicycle, which is the

profit per bicycle.

In business contexts, the word "marginal" usually means the derivative or rate of change of some quantity.

One of the strengths of calculus is that it provides a unity and economy of ideas among diverse applications. The vocabulary and problems may be different, but the ideas and even the notations of calculus are still useful.

## Business and Economics Terms

Suppose you are producing and selling some item. The profit you make is the amount of money you take in minus what you have to pay to produce the items. Both of these quantities depend on how many you make and sell. (So we have functions here.) Here is a list of definitions for some of the terminology, together with their meaning in algebraic terms and in graphical terms.

Your **cost** is the money you have to spend to produce your items.

The **Fixed Cost (FC)** is the amount of money you have to spend regardless of how many items you produce. FC can include things like rent, purchase costs of machinery, and salaries for office staff. You have to pay the fixed costs even if you don't produce anything.

The **Total Variable Cost (TVC)** for q items is the amount of money you spend to actually produce them. TVC includes things like the materials you use, the electricity to run the machinery, gasoline for your delivery vans, maybe the wages of your production workers. These costs will vary according to how many items you produce.

The **Total Cost (TC)** for q items is the total cost of producing them. It's the sum of the fixed cost and the total variable cost for producing q items.

The **Average Cost (AC)** for q items is the total cost divided by q, or TC/q. You can also talk about the average fixed cost, FC/q, or the average variable cost, TVC/q.

The **Marginal Cost (MC)** at q items is the cost of producing the *next* item. Really, it's

$$\mathbf{MC(q) = TC(q + 1) - TC(q).}$$

In many cases, though, it's easier to approximate this difference using calculus (see Example below). And some

sources define the marginal cost directly as the derivative,

$$\mathbf{MC(q) = TC'(q)}.$$

In this course, we will use both of these definitions as if they were interchangeable.

**Demand** is the functional relationship between the price **p** and the quantity **q** that can be sold (that is demanded). Depending on your situation, you might think of **p** as a function of **q**, or of **q** as a function of **p**.

Your **revenue** is the amount of money you actually take in from selling your products. Revenue is price × quantity.

The **Total Revenue (TR)** for q items is the total amount of money you take in for selling q items.

The **Average Revenue (AR)** for q items is the total revenue divided by q, or TR/q.

The **Marginal Revenue (MR)** at q items is the cost of producing the *next* item,

**MR(q) = TR(q + 1) – TR(q)**.

Just as with marginal cost, we will use both this definition and the derivative definition

**MR(q) = TR'(q).**

Your **profit** is what's left over from total revenue after costs have been subtracted.

The **Profit (π)** for q items is TR(q) – TC(q).

The average profit for q items is **π**/q. The marginal profit at q items is π(q + 1) – π(q), or $\pi'(q)$

**Example:** Why is it OK that are there two definitions for Marginal Cost (and Marginal Revenue, and Marginal Profit)?

We have been using slopes of secant lines over tiny intervals to approximate derivatives. In this example, we'll turn that around – we'll use the derivative to approximate the slope of the secant line.

Notice that the "cost of the next item" definition is actually the slope of a secant line, over an interval of 1 unit:

$$MC(q) = C(q+1) - 1 = \frac{C(q+1)-1}{1}$$

So this is approximately the same as the derivative of the cost function at *q*:

$$MC(q) = C'(q)$$

In practice, these two numbers are so close that there's no practical reason to make a distinction.  For our purposes, the marginal cost **is** the derivative **is** the cost of the next item.

**Graphical Interpretations of the Basic Business Math Terms**

**Illustration/Example:**

Here are the graphs of TR and TC for producing and selling a certain item.  The horizontal axis is the number of items, in thousands.  The vertical axis is the number of dollars, also in thousands.

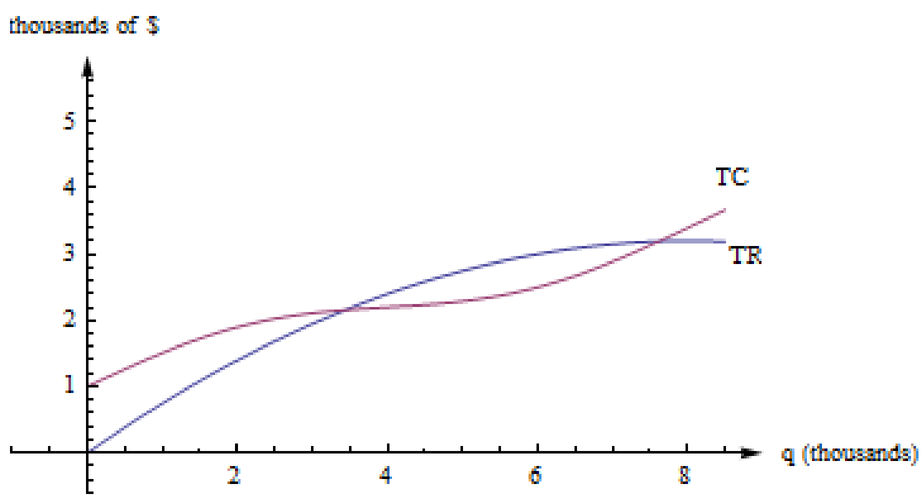



Figure 23

First, notice how to find the fixed cost and variable cost from the graph here. **FC is the y-intercept of the TC graph.** (FC = TC(0).) The graph of TVC would have the same shape as the graph of TC, shifted down. (TVC = TC – FC.)

We already know that we can find average rates of change by finding slopes of secant lines. AC, AR, MC, and MR are all rates of change, and we can find them with slopes, too.

**AC(q) is the slope of a diagonal line, from (0, 0) to (q, TC(q)). AR(q) is the slope of the line from (0, 0) to (q, TR(q)).**

MC(q) = TC(q + 1) – TC(q), but that's impossible to read on this graph. How could you distinguish between TC(4022) and TC(4023)? On this graph, that interval is too small to see, and our best guess at the secant line is actually the tangent line to the TC curve at that point. (This is the reason we want to have the derivative definition handy.)

**MC(q) is the slope of the tangent line to the TC curve at (q, TC(q)). In a similar way, MR(q) is the slope of the tangent line to the TR curve at (q, TR(q)).**

**Profit is the distance between the TR and TC curve.** If you experiment with your clear plastic ruler, you'll see that the biggest profit occurs exactly when the tangent lines to the TR and TC curves are parallel. This is the rule **"profit is maximized when MR = MC."**

## Rates in Real Life

**Example:** You can estimate a tree's age in years by multiplying its diameter (measured in inches) by its growth factor (a number that depends on the species). According to the Missouri Department of Conservation, the Growth factor for a cottonwood tree is 2.

a. Suppose you find a cottonwood tree in Missouri that is 6 inches in diameter. How old would you estimate it to be?
b. What are the units of the growth factor?
c. Is this growth factor a derivative?

**Solution:** a. The cottonwood tree should be about 6 × 2 = 12 years old.

b. The units of the growth factor are years per inch (because when we multiply the growth factor by inches, we get years).

c. Yes, the growth factor is a derivative. It has fractional units (years per inch), so it represents a rate. In this case, it's the derivative of the function that gives the age of a tree as a function of its diameter. The function is linear, so the derivative in this case is the constant slope, 2 years per inch.

**Example:** The length of day (that is, daylight) in Seattle is a function of the day of the year. For example, on August 12th, 2012, there were about 14 hours 24 minutes of daylight. In Seattle, August is the summer, approaching the autumnal equinox. The days are decreasing in length by about three minutes per day. So the derivative of this function is about −3 minutes per day. On January 15, 2012, which is wintertime in Seattle, there were about 8 hours 52 minutes of daylight, and the derivative was about (positive) 2 minutes per day; the length of the day was increasing by about 2 minutes a day.

## Section 4: Derivatives of Formulas

In this section, we'll get the derivative rules that will let us find formulas for derivatives when our function comes to us as a formula. This is a very algebraic section, and you should get lots of practice. When you tell someone you have studied calculus, this is the one skill they will expect you to have. There's not a lot of deep meaning here – these are strictly algebraic rules.

### Building Blocks

These are the simplest rules – rules for the basic functions. We won't prove these rules; we'll just use them. But first, let's look at a few so that we can see they make sense.

**Example:** Find the derivative of $y = f(x) = mx + b$

**Solution:** This is a linear function, so its graph is its own tangent line! The slope of the tangent line, the derivative, is the slope of the line: $f'(x) = m$

**Rule: The derivative of a linear function is its slope**

**Example:** Find the derivative of $f(x) = 135$.

**Solution:** Think about this one graphically, too. The graph of *f*(*x*) is a horizontal line. So its slope is zero.

$$f'(x) = 0$$

**Rule: The derivative of a constant is zero**

**Example:** I will just tell you that the derivative of $f(x) = x^3$ is $f'(x) = 3x^2$. Now think about the function $g(x) = 4x^3$. What will its derivative be?

**Solution:** Think about what this change means to the graph of *g* – it's now 4 times as tall as the graph of *f*. If we find the slope of a secant line, it will be $\frac{\Delta g}{\Delta x} = \frac{4\Delta f}{\Delta x} = 4\frac{\Delta f}{\Delta x}$; each slope will be 4 times the slope of the secant line on the *f* graph. This property will hold for the slopes of tangent lines, too:

$$\frac{d}{dx}\left(4x^3\right) = 4\frac{d}{dx}\left(x^3\right) = 4 \cdot 3x^2 = 12x^2$$

**Rule: Constants come along for the ride.**

OK, enough of that. Here are the basic rules, all in one place.

## Derivative Rules: Building Blocks

In what follows, *f* and *g* are differentiable functions of *x*.

(a) **Constant Multiple Rule:** $\frac{d}{dx}(kf)=kf'$

(b) **Sum (or Difference) Rule:** $\frac{d}{dx}(f+g)=f'+g'$ **(or** $\frac{d}{dx}(f-g)=f'-g'$**)**

(c) **Power Rule:** $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$

Special cases: $\frac{d}{dx}(k)=0$ (because $k=kx^0$ )

$\frac{d}{dx}(x)=1$ (because $x=x^1$)

(d) **Exponential Functions:** $\frac{d}{dx}\left(e^x\right)=e^x$

$\frac{d}{dx}\left(a^x\right)=\ln a\cdot a^x$

(e) **Natural Logarithm:** $\frac{d}{dx}(\ln x)=\frac{1}{x}$

The sum, difference, and constant multiple rule combined with the power rule allow us to easily find the derivative of any polynomial.

**Example:** Find the derivative of $p(x)=17x^{10}+13x^8-1.8x+1003$

**Solution:**

$$\frac{d}{dx}\left(17x^{10}+13x^8-1.8x+1003\right)$$
$$=\frac{d}{dx}\left(17x^{10}\right)+\frac{d}{dx}\left(13x^8\right)-\frac{d}{dx}(1.8x)+\frac{d}{dx}(1003)$$
$$=17\frac{d}{dx}\left(x^{10}\right)+13\frac{d}{dx}\left(x^8\right)-1.8\frac{d}{dx}(x)+\frac{d}{dx}(1003)$$
$$=17\left(10x^9\right)+13\left(8x^7\right)-1.8(1)+0$$
$$=170x^9+104x^7-1.8$$

No, you don't have to show every single step. Do be careful when you're first working with the rules, but pretty soon you'll be able to just write down the derivative directly:

**Example:** $\frac{d}{dx}\left(17x^2-33x+12\right)=34x-33.$

The power rule works even if the power is negative or a fraction. In order to apply it, first translate all roots and one-overs into exponents:

**Example:** Find the derivative of $y=3\sqrt{t}-\frac{4}{t^4}+5e^t$

**Solution:** First step – translate into exponents:

$$y=3\sqrt{t}-\frac{4}{t^4}+5e^t=3t^{1/2}-4t^{-4}+5e^t$$

Now you can take the derivative:

$$\frac{d}{dt}\left(3\sqrt{t}-\frac{4}{t^4}+5e^t\right)=\frac{d}{dt}\left(3t^{1/2}-4t^{-4}+5e^t\right)$$
$$=3\left(\frac{1}{2}t^{-1/2}\right)-4\left(-4t^{-5}\right)+5\left(e^t\right)=\frac{3}{2}t^{-1/2}+16t^{-5}+5e^t.$$

Be careful when finding the derivatives with negative exponents.

**Example :** The cost to produce x items is $\sqrt{x}$ hundred dollars.

(a) What is the cost for 100 items? 101 items? What is cost of the 101st item?

(b) For $f(x) = \sqrt{x}$, calculate $f'(x)$ and evaluate $f'$ at $x = 100$. How does $f'(100)$ compare with the last answer in part (a)?

**Solution:**

(a) Put $f(x) = \sqrt{x} = x^{1/2}$ hundred dollars, the cost for x items. Then f(100) = \$1000 and f(101) = \$1004.99, so it costs \$4.99 for that 101st item. Using this definition, the marginal cost is \$4.99.

(b) $f'(x) = \frac{1}{2}\ x^{-1/2} = \frac{1}{2\sqrt{x}}$ so $f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$ hundred dollars = \$5.00. Note how close these answers are! This shows (again) why it's OK that we use both definitions for marginal cost.

## Product and Quotient Rules

The basic rules will let us tackle simple functions. But what happens if we need the derivative of a combination of these functions?

**Example:** Find the derivative of $g(x) = (4x^3 - 11)(x + 3)$

**Solution:** This function is not a simple sum or difference of polynomials. It's a product of polynomials. We can simply multiply it out to find its derivative:

$$g(x) = (4x^3 - 11)(x + 3) = 4x^4 - 11x + 12x^3 - 33$$
$$g'(x) = 16x^3 - 11 + 36x^2$$

**Example:** Find the derivative of $f(x) = (4x^5 + x^3 - 1.5x^2 - 11)(x^7 - 7.25x^5 + 120x + 3)$

**Solution:** This function is not a simple sum or difference of polynomials. It's a product of polynomials. We **could** simply multiply it out to find its derivative as before – who wants to volunteer? Nobody?

We'll need a rule for finding the derivative of a product so we don't have to multiply everything out.

Is the rule what we hope it is, that we can just take the derivatives of the factors and multiply them? Unfortunately, no – that won't give the right answer.

**Example:** Find the derivative of $g(x) = (4x^3 - 11)(x + 3)$

**Solution:** We already worked out the derivative. It's $g'(x) = 16x^3 - 11 + 36x^2$. What if we try differentiating the factors and multiplying them? We'd get $(12x^2)(1) = 12x^2$, which is totally different from the correct answer.

The rules for finding derivatives of products and quotients are a little complicated, but they save us the much more complicated algebra we might face if we were to try to multiply things out. They also let us deal with products where the factors are not polynomials. We can use these rules, together with the basic rules, to find derivatives of many complicated looking functions.

## Derivative Rules: Product and Quotient Rules

In what follows, *f* and *g* are differentiable functions of *x*.

(f) **Product Rule:**

$$\frac{d}{dx}(fg) = f'g + fg'$$

The derivative of the first factor times the second left alone, plus the first left alone times the derivative of the second.

The product rule can extend to a product of several functions; the pattern continues – take the derivative of each factor in turn, multiplied by all the other factors left alone, and add them up.

(g) **Quotient Rule:**

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

The numerator of the result resembles the product rule, but there is a minus instead of a plus; the minus sign goes with the g'. The denominator is simply the square of the original denominator – no derivatives there.

**Example:** Find the derivative of $F(t) = e^t \ln t$

**Solution:** This is a product, so we need to use the product rule. I like to put down empty parentheses to remind myself of the pattern; that way I don't forget anything.

$$F'(t) = (\quad)(\quad) + (\quad)(\quad)$$

Then I fill in the parentheses – the first set gets the derivative of $e^t$, the second gets $\ln t$ left alone, the third gets $e^t$ left alone, and the fourth gets the derivative of $\ln t$.

$$F'(t) = (e^t)(\ln t) + (e^t)\left(\frac{1}{t}\right)$$

Notice that this was one we couldn't have done by "multiplying out."

**Example:** Find the derivative of $y = \dfrac{x^4 + 4^x}{3 + 16x^3}$

**Solution:** This is a quotient, so we need to use the quotient rule. Again, I find it helpful to put down the empty parentheses as a template:

$$y' = \frac{(\ )(\ ) - (\ )(\ )}{(\ )^2}$$

Then I fill in all the pieces:

$$y' = \frac{\left(4x^3 + \ln 4 \cdot 4^x\right)\left(3 + 16x^3\right) - \left(x^4 + 4^x\right)\left(48x^2\right)}{\left(3 + 16x^3\right)^2}$$

Now for goodness' sakes don't try to simplify that! Remember that "simple" depends on what you will do next; in this case, we were asked to find the derivative, and we've done that. Please STOP!

## Chain Rule

There is one more type of complicated function that we will want to know how to differentiate: composition. The Chain Rule will let us find the derivative of a composition. (This is the last derivative rule we will learn!)

**Example:** Find the derivative of $y = \left(4x^3 + 15x\right)^2$.

**Solution:** This is not a simple polynomial, so we can't use the basic building block rules yet. It is a product, so we could write it as $y = \left(4x^3 + 15x\right)^2 = \left(4x^3 + 15x\right)\left(4x^3 + 15x\right)$ and use the product rule. Or we could multiply it out and simply differentiate the resulting polynomial. I'll do it the second way:

$$y = \left(4x^3 + 15x\right)^2 = 16x^6 + 120x^4 + 225x^2$$
$$y' = 64x^5 + 480x^3 + 450x$$

**Example:** Find the derivative of $y = \left(4x^3 + 15x\right)^{20}$

**Solution:** We **could** write it as a product with 20 factors and use the product rule, or we **could** multiply it out. But I don't want to do that, do you?

We need an easier way, a rule that will handle a composition like this. The Chain Rule is a little complicated, but it saves us the much more complicated algebra of multiplying something like this out. It will also handle compositions where it wouldn't be possible to "multiply it out."

The Chain Rule is the most common place for students to make mistakes. Part of the reason is that the notation takes a little getting used to. And part of the reason is that students often forget to use it when they should. When should you use the Chain Rule? Almost every time you take a derivative.

## Derivative Rules: Chain Rule

In what follows, *f* and *g* are differentiable functions with $y = f(u)$ and $u = g(x)$

(h) **Chain Rule (Leibniz notation):**

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Notice that the du's seem to cancel. This is one advantage of the Leibniz notation; it can remind you of how the chain rule chains together.

(h) **Chain Rule (using prime notation):**

$$f'(x) = f'(u) \cdot g'(x) = f'(g(x)) \cdot g'(x)$$

(h) **Chain Rule (in words):**

The derivative of a composition is) the derivative of the outside TIMES the derivative of what's inside.

I recite the version in words each time I take a derivative, especially if the function is complicated.

**Example:** Find the derivative of $y=\left(4x^3+15x\right)^2$.

**Solution:** This is the same one we did before by multiplying out. This time, let's use the Chain Rule: The inside function is what appears inside the parentheses: $4x^3+15x$. The outside function is the first thing we find as we come in from the outside – it's the square function, something$^2$. We want the derivative of the outside

(2 something) TIMES the derivative of what's inside (which is $12x^2+15$):

$$y=\left(4x^3+15x\right)^2$$
$$y'=2\left(4x^3+15x\right)\cdot\left(12x^2+15\right)$$

(By the way, if you multiply this out, you get the same answer we got before. Hurray! Algebra works!)

**Example:** Find the derivative of $y=\left(4x^3+15x\right)^{20}$

**Solution:** Now we have a way to handle this one. It's the derivative of the outside TIMES the derivative of what's inside.

$$y=\left(4x^3+15x\right)^{20}$$
$$y'=20\left(4x^3+15x\right)^{19}\cdot\left(12x^2+15\right)$$

**Example :** Differentiate $e^{x^2+5}$ .

**Solution:** This isn't a simple exponential function; it's a composition. Typical calculator or computer syntax can help you see what the "inside" function is here. On a TI calculator, for example, when you push the $e^x$ key, it opens up parentheses: $\boxed{e\wedge(}$ This tells you that the "inside" of the exponential function is the exponent. Here, the inside is the exponent $x^2+5$. Now we can use the Chain Rule: We want the derivative of the outside TIMES the derivative of what's inside. The outside is the "e to the" function, so its derivative is the same thing. The derivative of what's inside is 2x. So

$$\frac{d}{dx}\left(e^{x^2+5}\right)=\left(e^{x^2+5}\right)\cdot(2x)$$

**Example:** The table gives values for f , f ' , g and g ' at a number of points. Use these values to determine ( f∘g )(x) and **(** f∘g **) '**(x) at x = –1 and 0.

| x | f(x) | g(x) | f '(x) | g '(x) | ( f∘g )(x) | **(** f∘g **) '**(x) |
|---|---|---|---|---|---|---|
| –1 | 2 | 3 | 1 | 0 | | |
| 0 | –1 | 1 | 3 | 2 | | |
| 1 | 1 | 0 | –1 | 3 | | |
| 2 | 3 | –1 | 0 | 1 | | |
| 3 | 0 | 2 | 2 | –1 | | |

**Solution:** ( f∘g )(–1) = f( g(–1) ) = f( 3 ) = 0 and ( f∘g )(0) = f( g(0) ) = f( 1 ) = 1.

**(** f∘g **) '**(–1) = **f '(** g(–1) **)·g '(** –1 **)** = **f '(** 3 **)·**(0) = (2)(0) = 0 and

**(** f∘g **) '**( 0 ) = **f '(** g( 0 ) **)·g '(** 0 **)** = **f '(** 1 **)·**( 2 ) = (–1)(2) = –2 .

I'll let you do the rest.

## Derivatives of Complicated Functions

You're now ready to take the derivative of some mighty complicated functions. But how do you tell what rule applies first? Come in from the outside – what do you encounter first? That's the first rule you need. Use the Product, Quotient, and Chain Rules to peel off the layers, one at a time, until you're all the way inside.

**Example:** Find $\frac{d}{dx}\left(e^{3x}\cdot\ln(5x+7)\right)$

**Solution:** Coming in from the outside, I see that this is a product of two (complicated) functions. So I'll need the Product Rule first. I'll fill in the pieces I know, and then I can figure the rest as separate steps and substitute in at the end:

$$\frac{d}{dx}\left(e^{3x}\cdot\ln(5x+7)\right)=\left(\frac{d}{dx}\left(e^{3x}\right)\right)\left(\ln(5x+7)\right)+\left(e^{3x}\right)\left(\frac{d}{dx}\left(\ln(5x+7)\right)\right)$$

Now as separate steps, I'll find

$\frac{d}{dx}\left(e^{3x}\right)=3e^{3x}$ (using the Chain Rule) and

$\frac{d}{dx}\left(\ln(5x+7)\right)=\frac{1}{5x+7}\cdot 5$ (also using the Chain Rule).

Finally, to substitute these in their places:

$$\frac{d}{dx}\left(e^{3x}\cdot\ln(5x+7)\right)=\left(3e^{3x}\right)\left(\ln(5x+7)\right)+\left(e^{3x}\right)\left(\frac{1}{5x+7}\cdot 5\right)$$

(And please don't try to simplify that!)

**Example:** Differentiate $z=\left(\frac{3t^3}{e^t(t-1)}\right)^4$

**Solution:** Don't panic! As you come in from the outside, what's the first thing you encounter? It's that 4th power. That tells you that this is a composition, a (complicated) function raised to the 4th power.

**Step One:** Use the Chain Rule. The derivative of the outside TIMES the derivative of what's inside.

$$\frac{dz}{dt} = \frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)^4 = 4\left(\frac{3t^3}{e^t(t-1)}\right)^3 \cdot \frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)$$

Now we're one step inside, and we can concentrate on just the $\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)$ part. Now, as you come in from the outside, the first thing you encounter is a quotient – this is the quotient of two (complicated) functions.

**Step Two:** Use the Quotient Rule:

$$\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right) = \frac{(9t^2)(e^t(t-1)) - (3t^3)\left(\frac{d}{dt}(e^t(t-1))\right)}{(e^t(t-1))^2}$$

Now we've gone one more step inside, and we can concentrate on just the $\frac{d}{dt}(e^t(t-1))$ part. Now we have a product.

**Step Three:** Use the Product Rule:

$$\frac{d}{dt}(e^t(t-1)) = (e^t)(t-1) + (e^t)(1)$$

And now we're all the way in – no more derivatives to take.

**Step Four:** Now it's just a question of substituting back – be careful now!

$\frac{d}{dt}(e^t(t-1)) = (e^t)(t-1) + (e^t)(1)$, so

$$\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)=\frac{(9t^2)(e^t(t-1))-(3t^3)((e^t)(t-1)+(e^t)(1))}{(e^t(t-1))^2}$$, so

$$\frac{dz}{dt}=\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)^4=4\left(\frac{3t^3}{e^t(t-1)}\right)^3\cdot\left(\frac{(9t^2)(e^t(t-1))-(3t^3)((e^t)(t-1)+(e^t)(1))}{(e^t(t-1))^2}\right).$$

**Phew!**

## What if the Derivative Doesn't Exist?

A function is called **differentiable** at a point if its derivative exists at that point.

We've been acting as if derivatives exist everywhere for every function. This is true for most of the functions that you will run into in this class. But there are some common places where the derivative doesn't exist.

Remember that the derivative is the slope of the tangent line to the curve. That's what to think about.

**Where can a slope not exist?** If the tangent line is vertical, the derivative will not exist.

**Example:** This is the graph of $f(x) = \sqrt[3]{x} = x^{1/3}$. Notice that the tangent line to this curve at $x = 0$ is vertical. So its slope does not exist, and so the derivative does not exist at $x = 0$.

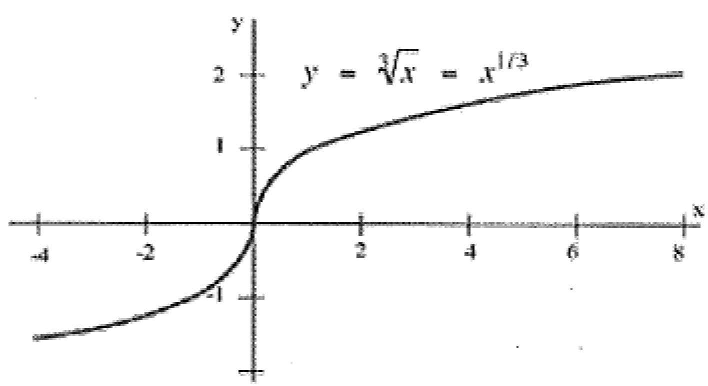



Figure 24

**Where can a tangent line not exist?** If there is a corner in the graph, the derivative will not exist at that point because there is no well-defined tangent line (a teetering tangent, if you will.) Or if there is a jump in the graph, the tangent line will be different on either side and the derivative can't exist.

**Example:** This is the graph of the Greatest Integer Function, a basic step function. There is no single

tangent line at $x = 1$; the tangent lines on either side are different. So the derivative does not exist at $x = 1$.

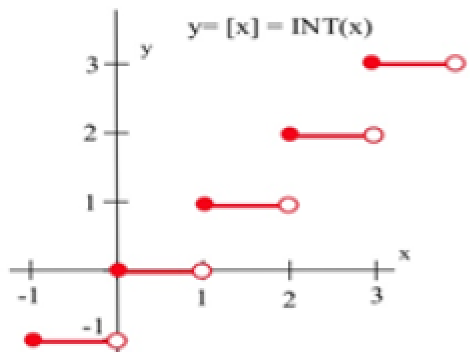



Figure 25

## Section 5: Second Derivative and Concavity

### Second Derivative and Concavity

Graphically, a function is **concave up** if its graph is curved with the opening upward (Fig. 17a). Similarly, a function is **concave down** if its graph opens downward (Fig. 17b).

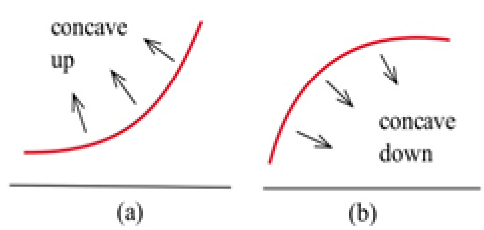



Figure 26

For example, **An Epidemic:** Suppose an epidemic has started, and you, as a member of congress, must decide whether the current methods are effectively fighting the spread of the disease or whether more drastic measures and more money are needed. In Fig. 18, f(x) is the number of people who have the disease at time x, and two different situations are shown. In both (a) and (b), the number of people with the disease, f(now), and the rate at which new people are getting sick, f '(now), are the same. The difference in the two situations is the concavity of f, and that difference in concavity might have a big effect on your decision.

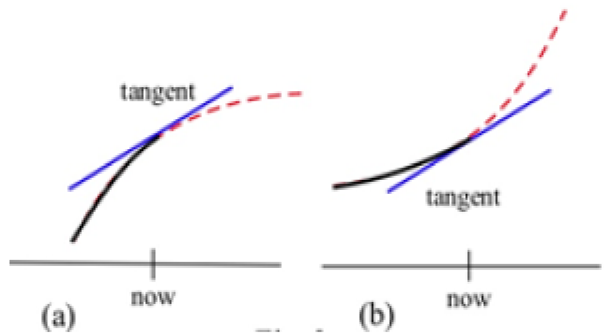



Figure 27

In (a), f is concave down at "now", the slopes are decreasing, and it looks as if it's tailing off. We can say "f is increasing at a decreasing rate." It appears that the current methods are starting to bring the epidemic under control.

In (b), f is concave up, the slopes are increasing, and it looks as if it will keep increasing faster and faster. It appears that the epidemic is still out of control.

The differences between the graphs come from whether the *derivative* is increasing or decreasing.

The derivative of a function f is a function that gives information about the slope of f. **The derivative tells us if the original function is increasing or decreasing.**

Because f ' is a function, we can take its derivative. This second derivative also gives us information about our original function f. The second derivative gives us a mathematical way to tell how the graph of a function is curved. **The second derivative tells us if the original function is concave up or down.**

**Second Derivative** Let $y = f(x)$

The **second derivative of f** is the derivative of $y' = f'(x)$.

Using prime notation, this is $f''(x)$ or $y''$. You can read this aloud as "y double prime."

Using Leibniz notation, the second derivative is written $\frac{d^2y}{dx^2}$ or $\frac{d^2f}{dx^2}$. This is read aloud as "the second derivative of f.

If $f''(x)$ is positive on an interval, the graph of $y = f(x)$ is **concave up** on that interval**.** We can say that *f* is increasing (or decreasing) **at an increasing rate**.

If $f''(x)$ is negative on an interval, the graph of $y = f(x)$ is **concave down** on that interval**.** We can say that *f* is increasing (or decreasing) **at a decreasing rate.**

**Example:** Find $f''(x)$ for $f(x) = 3x^7$

**Solution:** First, we need to find the first derivative:

$$f'(x) = 21x^6$$

Then we take the derivative of that function:

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}(21x^6) = 126x^5$$

If $f(x)$ represents the position of a particle at time $x$, then $v(x) = f'(x)$ will represent the velocity (rate of change of the position) of the particle and $a(x) = v'(x) = f''(x)$ will represent the acceleration (the rate of change of the velocity) of the particle.

**Example :** The height (feet) of a particle at time $t$ seconds is $t^3 - 4t^2 + 8t$. Find the height, velocity and acceleration of the particle when $t = 0, 1,$ and $2$ seconds.

**Solution:** $f(t) = t^3 - 4t^2 + 8t$ so $f(0) = 0$ feet, $f(1) = 5$ feet, and $f(2) = 8$ feet.

The velocity is $v(t) = f'(t) = 3t^2 - 8t + 8$ so $v(0) = 8$ ft/s , $v(1) = 3$ ft/s, and $v(2) = 4$ ft/s. At each of these times the velocity is positive and the particle is moving upward, increasing in height.

The acceleration is $a(t) = 6t - 8$ so $a(0) = -8$ ft/s$^2$ , $a(1) = -2$ ft/s$^2$ and $a(2) = 4$ ft/s$^2$ .

## Inflection Points

**Definition:** An **inflection point** is a point on the graph of a function where the concavity of the function changes, from concave up to down or from concave down to up.

**Example:** Which of the labeled points in Fig. 19 are inflection points?

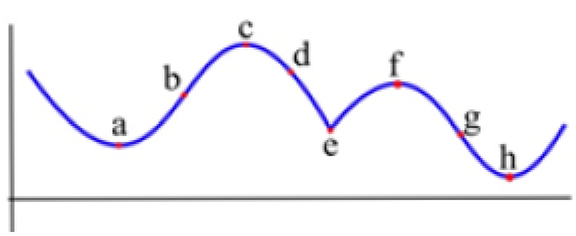



Figure 28

**Solution:** The concavity changes at points b and g. At points a and h, the graph is concave up on either side, so the concavity does not change. At points c and f, the graph is concave down on either side. And at point e, even though the graph looks strange there, the graph is concave down on both sides – the concavity does not change.

Inflection points happen when the concavity changes. Because we know the connection between the concavity of a function and the sign of its second derivative, we can use this to find inflection points.

**Working Definition:** An **inflection point** is a point on the graph where the second derivative changes sign.

In order for the second derivative to change signs, it must either be zero or be undefined. So to find the inflection points of a function we only need to check the points where $f''(x)$ is 0 or undefined.

Note that it is not enough for the second derivative to be zero or undefined. We still need to check that the sign of f'' changes sign. The functions in the next example illustrate what can happen.

**Example:** Let $f(x) = x^3$, $g(x) = x^4$ and $h(x) = x^{1/3}$ (Fig. 20). For which of these functions is the point (0,0) an inflection point?

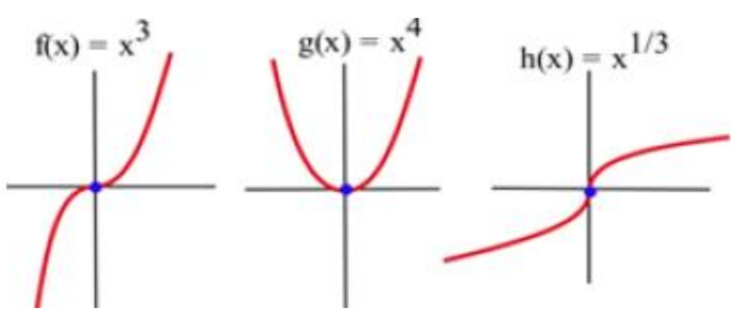



Figure 29

**Solution:** Graphically, it is clear that the concavity of $f(x) = x^3$ and $h(x) = x^{1/3}$ changes at (0,0), so (0,0) is an inflection point for f and h. The function $g(x) = x^4$ is concave up everywhere so (0,0) is not an inflection point of g.

We can also compute the second derivatives and check the sign change.

If $f(x) = x^3$, then $f'(x) = 3x^2$ and $f''(x) = 6x$. The only point at which $f''(x) = 0$ or is undefined ($f'$ is not differentiable) is at $x = 0$. If $x < 0$, then $f''(x) < 0$ so f is concave down. If $x > 0$, then $f''(x) > 0$ so f is concave up. At $x = 0$ the concavity changes so the point $(0,f(0)) = (0,0)$ is an inflection point of $x^3$.

If $g(x) = x^4$, then $g'(x) = 4x^3$ and $g''(x) = 12x^2$. The only point at which $g''(x) = 0$ or is undefined is at $x = 0$. If $x < 0$, then $g''(x) > 0$ so g is concave up. If $x > 0$, then $g''(x) > 0$ so g is also concave up. At $x = 0$ the concavity **does not change** so the point $(0, g(0)) = (0,0)$ is **not an inflection point** of $x^4$. Keep this example in mind!.

If $h(x) = x^{1/3}$, then $h'(x) = \frac{1}{3} x^{-2/3}$ and $h''(x) = -\frac{2}{9} x^{-5/3}$. $h''$ is not defined if $x = 0$, but $h''(\text{negative number}) > 0$ and $h''(\text{positive number}) < 0$ so h changes concavity at (0,0) and (0,0) is an inflection point of h.

**Example:** Sketch the graph of a function with $f(2) = 3$, $f'(2) = 1$, and an inflection point at (2,3).

**Solution:** Two solutions are given in Fig. 21.

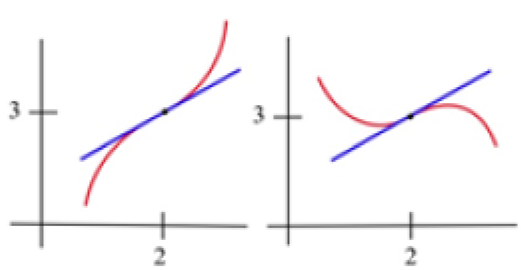



Figure 30

## Section 6: Optimization

In theory and applications, we often want to maximize or minimize some quantity. An engineer may want to maximize the speed of a new computer or minimize the heat produced by an appliance. A manufacturer may want to maximize profits and market share or minimize waste. A student may want to maximize a grade in calculus or minimize the hours of study needed to earn a particular grade.

Without calculus, we only know how to find the optimum points in a few specific examples (for example, we know how to find the vertex of a parabola). But what if we need to optimize an unfamiliar function?

The best way we have without calculus is to examine the graph of the function, perhaps using technology. But our view depends on the viewing window we choose – we might miss something important. In addition, we'll probably only get an approximation this way. (In some cases, that will be good enough.)

Calculus provides ways of drastically narrowing the number of points we need to examine to find the exact locations of maximums and minimums, while at the same time ensuring that we haven't missed anything important.

### Local Maxima and Minima

Before we examine how calculus can help us find maximums and minimums, we need to define the concepts we will develop and use.

**Definitions:** f has a **local maximum** at a if $f(a) \ge f(x)$ for all x near a

f has a **local minimum** at a if $f(a) \le f(x)$ for all x near a

f has a **local extreme** at a if f(a) is a **local maximum or minimum.**

The plurals of these are maxima and minima. We often simply say "max" or "min;" it saves a lot of syllables.

Some books say "relative" instead of "local."

The process of finding maxima or minima is called **optimization.**

A point is a local max (or min) if it is higher (lower) than all the **nearby points**. These points come from the shape of the graph.

**Definitions:** f has a **global maximum** at a if f(a) ≥ f(x) for all x in the domain of f.

f has a **global minimum** at a if f(a) **≤** f(x) for all x in the domain of f.

f has a **global extreme** at a if f(a) is a **global maximum or minimum.**

Some books say “absolute” instead of “global”

A point is a global max (or min) if it is higher (lower) than every point on the graph. These points come from the shape of the graph **and** the window through which we view the graph.

The local and global extremes of the function in Fig. 22 are labeled. You should notice that every global extreme is also a local extreme, but there are local extremes that are not global extremes.

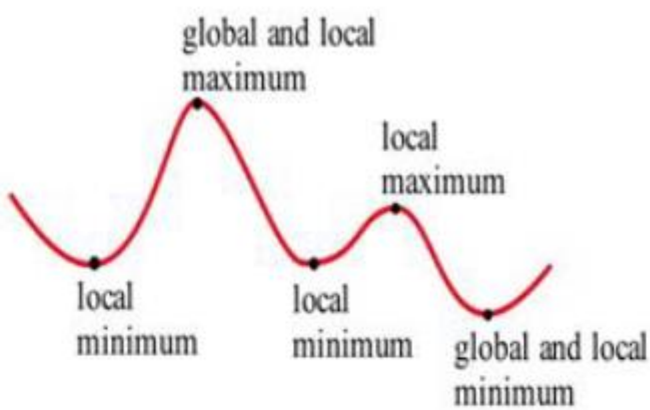



Figure 31

If h(x) is the height of the earth above sea level at the location x, then the global maximum of h is h(summit of Mt. Everest) = 29,028 feet. The local maximum of h for the United States is h(summit of Mt. McKinley) = 20,320 feet. The local minimum of h for the United States is h(Death Valley) = – 282 feet.

**Example:** The table shows the annual calculus enrollments at a large university. Which years had local maximum or minimum calculus enrollments? What were the global maximum and minimum enrollments in calculus?

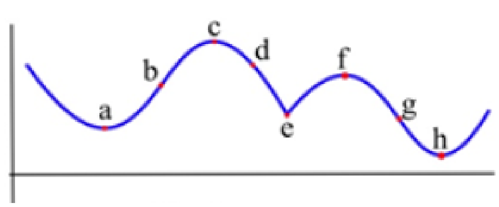
| year | 1980 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| enrollment | 1257 | 1324 | 1378 | 1336 | 1389 | 1450 | 1523 | 1582 | 1567 | 1545 | 1571 |

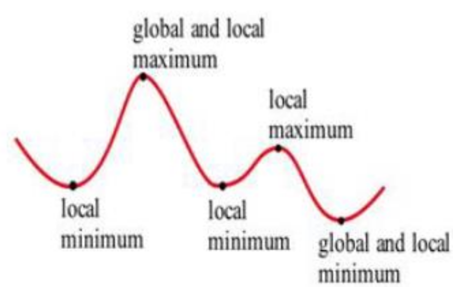
**Solution:** There were local maxima in 1982 and 1987; the global maximum was 1582 students in 1987. There were local minima in 1983 and 1989; the global minimum was 1336 students in 1983. I choose not to think of 1980 as a local minimum or 1990 as a local maximum. However, some books would include the endpoints.

## Finding Maxima and Minima of a Function

What must the tangent line look like at a local max or min? Look at these two graphs again – you'll see that at all the extreme points, the tangent line is horizontal (so f' = 0). There is one cusp in the blue graph – the tangent line if vertical there (so f' is undefined).

That gives us the clue how to find extreme values.



**Definition:** A **critical number** for a function *f* is a value x = a in the domain of f where either

***f'*(a) = 0** or ***f'*(a) is undefined.**

**Definition:** A **critical point** for a function f is a point (a, f(a)) where a is a critical number of f.

**Useful Fact: A local max or min of *f* can only occur at a critical point.**

**Example:** Find the critical points of $f(x) = x^3 - 6x^2 + 9x + 2$.

Solution: A critical number of f can occur only where $f'(x) = 0$ or where $f'$ does not exist.

$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$ so $f'(x) = 0$ only at $x = 1$ and $x = 3$.

There are no places where f' is undefined.

The critical numbers are x = 1 and x= 3. So the critical points are (1, 6) and (3, 2).

These are the only possible locations of local extremes of f. We haven't discussed yet how to tell whether either of these points is actually a local extreme of f, or which kind it might be. But we can be certain that no other point is a local extreme.

The graph of f (Fig. 23) shows that $(1, f(1)) = (1, 6)$ is a local maximum and $(3, f(3)) = (3, 2)$ is a local minimum. This function does not have a global maximum or minimum.

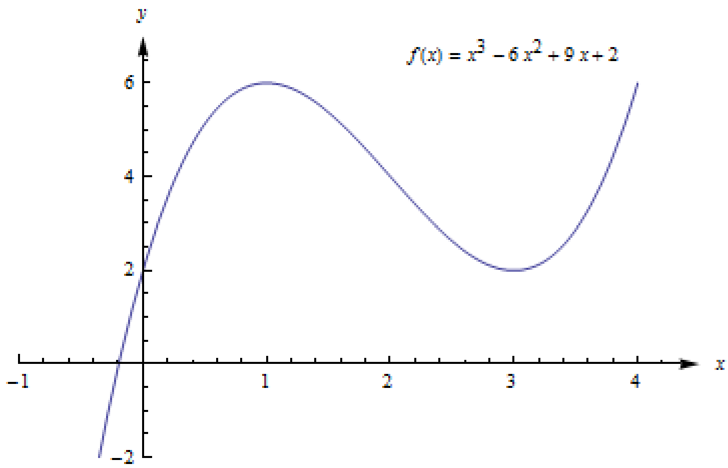



Figure 32

**Example :** Find all local extremes of $f(x) = x^3$ .

**Solution:** $f(x) = x^3$ is differentiable for all x, and $f'(x) = 3x^2$. The only place where

$f'(x) = 0$ is at $x = 0$, so the only candidate is the critical point (0,0). But if $x > 0$ then

$f(x) = x^3 > 0 = f(0)$, so $f(0)$ is not a local maximum. Similarly, if $x < 0$ then

$f(x) = x^3 < 0 = f(0)$ so $f(0)$ is not a local minimum. The critical point (0,0) is the only candidate to be a local extreme of f, and this candidate did not turn out to be a local

extreme of f. The function $f(x) = x^3$ does not have any local extremes. (Fig. 24)

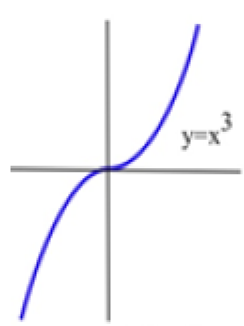



Figure 33

Remember this example! It is not enough to find the critical points -- we can only say that f **might have** a local extreme at the critical points.

# First and Second Derivative Tests

## Is that critical point a Maximum or Minimum (or Neither)?

Once we have found the critical points of *f*, we still have the problem of determining whether these points are maxima, minima or neither.

All of the graphs in Fig. 25 have a critical point at (2, 3). It is clear from the graphs that the point (2,3) is a local maximum in (a) and (d), (2,3) is a local minimum in (b) and (e), and (2,3) is not a local extreme in (c) and (f).

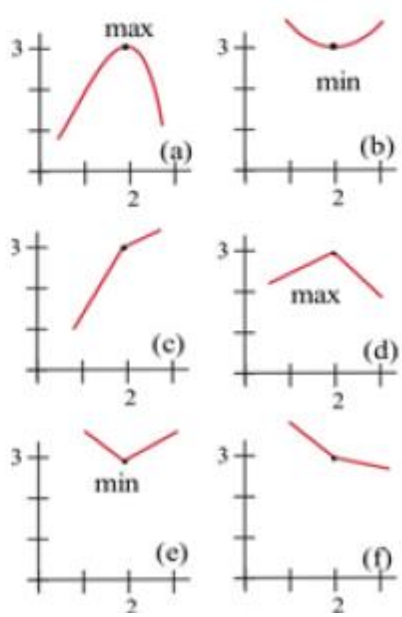



Figure 34

The critical numbers only give the **possible** locations of extremes, and some critical numbers are not the locations of extremes. The critical numbers are the **candidates** for the locations of maxima and minima.

## f '  and Extreme Values of  f

Four possible shapes of graphs are shown here – in each graph, the point marked by an arrow is a critical point, where f '(x) = 0.  What happens to the derivative near the critical point?

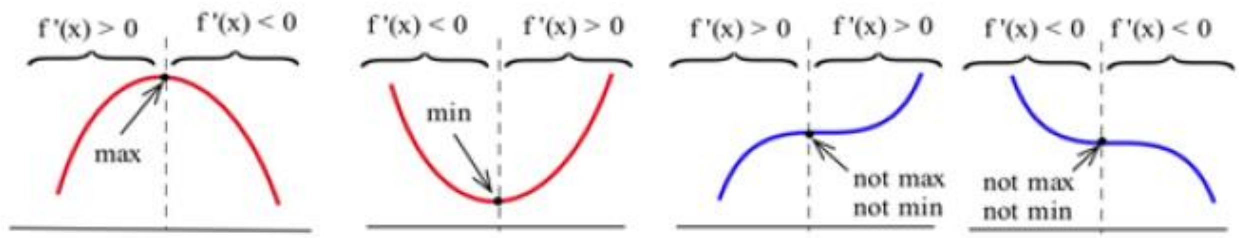



Figure 35

At a local max, such as in the graph on the left, the function increases on the left of the local max, then decreases on the right.  The derivative is first positive, then negative at a local max.  At a local min, the function decreases to the left and increases to the right, so the derivative is first negative, then positive. When there isn't a local extreme, the function continues to increase (or decrease) right past the critical point – the derivative doesn't change sign.

**The First Derivative Test for Extremes:**

Find the critical points of f.  For each critical number c, examine the sign of f' to the left and to the right of c.  What happens  to the sign as you move from left to right?

**If  f '(x) changes from positive to negative** at x = c, then f has a local **maximum** at (c, f(c)).

**If  f '(x) changes from negative to positive** at x = c, then f has a local **minimum** at (c, f(c)).

**If  f '(x) does not change sign** at x = c, then (c, f(c)) is **neither** a local max nor a local min.

**Example:** Find the critical points of $f(x) = x^3 - 6x^2 + 9x + 2$ and classify them as local max, local min, or neither.

**Solution:** We already found the critical points; they are (1, 6) and (3, 2).

Now we can use the first derivative test to classify each. Recall that $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$. The factored form is easiest to work with here, so let's use that.

(1, 6): You could choose a number slightly less than 1 to plug into the formula for f' – perhaps use x = 0, or x = 0.9. Then you could examine its sign. But I don't care about the numerical value, all I'm interested in is its sign. And for that, you don't have to do any plugging in:

If x is a little less than 1, then $x - 1$ is negative, and $x - 3$ is negative. So $f' = 3(x - 1)(x - 3)$ will be pos(neg)(neg) = positive.

For x a little more than 1, you can evaluate f' at a number more than 1 (but less than 3, you don't want to go past the next critical point!) – perhaps x = 2. Or you can make a quick sign argument like what I did above. For x a little more than 1, $f' = 3(x - 1)(x - 3)$ will be pos(pos)(neg) = negative.

f' changes from positive to negative, so there is a local max at (1, 6)

(3, 2): f' changes from negative to positive, so there is a local min at (3, 2).

This confirms what we saw before in the graph.

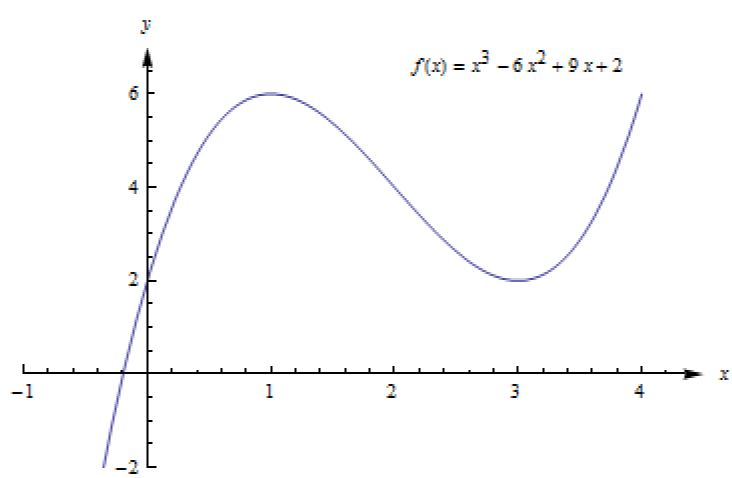



Figure 36

## f '' and Extreme Values of f

The concavity of a function can also help us determine whether a critical point is a maximum or minimum or neither. For example, if a point is at the bottom of a concave up function (Fig. 28), then the point is a minimum.

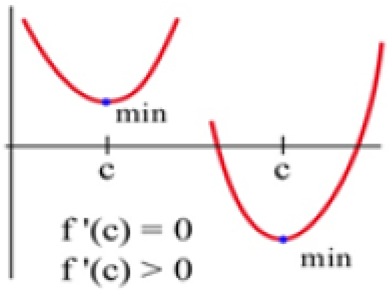



Figure 37

**The Second Derivative Test for Extremes:**

Find all critical points of f. For those critical points where f'(c) = 0, find f''(c).

(a) If f ''(c) < 0 then f is concave down and has a local maximum at x = c.

(b) If f ''(c) > 0 then f is concave up and has a local minimum at x = c.

(c) If f ''(c) = 0 then f may have a local maximum, a minimum or neither at x = c.

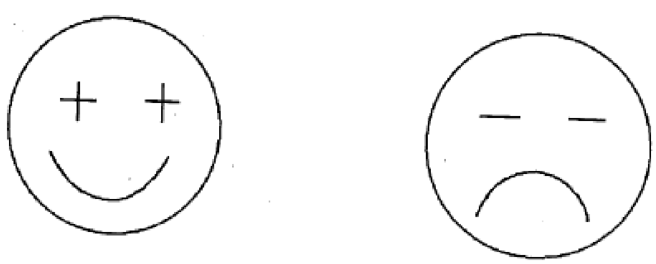


Figure 38

The cartoon faces can help you remember the Second Derivative Test.

**Example:** f(x) = $2x^3 - 15x^2 + 24x - 7$ has critical numbers x = 1 and 4. Use the Second Derivative

Test for Extremes to determine whether f(1) and f(4) are maximums or minimums or neither.

**Solution:** We need to find the second derivative:

$$f(x) = 2x^3 - 15x^2 + 24x - 7$$
$$f'(x) = 6x^2 - 30x + 24$$
$$f''(x) = 12x - 30$$

Then we just need to evaluate f ʺ at each critical number:

x = 1: $f''(1) = 12(1) - 30 < 0$; there is a local maximum at x = 1.

x = 4: $f''(4) = 12(4) - 30 > 0$; there is a local minimum at x = 4.

Many students like the Second Derivative Test. The Second Derivative Test is often easier to use than the First Derivative Test. You only have to find the sign of one number for each critical number rather than two. And if your function is a polynomial, its second derivative will probably be a simpler function than the derivative.

But if you needed a product rule, quotient rule, or chain rule to find the first derivative, finding the second derivative can be a lot of work. And, even if the second derivative is easy, the Second Derivative Test doesn't always give an answer. The First Derivative Test will always give you an answer.

Use whichever test you want to. But remember – you have to do some test to be sure that your critical point actually is a local max or min.

## Global Maxima and Minima

In applications, we often want to find the global extreme; knowing that a critical point is a local extreme is not enough. For example, if we want to make the greatest profit, we want to make the absolutely greatest profit of all. How do we find global max and min?

There are just a few additional things to think about.

### Endpoint Extremes

The local extremes of a function occur at critical points – these are points in the function that we can find by thinking about the shape (and using the derivative to help us). But if we're looking at a function on a closed interval, the endpoints could be extremes. These endpoint extremes are not related to the shape of the function; they have to do with the interval, the window through which we're viewing the function.

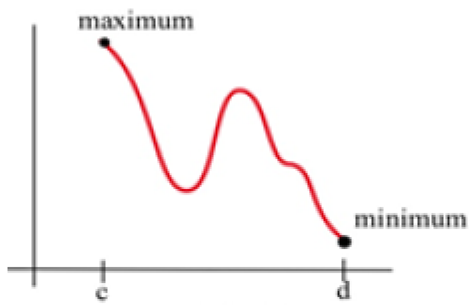



Figure 39

In Fig. 30, it appears that there are three critical points – one local min, one local max, and one that is neither one. But the global max, the highest point of all, is at the left endpoint. The global min, the lowest point of all, is at the right endpoint.

How do we decide if endpoints are global max or min? It's easier than you expected – simply plug in the endpoints, along with all the critical numbers, and compare y-values.

**Example 3:** Find the global max and min of $f(x) = x^3 - 3x^2 - 9x + 5$ for $-2 \le x \le 6$ .

**Solution:** $f'(x) = 3x^2 - 6x - 9 = 3(x + 1)(x - 3)$. We need to find critical points, and we need to check the endpoints.

(i) $f'(x) = 3(x + 1)(x - 3) = 0$ when $x = \mathbf{-1}$ and $x = \mathbf{3}$.

(ii) f is a polynomial so f' is defined everywhere.

(iii) The endpoints of the interval are x = **–2** and x = **6**.

Now we simply compare the values of f at these 4 values of x:

f(–2) = 3 , f(–1) = 10 , f(3) = –22 , and f(6) = 59.

The global minimum of f on [ –2, 6] is –22, when x = 3, and the global maximum of f on [ –2, 6] is 59, when x = 6.

### If there's only one critical point

If the function has only one critical point and it's a local max (or min), then it must be the global max (or min). To see this, think about the geometry. Look at the graph on the left – there is a local max, and the graph goes down on either side of the critical point. Suppose there was some other point that was higher – then the graph would have to turn around. But that turning point would have shown up as another critical point. If there's only one critical point, then the graph can never turn back around.

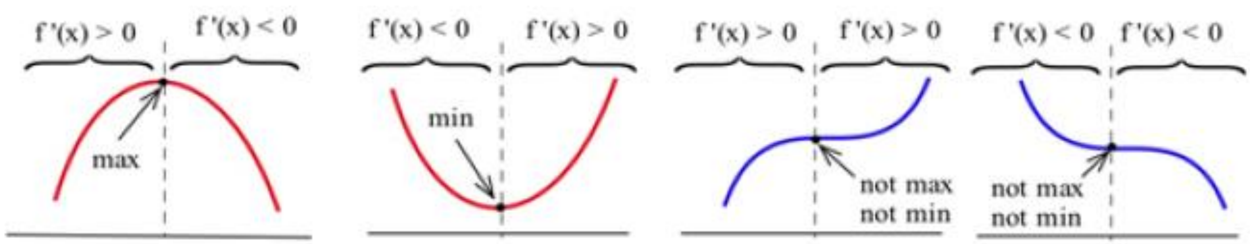



Figure 40

### When in doubt, graph it and look.

If you are trying to find a global max or min on an open interval (or the whole real line), and there is more than one critical point, then you need to look at the graph to decide whether there is a global max or min. Be sure that all your critical points show in your graph, and that you go a little beyond – that will tell you what you want to know.

**Example:** Find the global max and min of $f(x) = x^3 - 6x^2 + 9x + 2$.

**Solution:** We have previously found that (1, 6) is a local max and (3, 2) is a local min. This is not a closed interval, and there are two critical points, so we must turn to the graph of the function to find global max and min.

The graph of f (Fig. 32) shows that points to the left of x = 4 have y-values greater than 6, so (1, 6) is not a global max. Likewise, if x is negative, y is less than 2, so (3, 2) is not a global min. There are no endpoints, so we've exhausted all the possibilities. This function does not have a global maximum or minimum.

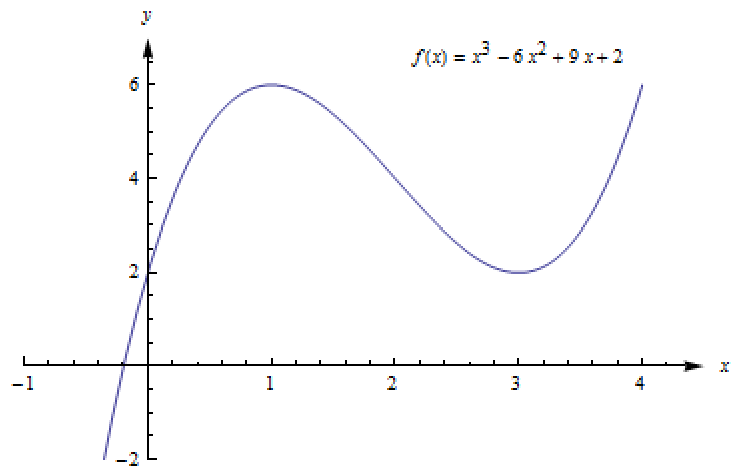



Figure 41

**To find Global Extremes:**

The only places where a function can have a global extreme are critical points or endpoints.

**(a)** If the function has only one critical point, and it's a local extreme, then it is also the global extreme.

(b) If there are endpoints, find the global extremes by comparing y-values at all the critical points and at the endpoints.

(c) When in doubt, graph the function to be sure.

## Section 7: Applied Optimization

We have used derivatives to help find the maximums and minimums of some functions given by equations,

but it is very unlikely that someone will simply hand you a function and ask you to find its extreme values. More typically, someone will describe a problem and ask your help in maximizing or minimizing something: "What is the largest volume package which the post office will take?"; "What is the quickest way to get from here to there?"; or "What is the least expensive way to accomplish some task?" In this section, we'll discuss how to find these extreme values using calculus.

### Max/Min Applications

> **Example:** The manager of a garden store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of $7 per running foot. The fourth side will be built of cement blocks, at a cost of $14 per running foot. Find the dimensions of the least costly such enclosure.

The process of finding maxima or minima is called optimization. The function we're optimizing is called the *objective function.* The objective function can be recognized by its proximity to "est" words (greatest, least, highest, farthest, most, …) Look at the garden store example; the cost function is the objective function.

In many cases, there are two (or more) variables in the problem. In the garden store example again, the length and width of the enclosure are both unknown. If there is an equation that relates the variables we can solve for one of them in terms of the others, and write the objective function as a function of just one variable. Equations that relate the variables in this way are called *constraint equations.* The constraint equations are always equations, so they will have equals signs. For the garden store, the fixed area relates the length and width of the enclosure. This will give us our constraint equation.

Once we have a function of just one variable, we can apply the calculus techniques we've just learned to find maxima or minima.

**Max-Min Story Problem Technique:**

**(a)** Translate the English statement of the problem line by line into a picture (if that applies) and into math. This is often the hardest step!

**(b)** Identify the objective function. Look for "est" words.

**(b1)** If you seem to have two or more variables, find the constraint equation. Think about the English meaning of the word "constraint," and remember that the constraint equation will have an = sign.

**(b2)** Solve the constraint equation for one variable and substitute into the objective function. Now you have an equation of one variable.

**(c)** Use calculus to find the optimum values. (Take derivative, find critical points, test. Don't forget to check the endpoints!)

**(d)** Look back at the question to make sure you answered what was asked. Translate your number answer back into English.

**Example:** The manager of a garden store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of \$7 per running foot. The fourth side will be built of cement blocks, at a cost of \$14 per running foot. Find the dimensions of the least costly such enclosure.

**Solution:** First, translate line by line into math and a picture:

| ***Text*** | ***Translation*** |
|---|---|
| The manager of a garden store wants to build a *600 square foot rectangular* enclosure on the store's parking lot in order to display some equipment. | Let $x$ and $y$ be the dimensions of the enclosure, with $y$ being the length of the side made of blocks. Then: |
| | *Area* $= A = xy = 600$ |
| *Three sides of the enclosure* will be built of redwood fencing, at a *cost of \$7 per running foot*. *The fourth side* will be built of cement blocks, at a *cost of \$14 per running foot.* | $2x + y$ *costs \$7 per foot*<br>$y$ *costs \$14 per foot*<br>So<br>*Cost* $= C = 7(2x + y) + 14y = 14x + 21y$ |
| Find the dimensions of the least costly such enclosure. | Find $x$ and $y$ so that $C$ is minimized. |

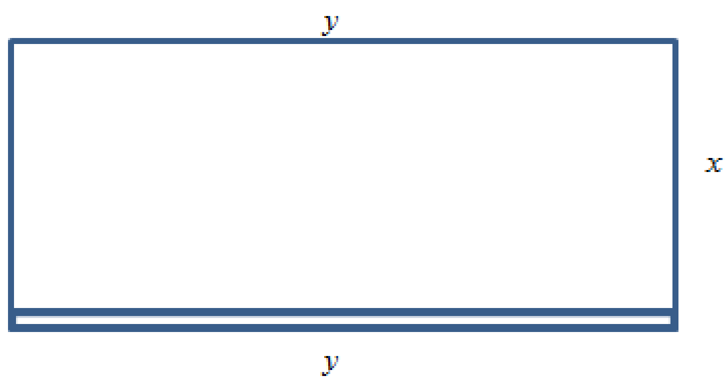



Figure 42

The objective function is the cost function, and we want to minimize it. As it stands, though, it has two variables, so we need to use the constraint equation. The constraint equation is the fixed area *A = xy = 600.* Solve *A* for *x* to get $x = \frac{600}{y}$, and then substitute into *C*:

$$C = 14\left(\frac{600}{y}\right) + 21y = \frac{8400}{y} + 21y.$$

Now we have a function of just one variable, so we can whack it with calculus (find the critical points, etc.)

$$C' = -\frac{8400}{y^2} + 21$$

*C′* is undefined for *y* = 0, and *C′* = 0 when *y* = 20 or *y* = −20.

Of these three critical numbers, only *y* = 20 makes sense (is in the domain of the actual function) – remember that *y* is a length, so it can't be negative. And *y* = 0 would mean there was no enclosure at all, so it couldn't have an area of 600 square feet.

Test *y* = 20: (I chose the second derivative test)

$C'' = \frac{16800}{y^3} > 0$, so this is a local minimum.

Since this is the only critical point in the domain, this must be the global minimum. When *y* = 20, *x* = 30. The dimensions of the enclosure that minimize the cost are 20 feet × 30 feet.

## "Marginal Revenue = Marginal Cost"

You've probably heard before that "profit is maximized when marginal cost and marginal revenue are equal." Now you can see why people say that! (Even though it's not completely true.)

**General Example:** Suppose we want to maximize profit. Now we know what to do – find the profit function, find its critical points, test them, etc.
But remember that Profit = Revenue – Cost. So Profit' = Revenue' – Cost'. That is, the derivative of the profit function is MR – MC.

Now let's find the critical points – those will be where Profit' = 0 or is undefined.

Profit' = 0 when MR – MC = 0, or where MR = MC.

That's where the saying comes from! Here's a more accurate way to express this:

**Profit has critical points when Marginal Revenue and Marginal Cost are equal.**

In all the cases we'll see in this class, Profit will be very well behaved, and we won't have to worry about looking for critical points where Profit' is undefined. But remember that not all critical points are local max! The places where MR = MC could represent local max, local min, or neither one.

**Example:** A company sells *q* ribbon winders per year at \$*p* per ribbon winder. The demand function for ribbon winders is given by:

$$p = 300 - 0.02q$$

The ribbon winders cost \$30 apiece to manufacture, plus there are fixed costs of \$9000 per year. Find the quantity where profit is maximized.

**Solution:** We want to maximize profit, but there isn't a formula for profit showing … yet. So let's make one. We can find a function for Revenue = *pq* using the demand function for *p*.

$$R(q) = (300 - 0.02q)q = 300q - 0.02q^2$$

We can also find a function for Cost, using the variable cost of $30 per ribbon winder, plus the fixed cost:

$$C(q) = 9000 + 30q$$

Putting them together, we get a function for Profit:

$$\pi(q) = R(q) - C(q) = (300q - 0.02q^2) - (9000 + 30q) = -0.02q^2 + 270q - 9000 .$$

Now we have two choices. We can find the critical points of Profit by taking the derivative of π($q$) directly, or we can find MR and MC and set them equal. (Naturally, you'll get the same answer either way.)

I'll use MR = MC this time.

$$\begin{aligned} &MR = 300 - 0.04q \\ &MC = 30 \\ &300 - 0.04q = 30 \\ &270 = 0.04q \\ &q = 6750 \end{aligned}$$

The only critical point is at $q = 6750$. Now we need to be sure this is a local max and not a local min. In this case, I'll look to the graph of π($q$) – it's a downward opening parabola, so this must be a local max. And since it's the only critical point, it must also be the global max.

Profit is maximized when they sell 6750 ribbon winders.

## "Average Cost = Marginal Cost"

"Average cost is minimized when average cost = marginal cost" is another saying that isn't quite true; in this case, the correct statement is:

**Average Cost has critical points when Average Cost and Marginal Cost are equal.**

Let's look at a geometric argument here:

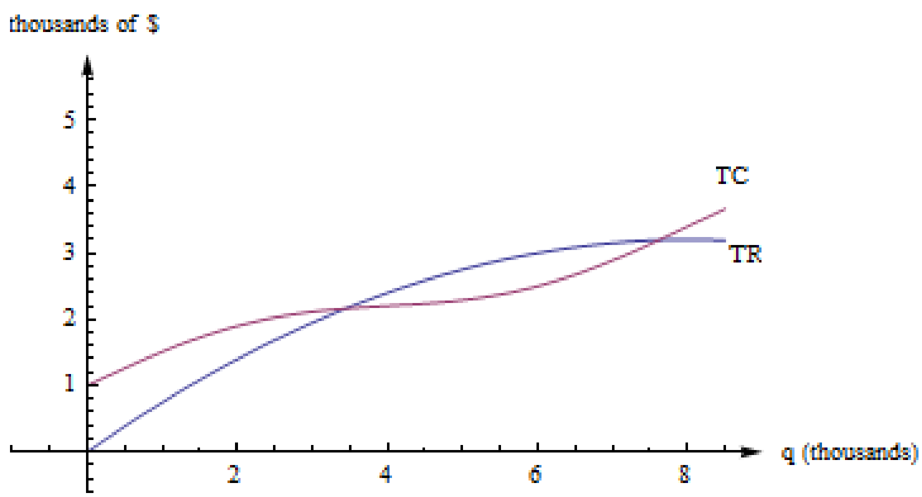



Figure 43

Remember that the average cost is the slope of the diagonal line, the line from the origin to the point on the total cost curve. If you move your clear plastic ruler around, you'll see (and feel) that the slope of the diagonal line is smallest when the diagonal line just touches the cost curve – when the diagonal line is actually a tangent line – when the average cost is equal to the marginal cost.

**Example:** The cost in dollars to produce *q* jars of gourmet salsa is given by $C(q)=160+2q+.1q^2$.
Find the quantity where the average cost is minimum.

**Solution:** $A(q)=\frac{C(q)}{q}=\frac{160}{q}+2+.1q$. We could find the critical points by finding $A'$, or by setting average cost to marginal cost; I'll do the latter this time. $MC(q)=2+.2q$. So I want to solve:

$$\frac{160}{q}+2+.1q=2+.2q$$
$$\frac{160}{q}=.1q$$
$$1600=q^2$$
$$q=40$$

The critical point of average cost is when *q* = 40.

Notice that we still have to confirm that the critical point is a minimum. For this, we can use the first or second derivative test on $A(q)$.

$$A'(q)=\frac{-160}{q^2}+.1$$
$$A''(q)=\frac{320}{q^3}>0$$

The second derivative is positive for all positive *q*, so that means this is a local min. Average cost is minimized when they produce 40 jars of salsa; at that quantity, the average cost is $10 per jar. (Mighty expensive salsa.)

## Section 8: Other Applications

### Tangent Line Approximation

Back when we first thought about the derivative, we used the slope of secant lines over tiny intervals to approximate the derivative:

$$f'(a)\cong\frac{\Delta y}{\Delta x}=\frac{f(x)-f(a)}{x-a}$$

Now that we have other ways to find derivatives, we can exploit this approximation to go the other way. Solve the expression above for *f*(*x*), and you'll get the tangent line approximation:

**The Tangent Line Approximation (TLA)**

To approximate the value of *f*(*x*) using TLA, find some *a* where

1. *a* and *x* are "close," and

2. You know the exact values of both *f* (*a*) and *f* '(*a*).

Then $f(x) \cong f(a) + f'(a)(x-a)$

Another way to look at the same formula:

$$\Delta y \cong f'(a)\Delta x$$

How close is close? It depends on the shape of the graph of *f*. In general, the closer the better.

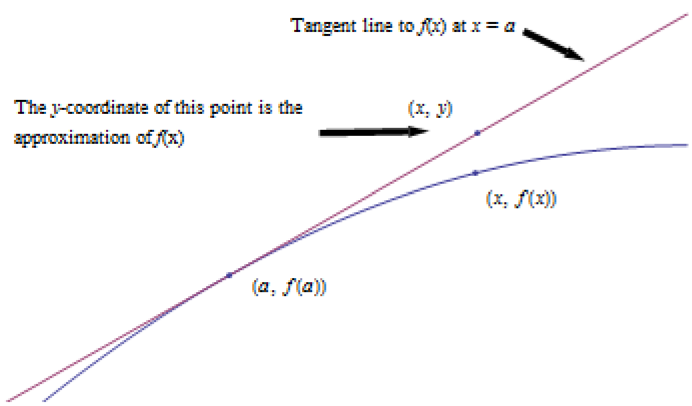



Figure 44

**Example:** Suppose we know that *g*(20) = 5 and *g*'(20) = 1.4. Using just this information, we can approximate the values of *g* at some nearby points:

$g(23) \approx 5 + (1.4)(23 - 20) = 9.2$

$g(18) \approx 5 + (1.4)(18 - 20) = 2.2$

Note that we don't know if these approximations are close – but they're the best we can do with the limited information we have to start with. Note also that 18 and 23 are sort of close to 20, so we can hope these approximations are pretty good. We'd feel more confident using this information to approximate *g*(20.003). We'd feel very unsure using this information to approximate *g*(55).

## Elasticity

We know that demand functions are decreasing, so when the price increases, the quantity demanded goes down. But what about revenue = price × quantity? Will revenue go down because the demand dropped so much? Or will revenue increase because demand didn't drop very much?

Elasticity of demand is a measure of how demand reacts to price changes. It's normalized – that means the particular prices and quantities don't matter, so we can compare onions and cars. The formula for elasticity of demand involves a derivative, which is why we're discussing it here ☺.

**Elasticity of Demand**

Given a demand function that gives q in terms of p,

**The elasticity of demand is** $E = \left| \frac{p}{q} \cdot \frac{dq}{dp} \right|$

(Note that since demand is a decreasing function of p, the derivative is negative. That's why we have the absolute values – so E will always be positive.)

If **E < 1,** we say demand is **inelastic.** In this case, raising prices increases revenue.

If **E > 1**, we say demand is **elastic.** In this case, raising prices decreases revenue.

If **E = 1**, we say demand is **unitary**. E = 1 at critical points of the revenue function.

**Example:** A company sells *q* ribbon winders per year at $*p* per ribbon winder. The demand function for ribbon winders is given by:

$$p = 300 - 0.02q$$

Find the elasticity of demand when the price is $70 apiece. Will an increase in price lead to an increase in revenue?

**Solution:** First, we need to solve the demand equation so it gives *q* in terms of *p*, so that we can find $\frac{dq}{dp}$:

$$p = 300 - 0.02q,\ so\ q = 15000 - 50p\,.$$

We need to find *q* when *p* = 70: *q* = 11500. We also need $\frac{dq}{dp} = -50$

Now compute $E = \left|\frac{p}{q}\cdot\frac{dq}{dp}\right| = \left|\frac{70}{11500}\cdot(-50)\right| \cong 0.3$.

E < 1, so demand is inelastic. Increasing the price would lead to an increase in revenue; it seems that the company should increase its price.

The demand for products that people have to buy, such as onions, tends to be inelastic. Even if the price goes up, people still have to buy about the same amount of onions, and revenue will not go down. The demand for products that people can do without, or put off buying, such as cars, tends to be elastic. If the price goes up, people will just not buy cars right now, and revenue will drop.

## Chapter 2 Exercises

**1.** What is the slope of the line through (3,9) and (x, y) for $y = x^2$ and $x = 2.97$? $x = 3.001$?

$x = 3+h$? What happens to this last slope when h is very small (close to 0)?

**2.** What is the slope of the line through (–2,4) and (x, y) for $y = x^2$ and $x = -1.98$? $x = -2.03$?

$x = -2+h$? What happens to this last slope when h is very small (close to 0)?

**3.** Fig. 36 shows the temperature during a day in Ames.

(a) What was the average change in temperature from 9 am to 1 pm?

(b) Estimate how fast the temperature was rising **at** 10 am and **at** 7 pm?

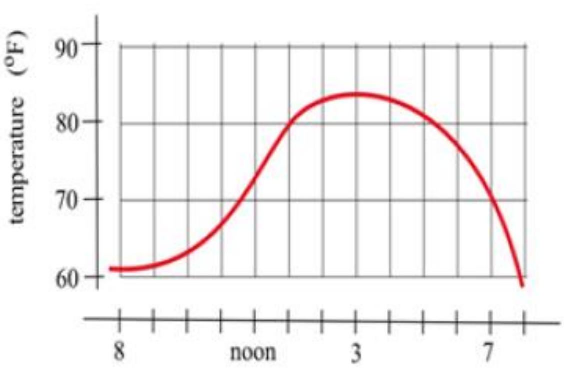



Figure 45

**4.** Fig. 37 shows the distance of a car from a measuring position located on the edge of a straight road.

(a) What was the average velocity of the car from t = 0 to t = 30 seconds?

(b) What was the average velocity of the car from t = 10 to t = 30 seconds?

(c) About how fast was the car traveling **at** t = 10 seconds? **at** t = 20 s ? **at** t = 30 s ?

(d) What does the horizontal part of the graph between t = 15 and t = 20 seconds mean?

(e) What does the negative velocity at t = 25 represent?

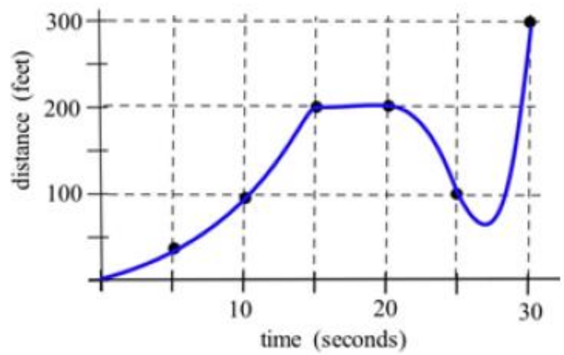



Figure 46

**5**. Fig. 38 shows the distance of a car from a measuring position located on the edge of a straight road.

(a) What was the average velocity of the car from t = 0 to t = 20 seconds?

(b) What was the average velocity from t = 10 to t = 30 seconds?

(c) About how fast was the car traveling **at** t = 10 seconds? **at** t = 20 s? **at** t = 30 s?

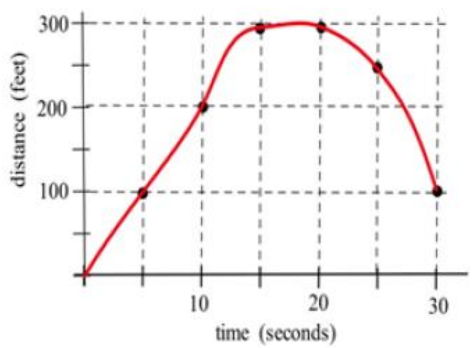



Figure 47

**6.** Use the function in Fig. 39 to fill in the table and then graph m(x).

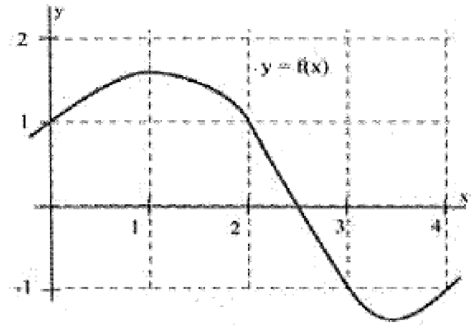



Figure 48

| $x$ | $y = f(x)$ | $m(x)$, the estimated slope of the tangent line to $y = f(x)$ at the point $(x, y)$ |
|---|---|---|
| 0 | | |
| 0.5 | | |
| 1.0 | | |
| 1.5 | | |
| 2.0 | | |
| 2.5 | | |
| 3.0 | | |
| 3.5 | | |
| 4.0 | | |

**7.** The graph of y = f(x) is given in Fig. 40. Set up a table of values for x and m(x) (the slope of the line tangent to the graph of y=f(x) at the point (x,y) ), and then graph the function m(x).

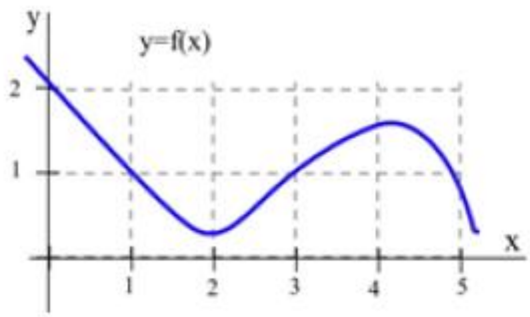



Figure 49

**8**. (a) At what values of x does the graph of f in Fig. 41 have a horizontal tangent line?

(b) At what value(s) of x is the value of f the largest? smallest?

(c) Sketch the graph of m(x) = the slope of the line tangent to the graph of f at the point (x,y).

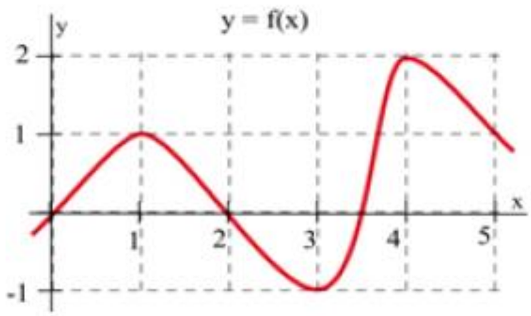



Figure 50

**9.** (a) At what values of x does the graph of g in Fig. 42 have a horizontal tangent line?

(b) At what value(s) of x is the value of g the largest? smallest?

(c) At what value(s) of x is the slope of g the largest? smallest?

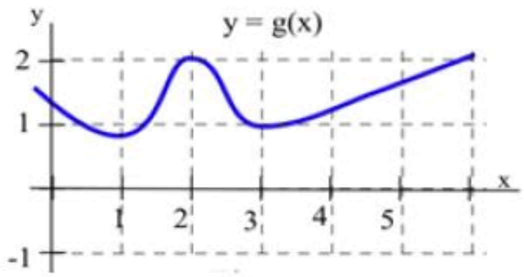



Figure 51

**10**. Match the situation descriptions with the corresponding **time–velocity** graphs in Fig. 16.

(a) A car quickly leaving from a stop sign.

(b) A car sedately leaving from a stop sign.

(c) A student bouncing on a trampoline.

(d) A ball thrown straight up.

(e) A student confidently striding across campus to take a calculus test.

(f) An unprepared student walking across campus to take a calculus test.

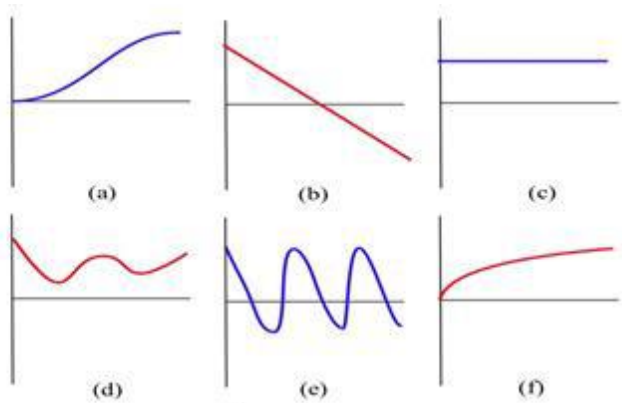



Figure 52

**11.** Fig. 44 shows the temperature during a summer day in Chicago. Sketch the graph of the **rate** at which the temperature is changing. (This is just the graph of the **slopes** of the lines which are tangent to the temperature graph.)

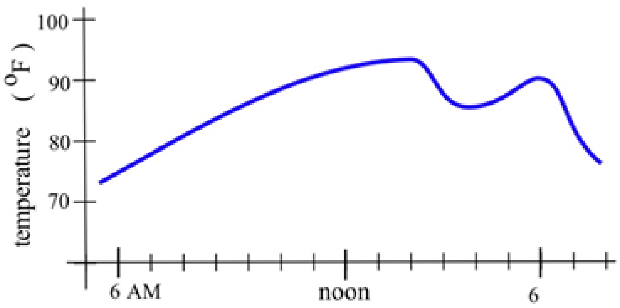



Figure 53

**12.** Fig. 45 shows six graphs, three of which are derivatives of the other three. Match the functions with their derivatives.

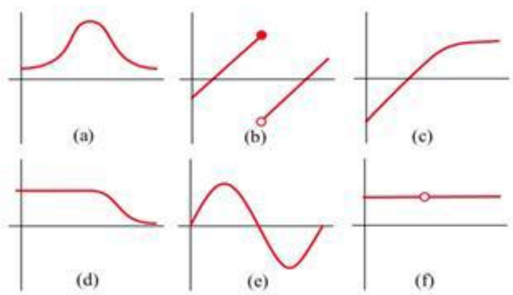



Figure 54

**13.** Match the graphs of the three functions in Fig. 46 with the graphs of their derivatives.

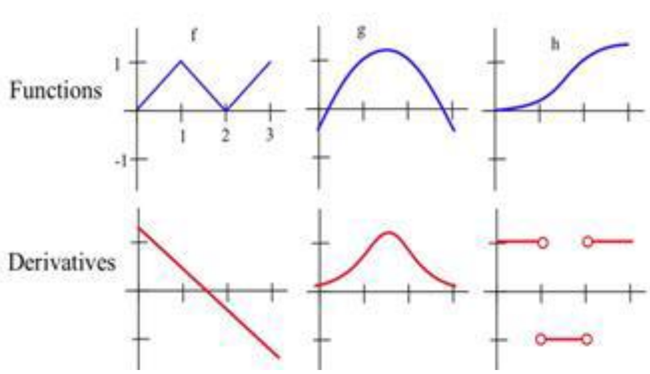



Figure 55

**14:** Fill in the values in the table for $\frac{d}{dx}(3f(x))$, $\frac{d}{dx}(2f(x)+g(x))$, and $\frac{d}{dx}(3g(x)-f(x))$.

| x | f(x) | f '(x) | g(x) | g '(x) | $\frac{d}{dx}(3f(x))$ | $\frac{d}{dx}(2f(x)+g(x))$ | $\frac{d}{dx}(3g(x)-f(x))$ |
|---|---|---|---|---|---|---|---|
| 0 | 3 | –2 | –4 | 3 | | | |
| 1 | 2 | –1 | 1 | 0 | | | |
| 2 | 4 | 2 | 3 | 1 | | | |

**15:** Use the values in the table to fill in the rest of the table.

| x | f(x) | f '(x) | g(x) | g '(x) | $\frac{d}{dx}(f(x)\cdot g(x))$ | $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$ | $\frac{d}{dx}\left(\frac{g(x)}{f(x)}\right)$ |
|---|---|---|---|---|---|---|---|
| 0 | 3 | –2 | –4 | 3 | | | |
| 1 | 2 | –1 | 1 | 0 | | | |
| 2 | 4 | 2 | 3 | 1 | | | |

**16.** Use the information in Fig. 47 to plot the values of the functions f + g, f•g and f/g and their derivatives at x = 1, 2, and 3.

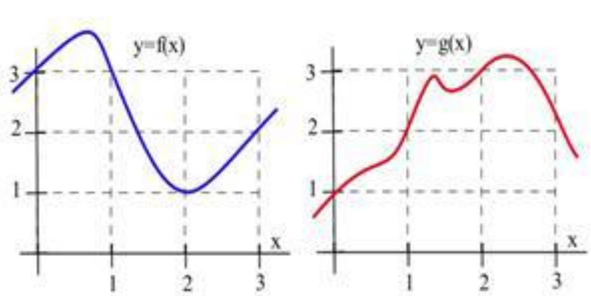



Figure 56

**17**. Calculate $\frac{d}{dx}\left((x-5)(3x+7)\right)$ by (a) using the product rule and (b) expanding the product and then differentiating. Verify that both methods give the same result.

**18.** If the product of f and g is a constant ( f(x) · g(x) = k for all x), then how are $\dfrac{\frac{d}{dx}\left(f(x)\right)}{f(x)}$ and $\dfrac{\frac{d}{dx}\left(g(x)\right)}{g(x)}$ related?

**19.** If the quotient of f and g is a constant ( $\frac{f(x)}{g(x)}=k$ for all x), then how are g · f ' and f · g ' related?

In problems **20 – 25**, (a) calculate f '(1) and (b) determine when f '(x) = 0.

**20**. f(x) = $x^2 - 5x + 13$

**21**. f(x) = $5x^2 - 40x + 73$

**22**. f(x) = $x^3 + 9x^2 + 6$

**23**. f(x) = $x^3 + 3x^2 + 3x - 1$

**24**. f(x) = $x^3 + 2x^2 + 2x - 1$

**25**. f(x) = $\frac{7x}{x^2 + 4}$

**26.**Determine $\frac{d}{dx}(x^2+1)(7x-3)$ and $\frac{\mathbf{d}}{\mathbf{dt}}\left( \frac{3t-2}{5t+1} \right)$ .

**27.** Find (a) $\frac{d}{dx}\left(x^3e^x\right)$ and (b) $\frac{d}{dx}\left(e^x\right)^3$.

**28**. Find (a) $\frac{d}{dt}\left(te^{t}\right)$, (b) $\frac{d}{dx}\left(e^{x}\right)^{5}$

**29:** Where do $f(x) = x^2 - 10x + 3$ and $g(x) = x^3 - 12x$ have horizontal tangent lines ?

**30.** $f(x) = x^3 + Ax^2 + Bx + C$ with constants A, B and C. Can you find conditions on the constants A, B and C which will guarantee that the graph of $y = f(x)$ has two distinct "vertices"? (Here a "vertex" means a place where the curve changes from increasing to decreasing or from decreasing to increasing.)

**31.** An arrow shot straight up from ground level with an initial velocity of 128 feet per second will be at height

$h(x) = -16x^2 + 128x$ feet after x seconds. (Fig.48)

(a) Determine the velocity of the arrow when x = 0, 1 and 2 seconds.

(b) What is the velocity of the arrow, v(x), at any time x?

(c) At what time x will the velocity of the arrow be 0?

(d) What is the greatest height the arrow reaches?

(e) How long will the arrow be aloft?

(f) Use the answer for the velocity in part (b) to determine the acceleration, a(x) = v '(x), at any time x.

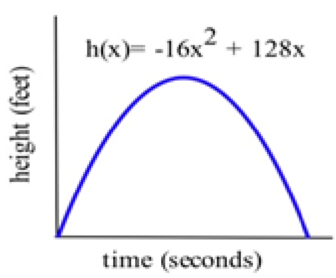



Figure 57

**32**. If an arrow is shot straight up from ground level on the moon with an initial velocity of 128 feet

per second, its height will be $h(x) = -2.65x^2 + 128x$ feet at $x$ seconds. Do parts (a) – (f) of problem 31 using this new equation for $h$.

In problems 33 - 38, differentiate each function and find the equation of the tangent line at $x = a$.

**33**. $y = \left(x^2 + 2x + 2\right)^{12}$; $a = 0$

**34**. $y = \left(1 - \dfrac{3}{x}\right)^4$; $a = 1$

**35**. $y = \left(x + \dfrac{1}{x}\right)^2$; $a = 0.5$

**36**. $y = \dfrac{5}{3 + e^t}$; $a = 0$

**37.** $y = e^x + e^{-x}$; $a = 0$

**38**. $y = e^x - e^{-x}$; $a = 0$.

**39.** A manufacturer has determined that an employee with d days of production experience will be able to

produce approximately $P(d) = 3 + 15(1 - e^{-0.2d})$ items per day.

(a) Approximately how many items will a beginning employee be able to produce each day?

(b) How many items will an experienced employee be able to produce each day?

(c) What is the marginal production rate of an employee with 5 days of experience? (What are the units of your answer, and what does this answer mean?)

**40**. The air pressure P(h) , in pounds per square inch, at an altitude of h feet above sea level is approximately $P(h) = 14.7\ e^{-0.0000385h}$ .

(a) What is the air pressure at sea level? What is the air pressure at an altitude of 30,000 feet?

(b) At what altitude is the air pressure 10 pounds per square inch?

(c) If you are in a balloon which is 2000 feet above the Pacific Ocean and is rising at 500 feet per minute, how fast is the air pressure on the balloon changing?

(d) If the temperature of the gas in the balloon remained constant during this ascent, what would happen to the volume of the balloon?

For problems 41 - 53, calculate $y' = \frac{dy}{dx}$. The letters A–D represent constants.

**41**. $y = Ax^3 - B$

**42**. $y = Ax^3 + Bx^2 + C$

**43**. $y = \sqrt{A + Bx^2}$

**44.** $y = \sqrt[3]{A - Bx^3}$

**45**. $y = Ae^{Bx}$

**46**. $y = xe^{Bx}$

**47**. $e^{Ax} + e^{-Ax}$

**48**. $y = \dfrac{1}{Ax + B}$

**49.** $y = \dfrac{Ax + B}{Cx + D}$

**50.** $y = \ln(Ax + B)$

**51.** $y = \dfrac{1}{\ln(Ax+B)}$

**52.** $y = \dfrac{\ln(Ax+B)}{Ax+B}$

**53.** $y = \ln\left(\dfrac{1}{Ax+B}\right)$

**54.** For $y = Ax^2 + Bx + C$, (a) find $y'$, (b) find the value(s) of $x$ so that $y' = 0$, and (c) find $y''$.

(You should recognize the part (b) answer from intermediate algebra. What is it?)

**55.** $y = Ax(B - x) = ABx - Ax^2$, (a) find $y'$, (b) find the value(s) of $x$ so that $y' = 0$, and (c) find $y''$.

**56.** $y = Ax^3 + Bx^2 + C$ . , (a) find $y'$, (b) find the value(s) of $x$ so that $y' = 0$, and (c) find $y''$.

In problems 57 and 58, each quotation is a statement about a quantity of something changing over time. Let $f(t)$ represent the quantity at time $t$. For each quotation, tell what $f$ represents and whether the first and second derivatives of $f$ are positive or negative.

**57**. (a) "Unemployment rose again, but the rate of increase is smaller than last month."

(b) "Our profits declined again, but at a slower rate than last month."

(c) "The population is still rising and at a faster rate than last year."

**58.** (a) "The child's temperature is still rising, but slower than it was a few hours ago."

(b) "The number of whales is decreasing, but at a slower rate than last year."

(c) "The number of people with the flu is rising and at a faster rate than last month."

**59**. On which intervals is the function in Fig. 49

(a) concave up? (b) concave down?

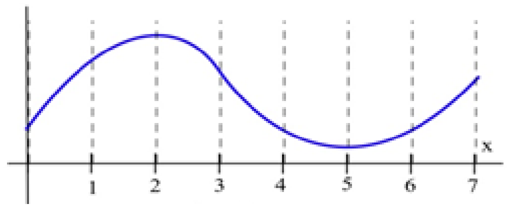



Figure 58

**60.** On which intervals is the function in Fig. 50

(a) concave up? (b) concave down?

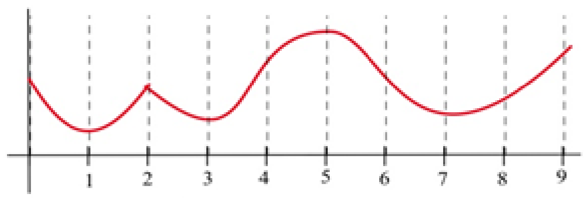



Figure 59

**61.** Sketch the graphs of functions which are defined and concave up everywhere and which have

(a) no roots. (b) exactly 1 root. (c) exactly 2 roots. (d) exactly 3 roots.

In problems 62 – 65, a function and values of x so that $f'(x) = 0$ are given. Use the Second Derivative Test to determine whether each point $(x, f(x))$ is a local maximum, a local minimum or neither .

**62.** $f(x) = 2x^3 - 15x^2 + 6$ , $x = 0, 5$ .

**63**. $g(x) = x^3 - 3x^2 - 9x + 7$ , $x = -1, 3$ .

**64**. $h(x) = x^4 - 8x^2 - 2$ , $x = -2, 0, 2$ .

**65**. $f(x) = x \cdot \ln(x)$ , $x = 1/e$ .

**66.** Which of the labeled points in Fig. 51 are inflection points?

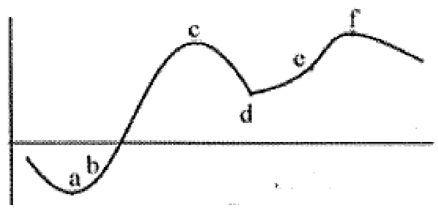



Figure 60

**67.** Which of the labeled points in Fig. 52 are inflection points?

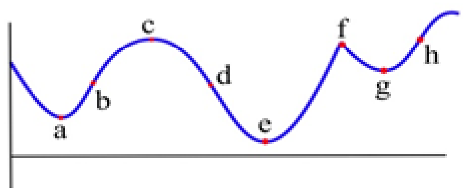



Figure 61

**68**. How many inflection points can a (a) quadratic polynomial have? (b) cubic polynomial have?

(c) polynomial of degree n have?

**69.** Fill in the table with "+", "–", or "0" for the function in Fig. 53.

| $x$ | $f(x)$ | $f'(x)$ | $f''(x)$ |
|---|---|---|---|
| 0 | | | |
| 1 | | | |
| 2 | | | |
| 3 | | | |

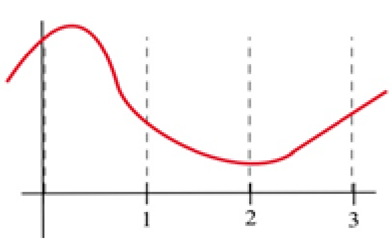



Figure 62

**70**. Fill in the table with "+", "–", or "0" for the function in Fig. 54

| $x$ | $f(x)$ | $f\,'(x)$ | $f\,''(x)$ |
|---|---|---|---|
| 0 | | | |
| 1 | | | |
| 2 | | | |
| 3 | | | |

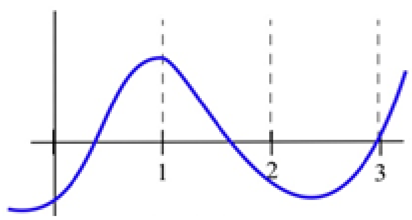



Figure 63

In problems 71 – 76 , find the derivative and second derivative of each function.

**71**. $f(x) = 7x^2 + 5x - 3$

**72**. $f(x) = (2x - 8)^5$

**73**. $f(x) = (6x - x^2)^{10}$

**74.** $f(x) = x \cdot (3x + 7)^5$

**75.** $f(x) = (2x^3 + 3)^6$

**76**. $f(x) = \sqrt{x^2 + 6x - 1}$

**77.** $f(x) = \ln(x^2 + 4)$

**78**. Find the equation of the line tangent to $f(x) = e^x$ at the point $(3, e^3)$. Where will this tangent line intersect the x–axis? Where will the tangent line to $f(x) = e^x$ at the point $(p, e^p)$ intersect the x–axis?

**79**. Find all extremes of $f(x) = 3x^2 - 12x + 7$ and use the First Derivative Test to determine if they are maximums, minimums or neither.

In problems 80 – 85, find all of the critical points and local maximums and minimums of each function.

**80.** $f(x) = x^2 + 8x + 7$

**81**. $f(x) = 2x^2 - 12x + 7$

**82**. $f(x) = x^3 - 6x^2 + 5$

**83**. $f(x) = (x-1)^2 (x-3)$

**84.** $f(x) = \ln( x^2 - 6x + 11 )$

**85.** $f(x) = 2x^3 - 96x + 42$

In problems 86 – 93 , find all critical points and global extremes of each function on the given intervals.

**86.** $f(x) = x^2 - 6x + 5$ on the entire real number line.

**87.** $f(x) = 2 - x^3$ on the entire real number line.

**88**. $f(x) = x^3 - 3x + 5$ on the entire real number line.

**89.** $f(x) = x - e^x$ on the entire real number line.

**90.** $f(x) = x^2 - 6x + 5$ on $[-2, 5]$ .

**91**. $f(x) = 2 - x^3$ on $[-2, 1]$ .

**92**. $f(x) = x^3 - 3x + 5$ on $[-2, 1]$ .

**93.** $f(x) = x - e^x$ on $[1, 2]$ .

**94**.Find all of the critical points of the function in Fig. 55 and identify them as local max, local min, or neither. Find the global max and min on the interval.

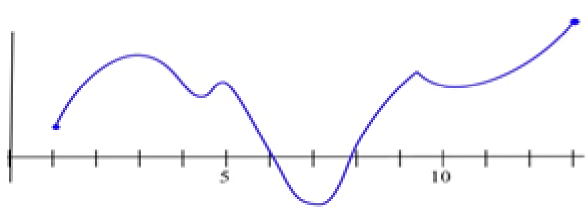



Figure 64

**95**. Find all of the critical points of the function in Fig. 56 and identify them as local max, local min, or neither. Find the global max and min on the interval.

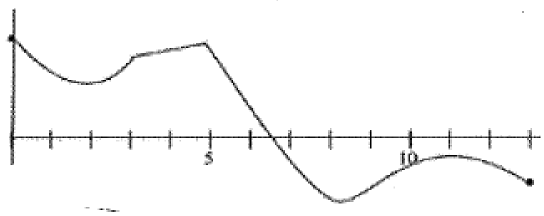



Figure 65

**96**. Suppose f(1) = 5 and f '(1) = 0. What can we conclude about the point (1,5) if

(a) f '(x) < 0 for x < 1, and f '(x) > 0 for x > 1? (b) f '(x) < 0 for x < 1, and f '(x) < 0 for x > 1?

(c) f '(x) > 0 for x < 1, and f '(x) < 0 for x > 1? (d) f '(x) > 0 for x < 1, and f '(x) > 0 for x > 1?

**97**. What will the $2^{nd}$ derivative of a quadratic polynomial be? The $3^{rd}$ derivative? The $4^{th}$ derivative?

**98**. What will the $3^{rd}$ derivative of a cubic polynomial be? The $4^{th}$ derivative?

**99.** What can you say about the $n^{th}$ and $(n+1)^{st}$ derivatives of a polynomial of degree n?

**100.** Sketch the graph of a continuous function f so that

(a) f(1) = 3, f '(1) = 0 , and the point (1,3) is a local maximum of f .

(b) f(2) = 1, f '(2) = 0 , and the point (2,1) is a local minimum of f .

(c) f(5) = 4, f '(5) = 0, and the point (5,4) is not a local minimum or maximum of f .

**101**. Define A(x) to be the **area** bounded between the x–axis, the graph of f, and a vertical line at x ( $0 \le x \le 10$ ). See Fig. 57.

(a) At what value of x is A(x) minimum?

(b) At what value of x is A(x) maximum?

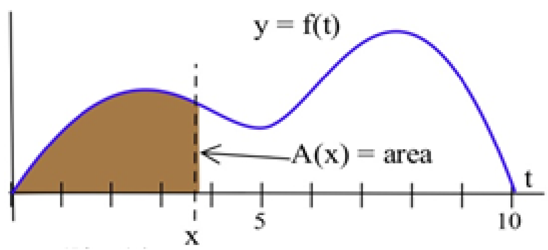



Figure 66

**102**. Define S(x) to be the **slope** of the line through the points (0,0) and ( x, f(x) ) ( $0 \le x \le 10$ ). See Fig. 58.

(a) At what value of x is S(x) minimum?

(b) At what value of x is S(x) maximum?

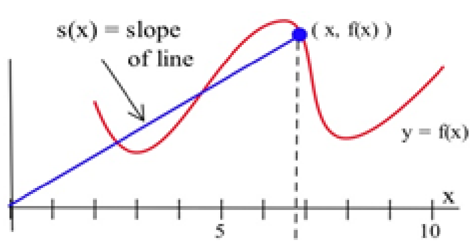



Figure 67

**103**. (a) You have 200 feet of fencing to enclose a rectangular vegetable garden. What should the dimensions of your garden be in order to enclose the largest area?

(b) Show that if you have P feet of fencing available, the garden of greatest area is a square.

(c) What are the dimensions of the largest rectangular garden you can enclose with P feet of fencing if one edge of the garden borders a straight river and does not need to be fenced?

(d) Just thinking — calculus will not help with this one: What do you think is the shape of the largest garden which can be enclosed with P feet of fencing if we do not require the garden to be rectangular? What do you think is the shape of the largest garden which can be enclosed with P feet of fencing if one edge of the garden borders a river and does not need to be fenced?

**104**. (a) You have 200 feet of fencing available to construct a rectangular pen with a fence divider down the middle (see Fig. 59). What dimensions of the pen enclose the largest total area?

(b) If you need 2 dividers, what dimensions of the pen enclose the largest area?

(c) What are the dimensions in parts (a) and (b) if one edge of the pen borders on a river and does not require any fencing?

**Figure 68**

**105.** You have 120 feet of fencing to construct a pen with 4 equal sized stalls. If the pen is rectangular and shaped like the one in Fig. 60, what are the dimensions of the pen of largest area and what is that area?

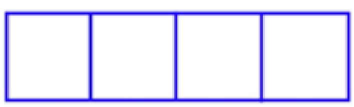


**Figure 69**

**106**. Suppose you decide to fence the rectangular garden in the corner of your yard. Then two sides of the garden are bounded by the yard fence which is already there, so you only need to use the 80 feet of fencing to enclose the other two sides. What are the dimensions of the new garden of largest area? What are the dimensions of the rectangular garden of largest area in the corner of the yard if you have F feet of new fencing available?

**107.** (a) You have been asked to bid on the construction of a square-bottomed box with no top which will hold 100 cubic inches of water. If the bottom and sides are made from the same material, what are the dimensions of the box which uses the least material? (Assume that no material is wasted.)

(b) Suppose the box in part (a) uses different materials for the bottom and the sides. If the bottom material costs 5¢ per square inch and the side material costs 3¢ per square inch, what are the dimensions of the least expensive box which will hold 100 cubic inches of water?

**108.** Problem 107 is a "classic" problem which has many variations. We could require that the box be twice as long as it is wide, or that the box have a top, or that the ends cost a different amount than the front and back, or even that it costs some amount of money to weld each inch of edge. Write and solve a variation of Problem 107.

**109**. U.S. postal regulations state that the sum of the length and girth (distance around) of a package must be no more than 108 inches. (Fig. 61)

(a) Find the dimensions of the acceptable box with a square end which has the largest volume.

(b) Find the dimensions of the acceptable box which has the largest volume if its end is a rectangle twice as long as it is wide.

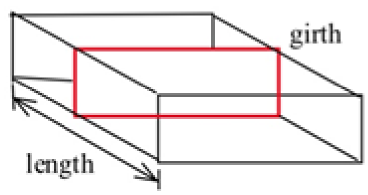



**Figure 70**

**110**. D. Simonton claims that the "productivity levels" of people in different fields can be described as a function of their "career age" $t$ by $p(t) = e^{-at} - e^{-bt}$ where $a$ and $b$ are constants which depend on the field of work, and career age is approximately 20 less than the actual age of the individual.

(a) Based on this model, at what ages do mathematicians (a=.03, b=.05), geologists (a=.02, b=.04), and historians (a=.02, b=.03) reach their maximum productivity?

(b) Simonton says "With a little calculus we can show that the curve ( p(t) ) maximizes at $t = \frac{1}{b-a} \ln( \frac{b}{a} )$." Use calculus to show that Simonton is correct.

Note: Models of this type have uses for describing the behavior of groups, but it is dangerous and usually invalid to apply group descriptions or comparisons to **individuals** in the group.

(Scientific Genius, by Dean Simonton, Cambridge University Press, 1988, pp. 69 – 73)

**111.** You own a small airplane which holds a maximum of 20 passengers. It costs you $100 per flight from St. Thomas to St. Croix for gas and wages plus an additional $6 per passenger for the extra gas required by the extra weight. The charge per passenger is $30 each if 10 people charter your plane (10 is the minimum number you will fly), and this charge is reduced by $1 per passenger for each passenger over 10 who goes (that is, if 11 go they each pay $29, if 12 go they each pay $28, etc.). What number of passengers on a flight will maximize your profits?

**112.** The function $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-c)^2}{2b^2}}$ is called the Gaussian distribution, and its graph is a bell–shaped curve (Fig. 62) that occurs commonly in statistics.

(i) Show that f has a maximum at x = c . ( The value c is called the mean of this distribution.)

(ii) Show that f has inflection points where x = c + b and x = c – b . (The value b is called the standard deviation of this distribution.)

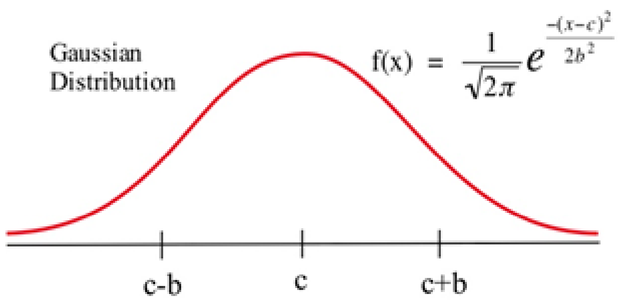



Figure 71

**113.** In the planning of a coffee shop, we estimate that if there is seating for between 40 and 80 people, the daily profit will be \$50 per seat. However, if the seating capacity is more than 80 places, the daily profit per seat will be decreased by \$1 for each additional seat over 80. What should the seating capacity be in order to maximize the coffee shop's total profit?

**114.** In the planning of a taco restaurant, we estimate that if there is seating for between 10 and 40 people, the daily profit will be \$10 per seat. However, if the seating capacity is more than 40 places, the daily profit per seat will be decreased by \$0.20 per seat. What should the seating capacity be in order to maximize the taco restaurant's total profit?

**115.** The total cost in dollars for Alicia to make *q* oven mitts is given by $C(q)=64+1.5q+.01q^2$.

(a) What is the fixed cost?

(b) Find a function that gives the marginal cost.

(c) Find a function that gives the average cost.

(d) Find the quantity that minimizes the average cost.

(e) Confirm that the average cost and marginal cost are equal at your answer to part (d).

**116.** Shaki makes and sells backpack danglies. The total cost in dollars for Shaki to make *q* danglies is given by $C(q) = 75 + 2q + .015q^2$. Find the quantity that minimizes Shaki's average cost for making danglies.

**117.** If $g(20) = 35$ and $g'(20) = -2$, estimate the value of $g(22)$.

**118**. If $g(1) = -17$ and $g'(1) = 5$, estimate the value of $g(1.2)$.

**119.** Use the Tangent Line Approximation to estimate the cube root of 9.

**120.** Use the Tangent Line Approximation to estimate the fifth root of 30.

**121.** The demand function for Alicia's oven mitts is given by $q = -8p + 80$ (*q* is the number of oven mitts, *p* is the price in dollars). Find the elasticity of demand when *p* = $7.50. Will revenue increase if Alicia raises her price from $7.50?

**122.** The demand function for Shaki's danglies is given by $q = -35p + 205$ (*q* is the number of danglies, *p* is the price in dollars per dangly). Find the elasticity of demand when *p* = $5. Should Shaki raise or lower his price to increase revenue?

# Chapter 3: The Integral

The previous chapters dealt with **Differential Calculus**. We started with the "simple" geometrical idea of the **slope of a tangent line** to a curve, developed it into a combination of theory about derivatives and their properties, techniques for calculating derivatives, and applications of derivatives. This chapter deals with **Integral Calculus** and starts with the "simple" geometric idea of **area**. This idea will be developed into another combination of theory, techniques, and applications.

## PreCalculus Idea – The Area of a Rectangle

If you look on the inside cover of nearly any traditional math book, you'll find a bunch of area and volume formulas – the area of a square, the area of a trapezoid, the volume of a right circular cone, and so on. Some of these formulas are pretty complicated. But you still won't find a formula for the area of a jigsaw puzzle piece or the volume of an egg. There are lots of things for which there is no formula. Yet we might still want to find their areas.

One reason areas are so useful is that they can represent quantities other than simple geometric shapes. If the units for each side of the rectangle are *meters*, then the area will have the units *meters*× *meters* = *square meters* = m$^2$. But if the units of the base of a rectangle are *hours* and the units of the height are *miles/hour*, then the units of the area of the rectangle are *hours* × *miles/hour* = *miles*, a measure of distance (Fig. 1a). Similarly, if the base units are *centimeters* and the height units are *grams* (Fig. 1b), then the area units are *gram-centimeters*, a measure of work.

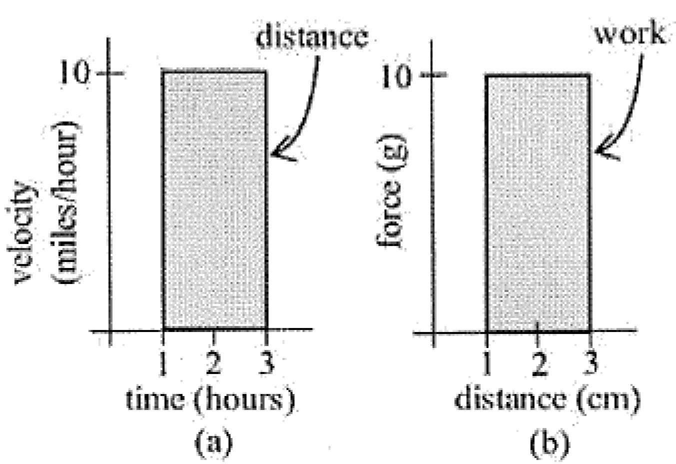



Figure 72

The basic shape we will use is the rectangle; the area of a rectangle is base × height. The only other area formulas I'll expect you to know are for triangles ( $A = \frac{1}{2}bh$ ) and for circles ( $A = \pi r^2$ ).

# Section 1: The Definite Integral

## Distance from Velocity

**Example**: Suppose a car travels on a straight road at a constant speed of 40 miles per hour for two hours. See the graph of its velocity in Fig. 2. How far has it gone?

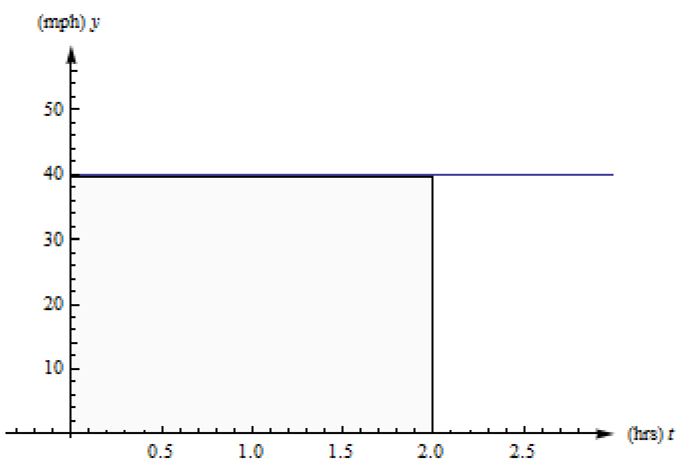



Figure 73

**Solution:** We all remember **distance = rate × time**, so this one is easy. The car has gone 40 miles per hour × 2 hours = 80 miles.

**Example:** But now suppose that a car travels so that its speed increases steadily from 0 to 40 miles per hour, for two hours. (Just be grateful you weren't stuck behind this car on the highway.) See the graph of its velocity in Fig. 3. How far has this car gone?

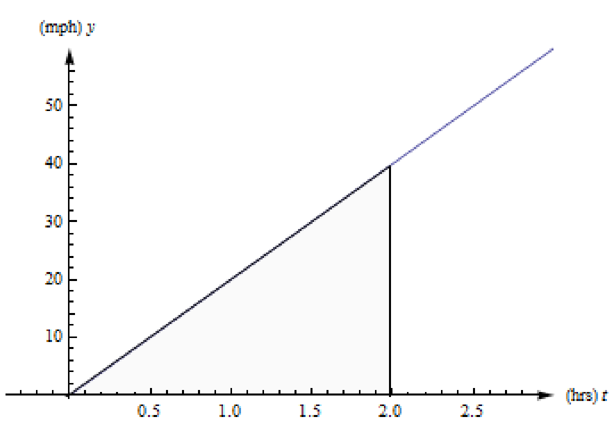



Figure 74

The trouble with our old reliable **distance = rate × time** relationship is that it only works if the rate is constant. If the rate is changing, there isn't a good way to use this formula. But look at Fig. 1 again. Notice that **distance = rate × time** also describes the area between the velocity graph and the t-axis, between t = 0 and t = 2 hours. The **rate** is the height of the rectangle, the **time** is the length of the rectangle, and the **distance** is the **area** of the rectangle. This is the way we can extend our simple formula to handle more complicated velocities: And this is the way we can answer the second example.

**Solution:** The distance the car travels is the area between its velocity graph, the t-axis, t = 0 and t = 2. This region is a triangle, so its area is ½bh = ½(2 hours)(40 miles per hour) = 40 miles. So the car travels 40 miles during its annoying trip.

In our distance/velocity examples, the function represented a **rate** of travel (miles per hour), and the area represented the **total** distance traveled. This principle works more generally:

For functions representing other **rates** such as the production of a factory (bicycles per day), or the flow of water in a river (gallons per minute) or traffic over a bridge (cars per minute), or the spread of a disease (newly sick people per week), the area will still represent the **total** amount of something.

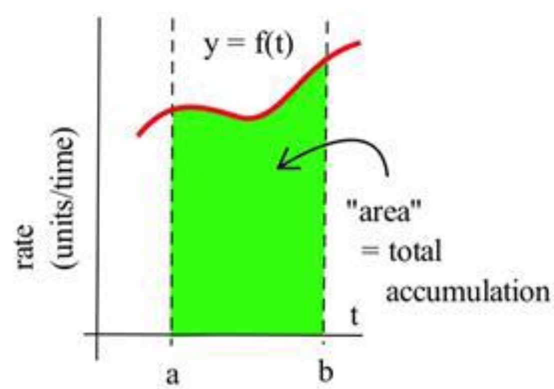



Figure 75

**Example:** Fig. 5 shows the flow rate (cubic feet per second) of water in the Skykomish river at the town of Goldbar in Washington state. (For comparison, the flow over Niagara Falls is about $2.12 \times 10^5$ cf/s.)

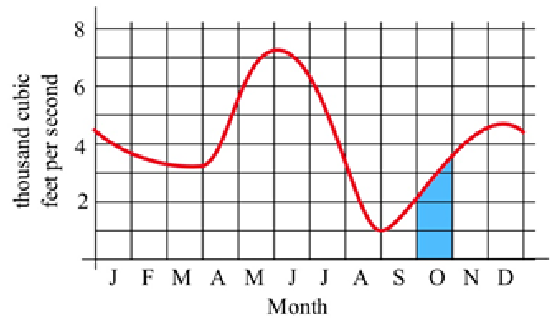



Figure 76

The area of the shaded region represents the total volume (cubic feet) of water flowing past the town during the month of October. We can approximate this area to approximate the total water by thinking of the shaded region as a rectangle with a triangle on top.

Total water = total area ≈ area of rectangle + area of the "triangle"

$$\approx (2000 \text{ cubic feet/sec})(30 \text{ days}) + \frac{1}{2}(1500 \text{ cf/s})(30 \text{ days}) = (2750 \text{ cubic feet/sec})(30 \text{ days})$$

Note that we need to convert the units to make sense of our result:

$$\text{Total water} \approx (2750 \text{ cubic feet/sec})(30 \text{ days}) = (2750 \text{ cubic feet/sec})(2{,}592{,}000 \text{ sec})$$

$$\approx 7.128 \times 10^9 \text{ cubic feet.}$$

About 7 billion cubic feet of water flowed past Goldbar in October.

## Approximating with Rectangles

How do we approximate the area if the rate curve is, well, curvy? We could use rectangles and triangles, like we did in the last example. But it turns out to be more useful (and easier) to simply use rectangles. The more rectangles we use, the better our approximation is.

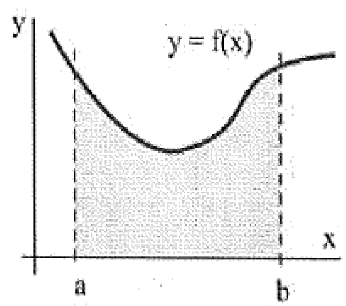



Figure 77

Suppose we want to calculate the area between the graph of a positive function $f$ and the interval [a, b] on the $x$–axis (Fig. 6). The **Riemann Sum method** is to build several rectangles with bases on the interval [a, b] and sides that reach up to the graph of $f$ (Fig. 7). Then the areas of the rectangles can be calculated and added together to get a number called a Riemann Sum of $f$ on [a, b]. The area of the region formed by the rectangles is an **approximation** of the area we want.

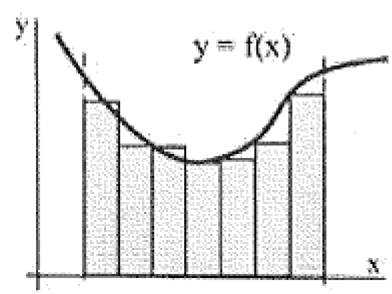



Figure 78

**Example**: Approximate the area in Fig. 8a between the graph of f and the interval [2, 5] on the x–axis by summing the areas of the rectangles in Fig. 8b.

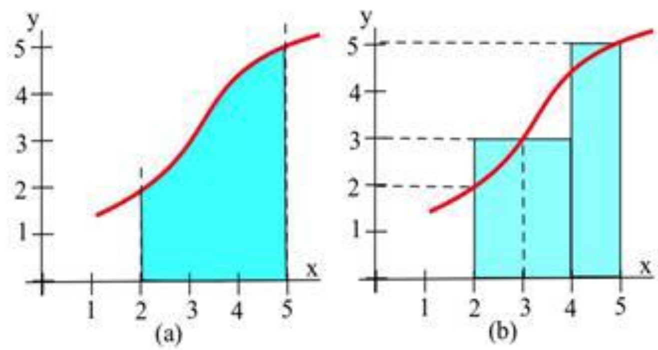



Figure 79

**Solution:** The total area of rectangles is (2)(3) + (1)(5) = 11 square units.

**Example:** Let A be the region bounded by the graph of f(x) = 1/x, the x–axis, and vertical lines at x = 1 and x = 5. We can't find the area exactly (with what we know now), but we can approximate it using rectangles.

When we make our rectangles, we have a lot of choices. We could pick any (non-overlapping) rectangles whose bottoms lie within the interval on the x-axis, and whose tops intersect with the curve somewhere. But it's easiest to choose rectangles that – (a) have all the same width, and (b) take their heights from the function at one edge. Figs. 9 and 10 below show two ways to use four rectangles to approximate this area. In Fig. 9, we used left-endpoints; the height of each rectangle comes from the function value at its left edge. In Fig 10, we used right-hand endpoints.

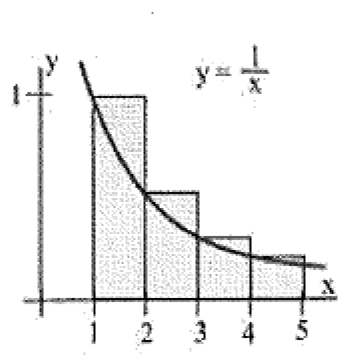



Figure 80

**Left-hand endpoints:** The area is approximately the sum of the areas of the rectangles. Each rectangle gets its height from the function $f(x) = \frac{1}{x}$ and each rectangle has width = 1.

You can find the area of each rectangle using area = height × width. So the total area of the rectangles, the left-hand estimate of the area under the curve, is

$$f(1)(1) + f(2)(1) + f(3)(1) + f(4)(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \cong 2.08$$

Notice that because this function is decreasing, all the left endpoint rectangles stick out above the region we want – using left-hand endpoints will overestimate the area.

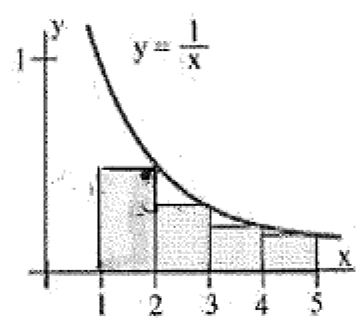



Figure 81

**Right-hand endpoints:** The right-hand estimate of the area is

$$f(2)(1) + f(3)(1) + f(4)(1) + f(5)(1) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} \cong 1.28$$

All the right-hand rectangles lie completely under the curve, so this estimate will be an underestimate.

We can see that the true area is actually in between these two estimates. So we could take their average:

**Average:** $\dfrac{25/12+77/60}{2}=\dfrac{101}{60}\cong 1.68$

In general, the average of the left-hand and right-hand estimates will be closer to the real area than either individual estimate.

My estimate of the area under the curve is about 1.68. (The actual area is about 1.61.)

If we wanted a better answer, we could use even more, even narrower rectangles. But there's a limit to how much work we want to do by hand. In practice, it's probably best to choose a manageable number of rectangles. We'll have better methods to get more accurate answers before long.

These sums of areas of rectangles are called **Riemann sums**. You may see a shorthand notation used when people talk about sums. We won't use it much in this book, but you should know what it means.

**Riemann sum:** A Riemann sum for a function f(x) over an interval [a, b] is a sum of areas of rectangles that approximates the area under the curve. Start by dividing the interval [a, b] into n subintervals; each subinterval will be the base of one rectangle. We usually make all the rectangles the same width Δx. The height of each rectangle comes from the function evaluated at some point in its sub interval. Then the Riemann sum is:

$$f(x_1)\Delta x+f(x_2)\Delta x+f(x_3)\Delta x+\cdots+f(x_n)\Delta x$$

**Sigma Notation:** The upper-case Greek letter Sigma Σ is used to stand for Sum. Sigma notation is a way to compactly represent a sum of many similar terms, such as a Riemann sum.

Using the Sigma notation, the Riemann sum can be written $\sum_{i=1}^{n} f(x_i)\Delta x$.

This is read aloud as "the sum as i = 1 to n of f of x sub i Delta x." The "i" is a counter, like you might have seen in a programming class.

## Definition of the Definite Integral

Because the area under the curve is so important, it has a special vocabulary and notation.

### The Definite Integral:

The **definite integral** of a positive function f(x) over an interval [a, b] is the area between f, the x-axis, x = a and x = b.

The **definite integral** of a positive function f(x) from a to b is the area under the curve between a and b.

If f(t) represents a positive rate (in y-units per t-units), then the **definite integral** of f from a to b is the total y-units that accumulate between t = a and t = b.

**Notation for the Definite Integral:**

The definite integral of f from a to b is written

$$\int_a^b f(x)\,dx$$

The $\int$ symbol is called an **integral sign**; it's an elongated letter S, standing for sum. (The $\int$ is actually the Σ from the Riemann sum, written in Roman letters instead of Greek letters.)

The dx on the end must be included; you can think of $\int$ and dx as left and right parentheses. The dx tells what the variable is – in this example, the variable is x. (The dx is actually the $\Delta x$ from the Riemann sum, written in Roman letters instead of Greek letters.)

The function f is called the **integrand**.

The a and b are called the limits **of integration**.

**Verb forms:**

We **integrate**, or **find the definite integral** of a function. This process is called **integration**.

**Formal Algebraic Definition:** $\int_a^b f(x)\,dx = \lim\limits_{\substack{n\to\infty \\ \Delta x\to 0}} \sum_{i=1}^{n} f(x_i)\Delta x$. **(*)**

**Practical Definition:**

The definite integral can be approximated with a Riemann sum (dividing the area into rectangles where the height of each rectangle comes from the function, computing the area of each rectangle, and adding them up). The more rectangles you use, the narrower the rectangles are, the better your approximation will be.

**Looking Ahead:**

We will have methods for computing exact values of some definite integrals from formulas soon. In many cases, including when the function is given to you as a table or graph, you will still need to approximate the definite integral with rectangles.

(* information about "lim" is in Chapter 5: Optional Topics)

**Example:** Fig. 11 shows y = r(t), the number of telephone calls made per hours (a rate!) on a Tuesday. Approximately how many calls were made between 9 pm and 11 pm? Express this as a definite integral and approximate with a Riemann sum.

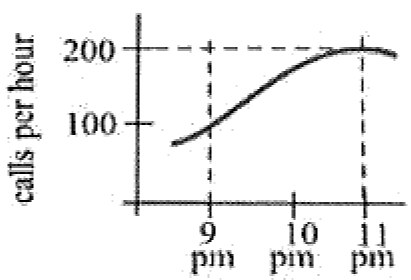



Figure 82

**Solution:** We know that the accumulated calls will be the area under this rate graph over that two-hour period, the definite integral of this rate from t = 9 to t = 11.

The total number of calls will be $\int_9^{11} r(t)dt$ .

The top here is a curve, so we can't get an exact answer. But we can approximate the area using rectangles. I'll choose to use 4 rectangles, and I'll choose left-endpoints:

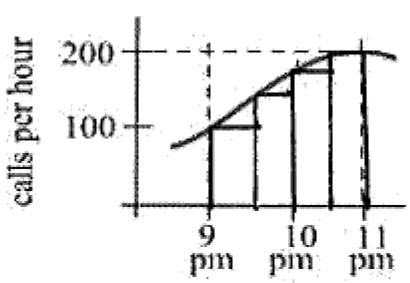



Figure 83

$$\int_9^{11} r(t)dt \cong 100(.5)+150(.5)+180(.5)+195(.5)=312.5.$$

The units are *calls per hour* × *hours* = *calls*. My estimate is that about 312 calls were made between 9 pm and 11 pm. Is this an under-estimate or an over-estimate?

**Example**: Describe the area between the graph of f(*x*) = 1/*x*, the *x*–axis, and the vertical lines at *x* = 1 and *x* = 5 as a definite integral.

**Solution:** This is the same area we estimated to be about 1.68 before. Now we can use the notation of the definite integral to describe it. Our estimate of $\int_1^5 \frac{1}{x}dx$ was 1.68. The true value of $\int_1^5 \frac{1}{x}dx$ is about 1.61.

**Example**: Using the idea of area, determine the value of $\int_1^3 (1+x)dx$.

**Solution:** $\int_1^3 (1+x)dx$ represents the area between the graph of f(*x*) = 1+*x*, the *x*–axis, and the vertical lines at 1 and 3 (Fig. 13).

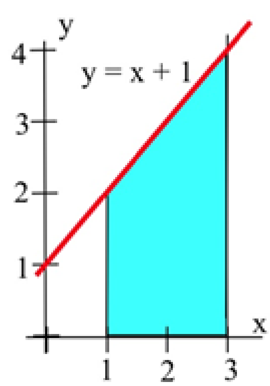



Figure 84

Since this area can be broken into a rectangle and a triangle, we can find the area exactly. The area equals 4 + ½ (2)(2) = 6 square units.

**Example:** The table shows rates of population growth for Berrytown for several years. Use this table to estimate the total population growth from 1970 to 2000:

| Year (t) | 1970 | 1980 | 1990 | 2000 |
|---|---|---|---|---|
| Rate of population growth R(<br>(thousands of people per yea | 1.5 | 1.9 | 2.2 | 2.4 |

**Solution:** The definite integral of this rate will give the total change in population over the thirty-year period. We only have a few pieces of information, so we can only estimate. Even though I haven't made a graph, we're still approximating the area under the rate curve, using rectangles. How wide are the rectangles? I have information every 10 years, so the rectangles have a width of 10 years. How many rectangles? Be careful here – this is a thirty-year span, so there are three rectangles.

Using left-hand endpoints: (1.5)(10) + (1.9)(10) + (2.2)(10) = 56;

Using right-hand endpoints: (1.9)(10) + (2.2)(10) + (2.4)(10) = 65;

Taking the average of these two: $\frac{56+65}{2} = 60.5$

My best estimate of the total population growth from 1970 to 2000 is 60.5 thousand people.

## Signed Area

You may have noticed that until this point, we've insisted that the integrand (the function we're integrating) be positive. That's because we've been talking about area, which is always positive. If the "height" (from the function) is a negative number, then multiplying it by the width doesn't give us actual area, it gives us the area with a negative sign.

But it turns out to be useful to think about the possibility of negative area. We'll expand our idea of a definite integral now to include integrands that might not always be positive. The "heights" of the rectangles, the values from the function, now might not always be positive.

## The Definite Integral and Signed Area:

The **definite integral** of a function f(x) over an interval [a, b] is the **signed area** between f, the x-axis, x = a and x = b.

The **definite integral** of a function f(x) from a to b is the **signed area** under the curve between a and b.

If the function is positive, the signed area is positive, as before (and we can call it area.)

If the function dips below the x-axis, the areas of the regions below the x-axis come in with a negative sign. In this case, we cannot call it simply "area." These negative areas take away from the definite integral.

$$\int_a^b f(x)\,dx = \text{(Area above x-axis)} - \text{(Area below x-axis)}.$$

If f(t) represents a positive rate (in y-units per t-units), then the **definite integral** of f from a to b is the **total** y-units that accumulate between t = a and t = b.

If f(t) represents any rate (in y-units per t-units), then the **definite integral** of f from a to b is the **net** y-units that accumulate between t = a and t = b.

**Example:** Find the definite integral of of f($x$) = –2 on the interval [1,4].

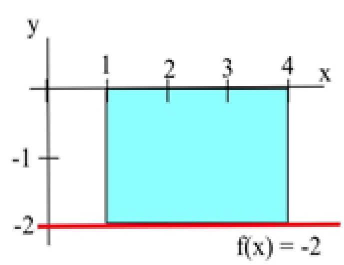



Figure 85

**Solution:** $\int_1^4 -2\,dx$ is the signed area of the region shown to the right. The region lies below the x-axis, so the area (6) comes in with a negative sign. So the definite integral is

$$\int_1^4 -2\,dx = -6.$$

Negative rates indicate that the amount is decreasing. For example, if f($t$) is the velocity of a car in the positive direction along a straight line at time $t$ (miles/hour) , then negative values of f indicate that the car is traveling in the negative direction, backwards. The definite integral of f is the change in position of the car during the time interval. If the velocity is positive, positive distance accumulates. If the velocity is negative, distance in the negative direction accumulates.

This is true of any rate. For example, if f($t$) is the rate of population change (people/year) for a town, then negative values of f would indicate that the population of the town was getting smaller, and the definite integral (now a negative number) would be the **change** in the population, a decrease, during the time interval.

**Example**: In 1980 there were 12,000 ducks nesting around a lake, and the **rate** of population change (in ducks per year) is shown in Fig. 15. Write a definite integral to represent the total change in the duck population from 1980 to 1990, and estimate the population in 1990.

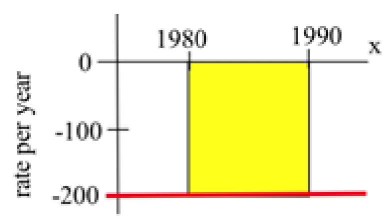



Figure 86

**Solution:** The **change** in population

$$= \int_{1980}^{1990} f(t)\,dt = -\{\text{area between f and axis}\}$$

$$\approx -\{200 \text{ ducks/year}\}\cdot\{10 \text{ years}\} = -2000 \text{ ducks}.$$

Then {1990 duck population} = {1980 population} + {change from 1980 to 1990}

= {12,000} + { –2000} = 10,000 ducks.

**Example**: A bug starts at the location x = 12 on the x–axis at 1 pm walks along the axis with the velocity *v*(*x*) shown in Fig. 16. How far does the bug travel between 1 pm and 3 pm, and where is the bug at 3 pm?

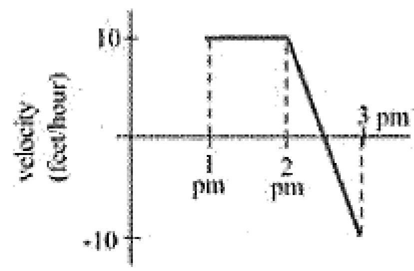



Figure 87

**Solution:** Note that the velocity is positive from 1 until 2:30, then becomes negative. So the bug moves in the positive direction from 1 until 2:30, then turns around and moves back toward where it started. The area under the velocity curve from 1 to 2:30 shows the total distance traveled by the bug in the positive direction; the bug moved 12.5 feet in the positive direction. The **area** between the velocity curve and the x-axis, between 2:30 and 3, shows the total distance traveled by the bug in the negative direction, back toward home; the bug traveled 2.5 feet in the negative direction. The definite integral of the velocity curve, $\int_1^3 v(t)dt$, shows the net change in distance:

$$\int_1^3 v(t)dt = 12.5 - 2.5 = 10$$

The bug ended up 10 feet further in the positive direction than he started. At 3 pm, the bug is at x = 22.

**Example**: Use Fig. 17 to calculate $\int_0^2 f(x)\,dx$ , $\int_2^4 f(x)\,dx$ , $\int_4^5 f(x)\,dx$ , and $\int_0^5 f(x)\,dx$ .

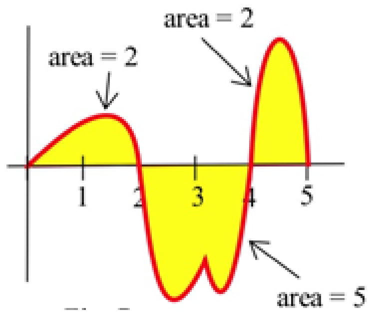



Figure 88

**Solution:** $\int_0^2 f(x)\,dx = 2$, $\int_2^4 f(x)\,dx = \mathbf{-5}$ , $\int_4^5 f(x)\,dx = 2$, and

$$\int_0^5 f(x)\,dx = \{\text{area above}\} - \{\text{area below}\} = \{2+2\} - \{5\} = -1.$$

## Approximating with Technology

If your function is given as a graph or table, you will still have to approximate definite integrals using areas, usually of rectangles. But if your function is given as a formula, you can turn to technology to get a better approximate answer. For example, most graphing calculators have some kind of numerical integration tool built in. You can also find many online tools that can do this; type numerical integration into any search engine to see a selection of these.

Most numerical integration tools use rectangles to estimate the signed area, just as you would do by hand. But they use many more rectangles than you would have the patience for, so they get a better answer. Some of them use computer algebra systems to find exact answers; we will learn how to do this ourselves later in this chapter.

When you turn to technology to find the value of a definite integral, be careful. Not every tool will be able to give you a correct answer for every integral. I have had good luck with my TI 84. You should make an estimate of the answer yourself first so you can judge whether the answer you get makes sense.

**Example:** Use technology to approximate the definite integral $\int_1^5 \frac{1}{x} dx$. (This is the same definite integral we approximated with rectangles before.)

**Solution:** I used my TI-84; the answer it gave me was 1.609437912. This agrees with the exact answer for all the decimal digits displayed. WebMath said the answer was 1.60944, which is accurate for all the decimal digits displayed. Microsoft Math said the answer was ln(5); that's exactly correct. Wolfram|Alpha says the answer is log(5); that's not how everyone writes the natural log, so that might trick you into writing the wrong answer.

**Example:** Use technology to approximate the definite integral $\int_1^2 e^{x^2+x}\, dx$

**Solution:** I asked WebMath, and it said the answer was zero – I know this is not correct, because the function here is positive, so there must be some area under the curve here. I asked Microsoft Math, and it simply repeated the definite integral; that's because there isn't an algebraic way to find the exact answer. I asked my TI-84, and it said the answer was 86.83404047; that makes sense with what I expected. Wolfram|Alpha also says the answer is about 86.834. So I believe: $\int_1^2 e^{x^2+x}\, dx \cong 86.834.$

## Accumulation in Real Life

We have already seen that the "area" under a graph can represent quantities whose units are not the usual geometric units of square meters or square feet. For example, if $t$ is a measure of time in seconds and f($t$) is a velocity with units feet/second, then the definite integral has units (feet/second) · (seconds) = feet.

In general, the units for the definite integral $\int_a^b f(x)\,dx$ are ($y$-units ) • ($x$-units). A quick check of the units can help avoid errors in setting up an applied problem.

In previous examples, we looked at a function represented a **rate** of travel (miles per hour); in that case, the area represented the **total** distance traveled. For functions representing other **rates** such as the production of a factory (bicycles per day), or the flow of water in a river (gallons per minute) or traffic over a bridge (cars per minute), or the spread of a disease (newly sick people per week), the area will still represent the **total** amount of something.

**Example:** Suppose MR(q) is the marginal revenue in dollars/item for selling q items. Then $\int_0^{150} MR(q)dq$ has units (dollars/item) · (items) = dollars, and represents the accumulated dollars for selling from 0 to 150 items. That is, $\int_0^{150} MR(q)dq = TR(150)$, the total revenue from selling 150 items.

**Example:** Suppose *r*(*t*), in centimeters per year, represents how the diameter of a tree changes with time. Then $\int_{T_1}^{T_2} r(t)dt$ has units of (centimeters per year) · (years) = centimeters, and represents the accumulated growth of the tree's diameter from $T_1$ to $T_2$. That is, $\int_{T_1}^{T_2} r(t)dt$ is the change in the diameter of the tree over this period of time.

## Section 2: The Fundamental Theorem and Antidifferentiation

### The Fundamental Theorem of Calculus

This section contains the most important and most used theorem of calculus, the Fundamental Theorem of Calculus. Discovered independently by Newton and Leibniz in the late 1600s, it establishes the connection between derivatives and integrals, provides a way of easily calculating many integrals, and was a key step in the development of modern mathematics to support the rise of science and technology. Calculus is one of the most significant intellectual structures in the history of human thought, and the Fundamental Theorem of Calculus is a most important brick in that beautiful structure.

**The Fundamental Theorem of Calculus:**

$$\int_a^b F'(x)dx = F(b) - F(a)$$

This is actually not new for us; we've been using this relationship for some time; we just haven't written it this way. This says what we've said before: the definite integral of a rate from a to b is the net y-units, the change in y, that accumulate between t = a and t = b. Here we've just made it plain that that the rate is a derivative.

Thinking about the relationship this way gives us the key to finding exact answers for some definite integrals. If the integrand is the derivative of some F, then maybe we could simply find F and subtract – that would be easier than approximating with rectangles. Going backwards through the differentiation process will help us evaluate definite integrals.

**Example:** Find f(x) if f '(x) = 2x.

**Solution:** Oooh, I know this one. It's $f(x) = x^2 + 3$. Oh, wait, you were thinking something else? Yes, I guess you're right -- $f(x) = x^2$ works too. So does $f(x) = x^2 - \pi$, and $f(x) = x^2 + 104{,}589.2$. In fact, there are lots of answers.

In fact, there are infinitely many functions that all have the same derivative. And that makes sense – the derivative tells us about the shape of the function, but it doesn't tell about the location. We could shift the graph up or down and the shape wouldn't be affected, so the derivative would be the same.

This leads to one of the trickiest definitions I ever give – pay careful attention to the articles, because they're important.

## Antiderivatives

**An antiderivative** of a function f(x) is any function F(x) where F '(x) = f(x).

**The antiderivative** of a function f(x) is a whole family of functions, written F(x) + C, where F '(x) = f(x) and C represents any constant.

The antiderivative is also called the **indefinite integral**.

**Notation for the antiderivative:**

The antiderivative of f is written

$$\int f(x)\,dx$$

This notation resembles the definite integral, because the Fundamental Theorem of Calculus says antiderivatives and definite integrals are intimately related. But in this notation, there are no limits of integration.

The $\int$ symbol is still called an **integral sign**; the dx on the end still must be included; you can still think of $\int$ and dx as left and right parentheses. The function f is still called the **integrand**.

**Verb forms:**

We **antidifferentiate, or integrate**, or **find the indefinite integral** of a function. This process is called **antidifferentiation** or **integration**.

There are no small families in the world of antiderivatives: if f has one antiderivative F, then f has an infinite number of antiderivatives and every one of them has the form F(*x*) + C.

**Example:** Find **an** antiderivative of 2x.

**Solution:** I can choose any function I like as long as its derivative is 2x, so I'll pick $F(x) = x^2 - 5.2$.

**Example:** Find **the** antiderivative of 2x.

**Solution:** Now I need to write the entire family of functions whose derivatives are 2x. I can use the notation:

$$\int 2x\,dx = x^2 + C$$

**Example:** Find $\int e^x\,dx$.

**Solution:** Luckily this one is one I remember -- $e^x$ is its own derivative, so it is also its own antiderivative. The integral sign tells me that I need to include the entire family of functions, so I need that + *C* on the end:

$$\int e^x\,dx = e^x + C.$$

## Antiderivatives Graphically or Numerically

Another way to think about the Fundamental Theorem of Calculus is to solve the expression for F(b):

**The Fundamental Theorem of Calculus:**

$$\int_a^b F'(x)dx = F(b) - F(a)$$

The definite integral of a derivative from a to b gives the net change in the original function.

$$F(b) = F(a) + \int_a^b F'(x)dx$$

The amount we end up is the amount we start with plus the net change in the function.

This lets us get values for the antiderivative – as long as we have a starting point, and we know something about the area.

**Example**: Suppose F(t) has the derivative f(t) shown in Fig. 18, and suppose that we know F(0) = 5. Find values for F(1), F(2), F(3), and F(4).

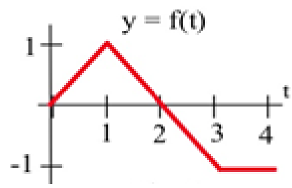



Figure 89

**Solution:** Using the second way to think about the Fundamental Theorem of Calculus, $F(b) = F(a) + \int_a^b F'(x)dx$ -- we can see that

$F(1) = F(0) + \int_0^1 f(x)dx$. We know the value of F(0), and we can easily find $\int_0^1 f(x)dx$ from the graph – it's just the area of a triangle.

So $F(1) = F(0) + \int_0^1 f(x)dx = 5 + .5 = 5.5$

$$F(2) = F(0) + \int_0^2 f(x)dx = 5 + 1 = 6$$

Note that we can start from any place we know the value of – now that we know F(2), we can use that:

$$F(3) = F(2) + \int_2^3 f(x)dx = 6 - .5 = 5.5$$

$$F(4) = F(3) + \int_3^4 f(x)dx = 5.5 - 1 = 4.5$$

**Example**: F '(t) = f(t) is shown in Fig. 19. Where does F(t) have maximum and minimum values on the interval [0, 4]?

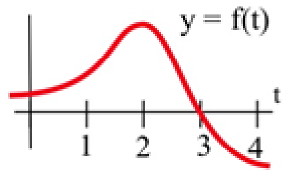



Figure 90

**Solution:** Since $F(b)=F(a)+\int_a^b f(t)dt$, we know that F is increasing as long as the area accumulating under F ' = f is positive (until t = 3), and then decreases when the curve dips below the x-axis so that negative area starts accumulating. The area between t = 3 and t = 4 is much smaller than the positive area that accumulates between 0 and 3, so we know that F(4) must be larger than F(0). The maximum value is when t = 3; the minimum value is when t = 0.

Note that this is a different way to look at a problem we already knew how to solve – in Chapter 2, we would have found critical points of F, where f = 0 – there's only one, when t = 3. f = F' goes from positive to negative there, so F has a local max at that point. It's the only critical point, so it must be a global max. Then we would look at the values of F at the endpoints to find which was the global min.

## Section 3: Antiderivatives of Formulas

Now we can put the ideas of areas and antiderivatives together to get a way of evaluating definite integrals that is exact and often easy. To evaluate a definite integral $\int_a^b f(t)\,dt$, we can find any antiderivative F of f and evaluate F(b) – F(a). The problem of finding the exact value of a definite integral reduces to finding some (any) antiderivative F of the integrand and then evaluating F(b) – F(a). Even finding one antiderivative can be difficult, and we will stick to functions that have easy antiderivatives.

## Building Blocks

Antidifferentiation is going backwards through the derivative process. So the easiest antiderivative rules are simply backwards versions of the easiest derivative rules. Recall from Chapter 2:

**Derivative Rules: Building Blocks**

In what follows, *f* and *g* are differentiable functions of *x* and *k* and *n* are constants.

(a) **Constant Multiple Rule:** $\frac{d}{dx}(kf) = kf'$

(b) **Sum (or Difference) Rule:** $\frac{d}{dx}(f+g) = f'+g'$ **(or** $\frac{d}{dx}(f-g) = f'-g'$**)**

(c) **Power Rule:** $\frac{d}{dx}(x^n) = nx^{n-1}$

Special cases: $\frac{d}{dx}(k) = 0$ (because $k = kx^0$ )

$\frac{d}{dx}(x) = 1$ (because $x = x^1$)

(d) **Exponential Functions:** $\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(a^x) = \ln a \cdot a^x$

(e) **Natural Logarithm:** $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Thinking about these basic rules was how we came up with the antiderivatives of 2x and $e^x$ before.

The corresponding rules for antiderivatives are next – each of the antiderivative rules is simply rewriting the derivative rule. All of these antiderivatives can be verified by differentiating.

There is one surprise – the antiderivative of 1/x is actually not simply ln(x), it's ln|x|. This is a good thing – the antiderivative has a domain that matches the domain of 1/x, which is bigger than the domain of ln(x), so we don't have to worry about whether our x's are positive or negative. But you must be careful to include those absolute values – otherwise, you could end up with domain problems.

## Antiderivative Rules: Building Blocks

In what follows, *f* and *g* are differentiable functions of *x* and *k*, *n*, and *C* are constants.

(a) **Constant Multiple Rule:** $\int kf(x)\,dx = k\int f(x)dx$

(b) **Sum (or Difference) Rule:** $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int f(x)dx$

(c) **Power Rule:** $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$, provided that n = −1

Special case: $\int k\,dx = kx + C$ (because $k = kx^0$ )

(d) **Exponential Functions:** $\int e^x\,dx = e^x + C$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

(e) **Natural Logarithm:** $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$

**Example:** Find the antiderivative of $3x^7 - 15\sqrt{x} + \dfrac{14}{x^2}$

**Solution:** $\displaystyle\int\left(3x^7 - 15\sqrt{x} + \frac{14}{x^2}\right)dx = \int\left(3x^7 - 15x^{1/2} + 14x^{-2}\right)dx = 3\frac{x^8}{8} - 15\frac{x^{3/2}}{3/2} + 14\frac{x^{-1}}{-1} + C$

That's a little hard to look at, so you might want to simplify a little:

$$\int\left(3x^7 - 15\sqrt{x} + \frac{14}{x^2}\right)dx = \frac{3x^8}{8} - 10x^{3/2} - 14x^{-1} + C.$$

**Example:** Find $\displaystyle\int\left(e^x + 12 - \frac{16}{x}\right)dx$

**Solution:** $\displaystyle\int\left(e^x + 12 - \frac{16}{x}\right)dx = e^x + 12x - 16\ln|x| + C$

**Example:** Find *F(x)* so that $F'(x) = e^x$ and $F(0) = 10$.

**Solution:** This time we are looking for a particular antiderivative; we need to find exactly the right constant. Let's start by finding the antiderivative:

$$\int e^x\,dx = e^x + C$$

So we know that $F(x) = e^x +$ some constant; we just need to find which one. For that, we'll use the other piece of information (the initial condition):

$$F(x) = e^x + C$$
$$F(0) = e^0 + C = 1 + C = 10$$
$$C = 9$$

The particular constant we need is 9; $F(x) = e^x + 9$.

The reason we are looking at antiderivatives right now is so we can evaluate definite integrals exactly. Recall the Fundamental Theorem of Calculus:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

If we can find an antiderivative for the integrand, we can use that to evaluate the definite integral. The evaluation F(b) – F(a) is represented by the symbol $F(x)\Big]_a^b$ or $F(x)\Big|_a^b$.

**Example**: Evaluate $\int_1^3 x\,dx$ in two ways:

(i) By sketching the graph of $y = x$ and geometrically finding the area.

(ii) By finding an antiderivative of F(x) of the integrand and evaluating F(3)–F(1).

**Solution:** (i) The graph of $y = x$ is shown in Fig. 20, and the shaded region has area 4.

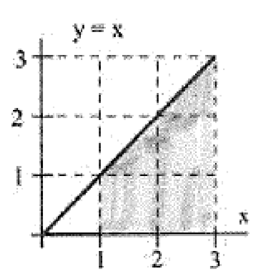



Figure 91

(ii) One antiderivative of $x$ is $F(x) = \frac{1}{2}x^2$ (check by differentiating ), and

$\int_1^3 x\,dx = \frac{1}{2}x^2\Big]_1^3 = \left[\frac{1}{2}(3)^2\right] - \left[\frac{1}{2}(1)^2\right] = \frac{9}{2} - \frac{1}{2} = 4.$ Note that this answer agrees with the answer we got geometrically.

If we had used another antiderivative of $x$, say $F(x) = \frac{1}{2}x^2 + 7$ ( check by differentiating ),

then $F(x)\Big|_1^3 = F(3) - F(1) = \{\frac{1}{2}(3^2) + 7\} - \{\frac{1}{2}(1^2) + 7\} = \frac{23}{2} - \frac{15}{2} = 4$ . Whatever constant you choose, it gets subtracted away during the evaluation; we might as well always choose the easiest one, where the constant = 0.

**Example**: Find the area between the graph of $y = 3x^2$ and the horizontal axis for $x$ between 1 and 2.

**Solution:** This is $\int_1^2 3x^2\,dx = x^3\Big]_1^2 = (2^3) - (1^3) = 7.$

**Example**: A robot has been programmed so that when it starts to move, its velocity after $t$ seconds will be $3t^2$ feet/second.

(a) How far will the robot travel during its first 4 seconds of movement?

(b) How far will the robot travel during its next 4 seconds of movement?

**Solution:** (a) The distance during the first 4 seconds will be the area under the graph (Fig. 21) of velocity, from $t = 0$ to $t = 4$.

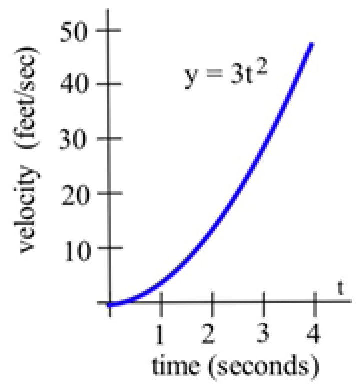



Figure 92

That area is the definite integral $\int_0^4 3t^2\,dt$. An antiderivative of $3t^2$ is $t^3$, so

$\int_0^4 3t^2\,dt = t^3\Big]_0^4 = 4^3 - 0^3 = 64$ feet.

(b) $\int_4^8 3t^2\,dt = t^3\Big]_4^8 = 8^3 - 4^3 = 512 - 64 = 448$ feet.

**Example 6**: Suppose that $t$ minutes after putting 1000 bacteria on a Petri plate the rate of growth of the population is $6t$ bacteria per minute.

(a) How many new bacteria are added to the population during the first 7 minutes?

(b) What is the total population after 7 minutes?

**Solution:** (a) The number of new bacteria is the area under the rate of growth graph (Fig. 22), and one antiderivative of $6t$ is $3t^2$.

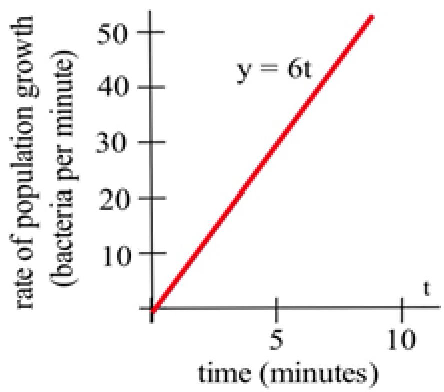



Figure 93

So new bacteria = $\int_0^7 6t\,dt = 3t^2 \Big|_0^7 = 3(7)^2 - 3(0)^2 = 147$.

(b) The new population = {old population} + {new bacteria} = 1000 + 147 = 1147 bacteria.

## Section 4: Substitution

We don't have many integration rules. For quite a few of the problems we see, the rules won't directly apply. We'll have to do some algebraic manipulation first. In practice, it is much harder to write down the antiderivative of a function than it is to find a derivative. (In fact, it's really easy to write a function that doesn't have any antiderivative you can find with algebra.)

The Substitution Method is one way of algebraically manipulating an integrand so that the rules apply. This is a way to unwind the Chain Rule for derivatives. When you find the derivative of a function using the Chain Rule, you end up with a product of something like the original function TIMES a derivative. Try Substitution when you see a product in your integral, especially if you recognize one factor as the derivative of some part of the other factor.

## The Substitution Method for Antiderivatives:

The goal is to turn $\int f(x)dx$ into $\int g(u)du$ , where *g* is much less messy than *f*.

1. Let *u* be some part of the integrand. A good first choice is "one step inside the messiest bit."

2. Compute $du = \frac{du}{dx}dx$

3. Translate all your x's into u's everywhere in the integral, including the dx. When you're done, you should have a new integral that is entirely in u. If you have any x's left, then that's an indication that the substitution didn't work; go back to step 1 and try a different choice for u.

4. Integrate the new u-integral, if possible. If you still can't integrate it, go back to step 1 and try a different choice for u.

5. Finally, substitute back x's for u's everywhere in your answer.

**Example:** Evaluate $\int \frac{x}{\sqrt{4-x^2}}dx$.

**Solution:** This integrand is more complicated than anything in our list of basic integral formulas, so we'll have to try something else. The only tool we have is substitution, so let's try that!

1. Let *u* be some part of the integrand. A good first choice is "one step inside the messiest bit."

   In this case, the square root in the denominator is the messiest part, so let's let u be one step inside:

   Let $u = 4 - x^2$

2. Compute $du = \frac{du}{dx}dx$

$du = -2xdx$

I see *x dx* in the integrand, so that's a good sign; that will be −½*du*.

3. Translate all your x's into u's everywhere in the integral, including the dx.

$$\int \frac{x}{\sqrt{4-x^2}}dx = \int \frac{1}{\sqrt{4-x^2}}(xdx) = \int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}du\right) = -\frac{1}{2}\int \frac{1}{\sqrt{u}}du = -\frac{1}{2}\int u^{-1/2}du$$

4. Integrate the new u-integral, if possible.

$$-\frac{1}{2}\int u^{-1/2}du = -\frac{1}{2}\frac{u^{1/2}}{1/2} + C = -u^{1/2} + C$$

5. Finally, substitute back x's for u's everywhere in your answer.

$-u^{1/2} + C = -\sqrt{4-x^2} + C$. So we have found

$$\int \frac{x}{\sqrt{4-x^2}}dx = -\sqrt{4-x^2} + C.$$

How would we check this? By differentiating:

$$\frac{d}{dx}\left(-\sqrt{4-x^2} + C\right) = \frac{d}{dx}\left(-\left(4-x^2\right)^{1/2} + C\right) = -\frac{1}{2}\left(4-x^2\right)^{-1/2}(-2x) = x\left(4-x^2\right)^{-1/2} = \frac{x}{\sqrt{4-x^2}}$$

.

Phew!

**Example:** Evaluate $\int \frac{e^x dx}{\left(e^x+15\right)^3}$

**Solution:** This integral is not in our list of building blocks. But notice that the derivative of $e^x+15$ (that we see in the denominator) is just $e^x$ (which I see in the numerator). So substitution will be a good choice for this.

Let $u=e^x+15$. Then $du=e^x\,dx$, and this integral becomes $\int \frac{du}{u^3}=\int u^{-3}du$. Luckily, that is on our list of building block formulas: $\int u^{-3}du=\frac{u^{-2}}{-2}+C=-\frac{1}{u^2}+C$. Finally, translating back:

$$\int \frac{e^x dx}{\left(e^x+15\right)^3}=-\frac{1}{\left(e^x+15\right)^2}+C$$

## Substitution and Definite Integrals

When you use substitution to help evaluate a definite integrals, you have a choice for how to handle the limits of integration. You can do either of these, whichever seems better to you. The important thing to remember is – the original limits of integration were values of the original variable (say, x), not values of the new variable (say, u).

(a) You can solve the antiderivative as a side problem, translating back to x's, and then use the antiderivative with the original limits of integration. Or

(b) You can substitute for the limits of integration at the same time as you're substituting for everything inside the integral, and then skip the "translate back into x" step. If the original integral had endpoints *x* =a and *x* =b, and we make the substitution *u* = g(*x* ) and du = g'(*x* )dx, then the new integral will have endpoints *u*= g(a) and *u*=g(b) and

$$\int_{x=a}^{x=b} (\text{original integrand})\,dx \quad \text{becomes} \quad \int_{\boldsymbol{u=g(a)}}^{\boldsymbol{u=g(b)}} (\text{new integrand})\,du\ .$$

Method (a) seems more straightforward for most students. But it can involve some messy algebra. Method (b) is often neater and usually involves fewer steps.

**Example:** Evaluate $\int_0^1 (3x-1)^4 \, dx$

**Solution:** We'll need substitution to find an antiderivative, so we'll need to handle the limits of integration carefully. I'll solve this example both ways.

(a) Doing the antiderivative as a side problem:

**Step One –** find the antiderivative, using substitution:

$$\int (3x-1)^4 \, dx$$

Let $u = 3x-1$. Then $du = 3dx$ and $\int (3x-1)^4 \, dx = \int u^4 \left(\frac{1}{3} du\right) = \frac{1}{3}\frac{u^5}{5} + C$

Translating back to x: $\int (3x-1)^4 \, dx = \frac{(3x-1)^5}{15} + C$

**Step Two –** evaluate the definite integral:

$$\int_0^1 (3x-1)^4 \, dx = \frac{(3x-1)^5}{15}\Bigg]_0^1 = \frac{(3(1)-1)^5}{15} - \frac{(3(0)-1)^5}{15} = \frac{32}{15} - \frac{-1}{15} = \frac{33}{15}.$$

(b) Substituting for the limits of integration:

$$\int_0^1 (3x-1)^4 \, dx$$

Let $u = 3x-1$. Then $du = 3dx$, and (substituting for the limits of integration) when x = 0, u = -1, when x = 1, u = 2.

$$\int_{x=0}^{x=1} (3x-1)^4 \, dx = \int_{u=-1}^{u=2} u^4 \left(\frac{1}{3} du\right) = \frac{u^5}{15}\Bigg]_{u=-1}^{u=2} = \frac{(2)^5}{15} - \frac{(-1)^5}{15} = \frac{32}{15} - \frac{-1}{15} = \frac{33}{15}.$$

**Example:** Evaluate $\int_{2}^{10} \frac{(\ln x)^6}{x}\,dx$

**Solution:** I can see the derivative of $\ln x$ in the integrand, so I can tell that substitution is a good choice. Let $u = \ln x$. Then $du = \frac{1}{x}dx$. When $x = 2$, $u = \ln 2$. When $x = 10$, $u = \ln 10$. So the new definite integral is

$$\int_{x=2}^{x=10} \frac{(\ln x)^6}{x}\,dx = \int_{u=\ln 2}^{u=\ln 10} u^6\,du = \frac{u^7}{7}\Bigg]_{u=\ln 2}^{u=\ln 10} = \frac{1}{7}\left((\ln 10)^7 - (\ln 2)^7\right) \cong 49.01.$$

## Section 5: Applications of the Definite Integral

### Area

We have already used integrals to find the area between the graph of a function and the horizontal axis. Integrals can also be used to find the area between two graphs.

If $f(x) \geq g(x)$ for all $x$ in $[a,b]$, then we can approximate the area between $f$ and $g$ by partitioning the interval $[a,b]$ and forming a Riemann sum (Fig. 23). The height of each rectangle is top – bottom, $f(c_i) - g(c_i)$ so the area of the $i^{th}$ rectangle is (height)·(base) = $\{f(c_i) - g(c_i)\}\cdot\Delta x$. This approximation of the total area is a Riemann sum.

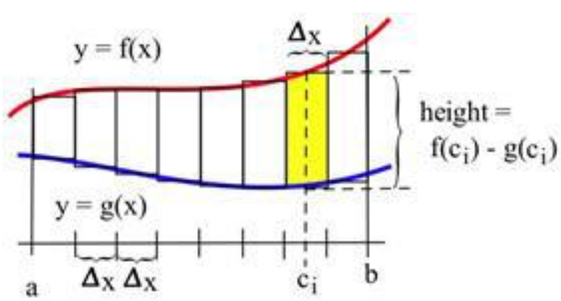



Figure 94

The limit of this Riemann sum, as the number of rectangles gets larger and their width gets smaller, is the definite integral $\int_a^b \{ f(x) - g(x) \}\,dx$ .

**The area between two curves f(x) and g(x), where f(x) ≥ g(x), between x = a and x = b is**

$$\int_a^b (f(x) - g(x))\,dx$$

**The integrand is "top – bottom." Make a graph to be sure which curve is which.**

**Example**: Find the area bounded between the graphs of f(x) = x and g(x) = 3 for 1 ≤ x ≤ 4. (Fig. 24)

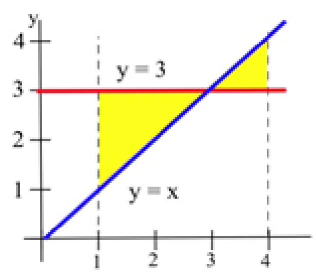



Figure 95

**Solution:** Always start with a graph so you can see which graph is the top and which is the bottom. In this example, the two curves cross, and they change positions; we'll need to split the area into two pieces. Geometrically, we can see that the area is 2 + ½ = 2.5.

Writing the area as a sum of definite integrals, we get:

Area = $\int_1^3 (3 - x)\,dx + \int_3^4 (x - 3)\,dx$

These integrals are easy to evaluate using antiderivatives:

$$\int_1^3 (3 - x)\,dx = \left(3x - \frac{x^2}{2}\right)\Bigg|_0^3 = \left(\left(9 - \frac{9}{2}\right) - \left(15 - \frac{25}{2}\right)\right) = 2.$$

$$\int_3^4 (x - 3)\,dx = \left(\frac{x^2}{2} - 3x\right)\Bigg|_3^4 = \left(\left(\frac{16}{2} - 12\right) - \left(\frac{9}{2} - 9\right)\right) = \frac{1}{2}.$$

The two integrals also tell us that the total area between f and g is 2.5 square units, which we already knew.

Note that the single integral $\int_1^4 (3-x)\,dx = 1.5$ is not the **area** we want in this problem. The value of the **integral is 1.5**, and the value of the **area is 2.5**. That's because for the triangle on the right, the graph of y = x is above the graph of y = 3, so the integrand 3 – x is negative; in the definite integral, the area of that triangle comes in with a negative sign.

In this example, it was easy to see exactly where the two curves crossed so we could break the region into the two pieces to figure separately. In other examples, you might need to solve an equation to find where the curves cross.

**Example**: Two objects start from the same location and travel along the same path with velocities $v_A(t) = t+3$ and $v_B(t) = t^2 - 4t + 3$ meters per second (Fig. 25). How far ahead is A after 3 seconds?

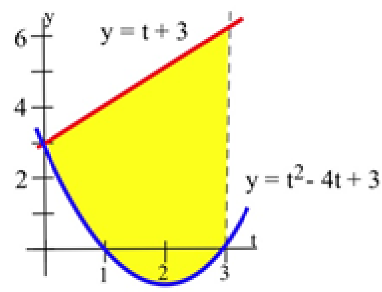



Figure 96

**Solution:** Since $v_A(t) \geq v_B(t)$ , the "area" between the graphs of $v_A(t)$ and $v_B(t)$ represents the distance between the objects.

After 3 seconds, the distance apart

$$= \int_0^3 (v_A(t) - v_B(t))\,dt = \int_0^3 ((t+3) - (t^2 - 4t + 3))\,dt = \int_0^3 (5t - t^2)\,dt$$

$$= \left(\frac{5}{2}t^2 - \frac{t^3}{3}\right)\Bigg|_0^3 = \left(\frac{5}{2}\cdot 9 - \frac{27}{3}\right) - (0) = 13.5 \text{ meters.}$$

## Average Value

We know the average of n numbers, $a_1, a_2, \ldots, a_n$, is their sum divided by n. But what if we need to find the average temperature over a day's time -- there are too many possible temperatures to add them up. This is a job for the definite integral.

**The average value of a function f(x) on the interval [a, b] is given by**

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

The average value of a positive $f$ has a nice geometric interpretation. Imagine that the area under $f$ (Fig. 26a) is a liquid that can "leak" through the graph to form a rectangle with the same area (Fig. 26b). If the height of the rectangle is $H$, then the area of the rectangle is $H \cdot (b-a)$. We know the area of the rectangle is the same as the area under $f$ so $H \cdot (b-a) = \int_a^b f(x)dx$. Then $H = \frac{1}{b-a}\int_a^b f(x)dx$, the average value of $f$ on [a,b].

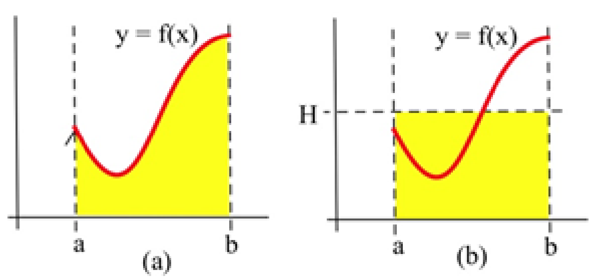



Figure 97

**The average value of a positive $f$ is the height H of the rectangle whose area is the same as the area under f.**

**Example**: During a 9 hour work day, the production rate at time $t$ hours after the start of the shift was given by the function r($t$)= 5 + $\sqrt{t}$ cars per hour. Find the average hourly production rate.

**Solution:** The average hourly production is $\frac{1}{9-0}\int_0^9 \left(5+\sqrt{t}\right)\, dt = 7$ cars per hour.

A note about the units – remember that the definite integral has units (cars per hour) · (hours) = cars. But the 1/(b-a) in front has units 1/hours – the units of the average value are cars per hour, just what we expect an average rate to be.

**In general, the average value of a function will have the same units as the integrand.**

Function averages, involving means and more complicated averages, are used to "smooth" data so that underlying patterns are more obvious and to remove high frequency "noise" from signals. In these situations, the original function f is replaced by some "average of f." If f is rather jagged time data, then the ten year average of f is the integral $g(x)=\frac{1}{10}\int_{x-5}^{x+5} f(t)dt$ , an average of f over 5 units on each side of x. For example, Fig. 27 shows the graphs of a Monthly Average (rather “noisy” data) of surface temperature data, an Annual Average (still rather “jagged), and a Five Year Average (a much smoother function). Typically the average function reveals the pattern much more clearly than the original data. This use of a “moving average” value of “noisy” data (weather information, stock prices) is a very common.

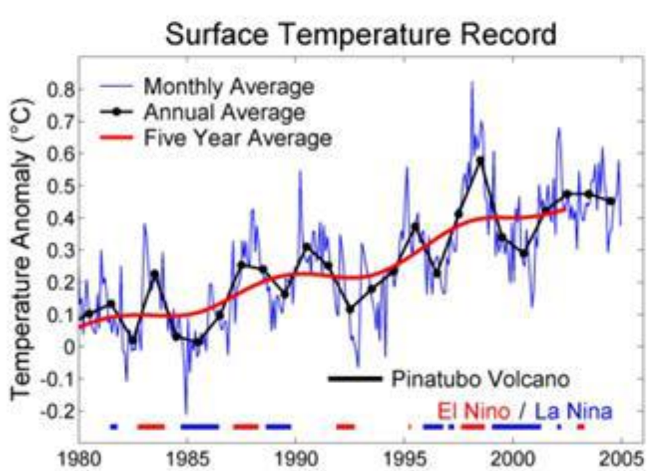



Figure 98

**Example:** The graph in Figure 28 shows the amount of water in a reservoir over a 12 hour period. Estimate the average amount of water in the reservoir over this period.

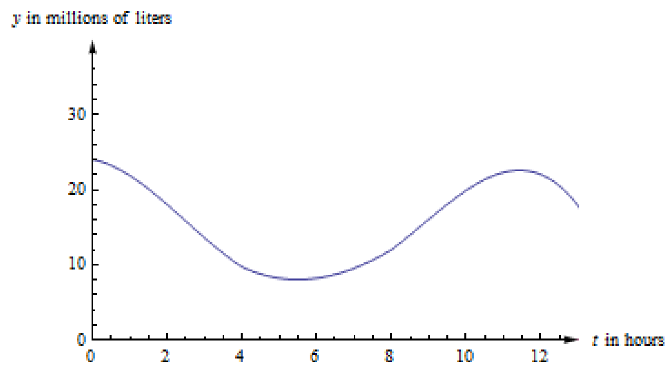



Figure 99

**Solution:** If $V(t)$ is the volume of the water (in millions of liters) after *t* hours, then the average amount is $\frac{1}{12}\int_0^{12} V(t)dt$ . In order to find the definite integral, we'll have to estimate. I'll use 6 rectangles, and I'll take the heights from their right edges.

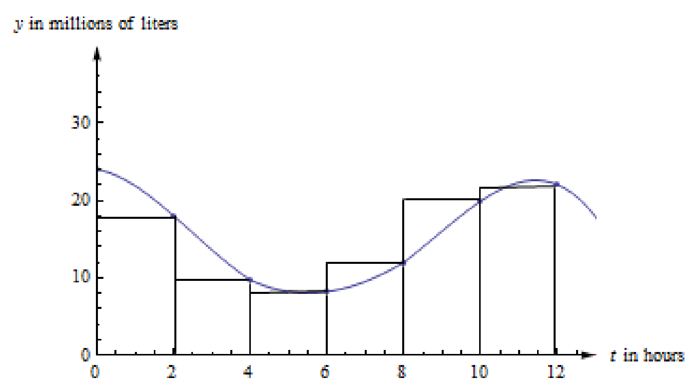



Figure 100

My estimate of the integral is

$$\int_0^{12} V(t)\,dt \cong (18)(2)+(9.7)(2)+(8.2)(2)+(12)(2)+(19.9)(2)+(22)(2)=179.6.$$

The units of this integral are millions of liters × feet. So my estimate of the average volume is $\frac{179.6}{12} \cong 15$ millions of liters. Your estimate might be a little different.

In Figure 30, you can see the same graph with the line $y = 15$ drawn in. The area under the curve and the area under the rectangle are (approximately) the same.

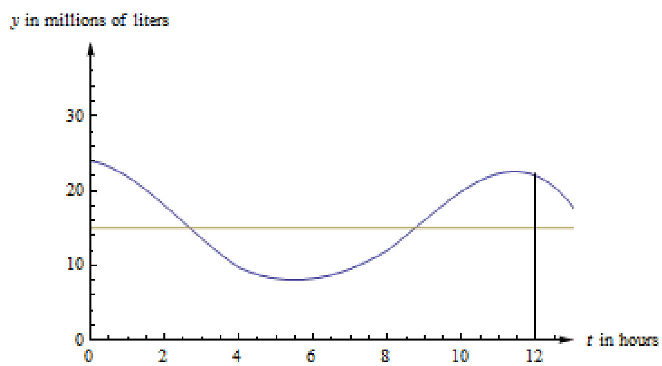



**Figure 101**

In fact, that would be a different way to estimate the average value. We could have estimated the placement of the horizontal line so that the area under the curve and under the line were equal.

## Consumer and Producer Surplus

Here are a demand and a supply curve for a product. Which is which?

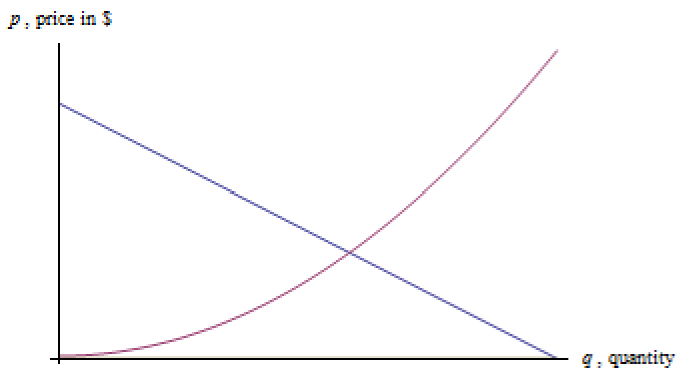



Figure 102

The demand curve is decreasing – lower prices are associated with higher quantities demanded, higher prices are associated with lower quantities demanded. Demand curves are often shown as if they were linear, but there's no reason they have to be.

The supply curve is increasing – lower prices are associated with lower supply, and higher prices are associated with higher quantities supplied.

The point where the demand and supply curve cross is called the equilibrium point (q*, p*).

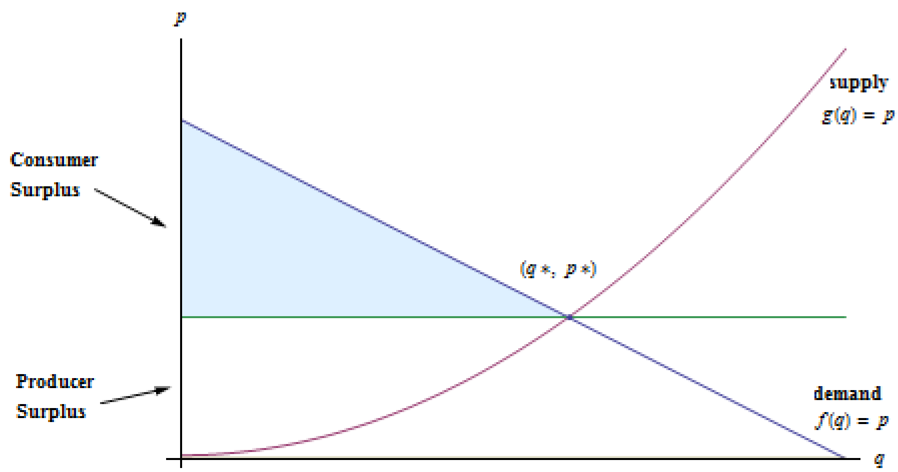



Figure 103

Suppose that the price is set at the equilibrium price, so that the quantity demanded equals the quantity supplied.  Now think about the folks who are represented on the left of the equilibrium point. The consumers on the left would have been willing to pay a higher price than they ended up having to pay, so the equilibrium price saved them money. On the other hand, the producers represented on the left would have been willing to supply these goods for a lower price – they made more money than they expected to. Both of these groups ended up with extra cash in their pockets!

Graphically, the amount of extra money that ended up in consumers' pockets is the area between the demand curve and the horizontal line at p*. This is the difference in price, summed up over all the consumers who spent less than they expected to – a definite integral.

The amount of extra money that ended up in producers' pockets is the area between the supply curve and the horizontal line at p*. This is the difference in price, summed up over all the producers who received more than they expected to.

## Consumer and Producer Surplus

Given a demand function p = f(q) and a supply function p = g(q), and the equilibrium point (q*, p*)

The **consumer surplus** = $\int_0^{q^*} f(q)dq - p^*q^*$

The **producer surplus** = $p^*q^* - \int_0^{q^*} g(q)dq$

The sum of the consumer surplus and producer surplus is the **total gains from trade**.

What are the units of consumer and producer surplus? The units are (price units)(quantity units) = money!

**Example:** Suppose the demand for a product is given by $p = -0.8q + 150$ and the supply for the same product is given by $p = 5.2q$. For both functions, *q* is the quantity and *p* is the price, in dollars.

a. Find the equilibrium point.
b. Find the consumer surplus at the equilibrium price.
c. Find the producer surplus at the equilibrium price.

**Solution:**

a. The equilibrium point is where the supply and demand functions are equal. Solving $-0.8q + 150 = 5.2q$ gives *q* = 25. The price when *q* = 25 is *p* = 130; the equilibrium point is (25, 130).

b. The consumer surplus is $\int_0^{25} (-0.8q + 150)dq - (130)(25) = \$250.$

c. The producer surplus is $(130)(25) - \int_0^{25} 5.2q\,dq = \$1625.$

**Example:** The tables below show information about the demand and supply functions for a product. For both functions, $q$ is the quantity and $p$ is the price, in dollars.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $q$ | 0 | 100 | 200 | 300 | 400 | 500 | 600 | 700 |
| $p$ | 70 | 61 | 53 | 46 | 40 | 35 | 31 | 28 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $q$ | 0 | 100 | 200 | 300 | 400 | 500 | 600 | 700 |
| $p$ | 14 | 21 | 28 | 33 | 40 | 47 | 54 | 61 |

a. Which is which? That is, which table represents demand and which represents supply?
b. What is the equilibrium price and quantity?
c. Find the consumer and producer surplus at the equilibrium price.

**Solution:** a. The first table shows decreasing price associated with increasing quantity; that is the demand function.

b. For both functions, $q$ = 400 is associated with $p$ = 40; the equilibrium price is \$40 and the equilibrium quantity is 400 units. Notice that we were lucky here, because the equilibrium point is actually one of the points shown. In many cases with a table, we would have to estimate.

c. The consumer surplus uses the demand function, which comes from the first table. We'll have to approximate the value of the integral using rectangles. There are 4 rectangles, and I choose to use left endpoints.

The consumer surplus = $\int_0^{400} demand\ dq - (40)(400) \cong$

$(70)(100) + (61)(100) + (53)(100) + (46)(100) - (40)(400) = 7000$. The consumer surplus is about \$7,000.

The producer surplus uses the supply function, which comes from the second table. I choose to use left endpoints for this integral also.

The producer surplus = $(40)(400) - \int_0^{400} \sup ply\ dq \cong$

$(40)(400) - [(14)(100) + (21)(100) + (28)(100) + (33)(100)] = 6400$. The producer surplus is about \$6400.

## Continuous Income Stream

In precalculus, you learned about compound interest in that really simple situation where you made a single deposit into an interest-bearing account and let it sit undisturbed, earning interest, for some period of time. Recall:

**Compound Interest Formulas**

Let P = the principal (initial investment), r = the annual interest rate expressed as a decimal, and let A(t) be the amount in the account at the end of t years.

**Compounding *n* times per year:** $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$

**Compounded continuously:** $A(t) = Pe^{rt}$

If you're using this formula to find what an account will be worth in the future, t > 0 and A(t) is called the **future value**.

If you're using the formula to find what you need to deposit today to have a certain value P sometime in the future, t < 0 and A(t) is called the **present value.**

You may also have learned somewhat more complicated annuity formulas to deal with slightly more complicated situations – where you make equal deposits equally spaced in time.

But real life is not usually so neat.

Calculus allows us to handle situations where "deposits" are flowing continuously into an account that earns interest. As long as we can model the flow of income with a function, we can use a definite integral to calculate the present and future value of a continuous income stream. The idea is – each little bit of income in the future needs to be multiplied by the exponential function to bring it back to the present, and then we'll add them all up (a definite integral).

## Continuous Income Stream

Suppose money can earn interest at an annual interest rate of r, compounded continuously. Let F(t) be a continuous income function (in dollars per year), that applies between year 0 and year T.

Then the present value of that income stream is given by $PV = \int_0^T F(t)e^{-rt}\,dt$.

The future value can be computed by the ordinary compound interest formula $FV = PVe^{rt}$

This is a useful way to compare two investments – find the present value of each to see which is worth more today.

**Example:** You have an opportunity to buy a business that will earn $75,000 per year continuously over the next eight years. Money can earn 2.8% per year, compounded continuously. Is this business worth its purchase price of $630,000?

**Solution:** First, please note that we still have to make some simplifying assumptions. We have to assume that the interest rates are going to remain constant for that entire eight years. We also have to assume that the $75,000 per year is coming in continuously, like a faucet dripping dollars into the business. Neither of these assumptions might be accurate. But moving on:

The present value of the $630,000 is … $630,000. This is one investment, where we put our $630,000 in the bank and let it sit there.

To find the present value of the business, we think of it as an income stream. The function F(t) in this case is $75,000 dollars per year, r = .028, and T = 8:

$$PV = \int_0^8 75000e^{-.028t}\,dt \cong 672{,}511.66$$

The present value of the business is about $672,500, which is more than the $630,000 asking price – this is a good deal.

I used technology to compute the value of this definite integral. For many of the integrals in this section, you won't be able to use antiderivatives. But technology will work quickly, and it will give you an answer that's good enough.

**Example:** A company is considering purchasing a new machine for its production floor. The machine costs \$65,000. The company estimates that the additional income from the machine will be a constant \$7000 for the first year, then will increase by \$800 each year after that. In order to buy the machine, the company needs to be convinced that it will pay for itself by the end of 8 years with this additional income. Money can earn 1.7% per year, compounded continuously. Should the company buy the machine?

**Solution:** Assumptions, assumptions. We'll assume that the income will come in continuously over the 8 years. We'll also assume that interest rates will remain constant over that 8-year time period.

We're interested in the present value of the machine, which we will compare to its \$65,000 price tag. Let $t$ be the time, in years, since the purchase of the machine. The income from the machine is different depending on the time: From $t$ = 0 to $t$ = 1 (the first year), the income is constant \$7000 per year. From $t$ = 1 to $t$ = 8, the income is increasing by \$800 each year; the income flow function $F(t)$ will be $F(t) = 7000 + 800(t-1) = 6200 + 800t$. To find the present value, we'll have to divide the integral into the two pieces, one for each of the functions:

$PV = \int_0^1 7000e^{-0.017t}\,dt + \int_1^8 (6200 + 800t)e^{-0.017t}\,dt \cong 70166$. (Again, I used technology to evaluate these integrals. This is an example where you can't use antiderivatives.)

The present value is greater than the cost of the machine, so the company should buy the machine.

## Chapter 3 Exercises

1. Let A($x$) represent the area bounded by the graph and the horizontal axis and vertical lines at $t$=0 and $t$=$x$ for the graph in Fig. 33. Evaluate A($x$) for $x$ = 1, 2, 3, 4, and 5.

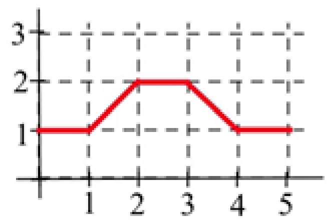



Figure 104

2. Let B($x$) represent the area bounded by the graph and the horizontal axis and vertical lines at $t$=0 and $t$=$x$ for the graph in Fig. 34. Evaluate B($x$) for $x$ = 1, 2, 3, 4, and 5.

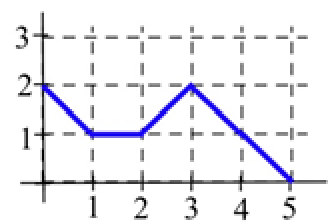



Figure 105

3. Let C($x$) represent the area bounded by the graph and the horizontal axis and vertical lines at $t$=0 and $t$=$x$ for the graph in Fig. 35. Evaluate C($x$) for $x$ = 1, 2, and 3 and find a formula for C($x$).

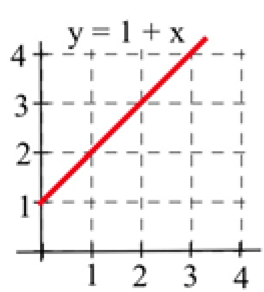



Figure 106

4. Let A($x$) represent the area bounded by the graph and the horizontal axis and vertical lines at $t$=0 and $t$=$x$ for the graph in Fig. 36. Evaluate A($x$) for $x$ = 1, 2, and 3 and find a formula for A($x$).

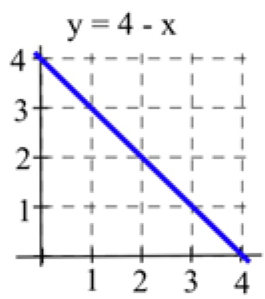



Figure 107

5. A car had the velocity shown in Fig. 37. How far did the car travel from $t$= 0 to $t$ = 30 seconds?

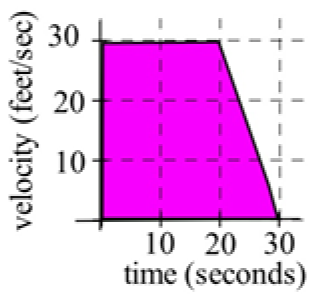



Figure 108

**6.** A car had the velocity shown in Fig. 38. How far did the car travel from $t$ = 0 to $t$ = 30 seconds?

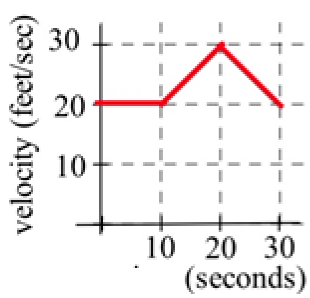



Figure 109

**7.** The velocities of two cars are shown in Fig. 39.

(a) From the time the brakes were applied, how many seconds did it take each car to stop?

(b) From the time the brakes were applied, which car traveled farther until it came to a complete stop?

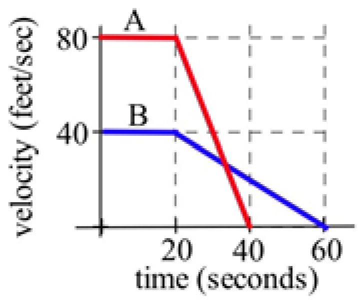



Figure 110

**8.** You and a friend start off at noon and walk in the same direction along the same path at the rates shown in Fig. 40.

(a) Who is walking faster at 2 pm? Who is ahead at 2 pm?

(b) Who is walking faster at 3 pm? Who is ahead at 3 pm?

(c) When will you and your friend be together? (Answer in words.)

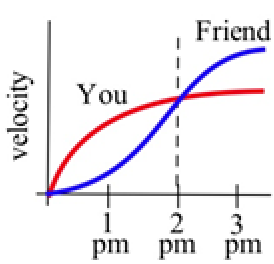



Figure 111

**9.** Police chase: A speeder traveling 45 miles per hour (in a 25 mph zone) passes a stopped police car which immediately takes off after the speeder. If the police car speeds up steadily to 60 miles/hour in 20 seconds and then travels at a steady 60 miles/hour, **how long** and **how far** before the police car catches the speeder who continued traveling at 45 miles/hour? (Fig. 41)

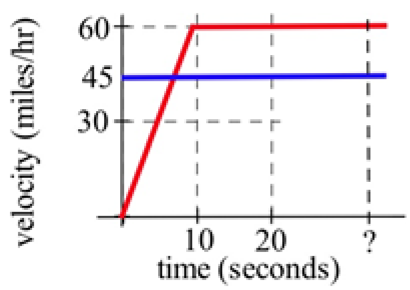



Figure 112

**10.** Water is flowing into a tub. The table shows the rate at which the water flows, in gallons per minute. The tub is initially empty.

| *t*, in minutes | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Flow rate, in gal/min | 0.5 | 1.0 | 1.2 | 1.4 | 1.7 | 2.0 | 2.3 | 1.8 | 0.7 | 0.5 | 0.2 |

Use the table to estimate how much water is in the tub after

a. five minutes

b. ten minutes

**11.** The table shows the speedometer readings for a short car trip.

| *t* , in minutes | 0 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| Speed, in mph | 0 | 30 | 40 | 65 | 40 |

a. Use the table to estimate how far the car traveled over the twenty minutes shown.

b. How accurate would you expect your estimate to be?

**12.** The table shows values of $f(t)$. Use the table to estimate $\int_0^{40} f(t)dt$.

| $t$ | 0 | 10 | 20 | 30 | 40 |
|---|---|---|---|---|---|
| $f(t)$ | 17 | 22 | 18 | 11 | 35 |

**13.** The table shows values of $g(x)$.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $g(x)$ | 140 | 142 | 144 | 152 | 154 | 165 | 200 |

Use the table to estimate

a. $\int_0^3 g(x)dx$

b. $\int_3^6 g(x)dx$

c. $\int_0^6 g(x)dx$

**14.** What are the units for the "area" of a rectangle with the given base and height units?

| Base units | Height units | "Area" units |
|---|---|---|
| miles per second | seconds | |
| hours | dollars per hour | |
| square feet | feet | |
| kilowatts | hours | |
| houses | people per house | |
| meals | meals | |

In problems 15 – 17 , represent the area of each bounded region as a definite integral, and use geometry to

determine the value of the definite integral.

**15**. The region bounded by y = 2*x* , the x–axis, the line *x* = 1, and *x* = 3.

**16.** The region bounded by y = 4 – 2*x* , the x–axis, and the y–axis.

**17.** The shaded region in Fig. 42.

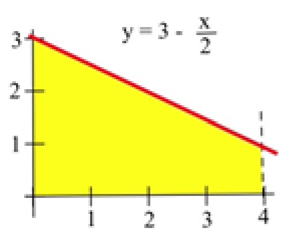



Figure 113

**18.** Fig. 43 shows the graph of f and the areas of several regions. Evaluate:

(a) $\int_0^3 f(x)\,dx$ (b) $\int_3^5 f(x)\,dx$ (c) $\int_3^7 f(x)\,dx$

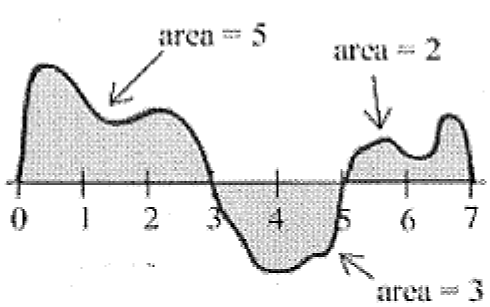



Figure 114

**19.** Fig. 44 shows the graph of g and the areas of several regions.

Evaluate : (a) $\int_{1}^{3} g(x)\,dx$ (b) $\int_{3}^{4} g(x)\,dx$

(c) $\int_{4}^{8} g(x)\,dx$ (d) $\int_{1}^{8} g(x)\,dx$ (e) $\int_{3}^{8} |\,g(x)\,|\,dx$

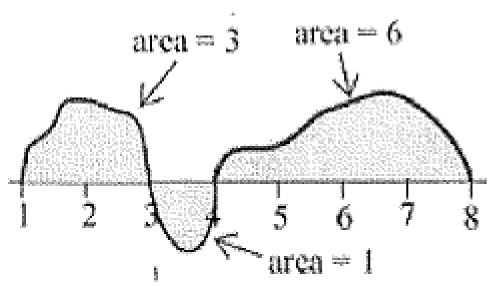



Figure 115

**20.** Fig. 45 shows the graph of h. Use the graph to evaluate:

(a) $\int_{-2}^{1} h(x)\,dx$ (b) $\int_{4}^{6} h(x)\,dx$ (c) $\int_{-2}^{6} h(x)\,dx$ (d) $\int_{-2}^{4} h(x)\,dx$

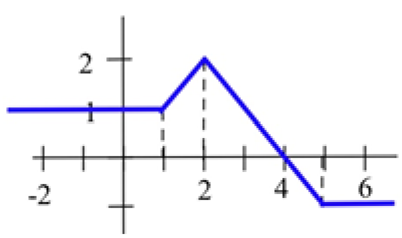



Figure 116

**21.** Your velocity along a straight road is shown in Fig. 46. How far did you travel in 8 minutes?

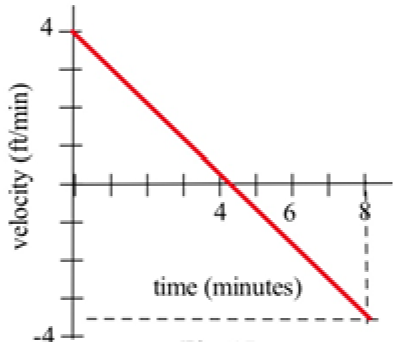



Figure 117

**22.** Your velocity along a straight road is shown in Fig. 47. How many feet did you walk in 8 minutes?

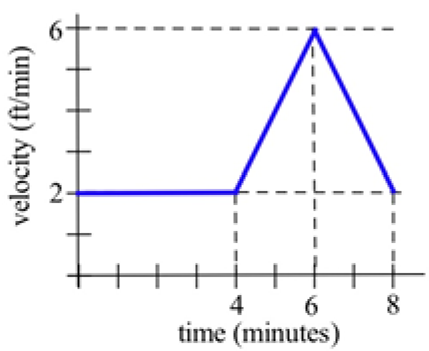



Figure 118

In problems 23 - 26, the units are given for *x* and for f(*x*). Give the units of $\int_a^b f(x)\,dx$ .

**23**. *x* is time in "seconds", and f(*x*) is velocity in "meters per second."

**24.** *x* is time in "hours", and f(*x*) is a flow rate in "gallons per hour."

**25.** *x* is a position in "feet", and f(*x*) is an area in "square feet."

**26**. *x* is a position in "inches", and f(*x*) is a density in "pounds per inch."

In problems 27 – 31, represent the area with a definite integral and use technology to find the approximate answer.

**27**. The region bounded by $y = x^3$, the x–axis, the line $x = 1$, and $x = 5$.

**28**. The region bounded by $y = \sqrt{x}$ , the x–axis, and the line $x = 9$.

**29.** The shaded region in Fig. 48.

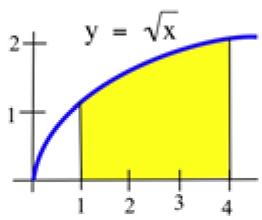



Figure 119

**30.** The shaded region in Fig. 49.

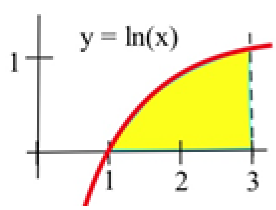



Figure 120

**31.** The shaded region in Fig. 49 for $2 \le x \le 3$.

**32**. Consider the definite integral $\int_0^3 (3+x)\,dx$.

(a) Using six rectangles, find the left-hand Riemann sum for this definite integral.

(b) Using six rectangles, find the right-hand Riemann sum for this definite integral.

(c) Using geometry, find the exact value of this definite integral.

**33.** Consider the definite integral $\int_0^2 x^3\,dx$.

(a) Using four rectangles, find the left-hand Riemann sum for this definite integral.

(b) Using four rectangles, find the right-hand Riemann sum for this definite integral.

Problems 34 – 41 refer to the graph of f in Fig. 50. Use the graph to determine the values of the definite integrals. (The bold numbers represent the **area** of each region.)

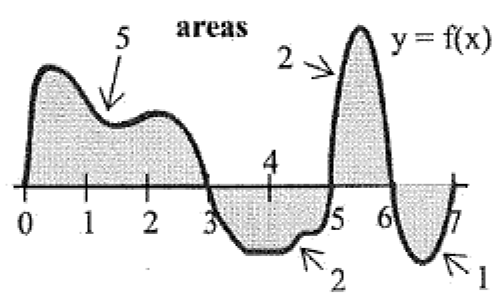



Figure 121

**34.** $\int_0^3 f(x)\,dx$ **35.** $\int_3^5 f(x)\,dx$ **36.** $\int_2^2 f(x)\,dx$ **37.** $\int_6^7 f(w)\,dw$ **38.** $\int_0^5 f(x)\,dx$

**39.** $\int_0^7 f(x)\,dx$ **40.** $\int_3^6 f(t)\,dt$ **41.** $\int_5^7 f(x)\,dx$

Problems 42 – 47 refer to the graph of g in Fig. 51. Use the graph to evaluate the integrals.

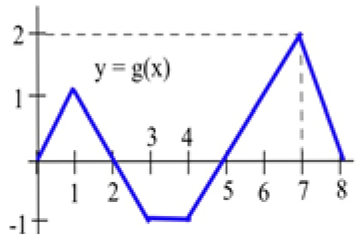



Figure 122

**42.** $\int_0^2 g(x)\, dx$ **43.** $\int_1^3 g(t)\, dt$ **44.** $\int_0^5 g(x)\, dx$

**45.** $\int_0^8 g(s)\, ds$ **46.** $\int_0^3 2g(t)\, dt$ **47.** $\int_5^8 1+g(x)\, dx$

**48**. Write the total distance traveled by the car in Fig. 52 between 1 pm and 4 pm as a definite integral and estimate the value of the integral.

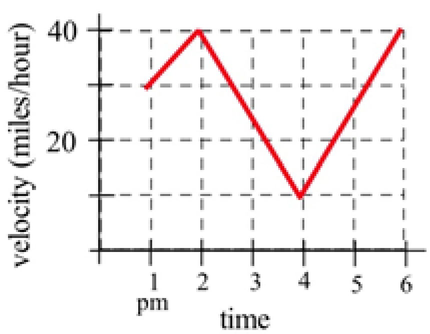



Figure 123

**49.** Write the total distance traveled by the car in Fig. 52 between 3 pm and 6 pm as a definite integral and estimate the value of the integral.

For problems 50 - 67, find the indicated antiderivative.

**50.** $\int \left(x^3 - 14x + 5\right) dx$

**51.** $\int \left(2.5x^5 - x - 1.25\right) dx$

**52.** $\int 12.3 dy$

**53.** $\int \pi^2 dw$

**54.** $\int e^P dP$

**55.** $\int \left(\sqrt{x} + e^x - \frac{1}{4x^3}\right) dx$

**56.** $\int \frac{1}{x} dx$

**57.** $\int \frac{1}{x^2} dx$

**58.** $\int (x-2)(x+2) dx$

**59.** $\int \frac{t^5 - t^2}{t} dt$

**60.** $\int \frac{1}{(4x+1)^3} dx$

**61**. $\int e^{100x} dx$

**62.** $\int (1.0003)^{12t} dt$

**63.** $\int \frac{e^{10/x}}{x^2} dx$

**64.** $\int \sqrt{w+5}\, dw$

**65.** $\int 6x^2 \sqrt{3x^3 - 1}\, dx$

**66.** $\int \frac{dx}{x \ln x}$

**67.** $\int \frac{x-3}{x^2 - 6x + 5} dx$

For problems 68 - 79, find an antiderivative of the integrand and use the Fundamental Theorem to evaluate the definite integral.

**68.** $\int_2^5 3x^2 \, dx$

**69.** $\int_{-1}^2 x^2 \, dx$

**70.** $\int_1^3 (x^2 + 4x - 3)\, dx$

**71.** $\int_1^e \frac{1}{x} dx$

**72.** $\int_{25}^{100} \sqrt{x}\, dx$

**73.** $\int_3^5 \sqrt{x}\, dx$

**74.** $\int_1^{10} \frac{1}{x^2} dx$

**75.** $\int_1^{1000} \frac{1}{x^2} dx$

**76.** $\int_0^1 e^x dx$ **77.** $\int_{-2}^{2} \frac{2x}{1+x^2} dx$ **78.** $\int_0^1 e^{2x} dx$ **79.** $\int_2^4 (x-2)^3 dx$

**80.** Find the area shown in Fig. 53

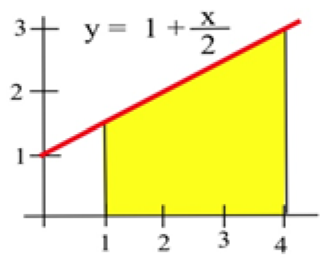



Figure 124

**81.** Find the area shown in Fig. 54

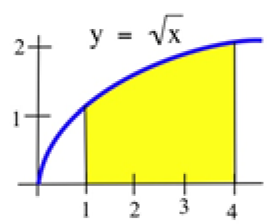



Figure 125

**82**. Find the area shown in Fig. 55

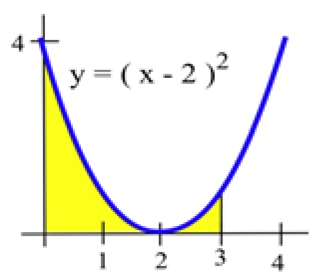



Figure 126

**83**. Find the area shown in Fig. 56

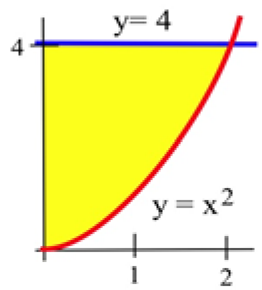



Figure 127

In problems 84 – 87, use the values in the table to estimate the areas.

| $x$ | $f(x)$ | $g(x)$ | $h(x)$ |
|---|---|---|---|
| 0 | 5 | 2 | 5 |
| 1 | 6 | 1 | 6 |
| 2 | 6 | 2 | 8 |
| 3 | 4 | 2 | 6 |
| 4 | 3 | 3 | 5 |
| 5 | 2 | 4 | 4 |
| 6 | 2 | 0 | 2 |

**84.** Estimate the area between *f* and *g*, between *x* = 0 and *x* = 4.

**85.** Estimate the area between *g* and *h*, between *x* = 0 and *x* = 6.

**86**. Estimate the area between *f* and *h*, between *x* = 0 and *x* = 4.

**87.** Estimate the area between *f* and *g*, between *x* = 0 and *x* = 6.

**88.** Estimate the area of the island in Fig. 57.

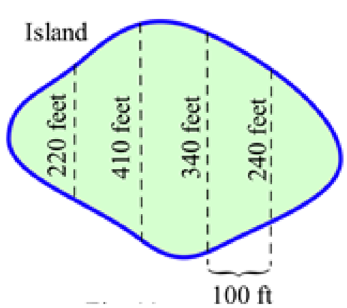



Figure 128

In problems 89 – 98, find the **area** between the graphs of f and g for x in the given interval. Remember to draw the graph!

**89**. $f(x) = x^2 + 3$, $g(x) = 1$ and $-1 \le x \le 2$.

**90.** $f(x) = x^2 + 3$, $g(x) = 1 + x$ and $0 \le x \le 3$.

**91.** $f(x) = x^2$, $g(x) = x$ and $0 \le x \le 2$.

**92.** $f(x) = (x - 1)^2$, $g(x) = x + 1$ and $0 \le x \le 3$.

**93.** $f(x) = \frac{1}{x}$, $g(x) = x$ and $1 \le x \le e$.

**94.** $f(x) = \sqrt{x}$, $g(x) = x$ and $0 \le x \le 4$.

**95.** $f(x) = 4 - x^2$, $g(x) = x + 2$ and $0 \le x \le 2$.

**96.** $f(x) = e^x$, $g(x) = x$ and $0 \le x \le 2$.

**97.** $f(x) = 3$, $g(x) = \sqrt{1 - x^2}$ and $0 \le x \le 1$.

**98.** $f(x) = 2$, $g(x) = \sqrt{4 - x^2}$ and $-2 \le x \le 2$.

In problems 99 and 100, use the values in the table to estimate the average values.

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 0 | 5 | 2 |
| 1 | 6 | 1 |
| 2 | 6 | 2 |
| 3 | 4 | 2 |
| 4 | 3 | 3 |
| 5 | 2 | 4 |
| 6 | 2 | 0 |

**99.** Estimate the average value of f on the interval [0, 6].

**100.** Estimate the average value of g on the interval [0, 6].

In problems 101 – 106, find the **average value** of f on the given interval.

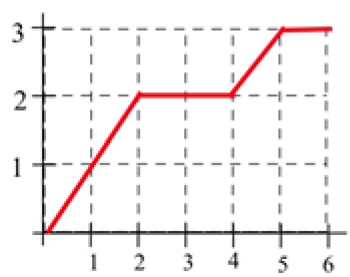



Figure 129

**101.** f($x$) in Fig. 58 for $0 \le x \le 2$.

**102**. f($x$) in Fig. 58 for $0 \le x \le 4$.

**103.** f($x$) in Fig. 58 for $1 \le x \le 6$.

**104**. f($x$) in Fig. 58 for $4 \le x \le 6$.

**105**. $f(x) = 2x + 1$ for $0 \le x \le 4$.

**106.** $f(x) = x^2$ for $0 \le x \le 2$.

**107.** Fig. 59 shows the velocity of a car during a 5 hour trip.

(a) Estimate how far the car traveled during the 5 hours.

(b) At what **constant** velocity should you drive in order to travel the same distance in 5 hours?

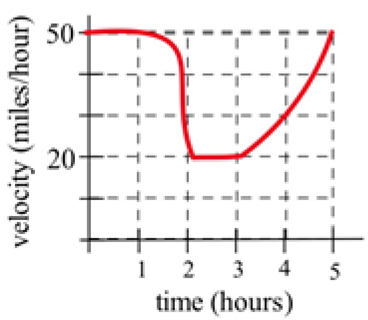



Figure 130

**108.** Fig. 60 shows the number of telephone calls per minute at a large company.

(a) Estimate the average number of calls per minute from 8 am to 5 pm.

(b) From 9 am to 1 pm.

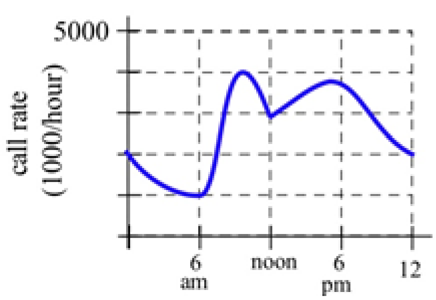



Figure 131

**109.** The demand and supply functions for a certain product are given by $p = 150 - .5q$ and $p = .002q^2 + 1.5$, where *p* is in dollars and *q* is the number of items.

(a) Which is the demand function?

(b) Find the equilibrium price and quantity

(c) Find the total gains from trade at the equilibrium price.

**110**. Still thinking about the product from Exercise 109, with its demand and supply functions, suppose the price is set artificially at $70 (which is above the equilibrium price).

**(a)** Find the quantity supplied and the quantity demanded at this price.

**(b)** Compute the consumer surplus at this price, using the quantity demanded.

**(c)** Compute the producer surplus at this price, using the quantity demanded (why?).

**(d)** Find the total gains from trade at this price.

**(e)** What do you observe?

**111.** When the price of a certain product is $40, 25 items can be sold. When the price of the same product costs $20, 185 items can be sold. On the other hand, when the price of this product is $40, 200 items will be produced. But when the price of this product is $20, only 100 items will be produced. Use this information to find supply and demand functions (assume for simplicity that the functions are linear), and compute the consumer and producer surplus at the equilibrium price.

**112**. Find the present and future values of a continuous income stream of $5000 per year for 12 years if money can earn 1.3% annual interest compounded continuously.

**113.** Find the present value of a continuous income stream of $40,000 per year for 35 years if money can earn

**(a)** 0.8% annual interest, compounded continuously,

**(b)** 2.5% annual interest, compounded continuously,

**(c)** 4.5% annual interest, compounded continuously.

**114.** Find the present value of a continuous income stream $F(t) = 20 + t$, where *t* is in years and *F* is in tens of thousands of dollars per year, for 10 years, if money can earn 2% annual interest, compounded continuously.

**115.** Find the present value of a continuous income stream $F(t) = 12 + 0.3t^2$, where *t* is in years and *F* is in thousands of dollars per year, for 8 years, if money can earn 3.7% annual interest, compounded continuously.

**116.** Find the future value of a continuous income stream $F(t) = 8500 + \sqrt{640t + 100}$, where *t* is in years and *F* is in dollars per year, for 15 years, if money can earn 6% annual interest, compounded continuously.

**117.** A business is expected to generate income at a continuous rate of $25,000 per year for the next eight years. Money can earn 3.4% annual interest, compounded continuously. The business is for sale for $153,000. Is this a good deal?

# Chapter 4: Functions of Two Variables

## PreCalculus Idea -- Topographical Maps

If you've ever hiked, you have probably seen a topographical map. Here is part of a topographic map of Stowe, Vermont, USA (courtesy of United States Geological Survey and http://en.wikipedia.org/wiki/File:Topographic_map_example.png).

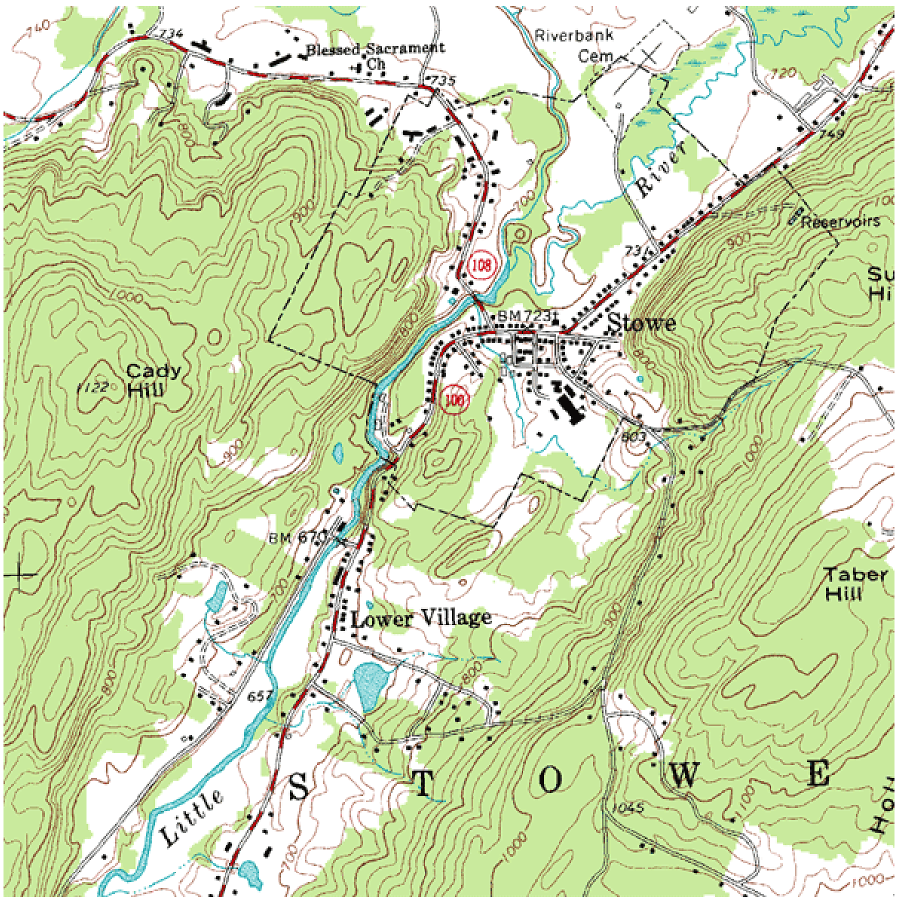



Figure 132

Points with the same elevation are connected with curves, so you can read not only your east-west and your north-south location, but also your elevation. You may have also seen weather maps that use the same principle – points with the same temperature are connected with curves (isotherms), or points with the same atmospheric pressure are connected with curves (isobars). These maps let you read not only a place's location but also its temperature or atmospheric pressure.

In this chapter, we'll use that same idea to make graphs of functions of two variables.

## Section 1: Functions of Two Variables

Real life is rarely as simple as one input – one output. Many relationships depend on lots of variables.

> **Example:** If I put a deposit into an interest-bearing account and let it sit, the amount I have at the end of 3 years depends on P (how much my initial deposit is), r (the annual interest rate), and n (the number of compoundings per year).
>
> **Example:** The air resistance on a wing in a wind tunnel depends on the shape of the wing, the speed of the wind, the wing's orientation (pitch, yaw, and roll), plus a myriad of other things that I can't begin to describe.
>
> **Example:** The amount of your television cable bill depends on which basic rate structure you have chosen and how many pay-per-view movies you ordered.

Since the real world is so complicated, we want to extend our calculus ideas to functions of several variables.

### Functions of Two Variables

If $x_1, x_2, x_3, \ldots, x_n$ are real numbers, then $(x_1, x_2, x_3, \ldots, x_n)$ is called an *n*-tuple. This is an extension of ordered pairs and triples. A function of *n* variables is a function whose domain is some set of *n*-tuples and whose range is some set of real numbers.

For much of what we do here, everything would work the same if we were working with 2, 3, or 47 variables. Because we're trying to keep things a little bit simple, we'll concentrate on functions of two variables.

**A Function of Two Variables**

A function of two variables is a function – that is, to each input is associated exactly one output.

The inputs are ordered pairs, (x, y). The outputs are real numbers. The domain of a function is the set of all possible inputs (ordered pairs); the range is the set of all possible outputs (real numbers).

The function can be written z = f(x,y).

Functions of two variables can be described numerically (a table), graphically, algebraically (a formula), or in English.

We will often now call the familiar y = f(x) a function of one variable.

**Example:** The cost of renting a car depends on how many days you keep it and how far you drive. Let d = the number of days you rent the car, and m = the number of miles you drive. Then the cost of the car rental C(d, m) is a function of two variables.

**Example:** The demand for hot dog buns depends on the price for the hot dog buns and also on the price for hot dogs. The demand $q_B = f\left(p_B, p_D\right)$ is a function of two variables. (The demand for hot dogs also depends on the price of both dogs and buns).

## Formulas and Tables

Just as in the case of functions of one variable, we can display a function of two variables in a table. The two inputs are shown in the margin (top row, left column), and the outputs are shown in the interior cells.

**Example:** Here is a table that shows the cost C(d, m) in dollars for renting a car for d days and driving it m miles:

| d ↓ m → | 100 | 200 | 300 | 400 |
|---|---|---|---|---|
| 1 | 55 | 70 | 85 | 100 |
| 2 | 95 | 110 | 125 | 140 |
| 3 | 135 | 150 | 165 | 180 |
| 4 | 175 | 190 | 205 | 220 |

a. What is the cost of renting a car for 3 days and driving it 200 miles?

b. What is C(100, 4)? What is C(4, 100)?

c. Suppose we rent the car for 3 days. Is C an increasing function of miles?

**Solution:**

a. **According** to the table, renting the car for 3 days (row with d = 3) and driving it 200 miles (column with m = 300) will cost $150 (highlighted in aqua).

b. Careful now – the input is an ordered pair, so in C(100, 4), the 100 has to be a value of d and the 4 has to be a value of m. C(100, 4) would be the cost of renting a car for 100 days and driving it 4 miles. That cost is not in the table. (And that would be a pretty silly way to rent a car.) On the other hand, C(4, 100) is the cost of renting for 4 days and driving 100 miles – the table says that would cost $175.

c. If we know that d is fixed at 3, we're looking at C(3, m). This is now a function of 1 variable, just m. We can see the table that displays values of this function by focusing our attention on just the row where d = 3:

| d ↓ m → | 100 | 200 | 300 | 400 |
|---|---|---|---|---|
| 3 | 135 | 150 | 165 | 180 |

Now we can see that if we rent for 3 days, the cost appears to be an increasing function of the number of miles we drive. (That shouldn't have been surprising.)

The idea of fixing one variable and watching what happens to the function as the other varies will come up again and again.

It's hard to display a function of more than two variables in a table. But it's convenient to work with formulas for functions of two variables, or as many variables as you like.

**Example:** The cost C(d,m) in dollars for renting a car for d days and driving it m miles is given by the formula $C(d,m) = 40d + .15m$

a. What is the cost of renting a car for 3 days and driving it 200 miles?

b. What is C(100, 4)? What is C(4, 100)?

c. Suppose we rent the car for 3 days. Is C an increasing function of miles?

**Solution:**

a. $C(3{,}200) = 40(3) + 15(200) = \$150$. This is the same value we got from the table. The formula will give us the same answers for any of the table values.

b. C(100, 4) makes perfect sense to the formula (even if it doesn't make sense for actually renting a car). So now we can get an answer. To rent the car for 100 days and drive it for 4 miles should cost $4000.60. C(4, 100) = $175, as before.

c. If we fix d = 3, then C(d, m) becomes C(3, m) = 40(3) + .15m = 120 + .15m. Yes, this is an increasing function of m; I can tell because it's linear and its slope is .15 > 0.

Reality check – the formula that gives the cost for the rental car makes sense for all values of d and m. But that's not how the real cost works – you can't rent the car for a negative number of days or drive a negative number of miles. (That is, there are domain restrictions.) In addition, most car rental agreements don't compute a charge for fractions of days; they round up to the next whole number of days.

**Example:** Let $f(x, y, z, w) = 35x^2w - \frac{1}{z} + yz^2$. Evaluate $f(0,1,2,3)$.

**Solution:** Remember that this is an ordered 4-tuple; make sure the numbers get substituted into the correct places:

$$f(0,1,2,3) = 35 \cdot 0^2 3 - \frac{1}{2} + 1 \cdot 2^2 = 3.5$$

## Graphs – Contour Diagrams

The graph of a function of two variables is a surface in three-dimensional space. Think of each input (x,y) as a location on the plane, and plot the point f(x,y) units above that point. But how do you draw a picture of the surface?

1. You can use a fancy computer program to draw beautiful perspective drawings.

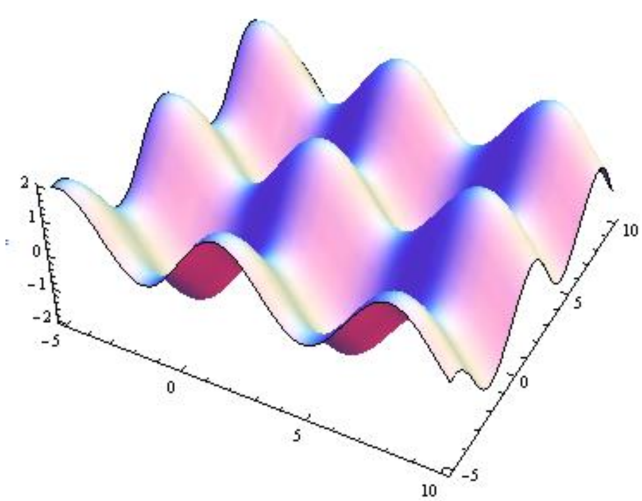



Figure 133

2. You can try to draw a perspective drawing by hand. I'm very bad at this.

3. But one of the best ways, the way I like best, the way we will concentrate on here, is using level curves to draw contour diagrams. A contour diagram is like a topographical map – points with the same elevation (outputs) are connected with curves. Each particular output is called a *level,* and these curves are called *level curves* or *contours*. The closer the curves are to each other, the steeper that section of the surface is. Topographical maps give hikers information about elevation, steep and shallow grades, peaks and valleys. Contour diagrams give us the same kind of information about a function.

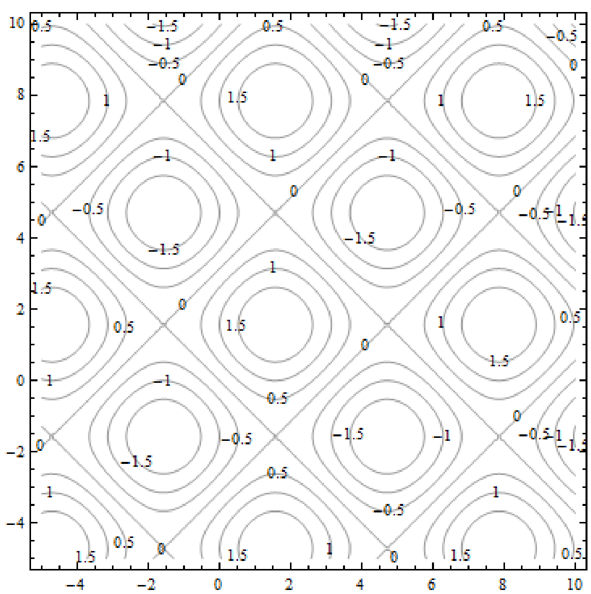



**Figure 134**

This is a contour diagram of the same surface shown in Fig. 2. The level curves are graphs in the xy-plane of curves f(x, y) = c for various constants c.

Each of the squares corresponds to one of the bumps on the surface. If the contours are positive, as highlighted in Fig. 4, the bump is above the xy-plane. If the contours are negative, the bump extends below the xy-plane.

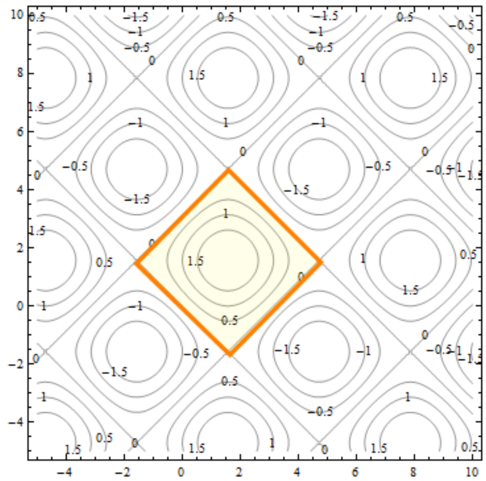


Figure 135

Everywhere on the crisscrossed pattern of diagonal lines, the height of the surface is 0, so the surface is on the xy-plane. This is a feature that I couldn't see when I looked at the perspective drawing.

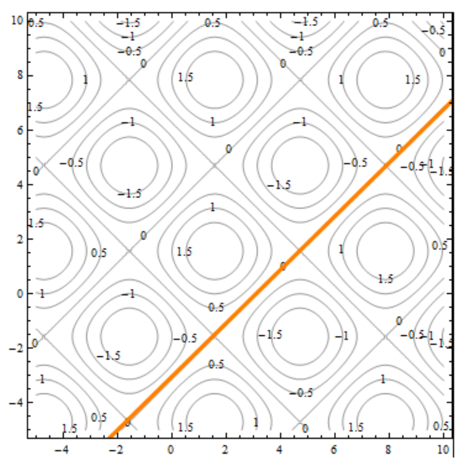


Figure 136

**Example:** Here is the contour diagram for our car rental example. I made it by setting the cost function $C(d,m)=40d+.15m=c$ for c = 0, 100, 200, 300, and 400 and drawing the curves in the dm-plane.

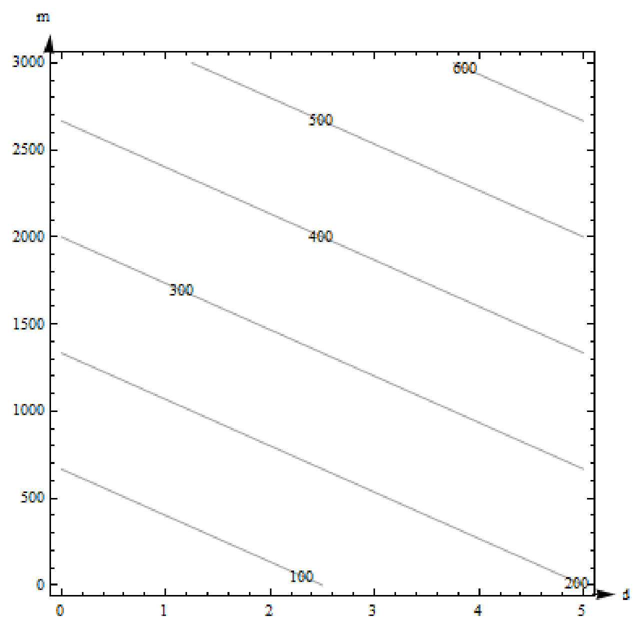



Figure 137

The first coordinate of the ordered pair is d, so the d-axis will be horizontal; the m-axis will be vertical. Remember that the domain for this function is really just where d ≥ 0 and m ≥ 0, so I only drew the curves in the first quadrant.

For c = 0:

$$C(d,m)=40d+.15m=0$$
$$.15m=-40d$$
$$m=-\frac{40}{.15}d\cong-267d$$

This is the equation of a line, with slope about $-267$, passing through the origin. Because of the domain restrictions, the "curve" I will draw for this level is simply the origin.

Putting this back into the car rental context, the only point where I pay $0 for renting the car is when I rent the car for 0 days and drive it 0 miles – that is, if I don't rent it at all.

For c = 100:

$$C(d,m) = 40d + .15m = 100$$
$$.15m = -40d + 100$$
$$m = -\frac{40}{.15}d + \frac{100}{.15} \cong -267d + 667$$

This is the equation of a line, with slope about $-267$, and d-intercept of about 667. This section of this line that lies in the first quadrant is shown with 100 labeling it.

Putting this into context, any point on that line represents a (d, m) combination of days and miles that will make the cost exactly $100. So, for example – if I rent the car for 0 days and drive it 667 miles, it will cost me $100. If I rent the car for 2.5 days and don't drive any miles, it will cost me $100.

For c = 200, c = 300, and so on? I can see the pattern now. Each of these level curves will have the same slope, but the m-intercept will increase each time. The contour diagram is a bunch of equally spaced parallel lines.

**Example:** The contour diagram for the cost C(d,m) in dollars for renting a car for d days and driving it m miles is shown in Fig. 6. Use the diagram to answer the following questions.

a. What is the cost of renting a car for 3 days and driving it 200 miles?

b. What is C(100, 4)? What is C(4, 100)?

c. Suppose we rent the car for 3 days. Is C an increasing function of miles?

**Solution:**

a. The point (3, 200) is between contours on this graph, so I can't get an exact answer for C(3, 200). (But it's typical for a graph that we would have to estimate). It looks to me as if (3, 200) is halfway between the 100 and the 200 contours, so I will estimate that C(3, 200) is about $150.

Estimates from the graph are necessarily very rough. The graph only shows a little information (in this way, a contour diagram is like a table), so I have to extrapolate in between. But for most graphs, I don't actually know what happens between the contours. All I know for sure is that the output at (3, 200) is between the two levels I see. For this car rental example, I also know a formula, and my table showed this particular input, so I have other ways to get a better answer.

b. I can't find (100, 4) on this diagram, so I can't make an estimate of C(100, 4) from this graph.

(4, 100) lies between the contours for 100 and 200. It looks closer to 200, so I'll estimate that
C(4, 100) is about $180.

c. If we fix d = 3, we get a vertical line. What happens as m increases on this vertical line? As m increases, the function values shown on the contours increase – C appears to be an increasing function of miles.

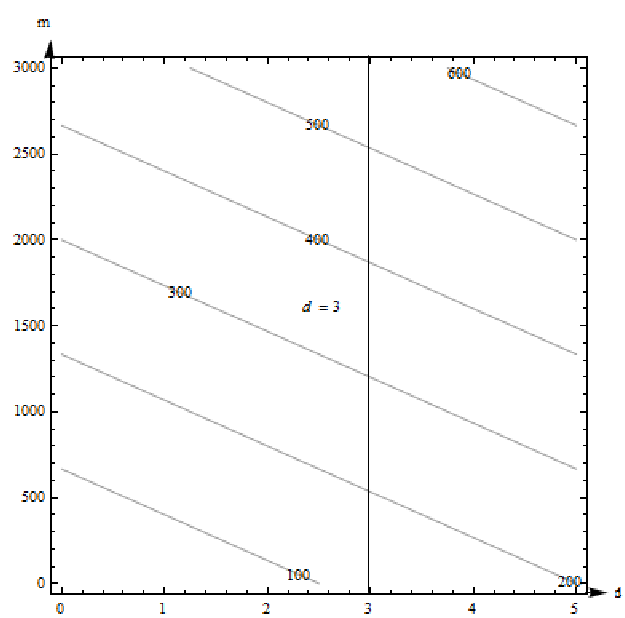



Figure 138

**Example:** Here is a contour diagram for a function g(x,y).

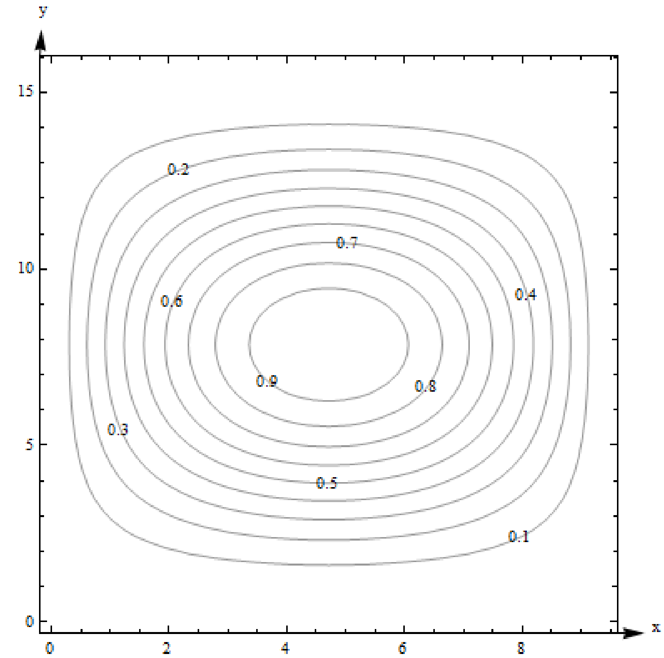



Figure 139

Use the diagram to answer the following questions:

a. What is g(3, 5)?

b. What is the highest point shown on the diagram? What is the lowest point shown?

c. If you start at (3, 5) and head in the positive x direction, do you go uphill or downhill first?

**Solution:** a. g(3, 5) is 0.6. I can tell because the point is right on one of the contours.

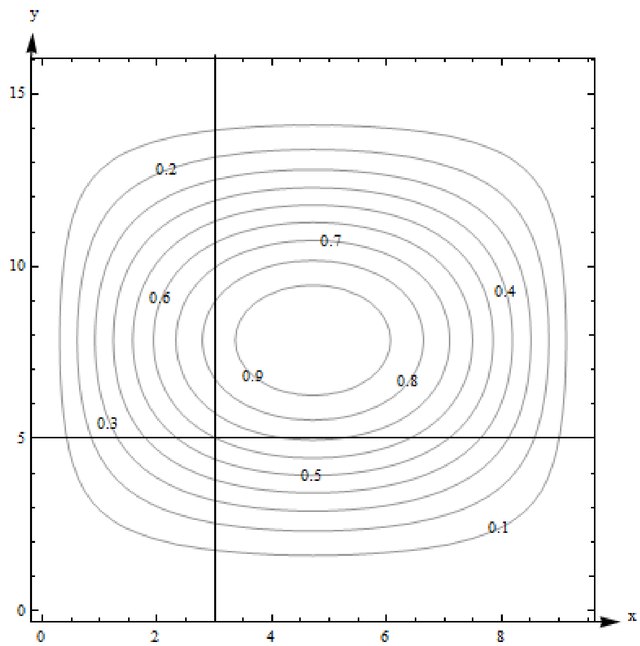



Figure 140

b. The highest contour shown is 0.9, and there would be a contour for 1.0 if the surface had ever got that high. However, the height seems to be increasing as we move in toward the center, so I'm guessing that the g gets to nearly 1 in the center. The lowest contour is 0.1. But again, I will guess that the height continues to decrease, so I think g is nearly 0 around the outside.

c. Starting at the point (3, 5, 0.6) on the surface and traveling to the right along the horizontal line shown in Fig. 9, you would cross the contour for 0.7 next. So the function increases first (we go uphill), and then decreases again.

Note one more time – we don't really know what happens between the contours. All we can do is estimate from the information in the graph.

**Example:** Here is a contour diagram for a function F(x,y).

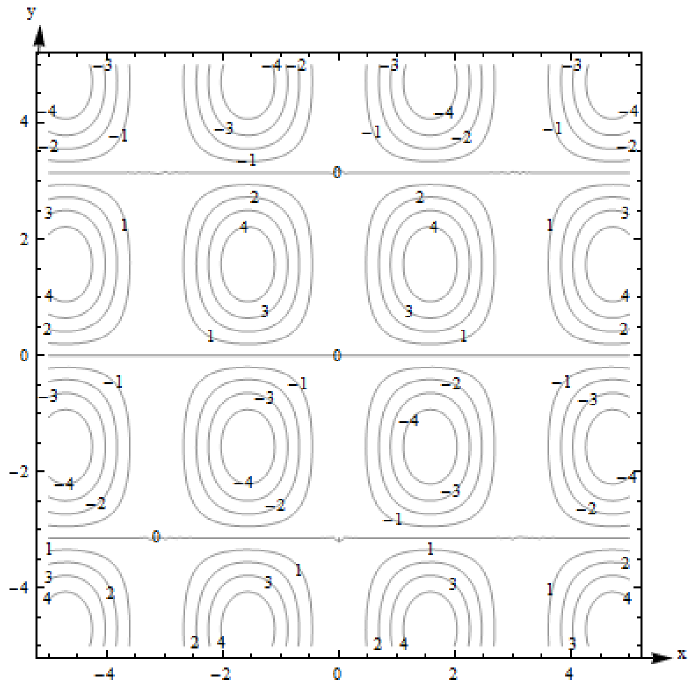


Figure 141

a) Describe the surface shown in Fig. 10.

b) Suppose you travel along the surface in the positive y-direction, starting on the surface at the point above (or below) the point (x, y) = (-1, 1). Describe your journey.

**Solution:**

a) The surface is bumpy, with regularly spaced oval bumps. Notice that some of the bumps go up (positive contours), but others go down. Between the bumps, there are horizontal lines that are completely level, with an elevation of 0.

b) It looks as if F(-1,1) is about 3. As I head in the positive y-direction along the line shown in Fig. 11, I first go uphill, nearly to 4, then I start going downhill. As I keep going north, I keep descending, going into the dip, until nearly -4. I'm starting to go uphill again just as I leave the graph.

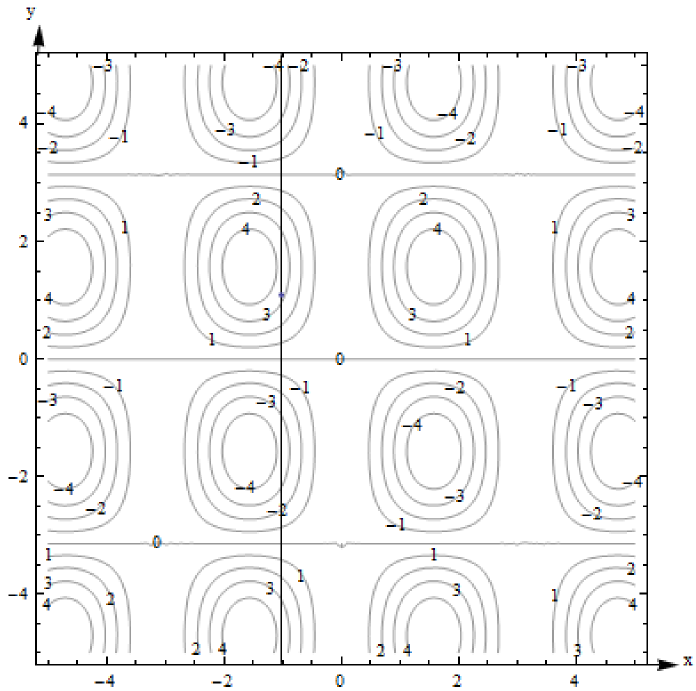


Figure 142

What happens if you have a function of more than two variables? Its graph will be a *hyper-surface*. For example, the graph of a function of four variables will be a hyper-surface in 5-dimensional space. This is hard (impossible for most of us) to visualize. Even the contours are hard to visualize – instead of curves in the plane, they're hyper-surfaces in 4-dimensional space. So – if you have more than two variables, the graph isn't usually very useful.

## Functions of Two Real-Life Variables

### Complementary goods and substitute goods

The demand for some pairs of goods have a relationship, where the quantity demanded for one product depends somehow on the prices for both.

Two goods are *complementary* if an increase in the price of either decreases the demand for both.

> **Example:** The demand for cars depends on both the price for cars and the price of gasoline.
>
> **Example:** The demand for hot dog buns depends on both the price for the buns and the price for the hot dogs.

Two goods are *substitutes* if an increase in the price of one increases the demand for the other.

**Example:** The demand for Brand A depends on its price and also on the price of its main competitor Brand B. If the Brand B raises its price, consumers will switch brands – substitute – and demand for Brand A will increase.

Think brands of soft drinks, detergent, or paper towels. A traditional example is coffee and tea – the idea is that consumers are simply looking for a hot drink and they'll buy whatever is cheaper. But this has always seemed fishy to me – I've never met any coffee- or tea-drinkers who would happily switch.

These demand functions are functions of two variables.

**Example:** The demand functions for two products are given below. $p_1$, $p_2$, $q_1$, and $q_2$ are the prices (in dollars) and quantities for products 1 and 2.

$$q_1 = 200 - 3p_1 - p_2$$
$$q_2 = 150 - p_1 - 2p_2$$

Are these two products complementary goods or substitute goods? What is the quantity demanded for each when the price for product 1 is \$20 per item and the price for product 2 is \$30 per item?

**Solution:** These products are complementary – an increase in either price decreases both demands. You can see that because the coefficients are both negative in each demand function.

When $p_1$ = 20 and $p_2$ = 30, we have

$$q_1 = 200 - 3(20) - (30) = 110$$
$$q_2 = 150 - (20) - 2(30) = 70$$

110 units are demanded for product 1 and 70 units are demanded for product 2 when the price for product 1 is \$20 per item and the price for product 2 is \$30 per item

**Cobb-Douglas Production function**

Production functions are used to model the total output of a firm for a variety of inputs (doesn't this sound like a function of several variables?). One example is a Cobb-Douglas Production function:

$$P = AL^{\alpha} K^{\beta}$$

In this function, P is the total production, A is a constant, α and β are constants between 0 and 1, L is the labor force, and K is the capital expenditure. (And the units must be massaged well.)

You can read more about Cobb-Douglas Production functions at http://en.wikipedia.org/wiki/Cobb-Douglas. You can read about other kinds of production functions at http://en.wikipedia.org/wiki/Production_function.

## Section 2: Calculus of Functions of Two Variables

Now that you have some familiarity with functions of two variables, it's time to start applying calculus to help us solve problems with them. In Chapter 2, we learned about the derivative for functions of two variables. Derivatives told us about the shape of the function, and let us find local max and min – we want to be able to do the same thing with a function of two variables.

First let's think. Imagine a surface, the graph of a function of two variables. Imagine that the surface is smooth and has some hills and some valleys. Concentrate on one point on your surface. What do we want the derivative to tell us? It ought to tell us how quickly the height of the surface changes as we move …. Wait, which direction do we want to move? This is the reason that derivatives are more complicated for functions of several variables – there are so many directions we could move from any point.

It turns out that our idea of fixing one variable and watching what happens to the function as the other changes is the key to extending the idea of derivatives to more than one variable.

# Partial Derivatives

## Partial Derivatives:

Suppose that z = f(x, y) is a function of two variables.

The **partial derivative of f with respect to x** is the ordinary derivative of the function f(x,y) where we think of x as the only variable and act as if y is a constant.

The **partial derivative of f with respect to y** is the ordinary derivative of the function f(x,y) where we think of y as the only variable and act as if x is a constant.

The "with respect to x" or "with respect to y" part is really important – you have to know and tell which variable you are thinking of as THE variable.

**Geometrically** – the partial derivative with respect to x gives the slope of the curve as you travel along a cross-section, a curve on the surface parallel to the x-axis. The partial derivative with respect to y gives the slope of the cross-section parallel to the y-axis.

**Notation for the Partial Derivative:**

The **partial derivative of y = f(x) with respect to x** is written as

$$f_x(x, y), \text{ or } z_x \text{ simply } f_x$$

The Leibniz notation is $\dfrac{\partial f}{dx}$, or $\dfrac{\partial z}{dx}$

We use an adaptation of the $\dfrac{\partial z}{dx}$ notation to mean "find the partial derivative of f(x,y) with respect to x:"

$$\frac{\partial}{\partial x}\left(f(x, y)\right) = \frac{\partial f}{\partial x}$$

**To estimate a partial derivative from a table or contour diagram:**

The partial derivative with respect to x can be approximated by looking at an average rate of change, or the slope of a secant line, over a very tiny interval in the x-direction (holding y constant). The tinier the interval, the closer this is to the true partial derivative.

**To compute a partial derivative from a formula:**

If f(x,y) is given as a formula, you can find the partial derivative with respect to x algebraically by taking the ordinary derivative thinking of x as the only variable (holding y fixed).

**Of course, everything here works the same way if we're trying to find the partial derivative with respect to y – just think of y as your only variable and act as if x is constant.**

The idea of a partial derivative works perfectly well for a function of several variables – you focus on one variable to be THE variable and act as if all the other variables are constants.

**Example:** Here is a contour diagram for a function g(x,y). Use the diagram to answer the following questions:

a. Estimate $g_x(3,5)$ and $g_y(3,5)$

b. Where on this diagram is $g_x$ greatest? Where is $g_y$ greatest?

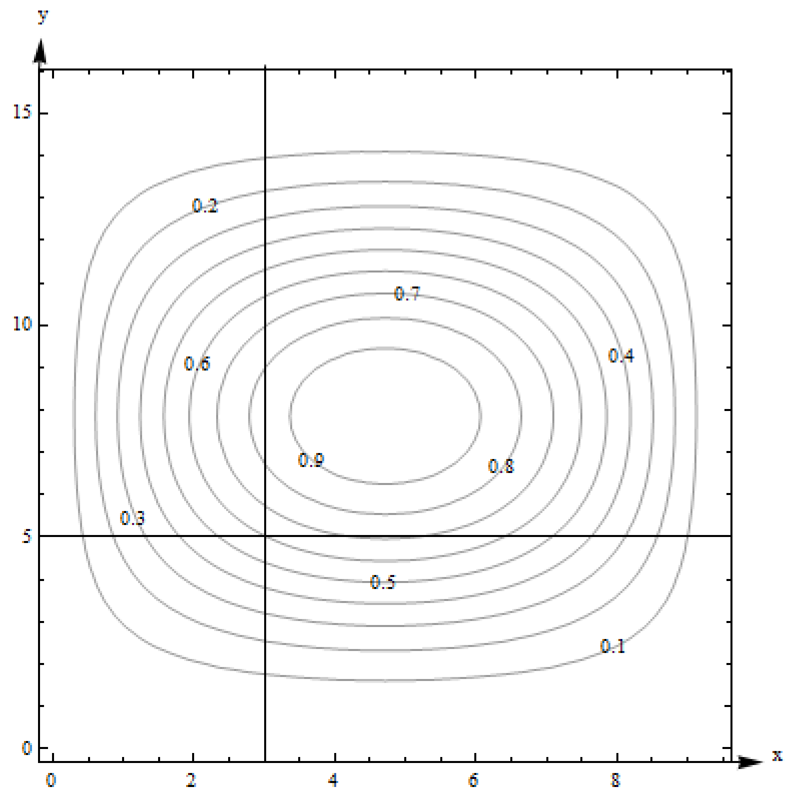



Figure 143

**Solution:**

a. $g_x(3,5)$ means I'm thinking of x as my only variable, so I'll hold y fixed at y = 5. That means I'll be looking along the horizontal line y = 5. (3, 5) lies on the contour line, so I know that g(3, 5) = 0.6. I'll use the next point that I can read as I move to the right – that would be g(4.2,5) = 0.7. Then I'll find the average rate of change:

Average rate of change = (change in output) / (change in input)

$$= \frac{\Delta g}{\Delta x} = \frac{0.7 - 0.6}{4.2 - 3} = \frac{1}{12} \cong .083.$$

I can do the same thing by going to the next point I can read to the left, which is g(2.4,5) = 0.5. Then the average rate of change is $=\frac{\Delta g}{\Delta x}=\frac{0.5-0.6}{2.4-3}=\frac{1}{6}\cong .167$. Either of these would be a fine estimate of $g_x(3,5)$ given the information we have, or you could take their average. I estimate that $g_x(3,5)\cong .125$.

Estimate $g_y(3,5)$ the same way, but moving on the vertical line. Using the next point up, I get the average rate of change $=\frac{\Delta g}{\Delta y}=\frac{0.7-0.6}{5.8-5}=.125$. Using the next point down, I get $\frac{\Delta g}{\Delta y}=\frac{0.5-0.6}{4.5-5}=.2$. Taking their average, I estimate $g_y(3,5)\cong .1625$.

b. $g_x$ means x is my only variable, and I'm thinking of y as a constant. So I'm thinking about moving across the diagram on horizontal lines. $g_x$ will be greatest when the contour lines are closest together, when the surface is steepest – then the denominator in $\frac{\Delta g}{\Delta x}$ will be small, so $\frac{\Delta g}{\Delta x}$ will be big. Scanning the graph, I can see that the contour lines are closest together when I head to the left or to the right from about (0.5, 8) and (9, 8). So $g_x$ is greatest at about (0.5, 8) and (9, 8). For $g_y$, I want to look at vertical lines. $g_y$ is greatest at about (5, 3.8) and (5, 12).

**Example:** Cold temperatures feel colder when the wind is blowing. Windchill is the perceived temperature, and it depends on both the actual temperature and the wind speed – a function of two variables! You can read more about windchill at http://www.nws.noaa.gov/om/windchill/. Fig. 13 shows a table (courtesy of the National Weather Service) that shows the perceived temperature for various temperatures and windspeeds. Note that they also include the formula, but I want to use the information in the table.

## Wind Chill Chart

Temperature (°F)

| Wind (mph) / Calm | 40 | 35 | 30 | 25 | 20 | 15 | 10 | 5 | 0 | -5 | -10 | -15 | -20 | -25 | -30 | -35 | -40 | -45 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 36 | 31 | 25 | 19 | 13 | 7 | 1 | -5 | -11 | -16 | -22 | -28 | -34 | -40 | -46 | -52 | -57 | -63 |
| 10 | 34 | 27 | 21 | 15 | 9 | 3 | -4 | -10 | -16 | -22 | -28 | -35 | -41 | -47 | -53 | -59 | -66 | -72 |
| 15 | 32 | 25 | 19 | 13 | 6 | 0 | -7 | -13 | -19 | -26 | -32 | -39 | -45 | -51 | -58 | -64 | -71 | -77 |
| 20 | 30 | 24 | 17 | 11 | 4 | -2 | -9 | -15 | -22 | -29 | -35 | -42 | -48 | -55 | -61 | -68 | -74 | -81 |
| 25 | 29 | 23 | 16 | 9 | 3 | -4 | -11 | -17 | -24 | -31 | -37 | -44 | -51 | -58 | -64 | -71 | -78 | -84 |
| 30 | 28 | 22 | 15 | 8 | 1 | -5 | -12 | -19 | -26 | -33 | -39 | -46 | -53 | -60 | -67 | -73 | -80 | -87 |
| 35 | 28 | 21 | 14 | 7 | 0 | -7 | -14 | -21 | -27 | -34 | -41 | -48 | -55 | -62 | -69 | -76 | -82 | -89 |
| 40 | 27 | 20 | 13 | 6 | -1 | -8 | -15 | -22 | -29 | -36 | -43 | -50 | -57 | -64 | -71 | -78 | -84 | -91 |
| 45 | 26 | 19 | 12 | 5 | -2 | -9 | -16 | -23 | -30 | -37 | -44 | -51 | -58 | -65 | -72 | -79 | -86 | -93 |
| 50 | 26 | 19 | 12 | 4 | -3 | -10 | -17 | -24 | -31 | -38 | -45 | -52 | -60 | -67 | -74 | -81 | -88 | -95 |
| 55 | 25 | 18 | 11 | 4 | -3 | -11 | -18 | -25 | -32 | -39 | -46 | -54 | -61 | -68 | -75 | -82 | -89 | -97 |
| 60 | 25 | 17 | 10 | 3 | -4 | -11 | -19 | -26 | -33 | -40 | -48 | -55 | -62 | -69 | -76 | -84 | -91 | -98 |

Frostbite Times: 30 minutes, 10 minutes, 5 minutes

$$\text{Wind Chill (°F)} = 35.74 + 0.6215T - 35.75(V^{0.16}) + 0.4275T(V^{0.16})$$

Where, T= Air Temperature (°F) V= Wind Speed (mph)

*Effective 11/01/01*

Figure 144

a. What is the perceived temperature when the actual temperature is 25˚F and the wind is blowing at 15 miles per hour?

b. Suppose the actual temperature is 25˚F. Use information from the table to describe how the perceived temperature would change if the wind speed increased from 15 miles per hour?

**Solution:**

a. Reading the table, we see that the perceived temperature is 13˚F.

b. This is a question about a partial derivative. We're holding the temperature (T) fixed at 25˚F, and asking what happens as wind speed (V) increases from 15 miles per hour. We're thinking of V as the only variable, so we want $\text{WindChill}_V = W_V$ when T = 25 and V = 15. We'll find the average rate of change by looking in the column where T = 25 and letting V increase, and use that to approximate the partial derivative.

$$W_V \cong \frac{\Delta W}{\Delta V} = \frac{11-13}{20-15} = -0.4$$

What are the units? W is measured in ˚F and V is measured in mph, so the units here are ˚F/mph. And that lets us describe what happens:

The perceived temperature would decrease by about .4˚F for each mph increase in wind speed.

**Example:** Find $f_x$ and $f_y$ at the points (0, 0) and (1, 1) if $f(x,y) = x^2 - 4xy + 4y^2$

**Solution:** To find $f_x$, take the ordinary derivative of *f* with respect to *x*, acting as if *y* is constant:

$$f_x(x,y) = 2x - 4y$$

Note that the derivative of the $4y^2$ term with respect to *x* is zero – it's a constant.

Similarly, $f_y(x,y) = -4x + 8y$. Now we can evaluate these at the points:

$f_x(0,0) = 0$ and $f_y(0,0) = 0$; this tells us that the cross sections parallel to the x- and y- axes are both flat at (0,0).

$f_x(1,1) = -2$ and $f_y(1,1) = 4$; this tells us that above the point (1, 1), the surface decreases if you move to more positive x values and increases if you move to more positive y values.

**Example:** Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if $f(x,y)=\frac{e^{x+y}}{y^3+y}+y(\ln y)$

**Solution:** $\frac{\partial f}{\partial x}$ means *x* is our only variable, we're thinking of *y* as a constant. Then we'll just find the ordinary derivative. From *x*'s point of view, this is an exponential function, divided by a constant, with a constant added. The constant pulls out in front, the derivative of the exponential function is the same thing, and we need to use the chain rule, so we multiply by the derivative of that exponent (which is just 1):

$$\frac{\partial f}{\partial x}=\frac{1}{y^3+y}e^{x+y}$$

$\frac{\partial f}{\partial y}$ means that we're thinking of *y* as the variable, acting as if *x* is constant. From *y*'s point of view, *f* is a quotient plus a product – we'll need the quotient rule and the product rule:

$$\begin{aligned}\frac{\partial f}{\partial y}&=\frac{(\ )(\ )-(\ )(\ )}{(\ )^2}+(\ )(\ )+(\ )(\ )\\&=\frac{\left(e^{x+y}(1)\right)\left(y^3+y\right)-\left(e^{x+y}\right)\left(3y^2+1\right)}{\left(y^3+y\right)^2}+(1)(\ln y)+(y)\left(\frac{1}{y}\right)\end{aligned}$$

For goodness' sake, don't try to simplify that. Just leave it.

**Example:** Find $f_z$ if $f(x,y,z,w)=35x^2w-\frac{1}{z}+yz^2$

**Solution:** $f_z$ means *z* is our only variable, so we'll act as if all the other variables (x, y and w) are constants and take the ordinary derivative.

$$f_z(x,y,z,w)=\frac{1}{z^2}+2yz$$

## Using Partial Derivatives to Estimate Function Values

We can use the partial derivatives to estimate values of a function. The geometry is similar to the tangent line approximation in one variable. Recall the one-variable case: if *x* is close enough to a known point *a*, then $f(x) \cong f(a) + f'(a)(x-a)$. In two variables, we do the same thing in both directions at once:

**Approximating Function Values with Partial Derivatives**

To approximate the value of *f*(*x, y*), find some point (*a, b*) where

1. (*x, y*) and (*a, b*) are close – that is, *x* and *a* are close and *y* and *b* are close.

2. You know the exact values of *f* (*a, b*) and both partial derivatives there.

Then $f(x, y) \cong f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Notice that the total change in *f* is being approximated by adding the approximate changes coming from the *x* and *y* directions. Another way to look at the same formula:

$$\Delta f \cong f_x \Delta x + f_y \Delta y$$

How close is close? It depends on the shape of the graph of *f*. In general, the closer the better.

**Example:** Use partial derivatives to estimate the value of $f(x, y) = x^2 - 4xy + 4y^2$ at (0.9, 1.1)

**Solution:** Note that the point (0.9, 1.1) is close to an "easy" point, (1, 1). In fact, we already worked out the partial derivatives at (1, 1): $f_x(x, y) = 2x - 4y$; $f_x(1,1) = -2$.
$f_y(x, y) = -4x + 8y$; $f_y(1,1) = 4$. We also know that $f(1,1) = 1$.

So $f(0.9,1.1) \cong 1 - 2(-0.1) + 4(0.1) = 1.6.$

Note that it would have been possible in this case to simply compute the exact answer; $f(0.9,1.1) = (0.9)^2 - 4(0.9)(1.1) + 4(1.1)^2 = 1.69$. Our estimate is not perfect, but it's pretty close.

**Example:** Here is a contour diagram for a function g(x,y). Use partial derivatives to estimate the value of g(3.2, 4.7).

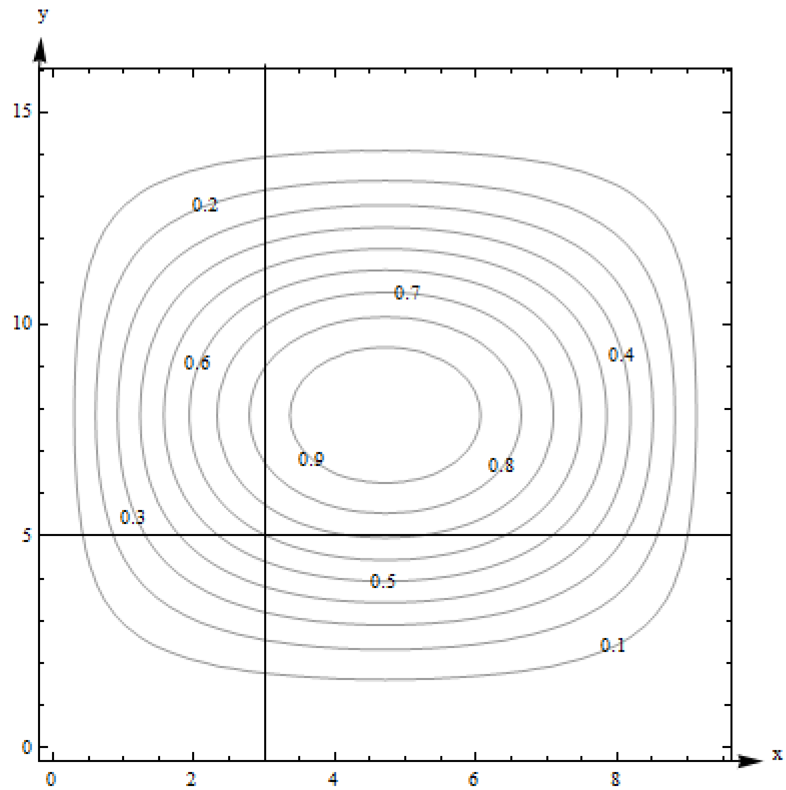



Figure 145

**Solution:** This is the same diagram from before, so we already estimated the value of the function and the partial derivatives at the nearby point (3,5). g(3, 5) is 0.6, our estimate of $g_x(3,5) \cong .125$, and our estimate of $g_y(3,5) \cong .1625$. So $g(3.2,4.7) \cong 0.6 + (.125)(.2) + (.1625)(-.3) = .57625$. Note that in this case, we have no way to know how close our estimate is to the actual value.

## Section 3: Optimization

The partial derivatives tell us something about where a surface has local maxima and minima. Remember that even in the one-variable cases, there were critical points which were neither maxima nor minima – this is also true for functions of many variables. In fact, as you might expect, the situation is even more complicated.

### Second Derivatives

When you find a partial derivative of a function of two variables, you get another function of two variables – you can take its partial derivatives, too. We've done this before, in the one-variable setting. In the one-variable setting, the second derivative gave information about how the graph was curved. In the two-variable setting, the second partial derivatives give some information about how the surface is curved, as you travel on cross-sections – but that's not very complete information about the entire surface.

Imagine that you have a surface that's ruffled around a point, like what happens near a button on an overstuffed sofa, or a pinched piece of fabric, or the wrinkly skin near your thumb when you make a fist. Right at that point, every direction you move, something different will happen – it might increase, decrease, curve up, curve down … A simple phrase like "concave up" or "concave down" can't describe all the things that can happen on a surface.

Surprisingly enough, though, there is still a second derivative test that can help you decide if a point is a local max or min or neither. So we still do want to find second derivatives.

**Second Partial Derivatives**

Suppose $f(x,y)$ is a function of two variables. Then it has four **second partial derivatives**:

$f_{xx} = \frac{\partial}{\partial x}(f_x) = (f_x)_x$; $f_{xy} = \frac{\partial}{\partial y}(f_x) = (f_x)_y$; $f_{yx} = \frac{\partial}{\partial x}(f_y) = (f_y)_x$; and
$f_{yy} = \frac{\partial}{\partial y}(f_y) = (f_y)_y$

$f_{xy}$ and $f_{yx}$ are called the **mixed (second) partial derivatives of *f***

Leibniz notation for the second partial derivatives is a bit confusing, and we won't use it as often:

$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$; $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$; $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$;
$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$

Notice that the order of the variables for the mixed partials goes from right to left in the Leibniz notation instead of left to right.

**Example:** Find all four partial derivatives of $f(x, y) = x^2 - 4xy + 4y^2$

**Solution:** We have to start by finding the (first) partial derivatives:

$$f_x(x, y) = 2x - 4y$$

$$f_y(x, y) = -4x + 8y$$

Now we're ready to take the second partial derivatives:

$$f_{xx}(x, y) = \frac{\partial}{\partial x}(2x - 4y) = 2$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}(2x - 4y) = -4$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x}(-4x + 8y) = -4$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}(-4x + 8y) = 8$$

You might have noticed that the two mixed partial derivatives were equal in this last example. It turns out that it's not a coincidence – it's a theorem.

**Mixed Partial Derivative Theorem**

If $f$ , $f_x$, $f_y$, $f_{xy}$, and $f_{yx}$ are all continuous (no breaks in their graph)

Then $f_{xy}$ = $f_{yx}$.

In fact, as long as *f* and all its appropriate partial derivatives are continuous, the mixed partials are equal even if they are of higher order, and even if the function has more than two variables.

This theorem means that the confusing Leibniz notation for second derivatives is not a big problem – in almost every situation, the mixed partials are equal, so it doesn't matter in which order we compute them.

**Example:** Find $\dfrac{\partial^2 f}{\partial x \partial y}$ for $f(x,y) = \dfrac{e^{x+y}}{y^3+y} + y(\ln y)$

**Solution:** We already found the first partial derivatives in an earlier example:

$$\frac{\partial f}{\partial x} = \frac{1}{y^3+y} e^{x+y}$$

$$\frac{\partial f}{\partial y} = \frac{\left(e^{x+y}(1)\right)\left(y^3+y\right) - \left(e^{x+y}\right)\left(3y^2+1\right)}{\left(y^3+y\right)^2} + (1)(\ln y) + (y)\left(\frac{1}{y}\right)$$

Now we need to find the mixed partial derivative – the Theorem says it doesn't matter whether we find the partial derivative of $\dfrac{\partial f}{\partial x} = \dfrac{1}{y^3+y} e^{x+y}$ with respect to *y* or the partial derivative of $\dfrac{\partial f}{\partial y} = \dfrac{\left(e^{x+y}(1)\right)\left(y^3+y\right) - \left(e^{x+y}\right)\left(3y^2+1\right)}{\left(y^3+y\right)^2} + (1)(\ln y) + (y)\left(\dfrac{1}{y}\right)$ with respect to *x*. Which would you rather do?

Yes, me too. I'll compute the mixed partial by finding the partial derivative of $\dfrac{\partial f}{\partial x} = \dfrac{1}{y^3+y} e^{x+y}$ with respect to *y* – it still looks messy, but it looks less messy:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{1}{y^3+y} e^{x+y}\right) = \frac{\left(e^{x+Y}\right)\left(y^3+y\right) - \left(e^{x+Y}\right)\left(3y^2+1\right)}{\left(y^3+y\right)^2}$$

If you'd decided to do this the other way, you'd end up in the same place. Eventually.

## Local Maxima, Minima, and Saddle Points

Let's briefly review max-min problems in one variable.

A local max is a point on a curve that is higher than all the nearby points. A local min is lower than all the nearby points. We know that local max or min can only occur at critical points, where the derivative is zero or undefined. But we also know that not all critical points are max or min, so we also need to test them, with the First Derivative or Second Derivative Test.

The situation with a function of two variables is much the same. Just as in the one-variable case, the first step is to find critical points, places where both the partial derivatives are either zero or undefined.

**Definition:**

f has a **local maximum** at (a, b) if f(a, b) ≥ f(x, y) for all points (x, y) near (a, b)

f has a **local minimum** at (a, b) if f(a, b) ≤ f(x, y) for all points (x, y) near (a, b)

A **critical point** for a function f(x, y) is a point (x, y) (or (x, y, f(x, y)) where **both** the following are true:

$f_x = 0$ **or is undefined**

**and**

$f_y = 0$ **or is undefined**

**Useful Fact: Just as in the one-variable case, a local max or min of f can only occur at a critical point.**

And then, just as in the one-variable setting, not all critical points are local max or min. For a function of two variables, the critical point could be a local max, local min, or a saddle point.

A point on a surface is a local maximum if it's higher than all the points nearby; a point is a local minimum if it's lower than all the points nearby.

A saddle point is a point on a surface that is a minimum along some paths and a maximum along some others. It's called this because it's shaped a bit like a saddle you might use to ride a horse. You can see a saddle point by making a fist – between the knuckles of your index and middle fingers, you can see a place that is a minimum as you go across your knuckles, but a maximum as you go along your hand toward your fingers.

Here is a picture of a saddle point from a few different angles. This is the surface $f(x,y)=5x^2-3y^2+10$, and there is a saddle point above the origin. The lines show what the surface looks like above the *x*- and *y*-axes. Notice how the point above the origin, where the lines cross, is a local minimum in one direction, but a local maximum in the other direction.

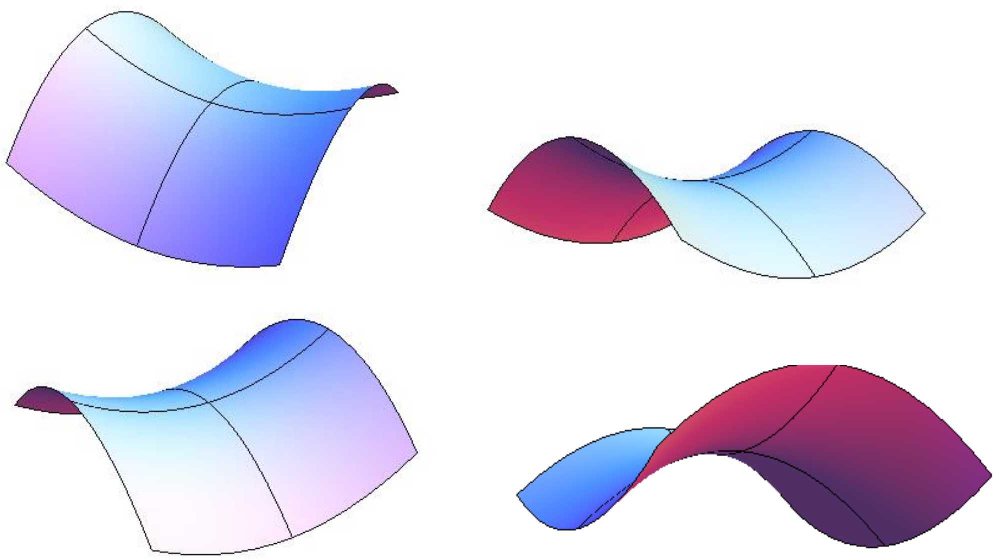


Figure 146

## Second Derivative Test

Just as in the one-variable case, we'll need a way to test critical points to see whether they are local max or min. There is a second derivative test for functions of two variables that can help – but, just as in the one-variable case, it won't always give an answer.

**The Second Derivative Test for Functions of Two Variables:**

Find all critical points of f(x,y).

Compute $D = (f_{xx})(f_{yy}) - (f_{xy})(f_{yx})$, and evaluate it at each critical point.

(a) If D > 0, then f has a local max or min at the critical point. To see which, look at the sign of $f_{xx}$ :

If $f_{xx} > 0$, then f has a local minimum at the critical point.

If $f_{xx} < 0$, then f has a local maximum at the critical point.

(b) If D < 0 then f has a saddle point at the critical point.

(c) If D = 0, there could be a local max, local min, or neither.

**Example:** Find all local maxima, minima, and saddle points for the function $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

**Solution:** First we need the critical points:

$f_x = 3x^2 + 6x$ and $f_y = 3y^2 - 6y$

Critical points are the places where both these are zero (neither is ever undefined):

$f_x = 3x^2 + 6x = 3x(x+2) = 0$ when *x* = 0 or when *x* = −2.

$f_y = 3y^2 - 6y = 3y(y-2) = 0$ when *y* = 0 or when *y* = 2.

Putting these together, we get four critical points: (0, 0), (−2, 0), (0, 2), and (−2, 2).

Now to classify them, we'll use the Second Derivative Test. We'll need all the second partial derivatives:

$$f_{xx} = 6x + 6,\ f_{yy} = 6y - 6,\ f_{xy} = 0 = f_{yx}$$

Then $D = (6x+6)(6y-6) - (0)(0) = (6x+6)(6y-6)$.

Now look at each critical point in turn:

At (0, 0): $D = (6(0)+6)(6(0)-6) = (6)(-6) = -36 < 0$; there is a saddle point at (0, 0).

At (−2, 0): $D = (6(-2)+6)(6(0)-6) = (-6)(-6) = 36 > 0$, and $f_{xx} = 6(-2)+6 = -6 < 0$; there is a local maximum at (−2, 0).

At (0, 2): $D = (6)(6) > 0$ and $f_{xx} = 6 > 0$; there is a local minimum at (0, 2).

At (−2, 2): $D = (-6)(6) < 0$; there is another saddle point at (−2, 2).

**Example:** Find all local maxima, minima, and saddle points for the function

$$z = 9x^3 + \frac{y^3}{3} - 4xy.$$

**Solution:** We'll need all the partial derivatives and second partial derivatives, so let's compute them all first:

$$z_x = 27x^2 - 4y;\ z_y = y^2 - 4x;$$

$$z_{xx} = 54x;\ z_{yy} = 2y;\ z_{xy} = -4 = z_{yx}$$

Now to find the critical points: We need both $z_x$ and $z_y$ to be zero (neither is ever undefined), so we need to solve this set of equations simultaneously:

$$z_x = 27x^2 - 4y = 0$$
$$z_y = y^2 - 4x = 0$$

Perhaps it's been a while since you solved systems of equations. Just remember the substitution method – solve one equation for one variable and substitute into the other equation:

$$\begin{aligned} 27x^2 - 4y &= 0 \\ y^2 - 4x &= 0 \end{aligned}$$ → solve $y^2 - 4x = 0$ for $x = \frac{y^2}{4}$, then substitute into the other equation →

$$27\left(\frac{y^2}{4}\right)^2 - 4y = 0$$

$$\frac{27}{16}y^4 - 4y = 0$$

Now we have just one equation in one variable to solve. You can use algebra (this one factors, it's not too bad), or you can use technology to find the solutions: $y = 0$ or $y = \frac{4}{3}$. Plugging back in to find x gives us the two critical points: (0, 0) and $\left(\frac{4}{9}, \frac{4}{3}\right)$.

Now to test them: Compute $D = (f_{xx})(f_{yy}) - (f_{xy})(f_{yx}) = (54x)(2y) - (-4)(-4)$, evaluate it at the two critical points, and see:

At (0,0): D = −16 < 0, so there is a saddle point at (0, 0).

At $\left(\frac{4}{9}, \frac{4}{3}\right)$: D = 48 > 0, and $f_{xx} > 0$, so there is a local minimum at $\left(\frac{4}{9}, \frac{4}{3}\right)$.

## Applied Optimization

**Example:** A company makes two products. The demand equations for the two products are given below. $p_1$, $p_2$, $q_1$,and $q_2$ are the prices and quantities for products 1 and 2.

$$\begin{aligned} q_1 &= 200 - 3p_1 - p_2 \\ q_2 &= 150 - p_1 - 2p_2 \end{aligned}$$

Find the price the company should charge for each product in order to maximize total revenue. What is that maximum revenue?

**Solution:** Revenue is still price × quantity. If we're selling two products, the total revenue will be the sum of the revenues from the two products:

$$p_1q_1 + p_2q_2 = p_1(200-3p_1-p_2)+p_2(150-p_1-2p_2)$$
$$R(p_1, p_2) = 200p_1 - 3p_1^{\ 2} - 2p_1p_2 + 150p_2 - 2p_2^{\ 2}$$

This is a function of two variables, the two prices, and we need to optimize it – just as in the previous examples.

Find critical points (now the notation here gets a bit hard to look at, but hang in there – this is the same stuff we've done before): $R_{p_1} = 200 - 6p_1 - 2p_2$ and $R_{p_2} = 150 - 2p_1 - 4p_2$

Solving these simultaneously gives the one critical point $(p_1, p_2) = (25,25)$.

To confirm that this gives maximum revenue, we need to use the Second Derivative Test. Find all the second derivatives:

$R_{p_1p_1} = -6$ , $R_{p_2p_2} = -4$ , and $R_{p_1p_2} = -2 = R_{p_2p_{12}}$

So $D = (-6)(-4)-(-2)(-2) > 0$ and $R_{p_1p_1} < 0$, so this really is a local maximum.

To maximize revenue, the company should charge \$25 per unit for both products. This will yield a maximum revenue of \$4375.

## Chapter 4 Exercises

**1.** $F(x,y)=x^2-y^2$. Find

a) $F(0,4)$ b) $F(4,0)$ c) $F(x,4)$ d) $F(4,y)$

e) $F(800,800)$ f) $F(x,x)$ g) $F(x,-x)$

**2.** $g(s,t)=\sqrt{st^2}$. Find

a) $g(1,9)$ b) $g(9,1)$ c) $g(1,t)$ d) $g(s,9)$ e) $g(w,z+1)$

**3.** Let $f(x,y,z,w)=x^2-\frac{1}{zw}+xyz^2$. Evaluate $f(1,2,3,4)$.

**4.** Let $f(x,y,z,w)=\sqrt{xy}-w^2+102yz$. Evaluate $f(1,2,3,4)$.

**5.** Here is a table showing the function $A(t,r)$

| t ↓ r → | .03 | .04 | .05 | .06 | .07 |
|---|---|---|---|---|---|
| 1 | 30.45 | 40.81 | 51.27 | 61.84 | 72.51 |
| 2 | 61.84 | 83.29 | 105.17 | 127.50 | 150.27 |
| 3 | 94.17 | 127.50 | 161.83 | 197.22 | 233.68 |

a) Find $A(2,.05)$

b) Find $A(.05,.2)$

c) Is $A(t,.06)$ an increasing or decreasing function of *t* ?

d) Is $A(3,r)$ an increasing or decreasing function of *r*?

**6.** Here is a table showing values for the function $H(t,h)$.

| t ↓    h → | 100 | 150 | 200 |
|---|---|---|---|
| 0 | 100 | 150 | 200 |
| 1 | 110.1 | 160.1 | 210.1 |
| 2 | 110.4 | 160.4 | 210.4 |
| 3 | 100.9 | 150.9 | 200.9 |
| 4 | 81.6 | 131.6 | 181.6 |
| 5 | 52.5 | 102.5 | 152.5 |

a) Is $H(t,150)$ an increasing or decreasing function of *t* ?

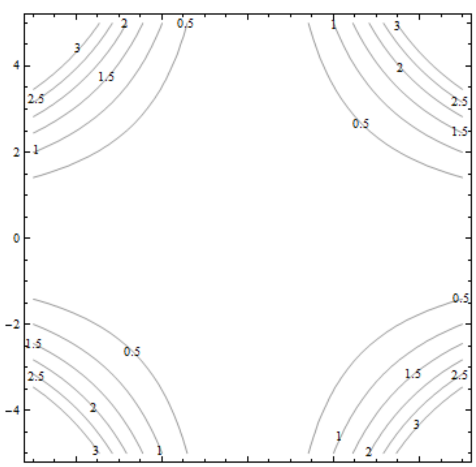
b) Is $H(4,h)$ an increasing or decreasing function of *h*?

c) Fill in the blanks: The maximum value shown on this table is $H(\_\_\_,\_\_\_)=\_\_\_\_$ .

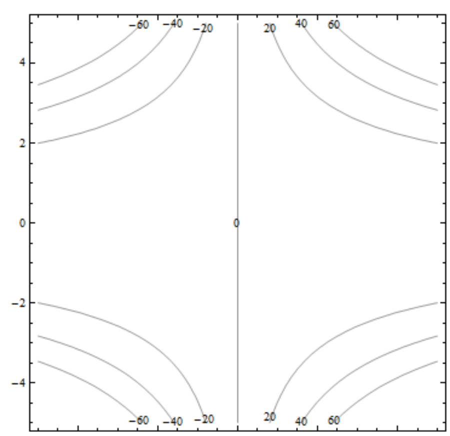
d) Fill in the blanks: The minimum value shown on this table is $H(\_\_\_,\_\_\_)=\_\_\_\_$ .

For problems 7 through 12. Match the contour diagram to the computer-generated, perspective drawing (a through f) it matches. Briefly explain your answer.

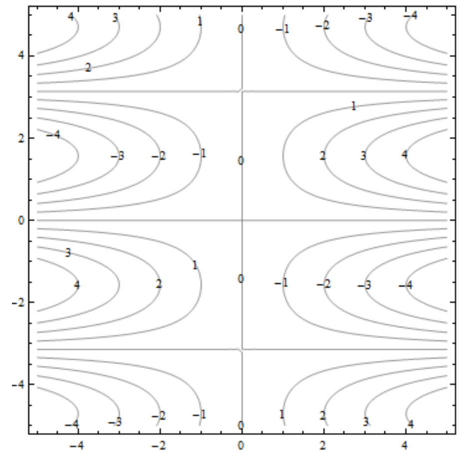
**7.**



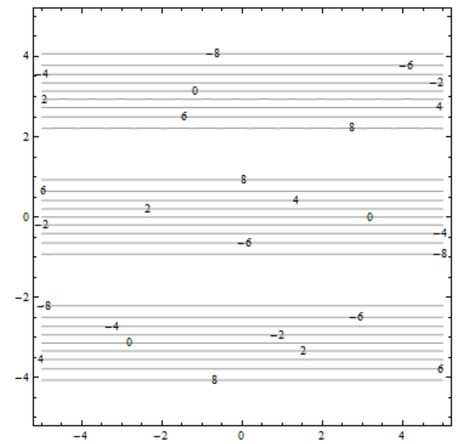
**8.**



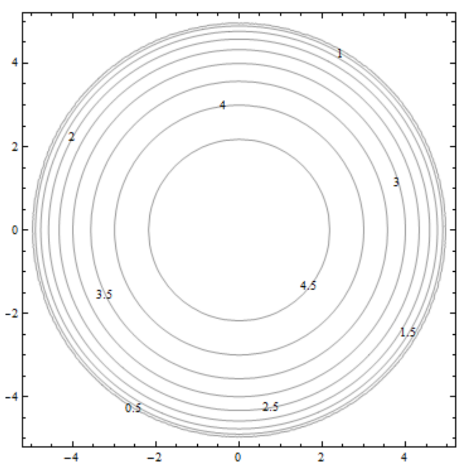
**9.**



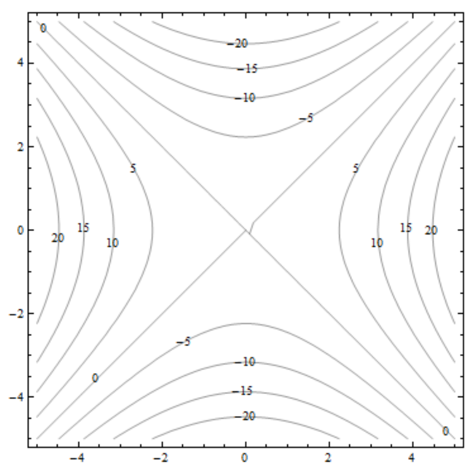
**10.**



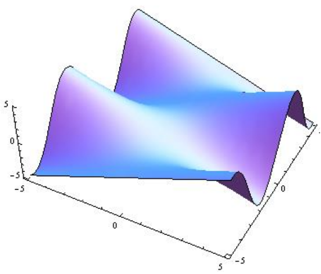
**11**.



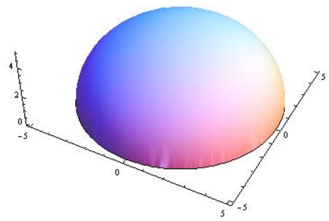
**12**.



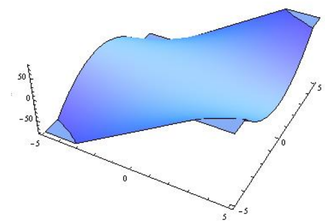
**a.**



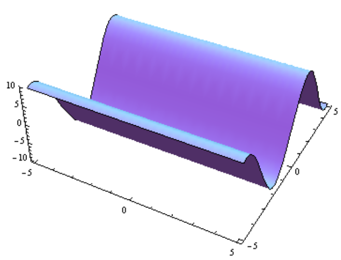
**b.**



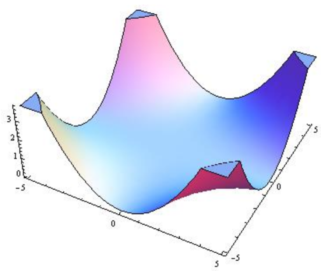
**c.**



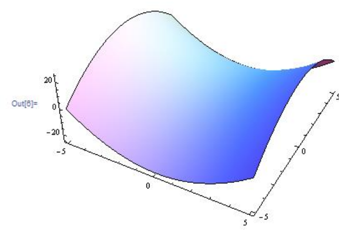
**d.**



**e.**

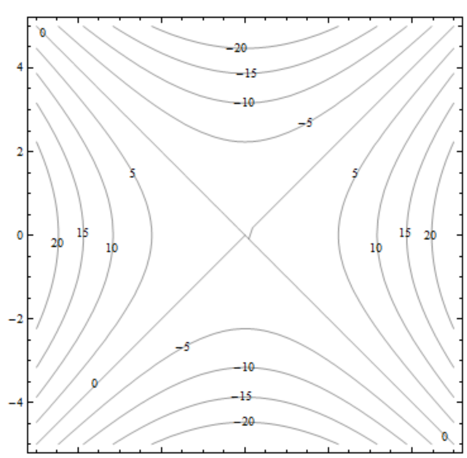


**f.**

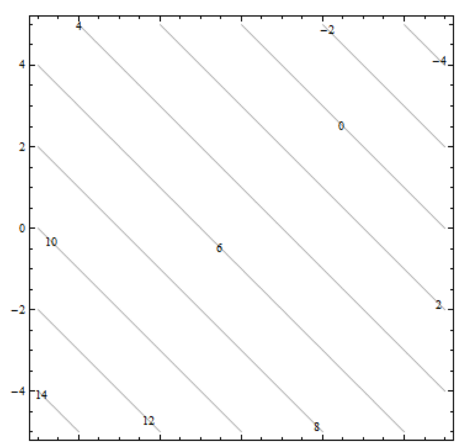


For problems 13 through 18. Match the contour diagram to the equation (a through f) it matches. Briefly explain your answer.

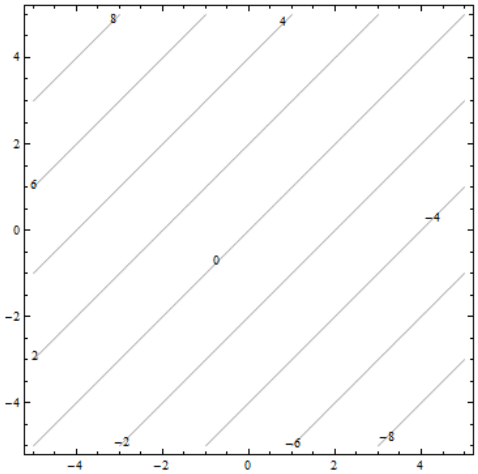
**13.**



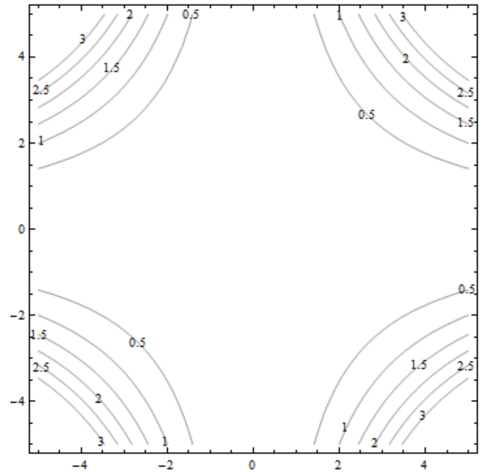
**14.**



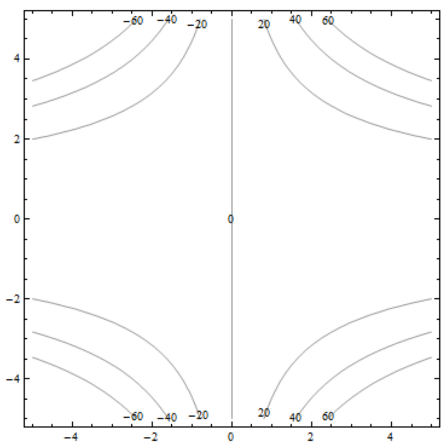
**15.**



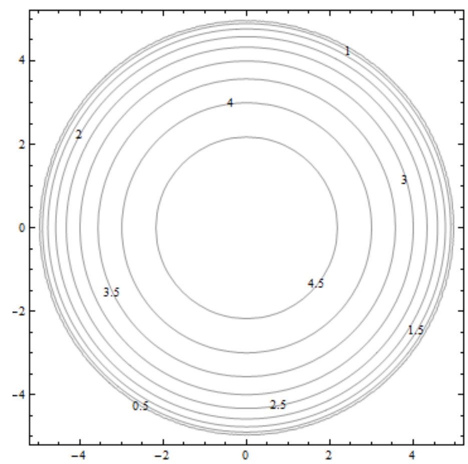

**16.**




**17.**




**18.**




a. $f(x,y)=y-x$

b. $f(x,y)=xy^2$

c. $f(x,y)=\sqrt{25-x^2-y^2}$

d. $f(x,y)=5-x-y$

e. $f(x,y)=0.01x^2y^2$

f. $f(x,y)=x^2-y^2$

**19.** The contour diagram shown is for a function M(x, y). Use the diagram to answer the following:

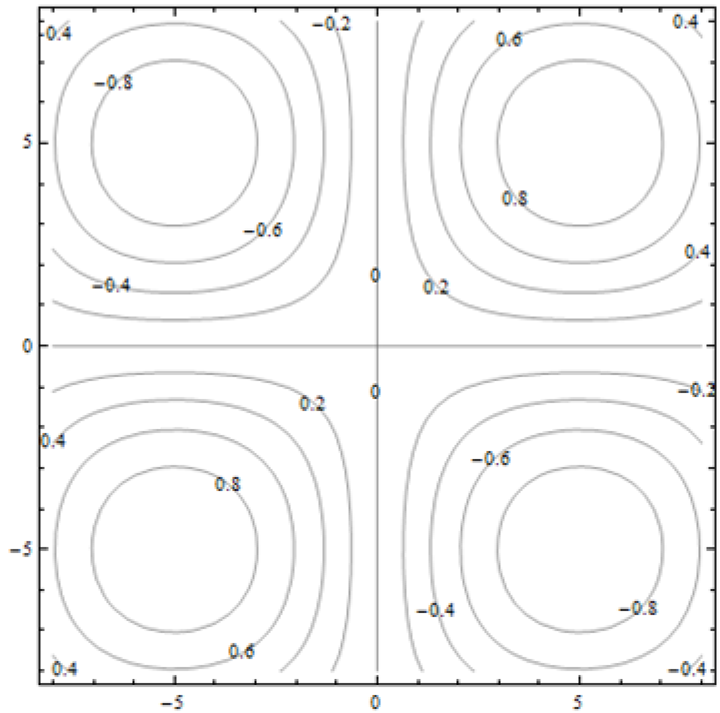



Figure 147

a) Estimate M(1, 3)

b) Estimate M(3, 1)

c) Is M(x, 3) an increasing or decreasing function of x?

d) Is M(3, y) an increasing or decreasing function of y?

e) Find a value of c so that M(c, y) is a constant function of y.

**20.** The contour diagram shown is for a function G(x, y). Use the diagram to answer the following:

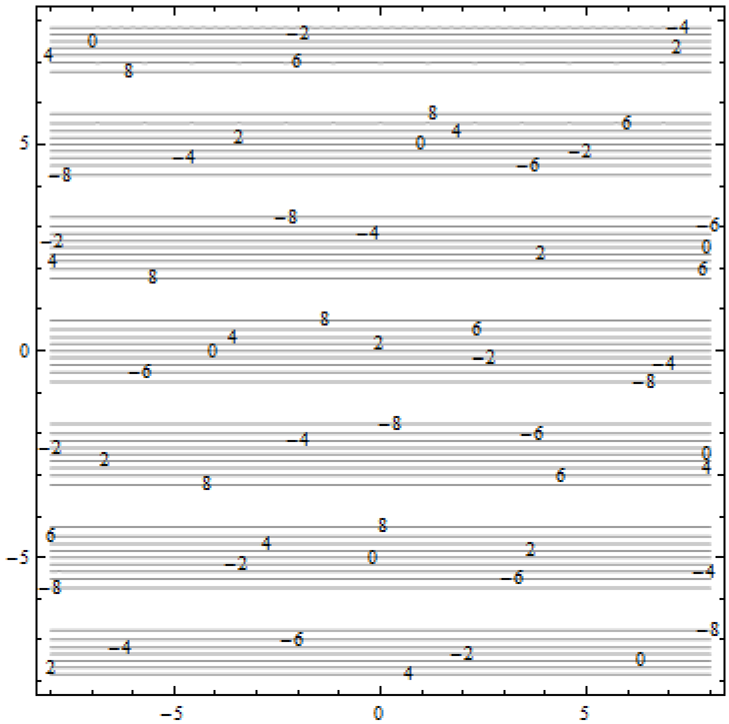


**Figure 148**

a) Estimate G(2, 3)

b) Suppose you travel north (in the direction of increasing y) along the surface, starting above (2, 3). Describe your journey.

c) Suppose you travel east (in the direction of increasing x) along the surface, starting above (2, 3). Describe your journey.

For problems 21 through 35, find $f_x$ and $f_y$ for the function given

**21.** $f(x,y)=x^2-5y^2$

**22.** $f(x,y)=\dfrac{x^2-5y^2}{x+4}$

**23.** $f(x,y)=e^{x+6y}$

**24.** $f(x,y)=(x^2-5y^2)e^x$

**25.** $f(x,y)=(x^2-5y^2)\left(\dfrac{1}{3y}+4\right)$

**26.** $f(x,y)=x$

**27.** $f(x,y)=6$

**28.** $f(x,y)=\ln(xy+2x-6y)$

**29.** $f(x,y)=\dfrac{x^2-5y^2}{y^4-5x^4}$

**30.** $f(x,y)=e^{\sqrt{x-4y}}(x-4y)$

**31.** $f(x,y)=y^5e^x$

**32.** $f(x,y)=\dfrac{1}{16xy}$

**33.** $f(x,y)=\left(x+e^y\right)^7$

**34.** $f(x,y)=x^4+4x^3y-6x^2y^2-4xy^3+y^4$

**35.** $f(x,y)=\sqrt{x+\sqrt{y}}$

For problems 36 through 46, find $f_{xx}$, $f_{yy}$, $f_{xy}$ and $f_{yx}$ for the function given. Confirm that $f_{xy}=f_{yx}$.

**36.** $f(x,y)=x^2-5y^2$

**37**. $f(x,y)=x^4+4x^3y-6x^2y^2-4xy^3+y^4$

**38**. $f(x,y)=5x^2y^2$

**39.** $f(x,y)=e^{x+6y}$

**40.** $f(x,y)=\ln(xy+2x-6y)$

**41.** $f(x,y)=\dfrac{x^2}{y^4-5}$

**42.** Given the function $g(x,y)=\left(x^4-e^x\right)\left(x-\dfrac{1}{x}\right)y^2$, find $g_{xxyyy}$. Hint – think about which order to find these partial derivatives in; is there a way to save yourself some work?

**43.** Here is a table showing the function $A(t,r)$

| t ↓ r → | .03 | .04 | .05 | .06 | .07 |
|---|---|---|---|---|---|
| 1 | 30.45 | 40.81 | 51.27 | 61.84 | 72.51 |
| 2 | 61.84 | 83.29 | 105.17 | 127.50 | 150.27 |
| 3 | 94.17 | 127.50 | 161.83 | 197.22 | 233.68 |

a. Estimate $A_t(2,.05)$.

b. Estimate $A_r(2,.05)$

c. Use your answers to parts a and b to estimate the value of $A(2.5,.054)$

d. The values in the table came from $A(t,r)=1000\left(e^{rt}-1\right)$, which shows the interest earned if 1000 dollars is deposited in an account earning *r* annual interest, compounded continuously, and left there for *t* years. How close are your estimates from parts a, b, and c?

**44**. Here is a table showing values for the function $H(t,h)$.

| t ↓ / h → | 100 | 150 | 200 |
|---|---|---|---|
| 0 | 100 | 150 | 200 |
| 1 | 110.1 | 160.1 | 210.1 |
| 2 | 110.4 | 160.4 | 210.4 |
| 3 | 100.9 | 150.9 | 200.9 |
| 4 | 81.6 | 131.6 | 181.6 |
| 5 | 52.5 | 102.5 | 152.5 |

a. Estimate the value of $\dfrac{\partial H}{dt}$ at (3, 150).

b. Estimate the value of $\dfrac{\partial H}{dh}$ at (3, 150).

c. Use your answers to parts a and b to estimate the value of $H(2.6,156)$.

d. The values in the table came from $H(t,h)=h+15t-4.9t^2$, which gives the height in meters above the ground after *t* seconds of an object that is thrown upward from an initial height of *h* meters with an initial velocity of 15 meters per second. How close are your estimates from parts a, b, and c?

**45**. Find the critical points of $f(x,y)=y^3-x^3+15x^2-12y+12$ and use the Second Derivative Test to classify them. If the test fails, say “the test fails.”

**46.** Find the critical points of $f(x,y)=2xy-x^2-2y^2+6x+4$ and use the Second Derivative Test to classify them. If the test fails, say “the test fails.”

**47.** Find the critical points of $f(x,y)=y^2-4\ln(x)+4x$ and use the Second Derivative Test to classify them. If the test fails, say “the test fails.”

**48**. Find the critical points of $f(x,y)=xy-6x^2+3x-y+2$ and use the Second Derivative Test to classify them. If the test fails, say “the test fails.”

**49.** The origin is a critical point for the function $f(x,y)=x^3+y^3$, and D = 0 there. That is, the Second Derivative Test fails. Use what you know about shapes of functions to decide if there is a local minimum, local maximum, or saddle point for this function at (0, 0).

**50.** The origin is a critical point for the function $f(x,y)=15-x^2y^2$, and D = 0 there. That is, the Second Derivative Test fails. Use what you know about shapes of functions to decide if there is a local minimum, local maximum, or saddle point for this function at (0, 0).

For problems 51 through 56, find all local maxima, minima, and saddle points for the function.

**51.** $f(x,y)=xy-5x^2-5y^2+33y$

**52.** $f(x,y)=10xy-x^2-y^2+3x$

**53.** $f(x,y)=x^3+y^3-3xy$

**54.** $f(x,y)=5x^2-4xy+2y^2+4x-4y+10$

**55.** $f(x,y)=y^2e^x+x^2$

**56.** $f(x,y)=xy+2x-\ln\left(x^2y\right)$, for $x>0$ and $y>0$.

**57**. The demand functions for two products are given below. $p_1$, $p_2$, $q_1$,and $q_2$ are the prices (in dollars) and quantities for products 1 and 2.

$$\begin{aligned} q_1 &= 200 - 3p_1 + p_2 \\ q_2 &= 150 + p_1 - 2p_2 \end{aligned}$$

a. Are these two products complementary goods or substitute goods?

b. What is the quantity demanded for each when the price for product 1 is $20 per item and the price for product 2 is $30 per item?

c. Write a function $R(p_1, p_2)$ that expresses the total revenue from these two products.

d. Find the price and quantity for each product that maximizes the total revenue.

**58.** Suppose the demand functions for two products are $q_1 = f(p_1, p_2)$ and $q_2 = g(p_1, p_2)$, where $p_1$, $p_2$, $q_1$, and $q_2$ are the prices (in dollars) and quantities for products 1 and 2. Consider the four partial derivatives $\dfrac{\partial q_1}{\partial p_1}, \dfrac{\partial q_1}{\partial p_2}, \dfrac{\partial q_2}{\partial p_1}$, and $\dfrac{\partial q_2}{\partial p_2}$. Tell the sign of each of these partial derivatives if

a. the products are complementary goods.

b. the products are substitute goods.

# Brief Answers to Selected Exercises

## Chapter 1

1. a. $\dfrac{4y^3}{12y^7} = \dfrac{1}{3y^4}$ b. $\left(2xy^{-3}z^0\right)^3 = \dfrac{8x^3}{y^9}$ f. $\sqrt[4]{\left(4ab^2\right)^4\left(b^{-3}\right)^{-2}} = 4ab^{7/2}$

3. $y = 3x + 2$

4. $C(n) = 5n + 350$, where *C* is the cost in dollars and *n* is the number of pounds of coffee produced.

5. $y^3(y-3)^2$

7. a. $g(x) = \dfrac{31}{8}x - \dfrac{49}{4}$. b. $g(x) = 5 \cdot 2^x$.

9. $u(x) = x + 1;\ v(x) = \dfrac{1}{x^2}$

11. The graph is the same basic shape, but is upside down, 3 times as tall, and shifted 2 units to the right. The four labeled points are now (-1, -6), (0, 3), (5, -12), and (9, 0).

13. a. This is exponential decay, so an exponential function with 0 < *b* < 1.
b. (You have to estimate here. Your answer might be different from mine.) It looks to me as if when *x* = -1, *y* is about 2.8, so I will guess that this is $y = \left(\dfrac{1}{e}\right)^x = e^x$.

## Chapter 2

**1.** m = $\dfrac{y-9}{x-3}$. If x = 2.97, then m = $\dfrac{-0.1791}{-0.03}$ = 5.97 . If x = 3.001, then m = $\dfrac{0.00601}{0.001}$ = 6.001.

If x = 3 + h, then m = $\dfrac{(3+h)^2 - 9}{(3+h) - 3} = \dfrac{9 + 6h + h^2 - 9}{h}$ = 6 + h. When h is very small (close to 0), 6 + h is very close to 6.

**3.** All of these answers are **approximate**. Your answers should be close to these numbers.

(a) average rate of temperature change ≈ $\dfrac{80^o - 64^o}{1 \text{ pm} - 9 \text{ am}} = \dfrac{16^o}{4 \text{ hours}} = 4^o$ per hour.

(b) Use the slope of the tangent line. At 10 am, temperature was rising about $5^o$ per hour.

At 7 pm, temperature was rising about **–**$10^o$ per hour (**falling** about $10^o$ per hour).

**5.** All of these answers are **approximate**. Your answers should be close to these numbers.

(a) average velocity ≈ $\dfrac{300 \text{ ft} - 0 \text{ ft}}{20 \text{ sec} - 0 \text{ sec}}$ = 15 feet per second.

(b) average velocity ≈ $\dfrac{100 \text{ ft} - 200 \text{ ft}}{30 \text{ sec} - 10 \text{ sec}}$ = –5 feet per second.

(c) at t = 10 seconds, velocity ≈ 30 feet per second (between 20 and 35 ft/s).

at t = 20 seconds, velocity ≈ –1 feet per second.

at t = 30 seconds, velocity ≈ –40 feet per second.

**7.** Fig. 1 is a graph of m(x). Approximate values of m(x) are in the table.

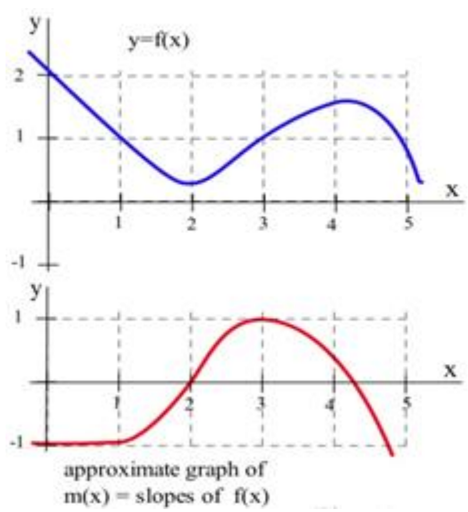



Figure 149

| *x* | *f*(*x*) | *m*(*x*)= the estimated slope of the tangent line to *y* = *f*(*x*) at the point (*x, f*(*x*)) |
|---|---|---|
| 0 | 2 | −1 |
| 1 | 1 | −1 |
| 2 | 1/3 | 0 |
| 3 | 1 | 1 |
| 4 | 3/2 | ½ |
| 5 | 1 | −2 |

**9:** a. The graph of g has a horizontal tangent line at about x = 1, x = 2, and x = 3.

b. g is largest when x = 2 and for x = -0.5 and x = 6 (the edges of the graph). g is smallest when x = 1 and x = 3.

c. The slope of g is the largest when x = about 1.5, when the graph of g is steepest while increasing. The slope of g is smallest when x = about 2.5, when the graph of g is steepest while decreasing.

**11.** Fig. 2 is a graph of the approximate **rate** of temperature change (slope).

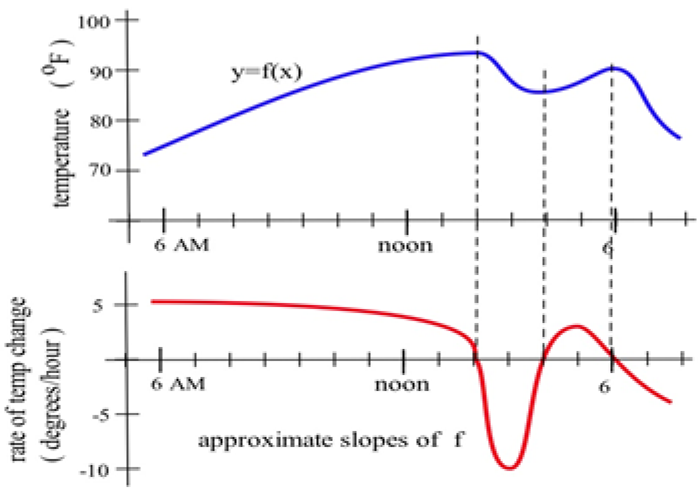



Figure 150

**13.** The left derivative graph is g'. The middle derivative graph is h'. The right derivative graph is f'.

**15:**

| $x$ | f(x) | f'(x) | g(x) | g'(x) | $\frac{d}{dx}(f(x)\cdot g(x))$ | $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$ | $\frac{d}{dx}\left(\frac{g(x)}{f(x)}\right)$ |
|---|---|---|---|---|---|---|---|
| 0 | 3 | −2 | −4 | 3 | ( −2)( −4) + (3)(3) = 17 | $\frac{(-2)(-4)-(3)(3)}{(-4)^2}$ $=\frac{17}{16}$ | $\frac{(3)(3)-(-2)(-4)}{(3)^2}$ $=\frac{17}{9}$ |
| 1 | 2 | −1 | 1 | 0 | −1 | −1 | 1/4 |
| 2 | 4 | 2 | 3 | 1 | 10 | 2/9 | −1/8 |

**17.** (a) $\frac{d}{dx}((x-5)(3x+7)) = (1)(3x+7) + (x-5)(3) = 6x-8.$

(b) $\frac{d}{dx}((x-5)(3x+7)) = \frac{d}{dx}(3x^2-8x-35) = 6x-8.$

**19.** If $\frac{f(x)}{g(x)} = k$, then $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = 0$. So the numerator of the rational function we'd get from the quotient rule must be zero; that is, $f'g - fg' = 0$. So $f'g$ and $fg'$ are equal. (Another way to think about this – if $\frac{f(x)}{g(x)} = k$, then $f(x) = kg(x)$ and $f'(x) = kg'(x)$. So $f'g = kg'g = kgg' = fg'$.)

**21.** $f(x) = x^2 - 5x + 13$, so $f'(x) = 2x - 5$. $f'(1) = 2 - 5 = -3$, and $f(x) = 2x - 5 = 0$ when $x = 2.5$.

**23.** $f(x) = x^3 + 3x^2 + 3x - 1$, so $f'(x) = 3x^2 + 6x + 3$. $f'(1) = 12$, and $f(x) = 0$ when $x = -1$.

**25.** $f(x) = \frac{7x}{x^2+4}$, so $f'(x) = \frac{(7)(x^2+4)-(7x)(2x)}{(x^2+4)^2} = \frac{-7x+28}{(x^2+4)^2}$. $f'(1) = \frac{21}{25}$, and $f(x) = 0$ when $x = 4$.

**27.** (a) $\frac{d}{dx}(x^3e^x) = (3x^2)(e^x) + (x^3)(e^x)$

(b) $\frac{d}{dx}(e^x)^3 = 3(e^x)^2(e^x) = 3e^{3x}$. Alternatively, $\frac{d}{dx}(e^x)^3 = \frac{d}{dx}(e^{3x}) = 3e^{3x}$.

**29**. $f(x) = x^2 - 10x + 3$, so $f'(x) = 2x - 10$. *f* has horizontal tangents where *f* ' = 0, so *f* has a horizontal tangent at *x* = 5. $g(x) = x^3 - 12x$, so $g'(x) = 3x^2 - 12$. *g* has horizontal tangents at *x* = −2 and at *x* = 2.

**31**. (a) h'(x) = −32x + 128. When x = 0, the velocity is 128 ft/sec (of course), when x = 1, the velocity is 96 ft/sec, and when x = 2, the velocity is 64 ft/sec.

(b) v(x) = h'(x) = −32x + 128.

(c) The velocity is zero when x = 4; the velocity is zero 4 seconds after the arrow is shot.

(d) The greatest height is the vertex of the parabola (also where velocity = 0), when x = 4 seconds. After 4 seconds, the arrow is 256 feet high.

(e) The arrow will be aloft until it hits the ground, until its height is zero. Solving h(x) = 0 gives x = 0 (when the arrow is shot) and x = 8 (when the arrow comes back to ground). The arrow is aloft for 8 seconds.

(f) a(x) = v'(x) = h''(x) = -32. The acceleration is a constant 32 ft/sec$^2$, the acceleration due to gravity.

**33**. $\frac{d}{dx}\left(\left(x^2+2x+2\right)^{12}\right)=12\left(x^2+2x+2\right)^{11}(2x+2)$; at *a* = 0, $y=2^{12}=4096$ and $y'=12\left(2^{11}\right)(2)=49152$. The equation of the tangent line is $y-4096=49152(x-0)$, or $y=49152x+4096.$

**35**. $\frac{d}{dx}\left(x+\frac{1}{x}\right)^2=2\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)$; at *a* =0.5, $y=6.25$ and $y'=-15$. The equation of the tangent line is $y-6.25=-15(x-0.5)$.

**37**. $\frac{d}{dx}\left(e^x+e^{-x}\right)=e^z+(-1)e^{-x}=e^x-e^{-x}$; at *a* = 0, $y=2$ and $y'=0$. The equation of the tangent line is $y=2$.

**39.** (a) A beginning employee has d = 0 days of production experience, so his production will be about $P(0)=3+15\left(1-e^{-0.2d}\right)=3$ items per day.

(b) As d → ∞, the exponential part of this function will → 0; an experienced employee will be approaching 18 items per day.

(c) Marginal production is the derivative, $P'(d) = 15(0.2e^{-0.2d})$. When $d$ = 5, $P$(5) = about 1.1 items per day per day. That is, when the employee has 5 days of experience, her production is increasing by a little more than one item per day each day.

**41.** $y' = 3Ax^2$

**43.** $y = \sqrt{A + Bx^2} = (A + Bx^2)^{1/2}$, so $\frac{dy}{dx} = \frac{1}{2}(A + Bx^2)^{-1/2}(2Bx) = \frac{Bx}{\sqrt{A + Bx^2}}$

**45.** $y' = ABe^{Bx}$

**47.** $\frac{dy}{dx} = Ae^{Ax} - Ae^{-Ax}$

**49.** $\frac{dy}{dx} = \frac{(A)(Cx + D) - (C)(Ax + B)}{(Cx + D)^2} = \frac{AD - CB}{(Cx + D)^2}$

**51.** $y = \frac{1}{\ln(Ax + B)} = (\ln(Ax + B))^{-1}$, so

$$y' = -1(\ln(Ax + B))^{-2}\left(\frac{1}{Ax + B}\right)(A) = \frac{-A}{(Ax + B)(\ln(Ax + B))^2}$$

**53.** $\frac{dy}{dx} = \frac{1}{(1/(Ax + B))} \cdot \frac{-1}{(Ax + B)^2} \cdot A = -\frac{A}{Ax + B}$. Alternatively, you can write $y = \ln\left(\frac{1}{Ax + B}\right) = \ln(Ax + B)^{-1} = -\ln(Ax + B)$; then $\frac{dy}{dx} = -\frac{1}{Ax + B} \cdot A = -\frac{A}{Ax + B}$.

**55**. (a) $y' = AB - 2Ax$, (b) $y' = 0$ when $x = \frac{B}{2}$ (provided $A \neq 0$), (c) $y'' = -2A$.

**57.** (a) The quantity that is changing is “unemployment.” The units of unemployment are not stated. Let f(t) be the unemployment after t months. The statement says that f’ is positive, but f’’ is negative.

(b) The quantity is profits. Let f(t) be profit in dollars after t months. The statement says f’ is negative, but f’’ is positive (f’ is getting less negative, so f’ is increasing).

(c) The quantity is population. Let f(t) be the population in millions of people after t years. The statement says f’ is positive, and f’’ is positive.

**59.** The function is concave down on (0, 3), concave up on (3, 7).

**61.** Answers will vary. Here are some examples:

(a)

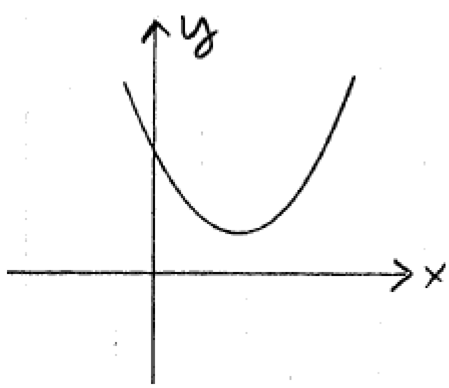



Figure 151

(b)

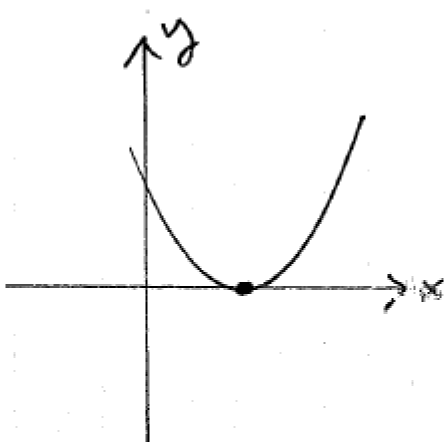



Figure 152

(c)

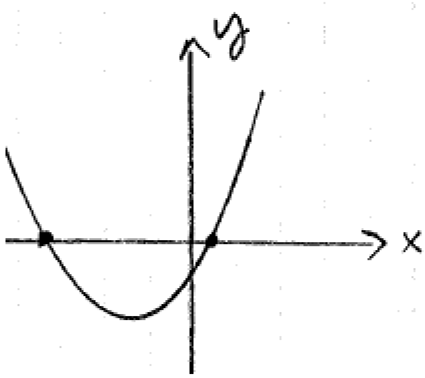



Figure 153

(d) This can't be done if the function really is concave up everywhere. Here is an example where the function is concave up everywhere except one point (the cusp).

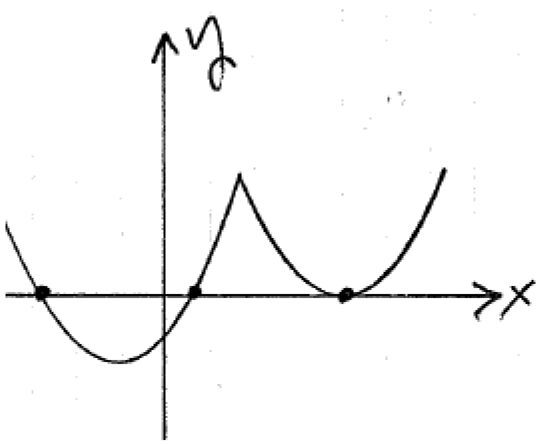



Figure 154

**63.** $g(x) = x^3 - 3x^2 - 9x + 7$; $g'(x) = 3x^2 - 6x - 9$; $g''(x) = 6x - 6$.

At *x* = −1: $g''(-1) = -12 < 0$; there is a local maximum at $(-1, 12)$.

At *x* = 3: $g''(3) = 12 > 0$; there is a local minimum at $(3, -20)$.

**65.** $f(x)=x\ln(x)$; $f\,'(x)=(1)(\ln x)+(x)\left(\frac{1}{x}\right)=\ln x+1$; $f\,''(x)=\frac{1}{x}$.

$f\,''\left(\frac{1}{e}\right)=e>0$ ; there is a local minimum at $\left(\frac{1}{e},-\frac{1}{e}\right)$.

**67.** The function seems to have points of inflection at b, d, and h. Note that the cusp f is not an inflection point, since the function is concave up on both sides.

**69.**

| $x$ | $f(x)$ | $f\,'(x)$ | $f\,''(x)$ |
|---|---|---|---|
| 0 | + | + | − |
| 1 | + | − | + |
| 2 | + | 0 | + |
| 3 | + | + | 0 |

**71.** $f\,'(x)=14x+5$; $f\,'(x)=14$.

**73.** $f\,'(x)=10(6x-x^2)^9(6-2x)$;

$f\,''(x)=10\left(9(6x-x^2)^8(6-2x)+(6x-x^2)^9(-2)\right)=20(6x-x^2)^8(x^2-15x+27)$. (I'm simplifying so you can recognize your answer if you did this problem a different way. )

**75.** $f\,'(x)=6(2x^3+3)^5(6x^2)$;

$f\,''(x)=6\left(5(2x^3+3)^4(6x^2)+(2x^3+3)^5(12x)\right)=6x(2x+3)^4(24x^3+30x+36)$.

**77.** $f\,'(x)=\frac{1}{x^2+4}\cdot 2x=\frac{2x}{x^2+4}$; $f\,''(x)=\frac{(2)(x^2+4)-(2x)(2x)}{(x^2+4)^2}=\frac{8-2x^2}{(x^2+4)}$.

**79.** First, find critical points: $f\,'(x)=6x-12=0$ when *x* = 2; this is the only critical point. As we move from left to right near *x* = 2, *f* ʻ goes from negative to positive; this point is a local minimum. The function has a local minimum at the point (2, −5).

**81.** $f\ '(x) = 4x - 12$; *f* has one critical point, where *x* = 3. At *x* = 3, *f ′* changes from negative to positive, so this critical point is a local minimum. The only local extremum of *f* is a local minimum at (3, −11).

**83.** $f\ '(x) = 2(x-1)(x-3) + (x-1)^2 = 3x^2 - 10x + 7$; $f\ '(x) = 0$ when $x = \frac{7}{3}$ or $x = 1$.

$f\ ''(x) = 6x - 10$;

At $x = \frac{7}{3}$: $f\ ''(x) = 4 > 0$; there is a local minimum at $\left(\frac{7}{3}, -\frac{32}{27}\right)$.

At $x = 1$: $f\ ''(x) = -4 < 0$; there is a local maximum at (1, 0).

**85.** $f\ '(x) = 6x^2 - 96 = 6(x^2 - 16)$; there are critical points at *x* = 4 and *x* = −4. $f\ ''(x) = 12x$.

At *x* = 4: $f\ ''(x) > 0$; there is a local minimum at (4, −214).

At *x* = −4: $f\ ''(x) < 0$; there is a local maximum at (−4, 298).

**87.** $f\ '(x) = -3x^2$, so the only critical point is (0, 2). The second derivative test fails here, but the first derivative test works – the first derivative is negative on both sides of the critical point, so this point is neither a maximum nor a minimum. This function has neither a global minimum nor a global maximum on the entire real number line.

**89.** $f\ '(x) = 1 - e^x$; $f\ ''(x) = -e^x$. The only critical point is (0, −1). The second derivative test tells us that this point is a local maximum. Since this is the only critical point, it is the global maximum and there is no global minimum on the entire real number line.

**91.** The only critical point is (0, 2), which we have seen is neither a min nor a max. So turn to the endpoints: $f(-2) = 10$ and $f(1) = 1$. On the interval [ −2, 1], *f* has a global max at *x* = −2 and a global min at *x* = 1.

**93.** The only critical point is (0, −1), but that point does not lie in the interval [1, 2]. So turn to the endpoints: $f(1)=1-e \cong -1.718$ and $f(2)=2-e^2 \cong -5.389$. On the interval [1, 2], *f* has a global max at *x* = 1 and *x* = 2.

**95.** The critical points are at *x* = 2, 3, 5, 8.2, and 11. There is a local min at *x* = 2 and *x* = 8.2, local max at *x* = 5 and *x* = 11; the critical point at *x* = 3 is neither a local max nor min. The global max occurs at *x* = 0 and *x* = 5; the global min is at *x* = 8.2.

**97.** The derivative of a quadratic is linear, the second derivative is constant, the third derivative (and all higher derivatives) are zero.

**99.** Each time you take the derivative of a polynomial, its degree decreases by one. If you take *n* derivatives of a polynomial of degree *n*, it has degree 0; it is constant. From that point on, each further derivative will be zero.

**101.** As *x* increases, the area increases. So (a) A(x) is minimum when *x* = 0, and (b) A(x) is maximum when *x* = 10.

**103. (a)** Let *x* be the width of the garden. Then the height of the garden is $\frac{1}{2}(200-2x)=100-x$. See Fig. 7. The area of the garden is $A(x)=x(100-x)=100x-x^2$; this is what we want to maximize. Whack it with calculus: $A'(x)=100-2x=0$ when *x* = 50; $A''(x)=-2<0$, so this is a maximum. It's the only critical point, so it is the global maximum. The area is maximized when *x* = 50 feet and *y* = 50 feet, or when the garden is square.

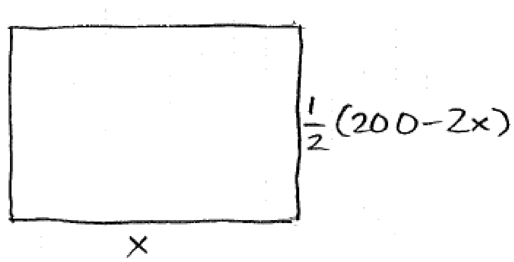



Figure 155

**(c)** Let *x* be the width of the garden. Then the height of the garden is $\frac{1}{2}(P-x)$. See Fig. 8. The area of the garden is $A(x)=\frac{1}{2}(Px-x^2)$; this is what we want to maximize. Whack it with calculus: $A'(x)=\frac{1}{2}(P-2x)=0$ when $x=\frac{P}{2}$. $A''(x)=-1<0$, so this is a local maximum. It's the only critical point, so it is the global maximum. The area is maximized when the dimensions are $\frac{P}{2}$ feet × $\frac{P}{4}$ feet.

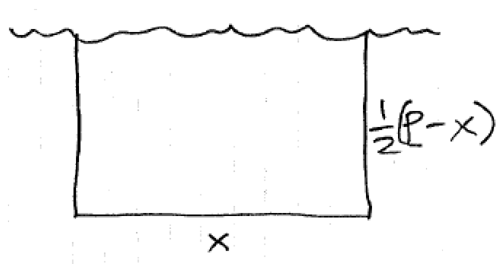



Figure 156

**105.** Let *x* be the height of the pen. There are 5 edges of length *x*. The width of the pen is $\frac{1}{2}(120-5x)=60-2.5x$. The area of the pen, which is what we want to maximize, is $A(x)=60x-2.5x^2$. $A'(x)=60-5x=0$ when *x* = 12. $A''(x)=-5<0$, so this is a local maximum. It's the only critical point, so it is the global maximum. The area is maximized when the outside dimensions are 12 feet × 30 feet; the maximum area is 360 square feet.

**107. (b)** The box is still square-bottomed, open top, and we still assume that no material is wasted. Let *x* be the length of one of the sides of the square base, and let *h* be the height of the box. See Fig. 9.

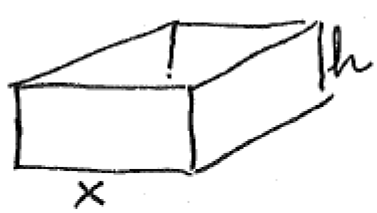



Figure 157

We know the volume is 100 cubic inches, so $100 = x^2 h$, or $h = \frac{100}{x^2}$. We're trying to minimize the cost here. The cost of the bottom is $.05x^2$ dollars, each of the 4 walls costs $.03xh = .03x\frac{100}{x^2} = \frac{3}{x}$ dollars. So the total cost of the box is $C(x) = .05x^2 + \frac{12}{x}$; this is the function we need to minimize. Whack it with calculus: $C'(x) = .1x - \frac{12}{x^2}$. *C* has critical points when *x* = 0 (there is no box, we can eliminate this one) or when $x = \sqrt[3]{120} \cong 4.93$. $C''(x) = .1 + \frac{24}{x^3} > 0$ for any positive *x*, so this critical point is a minimum. Cost is minimized when the base of the box is about 4.93" × 4.93" and the height is about 4.11".

**109. (b)** Let *x* be the shorter dimension of the rectangular end; then the longer dimension is 2*x*, the girth is *6x*, and the length of the box is $108 - 6x$. The volume, which is what we're trying to maximize, is $V(x) = 2x^2(108 - 6x) = 216x^2 - 12x^3$. Whack with calculus: $V'(x) = 432x - 36x^2 = 0$ when *x* = 0 (no box, we can ignore this one) and when *x* = 12. $V''(12) = 432 - 72(12) < 0$, so this is a maximum. The volume is maximized when the base is 12" × 24" × 36".

**111.** Profit is revenue minus costs. Let *q* be the number of passengers. Note the domain restrictions:

10 ≤ q ≤ 20. The total cost is $C(q) = 100 + 6q$ dollars. The revenue is $R(q) = (30 - (q - 10))q = 40q - q^2$ dollars. So the profit is $\pi(q) = R(q) - C(q) = -q^2 + 34q - 100$ dollars; this is the function we want to maximize. Calculus to the rescue: $\pi'(q) = -2q + 34 = 0$ when *q* = 17. At *q* = 17, π' changes from positive to negative, so this is a local maximum. This is the only critical point, so this is the global maximum. Profit is maximized if we carry 17 passengers.

**113.** First, note that if we have fewer than 80 people, adding another seat will definitely increase our profit. So any maximum must occur for a larger coffee shop. If we let *q* be the number of seats in our coffee shop, we can restrict the domain to 80 ≤ *q*. (That's good, because the entire profit function is a piecewise function, but we can now look just at one of the pieces.) For *q* ≥ 80, the daily profit will be $\pi(q) = (50 - (q - 80))q = 130q - q^2$.

On to calculus: $\pi'(q) = 130 - 2q$. The only critical point is when *q* = 65; this is not in the domain we're interested in. Time to consider the endpoints: the only endpoint is *q* = 80, where the profit is $4000 per day. Is this the maximum?

Geometry helps here. Notice that the derivative is negative for all $q$ bigger than 80, so the profit function decreases if we have more than 80 seats. (You can confirm this by checking some points – with 80 seats, the profit is \$4000, but with 81 seats the profit drops to \$3969.) Profit is maximized when there are 80 seats in the coffee shop.

**115. (a)** The fixed cost is \$64.

**(c)** The average cost is $AC(q) = \dfrac{64 + 1.5q + .01q^2}{q} = \dfrac{64}{q} + 1.5 + .01q$.

**(d)** To minimize average cost, use calculus: $AC'(q) = -\dfrac{64}{q^2} + .01$. *AC* has critical points at $q = 0$ (where the average cost is also undefined), and at $q = 80$. $AC''(80) > 0$, so this critical point is a minimum. Average cost is minimized when Alicia makes 80 oven mitts.

**117**. $g(22) \cong g(20) + g'(20)(22-20) = 35 - 2(2) = 31$.

**119.** The function we want to estimate values of is $f(x) = \sqrt[3]{x}$. We want to estimate the value of the function at $x = 9$. We know the value of the function at 8, so we'll use $a = 8$.

TLA says $f(x) \cong f(a) + f'(a)(x-a)$. We know $f(8) = 2$. We also know $f'(8) = \dfrac{1}{3}(8)^{-2/3} = \dfrac{1}{3 \cdot 4} = \dfrac{1}{12}$. We estimate that $f(9) = 2 + \dfrac{1}{12}(9-8) = \dfrac{25}{12} \cong 2.083$. (My calculator tells me that the cube root of 9 is about 2.080, so we are pretty close.)

**121.** Elasticity is given by $E = \left|\dfrac{p}{q} \cdot \dfrac{dp}{dq}\right|$, so we need $p$, $q$, and the derivative. The price is given, $p = 7.5$.

We can find $q$ from the demand function; when $p = 7.5$, $q = 20$. Finally, the derivative $\dfrac{dp}{dq} = -8$. So $E = \left|\dfrac{p}{q} \cdot \dfrac{dp}{dq}\right| = \left|\dfrac{7.5}{20} \cdot -8\right| = 3 > 1$. Demand is elastic; if she increases her price, her revenue will go down. In fact, if she were to decrease her price, her revenue would increase.

## Chapter 3

**1.** $A(1)=1$, $A(2)=2.5$, $A(3)=4.5$, $A(4)=6$, and $A(5)=7$.

**3.** $C(1)=1.5$, $C(2)=4$, $C(3)=7.5$. Note that each of these regions is a trapezoid, with base x and heights 1 and x+1. The area of that trapezoid is $C(x)=\frac{1}{2}x(x+2)=\frac{1}{2}x^2+x$.

**5.** The distance traveled is the area under the velocity curve. There are 7.5 shaded squares in the figure, and each square is 10 feet/second × 10 seconds = 100 feet, so the car traveled 750 feet.

**7. (a)** Car A was traveling a constant 80 feet per second until *t* = 20 when its velocity started decreasing – it applied its brakes at *t* = 20. It took another 20 seconds to stop. Car B also applied its brakes at *t* = 20, but it took 40 seconds to stop.

**(b)** The distance the cars traveled after they started braking are represented by the area under the graph from *t* = 20 until they stopped. For Car A, that area is 800 feet, so it traveled 800 feet before coming to a complete stop. Car B also traveled 800 feet.

**9.** The police car will catch the speeder when the distance it has traveled is equal to the distance traveled by the speeder. The area under the police car's velocity after *x* seconds is 300 + 60(x-10); the area under the speeder's velocity is 45x. These are equal when x = 20. The police car will catch the speeder after 20 seconds; they will each have traveled 900 feet.

**11.**

**a.** Let $v(t)$ be the speed of the car. The distance traveled is the definite integral $\int_0^{20} v(t)\,dt$.

Approximate this with rectangles (careful, there are only 4 rectangles). Be careful with the units! Each rectangle is 5 minutes = $\frac{1}{12}$ hour wide, and the height is the speed in miles per hour. Using left-hand endpoints, the estimate is $(0)\left(\frac{1}{12}\right)+(30)\left(\frac{1}{12}\right)+(40)\left(\frac{1}{12}\right)+(65)\left(\frac{1}{12}\right)=11.25$. Using right-hand endpoints, the estimate is $(30)\left(\frac{1}{12}\right)+(40)\left(\frac{1}{12}\right)+(65)\left(\frac{1}{12}\right)+(40)\left(\frac{1}{12}\right)\cong 14.58$. I'll use their average; I estimate that the car travelled about 12.91 miles.

**b.** We don't have very much speedometer information, so our estimate could be way off. I don't have much confidence in the accuracy of this estimate.

**13. a.** The left-hand estimate is 426, the right-hand estimate is 438, and their average is 432.

**b.** The left-hand estimate is 471, the right-hand estimate is 519, and their average is 495.

**c.** Note that you can simply add your answers from parts a and b. the left-hand estimate is 897, the right-hand estimate is 957, and their average is 927.

**15.** $\int_1^3 2x\,dx$ ; this is a trapezoid with base 2 units, left height 2 and right height 6. Its area is

$\frac{1}{2}(2)(2+6)=8$ square units.

**17.** $\int_0^4 \left(3-\frac{x}{2}\right)dx = 8$ square units.

**19. (a)** 3 **(b)** −1 **(c)** 6 **(d)** 8 **(e)** 10 (all in square units)

**21.** You traveled 8 feet forward in the first 4 minutes, and then you traveled 8 feet backwards in the second 4 minutes. You traveled a total of 16 feet, but you ended up the same place you started. (In case you were wondering why you made so little progress? You were practicing juggling while balancing on a tightrope.)

**23.** meters

**25.** cubic feet

**27.** $\int_1^5 x^3\,dx \cong 156$ .

**29.** $\int_1^4 \sqrt{x}\,dx \cong 4.67$ .

**31.** $\int_2^3 \ln x \, dx \cong 9.0954$.

**33.** See Fig. 1. Each rectangle has a width of 0.5, and its height comes from the function $y = x^3$.

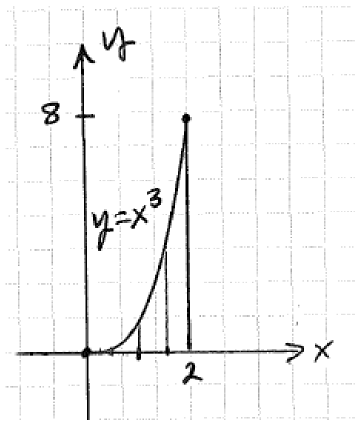



Figure 158

**(a)** Using the left endpoints for each rectangle, the Riemann sum is
$$(0)(0.5)+(.125)(0.5)+(1)(0.5)+(3.375)(0.5)=2.25.$$

**(b)** Using the right endpoints for each rectangle, the Riemann sum is
$$(.125)(0.5)+(1)(0.5)+(3.375)(0.5)+(8)(0.5)=6.25.$$

The actual area is 4, which lies between these two estimates.

**35.** $\int_3^5 f(x)dx = -2$.

**37.** $\int_6^7 f(x)dx = -1$.

**39.** $\int_0^7 f(x)dx = 4$.

**41.** $\int_5^7 f(x)dx = 1$.

**43.** $\int_1^3 g(t)dt = 0$.

**45.** $\int_0^8 g(t)dt = 2$.

**47.** The effect of adding 1 is to raise the graph up 1 unit; the integral is the area of the original triangle plus the area of the rectangle beneath.

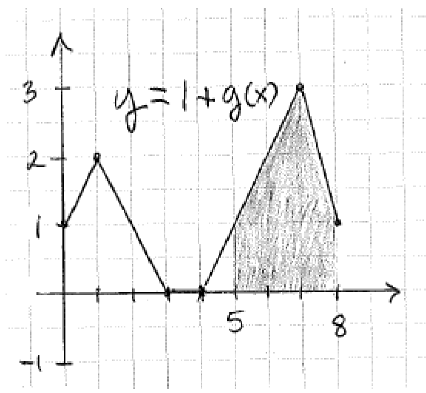



Figure 159

$$\int_5^8 1 + g(x)dx = 6.$$

**49.** Let's call the velocity function in Fig. 52 v(t). Then the total distance traveled between 3 pm and 6 pm is given by $\int_3^6 v(t)dt$, which is 67.5 miles.

**51.** $\int (2.5x^5 - x - 1.25)dx = 2.5\frac{x^6}{6} - \frac{x^2}{2} - 1.25x + C = \frac{5x^6}{12} - \frac{x^2}{2} - 1.25x + C$.

**53.** $\int \pi^2 dw = \pi^2 w + C$ (Remember that π is a constant!)

**55.**

$$\int\left(\sqrt{x}+e^x-\frac{1}{4x^3}\right)dx=\int\left(x^{1/2}+e^x-\frac{1}{4}x^{-3}\right)dx=\frac{x^{3/2}}{3/2}+e^x-\frac{1}{4}\frac{x^{-2}}{-2}+C=\frac{2}{3}x^{3/2}+e^x+\frac{1}{8x^2}+C.$$

**57.** $\int\frac{1}{x^2}\,dx=-\frac{1}{x}+C$.

**59.** $\int\frac{t^5-t^2}{t}\,dt=\int\left(t^4-t\right)dt=\frac{t^5}{5}-\frac{t^2}{2}+C$.

**61.** I can do this substitution in my head: $\int e^{100x}\,dx=\frac{e^{100x}}{100}+C$.

**63.** I need to write out the substitution for this one. Let $u=\frac{10}{x}$. Then $du=-\frac{10}{x^2}dx$, so

$\frac{1}{x^2}dx=-\frac{1}{10}du$. So $\int\frac{e^{10/x}}{x^2}dx=\int-\frac{1}{10}e^u\,du=-\frac{1}{10}e^u+C=-\frac{1}{10}e^{10/x}+C$.

**65.** Let $u=3x^3-1$. Then $du=9x^2dx$. So $6x^2dx=\frac{2}{3}du$.

$$\int 6x^2\sqrt{3x^3-1}\,dx=\int\frac{2}{3}\sqrt{u}\,du=\frac{2}{3}\frac{u^{3/2}}{3/2}+C=\frac{4}{9}\left(3x^3-1\right)^{3/2}+C.$$

**67.** Let $u=x^2-6x+5$. Then $du=(2x-6)dx=2(x-3)dx$. So

$$\int\frac{x-3}{x^2-6x+5}dx=\int\frac{1}{2}\frac{1}{u}du=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln\left|x^2-6x+5\right|+C.$$

**69.** $\int_{-1}^{2}x^2\,dx=\frac{1}{3}x^3\Big|_{-1}^{2}=\frac{1}{3}\left(2^3-(-1)^3\right)=9$.

**71.** $\int_1^e \frac{1}{x}\,dx = \ln|x|\Big]_1^e = \ln|e| - \ln|1| = 1$.

**73.** $\int_3^5 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_3^5 = \frac{2}{3}\left(5^{3/2} - 3^{3/2}\right) \cong 3.98946$.

**75.** $\int_1^{1000} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{1000} = -\frac{1}{1000} - \left(-\frac{1}{1}\right) = \frac{999}{1000}$.

**77.** This one needs substitution: Let $u = 1 + x^2$. Then $du = 2x\,dx$. When *x* = −2, *u* = 5, and when *x* = 2, *u* =5. So $\int_{-2}^{2} \frac{2x}{1+x^2} = \int_5^5 \frac{1}{u}\,du = 0$. (Look at the graph of the integrand to see why this integral is zero.)

**79.** I'll use substitution. Let $u = x - 2$. Then $du = dx$. When *x* = 2, *u* = 0, and when *x* = 4, *u* = 2. So $\int_2^4 (x-2)^3\,dx = \int_0^2 u^3 du = \frac{u^4}{4}\Big|_0^2 = 4$.

**81.** The area shown is $\int_1^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_1^4 = \frac{2}{3}\left(4^{3/2} - 1\right) = \frac{14}{3}$.

**83.** The area shown is $\int_0^2 \left(4 - x^2\right)dx = 4x - \frac{x^3}{3}\Big|_0^2 = \frac{16}{3}$.

**85.** I see that $h(x) \geq g(x)$ for the entire time from *x* = 0 to *x* = 6 (at least this is true for every entry in the table). So *h* is the "top" curve, and *g* is the "bottom" curve. The area would be $\int_0^6 (h(x) - g(x))dx$, which I will estimate using right hand rectangles:

$$\int_0^6 (h(x) - g(x))dx \cong (6-1)(1) + (8-2)(1) + (6-2)(1) + (5-3)(1) + (4-4)(1) + (2-0)(1) = 19.$$

**87.** The information in the table indicates that $f(x) \geq g(x)$ for $0 \leq x \leq 4$ and at $x = 6$, but $g(x) \geq f(x)$ for *x* = 5. So it's not the same "top" curve for the entire interval. We could write this as three integrals, or we could simply choose at each value of *x* which function is bigger. This is thinking of the area as $\int_0^6 |f(x) - g(x)|\,dx$. I will estimate this area using left hand rectangles:

$\int_0^6 |f(x) - g(x)|\,dx \cong (5-2)(1) + (6-1)(1) + (6-2)(1) + (4-2)(1) + (3-3)(1) + (4-2)(1) = 16$. Notice that in the last rectangle (the one from *x* = 5 to *x* = 6), I used *g* – *f*, because $g(5) > f(5)$.

**89.** $x^2 + 3$ is above 1 on the entire interval, so the area is $\int_{-1}^{2} (x^2 + 3 - 1)\,dx = 9$.

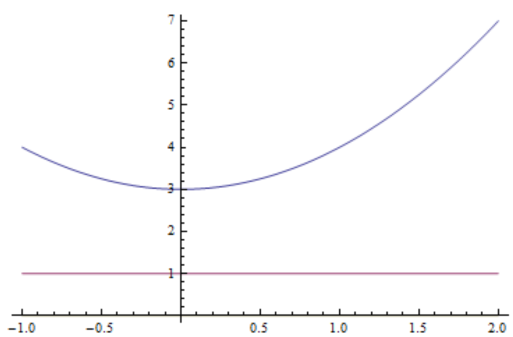



Figure 160

**91.** From *x* = 0 to *x* = 1, $g(x) = x$ is the top function. From *x* = 1 to *x* = 2, $f(x) = x^2$ is the top function. The area is $\int_0^1 (x - x^2)\,dx + \int_1^2 (x^2 - x)\,dx = 1$.

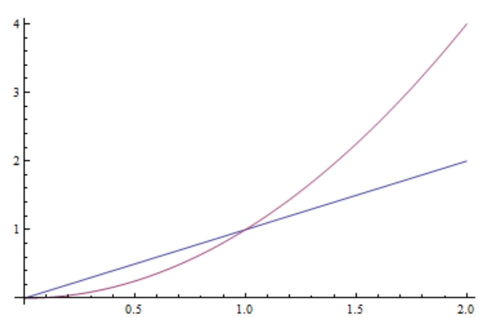



Figure 161

**93.** $g(x) = x$ is the top function over the entire interval, so the area is $\int_1^e \left(x - \frac{1}{x}\right)dx = \frac{e^2 - 3}{2} \cong 2.1945$.

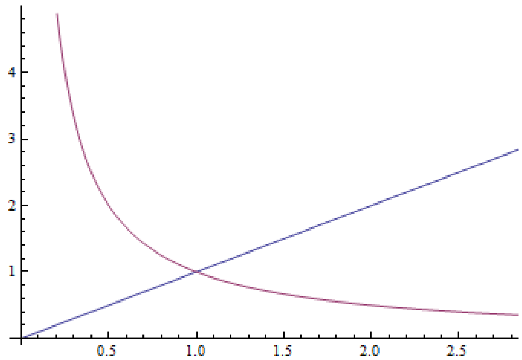



Figure 162

**95.** From *x* = 0 to *x* = 1, $f(x) = 4 - x^2$ is the top function. From *x* = 1 to *x* = 2, $g(x) = x + 2$ is the top function. The area is $\int_0^1 \left(4 - x^2 - (x+2)\right)dx + \int_1^2 \left(x + 2 - \left(4 - x^2\right)\right)dx = 3$.

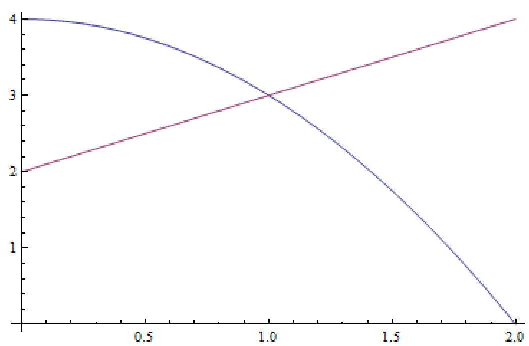



Figure 163

**97.** $f(x)=3$ is the top function over the entire interval, so the area is $\int_0^1 \left(3-\sqrt{1-x^2}\right)dx \cong 2.215$.

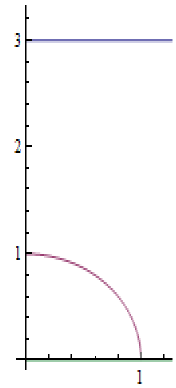


Figure 164

Note: this is one where you can't use antiderivatives to find the answer. I used technology to find this number. You could also use geometry – the square root function shown here is the top of a circle with radius = 1. The exact answer is $3-\pi/4$.

**99.** The average value of *f* on [0, 6] is $\frac{1}{6}\int_0^6 f(x)dx$. We need to estimate the definite integral using rectangles; I'll choose left-hand rectangles this time.

$\int_0^6 f(x)dx \cong (5)(1)+(6)(1)+(6)(1)+(4)(1)+(3)(1)+(2)(1)=26$. Dividing by 6 gives us an estimate for the average value of 13/3 ≈ 4.33.

**101.** The average value is $\frac{1}{2}\int_0^2 f(x)dx$. We can compute the definite integral as the area of a triangle; the average value is $\frac{1}{2}(2)=1$.

**103.** The average value is $\frac{1}{5}\int_1^6 f(x)dx = \frac{1}{5}(11)=2.2$.

**105.** The average value is $\frac{1}{4}\int_0^4 (2x+1)dx = 5$.

**107. (a)**Let $v(t)$ be the velocity function shown in Fig. 59. Then total distance traveled by the car is just $\int_0^5 v(t)dt$. I estimate the area under the curve to be about 183 miles. (Your estimate might be different.)

**(b)** The constant velocity asked for here is the average velocity over that 5-hour period. The average velocity is about 36.6 miles per hour.

**109. (a)** The demand function is $p = 150 - .5q$; that's the one that's decreasing.

**(b)** The equilibrium point is $(q, p) \cong (175, 62.6)$. (The actual *q* is about 174.79, the actual *p* is about 62.6).

**(c)** The total gains from trade is the sum of the consumer and producer surplus.

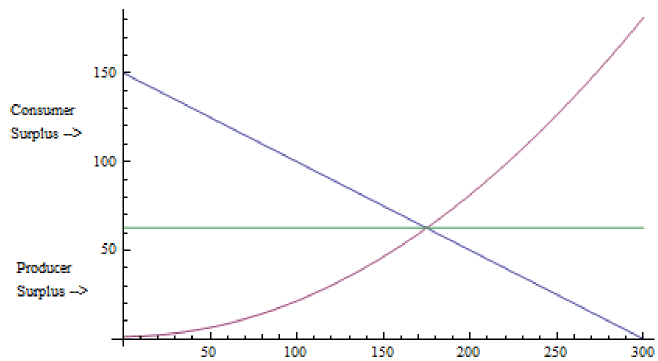



Figure 165

Consumer surplus: $\int_0^{175} (150 - 0.5q)dq - (175)(62.6) = 7638.75$

Producer surplus: $(175)(62.6) - \int_0^{175} (.002q^2 + 1.5)dq = 7119.58$

So the total gains from trade are about $14,758.

**111.** The two points from the demand function are $(q,p)=(25,40)$ and $(q,p)=(185,20)$; the demand function is $p=43.125-.125q$. The two points from the supply function are $(q,p)=(200,40)$ and $(q,p)=(100,20)$; the supply function is $p=0.2q$. The equilibrium point is $(q,p)\cong(133,26.50)$.

Consumer surplus: $\int_0^{133}(43.125-.125q)dq-(133)(26.5)\cong\$1106$.

Producer surplus: $(133)(26.5)-\int_0^{133}(.2q)dq\cong\$1756$.

**113. (a)** $PV=\int_0^{35}40000e^{-0.008t}\,dt\cong\$1{,}221{,}081$.

**(b)** $PV=\int_0^{35}40000e^{-0.025t}\,dt\cong\$933{,}021$.

**(c)** $PV=\int_0^{35}40000e^{-0.045t}\,dt\cong\$704{,}882$

**115.** $PV=\int_0^{8}\left(12+0.3t^2\right)e^{-0.037t}\,dt\cong124.171$. The present value is about \$124,171. Note that you must use technology to evaluate this integral.

**117.** The present value of the income stream is $PV=\int_0^{8}25000e^{-0.034t}\,dt\cong\$175{,}107$. So, yes, it looks like \$153,000 is a good deal.

## Chapter 4

**1.** a) $F(0,4)=0^2-4^2=-16$.

b) $F(4,0)=4^2-0^2=16$.

c) $F(x,4)=x^2-16$

d) $F(4,y)=16-y^2$

e) $F(800,800)=800^2-800^2=0$.

f) $F(x,x)=x^2-x^2=0$.

g) $F(x,-x)=x^2-(-x)^2=0$.

**3.** $f(1,2,3,4)=1^2-\frac{1}{3\cdot 4}+1\cdot 2\cdot 3^2=\frac{227}{12}\cong 18.92$.

**5.** a) $A(2,.05)=105.17$ b) $A(.05,.2)$ is not listed in this table. c) increasing. d) increasing.

**7.** The contour diagram shows that the surface increases in height as you head out toward any of the four corners; this matches e.

**9.** The contour diagram shows alternating negative and positive values on the loopy part – these correspond to the down and up ruffles in a.

**11.** The circular shape, with increasing height as you head inward, matches b.

**13.** The contour diagram increases as you head away in either x-direction from the center, and decreases as you head away in either y-direction. That matches the function in f; the positive $x^2$ makes the function increase as *x* gets further from zero, while the negative $y^2$ makes the function decrease as *y* gets further from zero. You can confirm your answer by plugging in a point or two.

**15.** If we increase in x, the values of the function decrease, while if we increase in y, the values of the function increase. That matches the function in a. Further confirmation comes from the fact that the contours are lines, which match the linear nature of the formula.

**17.** If we increase in x, the values of the function increase. If we move away from 0 in either y-direction, the values of the function do the same thing – they either increase together or decrease together – and the absolute value of the function values increase. That matches the function in b. You can confirm your answer by plugging in a point or two.

**19.** a. $M(1,3)$ lies between the contours 0.2 and 0.4, so $M(1,3) \cong 0.3$.

b. $M(3,1) \cong 0.3$

c. If we fix *y* = 3, we're moving along a horizontal line. As we go from the far left of the diagram, $M(x,3)$ first decreases, then increases, then decreases again.

d. $M(3,y)$ first decreases, then increases, then decreases again.

e. If we fix the x-value at c = 0, the vertical line we travel along is the contour for 0. $M(0,y) = 0$ is a constant function.

**21.** $f_x = 2x$; $f_y = -10y$

**23.** $f_x = e^{x+6y}$; $f_y = 6e^{x+6y}$

**25.** $f_x = (2x)e^x + (x^2 - 5y^2)e^x$; $f_y = (-10y)e^x$

**27.** $f_x = 0$; $f_y = 0$

**29.** $f_x = \dfrac{(2x)(y^4 - 5x^4) - (x^2 - 5y^2)(20x^3)}{(y^4 - 5x^4)^2}$; $f_y = \dfrac{(-10y)(y^4 - 5x^4) - (x^2 - 5y^2)(4y^3)}{(y^4 - 5x^4)^2}$

**31.** $f_x = y^5 e^x$; $f_y = 5y^4 e^x$

**33.** $f_x = 7(x + e^y)^6$; $f_y = 7(x + e^y)^6 e^y$

**35.** $f_x = \dfrac{1}{2}(x + \sqrt{y})^{-1/2}$; $f_y = \dfrac{1}{2}(x + y^{1/2})^{-1/2}\left(\dfrac{1}{2}y^{-1/2}\right)$

**37.** $f_{xx} = 12x^2 + 24xy - 12y^2$; $f_{yy} = -12x^2 - 24xy + 12y^3$; $f_{xy} = 12x^2 - 24xy - 12y^2 = f_{yx}$.

**39.** $f_{xx} = e^{x+6y}$; $f_{yy} = 36e^{x+6y}$; $f_{xy} = 6e^{x+6y} = f_{yx}$.

**41.** $f_{xx} = \dfrac{2}{y^4 - 5}$; $f_{yy} = -\dfrac{12x^2y^2}{\left(y^4-5\right)^2} + \dfrac{32x^2y^6}{\left(y^4-5\right)^3}$; $f_{xy} = -\dfrac{8xy^3}{\left(y^4-5\right)^2} = f_{yx}$.

**43.** a. We have to use the average rate of change over the tiniest interval we can to approximate this

instantaneous rate of change. We're thinking of *t* as the only variable, so we're looking in the column where *r* is fixed at *r* = .05.

From the left (for *t* less than 2): $A_t(2,.05) \cong \dfrac{105.17 - 51.27}{2-1} = 53.9$

From the right: $A_t(2,.05) \cong \dfrac{161.83 - 105.17}{3-2} = 56.66$

Either of these would be a fine estimate. The best estimate would probably be their average, or $A_t(2,.05) \cong 55.28$.

b. Now we're thinking of *r* as our only variable, and fixing *t* = 2.

From the left: $A_r(2,.05) \cong \dfrac{105.17 - 83.29}{.05 - .04} = 2188$.

From the right: $A_r(2,.05) \cong \dfrac{127.50 - 105.17}{.06 - .05} = 2233$.

My estimate is the average $A_r(2,.05) \cong 2210.5$.

c. We can estimate the value of the function using partial derivatives, even if we had to estimate the partial derivatives, too. But every time we estimate, we lose some precision; we should take our estimate with a grain of salt.

$$A(2.5,.054) \cong A(2,.05) + A_t(2,.05)(2.5-2) + A_r(2,.05)(.054-.05)$$
$$\cong 105.17 + (55.28)(.5) + (2210.5)(.004) = 141.652$$

d. $A_t = 1000\left(re^{rt}\right)$, so $A_t(2,.05) = 1000\left(.05e^{(.05)(2)}\right) \cong 55.25$

$A_r = 1000\left(te^{rt}\right)$, so $A_r(2,.05) = 1000\ \left(2e^{(.05)(2)}\right) \cong 2210.34$

Our estimates of the partial derivatives were close.

$A(2.5,.054)=1000\left(e^{(.054)(2.5)}-1\right)\cong 144.54$. Even this estimate wasn't too far off!

**45.** $f_x=-3x^2+30x=3x(10-x)=0 \rightarrow$ need *x* = 0 or *x* = 10

$f_y=3y^2-12=3(y+2)(y-2)=0 \rightarrow$ need *y* = 2 or *y* = −2.

There are four critical points: (0, 2), (0, −2), (10, 2) and (10, −2).

$f_{xx}=-6x+30$; $f_{yy}=6y$; $f_{xy}=0=f_{yx}$; $D=(-6x+30)(6y)-0^2$

For (0, 2): $D>0,\ f_{xx}>0$; there is a local minimum at (0, 2).

For (0, −2): $D<0$; there is a saddle point at (0, −2).

For (10, 2): $D<0$; there is a saddle point at (10, 2).

For (10, −2): $D>0,\ f_{xx}<0$; there is a local maximum at (10, −2).

**47.** $f_x=-\frac{4}{x}+4=0 \rightarrow$ need *x* = 1 (Note that *x* = 0, which makes this derivative undefined, is not in the domain of the original function).

$f_y=2y \rightarrow$ need *y* = 0.

There is one critical point, at (1, 0).

$f_{xx}=\frac{4}{x^2}$; $f_{yy}=2$; $f_{xy}=0=f_{yx}$; $D=\frac{8}{x^2}$

At (1, 0), $D>0,\ f_{xx}>0$; there is a local minimum at (1,0).

**49.** Look at $f(0,y)$ and $f(x,0)$; both of these functions continue to increase as we move past the origin. There are points nearby that are larger than $f(0,0)$, and there are points nearby that are smaller than $f(0,0)$. So this function has a saddle point at (0, 0).

**51.** $f_x = y - 10x = 0$, so $y = 10x$

$f_y = x - 10y + 33$. Substituting from above, we get $f_y = x - 10(10x) + 33 = -99x + 33 = 0$, so $x = \frac{1}{3}$ and $y = \frac{10}{3}$.

There is one critical point, at $\left(\frac{1}{3}, \frac{10}{3}\right)$.

$f_{xx} = -10$; $f_{yy} = -10$; $f_{xy} = 1 = f_{yx}$; $D = 99 > 0$

Since $f_{xx} < 0$; there is a local maximum at (1,0).

**53.** $f_x = 3x^2 - 3y = 0$, so $y = x^2$

$f_y = 3y^2 - 3x$. Substituting from above, we get $f_y = 3x^4 - 3x = 3x\left(x^3 - 1\right) = 0$, so $x = 0$ or $x = 1$.

There are two critical points, at (0, 0) and (1, 1).

$f_{xx} = 6x$; $f_{yy} = 6y$; $f_{xy} = -3 = f_{yx}$; $D = 36xy - 9$

At (0,0): $D < 0$, so there is a saddle point at (0, 0).

At (1, 1): $D > 0,\ f_{xx} > 0$, so there is a local minimum at (1, 1).

**55.** $f_x = y^2 e^x + 2x = 0$

$f_y = 2ye^x = 0$. The only way this can be zero is if *y* = 0. Substituting into the above gives *x* = 0 also. There is one critical point, at (0, 0).

$f_{xx} = y^2 + 2$; $f_{yy} = 2e^x$; $f_{xy} = 2ye^x = f_{yx}$; $D = \left(y^2 + 2\right)\left(2e^x\right) - \left(2ye^x\right)^2$

At (0, 0), $D = (2)(2) - 0 > 0,\ f_{xx} > 0$, so there is a local minimum at (0, 0).

**57**. a. These products are substitutes; you can see that increasing the price of one will increase the demand of the other.

b. When $p_1 = 20$ and $p_2 = 30$, $q_1 = 170$ and $q_2 = 130$

c.

$$R(p_1, p_2) = p_1 q_1 + p_2 q_2 = p_1(200 - 3p_1 + p_2) + p_2(150 + p_1 - 2p_2)$$
$$= 200p_1 - 3p_1{}^2 + 2p_1 p_2 + 150p_2 - 2p_2{}^2$$

d. To maximize revenue, find critical points and test:

$R_{p_1} = 200 - 6p_1 + 2p_2$ and $R_{p_2} = 150 + 2p_1 - 4p_2$

There is one critical point, where $p_1 = 55$ and $p_2 = 65$.

To confirm that this gives maximum revenue:

$R_{p_1 p_1} = -6$ , $R_{p_2 p_2} = -4$ , and $R_{p_1 p_2} = 2 = R_{p_2 p_{12}}$

So $D > 0$ and $R_{p_1 p_1} < 0$, so this really is a local maximum. Because it's our only critical point, this is the global maximum.

To maximize revenue, $p_1 = 55$ and $p_2 = 65$, and $q_1 = 100$ and $q_2 = 75$.