Calculus 1 for Team-Based Inquiry Learning

Live Development Edition

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Printable PDF

A PDF with all activities may be printed by visiting `pdf/main.pdf`.

Contents

Chapter 1

Limits (LT)

Learning Outcomes

How do we measure "close-by" values?

By the end of this chapter, you should be able to...

1. Find limits from the graph of a function.
2. Infer the value of a limit based on nearby values of the function.
3. Compute limits of functions given algebraically, using proper limit properties.
4. Determine where a function is and is not continuous.
5. Determine limits of functions at infinity.
6. Determine limits of functions approaching vertical asymptotes.

Readiness Assurance. Before beginning this chapter, you should be able to...

a Use function notation and evaluate functions

- Review: Khan Academy
- Practice:
  - Evaluate functions
  - Evaluate functions from their graphs
  - Function notation word problems

b Find the domain of a function

- Review: Khan Academy
- Practice: Determine the domain of functions

c Determine vertical asymptotes, horizontal asymptotes, and holes (removable discontinuities) of rational functions

- Review:
  - Discontinuities of rational functions
  - Finding horizonal asymptotes

- Practice:
  - Rational functions: zeros, asymptotes, and undefined points
  - Finding horizonal asymptotes

d Perform basic operations with polynomials

- Review:
  - Adding and subtracting polynomials
  - Multiplying polynomials
- Practice:
  - Add polynomials
  - Subtract polynomials
  - Multiply binomials by polynomials

e Factor quadratic expressions

- Review: Khan Academy
- Practice: Factoring quadratics intro

f Represent intervals using number lines, inequalities, and interval notation

- Review: Varsity Tutor

1.1 Limits graphically (LT1)

Learning Outcomes

- Find limits from the graph of a function.

Activity 1.1.1 In Figure .1 the graph of a function is given, but something is wrong. The graphic card failed and one portion did not render properly. We can't see what is happening in the neighborhood of $x = 2$.

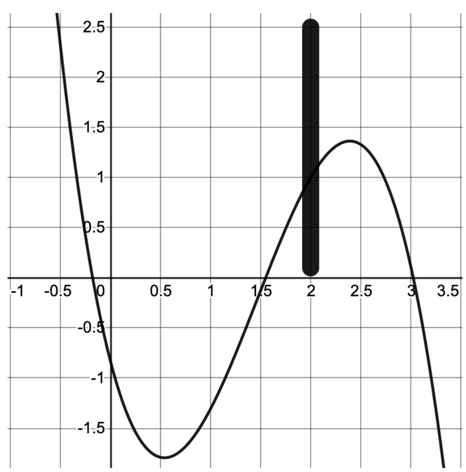


Figure .1 A graph of a function that has not been rendered properly.

(a) Imagine moving along the graph toward the missing portion from the left, so that you are climbing up and to the right toward the obscured area of the graph. What y-value are you approaching?

A 0.5

B 1

C 1.5

D 2

E 2.5

(b) Think of the same process, but this time from the right. You're falling down and to the left this time as you come close to the missing portion. What y-value are you approaching?

A 0.5

B 1

C 1.5

D 2

E 2.5

Activity 1.1.2 In Figure .2 the graphic card is working again and we can see more clearly what is happening in the neighborhood of $x = 2$.

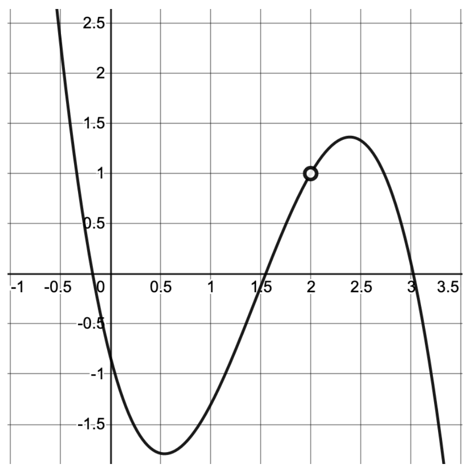


Figure .2 A graph of a function that has rendered properly

(a) What is the value of $f(2)$?

(b) What is the y-value that is approached as we move toward $x = 2$ from the left?

A 0.5

B 1

C 1.5

D 2

E 2.5

(c) What is the y-value that is approached as we move toward $x = 2$ from the right?

A 0.5

B 1

C 1.5

D 2

E 2.5

Remark 1.1.3 When studying functions in algebra, we often focused on the *value* of a function given a specific x-value. For instance, finding $f(2)$ for some function $f(x)$. In calculus, and here in Activity 1.1.1 and Activity 1.1.2, we have instead been exploring what is happening as we *approach* a certain value on a graph. This concept in mathematics is known as finding a limit.

Activity 1.1.4 Based on Activity 1.1.1 and Activity 1.1.2, write your first draft of the definition of a limit. What is important to include? (You can use concepts of limits from your daily life to motivate or define what a limit is.)

Definition 1.1.5 Given a function f, a fixed input $x = a$, and a real number L, we say that f **has limit** L **as** x **approaches** a, and write

$$\lim_{x \to a} f(x) = L$$

provided that we can make $f(x)$ as close to L as we like by taking x sufficiently close (but not equal) to a. If we cannot make $f(x)$ as close to a single value as we would like as x approaches a, then we say that f **does not have a limit as** x **approaches** a. ◊

Activity 1.1.6

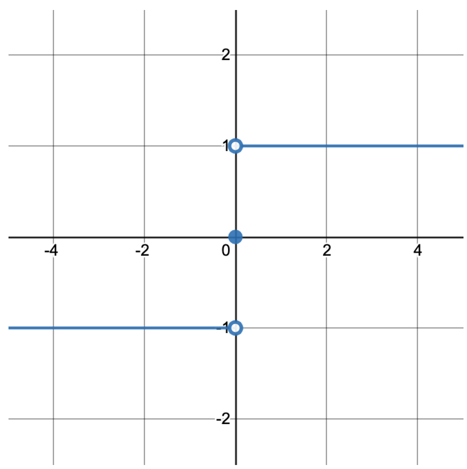


Figure .3 A piecewise-defined function

What is the limit as x approaches 0 in Figure .3?

A The limit is 1

B The limit is -1

C The limit is 0

D The limit is not defined

Definition 1.1.7 We say that f *has limit* L_1 *as* x *approaches* a *from the left* and write

$$\lim_{x \to a^-} f(x) = L_1$$

provided that we can make the value of $f(x)$ as close to L_1 as we like by taking x sufficiently close to a while always having $x < a$. We call L_1 the left-hand limit of f as x approaches a. Similarly, we say L_2 *is the right-hand limit of* f

as x approaches a and write

$$\lim_{x \to a^+} f(x) = L_2$$

provided that we can make the value of $f(x)$ as close to L_2 as we like by taking x sufficiently close to a while always having $x > a$. ◊

Activity 1.1.8 Refer again to Figure .3 from Activity 1.1.6.

(a) Which of the following best matches the definition of right and left limits? (Note that DNE is short for ”does not exist.”)

- A The left limit is -1. The right limit is 1.
- B The left limit is 1. The right limit is -1.
- C The left limit DNE. The right limit is 1.
- D The left limit is -1. The right limit DNE.
- E The left limit DNE. The right limit DNE.

(b) What do you think the overall limit equals?

- A The limit is 1
- B The limit is -1
- C The limit is 0
- D The limit is not defined

Activity 1.1.9 Consider the following graph:

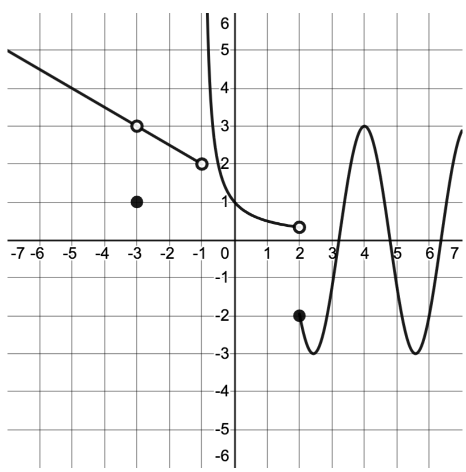


Figure .4 Another piecewise-defined function

(a) Find $\lim_{x \to -3^-} f(x)$ and $\lim_{x \to -3^+} f(x)$.

(b) Find $\lim_{x \to -1^-} f(x)$ and $\lim_{x \to -1^+} f(x)$.

(c) Find $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$.

(d) Find $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$.

(e) For which x-values does the *overall* limit exist? Select all. If the limit exists, find it. If it does not, explain why.

A -3

B -1

C 2

D 4

Activity 1.1.10 Sketch the graph of a function $f(x)$ that meets all of the following criteria. Be sure to scale your axes and label any important features of your graph.

1. $\lim_{x\to 5^-} f(x)$ is finite, but $\lim_{x\to 5^+} f(x)$ is infinite.
2. $\lim_{x\to -3} f(x) = -4$, but $f(-3) = 0$.
3. $\lim_{x\to -1^-} f(x) = -1$ but $\lim_{x\to -1^+} f(x) \neq -1$.

Theorem 1.1.11 *Suppose that:*

- *for an interval around $x = a$, we have that $f(x) \leq g(x) \leq h(x)$;*
- *the limit as x approaches a of $f(x)$ is equal to the same value L as the limit of $h(x)$, so $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$.*

Then, the limit of $g(x)$ as $x \to a$ is also L, so $\lim_{x\to a} g(x) = L$

Activity 1.1.12 In this activity we will explore a mathematical theorem, the Squeeze Theorem Theorem 1.1.11.

(a) The part of the theorem that starts with "Suppose…" forms the assumptions of the theorem, whilst the part of the theorem that starts with "Then…" is the conclusion of the theorem. What are the assumptions of the Squeeze Theorem? What is the conclusion?

(b) The assumptions of the Squeeze Theorem can be restated informally as "the function g is squeezed between the functions f and h around a". Explain in your own words how the two assumptions result into a "squeezing effect".

(c) Let's see an example of the application of this theorem. First examine the following picture. Explain why, from the picture, it seems that both assumptions of the theorem hold.

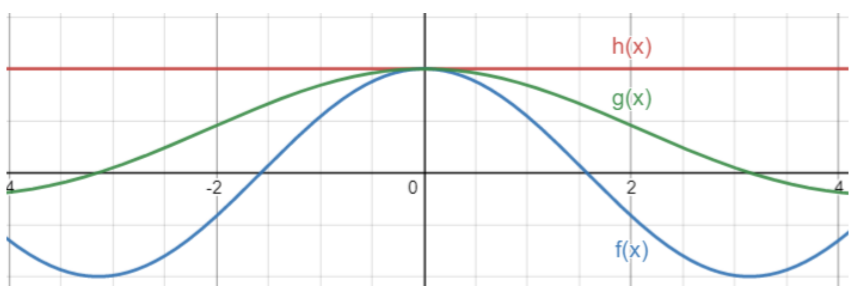


Figure .5 A pictorial example of the Squeeze Theorem.

(d) Match the functions $f(x), g(x), h(x)$ in the picture to the functions $\cos(x), 1, \frac{\sin(x)}{x}$.

(e) Using trigonometry, one can show algebraically that $\cos(x) \leq \frac{\sin(x)}{x} \leq 1$ for x values close to zero. Moreover, $\lim_{x\to 0} \cos(x) = \cos(0) = 1$ (we say that cosine is a continuous function). Use these facts and the Squeeze Theorem, to find the limit $\lim_{x\to 0} \frac{\sin(x)}{x}$.

1.2 Limits numerically (LT2)

Learning Outcomes

- Infer the value of a limit based on nearby values of the function.

Activity 1.2.1

Table .6

x	6.9	6.99	6.999	7	7.001	7.01	7.1
$f(x)$	0.1695	0.1699	0.1667	?	0.1667	0.1664	0.1639

Based on the values of Table .6, what is the best approximation for $\lim_{x\to 7} f(x)$?

A the limit is approximately 7

B the limit is approximately 0.17

C the limit is approximately 0.16

D the limit is approximately 0.1667

E the limit is approximately 6.9999

Remark 1.2.2 Notice that the value we obtained in Activity 1.2.1 is only an approximation, based on the trends that we have seen within the table.

Activity 1.2.3

Table .7

x	1.25	1.5	1.75	2.25	2.5	2.75
$f(x)$	-0.7606	-0.13	0.4881	1.3119	1.33	0.9606

In Activity 1.1.1 we got an approximation to the limit of the function as x tends to 2. Now let us say you are given the same plot as in Activity 1.1.1, but also given a table of numerical values (Table .7) for the function. Given this new information which of the choices below best describes the limit of the function as x tends to 2?

A There is not enough information because we do not know the value of the function at $x = 2$

B Can be approximated to be 1 because the data in the table and the graph show that from the left and the right the function approaches 1 as x goes to 2

C Can be approximated to be 1 because the values appear to approach 1 and the graph appears to approach 1, but we should zoom in on the graph to be sure

D Cannot be approximated because the function might not exist at $x = 2$.

Activity 1.2.4

Table .8

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	-0.4	-0.49	-0.499	0.499	0.49	0.4

Based on Table .8, what information can be inferred about $\lim_{x\to 1^-} f(x)$,

$\lim_{x\to 1^+} f(x)$, and $\lim_{x\to 1} f(x)$?

A $\lim_{x\to 1^-} f(x) = -0.5$, $\lim_{x\to 1^+} f(x) = 0.5$, and $\lim_{x\to 1} f(x) = 0$

B $\lim_{x\to 1^-} f(x) = -0.5$, $\lim_{x\to 1^+} f(x) = 0.5$, and $\lim_{x\to 1} f(x)$ does not exist

C $\lim_{x\to 1^-} f(x) = 0.5$, $\lim_{x\to 1^+} f(x) = -0.5$, and $\lim_{x\to 1} f(x)$ does not exist

D $\lim_{x\to 1^-} f(x) = 0.5$, $\lim_{x\to 1^+} f(x) = -0.5$, and $\lim_{x\to 1} f(x) = 0$

Activity 1.2.5 Consider the following function $f(x) = 3x^3 + 2x^2 - 5x + 20$.

(a) Of the following options, at which values given would you evaluate $f(x)$ to best determine $\lim_{x\to 2} f(x)$ numerically?

A 1.9, 1.99, 2.0, 2.01, 2.1

B 1.98, 1.99, 2.0, 2.01, 2.02

C 1.8, 1.9, 2.0, 2.1, 2.2

D 1.0, 1.5, 2.0, 2.5, 3.0

(b) Use the values that you chose in part A to calculate an approximation for $\lim_{x\to 2} f(x)$.

(c) Which value best describes the limit that you obtained in part (b)?

A the approximate value is 41.25

B the approximate value is 41.5

C approximate value is 41.75

D approximate value is 42

Activity 1.2.6 In Figure .9is the graph for $f(x) = \sin\left(\frac{1}{x}\right)$. Several values for $f(x)$ in the neighborhood of $x = 0$ are approximated in Table .10.

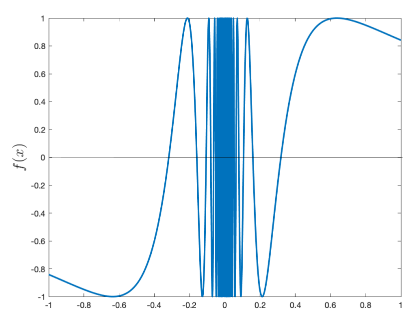


Figure .9 Graph of $f(x) = \sin(1/x)$.

Table .10

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.54402	0.50637	-0.82688	0.82688	-0.50637	-0.54402

(a) Based on the graph and table what is the best explanation for the limit as x tends to zero?

A the limit does not exist because the left and right limits have opposite values.

B the limit does not exist because we do not have enough information to answer the question.

C the limit does not exist because the function is oscillating between -1 and 1.

D the limit does not exist because you are dividing by zero when $x = 0$ for $f(x)$.

(b) Would your conclusion that resulted from Activity 1.2.6 change if the function was $f(x) = \cos(1/x)$ or $f(x) = \tan(1/x)$?

Activity 1.2.7 Use technology to complete the following table of values.

$$f(x) = \frac{x^2 - x - 12}{x^2 + 16\,x + 39}$$

x	-3.1	-3.01	-3.001	-3	-2.999	-2.99	-2.9
$f(x)$							

Then explain how to use it to make an educated guess as to the value of the limit

$$\lim_{x\to -3} \frac{x^2 - x - 12}{x^2 + 16\,x + 39}$$

Activity 1.2.8 In this activity you will study the velocity of Usain Bolt in his Beijing 100 meters dash. He completed 100 meters in 9.69 seconds for an overall average speed of $100/9.69 = 10.32$ meters per second (about 23 miles per hour). But this is the average velocity on the whole interval. How fast was he at different instances? What was his maximum velocity? Let's explore this. The table Table .11 shows his split times recorded every 10 meters.

Table .11

t (seconds)	1.85	2.87	3.78	4.65	5.5	6.32	7.14	7.96	8.79	9.69
d (meters)	10	20	30	40	50	60	70	80	90	100

(a) What was the average velocity on the first 50 meters? On the second 50 meters?

(b) What was the average velocity between 30 and 50 meters? Between 50 and 70 meters?

(c) What was the average velocity between 40 and 50 meters? Between 50 and 60 meters?

(d) What is your best estimate for the Usain's velocity at the instant when he passed the 50 meters mark? This is your estimate for the instantaneous velocity.

(e) Using the table of values, explain why 50 meters is NOT the best guess for when the instantaneous velocity was the largest. What other point would be more reasonable?

1.3 Limits analytically (LT3)

Learning Outcomes

- Compute limits of functions given algebraically, using proper limit properties.

Remark 1.3.1 Recall that in Activity 1.2.5 we used numerical methods and table of values to find the limit of a relatively simple degree three polynomial at a point. This was inefficient, "there's gotta be a better way!"

Activity 1.3.2 Given $f(x) = 3x^2 - \frac{1}{2}x + 4$, evaluate $f(2)$ and approximate $\lim_{x\to 2} f(x)$ numerically (or graphically). What do you think is more likely?

A $\lim_{x\to 2} f(x) = f(2)$

B $\lim_{x\to 2} f(x) \approx f(2)$

C $\lim_{x\to 2} f(x) \neq f(2)$

Activity 1.3.3 The table below gives values of a few different functions.

Table .12

x	6.99	6.999	7.001	7.01
f(x)	13.99	13.999	14.001	14.01
g(x)	22.97	22.997	23.003	23.03
3f(x)	41.97	41.997	42.003	42.03
f(x)+g(x)	36.96	36.996	37.004	37.04
f(x)g(x)	321.350	321.935	322.065	322.650

Using the table above, which of the following is *least* likely to be true?

A $\lim_{x\to 7} f(x) = 14$ and $\lim_{x\to 7} g(x) = 23$

B $\lim_{x\to 7} 3f(x) = 3\lim_{x\to 7} f(x)$

C $\lim_{x\to 7} (f(x) + g(x)) = \lim_{x\to 7} f(x) + \lim_{x\to 7} g(x)$

D $\lim_{x\to 7} (f(x)g(x)) = f(7)\left(\lim_{x\to 7} g(x)\right)$

Remark 1.3.4 In Activity 1.3.3 we observed that limits seem to be "well-behaved" when combined with standard operations on functions. The next theorems, known as **Limit Laws**, tell us how limits interact with combinations of functions.

Theorem 1.3.5 Limit Laws, I. *Let f and g be functions defined on an open interval I containing the real number c satisfying*

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K,$$

for L and K some real numbers. Then we have the following limits.

1. *Constant Law:* $\lim_{x\to c} b = b$*, for b any constant real number;*

2. *Identity Law:* $\lim_{x\to c} x = c$ *;*

3. *Sum/Difference Law:* $\lim_{x\to c}(f(x) \pm g(x)) = L \pm K$*;*

4. *Scalar Multiple Law:* $\lim_{x\to c} b \cdot f(x) = bL$, *for* b *any constant real number;*

5. *Product Law:* $\lim_{x\to c} f(x) \cdot g(x) = LK$

6. *Quotient Law: if* $K \neq 0$, *then* $\lim_{x\to c} f(x)/g(x) = L/K$

Activity 1.3.6 If $\lim_{x\to 2} f(x) = 2$ and $\lim_{x\to 2} g(x) = -3$, which of the following statements are true? Select all that apply!

A $\lim_{x\to 2}(f(x) \cdot g(x)) = -6$

B $\lim_{x\to 2}(f(x) + g(x)) = -1$

C $\lim_{x\to 2}(f(x) - g(x)) = -2$

D $\lim_{x\to 2}(f(x)/g(x)) = -2/3$

Theorem 1.3.7 Limit Laws, II. *Let* f *and* g *be functions defined on an open interval* I *containing* c *satisfying*

$$\lim_{x\to c} f(x) = L, \quad \lim_{x\to L} g(x) = K, \text{and}\, g(L) = K.$$

Then we have the following limits as well.

1. *Power Law:* $\lim_{x\to c} f(x)^n = L^n$, *for* n *a positive integer;*

2. *Root Law:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$, *for* n *a positive integer;*

3. *Composition Law:* $\lim_{x\to c} g(f(x)) = K$.

Activity 1.3.8 Below you are given the graphs of two functions. Compute the limits below (if possible).

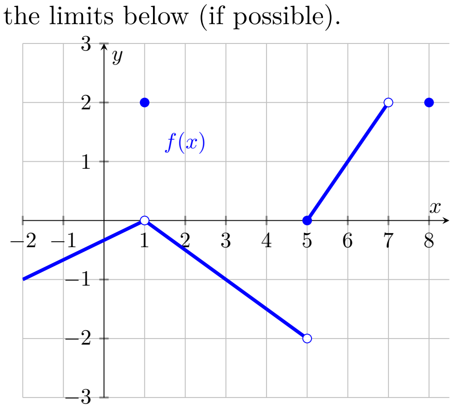


Figure .13 The graph of $f(x)$.

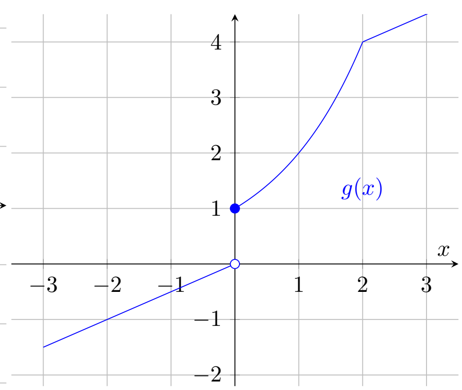


Figure .14 The graph of $g(x)$.

(a) $\lim_{x\to 1} f(x) + g(x)$.

(b) $\lim_{x\to 5^+} 3f(x)$.

(c) $\lim_{x\to 0^+} f(x)g(x)$.

(d) (Challenge) $\lim_{x\to 1} g(x)/f(x)$.

(e) (Challenge) $\lim_{x\to 0^+} f(g(x))$.

Activity 1.3.9 Given $p(x) = -3x^2 - 5x + 7$, which of the following limit laws would use to determine $\lim\limits_{x\to 2} p(x)$? Choose all that apply.

A Sums/Difference Law

B Scalar Multiple Law

C Product Law

D Identity Law

E Power Law

F Constant Law

Theorem 1.3.10 Limits of Polynomials. *If $p(x)$ is a polynomial and c is a real number, then $\lim\limits_{x\to c} p(x) = p(c)$. This is also known as the **Direct Substitution Property** for polynomials.*

Activity 1.3.11 Given $p(x) = -3x^2 - 5x + 7$ and $q(x) = x^4 - x^2 + 3$, which of the following describes the most efficient way to determine $\lim\limits_{x\to -1} \dfrac{p(x)}{q(x)}$?

A Sums/difference, scalar multiple, and product laws

B Theorem 1.3.10 and the quotient law

C Power, sums/difference, scalar multiple, and constant laws

D Quotient and root law

Theorem 1.3.12 Limits of Rational Functions. *If $p(x)$ and $q(x)$ are polynomials, c is a real number, and $q(c) \neq 0$ then $\lim\limits_{x\to c} \dfrac{p(x)}{q(x)} = \dfrac{p(c)}{q(c)}$.*

Activity 1.3.13 Consider taking the limit of a rational function $\frac{p(x)}{q(x)}$ as $x \to c$. If $q(c) = 0$, is it possible for $\lim_{x\to c} \frac{p(x)}{q(x)}$ to equal a number?

A No, because $\frac{p(x)}{q(x)}$ is not defined at $x = c$ since $q(c) = 0$.

B Yes, because if you graph $f(x) = \frac{x^2-1}{x-1}$, the value $f(1)$ is not defined, but the graph shows that the limit of $f(x)$ does exist as $x \to 1$.

C No, because if you graph $g(x) = \frac{x^2+1}{x-1}$, the value $g(1)$ is not defined and the graph shows that the limit of $\lim_{x\to c} g(x)$ does not exist.

D Yes, because we can use Theorem 1.3.12.

Activity 1.3.14 Let $f(x) = 2x$ and $g(x) = x$, which of the following statements is true?

A $\lim\limits_{x\to 0}(f(x)/g(x)) = 0$

B $\lim\limits_{x\to 0}(f(x)/g(x)) = 2$

C $\lim\limits_{x\to 0}(f(x)/g(x))$ cannot be determined

D $\lim\limits_{x\to 0}(f(x)/g(x))$ does not exist

Remark 1.3.15 When we compute the limit of a ratio where both the numerator and denominator have limit equal to zero, we have to compute the value of a $\frac{0}{0}$ **indeterminate form**. The value of an indeteminate form can be any real number or even infinity or not existent, we just do not know yet! We can usually determine the value of an indeterminate form using some algebraic manipulations of the expression given.

Definition 1.3.16 A function $f(x)$ has a **hole** at $x = c$ if $f(c)$ does not exist but $\lim\limits_{x\to c} f(x)$ does exist and is equal to a real number. ◊

Example 1.3.17 The function $f(x) = \frac{x^2-1}{x-1}$ has a hole at $x = 1$ because $f(1)$ is not defined but

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2,$$

so the limit exists and is equal to a real number. Notice that $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$ is also an example of a limit giving an indeterminate form $\frac{0}{0}$ which we could then compute using an algebraic manipulation of the function given. □

Activity 1.3.18 Determine the following limits and explain your reasoning.

$$\lim_{x\to -6} \frac{x^2-6\,x+5}{x^2-3\,x-18}$$

$$\lim_{x\to -1} \frac{x^2-1}{x^2+3\,x+2}$$

$$\lim_{x\to 5} \frac{x-5}{\sqrt{x+31}-6}$$

Activity 1.3.19 In activity Activity 1.2.8 you studied the velocity of Usain Bolt in his Beijing 100 meters dash. We will now study this situation analytically. To make our computations simpler, we will approximate that he could run 100 meters in 10 seconds and we will consider the model $d = f(t) = t^2$, where d is the distance in meters and t is the time in seconds.

Note 1.3.20 The average velocity is the ratio distance covered over time elapsed. If we consider the interval that starts at $t = a$ and has width h, written $[a, a+h]$, the average velocity on this interval is $\dfrac{f(a+h)-f(a)}{(a+h)-a} = \dfrac{f(a+h)-f(a)}{h}$. The instantaneous velocity at time $t = a$ is given by:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

(a) Compute the average velocity on the interval $[5, 6]$. We think of this interval as $[5, 5+h]$ for the value of $h = 1$.

(b) Compute the average velocity starting at 5 seconds, but now with $h = 0.5$ seconds.

(c) We want to study the instantaneous velocity at $a = 5$ seconds. Find an expression for the average velocity on the interval $[5, 5+h]$, where h is an unspecified value.

(d) Expand your expression. When $h \neq 0$, you can simplify it!

(e) Recall that the instantaneous velocity is the limit of your expression as $h \to 0$. Find the instantaneous velocity given by this model at $t = 5$ seconds.

(f) The model $d = f(t) = t^2$ does not really capture the real-world situation. Think of at least one reason why this model does not fit the scenario of Usain Bolt's 100 meters dash.

1.4 Continuity (LT4)

Learning Outcomes

- Determine where a function is and is not continuous.

Remark 1.4.1 A continuous function is one whose values change smoothly, with no jumps or gaps in the graph. We'll explore the idea first, and arrive at a mathematical definition soon.

Activity 1.4.2 Which of the following scenarios best describes a continuous function?

A The age of a person reported in years

B The price of postage for a parcel depending on its weight

C The volume of water in a tank that is gradually filled over time

D The number of likes on my latest TikTok depending on the time since I posted it

Remark 1.4.3 How would you use the language of limits to clarify the definition of continuity?

Activity 1.4.4 A function f defined on $-4 < x < 4$ has the graph pictured below. Use the graph to answer each of the following questions.

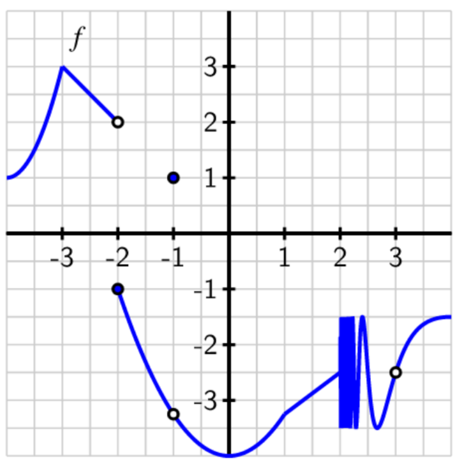


Figure .15

(a) For each of the values $a = -3, -2, -1, 0, 1, 2, 3$, determine whether the limit $\lim_{x\to a} f(x)$ exists. If the limit does not exist, be ready to explain why not.

(b) For each of the values of a where the limit of f exists, determine the value of $f(a)$ at each such point.

(c) For each such a value, is $f(a)$ equal to $\lim_{x\to a} f(x)$?

(d) Use your understanding of continuity to determine whether f is continuous at each value of a.

(e) Any revisions would you want to make to your definition of continuity that you arrived at toward the end of Remark 1.4.3?

Definition 1.4.5 A function f is **continuous** at $x = a$ provided that

1 f has a limit as $x \to a$

2 f is defined at $x = a$ (equivalently, a is in the domain of f), and

3 $\lim_{x\to a} f(x) = f(a)$.

♢

Activity 1.4.6 Suppose that some function $h(x)$ is continuous at $x = -3$. Use Definition 1.4.5 to decide which of the following quantities are equal to each other.

A $\lim\limits_{x\to -3^+} h(x)$

B $\lim\limits_{x\to -3^-} h(x)$

C $\lim\limits_{x\to -3} h(x)$

D $h(-3)$

Activity 1.4.7 Consider the function f whose graph is pictured below (it's the same graph from Activity 1.4.4). In the questions below, consider the values $a = -3, -2, -1, 0, 1, 2, 3$.

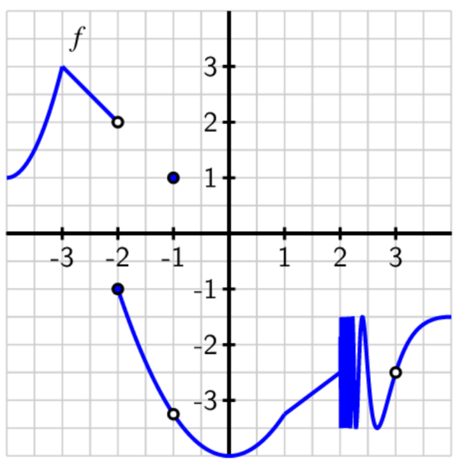


Figure .16

(a) For which values of a do we have $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$?

(b) For which values of a is $f(a)$ not defined?

(c) For which values of a does f have a limit at a, yet $f(a) \neq \lim_{x\to a} f(x)$?

(d) For which values of a does f fail to be continuous? Give a complete list of intervals on which f is continuous.

Activity 1.4.8 Which condition is *stronger*, meaning it implies the other?

A f has a limit at $x = a$, or

B f is continuous at $x = a$?

Activity 1.4.9 Previously, you have used graphs, tables, and formulas to answer questions about limits. Which of those are suitable for answering questions about continuity?

A Graphs only

B Formulas only

C Graphs and formulas only

D Tables and formulas only

Activity 1.4.10 Consider the function f whose graph is pictured below.

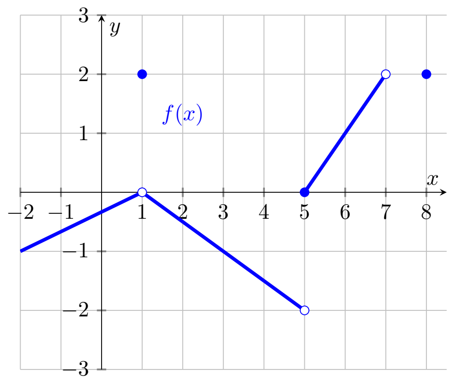


Figure .17 The graph of $f(x)$.

Give a list of x-values where $f(x)$ is not continuous. Be prepared to defend your answer based on Definition 1.4.5.

Remark 1.4.11 When $\lim_{x\to a} f(x)$ exists but is not equal to $f(a)$, we say that f has a **removable discontinuity** at $x = a$. This is because if $f(a)$ were redefined to be equal to $\lim_{x\to a} f(x)$, the redefined function would be continuous at $x = a$, thus "removing" the discontinuity.

When the left and right limit exist separately, but are not equal, the discontinuity is not removable and is called a **jump discontinuity**.

Activity 1.4.12

1. Determine the value of b to make $h(x)$ continuous at $x = 5$.

$$h(x) = \begin{cases} b - x, & x < 5 \\ -x^2 + 6\,x - 6, & x \geq 5 \end{cases}$$

2. Classify the type of discontinuity present at $x = -6$ for the function $f(x)$.

$$f(x) = \begin{cases} -8\,x - 46, & x < -6 \\ 6, & x = -6 \\ 4\,x + 30, & x > -6 \end{cases}$$

Answer.

1. To make $h(x)$ continuous at $x = 5$, let $b = 4$.

2. The function $f(x)$ has a jump discontinuity.

Theorem 1.4.13 *If f and g are continuous at $x = a$ and c is a real number, then the functions $f + g$, $f - g$, cf, and fg are also continuous at $x = a$. Moreover, f/g is continuous at $x = a$ provided that $g(a) \neq 0$.*

Activity 1.4.14 Answer the questions below about piecewise functions. It may be helpful to look at some graphs.

(a) Which values of c, if any, could make the following function continuous on the real line?

$$g(x) = \begin{cases} x + c & x \leq 2 \\ x^2 & x > 2 \end{cases}$$

(b) Which values of c, if any, could make the following function continuous on the real line?

$$h(x) = \begin{cases} 4 & x \leq c \\ x^2 & x > c \end{cases}$$

(c) Which values of c, if any, could make the following function continuous on the real line?

$$k(x) = \begin{cases} x & x \leq c \\ x^2 & x > c \end{cases}$$

Theorem 1.4.15 *Suppose that:*

- *the function f is continuous on the interval $[a, b]$;*

- *you pick a value N such that $f(a) \leq N \leq f(b)$ or $f(b) \leq N \leq f(a)$.*

Then, there is some input c in the interval $[a, b]$ such that $f(c) = N$.

Activity 1.4.16 In this activity we will explore a mathematical theorem, the Intermediate Value Theorem. Here is the statement of the theorem.

(a) To get an idea for the theorem, draw a continuous function $f(x)$ on the interval $[0, 10]$ such that $f(0) = 8$ and $f(10) = 2$. Find an input c where $f(c) = 5$.

(b) Now try to draw a graph similar to the previous one, but that does not have any input corresponding to the output 5. Then, find where your graph violates these conditions: $f(x)$ is continuous on $[0, 10]$, $f(0) = 8$, and $f(10) = 2$.

(c) The part of the theorem that starts with “Suppose…” forms the assumptions of the theorem, whilst the part of the theorem that starts with “Then…” is the conclusion of the theorem. What are the assumptions of the Intermediate Value Theorem? What is the conclusion?

(d) Apply the Intermediate Value Theorem to show that the function $f(x) = x^3 + x - 3$ has a zero (so crosses the x-axis) at some point between $x = -1$ and $x = 2$. (Hint: What interval of x values is being considered here? What is N? Why is N between $f(a)$ and $f(b)$?)

1.4.1 Slideshow

Slideshow of activities available at `LT4.slides.html`.

1.5 Limits with infinite inputs (LT5)

Learning Outcomes

- Determine limits of functions at infinity.

Activity 1.5.1 Consider the graph of the polynomial function $f(x) = x^3$. We want to think about what the long term behavior of this function might be. Which of the following best describes its behavior?

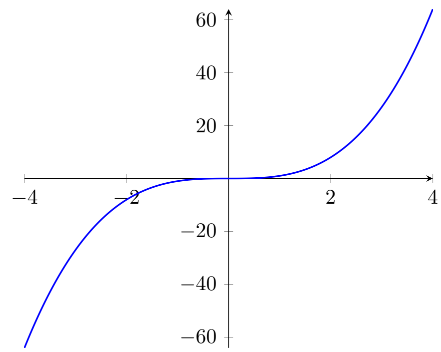


Figure .18 The graph of x^3.

A As x gets larger, the function x^3 gets smaller and smaller

B As x gets more and more negative, the function x^3 gets more and more negative

C As x gets more and more positive, the function x^3 gets more and more negative

D As x gets smaller, the function x^3 gets smaller and smaller

Remark 1.5.2 We say that "the limit as x tends to negative infinity of x^3 is negative infinity" and that "the limit as x tends to positive infinity of x^3 is positive infinity". In symbols, we write

$$\lim_{x\to+\infty} x^3 = +\infty\,, \lim_{x\to-\infty} x^3 = -\infty.$$

Activity 1.5.3 Consider the graph of the rational function $f(x) = 1/x^3$. We want to think about what the long term behavior of this function might be. Which of the following best describes its behavior?

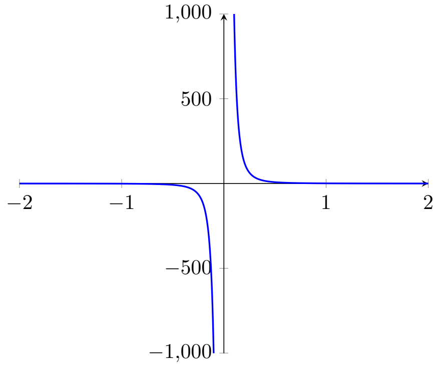


Figure .19 The graph of $1/x^3$.

A As x tends to positive infinity, the function $1/x^3$ tends to positive infinity

B As x tends to negative infinity, the function $1/x^3$ tends to 0

C As x tends to positive infinity, the function $1/x^3$ tends to negative infinity

D As x tends to 0, the function $1/x^3$ tends to 0

Definition 1.5.4 A function has a **horizontal asymptote** at $y = b$ when

$$\lim_{x \to +\infty} f(x) = b$$

or

$$\lim_{x \to -\infty} f(x) = b$$

This means that we can make the output of $f(x)$ as close as we want to b, as long as we take x a large enough positive number ($x \to \infty$) or a large enough negative number ($x \to -\infty$). $\Diamond$

Remark 1.5.5 We say that the function $1/x^3$ has horizontal asymptote y=0 because the limit as x tends to positive infinity of $1/x^3$ is 0. Alternatively, we could also justify it by saying that the limit as x goes to negative infinity is 0.

Activity 1.5.6 Which of the following functions have horizontal asymptotes? Select all!

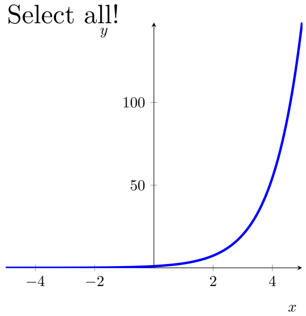


Figure .20 A

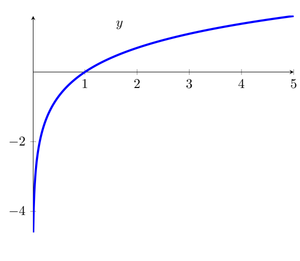


Figure .21 B

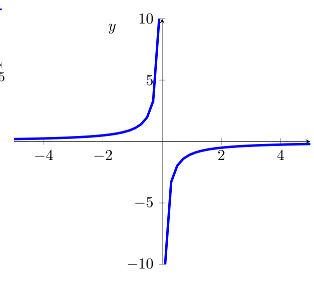


Figure .22 C

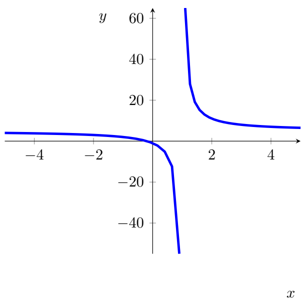


Figure .23 D

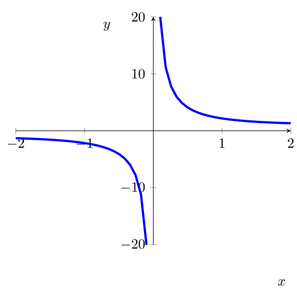


Figure .24 E

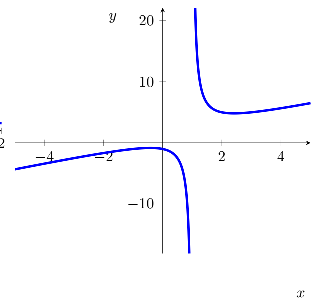


Figure .25 F

Activity 1.5.7 Recall that a rational function is a ratio of two polynomials. For any given rational function, what are all the possible behaviors as x tends to + or - infinity?

A The only possible limit is 0

B The only possible limits are 0 or $\pm\infty$

C The only possible limits are 0, 1 or $\pm\infty$

D The only possible limits are any constant number or $\pm\infty$

Activity 1.5.8 In this activity we will examine functions whose limit as x approaches positive and negative infinity is a nonzero constant.

(a) Graph the following functions and consider their limits as x approaches positive and negative infinity. Which function(s) have a limit that is nonzero and constant? Find each of these limits.

A $f(x) = \dfrac{x^3 - x + 3}{2x^3 - 6x + 1}$

B $f(x) = \dfrac{x^2 - 3}{5x^3 - 2x^2 + 5}$

C $f(x) = \dfrac{x^4 - 3x - 2}{3x^3 - 5x + 1}$

D $f(x) = \dfrac{10x^5 - 3x + 2}{5x^5 - 3x^2 + 1}$

E $f(x) = \dfrac{-8x^2 - 5x + 1}{2x^2 - 2x + 3}$

(b) Conjecture a rule for how to determine that a rational function has a nonzero constant limit as x approaches positive and negative infinity. Test your rule by creating a rational function whose limit as $x \to \infty$ equals 3 and then check it graphically.

Activity 1.5.9 What about when the limit is not a nonzero constant? How do we recognize those? In this activity you will first conjecture the general behavior of rational functions and then test your conjectures.

(a) Consider a rational function $r(x) = \frac{p(x)}{q(x)}$. Looking at the numerator $p(x)$ and the denominator $q(x)$, when does the function $r(x)$ have limit equal to 0 as $x \to \infty$?

A When the ratio of the leading terms is a constant.

B When the degree of the numerator is greater than the degree of the denominator.

C When the degree of the numerator is less than the degree of the denominator.

D When the degree of the numerator is equal to the degree of the denominator.

(b) Consider a rational function $r(x) = \frac{p(x)}{q(x)}$. Looking at the numerator $p(x)$ and the denominator $q(x)$, when does the function $r(x)$ have limit approaching infinity as $x \to \infty$?

A When the ratio of the leading terms is a constant.

B When the degree of the numerator is greater than the degree of the denominator.

C When the degree of the numerator is less than the degree of the denominator.

D When the degree of the numerator is equal to the degree of the denominator.

(c) Conjecture a rule for the each of the previous two parts of the activity. Test your rules by creating a rational function whose limit as $x \to \infty$ equals 0 and another whose limit as $x \to \infty$ is infinite. Then check them graphically.

Activity 1.5.10 Explain how to find the value of each limit.

1.
$$\lim_{x\to-\infty} -\frac{6\,x^4 + 7\,x^3 - 7}{6\,x - x^4 + 9} \text{ and } \lim_{x\to+\infty} -\frac{6\,x^4 + 7\,x^3 - 7}{6\,x - x^4 + 9}$$

2.
$$\lim_{x\to-\infty} -\frac{7\,x^4 - 5\,x^3 + 8}{3\,(2\,x^5 + 3\,x^2 - 3)} \text{ and } \lim_{x\to+\infty} -\frac{7\,x^4 - 5\,x^3 + 8}{3\,(2\,x^5 + 3\,x^2 - 3)}$$

3.
$$\lim_{x\to-\infty} \frac{3\,x^6 + x^3 - 8}{7\,x - 6\,x^5 + 7} \text{ and } \lim_{x\to+\infty} \frac{3\,x^6 + x^3 - 8}{7\,x - 6\,x^5 + 7}$$

Answer.

1.
$$\lim_{x\to-\infty} -\frac{6\,x^4 + 7\,x^3 - 7}{6\,x - x^4 + 9} = 6 \text{ and } \lim_{x\to+\infty} -\frac{6\,x^4 + 7\,x^3 - 7}{6\,x - x^4 + 9} = 6$$

2.
$$\lim_{x\to-\infty} -\frac{7\,x^4 - 5\,x^3 + 8}{3\,(2\,x^5 + 3\,x^2 - 3)} = 0 \text{ and } \lim_{x\to+\infty} -\frac{7\,x^4 - 5\,x^3 + 8}{3\,(2\,x^5 + 3\,x^2 - 3)} = 0$$

3.
$$\lim_{x\to-\infty} \frac{3\,x^6 + x^3 - 8}{7\,x - 6\,x^5 + 7} = +\infty \text{ and } \lim_{x\to+\infty} \frac{3\,x^6 + x^3 - 8}{7\,x - 6\,x^5 + 7} = -\infty$$

Activity 1.5.11 What is your best guess for the limit as x goes to $+\infty$ of the function graphed below?

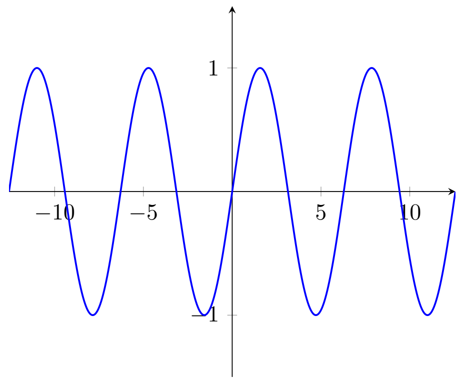


Figure .26 A mysterious periodic function.

A The limit is 0

B The limit is 1

C The limit is -1

D The limit is $+\infty$

E The limit DNE

Warning 1.5.12 For a periodic function, a function whose outputs repeat periodically, there is not one distinguished long term behavior, so the limit DNE. Notice that this is different from the limit being +infinity in which case the outputs have a clear behavior: they are getting larger and larger. Unfortunately, you will find in many cases that the notation DNE is used both for a limit equal to infinity and a limit that does not tend to one distinguished value. Beware!

Activity 1.5.13 Compute the following limits.

a $\displaystyle\lim_{x\to-\infty} \frac{x^3 - x + 83}{1}$

b $\displaystyle\lim_{x\to-\infty} \frac{1}{x^3 - x + 83}$

c $\displaystyle\lim_{x\to+\infty} \frac{x+3}{2-x}$

d $\displaystyle\lim_{x\to-\infty} \frac{\pi - 3x}{\pi x - 3}$

e (Challenge) $\displaystyle\lim_{x\to+\infty} \frac{3e^x + 2}{2e^x + 3}$

f (Challenge) $\displaystyle\lim_{x\to-\infty} \frac{3e^x + 2}{2e^x + 3}$

Activity 1.5.14 The graph below represents the function $f(x) = \dfrac{2(x+3)(x+1)}{x^2 - 2x - 3}$.

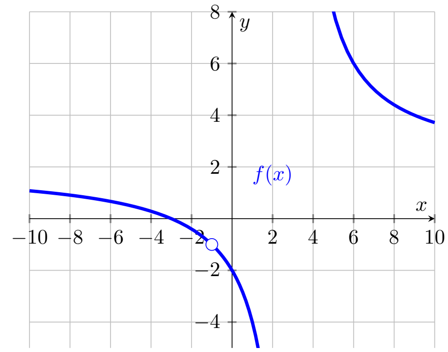


Figure .27 The graph of $f(x)$

(a) Find the horizontal asymptote of $f(x)$. First, guess it from the graph. Then, prove that your guess is right using algebra.

(b) Use limit notation to describe the behavior of $f(x)$ at its horizontal asymptotes.

(c) Come up with the formula of a rational function that has horizontal asymptote $y = 3$.

(d) What do you think is happening at $x = -1$? We will come back to this in the next section!

Activity 1.5.15 In this activity you will explore an exponential model for a cooling object.

Note 1.5.16 An exponential function $P(t) = a\,b^t$ exhibiting exponential decay will have the long term behavior $P(t) \to 0$ as $t \to \infty$. If we shift the graph up by c units, we obtain the new function $Q(t) = a\,b^t + c$, with the long term behavior $\lim_{t\to\infty} Q(t) = c$. A cooling object can be represented by the exponential decay model $Q(t) = a\,b^t + c$.

Consider a cup of coffee initially at 100 degrees Fahrenheit. The said cup of coffee was forgotten this morning on the kitchen counter where the thermostat is set at 72 degrees Fahrenheit. From previous observations, we can assume that a cup of coffee looses 10 percent of its temperature each minute.

(a) In the long run, what temperature do you expect the coffee to tend to? Write your observation with limit notation.

(b) In the model $Q(t) = a\,b^t + c$, your previous answer gives you the value of one of the parameters in this model. Which one?

(c) From the information given, we notice that the cup of coffee has decay rate of 10 % or $r = -0.1$. When an exponential model has decay rate r, its exponential base b has value $b = 1 + r$. Use this to find the value of b for the exponential model described in this scenario.

(d) Assume that the initial temperature corresponds to input $t = 0$. Use the data about the initial temperature to find the value of the parameter a in the model $Q(t) = a\,b^t + c$

(e) You should have found that this scenario has exponential model $Q(t) = 28\,(0.9)^t + 72$. If you go back to drink the cup of coffee 30 minutes after it was left on the counter, what temperature will the coffee have reached?

1.6 Limits with infinite outputs (LT6)

Learning Outcomes

- Determine limits of functions approaching vertical asymptotes.

Activity 1.6.1 Consider the graph in Figure .28.

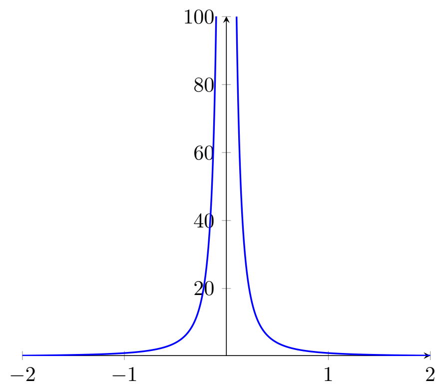


Figure .28 The graph of $1/x^2$.

(a) Which of the following best describes the limit as x approaches zero in the graph?

A The limit is 0

B The limit is positive infinity

C The limit does not exist

D This limit is negative infinity

(b) Which of the following best describes the relationship between the line $x = 0$ and the graph of the function?

A The line $x = 0$ is a horizontal asymptote for the function

B The function is not continuous at the point $x = 0$

C The function is moving away from the line $x = 0$

D The function is getting closer and closer to the line $x = 0$

E The function has a jump in outputs around $x = 0$

Definition 1.6.2 A function has a **vertical asymptote** at $x = a$ when

$$\lim_{x\to a} f(x) = +\infty$$

or

$$\lim_{x\to a} f(x) = -\infty$$

The limit being equal to positive infinity means that we can make the output of $f(x)$ as large a positive number as we want as long as we are sufficiently close to $x = a$. Similarly, the limit being equal to negative infinity means that we can make the output of $f(x)$ as large a negative number as we want as long as we are sufficiently close to $x = a$. ♢

Activity 1.6.3 Which of the following functions have vertical asymptotes? Select all!

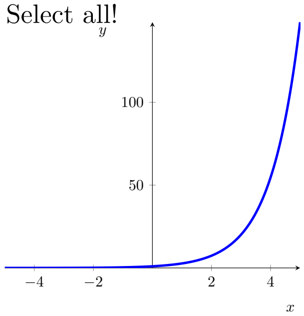


Figure .29 A

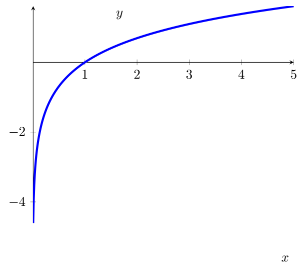


Figure .30 B

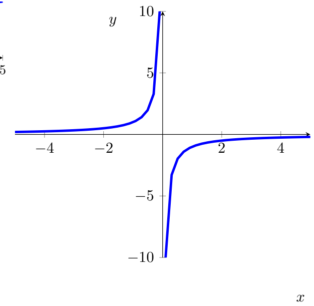


Figure .31 C

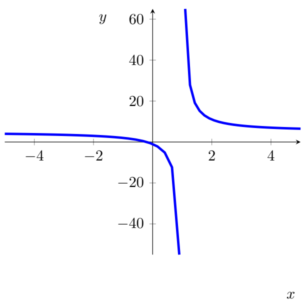


Figure .32 D

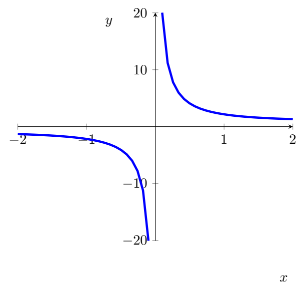


Figure .33 E

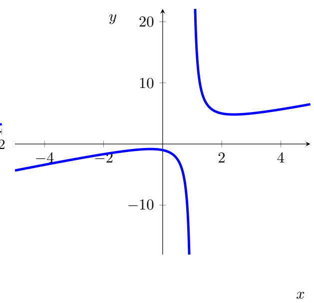


Figure .34 F

Remark 1.6.4 If $x = a$ is a vertical asymptote for the function $f(x)$, the function $f(x)$ is not defined at $x = a$. As $f(a)$ does not exist, the function is NOT continuous at $x = a$. Moreover, the function's output tends to plus or minus infinity and so the limit is not equal to a number.

Activity 1.6.5 Notice that as x goes to 0, the value of x^2 goes to 0 but the value of $1/x^2$ goes to infinity. What is the best explanation for this behavior?

A When dividing by an increasingly small number we get an increasing big number

B When dividing by an increasingly large number we get an increasing small number

C A rational function always has a vertical asymptote

D A rational function always has a horizontal asymptote

Remark 1.6.6 Informally, we say that the limit of "$\frac{1}{0}$" is infinite. Notice that this could be either positive or negative infinity, depending on how whether the outputs are becoming more and more positive or more and more negative as we approach zero.

Activity 1.6.7 Consider the rational function $f(x) = \frac{2}{x-3}$. Which of the following options best describes the limits as x approaches 3 from the right and from the left?

A As $x \to 3^+$, the limit DNE, but as $x \to 3^-$ the limit is $-\infty$

B As $x \to 3^+$, the limit is $+\infty$, but as $x \to 3^-$ the limit is $-\infty$

C As $x \to 3^+$, the limit is $+\infty$, but as $x \to 3^-$ the limit is $+\infty$

D As $x \to 3^+$, the limit is $-\infty$, but as $x \to 3^-$ the limit is $-\infty$

E As $x \to 3^+$, the limit DNE and as $x \to 3^-$ the limit DNE

Remark 1.6.8 When considering a ratio of functions $f(x)/g(x)$, the inputs a where $g(a) = 0$ are not in the domain of the ratio. If $g(a) = 0$ but $f(a)$ is not equal to 0, then $x = a$ is a vertical asymptote

Activity 1.6.9 Consider the function $f(x) = \frac{x^2-1}{x-1}$. The line $x = 1$ is NOT a vertical asymptote for $f(x)$. Why?

A When x is not equal to 1, we can simplify the fraction to $x - 1$, so the limit is 1.

B When x is not equal to 1, we can simplify the fraction to $x + 1$, so the limit is 2.

C The function is always equal to $x + 1$.

D The function is always equal to $x - 1$.

Remark 1.6.10 Recall the definition of a hole from Definition 1.3.16. In Activity 1.6.9 we have a hole at $x = 1$.

Activity 1.6.11 Find all the vertical asymptotes of the following rational function

(a) $y = \frac{3x-4}{7x+1}$

(b) $y = \frac{x^2+10x+24}{x^2-2x+1}$

(c) $y = \frac{(x^2-4)(x^2+1)}{x^6}$

(d) $y = \frac{2x+1}{2x^2+8x-10}$

Activity 1.6.12 Explain and demonstrate how to find the value of each limit.

1.
$$\lim_{x \to -3^-} \frac{(x+4)^2(x-2)}{(x+3)(x-5)}$$

2.
$$\lim_{x \to -3^+} \frac{(x+4)^2(x-2)}{(x+3)(x-5)}$$

3.
$$\lim_{x \to -3} \frac{(x+4)^2(x-2)}{(x+3)(x-5)}$$

Answer.

1.
$$\lim_{x \to -3^-} \frac{(x+4)^2(x-2)}{(x+3)(x-5)} = -\infty$$

2.
$$\lim_{x \to -3^+} \frac{(x+4)^2(x-2)}{(x+3)(x-5)} = +\infty$$

3.
$$\lim_{x \to -3} \frac{(x+4)^2(x-2)}{(x+3)(x-5)} \text{ does not exist}$$

Activity 1.6.13 The graph below represents the function $f(x) = \dfrac{(x+2)(x+4)}{x^2+3x-4}$.

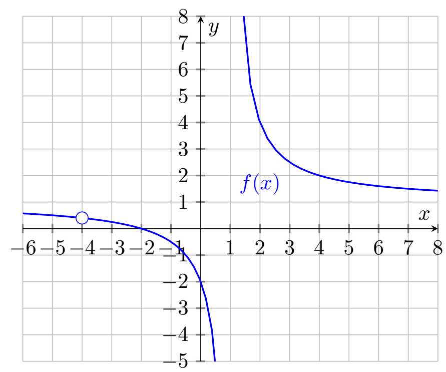


Figure .35 The graph of $f(x)$

(a) Explain the behavior of $f(x)$ at $x = -4$.

(b) Find the vertical asymptote(s) of $f(x)$. First, guess it from the graph. Then, prove that your guess is right using algebra.

(c) Find the horizontal asymptote(s) of $f(x)$. First, guess it from the graph. Then, prove that your guess is right using algebra.

(d) Use limit notation to describe the behavior of $f(x)$ at its asymptotes.

Activity 1.6.14 Consider the following rational function.

$$r(x) = \frac{5\,(x-3)(x-6)^3}{6\,(x+2)^3(x-3)}$$

1. Explain how to find the horizontal asymptote(s) of $r(x)$, if there are any. Then express your findings using limit notation.
2. Explain how to find the hole(s) of $r(x)$, if there are any. Then express your findings using limit notation.
3. Explain how to find the vertical asymptote(s) of $r(x)$, if there are any. Then express your findings using limit notation.
4. Draw a rough sketch of $r(x)$ that showcases all the limits that you have found above.

Answer.

1. The horizontal asymptote is $\frac{5}{6}$
2. The vertical asymptote is -2
3. There is a hole at $(3, -\frac{9}{50})$

Activity 1.6.15 You want to draw a function with all these properties.

- $\lim\limits_{x\to 3} f(x) = 5$
- $f(3) = 0$

- $\lim_{x \to 0^-} f(x) = -\infty$
- $\lim_{x \to 0^+} f(x) = 0$
- $\lim_{x \to +\infty} f(x) = 2$

Before you start drawing, consider the following guiding questions.

(a) At which x values will the limit not exist?

(b) What are the asymptotes of this function?

(c) At which x values will the function be discontinuous?

(d) Draw the graph of one function with all the properties above. Make sure that your graph is a function! You only need to draw a graph, writing a formula would be very challenging!

Chapter 2

Derivatives (DF)

Learning Outcomes

Need a question for this chapter

By the end of this chapter, you should be able to...

1. Estimate the value of a derivative using difference quotients, and draw corresponding secant and tangent lines on the graph of a function.
2. Find derivatives using the definition of derivative as a limit.
3. Compute basic derivatives using algebraic rules.
4. Compute derivatives using the Product and Quotient Rules.
5. Compute derivatives using the Chain Rule.
6. Compute derivatives using a combination of algebraic derivative rules.
7. Compute derivatives of implicitly-defined functions.
8. Compute derivatives of inverse functions.

Readiness Assurance. Before beginning this chapter, you should be able to...

a Write equations of lines using slope-intercept and/or point-slope form (Math is Fun)

b Find the average rate of change of a function over some interval (Khan Academy)

c Evaluate functions at variable expressions (Purple Math)

d Use the laws of exponents to rewrite a given expression (Khan Academy (1) and Khan Academy (2))

e Compose and decompose functions (Khan Academy)

f Recall special trig values on the unit circle (Khan Academy (1) and Khan Academy (2))

2.1 Derivatives graphically and numerically (DF1)

Learning Outcomes

- Estimate the value of a derivative using difference quotients, and draw corresponding secant and tangent lines on the graph of a function.

Activity 2.1.1 In this activity you will study the velocity of a ball falling under gravity. The height of the ball (in feet) is given by the formula $f(t) = 64-16(t-1)^2$, where t is measured in seconds. We want to study the velocity at the instant $t = 2$, so we will look at smaller and smaller intervals around $t = 2$. For your convenience, below you will find a table of values for $f(t)$. Recall that the average velocity is given by the change in height over the change in time.

Table .36

t (seconds)	1	1.5	1.75	2	2.25	2.5	3
$f(t)$ (feet)	64	60	55	48	39	28	0

(a) To start we will look at an interval of length one before $t = 2$ and after $t = 2$, so we consider the intervals $[1, 2]$ and $[2, 3]$. What was the average velocity on the interval $[1, 2]$? What about on the interval $[2, 3]$?

(b) Now let's consider smaller intervals of length 0.5. What was the average velocity on the interval $[1.5, 2]$? What about on the interval $[2, 2.5]$?

(c) What was the average velocity on the interval $[1.75, 2]$? What about on the interval $[2, 2.25]$?

(d) If we wanted to approximate the velocity at the instant $t = 2$, what would be your best estimate for this instantaneous velocity?

Observation 2.1.2 If we want to study the velocity at the instant $t = 2$, it is helpful to study the average velocity on small intervals around $t = 2$. If we consider the interval $[2, 2+h]$, where h is the width of the interval, the average velocity is given by the difference quotient

$$\frac{f(2+h)-f(2)}{(2+h)-2} = \frac{f(2+h)-f(2)}{h}.$$

Observation 2.1.3 We want to be able to consider intervals before and after $t = 2$. A positive value of h will give an interval after $t = 2$ for example, the interval $[2, 3]$ for $h = 1$. A negative value of h will give an interval before $t = 2$ for example, the interval $[1, 2]$ corresponds $h = -1$. In the formula above, it looks like the interval would be $[2, 1]$, but the standard notation in an interval is to write the smallest number first. This does not change the difference quotient because

$$\frac{f(2+h)-f(2)}{(2+h)-2} = \frac{f(2)-f(2+h)}{2-(2+h)}.$$

Activity 2.1.4 Consider the height of the ball falling under gravity as in Table .36 .

(a) What was the average velocity on the interval $[2, 2+h]$ for $h = 1$ and $h = -1$?

(b) What was the average velocity on the interval $[2, 2+h]$ for $h = 0.5$ and $h = -0.5$?

(c) What was the average velocity on the interval $[2, 2+h]$ for $h = 0.25$ and $h = -0.25$?

(d) What is your best estimate for the limiting value of these velocities as $h \to 0$? Notice that this is your estimate for the instantaneous velocity at $t = 2$!

Definition 2.1.5 The instanteous velocity at $t = a$ is the limit as $h \to 0$ of the difference quotient $\frac{f(a+h)-f(a)}{h}$. In the activity above the instantaneous velocity at $t = 2$ is given by the limit

$$v(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

◊

Definition 2.1.6 The slope of the secant line to $f(x)$ through the points $x = a$ and $x = b$ is given by the difference quotient

$$\frac{f(b) - f(a)}{b - a}.$$

◊

Activity 2.1.7 In this activity you will study the slope of a graph at a point. The graph of the function $g(x)$ is given below. For your convenience, below you will find a table of values for $g(x)$.

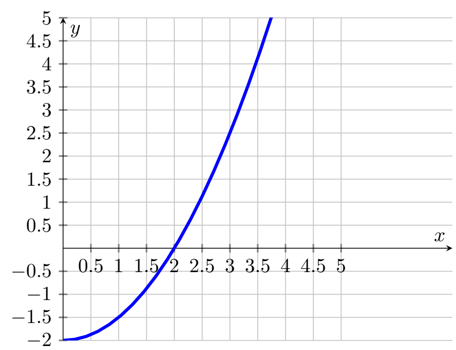


Figure .37 The graph of $g(x)$

Table .38

x	1	1.5	2	2.5	3
$g(x)$	-1.5	-0.875	0	1.125	2.5

(a) What is the slope of the line through $(1, g(1))$ and $(2, g(2))$? Draw this line on the graph of $g(x)$.

(b) What is the slope of the line through $(1.5, g(1.5))$ and $(2, g(2))$? Draw this line on the graph of $g(x)$.

(c) Draw the line tangent to $g(x)$ at $x = 2$. What would be your best estimate for the slope of this tangent line?

(d) Notice that the slope of the tangent line at $x = 2$ is positive. What feature of the graph of $f(x)$ around $x = 2$ do you think causes the tangent line to have positive slope?

A The function $f(x)$ is concave up

B The function $f(x)$ is increasing

C The function $f(x)$ is concave down

D The function $f(x)$ is decreasing

Observation 2.1.8 The slope of the secant line to $f(x)$ through the points $x = a$ and $x = b$ is given by the difference quotient $\frac{f(b)-f(a)}{b-a}$. As the point $x = b$ gets closer to $x = a$, the slope of the secant line tends to the slope of the tangent line. Letting $b = a + h$, we say that the slope of the tangent line at $x = a$ is given by the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Definition 2.1.9 The derivative of $f(x)$ at $x = a$, denoted $f'(a)$, is given by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

♢

Observation 2.1.10 In Activity 2.1.1 and Activity 2.1.4 you studied a ball falling under gravity and estimated the instantaneous velocity as a limiting value of average velocities on smaller and smaller intervals. Drawing the corresponding secant lines, we see how the secant lines approximate better the tangent line, showing graphically what we previouly saw numericaly. Here is a Desmos animation showing the secant lines approaching the tangent line `https://www.desmos.com/calculator/bzs1bxz7fa`.

Activity 2.1.11 Suppose that the function $f(x)$ gives the position of an object at time x. Which of the following quantities are the same? Select all that apply!

A The value of the derivative of $f(x)$ at $x = a$

B The slope of the tangent line to $f(x)$ at $x = a$

C The instantaneous velocity of the object at $x = a$

D The difference quotient $\frac{f(a+h)-f(a)}{h}$

E The limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

Observation 2.1.12 We can use the difference quotient $\frac{f(a+h)-f(a)}{h}$ for small values of h to estimate $f'(a)$, the value of the derivative at $x = a$.

Activity 2.1.13 Suppose that you know that the function $g(x)$ has values $g(-0.5) = 7$, $g(0) = 4$, and $g(0.5) = 2$. What is your best estimate for $g'(0)$?

A $g'(0) \approx -3$

B $g'(0) \approx -2$

C $g'(0) \approx -6$

D $g'(0) \approx -4$

E $g'(0) \approx -5$

Activity 2.1.14 Suppose that you know that the function $f(x)$ has value $f(1) = 3$ and has derivative at $x = 1$ given by $f'(1) = 2$. Which of the following scenarios is most likely?

A $f(2) = 3$ because the function is constant

B $f(2) = 2$ because the derivative is constant

C $f(2) \approx 1$ because the function's output decreases by about 2 units for each increase by 1 unit in the input

D $f(2) \approx 5$ because the function's output increases by about 2 units for each increase by 1 unit in the input

Observation 2.1.15 We can use the derivative at $x = a$ to estimate the increase/decrease of the function $f(x)$ close to $x = a$. A positive derivative at $x = a$ suggests that the output values are increasing around $x = a$ approximately at a rate given by the value of the derivative. A negative derivative at $x = a$ suggests that the output values are decreasing around $x = a$ approximately at a rate given by the value of the derivative.

Activity 2.1.16 In this activity you will study the abolute value function $f(x) = |x|$. The absolute value function is a piecewise defined function which outputs x when x is positive (or zero) and outputs $-x$ when x is negative. So the absolute value always outputs a number which is positive (or zero). Here is the graph of this function.

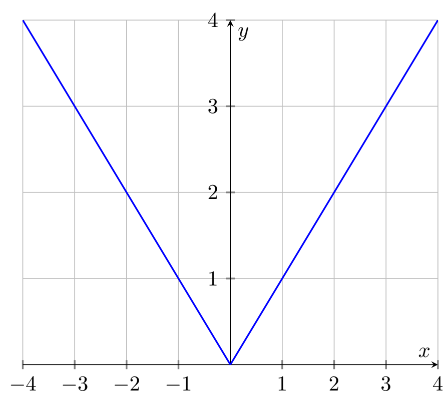


Figure .39 The graph of $|x|$

(a) What do you think is the slope of the function for any x value smaller than zero?

A 0

B 1

C -1

D DNE

(b) What do you think is the slope of the function for any x value greater than zero?

A 0

B 1

C -1

D DNE

(c) What do you think is the slope of the function at zero?

A 0

B 1

C -1

D DNE

Observation 2.1.17 Because the derivative at a point is defined in terms of a limit, the quantity $f'(a)$ might not exist! In that case we say that $f(x)$ is not differentiable at $x = a$. This might happen when the slope on the left of the point is different from the slope on the right, like in the case of the absolute value function. We call this behavior a corner in the graph.

Activity 2.1.18 Consider the graph of function $h(x)$.

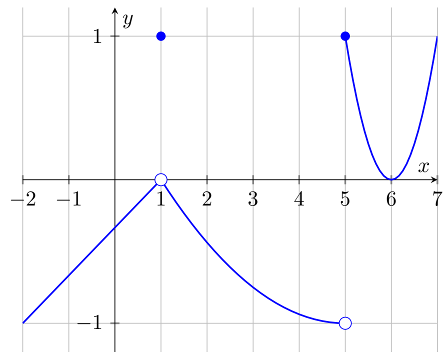


Figure .40 The graph of $h(x)$.

(a) For which of the following points a is $f'(a)$ positive? Select all that apply!

A -1

B 1

C 2

D 5

E 6

(b) For which of the following points a is $f'(a)$ negative? Select all that apply!

A -1

B 1

C 2

D 5

E 6

(c) For which of the following points a is $f'(a)$ zero? Select all that apply!

A -1

B 1

C 2

D 5

E 6

(d) For which of the following points a the quantity $f'(a)$ does NOT exist? Select all that apply!

A -1

B 1

C 2

D 5

E 6

Activity 2.1.19 Sketch the graph of a function $f(x)$ that satisfies the following criteria. (You do not need to define the function algebraically.)

- Defined and continuous on the interval $[-5, 5]$.
- $f'(x)$ does not exist at $x = 0$
- $\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h} < 0$
- The slope tangent to the graph of $f(x)$ at $x = 3$ is zero
- The rate of change of $f(x)$ when $x = -1$ is positive

Activity 2.1.20 You are given the graph of the function $f(x)$.

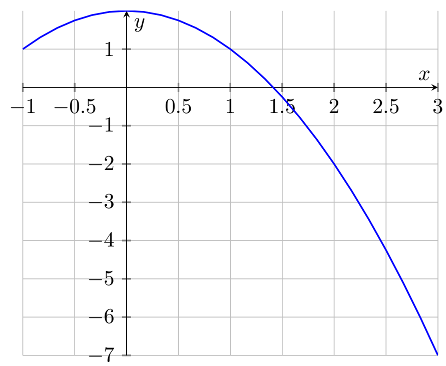


Figure .41 The graph of $f(x)$

(a) Using the graph, estimate the slope of the tangent line at $x = 2$. Make sure you can carefully describe your process for obtaining this estimate!

(b) If you call your approximation for the slope m, which one of the following expression gives you the equation of the tangent line at $x = 2$?

A $y - 2 = m(x - 2)$

B $y + 2 = m(x - 2)$

C $y - 2 = m(x + 2)$

D $y + 2 = m(x + 2)$

(c) Find the equation of the tangent line at $x = 2$.

2.2 Derivatives analytically (DF2)

Learning Outcomes

- Find derivatives using the definition of derivative as a limit.

Observation 2.2.1 Recall that $f'(a)$, the derivative of $f(x)$ at $x = a$, was defined as the limit as $h \to 0$ of the difference quotient on the interval $[a, a+h]$ as in Definition 2.1.9. If $f'(a)$exists, then say that $f(x)$ is differentiable at a. If for some interval I, $f'(x)$ exists for every point x in I, then we say that $f(x)$ is differentiable on I.

Activity 2.2.2 For the function $f(x) = x - x^2$ use the limit definition of the derivative at a point to compute $f'(2)$.

A $f'(2) = \lim_{h\to 0} \dfrac{(2+h)-(2+h)^2-2+4}{h} = -3$

B The limit $f'(2) = \lim_{h\to 0} \frac{(2+h)-(2+h)^2-2}{h}$ simplifies algebraically to $\lim_{h\to 0} \frac{-3h-h^2}{h}$ which does not exist, thus $f'(2)$ is not defined.

C The limit $f'(2) = \lim_{h\to 0} \frac{(2+h)-(2+h)^2-2}{h}$ simplifies algebraically to $\lim_{h\to 0} \frac{h-h^2}{h}$ which does not exist, thus $f'(2)$ is not defined.

D $f'(2) = \lim_{h\to 0} \dfrac{(2+h)-(2^2+h^2)-2+4}{h} = 1$

Activity 2.2.3 Consider the function $f(x) = 3 - 2x$. Which of the following best summarizes the average rates of changes of on f on the intervals $[1, 4]$, $[3, 7]$, and $[5, 5+h]$?

A The average rate of change on the intervals $[1, 4]$ and $[3, 7]$ are equal to the slope of $f(x)$, but the average rate of change of f cannot be determined on $[5, 5+h]$ without a specific value of h.

B The average rate of change on the intervals $[1, 4]$, $[3, 7]$, and $[5, 5+h]$ are all different values.

C The average rate of change on the intervals $[1, 4]$, $[3, 7]$, and $[5, 5+h]$ are all equal to -2.

Activity 2.2.4 Can you find $f'(\pi)$ when $f(x) = 3 - 2x$ without computing?

A No, because we cannot compute the value $f(\pi)$.

B No, because we cannot compute the average rate of change on the interval $[\pi, \pi+h]$.

C Yes, $f'(\pi) = 3$ because the intercept of the tangent line at any point is equal to the constant intercept of $f(x)$.

D Yes, $f'(\pi) = -2$ because the slope of the tangent line at any point is equal to the constant slope of $f(x)$.

Definition 2.2.5 Let $f(x)$ be function that is differentiable on an open interval I. The derivative function of $f(x)$, denoted $f'(x)$, is given by the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

At any particular input $x = a$, the derivative function outputs $f'(a)$, the value of derivative at the point $x = a$.

To specify the indendent variable of our functions, we say that $f'(x)$ is the derivative of $f(x)$ with respect to x. When $y = f(x)$ is a function that is differentiable on an interval I, each of the following are equivalent expressions of the derivative function of $y = f(x)$:

$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx}$. ◊

Observation 2.2.6 Notice that our notation for derivatives of functions is based on the names that we assign functions along with our choices of notations we use for indendent and dependent variables. For example, if we have a differentiable function $y = v(t)$, the derivative function of $v(t)$ with respect to t can be written as $v'(t) = y' = \frac{dy}{dt} = \frac{dv}{dt}$.

Activity 2.2.7 In this activity you will consider $f(x) = -x^2 + 4$ and compute its derivative function $f'(x)$ using the limit definition of the derivative function Definition 2.2.5.

(a) What expression do you get when you simplify the difference quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{(-(x+h)^2 + 4) - (-x^2 + 4)}{h}?$$

A $\dfrac{x^2 + h^2 + 4 - x^2 - 4}{h} = \dfrac{h^2}{h}$

B $\dfrac{-x^2 - h^2 + 4 + x^2 - 4}{h} = \dfrac{-h^2}{h}$

C $\dfrac{-x^2 - 2xh - h^2 + 4 + x^2 - 4}{h} = \dfrac{-2xh - h^2}{h}$

D $\dfrac{x^2 + 2xh + h^2 + 4 - x^2 - 4}{h} = \dfrac{2xh + h^2}{h}$

(b) After taking the limit as $h \to 0$, which of the following is your result for the derivative function $f'(x)$?

A $f'(x) = x$

B $f'(x) = -x$

C $f'(x) = 2x$

D $f'(x) = -2x$

Activity 2.2.8 Using the limit definition of the derivative, find $f'(x)$ for $f(x) = -x^2 + 2x - 4$. Which of the following is an accurate expression for $f'(x)$?

A $f'(x) = 2x + 2$

B $f'(x) = -2x$

C $f'(x) = -2x + 2$

D $f'(x) = -2x - 2$

Activity 2.2.9 Using the limit definition of the derivative, you want to find $f'(x)$ for $f(x) = \frac{1}{x}$. We will do this by first simplifying the difference quotient and then taking the limit as $h \to 0$.

(a) What expression do you get when you simplify the difference quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h}?$$

A $\dfrac{\frac{1}{x+h}}{h} = \dfrac{1}{(x+h)h}$

B $\dfrac{\frac{h}{x+h}}{h} = \dfrac{h}{h(x+h)}$

C $\dfrac{\frac{x-(x+h)}{(x+h)x}}{h} = \dfrac{-h}{h(x+h)x}$

D $\dfrac{\frac{x-(x+h)}{(x+h)x}}{h} = \dfrac{-h^2}{(x+h)x}$

E $\dfrac{\frac{h}{(x+h)x}}{h} = \dfrac{h}{h(x+h)x}$

(b) After taking the limit as $h \to 0$, which of the following is your result for the derivative function $f'(x)$?

A $f'(x) = 0$

B $f'(x) = 1/x$

C $f'(x) = -1/x$

D $f'(x) = 1/x^2$

E $f'(x) = -1/x^2$

Observation 2.2.10 We now have two ways to compute the derivative at a point! For example, to compute the derivative of $f(x)$ at $x = 1$, we have two methods

1. We can directly compute $f'(1)$ by finding the difference quotient on the interval $[1, 1+h]$ and then taking the limit as $h \to 0$.

2. We can first find the derivative function $f'(x)$ by finding the difference quotient on the interval $[x, x+h]$, taking the limit as $h \to 0$, and then finally plugging in $x = 1$ for the resulting expression for $f'(x)$.

The latter approach is more convenient when you want to consider the value of the derivative function at multiple points!

Activity 2.2.11 Consider the function $f(x) = \frac{1}{x^2}$. You will find $f'(1)$ in two ways!

(a) Using the limit definition of the derivative at a point, compute the difference quotient on the interval $[1, 1+h]$ and then take the limit as $h \to 0$. What do you get?

A -1

B 1

C 2

D -2

(b) Now, using the limit definition of the derivative function, find $f'(x)$. Which of the following is your result for the derivative function $f'(x)$?

A $f'(x) = -1/x^3$

B $f'(x) = 1/x^3$

C $f'(x) = -2/x^3$

D $f'(x) = 2/x^3$

(c) Make sure that your answers match! So if you plug in $x = 1$ in $f'(x)$, you should get the same number you got when you computed $f'(1)$.

Activity 2.2.12 In this activity you will study (again!) the velocity of a ball falling under gravity. A ball is tossed vertically in the air from a window. The height of the ball (in feet) is given by the formula $f(t) = 64 - 16(t-1)^2$, where t is the seconds after the ball is launched. Recall that in Activity 2.1.1, you used numerical methods to approxmiate the instantaneous velocity of $f(t)$ to calculate $v(2)$!.

(a) Using the limit definition of the derivative function, find the velocity function of the ball $v(t) = f'(t)$.

(b) Using the velocity function $v(t)$, what is $v'(1)$, the instantaneous velocity at $t = 1$?

A -32 feet per second

B 32 feet per second

C 0 feet per second

D -16 feet per second

E 16 feet per second

(c) What behavior would explain your finding?

A After 1 second the ball is falling at a speed of 32 meters per second.

B After 1 second the ball is moving upwards at a speed of 32 meters per second.

C After 1 second the ball reaches its highest point and it stops for an instant.

D After 1 second the ball is falling at a speed of 16 meters per second.

E After 1 second the ball is moving upwards at a speed of 16 meters per second.

Activity 2.2.13 In Observation 2.1.17, we said that a function is not differentiable when the limit that defines it does not exist. In this activity we will study differentiability analytically.

(a) Consider the following continuous function

$$g(x) = \begin{cases} x+2 & x \leq 2 \\ x^2 & x > 2 \end{cases}$$

Consider the interval $[2, 2+h]$. When $h < 0$, the interval falls under the first definition of $g(x)$ and the derivative is always equal to 1. What is the derivative function for x values greater than 2? Show that at $x = 2$ the value of this derivative is not equal to 1 and so $g(x)$ is not differentiable at $x = 2$.

(b) Consider the following discontinuous function

$$g(x) = \begin{cases} x+2 & x \leq 2 \\ x & x > 2 \end{cases}$$

On both sides of $x = 2$ it seems that the slope is the same, but this function is still not differentiable at $x = 2$. Notice that $g(2) = 4$. When $h > 0$, the interval $[2, 2+h]$falls under the second definition of $g(x)$, but $g(2)$ is always fixed at 4. Compute the difference quotient $\frac{g(2+h)-g(2)}{h}$ assuming that $h > 0$ and notice that this does not simplify as expected! Moreover, if you take the limit as $h \to 0$, you will get infinity and not the expected slope of 1!

(c) Consider the following function

$$g(x) = \begin{cases} ax+2 & x \leq 2 \\ bx^2 & x > 2 \end{cases}$$

where a, b are some nonzero parameters you will find. Find an equation in a, b that needs to be true if we want the function to be continuous at $x = 2$. Also, find an equation in a, b that needs to be true if we want the function to be differentiable at $x = 2$. Solve the system of two linear equations... you should find that $a = -2$ and $b = -1/2$ are the only values that make the function differentiable (and continuous!).

Observation 2.2.14 A function can only be differentiable at $x = a$ if it is also continuous at $x = a$. But not all continuous functions are differentiable: when we have a corner in the graph of a the function, the function is continuous at the corner point, but it is not differentiable at that point!

2.3 Elementary derivative rules (DF3)

Learning Outcomes

- Compute basic derivatives using algebraic rules.

Observation 2.3.1 If we recall from the previous section we learned about how to find the derivative of functions using the limit definition of a derivative. From the activities we see that this gets cumbersome when the functions are more complicated. In this section we will discuss a more convenient way to to calculate derivatives.

Activity 2.3.2 In this activity we will try to deduce a rule for finding the derivative of a power function. Note, a power function is one in which the function is of the form $f(x) = x^n$ where n is any real number.

(a) Using the limit definition of the derivative, what is $f'(x)$ for the power function $f(x) = x$?

A -1

B 1

C 0

D Does not exist

(b) Using the limit definition of the derivative, what is $f'(x)$ for the power function $f(x) = x^2$?

A 0

B $-2x$

C $2x$

D $2x + 1$

(c) Using the limit definition of the derivative, what is $f'(x)$ for the power function $f(x) = x^3$?

A $3x^2$

B $-3x^2$

C $3x^2 - 3x$

D $-3x^2 + 3x$

(d) WITHOUT using the limit definition of the derivative, what is $f'(x)$ for the power function $f(x) = x^4$? (See if you can find a pattern from the first three tasks of this activity.)

A $3x^2$

B $3x^3$

C $4x^2$

D $4x^3$

Theorem 2.3.3 The Power Rule. *The derivative of the functionf $(x) = x^n$, for any real number n is*

$$f'(x) = nx^{n-1}.$$

Observation 2.3.4 So far we have been using $f'(x)$ or f prime to denote a derivative of the function $f(x)$. However there are other ways to denote the derivative of a function. Besides $f'(x)$ there is also $\frac{df}{dx}$, pronounced "dee-f dee-x." Both notations can be used interchangeably. If you want to take the derivative of $f'(x)$ or $\frac{df}{dx}$, the notation is $f''(x)$ or $\frac{d^2f}{dx^2}$; this is known as second derivative of $f(x)$.

Activity 2.3.5 Using Theorem 2.3.3, which of the following statement(s) are true? For those statements that are wrong, give the correct derivative.

A The derivative of $y = x^{10}$ is $y' = 10x^{11}$.

B The derivative of $y = x^{-8}$ is $y' = -8x^{-9}$.

C The derivative of $y = x^{100}$ is $y' = 100x^{99}$.

D The derivative of $y = x^{-17}$ is $y' = -17x^{-16}$..

Theorem 2.3.6 The Derivative of a Constant Function. *The If $f(x) = c$, for some constant $c \in \mathbb{R}$, then $f'(x) = 0$.*

Activity 2.3.7 Using Theorem 2.3.6, which of the following statement(s) are true? Note: Pay attention to the dependent variable of the function.

A The derivative of $y(x) = 10$ is $y'(x) = 9$.

B The derivative of $y(t) = x$ is $y'(t) = 0$.

C The derivative of $y(a) = x^2$ is $y'(a) = 2x$.

D The derivative of $y(x) = -5$ is $y'(x) = -4$..

Theorem 2.3.8 The Constant Multiple Rule. *If c is a constant and $f(x)$ is a differentiable function, then*

$$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right]$$

Activity 2.3.9 What is the derivative of the function $y(x) = 12x^{2/3}$?

A $y'(x) = 8x^{5/3}$.

B $y'(x) = 18x^{-1/3}$.

C $y'(x) = 8x^{-1/3}$.

D $y'(x) = 18x^{5/3}$.

Theorem 2.3.10 The Sum and Difference Rule. *If $f(x)$ and $g(x)$ are both differentiable, then*

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

Activity 2.3.11 What is the derivative for the arbitrary parabola given by $f(x) = ax^2 + bx + c$, where a, b, c are any real numbers and $a \neq 0$?

A $f'(x) = 2ax + bx + c$.

B $f'(x) = 2x + 1$.

C $f'(x) = 2ax + b$.

D $f'(x) = ax + b$.

Theorem 2.3.12 The Derivative of an Exponential Function. *For any real number a, if $f(x) = a^x$, then $f'(x) = a^x \ln(a)$.*

Observation 2.3.13 A special case of Theorem 2.3.12 is when $a = e$, where e is the base of the natural logarithm function. In this case let $f(x) = e^x$. Then

$$f'(x) = e^x \ln(e) = e^x.$$

Activity 2.3.14 The first derivative of the function $g(x) = x^e + e^x$ is given by $g'(x) = ex^{e-1} + e^x$. What is the second derivative of $g(x)$?

A $g''(x) = x^e + e^x$.

B $g''(x) = e(e-1)x^{e-2} + e^x$.

C $g''(x) = ex^{e-1} + e^x$.

D $g''(x) = e^x$.

Theorem 2.3.15 The Derivative of the Sine and Cosine Functions. *If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$. If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.*

Activity 2.3.16 The derivative of $f(x) = 7\sin(x) + 2e^x + 3x^{1/3} - 2$ is,

A $f'(x) = 7\cos(x) + 2e^x + x^{-2/3} - 2x$.

B $f'(x) = 7\cos(x) + 2e^x + -2x^{-2/3} - 2$.

C $f'(x) = -7\sin(x) + e^x + x^{-2/3}$.

D $f'(x) = -7\cos(x) + 2e^x \ln(x) + x^{-2/3}$.

E $f'(x) = 7\cos(x) + 2e^x + x^{-2/3}$.

Theorem 2.3.17 The Derivative of the Natural Log Function. *If $f(x) = \ln(x)$, then*

$$f'(x) = \frac{1}{x}.$$

Activity 2.3.18 Which of the following statements is NOT true?

A The derivative of $y = 2\ln(x)$ is $y' = \frac{2}{x}$.

B The derivative of $y = \frac{\ln(x)}{2}$ is $y' = \frac{1}{2x}$.

C The derivative of $y = \frac{2}{3}\ln(x)$ is $y' = \frac{3}{2x}$.

D The derivative of $y = \ln(x^2)$ is $y' = \frac{2}{x}$.

2.4 The product and quotient rules (DF4)

Learning Outcomes

- Compute derivatives using the Product and Quotient Rules.

Activity 2.4.1 Let f and g be the functions defined by $f(t) = 2t^2$ and $g(t) = t^3 + 4t$.

(a) Determine $f'(t)$ and $g'(t)$.

(b) Let $p(t) = 2t^2(t^3 + 4t)$ and observe that $p(t) = f(t) \cdot g(t)$. Rewrite the formula for p by distributing the $2t^2$ term. Then, compute $p'(t)$ using the sum and constant multiple rules.

(c) True or false: $p'(t) = f'(t) \cdot g'(t)$.

Theorem 2.4.2 Product Rule. *If f and g are differentiable functions, then their product $P(x) = f(x) \cdot g(x)$ is also a differentiable function, and*

$$P'(x) = f(x)g'(x) + g(x)f'(x).$$

Activity 2.4.3 The product rule is a powerful tool, but sometimes it isn't necessary; a more elementary rule may suffice. Which of the following products could you find the derivative of without using the product rule? Select all that apply.

A $f(x) = e^x \sin x$

B $f(x) = \sqrt{x}(x^3 + 3x - 3)$

C $f(x) = (4)(x^5)$

D $f(x) = x \ln x$

Activity 2.4.4 Find the derivative of the following functions by using the product rule.

(a) $f(x) = (x^2 + 3x) \sin x$

(b) $f(x) = e^x \cos x$

(c) $f(x) = x^2 \ln x$

Activity 2.4.5 Let f and g be the functions defined by $f(t) = 2t^2$ and $g(t) = t^3 + 4t$.

(a) Determine $f'(t)$ and $g'(t)$. (These were found previously in Activity 2.4.1.)

(b) Let $q(t) = \frac{t^3+4t}{2t^2}$ and observe that $q(t) = \frac{g(t)}{f(t)}$. Rewrite the formula for q by dividing each term in the numerator by the denominator and simplify to write q as a sum of constant multiples of powers of t. Then, compute $q'(t)$ using the sum and constant multiple rules.

(c) True or false: $q'(t) = \frac{g'(t)}{f'(t)}$.

Theorem 2.4.6 Quotient Rule. *If f and g are differentiable functions, then their quotient $Q(x) = \frac{f(x)}{g(x)}$ is also a differentiable function for all x where $g(x) \neq 0$ and*

$$Q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

Activity 2.4.7 Just like with the product rule, there are times where we can find the derivative of a quotient using elementary rules rather than the quotient rule. Which of the following quotients could you find the derivative of without using the quotient rule? Select all that apply.

A $f(x) = \dfrac{6}{x^3}$

B $f(x) = \dfrac{2}{\ln x}$

C $f(x) = \dfrac{e^x}{\sin x}$

D $f(x) = \dfrac{x^3 + 3x}{x}$

Activity 2.4.8 Find the derivative of the following functions using the quotient rule (or, if applicable, an elementary rule).

(a) $f(x) = \frac{6}{x^3}$

(b) $f(x) = \frac{2}{\ln x}$

(c) $f(x) = \frac{e^x}{\sin x}$

(d) $f(x) = \frac{x^3+3x}{x}$

Activity 2.4.9 We have found the derivatives of $\sin x$ and $\cos x$, but what about the other trigonometric functions? It turns out that the quotient rule along with some trig identities can help us! (See `https://www.khanacademy.org/math/trigonometry/trig-equations-and-identities/using-trig-identities/a/trig-identity-reference` for a reminder of the reciprocal and Pythagorean identities.) Consider the function $f(x) = \tan x$, and remember that $\tan x = \frac{\sin x}{\cos x}$.

(a) What is the domain of f?

(b) Use the quotient rule to show that one expression for $f'(x)$ is

$$f'(x) = \frac{(\cos x)(\cos x) + (\sin x)(\sin x)}{(\cos x)^2}.$$

(c) Which Pythagorean identity might be useful here? How can this identity be used to find a simpler form for $f'(x)$?

(d) Recall that $\sec x = \frac{1}{\cos x}$. How can we express $f'(x)$ in terms of the secant function?

(e) For what values of x is $f'(x)$ defined? How does this set compare to the domain of f?

Activity 2.4.10 Let $g(x) = \cot x$, and recall that $\cot x = \frac{\cos x}{\sin x}$.

(a) What is the domain of $g(x)$?

(b) Use the quotient rule to develop a formula for $g'(x)$ that is expressed completely in terms of $\sin x$ and $\cos x$.

(c) Use other relationships among trigonometric functions to write $g'(x)$ only in terms of the cosecant function.

(d) What is the domain of $g'(x)$? How does this compare to the domain of $g'(x)$?

Activity 2.4.11 Let $h(x) = \sec x$, and recall that $\sec x = \frac{1}{\cos x}$.

(a) What is the domain of $h(x)$?

(b) Use the quotient rule to develop a formula for $h'(x)$ that is expressed completely in terms of $\sin x$ and $\cos x$.

(c) Use other relationships among trigonometric functions to write $h'(x)$ only in terms of the the tangent and secant functions.

(d) What is the domain of $h'(x)$? How does this compare to the domain of $h'(x)$?

Activity 2.4.12 Let $p(x) = \csc x$, and recall that $\csc x = \frac{1}{\sin x}$.

(a) What is the domain of $p(x)$?

(b) Use the quotient rule to develop a formula for $p'(x)$ that is expressed completely in terms of $\sin x$ and $\cos x$.

(c) Use other relationships among trigonometric functions to write $h'(x)$ only in terms of the the cotangent and cosecant functions.

(d) What is the domain of $p'(x)$? How does this compare to the domain of $p'(x)$?

Remark 2.4.13 We can now summarize the derivatives for all six trigonometric functions.

- $\dfrac{d}{dx} \sin x = \cos x$
- $\dfrac{d}{dx} \cos x = -\sin x$
- $\dfrac{d}{dx} \tan x = (\sec x)^2$
- $\dfrac{d}{dx} \cot x = -(\csc x)^2$
- $\dfrac{d}{dx} \sec x = \sec x \tan x$
- $\dfrac{d}{dx} \csc x = -\csc x \cot x$

2.5 The chain rule (DF5)

Learning Outcomes

- Compute derivatives using the Chain Rule.

Note 2.5.1 When we consider the consider the composition $f \circ g$ of the function f with the function g, we mean the composite function $f(g(x))$, where the function g is applied first and then f is applied to the output of g. We also call f the outside function whilst g is the inside function.

Activity 2.5.2

(a) Consider the function $f(x) = -x^2 + 5$ and $g(x) = 2x - 1$. Which of the following is a formula for $f(g(x))$?

A $-4x^2+4x+4$

B $4x^2-4x+5$

C $-2x^2+9$

D $-2x^2+4$

(b) One of the options above is a formula for $g(f(x))$. Which one?

Activity 2.5.3

(a) Consider the composite function $f(g(x)) = \sqrt{e^x}$. Which function is the outside function $f(x)$ and which one is the inside function $g(x)$?

A $f(x) = x^2$, $g(x) = e^x$

B $f(x) = \sqrt{x}$, $g(x) = e^x$

C $f(x) = e^x$, $g(x) = \sqrt{x}$

D $f(x) = e^x$, $g(x) = x^2$

(b) Using properties of the exponents, we can rewrite the original function as $e^{\frac{x}{2}}$. Using this new expression, what is your new inside function and your new outside function?

(c) Consider the function $e^{\sqrt{x}}$. In this case, what are the inside and outside functions in this case?

Activity 2.5.4 In this activity we will build the intuition for the chain rule using a real-world scenario and differential notation for derivatives. Consider the following scenario.

My neighborhood is being invaded! The squirrel population grows based on acorn availability, at a rate of 2 squirrels per bushel of acorns. Acorn availability grows at a rate of 100 bushels of acorns per week. How fast is the squirrel population growing per week?

(a) The scenario gives you information regarding the rate of growth of $s(a)$, the squirrel population as a function of acorn availability (measured in bushels). What is the current value of $\frac{ds}{da}$?

A 2

B 100

C 200

D 50

(b) The scenario gives you information regarding the rate of growth of $a(t)$, the acorn availability as a function of time (measured in weeks). What is the current value of $\frac{da}{dt}$?

A 2

B 100

C 200

D 50

(c) Given all the information provided, what is your best guess for the value of $\frac{ds}{dt}$, the rate at which the squirrel population is growing per week?

A 2

B 100

C 200

D 50

(d) Given your answers above, what is the relationship between $\frac{ds}{da}, \frac{da}{dt}, \frac{ds}{dt}$?

Definition 2.5.5 When looking at the composite function $f(g(x))$, we have that

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'(g(x)) \cdot g'(x).$$

Using differential notation, if we consider the composite function $(v \circ u)(x)$, we have that

$$\frac{dv}{dx} = \frac{dv}{du} \cdot \frac{du}{dx}$$

◊

Warning 2.5.6 It's important to consider the input of a function when taking the derivative! In fact, $f'(g(x))$ and $f'(x)$ are different functions. Also computing $\frac{dv}{dx}$ gives a different result than computing $\frac{dv}{du}$.

Activity 2.5.7

(a) Consider the function $f(x) = -x^2 + 5$ and $g(x) = 2x - 1$. Notice that $f(g(x)) = -4x^2+4x+4$. Which of the following is the derivative function of the composite function $f(g(x))$?

A $-8x + 4$

B $-4x$

C $-2x$

D 2

(b) One of the options above is a formula for $f'(x) \cdot g'(x)$. Which one? Notice that this is not the same as the derivative of $f(g(x))$!

Activity 2.5.8 Consider the composite function $h(x) = \sqrt{e^x} = e^{\frac{x}{2}}$. For each of the two expressions, find the derivative using the chain rule. Which of the following expressions are equal to $h'(x)$? Select all!

A $\frac{1}{2}\left(e^x\right)^{\frac{-1}{2}} \cdot e^x$

B $\frac{1}{2}\left(e^x\right)^{\frac{3}{2}} \cdot e^x$

C $\frac{1}{2}e^{\frac{-x}{2}}$

D $e^{\frac{x}{2}} \cdot \frac{1}{2}$

E $\frac{1}{2}\sqrt{e^x}$

F $\sqrt{e^x} \cdot e^x$

Activity 2.5.9 Below you are given the graphs of two functions: $a(x)$ and $b(x)$. Use the graphs to compute the required vaules of the functions and of their derivatives, when possible (there are points where the derivative of these functions is not defined!). Notice that to compute the derivative at a point, you first want to find the general formula for the derivative as a function of x and then plug in the input you want to study.

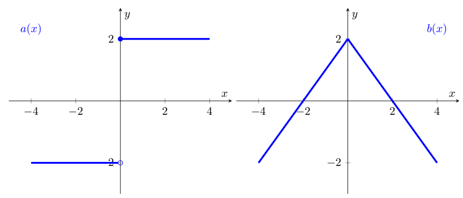


Figure .42 The graphs of $a(x)$ and $b(x)$

(a) Notice that the derivative of $a \circ b$ is given by $a'(b(x)) \cdot b'(x)$, so the derivative of $a \circ b$ at $x = 4$ is given by the quantity $a'(b(4)) \cdot b'(4) = a'(-2) \cdot b'(4)$ because $b(4) = -2$. Using the graphs to find the slopes, what is the final value of the derivative of $a \circ b$ at $x = 4$?

A 0

B -1

C 1

D -2

E 2

F The derivative does not exist at this point.

(b) Which of the following values is equal to the derivative of $a \circ b$ at $x = 2$?

A 0

B -1

C 1

D -2

E 2

F The derivative does not exist at this point.

(c) Which of the following values is equal to the derivative of $b \circ a$ (different order!) at $x = -2$?

A 0

B -1

C 1

D -2

E 2

F The derivative does not exist at this point.

Activity 2.5.10 In this activity you will study the derivative of $\cos^n(x)$ for different powers n.

(a) Consider the function $\cos^2(x) = (\cos(x))^2$. Combining power and chain rule, what do you get if you differentiate $\cos^2(x)$?

A $-\cos^2(x)\sin(x)$

B $-\cos^2(x)\sin(x)$

C $2\cos(x)\sin(x)$

D $-2\cos(x)\sin(x)$

(b) Consider the function $\cos^3(x)$. Find its derivative.

(c) Consider the function $\cos^n(x)$, for n any number. Find the general formula for its derivative.

Activity 2.5.11 In this activity you will study the derivative of $b^{\cos(x)}$ for different bases b.

(a) Consider the function $e^{\cos(x)}$. Combining exponential and chain rule, what do you get if you differentiate $e^{\cos(x)}$?

A $e^{\cos(x)}$

B $-e^{\cos(x)}\sin(x)$

C $e^{-\sin(x)}$

D $e^{\cos(x)}\sin(x)$

(b) Consider the function $2^{\cos(x)}$. Find its derivative.

(c) Consider the function $b^{\cos(x)}$, for b any positive number. Find the general formula for its derivative.

Remark 2.5.12 Remember that exponential and power functions obey very different differentiation rules. This behavior continues when we consider composite function. The composite power function $f(x)^3$ has derivative

$$3[f(x)]^2 \cdot f'(x)$$

but the composite exponential function $3^{f(x)}$ has derivative

$$\ln(3)\,3^{f(x)} \cdot f'(x)$$

Activity 2.5.13 Use the chain rule and all the rest of the rules you know to find the derivatives of the following functions.

(a)

$$f(x) = -2\,\sin\left(x^{\frac{7}{4}}\right)$$

(b)

$$h(y) = 7\,e^{\left(-4\,y^3-2\,y-2\right)}$$

(c)

$$k(t) = \left(4\,t - 6\,2^t - 2\right)^4$$

(d)

$$g(w) = -3\,\sin(w)^{\frac{3}{11}}$$

Activity 2.5.14 Remember my neighborhood squirrel invasion? The squirrel population grows based on acorn availability, at a rate of 2 squirrels per bushel of acorns. Acorn availability grows at a rate of 100 bushels of acorns per week. Considering this information as pertaining to the moment $t = 0$, you are given the following possible model for the squirrel:

$$s(a(t)) = 2a(t) + 10 = 2\,[50\sin(2t) + 60] + 10.$$

(a) Check that the model satisfies the data $\frac{ds}{da} = 2$ and $\left.\frac{da}{dt}\right|_{t=0} = 100$

(b) Find the derivative function $\frac{ds}{dt}$ and check that $\frac{ds}{dt}|_{t=0} = 200$.

(c) According to this model, what is the maximum and minimum squirrel population? What is the fastest rate of increase and decrease of the squirrel population? When will these extremal scenarions occur?

2.6 Differentiation strategy (DF6)

Learning Outcomes

- Compute derivatives using a combination of algebraic derivative rules.

Activity 2.6.1 Consider the functions defined below:

$$f(x) = \sin((x^2 + 3x)\cos(2x))$$

$$g(x) = \sin(x^2 + 3x)\cos(2x)$$

(a) What do you notice that is similar about these two functions?

(b) What do you notice that is different about these two functions?

(c) Imagine that you are sorting functions into different categories based on how you would differentiate them. In what category (or categories) might these functions fall?

Remark 2.6.2 We've learned a lot of rules to help us take derivatives of functions, and these rules are applied based on the algebraic structure of the function. (Note that we often see that algebraic structure come up in the names of the rules. Sum/difference, power, product, and quotient rules for example!) To take a derivative, we need to examine how the function is built and then proceed accordingly. Up until now we have not seen many situations where these rules were mixed together too much. Now we turn our attention to those types of situations.

Below are some questions you might ask yourself as you take the derivative of a function, especially one where multiple rules might need to be used:

1. How is this function built algebraically? What kind of function is this? What is the big picture?
2. Where do you start?
3. Is there an easier or more convenient way to write the function?
4. Are there products or quotients involved?
5. Is it a composition of two (or more) functions? If so, what are the outside and inside functions?
6. What derivatives rules will be needed along the way?

Activity 2.6.3 Consider the function $f(x) = x^3\sqrt{3 - 8x^2}$.

(a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

A Chain rule

B Power rule

C Product rule

D Quotient rule

E Sum and difference rule

(b) What other rules would be needed along the way? Select all that apply.

A Chain rule

B Power rule

C Product rule

D Quotient rule

E Sum and difference rule

(c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.4 Consider the function $f(x) = \left(\frac{\ln x}{(3x-4)^3}\right)^5$.

(a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

A Chain rule

B Power rule

C Product rule

D Quotient rule

E Sum and difference rule

(b) What other rules would be needed along the way? Select all that apply.

A Chain rule

B Power rule

C Product rule

D Quotient rule

E Sum and difference rule

(c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.5 Consider the function $f(x) = \sin(\cos(\tan(2x^3 - 1)))$.

(a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

A Chain rule

B Power rule

C Product rule

D Quotient rule

E Sum and difference rule

(b) What other rules would be needed along the way? Select all that apply.

A Chain rule

B Power rule

C Product rule

D Quotient rule

E Sum and difference rule

(c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.6 Consider the function $f(x) = \frac{x^2e^x}{2x^3-5x+\sqrt{x}}$.

(a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

A Chain rule

B Power rule

C Product rule

D Quotient rule

E Sum and difference rule

(b) What other rules would be needed along the way? Select all that apply.

A Chain rule

B Power rule

C Product rule

D Quotient rule

E Sum and difference rule

(c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.7 Find the derivative of the following functions. For each, include an explanation of the steps involved that references the algebraic structure of the function.

(a) $f(x) = e^{5x}(x^2 + 7^x)^3$

(b) $f(x) = \left(\frac{3x+1}{2x^6-1}\right)^5$

(c) $f(x) = \sqrt{\cos(2\,x^2 + x)}$

(d) $f(x) = \tan(xe^x)$

2.7 Differentiating implicitly defined functions (DF7)

Learning Outcomes

- Compute derivatives of implicitly-defined functions.

Observation 2.7.1 Many of the equations that has been discussed so far fall under the category of an explicit equation. An explicit equation is one in which the relationship between x and y is given explicitly, such as $y = f(x)$. In this section we will examine when the relationship between x and y is given implicity. An implicity expression is when the relationship has some equation like $f(x,y) = g(x,y)$ where both sides may depend on both x and y.

Observation 2.7.2 Note that if we are taking the derivative with respect to x, then

$$\frac{d}{dx}(f(x)) = f'(x).$$

However,

$$\frac{d}{dx}(g(y)) = g'(y)\frac{dy}{dx}.$$

Activity 2.7.3 For this activity we want to find the equation of a tangent line for a circle with radius 5 centered at the origin, $x^2 + y^2 = 25$, at the point $(-3, -4)$.

(a) The derivative with respect to x for the equation of the circle is given by which expression.

A $2x + 2y\dfrac{dy}{dx} = 25$

B $2x + y\dfrac{dy}{dx} = 0$

C $2x + 2y\dfrac{dy}{dx} = 0$

D $2x + 2\dfrac{dy}{dx} = 25$

(b) Solving for $\frac{dy}{dx}$ gives?

A $\dfrac{dy}{dx} = \dfrac{25 - 2x}{2y}$

B $\dfrac{dy}{dx} = -\dfrac{2x}{y}$

C $\dfrac{dy}{dx} = -\dfrac{x}{y}$

D $\dfrac{dy}{dx} = \dfrac{25 - 2x}{2}$

(c) Plug the point $(-3, -4)$ into the expression found above for the derivative to get the slope of the tangent line.

(d) Use the value for the slope of the tangent line to obtain the equation of the tangent line.

Activity 2.7.4 The curve given in Figure .43 is an example of an astroid. The equation of this astroid is $x^{2/3} + y^{2/3} = 3^{2/3}$. What is the derivative with respect x for this astroid? (Solve for $\frac{dy}{dx}$).

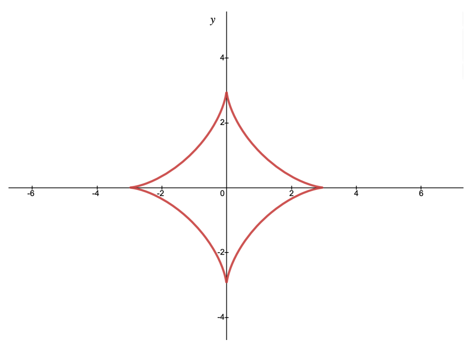


Figure .43 Graph of $x^{2/3} + y^{2/3} = 3^{2/3}$.

A $\dfrac{dy}{dx} = \dfrac{x^{-1/3}}{y^{-1/3}}$

B $\dfrac{dy}{dx} = \dfrac{y^{-1/3}}{x^{-1/3}}$

C $\dfrac{dy}{dx} = \dfrac{3^{-1/3} - x^{-1/3}}{y^{-1/3}}$

D $\dfrac{dy}{dx} = -\dfrac{x^{-1/3}}{y^{-1/3}}$

Activity 2.7.5 An example of a lemniscate is given in Figure .44. The equation of this lemniscate is $(x^2+y^2)^2 = x^2 - y^2$. What is the derivative with respect x for this lemniscate? (Solve for $\frac{dy}{dx}$).

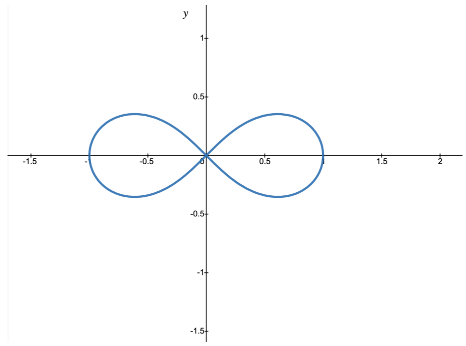


Figure .44 Graph of $(x^2+y^2)^2 = x^2 - y^2$.

A $\dfrac{dy}{dx} = \dfrac{x(1-2(x^2+y^2))}{y+2(x^2+y^2)}$

B $\dfrac{dy}{dx} = \dfrac{x(1-2(x^2+y^2))}{y(1+2(x^2+y^2))}$

C $\dfrac{dy}{dx} = \dfrac{y(1+2(x^2+y^2))}{x(1-2(x^2+y^2))}$

D $\dfrac{dy}{dx} = \dfrac{y+2(x^2+y^2)}{x(1-2(x^2+y^2))}$

Activity 2.7.6 To take the derivative of some explicit equations you might need to make it an implicit equation. For this activity we will find the derivative of $y = x^x$. Make the equation an implicit equation by taking natural logarithm of both sides, this gives $\ln(y) = x\ln(x)$. Knowing this, what is $\frac{dy}{dx}$?

A $\dfrac{dy}{dx} = x^x(\ln(x)+1)$

B $\dfrac{dy}{dx} = \dfrac{(\ln(x)+1)}{x^x}$

C $\dfrac{dy}{dx} = x^x(\ln(x)+x)$

D $\dfrac{dy}{dx} = \dfrac{(\ln(x)+x)}{x^x}$

2.8 Differentiating inverse functions (DF8)

Learning Outcomes

- Compute derivatives of inverse functions.

Remark 2.8.1 Let f^{-1} be the inverse function of f. The relationship between a function and its inverse can be expressed with the equation

$$f(f^{-1}(x)) = x.$$

Activity 2.8.2 In this activity you will use implicit differentiation and the inverse function identity above to find the derivative of $y = \ln(x)$.

(a) Suppose that $y = \ln(x)$. Then we have that

$$e^y = x.$$

Using implicit implicit differentiation, what do you get?

A $\dfrac{dy}{dx} = \dfrac{x}{y}$

B $\dfrac{dy}{dx} = \dfrac{1}{e^x}$

C $\dfrac{dy}{dx} = \dfrac{x}{e^y}$

D $\dfrac{dy}{dx} = \dfrac{1}{e^y}$

(b) Notice that we started with the relationship $e^y = x$. Use this to simplify $\frac{dy}{dx}$. You should get that when $y = \ln(x)$ we have that $\frac{dy}{dx} = \frac{1}{x}$... as expected!

Activity 2.8.3 In this activity we will try to find a general formula for the derivative of the inverse function. Let g be the inverse function of f. We have also used the notation f^{-1} before, but for the purpose of this problem, let's use g to avoid too many exponents. We can express the relationship "g is the inverse of f" with the equation

$$f(g(x)) = x.$$

(a) Looking at the equation $f(g(x)) = x$, what is the derivative with respect to x of the right hand side of the equation?

A x

B 1

C 0

D x^2

(b) Looking at the equation $f(g(x)) = x$, what is the derivative with respect to x of the left hand side of the equation?

A $f'(g(x))$

B $f'(g'(x))$

C $f(g(x))\, g'(x)$

D $f'(g(x))\, g'(x)$

(c) Setting the two sides of the equation equal after differentiating, we can solve for $g'(x)$. What do you get?

A $g'(x) = \dfrac{x}{f(g(x))}$

B $g'(x) = \dfrac{x}{f'(g(x))}$

C $g'(x) = \dfrac{1}{f(g(x))}$

D $g'(x) = \dfrac{1}{f'(g(x))}$

Observation 2.8.4 In the above activity you should have found that the derivative of $g = f^{-1}$, the inverse function of f, is given by

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Notice that because of the chain rule, the derivative of f has to be evaluated at $\boldsymbol{f^{-1}(x)}$

Activity 2.8.5 In this problem you will apply the general formula for the derivative of the inverse function to find the values of some derivatives graphically.

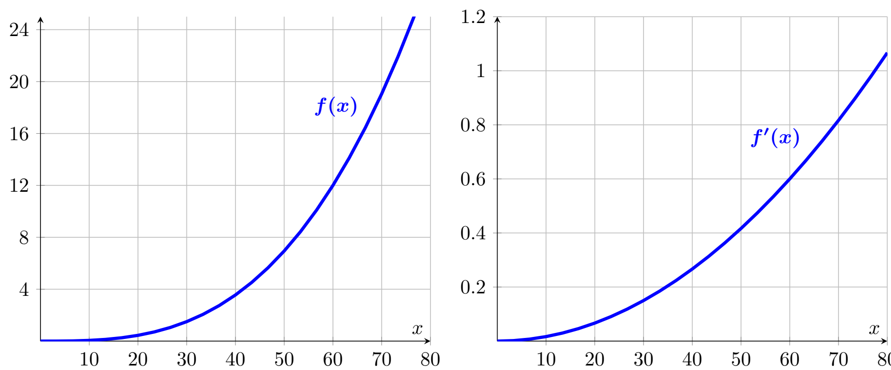


Figure .45 The graphs of $f(x)$ and $f'(x)$.

(a) The derivative of the inverse function at $x = 12$ given by $(f^{-1})'(12) = \frac{1}{f'(f^{-1}(12))}$. Using the graphs, what is your best approximation for this quantity?

A $(f^{-1})'(12) \approx \dfrac{1}{0.2} = 5$

B $(f^{-1})'(12) \approx \dfrac{1}{0.6} \approx 1.67$

C $(f^{-1})'(12) \approx \dfrac{1}{0.4} = 2.5$

D $(f^{-1})'(12) \approx \dfrac{1}{0.1} = 10$

(b) What is your best estimate for $(f^{-1})'(6)$?

A $(f^{-1})'(6) \approx \dfrac{1}{0.2} = 5$

B $(f^{-1})'(6) \approx \dfrac{1}{0.6} \approx 1.67$

C $(f^{-1})'(6) \approx \dfrac{1}{0.4} = 2.5$

D $(f^{-1})'(6) \approx \dfrac{1}{0.1} = 10$

Activity 2.8.6 Use the general formula for the derivative of the inverse function to find...

(a) The derivative of the inverse function of $f(x) = e^x$... This should match what you found before!

(b) The derivative of the inverse function of $f(x) = \frac{1}{x}$... This should match something you have seen before! See if you can explain why this holds.

Definition 2.8.7 We can only invert the function $y = \sin(x)$ on the restricted domain $[-\pi/2, \pi/2]$ (Why?). On this domain we define arcsine by the condition

$$x = \sin^{-1}(y) \quad \text{when} \quad y = \sin(x).$$

◊

Activity 2.8.8 In this activity you will study the arcsine function.

(a) Consider the values of $y = \sin(x)$ given in the table below for an angle x between $-\pi/2$ and $\pi/2$. Fill in the corresponding values for the inverse function arcsine $x = \sin^{-1}(y)$. In other words, you need to provide the angle whose sine is given. You can use the unit circle to help you remember which angles yield the given values of sine. The first entry is provided: a sine value of -1 corresponds to the angle $-\pi/2$.

Table .46

$y = \sin(x)$	-1	$-\sqrt{3}/2$	$-1/2$	0	$1/2$	$\sqrt{3}/2$	1
$x = \sin^{-1}(y)$	$-\pi/2$						

(b) From the graph of $y = \sin(x)$ and your table above, graph the arcsine function $y = \sin^{-1}(x)$

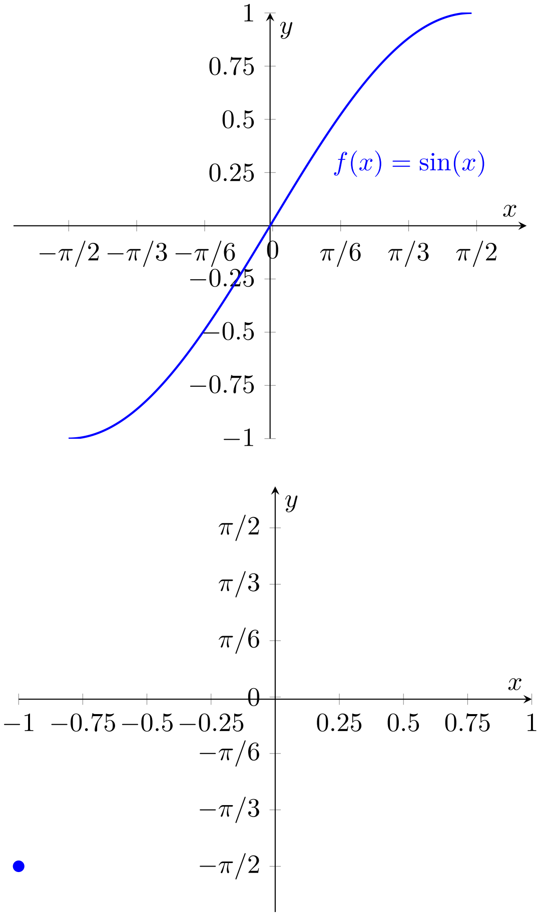


Figure .47 The graphs of $\sin(x)$ and one point on $\sin^{-1}(x)$.

(c) Let's now work with the function arccosine. Again, we need to restrict the domain of cosine to be able to invert the function (Why?). The convention is to restrict cosine to the domain $[0, \pi]$ in order to define arccosine. Given this restriction, what are the domain and range of arccosine? Create a table of values and graph the function arccosine.

(d) Let's now work with the function arctangent. Again, we need to restrict the domain of tangent to be able to invert the function (Why?). The convention is to restrict tangent to the domain $(-\pi/2, \pi/2)$ in order to define arctangent. Given this restriction, what are the domain and range of arctangent? Create a table of values and graph the function arctangent.

Activity 2.8.9 In this activity you will find a formula for the derivative of arctangent.

(a) Differentiate the implicit equation $\tan(y) = x$, what do you get for $\frac{dy}{dx}$?

A $\dfrac{dy}{dx} = \dfrac{x}{\tan(y)}$

B $\dfrac{dy}{dx} = \dfrac{1}{\tan(y)}$

C $\dfrac{dy}{dx} = \dfrac{x}{\sec^2(y)}$

D $\dfrac{dy}{dx} = \dfrac{1}{\sec^2(y)}$

(b) For what function $y = g(x)$ have you found the derivative $\frac{dy}{dx}$?

(c) We want to rewrite $\frac{dy}{dx}$ only in terms of x. Notice that

$$\tan^2(y) = \frac{\sin^2(y)}{\cos^2(y)} = \frac{1-\cos^2(y)}{\cos^2(y)}.$$

Multiplying out by the denominator, isolating, and solving for $\cos^2(y)$, we get that

A $\cos^2(y) = \dfrac{\tan^2(y)}{\cos^2(y)}$

B $\cos^2(y) = \dfrac{1}{\tan^2(y)+1}$

C $\cos^2(y) = \dfrac{1-\cos^2(y)}{\tan^2(y)}$

D $\cos^2(y) = \dfrac{1}{\tan^2(y)-1}$

(d) Finally, rewrite $\frac{dy}{dx}$ as $\frac{dy}{dx} = \cos^2(y)$ and use the fact that $\tan(y) = x$ to get a nice formula for the derivative of the arctangent function of x.

Remark 2.8.10 Consider the functions $y = \tan^{-1}(x)$. Using your algebra above, you should have found that

$$\frac{d}{dx}\Big(\tan^{-1}(x)\Big) = \frac{1}{1+x^2}.$$

In a similar fashion, one can find that

$$\frac{d}{dx}\Big(\sin^{-1}(x)\Big) = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\Big(\cos^{-1}(x)\Big) = -\frac{1}{\sqrt{1-x^2}}.$$

Activity 2.8.11

(a) Find the equation of the tangent line to $y = \tan^{-1}(x)$ at $x = 0$. Draw the function and the tangent on Desmos to check your work!

(b) Find the equation of the tangent line to $y = \sin^{-1}(x)$ at $x = 0.5$. Draw the function and the tangent on Desmos to check your work!

(c) Find the equation of the tangent line to $y = \cos^{-1}(x)$ at $x = -0.5$. Draw the function and the tangent on Desmos to check your work!

Activity 2.8.12 Demonstrate and explain how to find the derivative of the following functions. Be sure to explicitly denote which derivative rules (product, quotient, sum and difference, etc.) you are using in your work.

(a)

$$k(t) = \frac{\arctan(-4\,t)}{\ln(-4\,t)}$$

(b)

$$j(u) = -5\,\arcsin(u)\log\left(u^6+2\right)$$

(c)

$$n(x) = \ln\left(-\arcsin(x) + 4\arctan(x)\right)$$

Chapter 3

Applications of Derivatives (AD)

Learning Outcomes

How can we use derivatives to solve application questions?

By the end of this chapter, you should be able to...

1. Use derivatives to answer questions about rates of change and equations of tangents.
2. Use tangent lines to approximate functions.
3. Model and analyze scenarios using related rates.
4. Use the Extreme Value Theorem to find the absolute maximum and minimum values of a continuous function on a closed interval.
5. Determine where a differentiable function is increasing and decreasing and classify the critical points as local extrema.
6. Determine the intervals of concavity of a twice differentiable function and find all of its points of inflection.
7. Sketch the graph of a differentiable function whose derivatives satisfy given criteria.
8. Apply optimization techniques to solve various problems.
9. Compute the values of indeterminate limits using L'Hopital's Rule.

Readiness Assurance. Before beginning this chapter, you should be able to...

a Find the derivative of a function using methods from Section 2.3, Section 2.4, Section 2.5, and Section 2.6.

b Find the second derivative of a function. (Khan Academy)

c Determine whether values are in the domain of a function (Khan Academy)

d Find relative and absolute extrema given the graph of a function (Khan Academy) e Find the intervals where a graph is increasing or decreasing (Khan Academy)

f Determine the zeros of a polynomial (Khan Academy)

g Modeling with multiple variables (Khan Academy)

3.1 Tangents, motion, and marginals (AD1)

Learning Outcomes

- Use derivatives to answer questions about rates of change and equations of tangents.

Definition 3.1.1 The tangent line of a function $f(x)$ at $x = a$ is the linear function $L(x)$

$$L(x) = f'(a)(x - a) + f(a).$$

Notice that this is the linear function with slope $f'(a)$ and passing through $(a, f(a))$ in point-slope form. ♢

Activity 3.1.2 For the following functions, find the required tangent line.

(a) Find the tangent line to $f(x) = \ln(x)$ at $x = 1$

A $L(x) = x$

B $L(x) = x + 1$

C $L(x) = x - 1$

D $L(x) = -x + 1$

(b) Find the tangent line to $f(x) = e^x$ at $x = 0$

A $L(x) = x$

B $L(x) = x + 1$

C $L(x) = x - 1$

D $L(x) = -x + 1$

Activity 3.1.3 Let $f(x) = -2\,x^4 + 4\,x^2 - x + 5$. Find an equation of the line tangent to the graph at the point $(-2, -9)$.

Definition 3.1.4 If a particle has position function $s = f(t)$, where t is measured in seconds and s is measured in meters, then

- $v(t) = f'(t)$ is the velocity of the particle
- $a(t) = f''(t)$ is the acceleration of the particle

♢

Activity 3.1.5 A particle moves on a vertical line so that its y coordinate at time t is

$$y = t^3 - 9t^2 + 24t + 3$$

for $t \geq 0$. Here t is measured in seconds and y is measured in feet.

(a) Find the velocity and acceleration functions.

(b) Sketch graphs of the position, velocity and acceleration functions for $0 \leq t \leq 5$.

(c) When is the particle moving upward and when is it moving downward?

(d) When is the particle's velocity increasing?

(e) Find the total distance that the particle travels in the time interval $0 \leq t \leq 5$. Careful: the total distance is not the same as the displacement (the change in position)! Compute how much the particle moves up and add it to how much the particle moves down.

Observation 3.1.6 In some cases, we want to also consider the speed of a particle, which is the absolute value of the velocity. In symbols $|v(t)| = |f'(t)|$ is the speed of the particle. A particle is speeding up when the speed is increasing.

Activity 3.1.7 Consider the speed of a particle. What is the behavior of the speed in relation to velocity and acceleration?

A The speed is always positive and it is increasing when the velocity and the acceleration have the same sign.

B The speed is positive when the velocity is positive and negative when the velocity is negative.

C The speed is positive when the acceleration is positive and negative when the acceleration is negative.

D The speed is always positive and it is increasing when the velocity and the acceleration have opposite signs.

Definition 3.1.8 In a parametric motion on a curve C given by $x = f(t)$ and $y = g(t)$ we have that

- $\frac{dx}{dt} = f'(t)$ is the rate of change of $f(t)$, one component of the slope (or velocity)
- $\frac{dy}{dt} = g'(t)$ is the rate of change of $g(t)$, one component of the slope (or velocity)
- $\frac{dy}{dx}$ is the actual slope (or velocity) of the object and by the chain rule $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$

◊

Activity 3.1.9 An airplane is cruising at a fixed height and traveling in a pattern described by the parametric equations

$$x = 4t, \quad y = -t^4 + 4t - 1,$$

where x, y have units of miles, and t is in hours.

(a) Find the slope of the curve.

(b) What is the slope of the curve at $(0, -1)$.

(c) Write the equation of the tangent line to the curve at $(0, -1)$.

Definition 3.1.10 If $C(x)$ is the cost of producing x items and $R(x)$ is the revenue from selling x items, then $P(x) = R(x) - C(x)$ is the profit. We can study their derivatives, the marginals

- $C'(x)$ is the marginal cost
- $R'(x)$ is the marginal revenue
- $P'(x) = R'(x) - C'(x)$ is the marginal profit

◊

Activity 3.1.11 The manager of a computer shop has to decide how many computers to store in the back of the shop. If she stores a large number, she has to pay extra in storage costs. If she stores only a small number, she will have to reorder more often, which will involve additional handling costs. She has found that if she stores x computers, the storage and handling costs will be C dollars, where

$$C(x) = 10x^3 - 900x^2 + 16000x + 210000$$

(a) What is the fixed cost of the computer shop, the cost when no computers are in storage? In practical terms this may account for rent and utilities expenses.

(b) Find the marginal cost

(c) Now suppose that x computers give revenue $R(x) = 1000x$. What is the marginal revenue? What is the real world interpretation of your finding?

(d) Find a formula for the profit function $P(x)$ and find the marginal profit using the marginal revenue and the marginal cost.

Activity 3.1.12 The quantity q of skateboards sold depends on the selling price p of a skateboard, so we write $q = f(p)$. You are given that

$$f(140) = 15000, \quad f'(140) = -100$$

(a) What does the data above tell you about the change in demand for skateboards depending on the selling price?

(b) The total revenue earned by the sale of skateboards $R(p)$ is given by , $R(p) = q(p) \cdot p$. Explain why.

(c) Find

$$\left.\frac{dR}{dp}\right|_{p=140}$$

(d) What is the sign of the quantity above? What do you think would happen to the revenue if the price was changed from \$140 to \$141?

Definition 3.1.13 A cooling object has temperature modelled by

$$y = ae^{-kt} + c,$$

where a, c, k are positive constants determined by the local conditions. ◊

Activity 3.1.14 Consider a cup of coffee initially at 100°F. The said cup of coffee was forgotten this morning in my living room where the thermostat is set at 72°F. I also observed that when I initially prepared the coffee, the temperature was decreasing at a rate of 3.8 degrees per minute.

(a) In the long run, what temperature do you expect the coffee to tend to? Use this information in the model $y = ae^{-kt}t + c$ to determine the value of c.

(b) Using the initial temperature of the coffee and your value of c, find the value of a in the model $y = ae^{-kt}t + c$.

(c) The scenario also gives you information about the value of the rate of change at $t = 0$. Use this additional information to determine the model $y = ae^{-kt}t + c$ completely.

(d) You should find that the temperature model for this coffee cup is $y = 72 + 38e^{-0.1t}$. Explain how the values of each parameter connects to the information given.

3.2 Linear approximation (AD2)

Learning Outcomes

- Use tangent lines to approximate functions.

Definition 3.2.1 The tangent line approximation (or linear approximation, or linearization) of a function $f(x)$ at $x = a$ is the tangent line $L(x)$ at $x = a$. In formulas, $L(x)$ is the linear function

$$L(x) = f'(a)(x - a) + f(a).$$

Notice that this is obtained by writing the tangent line to $f(x)$ at $(a, f(a))$ in point-slope form and calling the resulting linear function $L(x)$. ◊

Activity 3.2.2 Without using a calculator, we will use calculus to approximate $\ln(1.1)$.

(a) Find the equation of the tangent line to $\ln(x)$ at $x = 1$. This will be your linear approximation $L(x)$. What do you get for $L(x)$?

A $L(x) = x$

B $L(x) = x + 1$

C $L(x) = x - 1$

D $L(x) = -x + 1$

(b) As 1.1 is close to 1, we can use $L(1.1)$ to approximate $\ln(1.1)$. What approximation do you get?

A $\ln(1.1) \approx 1.1$

B $\ln(1.1) \approx 2.1$

C $\ln(1.1) \approx 0.1$

D $\ln(1.1) \approx -0.1$

(c) Sketch the tangent line $L(x)$ on the same plane as the graph of $\ln(x)$. What do you notice? What's the relationship between the two graphs?

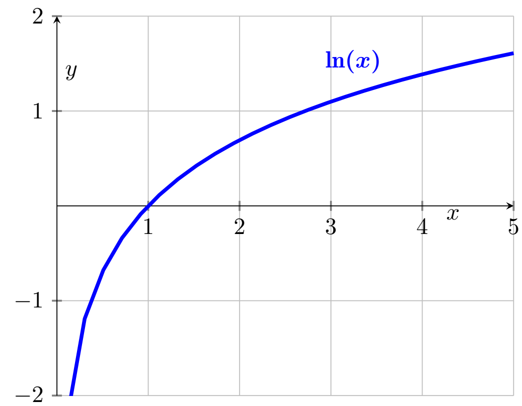


Figure .48 The graph of $\ln(x)$

Activity 3.2.3 Using the equation of the tangent line to the graph of $\ln(x)$ at $x = 1$ and the shape of this graph, you can show that for all values of x, we have that $\ln(x) \leq x - 1$.

(a) Compute the second derivative of $\ln(x)$. What do you notice about the sign of the second derivative of $\ln(x)$? What does this tell you about the shape of the graph?

(b) Conclude that because the graph of $\ln(x)$ has a certain shape, the graph will bend below the tangent line and so the tangent line approximation $L(x) = x - 1$ is always greater than $\ln(x)$.

Activity 3.2.4 In this activity you will approximate power functions near $x = 1$.

(a) Find the tangent line approximation to x^2 at $x = 1$.

A $L(x) = 2x$

B $L(x) = 2x + 1$

C $L(x) = 2x - 1$

D $L(x) = -2x + 1$

(b) Show that for any constant k, the tangent line approximation to x^k at $x = 1$ is $L(x) = kx - k + 1$.

(c) Someone claims that the square root of 1.1 is about 1.05. Use the linear approximation to check this estimate. Do you think this estimate is about right? Why or why not?

(d) Is the actual value $\sqrt{1.1}$ above or below 1.05? What feature of the graph of $\sqrt{x}$ makes this an over or under estimate?

Remark 3.2.5 If a function $f(x)$ is *concave up* around $x = a$, then the function is turning upwards from its tangent line. So when we use a linear approximation, the value of the approximation will be below the actual value of the function and the approximation is an underestimate. If a function $f(x)$ is *concave down* around $x = a$, then the function is turning downwards from its tangent line. So when we use a linear approximation, the value of the approximation will be above the actual value of the function and the approximation is an overestimate.

Activity 3.2.6 Suppose f has a continuous positive second derivative and Δx is a small increment in x (like h in the limit definition of the derivative). Which one is larger...

$$f(1+\Delta x) \quad \text{or} \quad f'(1)\Delta x + f(1) \quad ?$$

Activity 3.2.7 A certain function $p(x)$ satisfies $p(7) = 49$ and $p'(7) = 8$.

1. Explain how to find the local linearization $L(x)$ of $p(x)$ at 7.
2. Explain how to estimate the value of $p(6.951)$.
3. Suppose that $p'(7) = 0$ and you know that $p''(x) < 0$ for $x < 7$. Explain how to determine if your estimate of $p(6.951)$ is too large or too small.
4. Suppose that $p''(x) > 0$ for $x > 7$. Use this fact and the additional information above to sketch an accurate graph of $y = p(x)$ near $x = 7$.

Activity 3.2.8 Let's find the quadratic polynomial

$$q(x) = ax^2 + bx + c$$

where a, b, c are parameters to be determined so that $q(x)$ best approximates the graph of $f(x) = \ln(x)$ at $x = 1$.

(a) We want to choose a, b, c such that our quadratic polynomial resembles $f(x)$ at $x = 1$. First thing, we want $f(1) = q(1)$. What equation in a, b, c does this condition give you?

A $a + b + c = 1$

B $a + b + c = 0$

C $c = 0$

D $c = 1$

(b) We also want $f'(1) = q'(1)$. What equation in a, b, c does this condition give you?

(c) Finally, we want $f''(1) = q''(1)$. What equation in a, b, c does this condition give you?

(d) Find a solution to this system of linear equations! Your answer will give you values of a, b, c that can be used to draw a quadratic approximating the natural logarithm. You can check your answer on Desmos https://www.desmos.com/calculator/bad2xrwmvl

Observation 3.2.9 A linear approximation $L(x)$ to $f(x)$ at $x = a$ is a linear function with

$$L(a) = f(a), \quad L'(a) = f'(a).$$

A quadratic approximation $Q(x)$ to $f(x)$ at $x = a$ is a quadratic function with

$$Q(a) = f(a), \quad Q'(a) = f'(a), \quad Q''(a) = f''(a).$$

Activity 3.2.10 Find the linear approximation $L(x)$ of $\cos(x)$ at $x = 0$. Then find the quadratic approximation $Q(x)$ of $\cos(x)$ at $x = 0$. Graph both and compare the two approximations!

3.3 Related rates (AD3)

Learning Outcomes

- Model and analyze scenarios using related rates.

Remark 3.3.1 In most of our applications of the derivative so far, we have been interested in the instantaneous rate at which one variable, say y, changes with respect to another, say x, leading us to compute and interpret $\frac{dy}{dx}$. We next consider situations where several variable quantities are related, but where each quantity is implicitly a function of time, which will be represented by the variable t. Through knowing how the quantities are related, we will be interested in determining how their respective rates of change with respect to time are related.

Example 3.3.2 In a sense, the chain rule is our first example of related rates: recall that when y is a function of x, which in turn is a function of t, we are considering the composite function $y(x(t))$, and we learned that by the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Notice that the chain rule gives a relationship between three rates: $\frac{dy}{dt}, \frac{dy}{dx}, \frac{dx}{dt}$.
□

Activity 3.3.3 Remember the squirrels taking over my neighborhood? The population s grows based on acorn availability a, at a rate of 2 squirrels per bushel. The acorn availability a is currently growing at a rate of 100 bushels per week. What is $\frac{ds}{dt}$ in this situation?

A 2

B 100

C 200

D Not enough information

Example 3.3.4 In a more serious example, suppose that air is being pumped into a spherical balloon so that its volume increases at a constant rate of 20 cubic inches per second. Since the balloon's volume and radius are related, by knowing how fast the volume is changing, we ought to be able to discover how fast the radius is changing. We are interested in questions such as: can we determine how fast the radius of the balloon is increasing at the moment the balloon's diameter is 12 inches? □

Activity 3.3.5 A spherical balloon is being inflated at a constant rate of 20 cubic inches per second. How fast is the radius of the balloon changing at the instant the balloon's diameter is 12 inches? Is the radius changing more rapidly when $d = 12$ or when $d = 16$? Why? Draw several spheres with different radii, and observe that as volume changes, the radius, diameter, and surface area of the balloon also change.Recall that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Note as well that in the setting of this problem, *both* V and r are changing as time t changes, and thus both V and r may be viewed as *implicit* functions of t, with respective derivatives $\frac{dV}{dt}$ and $\frac{dr}{dt}$. Differentiate both sides of the equation $V = \frac{4}{3}\pi r^3$ with respect to t (using the chain rule on the right) to find a formula for $\frac{dV}{dt}$ that depends on both r and $\frac{dr}{dt}$.At this point in the problem, by differentiating we have "related the rates" of change of V and r. Recall that we are given in the problem that the balloon is being inflated at a constant *rate* of 20 cubic inches per second. Is this rate the value of $\frac{dr}{dt}$ or $\frac{dV}{dt}$? Why?From part (c), we know the value of $\frac{dV}{dt}$ at every value of t. Next, observe that when the diameter of the balloon is 12, we know the value of the radius. In the equation $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, substitute these values for the relevant quantities and solve for the remaining unknown quantity, which is $\frac{dr}{dt}$. How fast is the radius changing at the instant $d = 12$?How is the situation different when $d = 16$? When is the radius changing more rapidly, when $d = 12$ or when $d = 16$?

Remark 3.3.6 In problems where two or more quantities can be related to one another, and all of the variables involved are implicitly functions of time, t, we are often interested in how their rates are related; we call these *related rates* problems. Once we have an equation establishing the relationship among the variables, we differentiate implicitly with respect to time to find connections among the rates of change.

Remark 3.3.7 A guide to solving related rated problems.

1. *Picture it!* Draw a diagram to represent the situation.
2. *What do we know?* Make a list of all quantities you are given in the problem, choosing clearly defined variable names for them. If a quantity is changing (a rate), then it should be labeled as a derivative.
3. *What do we want to know?* Make a list of all quantities to be determined. Again, choose clearly defined variable names.
4. *How are the variables related to each other?* Find an equation that relates the variables whose rates of change are known to those variables whose rates of change are to be found.
5. *How are the rates related?* Differentiate implicitly with respect to time. This will give an equation that relates the rates together.
6. *Time to evaluate!* Evaluate the derivatives and variables at the information relevant to the instant at which a certain rate of change is sought.

Remark 3.3.8 Volume formulas.

- A sphere of radius r has volume $V = \frac{4}{3}\pi r^3$
- A vertical cylinder of radius r and height h has volume $V = \pi r^2 h$
- A cone of radius r and height h has volume $V = \frac{\pi}{3} r^2 h$

Activity 3.3.9 A vertical cylindrical water tank has a radius of 1 meter. If water is pumped out at a rate of 3 cubic meters per minute, at what rate will the water level drop?

(a) Draw a figure to represent the situation. Introduce variables that measure the radius of the water's surface, the water's depth in the tank, and the volume of the water. Label your diagram.

(b) What information about rates of changes does the problem give you?

(c) Recall that the volume of a cylinder of radius r and height h is $V = \pi r^2 h$. What is the related rates equation in the context of the vertical cylindrical tank? What derivative rules did you use to find this equation?

A $\dfrac{dV}{dt} = \pi 2r \dfrac{dh}{dt}$

B $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$

C $\frac{dV}{dt} = \pi \frac{dr}{dt} h$

D $\frac{dV}{dt} = \pi 2r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$

E $\frac{dV}{dt} = \pi 2rh + \pi r^2$

(d) Which variable(s) have a constant value in this situation? Why?

A The variable measuring the radius of the water's surface

B The variable measuring the depth of the water

C The variable measuring the volume of the water

(e) Which variable(s) have a constant rate of change in this situation? Why?

A The variable measuring the radius of the water's surface

B The variable measuring the depth of the water

C The variable measuring the volume of the water

(f) Using your finding above, find at what rate the water level is dropping.

(g) If the full tank contains 12 cubic centimeters of water, how long does it take to empty the tank?

(h) Confirm your finding in the previous part by finding the initial water level for 12 cubic centimeters of water and determine how long it takes for the water level to reach 0.

Activity 3.3.10 A water tank has the shape of an inverted circular cone (the cone points downwards) with a base of radius 6 feet and a depth of 8 feet. Suppose that water is being pumped into the tank at a constant instantaneous rate of 4 cubic feet per minute.

(a) Draw a picture of the conical tank, including a sketch of the water level at a point in time when the tank is not yet full. Introduce variables that measure the radius of the water's surface and the water's depth in the tank, and label them on your figure.

(b) Say that r is the radius and h the depth of the water at a given time, t. What proportional equation relates the radius and height of the water, and why?

(c) Determine an equation that relates the volume of water in the tank at time t to the depth h of the water at that time.

(d) Through differentiation, find an equation that relates the instantaneous rate of change of water volume with respect to time to the instantaneous rate of change of water depth at time t.

(e) Find the instantaneous rate at which the water level is rising when the water in the tank is 3 feet deep.

(f) When is the water rising most rapidly?

A $h = 3$

B $h = 4$

C $h = 5$

D The water level rises at a constant rate

Remark 3.3.11 Recall that in a right triangle with sides a, b and hypotenuse c we have the relationship

$$a^2 + b^2 = c^2,$$

also known in the western world as Pythagoras theorem (even though this result was well know well before his time by other civilizations).

Activity 3.3.12 Notice that by differentiating each variable in the equation above with respect to t, we get a relationship between $\frac{da}{dt}, \frac{db}{dt}, \frac{dc}{dt}$. Find this related rates equation!

Activity 3.3.13 A rectangle has one side of 8 cm. How fast is the diagonal of the rectangle changing at the instant when the other side is 6 cm and increaing at a rate of 3 cm per minute?

Activity 3.3.14 A 10 m ladder leans against a vertical wall and the bottom of the ladder slides away at a rate of 0.5 m/sec. When is the top of the ladder sliding the fastest down the wall?

A When the bottom of the ladder is 4 meters from the wall

B When the bottom of the laddre is 8 meters from the wall

C The top of the ladder is sliding down at a constant rate

3.4 Extreme values (AD4)

Learning Outcomes

- Use the Extreme Value Theorem to find the absolute maximum and minimum values of a continuous function on a closed interval.

Remark 3.4.1 In many different settings, we are interested in knowing where a function achieves its least and greatest values. These can be important in applications—say to identify a point at which maximum profit or minimum cost occurs—or in theory to characterize the behavior of a function or a family of related functions.

Example 3.4.2 Consider the simple and familiar example of a parabolic function such as $s(t) = -16t^2 + 32t + 48$ that represents the height of an object tossed vertically: its maximum value occurs at the vertex of the parabola and represents the greatest height the object reaches. This maximum value is an especially important point on the graph, the point at which the curve changes from increasing to decreasing.

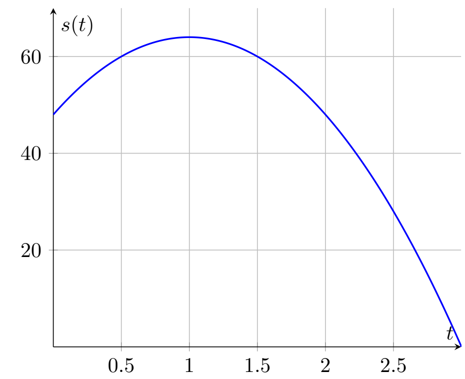


Figure .49 The graph of $s(t) = -16t^2 + 32t + 48$

□

Definition 3.4.3 Given a function f, we say that $f(c)$ is a **global** or **absolute maximum** of f provided that $f(c) \geq f(x)$ for all x in the domain of f, and similarly we call $f(c)$ a **global** or **absolute minimum** of f whenever $f(c) \leq f(x)$ for all x in the domain of f.The absolute maxima and minima are also called the absolute extrema or **extreme values** of the function. ♢

Activity 3.4.4 According to Definition 3.4.3, which of the following statements best describes the absolute extrema of the function in Figure .49?

A The absolute maximum is $t = 1$, because this is where the function goes from increasing to decreasing.

B The absolute maximum is $s(1) = 64$, because $s(t) \leq 64$ for every other input t.

C The graph has two absolute minima at the endpoints because the endpoints must be absolute extrema.

D The graph has no absolute minimum.

Observation 3.4.5 From Activity 3.4.4 we notice that there are some issues when determining the absolute mimimum and maximum values of a function from its graph alone. The Extreme Value Theorem will guarantee the existence of extrema on a closed interval. Thne we will learn about critical points to be able to detect extrema algebraically.

Theorem 3.4.6 *If f is continuous on a closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on the interval.*

Activity 3.4.7 For each of the following figures, decide where the absolute extrema are located.

(a)

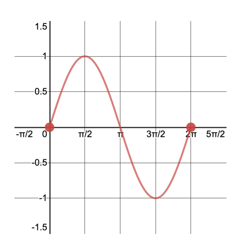


Figure .50

(b)

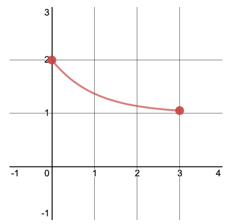


Figure .51

(c)

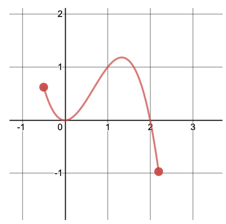


Figure .52

(d)

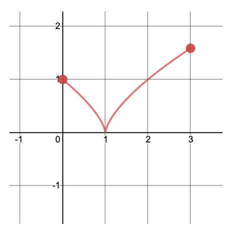


Figure .53

Activity 3.4.8 The Extreme Value Theorem (EVT) guarantees an absolute maximum and absolute minumum for which of the following?

A $f(x) = \dfrac{x^2}{x^2 - 4x - 5}$ on $[-5, 0]$.

B $f(x) = \dfrac{x^2}{x^2 - 4x - 5}$ on $[0, 4]$.

C $f(x) = \dfrac{x^2}{x^2 - 4x - 5}$ on $[4, 6]$.

D $f(x) = \dfrac{x^2}{x^2 - 4x - 5}$ on $[6, 10]$.

Activity 3.4.9 For the following activity, draw a sketch of a function that has the following properties.

(a) The function is continuous and has an absolute minimum but no absolute maximum.

(b) The function is continuous and has an absolute maximum but no absolute minimum.

Definition 3.4.10 We say that $f(c)$ is a **relative maximum** (or local maximum) of f provided that $f(c) \geq f(x)$ for all x near c. We say that $f(c)$ is a

relative minimum (or local minimum) of f provided that $f(c) \leq f(x)$ for all x near c. ♢

Observation 3.4.11 A strategy for finding the extreme values of a function will be to consider all the relative maxima and minima and compare them to decide ultimately which ones give the largest and smallest values. But now we have a new problem: how do you find the relative extrema? We can detect relative extrema by computing the first derivative and finding the critical points (or critical numbers) of the function. By finding the critical points, we will produce a list of candidates for the extrema (local and global) of the function.

Definition 3.4.12 We say that x is a **critical point** (or critical number) of $f(x)$ if x is in the domain of $f(x)$ and either $f'(x) = 0$ or $f'(x)$ does not exist. ♢

Activity 3.4.13 Which of the following are critical numbers for $f(x) = \frac{1}{3}x^3 - 2x + 2$?

A $x = \sqrt{2}$ and $x = -\sqrt{2}$.

B $x = \sqrt{2}$.

C $x = 2$ and $x = 0$.

D $x = 2$.

Activity 3.4.14 We have encountered several terms in this section, so we should make sure that we understand how they are related. Which of the following statements are true?

A In a closed interval an endpoint is always a local extrema but it might or might not be a global extrema.

B In a closed interval an endpoint is always a global extrema.

C A crtical point is always a local extrema but it might or might not be a global extrema.

D A local extrema only occurs where the first derivative is equal to zero.

E A local extrema always occurs at a critical point.

F A local extrema might occur at a critical point or at an endpoint of a closed interval.

Remark 3.4.15 The Closed Interval Method. The following is a way of finding the absolute extrema of a continuous f on a closed interval.

1 Make a list of all critical points of f in (a, b). (Do not include any critical points outside of the interval).

2 Add the endpoints a and b to the list.

3 Evaluate f at all points on your list.

4 The smallest value is the absolute minimum. The largest is the absolute maximum.

Activity 3.4.16 What are the absolute extrema for $f(x) = 3x^4 - 4x^3$ on $[-1, 2]$.

A Absolute maximum is when $x = 0$ and absolute minimum when $x = 1$.

B Absolute maximum is when $x = 2$ and absolute minimum when $x = -1$.

C Absolute maximum is when $x = 2$ and absolute minimum when $x = 1$.

D Absolute maximum is when $x = 0$ and absolute minimum when $x = -1$.

Activity 3.4.17 What is the absolute extrema for $f(x) = x\sqrt{4-x}$ on $[-2, 4]$.

A Absolute maximum is when $x = -2$ and absolute minimum when $x = \frac{8}{3}$.

B Absolute maximum is when $x = 4$ and absolute minimum when $x = \frac{8}{3}$.

C Absolute maximum is when $x = \frac{8}{3}$ and absolute minimum when $x = -2$.

D Absolute maximum is when $x = 4$ and absolute minimum when $x = -2$.

3.5 Derivative tests (AD5)

Learning Outcomes

- Determine where a differentiable function is increasing and decreasing and classify the critical points as local extrema.

Activity 3.5.1 What can we say about the graph of the function $f(x)$ if $f(x_1)$ is always less than $f(x_2)$ on the interval (a, b) ?

A The function is decreasing on (a, b).

B The function is increasing on (a, b).

C The function is constant on (a, b).

D The function could be some combination of increasing, decreasing, and/or constant on (a, b)

Activity 3.5.2

(a) Sketch a graph of a continuous function that is increasing on $(-\infty, -2)$ and constant on $(3, 5)$ and decreasing on $(-2, 1)$.

(b) How would you describe the derivative of the function on each interval?

A $f'(x) < 0$ on $(-2, 1)$ and $(3, 5)$ and $f'(x) > 0$ on $(-2, 1)$.

B $f'(x) > 0$ on $(-2, 1)$, $f'(x) < 0$ on $(-2, 1)$ and $f'(x)$ is undefined on $(3, 5)$.

C $f'(x) > 0$ on $(-2, 1)$, $f'(x) < 0$ on $(-2, 1)$ and $f'(x) = 0$ on $(3, 5)$.

D $f'(x) > 0$ on $(-2, 1)$, $f'(x) < 0$ on $(-2, 1)$ and $f'(x)$ is constant on $(3, 5)$.

Activity 3.5.3 Look back at the graph you made for Activity 3.5.2.

Which of the following best describes what is occurring when graph changes from increasing to decreasing or decreasing to increasing?

A There is a critical point.

B There is a relative maximum or minimum.

C The derivative is undefined.

D The derivative is equal to zero.

Theorem 3.5.4 The First Derivative Test. *Suppose $f(x)$ is continuous* *at $x = c$ and that $x = c$ is a critical point of $f(x)$.*

1. *If $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c, then a relative maximum occurs when $x = c$.*

2. *If $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c, then a relative minimum occurs when $x = c$.*

3. *If $f'(x)$ is the same sign on both sides of $x = c$, then neither a relative maximum nor a relative minimum occur when $x = c$.*

Activity 3.5.5 Let $f(x) = x^4 - 4x^3 + 4x^2$

(a) Find all critical points of $f(x)$. Draw them on the same number line.

(b) What intervals have been created by subdividing the number line at the critical points?

(c) Pick an x-value that lies in each interval. Determine whether $f'(x)$ is positive or negative at each point.

(d) On which intervals is $f(x)$ increasing? On which intervals is $f(x)$ decreasing?

(e) List all relative maxima and relative minima.

Remark 3.5.6 Dealing with discontinuities. Our previous activity dealt with a function that was continuous for all real numbers. Because of that, we could trust our chart to point out relative extrema. Let's now consider what might happen if a function has any discontinuities.

Activity 3.5.7 Draw a function that is increasing on the left of $x = 1$, discontinuous at $x = 1$,$f(1) = \lim_{x\to 1^+} f(x)$, and decreasing to the right of $x = 1$. Does the derivative of $f(x)$ exist at $x = 1$? Does your graph have a local maximum or minimum at $x = 1$?

Activity 3.5.8 Let $f(x) = \frac{x}{(x-2)^2}$.

Note that $f(x)$ is not defined for $x = 2$. So $x = 2$ cannot be a critical point and thus cannot be the location of a relative extrema. However, the function may be increasing on one side of $x = 2$ and decreasing on the other, so we must still include it on our number line.

(a) Find all critical points of $f(x)$. Plot them and any discontinuities for $f(x)$ on the same number line.

(b) What intervals have been created by subdividing the number line at the critical points and discontinuities?

(c) Pick an x-value that lies in each interval. Determine whether $f'(x)$ is positive or negative each point.

(d) On which intervals is $f(x)$ increasing? On which intervals is $f(x)$ decreasing?

(e) List all relative maxima and relative minima.

Activity 3.5.9 For each of the following functions, find the intervals on which $f(x)$ is increasing or decreasing. Then identify any relative extrema.

(a) $f(x) = x^3 + 3x^2 + 3x + 1$

(b) $f(x) = \frac{1}{2}x + \cos x$ on $(0, 2\pi)$

(c) $f(x) = (x^2 - 9)^{2/3}$

(d) $f(x) = \ln(2x - 1)$. (Hint: think about the domain of this one before you get started!)

(e) $f(x) = \frac{x^2}{x^2-4}$

Theorem 3.5.10 The Second Derivative Test. *Suppose that $x = c$ is a critical point of $f(x)$ and that $f''(x)$ is continuous at $x = c$.*

1. *If $f'(c) = 0$ and $f''(c) < 0$, then a relative maximum occurs when $x = c$.*
2. *If $f'(c) = 0$ and $f''(c) > 0$, then a relative minimum occurs when $x = c$.*
3. *If $f'(c) = 0$ and $f''(c) = 0$, then this test is inconclusive.*

In case of an inconclusive Seconf Derivative Test, we need to go back to the First Derivative Test to classify the critical point.

Activity 3.5.11 For each of the following functions, now identify any relative extrema using the Second Derivative Test, if possible.

(a) $f(x) = x^3 + 3x^2 + 3x + 1$

(b) $f(x) = \frac{1}{2}x + \cos x$ on $(0, 2\pi)$

(c) $f(x) = (x^2 - 9)^{2/3}$

(d) $f(x) = \ln(2x - 1)$. (Hint: think about the domain of this one before you get started!)

(e) $f(x) = \frac{x^2}{x^2-4}$

Theorem 3.5.12 The Mean Value Theorem. *If f is continuous and differentiable on the closed interval $[a, b]$, then there is some point c in the interval where $f'(c)$ is equal to the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$. In symbols, for some c in (a, b) we have that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Activity 3.5.13

(a) Suppose f is continuous and differentiable on $[a, b]$ and also suppose that $f(a) = f(b)$. What is the average rate of change of $f(x)$ on $[a, b]$? What does the MVT (Mean Value Theorem) tell you?

(b) Use part (a) to show with the MWT that $f(x) = (x - 1)^2 + 3$ has a critical point on $[0, 2]$.

3.6 Concavity and inflection (AD6)

Learning Outcomes

- Determine the intervals of concavity of a twice differentiable function and find all of its points of inflection.

In addition to asking *whether* a function is increasing or decreasing, it is also natural to inquire *how* a function is increasing or decreasing. Activity 3.6.1 describes three basic behaviors that an increasing function can demonstrate on an interval, as pictured in Figure .54: the function can increase more and more rapidly, it can increase at the same rate, or it can increase in a way that is slowing down. We are beginning to think about how a particular curve bends, with the natural comparison being made to lines, which don't bend at all. More than this, we want to understand how the bend in a function's graph is tied to behavior characterized by the first derivative of the function.

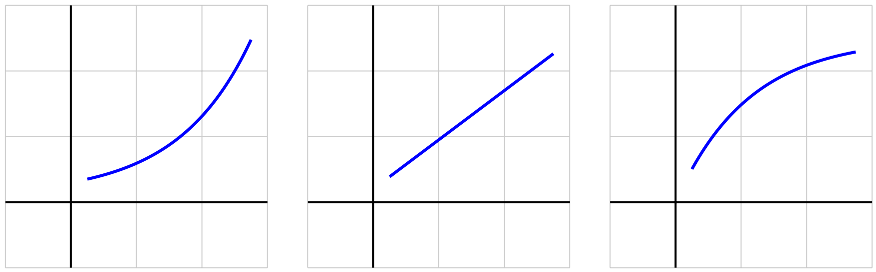


Figure .54 Three increasing functions

Activity 3.6.1 Sketch a sequence of tangent lines at various points to each of the following curves in Figure .54.

(a) Look at the curve pictured on the left. How would you describe the slopes of the tangent lines as you move from left to right?

A The slopes of the tangent lines decrease as you move from left to right.

B The slopes of the tangent lines remain constant as you move from left to right.

C The slopes of the tangent lines increase as you move from left to right.

(b) Look at the curve pictured in the middle. How would you describe the slopes of the tangent lines as you move from left to right?

A The slopes of the tangent lines decrease as you move from left to right.

B The slopes of the tangent lines remain constant as you move from left to right.

C The slopes of the tangent lines increase as you move from left to right.

(c) Look at the curve pictured on the right. How would you describe the slopes of the tangent lines as you move from left to right?

A The slopes of the tangent lines decrease as you move from left to right.

B The slopes of the tangent lines remain constant as you move from left to right.

C The slopes of the tangent lines increase as you move from left to right.

Remark 3.6.2 On the leftmost curve in Figure .54, draw a sequence of tangent lines to the curve. As we move from left to right, the slopes of those tangent lines will increase. Therefore, the rate of change of the pictured function is increasing, and this explains why we say this function is *increasing at* *an increasing rate*. For the rightmost graph in Figure .54, observe that as x increases, the function increases, but the slopes of the tangent lines decrease. This function is *increasing at a decreasing rate*.

Similar options hold for how a function can decrease. Here we must be extra careful with our language, because decreasing functions involve negative slopes. Negative numbers present an interesting tension between common language and mathematical language. For example, it can be tempting to say that "-100 is bigger than -2." But we must remember that "greater than" describes how numbers lie on a number line: $x > y$ provided that x lies to the right of y. So of course, -100 is less than -2. Informally, it might be helpful to say that "-100 is more negative than -2." When a function's values are negative, and those values get more negative as the input increases, the function must be decreasing.

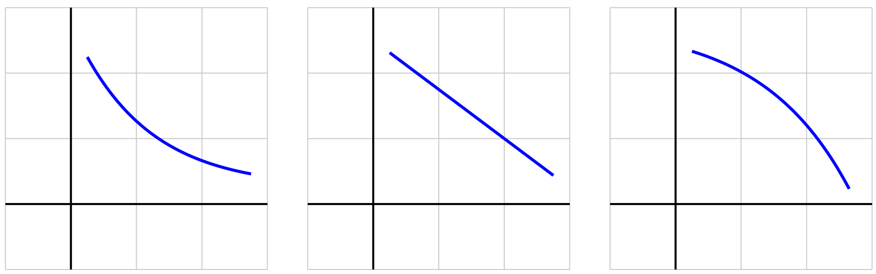


Figure .55 From left to right, three functions that are all decreasing.

Activity 3.6.3 Sketch a sequence of tangent lines at various points to each of the following curves in Figure .55.

(a) Look at the curve pictured on the left. How would you describe the slopes of the tangent lines as you move from left to right?

A The slopes of the tangent lines decrease as you move from left to right.

B The slopes of the tangent lines remain constant as you move from left to right.

C The slopes of the tangent lines increase as you move from left to right.

(b) Look at the curve pictured in the middle. How would you describe the slopes of the tangent lines as you move from left to right?

A The slopes of the tangent lines decrease as you move from left to right.

B The slopes of the tangent lines remain constant as you move from left to right.

C The slopes of the tangent lines increase as you move from left to right.

(c) Look at the curve pictured on the right. How would you describe the slopes of the tangent lines as you move from left to right?

A The slopes of the tangent lines decrease as you move from left to right.

B The slopes of the tangent lines remain constant as you move from left to right.

C The slopes of the tangent lines increase as you move from left to right.

We now introduce the notion of *concavity* which provides simpler language to describe these behaviors. When a curve opens upward on a given interval, like the parabola $y = x^2$ or the exponential growth function $y = e^x$, we say that the curve is *concave up* on that interval. Likewise, when a curve opens down, like the parabola $y = -x^2$ or the opposite of the exponential function $y = -e^x$, we say that the function is *concave down*. Concavity is linked to both the first and second derivatives of the function.

In Figure .56, we see two functions and a sequence of tangent lines to each. On the lefthand plot, where the function is concave up, observe that the tangent lines always lie below the curve itself, and the slopes of the tangent lines are increasing as we move from left to right. In other words, the function f is concave up on the interval shown because its derivative, f', is increasing on that interval. Similarly, on the righthand plot in Figure .56, where the function shown is concave down, we see that the tangent lines alway lie above the curve, and the slopes of the tangent lines are decreasing as we move from left to right. The fact that its derivative, f', is decreasing makes f concave down on the interval.

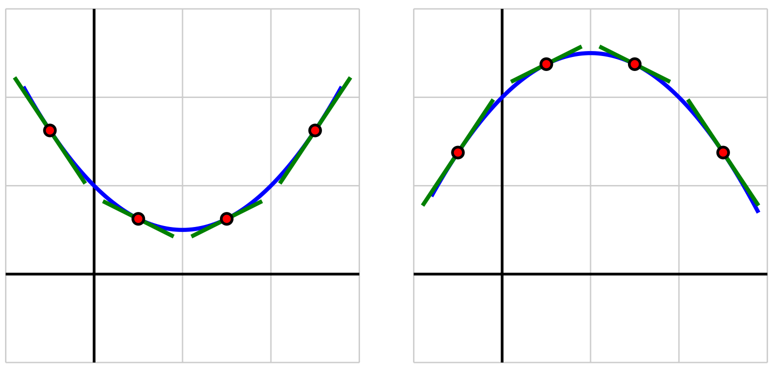


Figure .56 At left, a function that is concave up; at right, one that is concave down.

We state these most recent observations formally as the definitions of the terms *concave up* and *concave down*.

Definition 3.6.4 Let f be a differentiable function on an interval (a, b). Then f is **concave up** on (a, b) if and only if f' is increasing on (a, b); f is **concave down** on (a, b) if and only if f' is decreasing on (a, b). ♢

In Section 3.5, we used Theorem 3.5.4 as a tool to determine when continuous functions had local maxima and local minima. In this section, we would like to develop a similar test that can identify where continous fucntions are concave up and concave down.

Activity 3.6.5 Look at how the slopes of the tangent lines change from left to right for each of the two graphs in Figure .56

(a) Look at the curve pictured on the left. How would you describe the slopes of the tangent lines as you move from left to right?

A The slopes of the tangent lines decrease as you move from left to right.

B The slopes of the tangent lines increase as you move from left to right.

C The slopes of the tangent lines go from increasing to decreasing as you move from right to left.

D The slopes of the tangent lines go from decreasing to increasing as you move from right to left.

(b) Which of the following statements is true about the function on the left in Figure .56?

A $f'(x) > 0$ on the entire interval shown.

B $f'(x) < 0$ on the entire interval shown.

C $f''(x) > 0$ on the entire interval shown.

D $f''(x) < 0$ on the entire interval shown.

(c) Look at the curve pictured on the right. How would you describe the slopes of the tangent lines as you move from left to right?

A The slopes of the tangent lines decrease as you move from left to right.

B The slopes of the tangent lines increase as you move from left to right.

C The slopes of the tangent lines go from increasing to decreasing as you move from right to left.

D The slopes of the tangent lines go from decreasing to increasing as you move from right to left.

(d) Which of the following statements is true about the function on the right in Figure .56?

A $f'(x) > 0$ on the entire interval shown.

B $f'(x) < 0$ on the entire interval shown.

C $f''(x) > 0$ on the entire interval shown.

D $f''(x) < 0$ on the entire interval shown.

Theorem 3.6.6 Test for Concavity. *Suppose that $f(x)$ is twice differentiable on some interval (a, b). If $f'' > 0$ on (a, b), then f is concave up on (a, b). If $f'' < 0$ on (a, b), then f is concave down on (a, b).*

1. *If $f'(x) > 0$ and $f''(x) > 0$ on (a, b), then f is increasing and concave up on (a, b).*

2. *If $f'(x) > 0$ and $f''(x) < 0$ on (a, b), then f is increasing and concave down on (a, b).*

3. *If $f'(x) < 0$ and $f''(x) > 0$ on (a, b), then f is decreasing and concave up on (a, b).*

4. *If $f'(x) < 0$ and $f''(x) < 0$ on (a, b), then f is decreasing and concave down on (a, b).*

Activity 3.5.5 highlighted how to use the first derivative to identify intervals of increase and intervals of decrease of a function. Activity 3.6.7 uses Theorem 3.6.6 to identify where a function is concave up and concave down.

Activity 3.6.7 Let $f(x) = x^4 - 4x^3 + 4x^2$.

(a) Find all the zeros of $f'(x)$.

(b) What intervals have been created by subdividing the number line at zeros of $f'(x)$?

(c) Pick an x-value that lies in each interval. Determine whether $f'(x)$ is positive or negative at each point.

(d) On which intervals is $f'(x)$ increasing? On which intervals is $f'(x)$ decreasing?

(e) List all the intervals where $f(x)$ is concave up and all the intervals where $f(x)$ is concave down.

Remark 3.6.8 Just like the first derivative tells us when a function goes from increasing to decreasing and vice versa, we want to know when a function goes from concave up to concave down and vice versa.

Definition 3.6.9 If $x = c$ is a point where the concavity of graph of f changes from concave up to concave down or from concave down to concave up, then c is an **inflection point** of f. ♢

Activity 3.6.10 Use the results from Activity 3.6.7 to identify all of the inflection points of $f(x) = x^4 - 4x^3 + 4x^2$.

3.7 Graphing with derivatives (AD7)

Learning Outcomes

- Sketch the graph of a differentiable function whose derivatives satisfy given criteria.

In Section 3.5 and Section 3.6 we learned how the first and second derivatives give us information about the graph of a function. Specifically, we can determine the intervals where a function is increasing, decreasing, concave up, or concave down as well as any relative extrema or inflection points. Now we will put that information together to draw a sketch of the graph.

Activity 3.7.1 Which of the following best describes the curve graphed below?

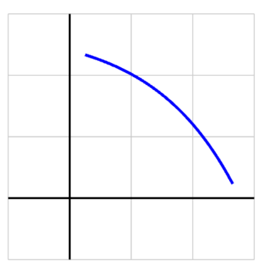


Figure .57

A Increasing and concave up

B Increasing and concave down

C Decreasing and concave up

D Decreasing and concave down

Activity 3.7.2

(a) Which of the following best describes the curve graphed below?

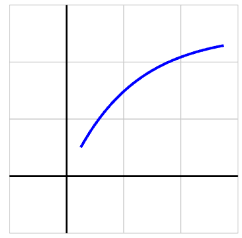


Figure .58

A $f' > 0$ and $f'' > 0$

B $f' > 0$ and $f'' < 0$

C $f' < 0$ and $f'' > 0$

D $f' < 0$ and $f'' < 0$

(b) For the other three answer choices above that you did not select, sketch a portion of a curve that matches each description.

Activity 3.7.3 For each prompt that follows, sketch a possible graph of a function on the interval $-3 < x < 3$ that satisfies the stated properties.

(a) $y = f(x)$ such that f is increasing on $-3 < x < 3$, concave up on $-3 < x < 0$, and concave down on $0 < x < 3$.

(b) $y = g(x)$ such that g is increasing on $-3 < x < 3$, concave down on $-3 < x < 0$, and concave up on $0 < x < 3$.

(c) $y = h(x)$ such that h is decreasing on $-3 < x < 3$, concave up on $-3 < x < -1$, neither concave up nor concave down on $-1 < x < 1$, and concave down on $1 < x < 3$.

(d) $y = p(x)$ such that p is decreasing and concave down on $-3 < x < 0$ and is increasing and concave down on $0 < x < 3$.

We must also be mindful of other characteristics of a function, such as the domain and the existence (or non-existence) of horizontal and vertical asymptotes. Activity 3.7.4 includes those aspects in addition to increasing, decreasing, and concavity.

Activity 3.7.4 The following chart describes the values of $f(x)$ and its first and second derivatives at or between a few given values of x, where $\nexists$ denotes that $f(x)$ does not exist at that value of x.

x		-8		-6		-3		0		2		5		8		11		13	
$f(x)$		3		5		$\nexists$		-5		$\nexists$		4		$\nexists$		-5		-3	
$f'(x)$	$+$		$+$		$-$		$-$		$-$		$-$		$+$		$+$		$+$		$+$
$f''(x)$	$+$		$-$		$-$		$+$		$-$		$+$		$+$		$-$		$-$		$-$

Assume that $f(x)$ has vertical asymptotes at each x-value where $f(x)$ does not exist, that $\lim_{x\to-\infty} f(x) = 1$, and that $\lim_{x\to\infty} f(x) = -1$.

(a) Use this information to sketch a reasonable graph of $f(x)$.

(b) Does $f(x)$ have any relative maxima or relative minima? If so, at what point(s)?

(c) Does $f(x)$ have any inflection points? If so, at what point(s)?

Now we will practice sketching the graph of a function from start to finish. We'll begin with an overview of the process.

Remark 3.7.5 A guide to curve sketching.

1. Identify the domain of the function.
2. Identify any vertical or horizontal asymptotes, if they exist.
3. Find $f'(x)$. Then use it to determine the intervals where the function is increasing and the intervals where the function is decreasing. State any relative extrema.
4. Find $f''(x)$. Then use it to determine the intervals where the function is concave up and the intervals where the function is concave down. State any inflection points.
5. Put everything together and draw sketch.

Activity 3.7.6 Sketch the graph of each of the following functions using the guide to curve sketching found in Remark 3.7.5

(a) $f(x) = x^4 - 4x^3 + 10$

(b) $f(x) = \frac{x^2-4}{x^2-9}$

(c) $f(x) = x + 2\cos x$ on the interval $[0, 2\pi]$

(d) $f(x) = \frac{x^2+x-2}{x+3}$

(e) $f(x) = \frac{x}{\sqrt{x^2+2}}$

(f) $f(x) = x^6 + \frac{12}{5}x^5 - 12x^4 + 10$

3.8 Applied optimization (AD8)

Learning Outcomes

- Apply optimization techniques to solve various problems.

Activity 3.8.1 The box. Help your company design a box with maximum volume given the following constraints:

- The box must be made from the following material: an 8 by 8 inches piece of cardboard.
- To create the box, you are asked to cut the same size square from each corner of the 8 by 8 inches piece of cardboard and to fold the remaining cardboard.

(a) Draw a diagram illustrating how the box is created.

(b) Explain why the volume of the box is a *function* of the side length x of the cutout squares.

(c) Express the volume of the box V as a function of the length of the cuts x.

(d) What is a realistic the domain of the function $V(x)$?

(e) What cut length x maximizes the volume of the box?

Remark 3.8.2 Approaching optimization problems.

- Draw a diagram and introduce variables. First understand what quantities are allowed to vary in the problem and then to represent those values with variables. Constructing a diagram with the variables labeled is almost always an essential first step. Sometimes drawing several diagrams can be especially helpful to get a sense of the situation.

- Identify the quantity to be optimized as well as any key relationships among the variables. Write down equations that involve the variables that have been introduced: one should represent the quantity whose minimum or maximum is sought, and others could show how multiple variables in the problem are interrelated.

- Determine a function of a single variable that models the quantity to be optimized; this may involve using other relationships among variables to eliminate one or more variables in the function formula. For example, in Activity 3.8.1, we initially found that $V = l \times w \times h$, but then the additional relationships that $l = w = 8 - 2x$ and $h = x$ yield $V(x) = (8 - 2x)^2 x$, and thus we have written the volume as a function of the single variable x.

- Decide the domain on which to consider the function being optimized. Often the physical constraints of the problem will limit the possible values that the independent variable can take on. Thinking back to the diagram describing the overall situation and any relationships among variables in the problem often helps identify the smallest and largest values of the input variable.

- Use calculus to identify the absolute maximum and/or minimum of the quantity being optimized. This always involves finding the critical numbers of the function first. Then, depending on the domain, we either construct a first derivative sign chart (for an open or unbounded interval we use the First or Second Derivative Tests) or evaluate the function at the endpoints and critical numbers (for a closed, bounded interval and a continuous function we use the Closed Interval Method).

- Finally, we make certain we have answered the question: what are the optimal points and what optimal values do we obtain at these points?

Activity 3.8.3 According to U.S. postal regulations, the girth plus the length of a parcel sent by mail may not exceed 108 inches, where by "girth" we mean the perimeter of the smallest end. What is the largest possible volume of a rectangular parcel with a square end that can be sent by mail? What are the dimensions of the package of largest volume?

(a) Let x represent the length of one side of the square end and y the length of the longer side. Label these quantities appropriately on the image shown in Figure .59.

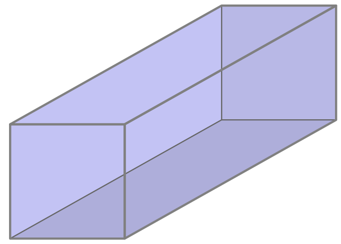


Figure .59 A rectangular parcel with a square end.

(b) What is the quantity to be optimized in this problem?

A maximize volume (V)

B maximize the girth plus length (L)

C minimize volume (V)

D minimize the girth plus length (L)

(c) Which formula below represents the quantity you want to optimize in terms of x and y?

A $V = x^2y$

B $V = xy^2$

C $L = 2x + y$

D $L = 4x + y$

(d) The problem statement tells us that the parcel's girth plus length (L) may not exceed 108 inches. In order to maximize volume, we assume that we will actually need the girth plus length (L) to equal 108 inches. What equation do we have involving x and y?

A $108 = 4x + y$

B $108 = 2x + y$

C $108 = x^2 + y$

D $108 = xy^2$

(e) The equation above gives the relationship between x and y. For ease of notation, solve this equation for y as a function on x and then find a formula for the volume of the parcel as a function of a the single variable x. What is the formula for $V(x)$?

A $V(x) = x^2(108 - 4x)$

B $V(x) = x(108 - 4x)^2$

C $V(x) = x^2(108 - 2x)$

D $V(x) = x(108 - 2x)^2$

(f) Over what domain should we consider this function? To answer this question, notice that the constraints that girth plus length is 108 inches produces intervals of possible values for x and y.

A $0 \le x \le 108$

B $0 \le y \le 108$

C $0 \le x \le 27$

D $0 \leq y \leq 27$

(g) Find the absolute maximum of the volume of the parcel on the domain you just established. Make sure to justify that you have found the absolute maximum using calculus!

Remark 3.8.4 Notice that a critical point might or might not be an absolute maximum or minimum, so finding the critical points is not enough to answer an optimization problem. Moreover, some of the critical points might be outside of the domain imposed by the context and thus they cannot be feasible optimal points.

Activity 3.8.5 Revenue = Number of tickets × Price of ticket. Waterford movie theater currently charges \$8 for a ticket. At this price, the theater sells 200 tickets daily. The general manager wonders if they can generate more revenue by increasing the price of a tickets. A survey shows that they will lose 20 customers for every dollar increase in the ticket price.

(a) If the price of a movie ticket is increased by d dollars, write a formula for the price P in terms of d.

(b) If the price of a ticket is increased by one dollar, how many many customers will the theater lose?

(c) Write a formula for the number of tickets sold T as a function of a ticket price increase of d dollars.

(d) Consider the new price of a ticket P and the new number of tickets sold T. Write a formula for the revenue R earned by ticket sales as a function of a price increase of d dollars.

(e) What is a realistic domain for $R(d)$?

(f) What increase in price d should the general manager choose to maximize the revenue? What would the movie ticket cost and what would be the revenue at that price?

(g) Now suppose that the cost of running the business if the price is increased by d dollars is given by $C(d) = 10d^3 - 40d^2 + 40d + 600$. If the manager decides that they will definitely increase the price, what price increase d maximizes the profit? (Recall that Profit = Revenue - Cost).

Activity 3.8.6 Modeling given a geometric shape. The city council is planning to construct a new sports ground in the shape of a rectangle with semicircular ends. A running track 400 meters long is to go around the perimeter.

(a) What choice of dimensions will make the rectangular area in the center as large as possible?

(b) What should the dimensions so the total area enclosed by the running track is maximized?

Activity 3.8.7 Modeling in algebraic situations.

(a) Find the coordinates of the point on the curve $y = \sqrt{x}$ closest to the point $(1, 0)$

(b) The sum of two positive numbers is 48. What is the smallest possible value of the sum of their squares?

3.9 Limits and Derivatives (AD9)

Learning Outcomes

- Compute the values of indeterminate limits using L'Hopital's Rule.

Remark 3.9.1 When we compute a limit algebraically, we often encounter the indeterminate form

$$\frac{0}{0}$$

but this means that limit can equal any number, infinity, or might even not exist. When we encounter an indeterminate form, we just do not know (yet) what the value of the limit is.

Activity 3.9.2 We can compute limits that give indeterminate forms via algebraic manipulations. Consider

$$\lim_{x\to 1} \frac{2x-2}{x^2-1}.$$

(a) Verify that this limit gives an indeterminate form of the type $\frac{0}{0}$.

(b) In the limit, you can cancel common factors. After you simplify the fraction, what is this limit equal to?

A 2

B 1

C $\dfrac{1}{2}$

D The limit does not exist.

Remark 3.9.3 Consider the limits

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a).$$

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a).$$

Notice that these limits give indeterminate forms of the type $\frac{0}{0}$. However, these limits are equal to the limit defintion of $f'(a)$, the derivative of $f(x)$ at $x = a$. If you can compute $f'(a)$, then you can compute the value of the limit.

Activity 3.9.4 Use the limit definition of the derivative to compute the following limits. Each limit is $f'(a)$, the derivative of some function $f(x)$ at $x = a$. You need to determine the function and the point to then find the value of the limit: $f'(a)$.

(a) Notice that $\displaystyle\lim_{x\to 0} \frac{e^{2+x}-e^2}{x}$ is the limit definition of the derivative of e^x at $x = 2$ (where x was used for h). Given these observations, what is this limit equal to?

A 2

B e

C e^2

D The limit does not exist.

(b) Consider $\lim_{x\to 0} \dfrac{\ln(1+x)}{x}$. This limit is also the limit definition of some derivative at a point. What is the value of this limit?

A 1

B 0

C $\ln(2)$

D The limit does not exist.

Activity 3.9.5 Compute the following limits using the limit defintion of the derivative at a point.

(a) $\lim_{x\to 0} \dfrac{\sin(x)}{x}$

(b) $\lim_{x\to 0} \dfrac{\tan(x)}{x}$

(c) $\lim_{x\to 0} \dfrac{\cos(\frac{\pi}{3}+x)-\frac{1}{2}}{x}$

Remark 3.9.6 When we compute a limit algebraically, we might encounter the indeterminate form

$$\frac{\infty}{\infty}$$

but this means that limit can equal any number, infinity, or might even not exist. When we encounter an indeterminate form, we just do not know (yet) what the value of the limit is.

Activity 3.9.7 We can compute limits that give indeterminate forms via algebraic manipulations. Consider

$$\lim_{x\to +\infty} \frac{2x^2+1}{x^2-1}.$$

(a) Verify that this limit gives an indeterminate form of the type $\frac{\infty}{\infty}$.

(b) You can manipulate this fraction algebraically by dividing numerator and denominator by x^2. Notice that $\pm\frac{1}{x^2} \to 0$ as $x \to \infty$. So after you manipulate and simplify the fraction, what is this limit equal to?

A 2

B 1

C $\dfrac{1}{2}$

D The limit does not exist.

Theorem 3.9.8 L' Hospital's Rule. *If for the limit of $\frac{f(x)}{g(x)}$ as $x \to a$ we have one of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f(x), g(x)$ are both differentiable around $x = a$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that the limit exists!

Activity 3.9.9 Look back at some limits that gave you an indeterminate form. Can you use l'Hospital's Rule to find the limit? If using the L'Hospital's Rule is appropriate, then try to compute the limit this way. It should give you the same result.

Activity 3.9.10 For the following limits, check if they give an indeterminate form. If they do, try to use l'Hospital's Rule. Does it help? It may or may not, or you may just need to use the rule repeatedly. Either way, try to compute the value of the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x}$

(b) $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x}$

(c) $\displaystyle\lim_{x\to \infty} \frac{3x^2 + 3}{x^2 + 2x}$

(d) $\displaystyle\lim_{x\to 0^+} \frac{\ln(x)}{-x}$

(e) $\displaystyle\lim_{x\to 0^+} \frac{\ln(x)}{1/x}$

(f) $\displaystyle\lim_{x\to 0} \frac{\sin^2(3x)}{5x^3 - 3x^2}$

Chapter 4

Definite and Indefinite Integrals (IN)

Learning Outcomes

By the end of this chapter, you should be able to...

1. Use geometric formulas to compute definite integrals.
2. Approximate definite integrals using Riemann sums.
3. Determine basic antiderivatives.
4. Solve basic initial value problems.
5. Evaluate a definite integral using the Fundamental Theorem of Calculus.
6. Find the derivative of an integral using the Fundamental Theorem of Calculus.
7. Use definite integrals to find area under a curve.
8. Use definite integral(s) to compute the area bounded by several curves.

Readiness Assurance. Before beginning this chapter, you should be able to...

a Find the derivative of a function using elementary derivative rules, as in Section 2.3. (CheckIt)

b Use sigma notation for sums. (Khan Academy and Active Calculus)

c Find the area of a rectangle in the coordinate plane. (Example at Khan AcademyExample at virtual nerd)

d Find the intersection of two graphs. (Khan Academy)

e Find the area of plane shapes, such as rectangles, triangles, circles, and trapezoids. (Math is fun)

4.1 Geometry of definite integrals (IN1)

Learning Outcomes

- Use geometric formulas to compute definite integrals.

Definition 4.1.1 The **definite integral** for a positive function $f(x) \geq 0$ between the points $x = a$ and $x = b$ is the area between the function and the x-axis. We denote this quantity as $\displaystyle\int_a^b f(x)\,dx$ ◊

Remark 4.1.2 For some functions which have known geometric shapes (like pieces of lines or circles) we can already compute these area exactly and we will do so in this section. But for most functions we do not know quite yet how to compute these areas. In the next section, we will see that because we can compute the areas of rectangles quite easily, we can always try to approximate a shape with rectangles, even if this could be a very coarse approximation.

Activity 4.1.3 Consider the linear function $f(x) = 2x$. Sketch a graph of this function. Consider the area between the x-axis and the function on the interval $[0, 1]$. What is $\int_0^1 f(x)\,dx$?

A 1

B 2

C 3

D 4

Activity 4.1.4 Consider the linear function $f(x) = 4x$. What is $\int_0^1 f(x)\,dx$?

A 1

B 2

C 3

D 4

Activity 4.1.5 Consider the linear function $f(x) = 2x+2$. Notice that on the interval $[0, 1]$, the shape formed between the graph and the x-axis is a trapezoid. What is $\int_0^1 f(x)\,dx$?

A 1

B 2

C 3

D 4

Activity 4.1.6 Consider the function $f(x) = \sqrt{4 - x^2}$. Notice that on the domain $[-2, 2]$, the shape formed between the graph and the x-axis is a semicircle. What is $\int_{-2}^2 f(x)\,dx$?

A π

B 2π

C 3π

D 4π

Definition 4.1.7 If a function $f(x) \leq 0$ on $[a, b]$, then we define the integral between a and b to be

$$\int_a^b f(x)\,dx \;=\; (-1) \times \text{area between the graph of and the axis on the interval }.$$

So the definite integral for a negative function is the "negative" of the area between the graph and the x-axis. ♢

Activity 4.1.8 Explain how to use geometric formulas for area to compute the following definite integrals. For each part, sketch the function to support your explanation.

1. $$\int_1^6 (-3\,x + 6)\,dx$$

2. $$\int_2^6 (-3\,x + 6)\,dx$$

3. $$\int_1^5 \left(-\sqrt{-(x-1)^2 + 16}\right) dx$$

Activity 4.1.9 The graph of $g(t)$ and the areas A_1, A_2, A_3 are given below.

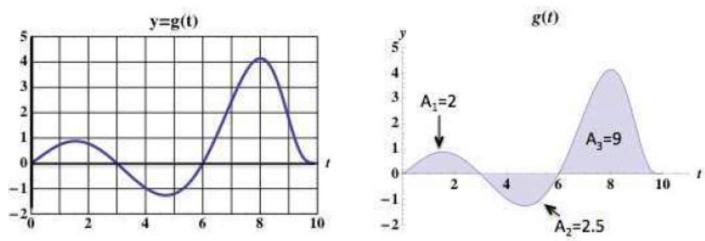


Figure .60

(a) Find $\int_3^3 g(t)\,dt$

(b) Find $\int_3^6 g(t)\,dt$

(c) Find $\int_0^{10} g(t)\,dt$

(d) Suppose that $g(t)$ gives the velocity in fps at time t (in seconds) of a particle moving in the vertical direction. A positive velocity indicates that the particle is moving up, a negative velocity indicates that the particle is moving down. If the particle started at a height of 3ft, at what height would it been after 3 seconds? After 6 seconds? After 10 seconds? At what time does the particle reach the highest point in this time interval?

4.2 Approximating definite integrals (IN2)

Learning Outcomes

- Approximate definite integrals using Riemann sums.

Activity 4.2.1 Suppose that a person is taking a walk along a long straight path and walks at a constant rate of 3 miles per hour.

a. On the left-hand axes provided in Figure .61, sketch a labeled graph of the velocity function $v(t) = 3$.

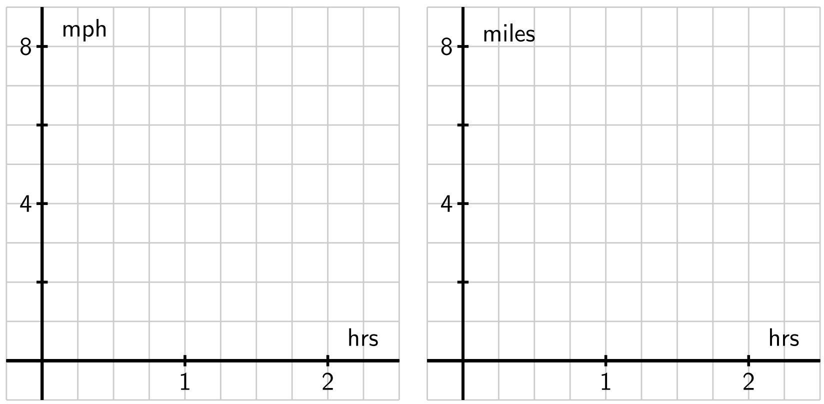


Figure .61 At left, axes for plotting $y = v(t)$; at right, for plotting $y = s(t)$.

Note that while the scale on the two sets of axes is the same, the units on the right-hand axes differ from those on the left. The right-hand axes will be used in question (d).

b. How far did the person travel during the two hours? How is this distance related to the area of a certain region under the graph of $y = v(t)$?

c. Find an algebraic formula, $s(t)$, for the position of the person at time t, assuming that $s(0) = 0$. Explain your thinking.

d. On the right-hand axes provided in Figure .61, sketch a labeled graph of the position function $y = s(t)$.

e. For what values of t is the position function s increasing? Explain why this is the case using relevant information about the velocity function v.

4.2.1 Area under the graph of the velocity function

In Activity 4.2.1, we learned that when the velocity of a moving object's velocity is constant (and positive), the area under the velocity curve over an interval of time tells us the distance the object traveled.

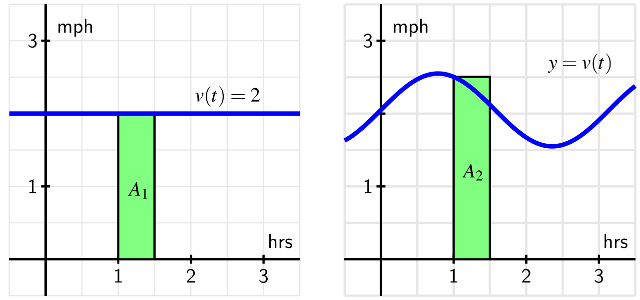


Figure .62 At left, a constant velocity function; at right, a non-constant velocity function.

The left-hand graph of Figure .62 shows the velocity of an object moving at 2 miles per hour over the time interval $[1, 1.5]$. The area A_1 of the shaded region under $y = v(t)$ on $[1, 1.5]$ is

$$A_1 = 2\,\frac{\text{miles}}{\text{hour}} \cdot \frac{1}{2}\,\text{hours} = 1\,\text{mile}.$$

This result is simply the fact that distance equals rate times time, provided the rate is constant. Thus, if $v(t)$ is constant on the interval $[a, b]$, the distance traveled on $[a, b]$ is equal to the area A given by

$$A = v(a)(b - a) = v(a)\Delta t,$$

where Δt is the change in t over the interval. (Since the velocity is constant, we can use any value of $v(t)$ on the interval $[a, b]$, we simply chose $v(a)$, the value at the interval's left endpoint.) For several examples where the velocity function is piecewise constant, see http://gvsu.edu/s/9T.[1]

The situation is more complicated when the velocity function is not constant. But on relatively small intervals where $v(t)$ does not vary much, we can use the area principle to estimate the distance traveled. The graph at right in Figure .62 shows a non-constant velocity function. On the interval $[1, 1.5]$, the velocity varies from $v(1) = 2.5$ down to $v(1.5) \approx 2.1$. One estimate for the distance traveled is the area of the pictured rectangle,

$$A_2 = v(1)\Delta t = 2.5\,\frac{\text{miles}}{\text{hour}} \cdot \frac{1}{2}\,\text{hours} = 1.25\,\text{miles}.$$

Note that because v is decreasing on $[1, 1.5]$, $A_2 = 1.25$ is an over-estimate of the actual distance traveled.

To estimate the area under this non-constant velocity function on a wider interval, say $[0, 3]$, one rectangle will not give a good approximation. Instead, we could use the six rectangles pictured in Figure .63, find the area of each rectangle, and add up the total. Obviously there are choices to make and issues to understand: How many rectangles should we use? Where should we evaluate the function to decide the rectangle's height? What happens if the velocity is sometimes negative? Can we find the exact area under any non-constant curve?

[1] Marc Renault, calculus applets.

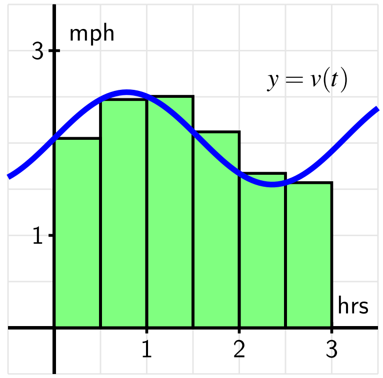


Figure .63 Using six rectangles to estimate the area under $y = v(t)$ on $[0, 3]$.

We will study these questions and more in what follows; for now it suffices to observe that the simple idea of the area of a rectangle gives us a powerful tool for estimating distance traveled from a velocity function, as well as for estimating the area under an arbitrary curve. To explore the use of multiple rectangles to approximate area under a non-constant velocity function, see the applet found at http://gvsu.edu/s/9U.[2]

Activity 4.2.2 Suppose that a person is walking in such a way that her velocity varies slightly according to the information given in Table .64 and graph given in Figure .65.

Table .64 Velocity data for the person walking.

t	$v(t)$
0.00	1.500
0.25	1.789
0.50	1.938
0.75	1.992
1.00	2.000
1.25	2.008
1.50	2.063
1.75	2.211
2.00	2.500

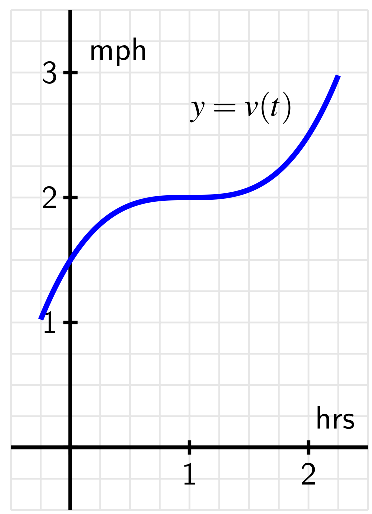


Figure .65 The graph of $y = v(t)$.

a. Using the grid, graph, and given data appropriately, estimate the distance traveled by the walker during the two hour interval from $t = 0$ to $t = 2$. You should use time intervals of width $\Delta t = 0.5$, choosing a way to use the function consistently to determine the height of each rectangle in order to approximate distance traveled.

b. How could you get a better approximation of the distance traveled on $[0, 2]$? Explain, and then find this new estimate.

[2]Marc Renault, calculus applets.

c. Now suppose that you know that v is given by $v(t) = 0.5t^3 - 1.5t^2 + 1.5t + 1.5$. Remember that v is the derivative of the walker's position function, s. Find a formula for s so that $s' = v$.

d. Based on your work in (c), what is the value of $s(2) - s(0)$? What is the meaning of this quantity?

Hint.

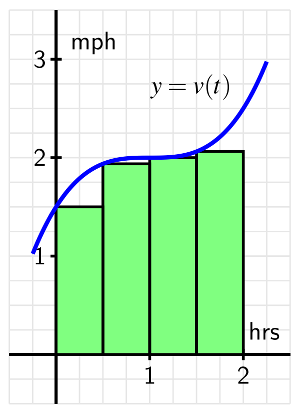
a. For instance, the approximate distance traveled on $[0, 0.5]$ can be computed by $v(0) \cdot 0.5 = 1.5 \cdot 0.5 = 0.75$ miles.

b. Think about the possibility of using a larger number of rectangles.

c. If $v(t) = t^3$ and we seek a function s such that $s' = v$, note that s has to involve t^4.

d. Observe that this quantity is measuring a change in position.

Answer.

a. 

$$A = v(0.0) \cdot 0.5 + v(0.5) \cdot 0.5 + v(1.0) \cdot 0.5 + v(1.5) \cdot 0.5$$
$$= 1.500 \cdot 0.5 + 1.9375 \cdot 0.5 + 2.000 \cdot 0.5 + 2.0625 \cdot 0.5$$
$$= 3.75$$

Thus, $D \approx 3.75$ miles.

b. Using 8 rectangles of width 0.25, $D \approx 3.875$.

c. $s(t) = \frac{1}{8}t^4 - \frac{1}{2}t^3 + \frac{3}{4}t^2 + \frac{3}{2}t$.

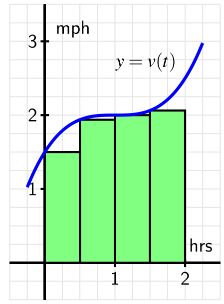
d. $s(2) - s(0) = \frac{1}{8}2^4 - \frac{1}{2}2^3 + \frac{3}{4}2^2 + \frac{3}{2}2 = 4$.

Solution.

a. Using rectangles of width $\Delta t = 0.5$ and choosing to set the heights of the rectangles from the function value at the left end of the interval, we see the following graph and find the sum of the areas of the rectangles to be

$$A = v(0.0) \cdot 0.5 + v(0.5) \cdot 0.5 + v(1.0) \cdot 0.5 + v(1.5) \cdot 0.5$$
$$= 1.500 \cdot 0.5 + 1.9375 \cdot 0.5 + 2.000 \cdot 0.5 + 2.0625 \cdot 0.5$$
$$= 3.75$$

Thus, the distance traveled is approximately $D \approx 3.75$ miles.



b. It appears that a better approximation could be found using narrower rectangles. If we move to 8 rectangles of width 0.25, similar computations show that $D \approx 3.875$.

c. By thinking about how the power rule for differentiation works, we can undo this rule and find a position function s whose derivative is v. For instance, since $\frac{d}{dt}[t^4] = 4t^3$, we see that $\frac{d}{dt}[\frac{1}{8}t^4] = \frac{1}{2}t^3$. Thus, if we let

$$s(t) = \frac{1}{8}t^4 - \frac{1}{2}t^3 + \frac{3}{4}t^2 + \frac{3}{2}t,$$

then it is straightforward to check that $s'(t) = \frac{1}{2}t^3 - \frac{3}{2}t^2 + \frac{3}{2}t + \frac{3}{2}$, which is precisely the formula for $v(t)$ that we were given.

d. By the rule found in (c) for s, we have that $s(2) - s(0) = \frac{1}{8}2^4 - \frac{1}{2}2^3 + \frac{3}{4}2^2 + \frac{3}{2}2 = 2 - 4 + 3 + 3 = 4$. This is the change in the walker's position over the time interval $[0, 2]$, and since the velocity is always positive, this is actually the exact distance traveled. We see how both earlier estimates (3.75 and 3.785) are good approximations to this value.

4.3 Elementary antiderivatives (IN3)

Learning Outcomes

- Determine basic antiderivatives.

In Activity 4.2.1, we established that whenever v is constant on an interval, the exact distance traveled is the area under the velocity curve. When v is not constant, we can estimate the total distance traveled by finding the areas of rectangles that approximate the area under the velocity curve.

Thus, we see that finding the area between a curve and the horizontal axis is an important exercise: besides being an interesting geometric question, if the curve gives the velocity of a moving object, the area under the curve tells us the exact distance traveled on an interval. We can estimate this area if we have a graph or a table of values for the velocity function.

In Activity 4.2.2, we encountered an alternate approach to finding the distance traveled. If $y = v(t)$ is a formula for the instantaneous velocity of a moving object, then v must be the derivative of the object's position function, s. If we can find a formula for $s(t)$ from the formula for $v(t)$, we will know the position of the object at time t, and the change in position over a particular time interval tells us the distance traveled on that interval.

For example, let's consider the situation from Activity 4.2.1, where a person is walking along a straight line with velocity function $v(t) = 3$ mph.

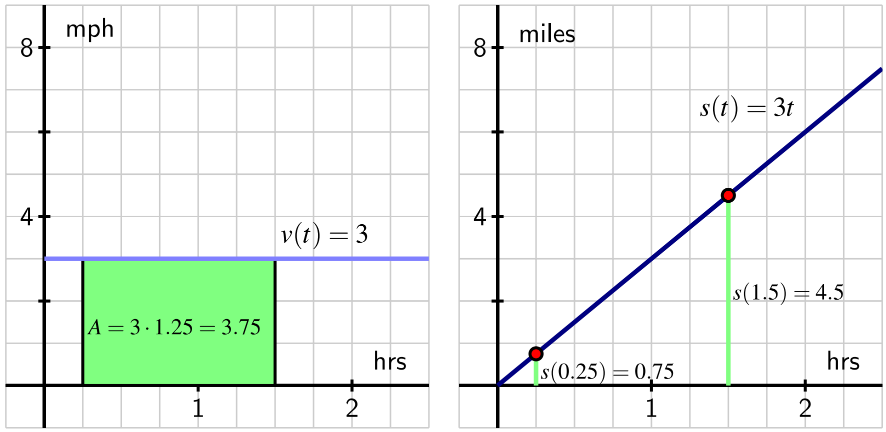


Figure .66 The velocity function $v(t) = 3$ and corresponding position function $s(t) = 3t$.

On the left-hand graph of the velocity function in Figure .66, we see the relationship between area and distance traveled,

$$A = 3\,\frac{\text{miles}}{\text{hour}} \cdot 1.25\,\text{hours} = 3.75\,\text{miles}.$$

In addition, we observe[1] that if $s(t) = 3t$, then $s'(t) = 3$, so $s(t) = 3t$ is the position function whose derivative is the given velocity function, $v(t) = 3$. The respective locations of the person at times $t = 0.25$ and $t = 1.5$ are $s(1.5) = 4.5$ and $s(0.25) = 0.75$, and therefore

$$s(1.5) - s(0.25) = 4.5 - 0.75 = 3.75 \text{ miles}.$$

This is the person's change in position on $[0.25, 1.5]$, which is precisely the distance traveled.

Observe that if we know a formula for a velocity function v, it can be very helpful to find a function s that satisfies $s' = v$. We say that s is an *antiderivative* of v. More generally, we have the following formal definition.

Definition 4.3.1 If g and G are functions such that $G' = g$, we say that G is an **antiderivative** of g. ♢

For example, if $g(x) = 3x^2 + 2x$, $G(x) = x^3 + x^2$ is an antiderivative of g, because $G'(x) = g(x)$. Note that we say "an" antiderivative of g rather than "the" antiderivative of g, because $H(x) = x^3 + x^2 + 5$ is also a function whose derivative is g, and thus H is another antiderivative of g.

The general problem of finding an antiderivative is difficult. In part, this is due to the fact that we are trying to undo the process of differentiating, and the undoing is much more difficult than the doing. However, some antiderivatives can be fairly simple to find if we remember our elementary derivative rules. (See Section 2.3.)

Activity 4.3.2 Consider the function $f(x) = \cos x$. Which of the following could be $F(x)$, an antiderivative of $f(x)$?

A $\sin x$

B $\cos x$

C $\tan x$

[1]Here we are making the implicit assumption that $s(0) = 0$.

D $\sec x$

Activity 4.3.3 Consider the function $f(x) = x^2$. Which of the following could be $F(x)$, an antiderivative of $f(x)$?

A $2x$

B $\frac{1}{3}x^3$

C x^3

D $\frac{2}{3}x^3$

We now note that whenever we know the derivative of a function, we have a *function-derivative pair*, so we also know the antiderivative of a function. For instance, in Activity 4.3.2 we could use our prior knowledge that

$$\frac{d}{dx}[\sin(x)] = \cos(x),$$

to determine that $F(x) = \sin(x)$ is an antiderivative of $f(x) = \cos(x)$. F and f together form a function-derivative pair. Every elementary derivative rule leads us to such a pair, and thus to a known antiderivative.

In the following activity, we work to build a list of basic functions whose antiderivatives we already know.

Activity 4.3.4 Use your knowledge of derivatives of basic functions to complete Table .67 of antiderivatives. For each entry, your task is to find a function F whose derivative is the given function f.

Table .67 Familiar basic functions and their antiderivatives.

given function, $f(x)$	antiderivative, $F(x)$
k, (k is constant)	
x^n, $n \neq -1$	
$\frac{1}{x}$, $x > 0$	
$\sin(x)$	
$\cos(x)$	
$\sec(x)\tan(x)$	
$\csc(x)\cot(x)$	
$\sec^2(x)$	
$\csc^2(x)$	
e^x	
a^x $(a > 1)$	
$\frac{1}{1+x^2}$	
$\frac{1}{\sqrt{1-x^2}}$	

Notice that in Definition 4.3.1 and Activity 4.3.4 that we were asked to find an antiderivative, not the antiderivative. Because the derivative of any constant is zero, we may add a constant of our choice to any antiderivative. For instance, we know that $G(x) = \frac{1}{3}x^3$ is an antiderivative of $g(x) = x^2$. But we could also have chosen $G(x) = \frac{1}{3}x^3 + 7$, since in this case as well, $G'(x) = x^2$. If $g(x) = x^2$, we say that the *general antiderivative* of g is

$$G(x) = \frac{1}{3}x^3 + C,$$

where C represents an arbitrary real number constant. Regardless of the formula for g, including $+C$ in the formula for its antiderivative G results in the most general possible antiderivative.

Activity 4.3.5 Using this information, which of the following is an antiderivative for $f(x) = 5\sin(x) - 4x^2$?

A $F(x) = -5\cos(x) + \frac{4}{3}x^3$.

B $F(x) = 5\cos(x) + \frac{4}{3}x^3$.

C $F(x) = -5\cos(x) - \frac{4}{3}x^3$.

D $F(x) = 5\cos(x) - \frac{4}{3}x^3$.

It is useful to have shorthand notation that indicates the instruction to find an antiderivative. Thus, in a similar way to how the notation

$$\frac{d}{dx}[f(x)]$$

represents the derivative of $f(x)$ with respect to x, we use the notation of the *indefinite integral*,

$$\int f(x)\,dx$$

to represent the general antiderivative of f with respect to x. Returning to the earlier example with $f(x) = 5\sin(x) - 4x^2$, we can rephrase the relationship between f and its antiderivative F through the notation

$$\int (5\sin(x) - 4x^2)\,dx = -5\cos(x) - \frac{4}{3}x^3 + C.$$

When we find an antiderivative, we will often say that we *evaluate an indefinite integral*. Just as the notation $\frac{d}{dx}[\square]$ means "find the derivative with respect to x of $\square$," the notation $\int \square\,dx$ means "find a function of x whose derivative is $\square$."

Activity 4.3.6 Find the general antiderivative for each function.

(a)

$$f(x) = -4\sec^2(x)$$

(b)

$$f(x) = \frac{8}{\sqrt{x}}$$

Activity 4.3.7 Find each indefinite integral.

(a)

$$\int (-9x^4 - 7x^2 + 4)\,dx$$

(b)

$$\int 3e^x\,dx$$

4.4 Initial Value Problems (IN4)

Learning Outcomes

- Solve basic initial value problems.

In this section we will discuss the relationship between antiderivatives and solving simple differential equations. A differential equation is an equation that has a derivative. For this section we will focus on differential equations of the form

$$\frac{dy}{dx} = f(x).$$

Our goal is to find a relationship of $y(x)$ that satisfies the differential equation. We can solve for $y(x)$ by finding the antiderivative of $f(x)$.

Activity 4.4.1 Which of the following equations for $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = x^2 + 2x.$$

A $y(x) = \frac{x^3}{3} + x^2 + 4$

B $y(x) = 2x + 2$

C $y(x) = \frac{x^3}{3} + x^2 + 10$

D $y(x) = \frac{x^3}{3} + x^2$

E $y(x) = 2x$

In Activity 4.4.1 there are more than one solution that satisfies the differential equation. In fact their is a family of functions that satisfies the differential equation, that is

$$f(x) = \frac{x^3}{3} + x^2 + c_1,$$

where c_1 is an arbitrary constant yet to be defined. To find c_1 we have to have some initial value for the differential equation, $y(x_0) = y_0$, where the point (x_0, y_0) is the starting point for the differential equation. In general this section we will focus on solving initial value problems (differential equation with an initial condition) of the form,

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0.$$

Activity 4.4.2 Which of the following equations for $y(x)$ satisfies the differential equation and initial condition,

$$\frac{dy}{dx} = x^2 + 2x, \;\; y(3) = 16.$$

A $y(x) = \frac{x^3}{3} + x^2 - 4$

B $y(x) = \frac{x^3}{3} + x^2 + 2$

C $y(x) = \frac{x^3}{3} + x^2 - 2$

D $y(x) = \frac{x^3}{3} + x^2 + 16$

Activity 4.4.3 Which of the following functions satisfies the initial value problem,

$$\frac{dy}{dx} = \sin(x), \;\; y(0) = 1.$$

A $y(x) = \cos(x)$

B $y(x) = \cos(x) + 2$

C $y(x) = \cos(x) + 1$

D $y(x) = -\cos(x)$

E $y(x) = -\cos(x) + 2$

Activity 4.4.4 One of the applications of initial value problems is calculating the distance traveled from a point based on the velocity of the object. Given that the velocity of the of an object is approximated by $v(t) = \cos(t)+1$ km/hr, what is the approximate distance travelled by the object after 1 hour?

A $s(1) \approx 1$ km

B $s(1) \approx 0.1585$ km

C $s(1) \approx 1.8415$ km

D $s(1) \approx 2.3415$ km

Activity 4.4.5 So far we have only been going from velocity to position of an object. Recall that to find the acceleration of an object, you can take the derivative of the velocity of an object. Let use say we have the acceleration of a falling object given by $a(t) = -9.8$ m/s^2.What is the velocity of the falling object, if the initial velocity is given by $v(0) = 0$ m/s.

A $v(t) = -9.8t$ m

B $v(t) = -9.8t$ m/s

C $v(t) = 9.8t$ m/s

D $v(t) = 9.8t + 1$ m

What is the function for the position of the object, if the initial position is given by $s(0) = 10$ m.

A $s(t) = 4.9t + 10$ m

B $s(t) = -4.9t^2 + 14.9$ m

C $s(t) = -4.9t^2 + 10$ m

D $s(t) = 4.9t + 5.1$ m

4.5 FTC for definite integrals (IN5)

Learning Outcomes

- Evaluate a definite integral using the Fundamental Theorem of Calculus.

Activity 4.5.1 Find the area beteween $f(x) = \frac{1}{2}x + 2$ and the x-axis from $x = 2$ to $x = 6$.

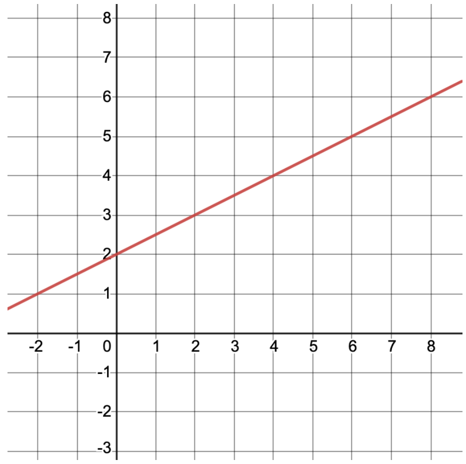


Figure .68

Activity 4.5.2 Approximate the area under the curve $f(x) = (x-1)^2+2$m> on the interval $[1,5]$m> using a left Riemann sum with four uniform subdivisions. Draw your rectangles on the graph.

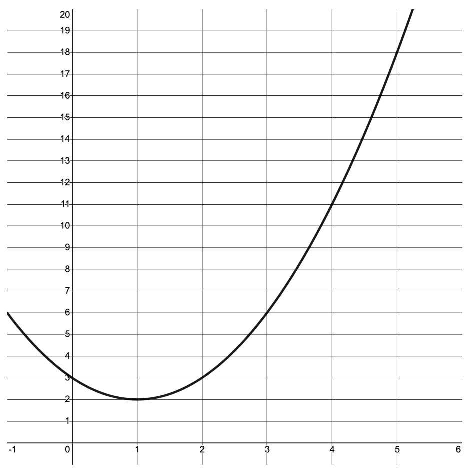


Figure .69

4.5.1 Definite Integrals

Definition 4.5.3 Let $f(x)$ be a continuous function on the interval $[a,b]$. Divide the interval into n subdivisions of equal width, Δx, and choose a point x_i in each interval. Then, the definite integral of $f(x)$ from a to b is

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

$\Diamond$

Activity 4.5.4 How does $\int_2^6 \left(\frac{1}{2}x+2\right)\, dx$ relate to Activity 4.5.1? Could you use Activity 4.5.1 to find $\int_0^4 \left(\frac{1}{2}x+2\right)\, dx$? What about $\int_1^7 \left(\frac{1}{2}x+2\right)\, dx$?

Remark 4.5.5 Properties of Definite Integrals.

1. If f is defined at $x = a$, then $\displaystyle\int_a^a f(x)\,dx = 0$.
2. If f is integrable on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.
3. If f is integrable on $[a, b]$ and c is in $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.
4. If f is integrable on $[a, b]$ and k is a constant, then kf is integrable on $[a, b]$ and $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$.
5. If f and g are integrable on $[a, b]$, then $f \pm g$ are integrable on $[a, b]$ and $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

Activity 4.5.6 Suppose that $\displaystyle\int_1^5 f(x)\,dx = 10$ and $\displaystyle\int_5^7 f(x)\,dx = 4$.Find each of the following.

(a) $\displaystyle\int_1^7 f(x)\,dx$

(b) $\displaystyle\int_5^1 f(x)\,dx$

(c) $\displaystyle\int_7^7 f(x)\,dx$

(d) $\displaystyle 3\int_5^7 f(x)\,dx$

4.5.2 Fundamental Theorem of Calculus

We've been looking at two big things in this chapter: antiderivatives and the area under a curve. In the early days of the development of calculus, they were not known to be connected to one another. The integral sign wasn't originally used in both instances. (Gottfried Leibniz introduced it as an elongated S to represent the sum when finding the area.) Connecting these two seemingly separate problems is done by the Fundamental Theorem of Calculus

Theorem 4.5.7 The Fundamental Theorem of Calculus. *If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Activity 4.5.8 Evaluate the following definite integrals. Include a sketch of the graph with the area you've found shaded in. Approximate the area to check to see if your definite integral answer makes sense. (Note: Just a guess, you don't have to use Riemann sums. Use the grid to help.)

(a) $\displaystyle\int_0^2 \left(x^2+3\right)\, dx$

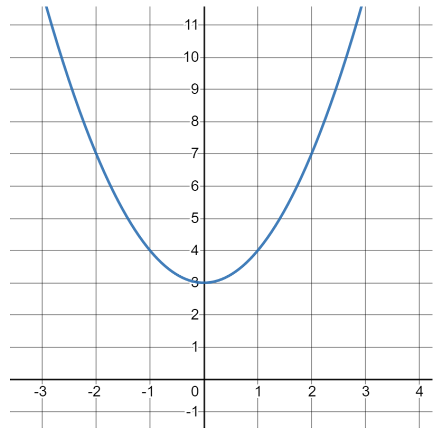


Figure .70

(b) $\displaystyle\int_1^9 \left(\sqrt{x}\right)\, dx$

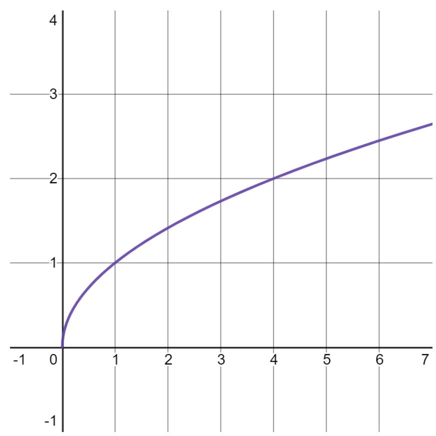


Figure .71

(c) $\displaystyle\int_{-\pi/4}^{\pi/2} (\cos x)\, dx$

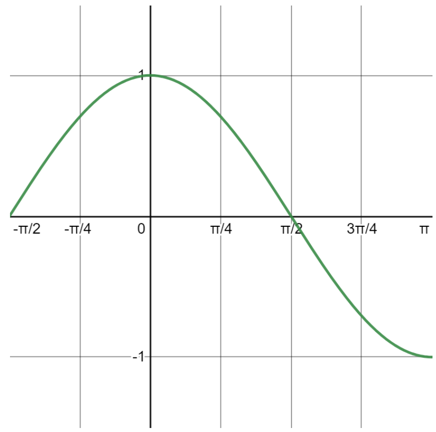


Figure .72

4.5.3 Area

Activity 4.5.9 Find the area between $f(x) = 2x - 6$ on the interval $[0, 8]$ using

1. geometry
2. the definite integral

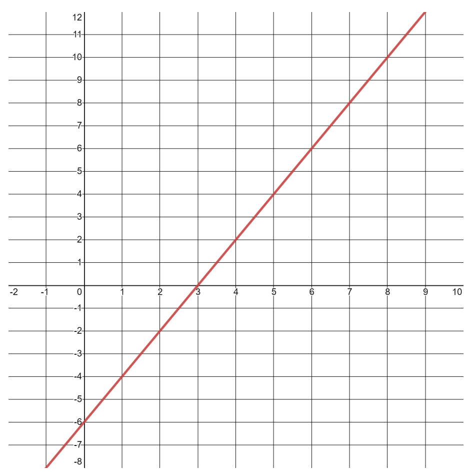


Figure .73

What do you notice?

Activity 4.5.10 Find the area bounded by the curves $f(x) = e^x - 2$, the x-axis, $x = 0$, and $x = 1$.

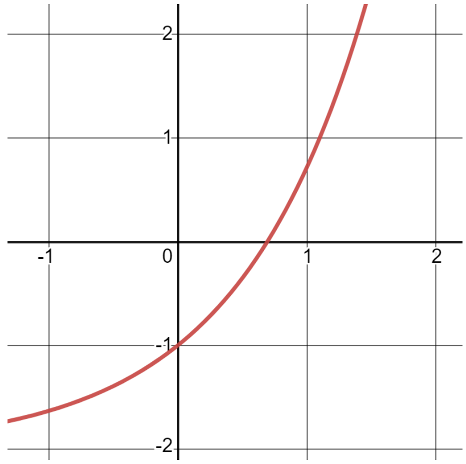


Figure .74

Activity 4.5.11 Set up a definite integral that represents the shaded area. Then find the area of the given region using the definite integral.

(a) $y = \frac{1}{x^2}$

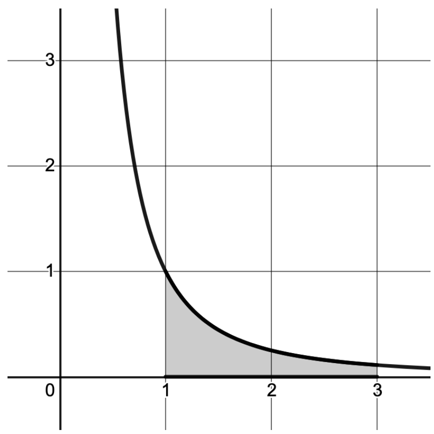


Figure .75

(b) $y = 3x^2 - x^3$

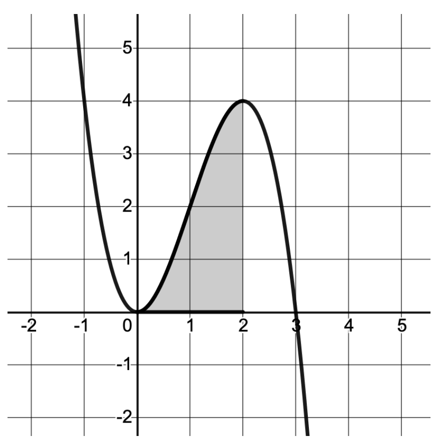


Figure .76

4.6 FTC for derivatives of integrals (IN6)

Learning Outcomes

- Find the derivative of an integral using the Fundamental Theorem of Calculus.

In this section we extend the Fundamental Theorem of Calculus discussed in Section 4.5 to include taking the derivatives of integrals. We will call this addition to the Fundamental Theorem of Calculus (FTC) part II. First we will introduce part II and then discuss the implications of this addition.

Theorem 4.6.1 The Fundamental Theorem of Calculus (Part II). *If a function f is continuous on the closed interval $[a, b]$, then the area function*

$$A(x) = \int_a^x f(t)\,dt \quad \text{for } \ a \leq x \leq b,$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.

Activity 4.6.2 For the following

4.7 Area under curves (IN7)

Learning Outcomes

- Use definite integrals to find area under a curve.

4.8 Area between curves (IN8)

Learning Outcomes

- Use definite integral(s) to compute the area bounded by several curves.

In Section 4.7, we learned how to find the area between a curve and the x-axis ($f(x) = 0$) using a definite integral. What if we want the area between any two functions? What if the x-axis is not one of the boundaries?

In this section, we'll investigate how a definite integral may be used to represent the area between two curves.

Activity 4.8.1 Consider the functions given by $f(x) = 5 - (x-1)^2$ and $g(x) = 4 - x$.

(a) Use algebra to find the points where the graphs of f and g intersect.

(b) Sketch an accurate graph of f and g on the axes provided, labeling the curves by name and the intersection points with ordered pairs.

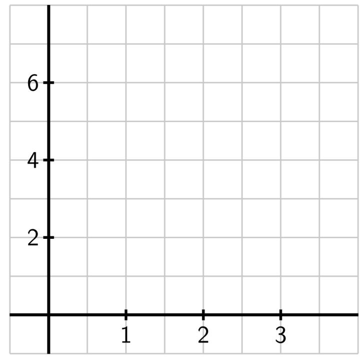


Figure .77 Axes for plotting f and g in Activity 4.8.1

(c) Find and evaluate exactly an integral expression that represents the area between $y = f(x)$ and the x-axis on the interval between the intersection points of f and g. Shade in this area on the axes provided in Figure .77

(d) Find and evaluate exactly an integral expression that represents the area between $y = g(x)$ and the x-axis on the interval between the intersection points of f and g. Shade in this area on the axes provided in Figure .77 in a different color or pattern than you used for the area between $y = f(x)$ and the x-axis.

(e) Let's denote the area between $y = f(x)$ and the x-axis as A_f and the area between $y = g(x)$ and the x-axis as A_g. How could we use A_f and A_g to find exact area between f and g between their intersection points?

A We could find $A_f + A_g$ to find the area between the curves.

B We could find $A_f - A_g$ to find the area between the curves.

C We could find $A_g - A_f$ to find the area between the curves.

We've seen from Activity 4.8.1 that a natural way to think about the area between two curve is as the area beneath the upper curve minus the area beneath the lower curve.

Activity 4.8.2 We now look for a general way of writing definite integrals for the area between two given curves, $f(x)$ and $g(x)$. Consider this area, illustrated in Figure .78.

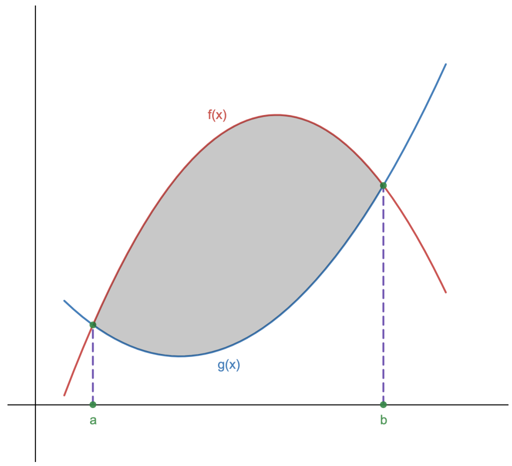


Figure .78 Area between $f(x)$ and $g(x)$.

(a) How could we represent the shaded area in Figure .78?

A $\int_b^a f(x)\,dx - \int_b^a g(x)\,dx$

B $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

C $\int_b^a g(x)\,dx - \int_b^a f(x)\,dx$

D $\int_a^b g(x)\,dx - \int_a^b f(x)\,dx$

(b) The two definite integrals above can be rewritten as one definite integral using the sum and difference property of definite integrals:

If f and g are continuous functions, then

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Use the property above to represent the shaded area in Figure .78 using one definite integral.

A $\int_b^a (f(x) - g(x))\,dx$

B $\int_a^b (f(x) - g(x))\,dx$

C $\int_b^a (g(x) - f(x))\,dx$

D $\int_a^b (g(x) - f(x))\,dx$

We can also think of the area this way: if we slice up the region between two curves into thin vertical rectangles, we see (as shown in Figure .79) that the height of a typical rectangle is given by the difference between the two functions, $f(x) - g(x)$, and its width is Δx. Thus the area of the rectangle is

$$A_{\text{rect}} = (f(x) - g(x))\Delta x.$$

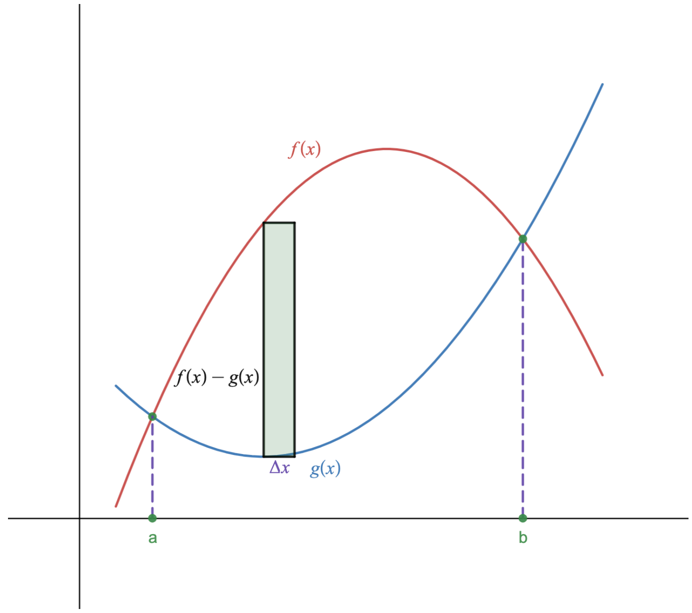


Figure .79 The area bounded by the functions $f(x)$ and $g(x)$ on the interval $[a, b]$.

The area between the two curves on $[a, b]$ is thus approximated by the Riemann sum

$$A \approx \sum_{i=1}^{n} (f(x_i) - g(x_i))\Delta x,$$

and as we let $n \to \infty$, it follows that the area is given by the single definite integral

$$A = \int_a^b (f(x) - g(x))\, dx.$$

In many applications of the definite integral, we will find it helpful to think of a "representative slice" and use the definite integral to add these slices. Here, the integral sums the areas of thin rectangles.

Finally, it doesn't matter whether we think of the area between two curves as the difference between the area bounded by the individual curves (as in Activity 4.8.2) or as the limit of a Riemann sum of the areas of thin rectangles between the curves (as in Figure .79). These two results are the same, since the difference of two integrals is the integral of the difference:

$$\int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b (f(x) - g(x))\, dx.$$

Our work so far in this section illustrates the following general principle.

If two curves $y = f(x)$ and $y = g(x)$ intersect at $(a, g(a))$ and $(b, g(b))$, and for all x such that $a \le x \le b$, $f(x) \ge g(x)$, then the area between the curves is $A = \int_a^b (f(x) - g(x))\, dx$.

Activity 4.8.3 In each of the following problems, our goal is to determine the area of the region described. For each region, (i) determine the intersection points of the curves, (ii) sketch the region whose area is being found, (iii) draw and label a representative slice, and (iv) state the area of the representative slice. Then, state a definite integral whose value is the exact area of the region, and evaluate the integral to find the numeric value of the region's area.

(a) The finite region bounded by $y = \sqrt{x}$ and $y = \frac{1}{4}x$.

(b) The finite region bounded by $y = 12 - 2x^2$ and $y = x^2 - 8$.

(c) The area bounded by the y-axis, $f(x) = \cos(x)$, and $g(x) = \sin(x)$, where we consider the region formed by the first positive value of x for which f and g intersect.

(d) The finite regions between the curves $y = x^3 - x$ and $y = x^2$.

4.9 Substitution method (IN9)

Learning Outcomes

- Evaluate various integrals via the substitution method.

Activity 4.9.1 Answer the following.

(a) Using the chain rule, which of these is derivative of e^{x^3} with respect to x?

A e^{3x^2}

B $x^3e^{x^3-1}$

C $3x^2e^{x^3}$

D $\dfrac{1}{4}e^{x^4}$

(b) Based on this result, which of these would you suspect to equal $\int x^2e^{x^3}\,dx$?

A $e^{x^3+1} + C$

B $\dfrac{1}{3x}e^{x^3+1} + C$

C $3e^{x^3} + C$

D $\dfrac{1}{3}e^{x^3} + C$

Activity 4.9.2 Recall that if u is a function of x, then $\frac{d}{dx}[u^7] = 7u^6u'$ by the Chain Rule.

For each question, choose from the following.

A $\dfrac{1}{7}u^7 + C$ B $u^7 + C$ C $7u^7 + C$ D $\dfrac{6}{7}u^7 + C$

(a) What is $\int 7u^6u'\,dx$?

(b) What is $\int u^6u'\,dx$?

(c) What is $\int 6u^6u'\,dx$?

Activity 4.9.3 Based on these activities, which of these choices seems to be a viable strategy for integration?

A Memorize an integration formula for every possible function.

B Attempt to rewrite the integral in the form $\int g'(u)u'dx = g(u) + C$.

C Keep differentiating functions until you come across the function you want to integrate.

Fact 4.9.4 *By the chain rule,*

$$\frac{d}{dx}[g(u) + C] = g'(u)u'.$$

There is an dual integration technique reversing this process, known as the ***substitution method****.*

This technique involves choosing an appropriate function u in terms of x to rewrite the integral as follows:

$$\int f(x)\,dx = \cdots = \int g'(u)u'dx = g(u) + C.$$

Observation 4.9.5 Recall that $u' = \frac{du}{dx}$. This allows for the following:

$$\int g'(u)u'dx = \int g'(u)\frac{du}{dx}\,dx = \int g'(u)du = g(u) + C.$$

Therefore rather than dealing with equations like $u' = \frac{du}{dx} = x^2$, we will prefer to write $du = x^2\,dx$.

Activity 4.9.6 Consider $\int x^2 e^{x^3}\,dx$, which we conjectured earlier to be $\frac{1}{3}e^{x^3} + C$.

Suppose we decided to let $u = x^3$.

(a) Compute $\frac{du}{dx} = ?$, and rewrite it as $du = ?\,dx$.

(b) Multiply both sides of the equation $du = ?\,dx$ so that its right-hand side appears in $\int x^2 e^{x^3}\,dx$.

(c) Show why $\int x^2 e^{x^3}\,dx$ may now be rewritten as $\int \frac{1}{3}e^u\,du$.

(d) Solve $\int \frac{1}{3}e^u\,du$ in terms of u, then replace u with x^3 to confirm our original conjecture.

Example 4.9.7 Here is how one might write out the explanation of how to find $\int x^2 e^{x^3}\,dx$ from start to finish:

$$\int x^2 e^{x^3}\,dx \qquad\qquad \text{Let } u = x^3$$

$$du = 3x^2\,dx$$

$$\frac{1}{3}du = x^2\,dx$$

$$\begin{aligned}\int x^2 e^{x^3}\,dx &= \int e^{(x^3)}(x^2\,dx)\\ &= \int e^u \frac{1}{3}du\\ &= \frac{1}{3}e^u + C\\ &= \frac{1}{3}e^{x^3} + C\end{aligned}$$

□

Activity 4.9.8 Which step of the previous example do you think was the most important?

A Choosing $u = x^3$.

B Finding $du = 3x^2\,dx$ and $\frac{1}{3}du = x^2\,dx$.

C Substituting $\int x^2 e^{x^3}\,dx$ with $\int \frac{1}{3}e^u\,du$.

D Integrating $\int \frac{1}{3}e^u\,du = \frac{1}{3}e^u + C$.

E Unsubstituting $\frac{1}{3}e^u + C$ to get $\frac{1}{3}e^{x^3} + C$.

Activity 4.9.9 Below are two correct solutions to the same integral, using two different choices for u. Which method would you prefer to use yourself?

$$\int x\sqrt{4x+4}\,dx \quad \text{Let } u = x+1$$

$$\begin{aligned} 4u &= 4x+4 \\ x &= u-1 \\ du &= dx \end{aligned}$$

$$\begin{aligned} \int x\sqrt{4x+4}\,dx &= \int (u-1)\sqrt{4u}\,du \\ &= \int (2u^{3/2} - 2u^{1/2})\,du \\ &= \frac{4}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C \\ &= \frac{4}{5}(x+1)^{5/2} \\ &\quad - \frac{4}{3}(x+1)^{3/2} + C \end{aligned}$$

$$\int x\sqrt{4x+4}\,dx \quad \text{Let } u = \sqrt{4x+4}$$

$$\begin{aligned} u^2 &= 4x+4 \\ x &= \frac{1}{4}u^2 - 1 \\ dx &= \frac{1}{2}u\,du \end{aligned}$$

$$\begin{aligned} \int x\sqrt{4x+4}\,dx &= \int \left(\frac{1}{4}u^2 - 1\right)(u)\left(\frac{1}{2}u\,du\right) \\ &= \int \left(\frac{1}{8}u^4 - \frac{1}{2}u^2\right) du \\ &= \frac{1}{40}u^5 - \frac{1}{3}u^3 + C \\ &= \frac{1}{40}(4x+4)^{5/2} \\ &\quad - \frac{1}{3}(4x+4)^{3/2} + C \end{aligned}$$

Activity 4.9.10 Suppose we wanted to try substitution method to find $\int e^x \cos(e^x + 3)\,dx$. Which of these choices for u appears to be most useful?

A $u = x$, so $du = dx$

B $u = e^x$, so $du = e^x\,dx$

C $u = e^x + 3$, so $du = e^x\,dx$

D $u = \cos(x)$, so $du = -\sin(x)\,dx$

E $u = \cos(e^x + 3)$, so $du = e^x \sin(e^x + 3)\,dx$

Activity 4.9.11 Complete the following solution using your choice from the previous activity to find $\int e^x \cos(e^x + 3)\,dx$.

$$\int e^x \cos(e^x + 3)\,dx \qquad \text{Let } u = ?$$

$$du = ?\,dx$$

$$\begin{aligned} \int e^x \cos(e^x + 3)\,dx &= \int ?\,du \\ &= \cdots \\ &= \sin(e^x + 3) + C \end{aligned}$$

Activity 4.9.12 Complete the following integration by substitution to find $\int \frac{x^3}{x^4+4}\,dx$.

$$\int \frac{x^3}{x^4+4}\,dx \qquad \text{Let } u = ?$$

$$\begin{aligned} du &= ?\,dx \\ ?du &= ?\,dx \end{aligned}$$

$$\begin{aligned}\int \frac{x^3}{x^4+4}\,dx &= \int \frac{?}{?}\,du\\ &= \cdots\\ &= \frac{1}{4}\ln|x^4+4|+C\end{aligned}$$

Activity 4.9.13 Given that $\int \frac{x^3}{x^4+4}\,dx = \frac{1}{4}\ln|x^4+4|+C$, what is the value of $\int_0^2 \frac{x^3}{x^4+4}\,dx$?

A $\frac{8}{20}$

B $-\frac{8}{20}$

C $\frac{1}{4}\ln(20)-\frac{1}{4}\ln(4)$

D $\frac{1}{4}\ln(4)-\frac{1}{4}\ln(20)$

Activity 4.9.14 What's wrong with the following computation?

$$\int_0^2 \frac{x^3}{x^4+4}\,dx$$

$$\begin{aligned}\text{Let } u &= x^4+4\\ du &= 4x^3\,dx\\ \frac{1}{4}du &= x^3\,dx\end{aligned}$$

$$\begin{aligned}\int_0^2 \frac{x^3}{x^4+4}\,dx &= \int_0^2 \frac{1/4}{u}\,du\\ &= \left[\frac{1}{4}\ln|u|\right]_0^2\\ &= \frac{1}{4}\ln 2-\frac{1}{4}\ln 0\end{aligned}$$

A The wrong u substitution was made.

B The antiderivative of $\frac{1/4}{u}$ was wrong.

C The x values $0, 2$ were plugged in for the variable u.

Activity 4.9.15 When $x=2$, we know that $u=2^4+4=20$. Likewise when $x=0$, we know that $u=0^4+4=4$. Use these facts to complete the below computation.

$$\int_0^2 \frac{x^3}{x^4+4}\,dx$$

$$\begin{aligned}\text{Let } u &= x^4+4\\ du &= 4x^3\,dx\\ \frac{1}{4}du &= x^3\,dx\end{aligned}$$

$$\begin{aligned}\int_{x=0}^{x=2} \frac{x^3}{x^4+4}\,dx &= \int_{u=?}^{u=?} \frac{1/4}{u}\,du\\ &= \ldots\\ &= \frac{1}{4}\ln(20)-\frac{1}{4}\ln(4)\end{aligned}$$

Example 4.9.16 Here is how one might write out the explanation of how to find $\int_1^3 x^2e^{x^3}\,dx$ from start to finish by leaving bounds in terms of x:

$$\int_1^3 x^2e^{x^3}\,dx \qquad\qquad \text{Let } u = x^3$$

$$du = 3x^2\,dx$$
$$\frac{1}{3}du = x^2\,dx$$

$$\begin{aligned}
\int_1^3 x^2 e^{x^3}\,dx &= \int_{x=1}^{x=3} e^{(x^3)}(x^2\,dx)\\
&= \int_{x=1}^{x=3} e^u \frac{1}{3}du\\
&= \left[\frac{1}{3}e^u\right]_{x=1}^{x=3}\\
&= \left[\frac{1}{3}e^{x^3}\right]_{x=1}^{x=3}\\
&= \frac{1}{3}e^{3^3} - \frac{1}{3}e^{1^3}\\
&= \frac{1}{3}e^{27} - \frac{1}{3}e
\end{aligned}$$

□

Example 4.9.17 Here is how one might write out the explanation of how to find $\int_1^3 x^2 e^{x^3}\,dx$ from start to finish by transforming bounds into terms of u:

$$\int_1^3 x^2 e^{x^3}\,dx \qquad\qquad \text{Let } u = x^3$$
$$du = 3x^2\,dx$$
$$\frac{1}{3}du = x^2\,dx$$

$$\begin{aligned}
\int_1^3 x^2 e^{x^3}\,dx &= \int_{x=1}^{x=3} e^{(x^3)}(x^2\,dx)\\
&= \int_{u=1^3}^{u=3^3} e^u \frac{1}{3}du\\
&= \left[\frac{1}{3}e^u\right]_1^{27}\\
&= \frac{1}{3}e^{27} - \frac{1}{3}e
\end{aligned}$$

□

Activity 4.9.18 Use substitution to show that

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^2 - 2e.$$

Activity 4.9.19 Use substitution to show that

$$\int_0^{\pi/4} \sin(2\theta)\,d\theta = \frac{1}{2}.$$

Activity 4.9.20 Use substitution to show that

$$\int u^5(u^3+1)^{1/3}\,du = \frac{1}{7}(u^3+1)^{7/3} - \frac{1}{4}(u^3+1)^{4/3} + C$$

Activity 4.9.21 Consider $\int (3x-5)^2\,dx$.

(a) Solve this integral using substitution.

(b) Replace $(3x-5)^2$ with $(9x^2-30x+25)$ in the original integral, the solve using the reverse power rule.

(c) Which method did you prefer?

Activity 4.9.22 Consider $\int \tan(x)\,dx$.

(a) Replace $\tan(x)$ in the integral with a fraction involving sine and cosine.

(b) Use substitution to solve the integral.

Colophon

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Colophon

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