Steven Clontz

Research Statement

My emerging career in mathematical research is punctuated with several results, invited talks, and multiple accepted publications with several more in perparation. My main work involves the areas of set-theoretic topology, continuum theory, and infinite combinatorial games. I am passionate not only about the creation of my research, but also about the cyberinfrastructure which we use to share and collaborate as 21st-century mathematicians. My experience with the open web provides me with a unique perspective on academic collaboration in the information age, which will be changing drastically in the coming decade. In addition, I'm excited to apply my history of leadership training and mentorship in order to guide young mathematicians in their own research experience, beginning at the undergraduate level.

Undergraduate Research

My calling to research began in my undergraduate career. Auburn University (regrettably, in my opinion) does not require any sort of thesis or capstone project of its undergraduate mathematics majors. Nonetheless, during my junior year, I worked on some toy problems suggested by two professors at Auburn: Phil Zenor in topology, and Andras Bezdek in geometry. While I would ultimately return to topology for my graduate-level research, I was fascinated with a very tangible and surprisingly deep question on the unfolding of polyhedral surfaces into the plane, which eventually became the topic of my undergraduate thesis written as part of the requirements to be recognized as a University Honors Scholar [5].

Suppose that a polyhedron can be cut along its edges and unfolded into the plane; we refer to such an unfolding as a *net* of the polyhedron. It is easy to construct polyhedra with non-convex faces which do not yield a net. A classic example of a polyhedron with convex faces which does not admit a net involves gluing so-called *witch's hats* to the faces of a regular octahedron; however, this polyhedron is not convex [2].

G. C. Shephard conjectured in 1975 that every convex polyhedron yields at least one net [18]. If it is surprising that this question had not been explicitly posed previously, then it is perhaps more shocking that such a seemingly simple statement remains open to this day. For a beginning mathematician, this was an exciting opportunity to engage a research-level problem which was accessible without graduate-level coursework. Indeed, the basic premise requires no mathematical background to appreciate, and I successfully applied and interviewed for undergraduate research fellowships at the university and college levels.

My thesis investigated a small subclass of convex polyhedra which I denoted *generalized pyramids*: the convex hull of two parallel polygons, one of which may be degenerate and whose projection must lay within the other. In addition to a survey of the problem of unfolding polyhedra, I outlined two techniques for unfolding certain generalized pyramids, based upon techniques which work for typical pyramids.

Due to my experience as an undergraduate researcher in mathematics, I gained my initial excitement for discovering new mathematical ideas. But just as importantly, I also learned a lot about how I would want to mentor other undergraduate researchers beginning their own mathematical journeys. In my first year at UNC Charlotte, I've already begun to do so; I have taken on my first student and will mentor her during Fall 2015 as she writes a survey on finite and infinite combinatiral game theory. We chose this topic as it relates to my own current research on topological games, but it also is accessible to both an undergraduate student with a generalized background and to the layman.

Current Research

I remained at Auburn to pursue a fifth-year mathematics education program, but ultimately committed full-time to mathematics, including coursework in general topology and combinatorial game theory. Before beginning my doctoral work, I wrote a masters thesis under Gary Gruenhage surveying known applications of stationary subsets of regular uncountable cardinals [6].

I was first introduced to infinite combinatorial games during a descriptive set theory course. Eager to learn more, I translated a paper by G. Debs [9] constructing a space for which the second player has a winning strategy in the Banach-Mazur game, but not a winning tactical strategy. This paper served as the inspiration for my doctoral work at Auburn under Gary Gruenhage.

Abstract. The majority of my research involves studying the applications of limited information strategies in various topological and set-theoretic games. More recently, I've also investigated generalized inverse limits with an ordinal-valued index set. Particular results of note include:

- extending results of Peter Nyikos [14] on limited information strategies in Gruenhage's open-point convergence game,
- answering a question of Nyikos [13] showing that Bell's proximal game characterizes the class of Corson compact spaces (joint work with Gary Gruenhage [7])
- characterizing strong Eberlein compactness with the existence of a tactical winning strategy for Bell's proximal game [4]
- showing an example of a locally compact, non-metacompact space for which the first player has a winning strategy but no winning k-tactical strategy for Gruenhage's compact-point game [3]
- extending results of Telgarsky [19] and Scheepers [16] to characterize game-theoretic strengthenings of the Menger property and other selection properties, and
- proving that a generalized invesrse limit with index set κ and an idempotent, continuum-valued, surjective u.s.c. bonding map is not Corson compact (joint work with Scott Varagona [8]).

Results and Open Questions. G. Gruenhage introduced the ω-length game $Gru_{O,P}(X, x)$ where during each round the first player $\mathscr{O}$ chooses a neighborhood of x in X, and the second player $\mathscr{P}$ chooses a point within all the previously chosen neighborhoods [10]. $\mathscr{O}$ wins this game if the points chosen by $\mathscr{P}$ converge to x. It's easy to see that if x has a countable base, then

$\mathscr{O} \underset{\text{0-mark}}{\uparrow} Gru_{O,P}(X, x)$, that is, $\mathscr{O}$ has a winning 0-Markov strategy using only knowledge of the round number; this in fact is a game-theoretic characterization of first-countability at x. The well-known class of W spaces are the spaces for which $\mathscr{O} \uparrow Gru_{O,P}(X, x)$ ($\mathscr{O}$ has a winning strategy using full information) for all $x \in X$; all first-countable spaces are then W spaces.

To investigate the limited-information setting for $Gru_{O,P}(X, x)$, we look to non-first-countable W spaces. Examples which I have studied include the one-point compactification of a discrete cardinal $\kappa^\star = \kappa \cup \{\infty\}$ and the Σ-product of κ real lines $\Sigma\mathbb{R}^\kappa$. (Note that $\kappa^\star$ is a subspace of $\Sigma\mathbb{R}^\kappa$ where $\infty \in \kappa^\star$ corresponds to $\vec{0} \in \Sigma\mathbb{R}^\kappa$.)

P. Nyikos first showed that $\mathscr{O} \underset{\text{mark}}{\not\uparrow} Gru_{O,P}(\omega_1^\star, \infty)$ when $\mathscr{P}$ is only required to play within the latest open set played [14]. When $\mathscr{P}$ must play within all previously played open sets, I have shown that $\mathscr{O} \underset{k\text{-mark}}{\not\uparrow} Gru_{O,P}(\kappa^\star, \infty)$ for $\kappa > \omega_1$ and $\mathscr{O} \underset{k\text{-tact}}{\not\uparrow} Gru_{O,P}(\omega_1^\star, \infty)$, but knowledge of the round number allows $\mathscr{O} \underset{\text{mark}}{\uparrow} Gru_{O,P}(\omega_1^\star, \infty)$. When $\mathscr{O}$ is allowed to use her most recent move as well as her opponents, she may win by encoding information on the history of the game into her own moves, that is, $\mathscr{O} \underset{\text{code}}{\uparrow} Gru_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$ for all cardinals κ.

This result suggests an analogous question to those asked of the Banach-Mazur game.

Question 1. *Does $\mathscr{O} \underset{code}{\uparrow} Gru_{O,P}(X, x)$ imply $\mathscr{O} \uparrow Gru_{O,P}(X, x)$?*

Recently, J. Bell introduced a related game $Bell_{D,P}(X)$ for uniformizable spaces [1]. During each round of this game, $\mathscr{D}$ chooses an entourage of the diagonal, and $\mathscr{P}$ chooses a point within the previous entourage of the previous point. $\mathscr{D}$ wins this game if either the points chosen by $\mathscr{P}$ converge, or the intersection of the entourage neighborhoods have empty intersection. Spaces for which $\mathscr{D}$ has a winning strategy are said to be *proximal*.

Bell demonstrated that all proximal spaces are W spaces; I've extended this by showing that for any $k < \omega$, a winning $2k$-Markov (resp. $2k$-tactical) strategy for Bell's game can be used to construct a winning k-Markov (resp. k-tactical) strategy for $Gru_{O,P}(X, x)$. In addition, if X only has a single non-isolated point, then a winning k-Markov (resp. k-tactical) strategy in either game can be used to construct a winning k-Markov (resp. k-tactical) strategy in the other game.

Compact subspaces of Σ-products of real lines are called *Corson compact*. Since Σ-products of proximal spaces and closed subspaces of proximal spaces are proximal, all Corson compacts are easily seen to be proximal. I showed with Gruenhage in [7] that the converse also holds. To see this, we gave a non-trivial proof that when $\mathscr{D} \uparrow Bell_{D,P}(X)$ and X is compact, then $\mathscr{O} \uparrow Gru_{O,P}(X, H)$ for any closed subset H of X. This yields the desired result when combined with a lemma due to Gruenhage in [11]: a compact space is Corson compact if and only if $\mathscr{O} \uparrow Gru_{O,P}(X^2, \Delta)$, where Δ is the diagonal of X^2.

Since $\mathscr{O} \underset{\text{mark}}{\uparrow} Gru_{O,P}(X^2, \Delta)$ characterizes *Eberlein compactness* in the category of compact spaces, and $\mathscr{D} \underset{\text{mark}}{\uparrow} Bell_{D,P}(X)$ for any Eberlein compact space, a natural question arises concerning the relationship between Bell's game and Gruenhage's game.

Question 2. *Does $\mathscr{D} \underset{mark}{\uparrow} Bell_{D,P}(X)$ characterize Eberlein compactness in the category of compact spaces?*

Question 3. *Does $\mathscr{D} \uparrow Bell_{D,P}(X)$ characterize all closed subspaces of a Σ-product of real lines (as posed by P. Nyikos)?*

More recently, I have shown that $\mathscr{O} \underset{\text{tact}}{\uparrow} Bell_{D,P}(X)$ characterizes *strong Eberlein compactness* [4]. This has shown that $Bell_{D,P}(X)$ and $Gru_{O,P}(X^2, \Delta)$ are not equivalent games for compact spaces when considering limited information: $\mathscr{D} \underset{\text{tact}}{\uparrow} Gru_{O,P}(X^2, \Delta)$ for any compact metrizable space, but many compact metrizable spaces (e.g. $I = [0, 1]$) are not strong Eberlein compact.

Another game related to $Gru_{O,P}(X, x)$ is $Gru_{K,P}(X)$. During each round of this game, $\mathscr{K}$ chooses a compact set in X, followed by $\mathscr{P}$ choosing a point outside every previously chosen compact set. $\mathscr{K}$ wins this game if the points chosen by $\mathscr{P}$ are locally finite in the space. In the case that X is locally compact, this is essentially the same game as $Gru_{O,P}(X^\star, \infty)$ where neighborhoods of ∞ in $X^\star = X \cup \{\infty\}$ are complements of compact sets in X.

Gruenhage showed in [12] that for locally compact spaces, $\mathscr{K} \underset{\text{tact}}{\uparrow} Gru_{K,P}(X)$ if and only if X is metacompact, and $\mathscr{K} \underset{\text{mark}}{\uparrow} Gru_{K,P}(X)$ if and only if X is σ-metacompact. I've extended this to show that for locally compact or Hausdorff compactly-generated spaces, $\mathscr{K} \underset{\text{0-mark}}{\uparrow} Gru_{K,P}(X)$ if and only if X is hemicompact.

In fact, for locally compact or Hausdorff compactly-generated spaces, this game is equivalent with respect to predetermined strategies to a variation $Gru_{K,L}(X)$ where the second player is able to choose compact sets instead of points. However, there exists an ultrafilter $\mathcal{F} \in \beta\omega \setminus \omega$ such that the single-ultrafilter space $\omega \cup \{\mathcal{F}\}$ is an example of a Hausdorff non-compactly-generated space with $\mathscr{K} \underset{\text{0-mark}}{\uparrow} Gru_{K,P}(X)$ but $\mathscr{K} \underset{\text{0-mark}}{\not\uparrow} Gru_{K,L}(X)$. Due to a lack of counter-examples, it's natural to ask:

Question 4. *Does $\mathscr{K} \underset{\text{0-mark}}{\uparrow} Gru_{K,L}(X)$ imply X is compactly generated?*

Gruenhage suggested a consistent example of a locally compact non-metacompact space for which $\mathscr{K}$ has a perfect information strategy which may allow a winning 2-tactical strategy for $\mathscr{K}$. However, I have shown that this space does not allow a winning k-tactical strategy for any $k < \omega$, which leaves the following question.

Question 5. *For locally compact spaces, does $\mathscr{K} \underset{\text{2-tact}}{\uparrow} Gru_{K,P}(X)$ imply X is metacompact?*

I have investigated the classic topological game $Men_{C,F}(X)$ characterizing the *Menger* property, in addition to several other selection property games of the form $G_{fin}(\mathcal{A}, \mathcal{B})$ as studied by Scheepers [17] and others. During round n of such games, $\mathscr{A}$ chooses an element A_n of $\mathcal{A}$, and $\mathscr{B}$ chooses some finite subset of A_n. $\mathscr{B}$ wins if the union of the chosen subsets is a member of $\mathcal{B}$. In this language, $Men_{C,F}(X)$ is the special case where $\mathcal{A} = \mathcal{B} = \mathcal{O}$, the collection of open covers of X, and denoting $\mathscr{A}$ by $\mathscr{C}$ and $\mathscr{B}$ by $\mathscr{F}$.

Telgarsky [19] and Scheepers [16] provided proofs of the fact that for metrizable spaces, $\mathscr{F} \uparrow Men_{C,F}(X)$ characterizes σ-compactness. I have broken this down to show that $\mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$ characterizes σ-compactness for regular spaces, and a winning perfect information strategy for a second-countable space may be improved to a winning Markov strategy.

Since a winning $(k+2)$-Markov strategy can always be used to construct a winning 2-Markov strategy, I have investigated the class of non-σ-compact spaces for which $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$. The one-point Lindelöfication $\omega_1^\dagger = \omega_1 \cup \{\infty\}$ of a discrete first-uncountable space (neighborhoods of ∞ have countable complements) is a ZFC example. Assuming an axiom $S(\kappa)$ independent of ZFC for $\omega_1 < \kappa \leq 2^\omega$ and introduced by Scheepers in [15] to study a set-theoretic game similar to $Men_{C,F}(\kappa^\dagger)$, I have shown that $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(\kappa^\dagger)$. Several games similar to $Men_{C,F}(\kappa^\dagger)$ can be won with a 2-Markov or 2-tactical strategy assuming $S(\kappa)$, so it would be interesting to construct a winning strategy when $S(\kappa)$ fails.

Question 6. *When does there exist a cardinal κ such that $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(\kappa^\dagger)$ but $\neg S(\kappa)$?*

It also hasn't been shown if there exists a space which actually requires perfect information for $\mathscr{F}$ to win $Men_{C,F}(X)$.

Question 7. *Is there a space X with $\mathscr{F} \uparrow Men_{C,F}(X)$ but $\mathscr{F} \underset{\text{2-mark}}{\not\uparrow} Men_{C,F}(X)$?*

Recent developments in continuum theory have involved the study of generalized inverse limits with set-valued bonding maps and linearly ordered index sets: $\varprojlim\{X_i, f_{ij}, L\} \subseteq \prod_{i\in L} X_i$ satisfying $x(i) \in f_{ij}(x(j))$ for all $i < j$ in L. Since L need not be countable, metrizability of the space is not guaranteed, and many other questions in general topology may arise.

I showed with Scott Varagona in [8] that when $X_i = [0,1]$ and $f_{ij} = f$ for some surjective, idempotent, upper-semicontinuous $f : [0,1] \to C([0,1])$ (where $C([0,1])$ is the collection of all subcontinua of $[0,1]$) distinct from the trivial map, the graph of f must satisfy the so-called condition Γ: there exist x, y with $\langle x, x\rangle$, $\langle y, y\rangle$, and $\langle x, y\rangle$ on the graph of f. It's not hard to then see that when L is an uncountable ordinal, the inverse limit must then contain a copy of $\omega_1 + 1$, which is not metrizable or even Corson compact.

Question 8. *Do all compactum-valued, idempotent, surjective, u.s.c. $f : [0,1] \to 2^{[0,1]}$ satsify condition Γ?*

Question 9. *Does there exist a nontrivial idempotent map f and uncountable index L such that $\varprojlim\{[0,1], f, L\}$ is metrizable?*

Mathematical Research and Cyberinfrastructure in the 21st Century

In addition to researching mathematics, I am also interested in the mechanics of how we research and collaborate as mathematicians in the 21st century. While we have made progress in recent years as journals move to make their archives available online and sites like arXiv.org provide convenient platforms for sharing preprints of cutting-edge results, we are still inefficiently passing around virtual sheets of paper.

Mathematics research currently operates as a waterfall process: results are found, results are written up in a paper (and possibly shared in a preprint), this paper is submitted for publication, and following the refereeing and copyediting process, the paper may be published and accepted by the community. This workflow moves in one direction, and results in a static product (occasionally with errors).

Thanks to the open web, we have the opportunity to adopt a more agile workflow. This process is perhaps most easily seen by looking at collaboration on open source software. Social coding platforms like GitHub and BitBucket have given software developers the ability to self-publish cutting-edge software projects for review by and possible collaboration with the open source community. These projects are tracked using revision control, with a public record of contributions made to each project. The process of creating, reviewing, and publishing such code is a continuous and democratic process, and results in living projects which can be refined and expanded over time. By taking advatnage of the tools provided by the modern web not only to assist with our research, but also with the publication and collaboration of our research, we will be able to most efficiently produce and share mathematical knowledge as we move forward in the modern era.

Projects such as the π-*Base Topology Database* (to which I am a contributor and consultant) and *Homotopy Type Theory: Univalent Foundations of Mathematics* [20] (licensed under Creative Commons and accepting pull requests on GitHub) are great examples of how mathematicians can collaborate in producing high quality resources to assist researchers and students. Likewise, I am actively developing *Online Seminars in Mathematics*, an initiative to assist mathematical seminars in streaming their content online so that researchers may be engaged in such scholarly activity regardless of their geographical location. Through these programs and others, I am excited to continue innovating in this area as a mathematician and faculty member.

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