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Research Statement

With my experience in software development and entrepreneurship, I'm often asked variations of a certain question. *"Why go into academia in lieu of a more lucrative career in industry?"* The reason is simple. While there are several intriguing puzzles in industry to be solved, the primary objective ultimately is profit. In comparison, as academics our foremost duty is to research open problems, create new knowledge, and pass on this knowledge through instruction and outreach, for the betterment of society.

My emerging career as a mathematical researcher is punctuated with several results, invited talks, and a co-authored publication in Topology and its Applications with several more papers in preparation. I am passionate not only about the creation of my research, but also about how we share and collaborate as 21st-century mathematicians. My experience with the open web provides me with a unique perspective on academic collaboration in the information age, which will be changing drastically in the coming decade. In addition, I'm excited to apply my history of leadership training and mentorship in order to guide young mathematicians in their own research experience, beginning at the undergraduate level.

Undergraduate Research

My calling to research began in my undergraduate career. Auburn University (regrettably, in my opinion) does not require any sort of thesis or capstone project of its undergraduate mathematics majors. Nonetheless, during my junior year, I worked on some toy problems suggested by two professors at Auburn: Phil Zenor in topology, and Andras Bezdek in geometry. While I would ultimately return to topology for my graduate-level research, I was fascinated with a very tangible and surprisingly deep question on the unfolding of polyhedral surfaces into the plane, which eventually became the topic of my undergraduate thesis written as part of the requirements to be recognized as a University Honors Scholar [3].

Suppose that a polyhedron can be cut along its edges and unfolded into the plane; we refer to such an unfolding as a *net* of the polyhedron. It is easy to construct polyhedra with non-convex faces which do not yield a net. A classic example of a polyhedron with convex faces which does not admit a net involves gluing so-called *witch's hats* to the faces of a regular octahedron; however, this polyhedron is not convex [2].

G. C. Shephard conjectured in 1975 that every convex polyhedron yields at least one net [15]. If it is surprising that this question had not been explicitly posed previously, then it is perhaps more shocking that such a seemingly simple statement remains open to this day. For a beginning mathematician, this was an exciting opportunity to engage a research-level problem which was accessible without graduate-level coursework. Indeed, the basic premise requires no mathematical background to appreciate, and I successfully applied and interviewed for undergraduate research fellowships at the university and college levels.

My thesis investigated a small subclass of convex polyhedra which I denoted *generalized pyramids*: the convex hull of two parallel polygons, one of which may be degenerate and whose projection must lay within the other. In addition to a survey of the problem of unfolding polyhedra, I outlined two techniques for unfolding certain generalized pyramids, based upon techniques which work for typical pyramids.

Due to my experience as an undergraduate researcher in mathematics, I gained my initial excitement for discovering new mathematical ideas. But just as importantly, I also learned a lot about how I would want to mentor other undergraduate researchers beginning their own mathematical journeys.

Graduate and Current Research

I remained at Auburn to pursue a fifth-year mathematics education program, but ultimately committed full-time to mathematics, including coursework in general topology and combinatorial game theory. Before beginning my doctoral work, I wrote a masters thesis under Gary Gruenhage surveying known applications of stationary subsets of regular uncountable cardinals [4].

I was first introduced to infinite combinatorial games during a descriptive set theory course. Eager to learn more, I translated a paper by G. Debs [6] constructing a space for which the second player has a winning strategy in the Banach-Mazur game, but not a winning tactical strategy. This paper served as the inspiration for my doctoral work at Auburn under Gary Gruenhage.

Abstract. In my doctoral and current research, I have studied the applications of limited information strategies in various topological games. Particular results of note include:

- extending results of Peter Nyikos [11] on limited information strategies in Gruenhage's open-point convergence game,
- answering a question of Nyikos [12] showing that Bell's proximal game characterizes the class of Corson compact spaces (joint work with Gary Gruenhage [5])
- characterizing strong Eberlein compactness with the existence of a tactical winning strategy for Bell's proximal game
- showing an example of a locally compact, non-metacompact space for which the first player has a winning strategy but no winning k-tactical strategy for Gruenhage's compact-point game [9], and
- extending results of Telgarsky [16] and Scheepers [14] to characterize game-theoretic strengthenings of the Menger property.

Definitions. Intuitively, an *ω-length game* involves two players exchanging *moves* by selecting elements from a given set over the course of $\omega = \{0, 1, 2, \dots\}$ rounds. At the "end" of the game, the ω-length sequence of choices made by the two players are inspected to see if it fits the game's *winning condition* for a particular player; if so, that player wins, and if not, the opponent wins. Rigorously, this may be modeled by a tuple $\langle M, W\rangle$ where M is the set of moves from which the players may choose from, and W is the set of ω-length sequences of moves for which the first player wins the game, called the *winning playthroughs* for the first player. Thus $M^\omega \setminus W$ would be the set of winning playthroughs for the second player, as no ties are allowed. The introduction of such games is commonly attributed to Gale and Stewart [7].

A *topological game* is an ω-length game where the moveset M is related to the topological structure of a given space X. Perhaps the most prolific example of such a game is the *Banach-Mazur game* $BM_{E,N}(X)$ played on a topological space X. During each round of this game, each player must choose an open subset of all previously played open sets, starting with player $\mathscr{E}$. Player $\mathscr{E}$ wins the game if the intersection of all the open sets is empty, and player $\mathscr{N}$ wins the game otherwise.

A function $\sigma : M^{<\omega} \to M$ is known as a *strategy* for a game with moveset M: intuitively, it determines the moves for a player given all the previous moves of her opponent. If there exists a strategy σ such that a player $\mathscr{A}$ will always win a game G while using it, then we say σ is a *winning strategy* and write $\mathscr{A} \uparrow G$ ("$\mathscr{A}$ wins G"). (All finite-length games have a player with a winning strategy; however, under ZFC this need not hold for an infinite game.) The presence or absence of a winning strategy for a player in a topological game on X characterizes a topological property of X. For example, a space X is *Baire* (the countable intersection of open dense sets is dense) if and only if $\mathscr{E} \not\uparrow BM_{E,N}(X)$.

Sometimes a player may not need to use the complete history of her opponent's moves to win a game. We call such strategies *limited information strategies.* A strategy which only uses the latest move (last k moves) of the opponent is called a *tactical* (k-*tactical*) strategy, and if it is winning we write $\mathscr{A} \underset{\text{tact}}{\uparrow} G$ ($\mathscr{A} \underset{k\text{-tact}}{\uparrow} G$). A strategy which only uses the latest move (last k moves, no moves) of the opponent and the number of the current round is called a *Markov* (k-*Markov*, *predetermined*) strategy, and if it is winning we write $\mathscr{A} \underset{\text{mark}}{\uparrow} G$ ($\mathscr{A} \underset{k\text{-mark}}{\uparrow} G$, $\mathscr{A} \underset{\text{pre}}{\uparrow} G$). A strategy which uses the latest move of both players is called a *coding* strategy, and if it is winning we write $\mathscr{A} \underset{\text{code}}{\uparrow} G$. The presence or absence of a winning limited information strategy for a player in a topological game on X may characterize a stronger property of X than the one characterized by a winning perfect information strategy. For example, $\mathscr{N} \uparrow BM_{E,N}(X)$ if and only if $\mathscr{N} \underset{\text{code}}{\uparrow} BM_{E,N}(X)$; however, $\mathscr{N} \underset{\text{tact}}{\uparrow} BM_{E,N}(X)$ if and only if X is a *siftable* space (all siftable spaces are thus Baire, but the converse does not hold). A famous question asks if for each $k < \omega$ there exists X_k such that $\mathscr{N} \underset{k+1\text{-tact}}{\uparrow} BM_{E,N}(X_k)$ but $\mathscr{N} \underset{k\text{-tact}}{\not\uparrow} BM_{E,N}(X_k)$. (The non-trivial example of Debs referenced earlier witnesses $k = 1$.)

For a more complete overview of the history of topological games, I recommend R. Telgarsky's excellent survey dedicated to the Banach Mazur game's 50th anniversary [17].

Results and Open Questions. G. Gruenhage introduced the ω-length game $Gru_{O,P}(X,x)$ such that during each round the first player $\mathscr{O}$ chooses a neighborhood of x in X, and the second player $\mathscr{P}$ chooses a point within all the previously chosen neighborhoods [8]. $\mathscr{O}$ wins this game if the points chosen by $\mathscr{P}$ converge to x. It's easy to see that if x has a countable base, then $\mathscr{O} \underset{\text{pre}}{\uparrow} Gru_{O,P}(X,x)$, and this in fact is a game-theoretic characterization of first-countability at x. The well-known class of W spaces are the spaces for which $\mathscr{O} \uparrow Gru_{O,P}(X,x)$ for all $x \in X$; all first-countable spaces are then W spaces.

To investigate the limited-information setting for $Gru_{O,P}(X,x)$, we look to non-first-countable W spaces. Examples which I have studied include the one-point compactification of a discrete cardinal $\kappa^* = \kappa \cup \{\infty\}$ and the Σ-product of κ real lines $\Sigma\mathbb{R}^\kappa$. (Note that κ^* is a subspace of $\Sigma\mathbb{R}^\kappa$ where $\infty \in \kappa^*$ corresponds to $\vec{0} \in \Sigma\mathbb{R}^\kappa$.)

P. Nyikos first showed that $\mathscr{O} \underset{\text{mark}}{\not\uparrow} Gru_{O,P}(\omega_1^*, \infty)$ when $\mathscr{P}$ is only required to play within the latest open set played [11]. When $\mathscr{P}$ must play within all previously played open sets, I have shown that $\mathscr{O} \underset{k\text{-mark}}{\not\uparrow} Gru_{O,P}(\kappa^*, \infty)$ for $\kappa > \omega_1$ and $\mathscr{O} \underset{k\text{-tact}}{\not\uparrow} Gru_{O,P}(\omega_1^*, \infty)$, but knowledge of the round number in fact allows $\mathscr{O} \underset{\text{mark}}{\uparrow} Gru_{O,P}(\omega_1^*, \infty)$. When given the ability to encode information within $\mathscr{O}$'s own previous move, $\mathscr{O} \underset{\text{code}}{\uparrow} Gru_{O,P}(\Sigma\mathbb{R}^\kappa, \vec{0})$ for all cardinals κ.

These results suggest analogous questions to those asked of the Banach-Mazur game $BM_{E,N}(X)$.

Question 1. *For any $k < \omega$, does there exist a space X and point $x \in X$ such that $\mathscr{O} \underset{k+1\text{-mark}}{\uparrow} Gru_{O,P}(X,x)$ but $\mathscr{O} \underset{k\text{-mark}}{\not\uparrow} Gru_{O,P}(X,x)$?*

Question 2. *Does $\mathscr{O} \underset{code}{\uparrow} Gru_{O,P}(X,x)$ imply $\mathscr{O} \uparrow Gru_{O,P}(X,x)$?*

Recently, J. Bell introduced a related game $Bell_{D,P}(X)$ for uniformizable spaces [1]. During each round of this game, $\mathscr{D}$ chooses an entourage of the diagonal, and $\mathscr{P}$ chooses a point within the previous entourage of the previous point. $\mathscr{D}$ wins this game if either the points chosen by $\mathscr{P}$ converge, or the intersection of the entourage neighborhoods have empty intersection. Spaces for which $\mathscr{D}$ has a winning strategy are said to be *proximal*.

Bell demonstrated that all proximal spaces are W spaces; I've extended this by showing that for any $k < \omega$, a winning $2k$-Markov (resp. $2k$-tactical) strategy for Bell's game can be used to construct a winning k-Markov (resp. k-tactical) strategy for $Gru_{O,P}(X,x)$. In addition, if X only has a single non-isolated point, then a winning k-Markov (resp. k-tactical) strategy in either game can be used to construct a winning k-Markov (resp. k-tactical) strategy in the other game.

Compact subspaces of Σ-products of real lines are called *Corson compact*. Since Σ-products of proximal spaces and closed subspaces of proximal spaces are proximal, all Corson compacts are easily seen to be proximal. I showed with Gruenhage in [5] that the converse also holds. To see this, we gave a non-trivial proof that when $\mathscr{D} \uparrow Bell_{D,P}(X)$ and X is compact, then $\mathscr{O} \uparrow Gru_{O,P}(X,H)$ for any closed subset H of X. This yields the desired result when combined with a lemma due to Gruenhage in [9]: a compact space is Corson compact if and only if $\mathscr{O} \uparrow Gru_{O,P}(X^2, \Delta)$, where Δ is the diagonal of X^2.

More recently, I have shown that $\mathscr{O} \underset{\text{tact}}{\uparrow} Bell_{D,P}(X)$ characterizes *strong Eberlein compactness.* Since $\mathscr{O} \underset{\text{mark}}{\uparrow} Gru_{O,P}(X^2, \Delta)$ characterizes *Eberlein compactness* in the category of compact spaces, some natural questions arise concerning the relationship between Bell's game and Gruenhage's game.

Question 3. *Does $\mathscr{D} \underset{k\text{-mark}}{\uparrow} Bell_{D,P}(X)$ characterize Eberlein compactness in the category of compact spaces for some k?*

Question 4. *For what conditions on X and types of limited information strategies can we conclude $Gru_{O,P}(X^2, \Delta)$ and $Bell_{D,P}(X)$ are equivalent?*

Another game related to $Gru_{O,P}(X,x)$ is $Gru_{K,P}(X)$. During each round of this game, $\mathscr{K}$ chooses a compact set in X, followed by $\mathscr{P}$ choosing a point outside every previously chosen compact set. $\mathscr{K}$ wins this game if the points chosen by $\mathscr{P}$ are locally finite in the space. In the case that X is locally compact, this is essentially the same game as $Gru_{O,P}(X^*, \infty)$ where neighborhoods of ∞ in $X^* = X \cup \{\infty\}$ are complements of compact sets in X.

Gruenhage showed in [10] that for locally compact spaces, $\mathscr{K} \underset{\text{tact}}{\uparrow} Gru_{K,P}(X)$ if and only if X is metacompact, and $\mathscr{K} \underset{\text{mark}}{\uparrow} Gru_{K,P}(X)$ if and only if X is σ-metacompact. I've extended this to show that for locally compact or Hausdorff compactly-generated spaces, $\mathscr{K} \underset{\text{pre}}{\uparrow} Gru_{K,P}(X)$ if and only if X is hemicompact.

In fact, for locally compact or Hausdorff compactly-generated spaces, this game is equivalent with respect to predetermined strategies to a variation $Gru_{K,L}(X)$ where the second player is able to choose compact sets instead of points. However, there exists an ultrafilter $\mathcal{F} \in \beta\omega \setminus \omega$ such that the single-ultrafilter space $\omega \cup \{\mathcal{F}\}$ is an example of a Hausdorff non-compactly-generated space with $\mathscr{K} \underset{\text{pre}}{\uparrow} Gru_{K,P}(X)$ but $\mathscr{K} \underset{\text{pre}}{\not\uparrow} Gru_{K,L}(X)$. Due to a lack of counter-examples, it's natural to ask:

Question 5. *Does $\mathscr{K} \underset{pre}{\uparrow} Gru_{K,L}(X)$ imply X is compactly generated?*

Gruenhage suggested a consistent example of a locally compact non-metacompact space for which $\mathscr{K}$ has a perfect information strategy which may allow a winning 2-tactical strategy for $\mathscr{K}$. However, I have shown that this space does not allow a winning k-tactical strategy for any $k < \omega$, which leaves the following question.

Question 6. *For locally compact spaces, does $\mathscr{K} \underset{2\text{-}tact}{\uparrow} Gru_{K,P}(X)$ imply X is metacompact?*

Finally, I have investigated the classic topological game $Men_{C,F}(X)$ characterizing the *Menger* property. In these games, $\mathscr{C}$ chooses an open cover of the space X, and $\mathscr{F}$ chooses a subset of X which is covered by a finite subset of the open cover. $\mathscr{F}$ wins if the union of the chosen subsets equals the space.

Telgarsky [16] and Scheepers [14] provided proofs of the fact that for metrizable spaces, $\mathscr{F} \uparrow Men_{C,F}(X)$ characterizes σ-compactness. I have broken this down to show that $\mathscr{F} \underset{\text{mark}}{\uparrow} Men_{C,F}(X)$ characterizes σ-compactness for regular spaces, and a winning perfect information strategy can be used to construct a winning Markov strategy for second-countable spaces.

Since a winning $(k+2)$-Markov strategy can always be used to construct a winning 2-Markov strategy, I have investigated the class of non-σ-compact spaces for which $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(X)$. The one-point Lindelöfication $\omega_1^\dagger = \omega_1 \cup \{\infty\}$ of a discrete first-uncountable space (neighborhoods of ∞ have countable complements) is a ZFC example. Assuming an axiom $S(\kappa)$ independent of ZFC for $\omega_1 < \kappa \leq 2^\omega$ and introduced by Scheepers in [13] to study a set-theoretic game similar to

$Men_{C,F}(\kappa^\dagger)$, I have shown that $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(\kappa^\dagger)$. Several games similar to $Men_{C,F}(\kappa^\dagger)$ can be won with a 2-Markov or 2-tactical strategy assuming $S(\kappa)$, so the axiom may in fact be necessary.

Question 7. *Does $\mathscr{F} \underset{\text{2-mark}}{\uparrow} Men_{C,F}(\kappa^\dagger)$ imply $S(\kappa)$?*

It also hasn't been shown if there exists a space which actually requires perfect information for $\mathscr{F}$ to win $Men_{C,F}(X)$.

Question 8. *Is there a space X with $\mathscr{F} \uparrow Men_{C,F}(X)$ but $\mathscr{F} \underset{\text{2-mark}}{\not\uparrow} Men_{C,F}(X)$?*

Mathematical Research and Cyberinfrastructure in the 21st Century

In addition to researching mathematics, I am also interested in the mechanics of how we research and collaborate as mathematicians in the 21st century. While we have made progress in recent years as journals move to make their archives available online and sites like arXiv.org provide convenient platforms for sharing preprints of cutting-edge results, we have yet to move beyond inefficiently passing around virtual sheets of paper.

Mathematics research currently operates as a *waterfall* process: results are found, results are written up in a paper (and possibly shared in a preprint), this paper is submitted for publication, and following the refereeing and copyediting process, the paper may be published and accepted by the community. This workflow moves in one direction, and results in a static product (occasionally with errors).

Thanks to the open web, we have the opportunity to adopt a more *agile* workflow. This process is perhaps most easily seen by looking at collaboration on open source software. Social coding platforms like GitHub and BitBucket have given software developers the ability to self-publish cutting-edge software projects for review by and possible collaboration with the open source community. These projects are tracked using revision control, with a public record of contributions made to each project. The process of creating, reviewing, and publishing such code is a continuous and democratic process, and results in living projects which can be refined and expanded over time.

Mathematics is not words on a page. By opening ourselves up to the tools of modern technology not only to assist with our research, but also with the publication and collaboration of our research, we will be able to most efficiently produce and share mathematical knowledge as we move forward in the modern era. Projects such as the *π-Base Topology Database* (to which I am a contributor) and *Homotopy Type Theory: Univalent Foundations of Mathematics* [18] (licensed under Creative Commons and accepting pull requests on GitHub) anticipate this paradigm shift, and as more digital natives grow up to be mathematicians, such tools and openness will be soon be expected as a matter of fact. I am excited to continue innovating in this area, not only as a mathematician and software developer, but also as a future faculty member.

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