Packet 4

Packet 4.2: Sections 16.5-16.9

16.5 Curl and Divergence

Definition 1. The **curl** of a vector field $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is given by the expression

$$\text{curl } \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

Problem 2. Prove that if $\vec{\mathbf{F}}$ is conservative, then curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$.

Solution. $\Diamond$

Contributors.

Remark 3. For a vector field $\vec{\mathbf{F}}$ and direction $\vec{\mathbf{u}}$, (curl $\vec{\mathbf{F}}$) $\cdot \vec{\mathbf{u}}$ may be thought of as the tendency of $\vec{\mathbf{F}}$ to "spin" counter-clockwise around $\vec{\mathbf{u}}$.

Problem 4. Compute the curl of $\langle x + y, z^2 - 3, yz \rangle$ around the point $(2, 0, -1)$.

Solution. $\Diamond$

Contributors.

Theorem 5. Green's Theorem may be rewritten in terms of curl as follows:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D (\text{curl } \vec{\mathbf{F}}) \cdot \widehat{\mathbf{k}}\, dA$$

Problem 6. Prove the previous theorem.

Solution. $\Diamond$

Contributors.

Definition 7. The **divergence** of a vector field $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is given by the expression

$$\text{div } \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R\rangle = P_x + Q_y + R_z$$

Problem 8. Prove that the divergence of a curl vector field is always 0. Put another way, show that div (curl $\vec{\mathbf{F}}$) $= 0$.

Solution. ◊

Contributors.

Remark 9. Divergence measures the tendency of a vector field to diverge away from a point.

Problem 10. Compute the divergence of $\langle x + y, z^2 - 3, yz\rangle$ away from the point $(2, 0, -1)$.

Solution. ◊

Contributors.

Definition 11. The **flux** of a velocity vector field $\vec{\mathbf{F}}$ across a closed curve C is given by

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds$$

where $\vec{\mathbf{n}}$ yields outward unit normal vectors to C.

Remark 12. Flux measures the tendency of a vector field to flow outward from a closed and bounded region (or inward if the flux is negative).

Theorem 13. Green's Theorem may be rewritten in terms of divergence as follows:

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = \iint_D \text{div } \vec{\mathbf{F}}\, dA$$

Problem 14. Compute the flux of the velocity vector field $\langle x+y, x^2+y^2\rangle$ across the boundary of the unit square.

Solution. ◊

Contributors.

16.6 Parametric Surfaces

Remark 15. Just like a curve may be parameterized by $\vec{\mathbf{r}}(t)$ for an interval $a \leq t \leq b$, a surface may be parameterized by $\vec{\mathbf{r}}(u, v)$ for a region R in the uv plane.

Theorem 16. Following are some common surface parameterizations.

- The surface $z = f(x, y)$ may be parametrized by

$$\vec{\mathbf{r}}(x, y) = \langle x, y, f(x, y) \rangle$$

- A surface determined by a cylindrical coordinate equation may be parametrized by substituting into

$$\vec{\mathbf{r}} = \langle r\cos\theta, r\sin\theta, z \rangle$$

- A surface determined by a spherical coordinate equation may be parametrized by substituting into

$$\vec{\mathbf{r}} = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi \rangle$$

Problem 17. Find a parameterization from the xy plane to the plane $2x - y + z = 7$ in xyz space.

Solution. ◊

Contributors.

Problem 18. Find the parameterization from the rectangle $0 \leq z \leq 3$ and $0 \leq \theta \leq 2\pi$ to the conical surface $z = \sqrt{x^2 + y^2}$ below the plane $z = 3$ in xyz space. (Hint: find the cylindrical coordinate equation for the surface.)

Solution. ◊

Contributors.

Problem 19. Find the parameterization from the rectangle $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$ to the spherical surface $x^2 + y^2 + z^2 = 9$ in xyz space. (Hint: find the spherical coordinate equation for the surface.)

Solution. ◊

Contributors.

16.7 Surface Integrals

Definition 20. The **surface integral** of a function $f(x, y, z)$ over a surface S in xyz space is given by

$$\iint_S f(\vec{\mathbf{r}})\, d\sigma = \iint_R f(\vec{\mathbf{r}}(u, v))|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA$$

where $\vec{\mathbf{r}}(u, v)$ is a parameterization from the region R in the uv plane to the surface S.

Theorem 21. The surface area of S is given by

$$\iint_S d\sigma = \iint_S 1\, d\sigma$$

Problem 22. Use the parameterization

$$\vec{\mathbf{r}}(\phi, \theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$$

from $0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$ to the unit sphere to show that the surface area of the unit sphere is 4π. (Note that this matches the formula $SA = 4\pi r^2$ used in high school geometry.)

Solution. ♢

Contributors.

Problem 23. Show that the area of the parallelogram with vertices $(0, 0, 0)$, $(2, 1, 2)$, $(0, 2, -1)$, and $(2, 3, 1)$ is $3\sqrt{5}$ using a surface integral. (Hint: use $\vec{\mathbf{r}}(u, v) = \langle 2u, u + 2v, 2u - v\rangle$.)

Solution. ♢

Contributors.