Calculus III - Spring 2015 - Mr. Clontz - Test 2

Name: Solutions Class: ______ Type: [illegible]

- This exam is open-"everything", provided you do not plagiarize. *Do not leave any unsupported answers!*
- Write your solutions so that a fellow student who has a perfect understanding of previous math classes and packets but has never seen that type of problem before could follow your work.
- Individual Test: You will have 40 minutes to complete this test on your own. You may not communicate with anyone during this period.
- Group Test: You will have 40 minutes to complete an identical test. You may collaborate with your group members during this time. All solutions must still be written by yourself, and may not be directly copied from another student.

1. (6 points) Prove that the helix given by the vector equation $\vec{\mathbf{r}}(t) = \langle \cos(2t), 3t, -\sin(2t) \rangle$ has constant curvature $\kappa = \frac{4}{13}$.

$$\vec{r}\,' = \langle -2\sin(2t), 3, -2\cos(2t) \rangle$$

$$\vec{r}\,'' = \langle -4\cos(2t), 0, 4\sin(2t) \rangle$$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2\sin(2t) & 3 & -2\cos(2t) \\ -4\cos(2t) & 0 & 4\sin(2t) \end{vmatrix} = \left\langle \begin{vmatrix} 3 & -2\cos(2t) \\ 0 & 4\sin(2t) \end{vmatrix}, -\begin{vmatrix} -2\sin(2t) & -2\cos(2t) \\ -4\cos(2t) & 4\sin(2t) \end{vmatrix}, \begin{vmatrix} -2\sin(2t) & 3 \\ -4\cos(2t) & 0 \end{vmatrix} \right\rangle$$

$$= \langle 12\sin(2t), -(-8\sin^2(2t) - 8\cos^2(2t)), 12\cos(2t) \rangle$$

$$= \langle 12\sin(2t), 8, 12\cos(2t) \rangle$$

By the alternate formula for curvature:

$$\kappa = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{\sqrt{144\sin^2(2t) + 64 + 144\cos^2(2t)}}{\left(\sqrt{4\sin^2(2t) + 9 + 4\cos^2(2t)}\right)^3} = \frac{\sqrt{208}}{(\sqrt{13})^3} = \frac{4}{13}$$

2. (6 points) Prove that the limit

$$\lim_{(x,y)\to(0,2)} \frac{\cancel{y}\cos(3x) - \cancel{y}}{xy^{\cancel{2}}} = 0$$

$$\lim_{(x,y)\to(0,2)} \frac{\cancel{y}\cos(3x) - \cancel{y}}{xy^{\cancel{2}}} = \lim_{(x,y)\to(0,2)} \frac{\cos(3x) - 1}{xy}$$

$$= \left(\lim_{y\to 2} \frac{1}{y}\right)\left(\lim_{x\to 0} \frac{\cos(3x)-1}{x}\right)$$

$$\overset{(LH)}{=} \frac{1}{2}\left(\lim_{x\to 0} \frac{-3\sin(3x)}{1}\right)$$

$$= \frac{1}{2}(0)$$

$$= 0$$

3. (6 points) Find the rate of change of $f(x, y, z) = y^2z - 3xz^2$ in the direction $\vec{\mathbf{u}} = \langle \frac{2}{7}, -\frac{6}{7}, \frac{3}{7} \rangle$ at the point $P_0 = (0, -2, 2)$.

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

$$= \langle -3z^2, 2yz, y^2 - 6xz \rangle \cdot \langle \tfrac{2}{7}, -\tfrac{6}{7}, \tfrac{3}{7} \rangle$$

$$= \langle -12, -8, 4 \rangle \cdot \langle \tfrac{2}{7}, -\tfrac{6}{7}, \tfrac{3}{7} \rangle$$

$$= \frac{-24 + 48 + 12}{7}$$

$$= \boxed{\frac{36}{7}}$$

4. (6 points) Prove that $f(x, y) = x^3 + y^3 - 3xy - 2$ has two critical points, one of which yields a local minimum, and another which is a saddle point.

To find the crit pts, set $\nabla f = \vec{0}$.

$\nabla f = \langle 3x^2 - 3y, 3y^2 - 3x \rangle = \vec{0}$

$3x^2 - 3y = 0$

$x^2 = y$

$3y^2 - 3x = 0$

$3(x^2)^2 - 3x = 0$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0$ OR $x = 1$

$y = 0^2 = 0$ $\quad$ $y = 1^2 = 1$

So $(0,0)$ and $(1,1)$ are the crit. pts.

$f_{xx} = 6x$

$f_{yy} = 6y$

$f_{xy} = -3$

$f_D = (6x)(6y) - (-3)^2$

$= 36xy - 9$

$(0,0)$

$f_D(0,0) = -9 < 0$

So $(0,0)$ is a saddle point

(by 2nd D. Test)

$(1,1)$

$f_D(1,1) = 36 - 9 > 0$

$f_{xx}(1,1) = 6(1) > 0$

So $(1,1)$ is a local min

(by 2nd D. Test)

5. (6 points) Choose functions $f(x,y)$ and $g(x,y)$ which could be used with the method of Lagrange Multipliers to find the point on the circle with center $(0,0)$ and radius 10 furthest away from the point $(-3,-4)$. Be sure to clearly **explain** how you chose f and g, but do not actually solve the problem.[1]

$f(x,y)$ should equal the distance of a point (x,y) to the point $(-3,-4)$, since this is what we wish to maximize. Therefore:

$$f(x,y) = \sqrt{(x+3)^2 + (y+4)^2}$$

$g(x,y)=k$ should give a constraint on the points (x,y) we wish to consider. Since they must lay on the described circle, we may use

$$g(x,y) = x^2 + y^2 = 100$$

(Note: This would be a tough system to solve, but

$$f(x,y) = (x+3)^2 + (y+4)^2$$

would also give the correct answer and be simpler algebraically.)

[1]The point on the circle furthest away is $(6,8)$ for what it's worth.