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Packet 2

Part 3: Sections 14.4-14.6

14.4 Tangent Planes and Linear Approximations

Definition 1. A **normal vector** to a surface is a vector normal to any vector tangent to a curve on the surface.

Theorem 2. Let $f(x, y)$ be a function of two variables with continuous partial derivatives, and let (a, b) be a point in the interior of f's domain. Then $\langle f_x(a, b), f_y(a, b), -1\rangle$ is normal to the surface at the point $(a, b, f(a, b))$.

Problem 3. OPTIONAL. Prove the previous theorem by using the curves $\vec{\mathbf{r}}(t) = \langle a, t, f(a, t)\rangle$ and $\vec{\mathbf{q}}(t) = \langle t, b, f(t, b)\rangle$ to yield the tangent vectors $\langle 0, 1, f_y(a, b)\rangle$ and $\langle 1, 0, f_x(a, b)\rangle$.

Solution. $\Diamond$

Definition 4. The **tangent plane** to a surface at a point is the plane passing through that point sharing the same normal vectors as the surface.

Theorem 5. The tangent plane to the surface $z = f(x, y)$ above the point (a, b) is given by the equation

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

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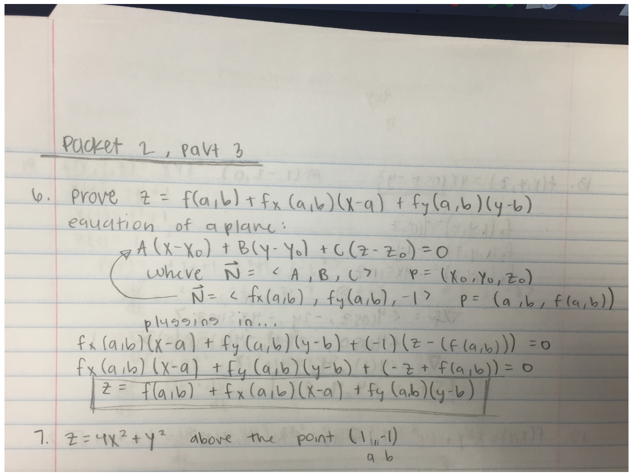
Problem 6. Prove the previous theorem.

Solution.



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6. Prove $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

equation of a plane:

$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

where $\vec{N} = \langle A, B, C \rangle$ $P = (x_0, y_0, z_0)$

$\vec{N} = \langle f_x(a,b), f_y(a,b), -1 \rangle$ $P = (a, b, f(a,b))$

plugging in...

$f_x(a,b)(x-a) + f_y(a,b)(y-b) + (-1)(z - (f(a,b))) = 0$

$f_x(a,b)(x-a) + f_y(a,b)(y-b) + (-z + f(a,b)) = 0$

$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$

7. $z = 4x^2 + y^2$ above the point $(\underset{a}{1}, \underset{b}{-1})$

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Problem 7. Find an equation for the plane tangent to the surface $z = 4x^2 + y^2$ above the point $(1, -1)$.

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Solution.

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$z = f(1,-1) + f_x(1,-1)(x-1) + f_y(1,-1)(y-(-1))$$
$$f = 4x^2 + y^2$$
$$\textit{and}$$
$$f_x = 8x$$
$$\textit{and}$$
$$f_y = 2y$$

$$z = 4(1)^2 + (-1)^2 + 8(1)(x-1) + 2(-1)(y+1)$$
$$z = 5 + 8x - 8 - 2y - 2$$
$$z = 8x - 2y - 5$$

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Definition 8. The **linearization** $L(x,y)$ of a function $f(x,y)$ at the point (a,b) is given by the formula:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Definition 9. A function f is **differentiable** at a point if its linearization at that point approximates the value of the function nearby.

Remark 10. Basically, a differentiable function is one which looks similar to its tangent planes when zoomed in sufficiently far.

Problem 11. Approximate the value of the differentiable function $f(x,y) = 4xy + 3y^2$ at $(1.1, -2.05)$ by using its linearization at the point $(1, -2)$. Then use a calculator to approximate $f(1.1, -2.05)$.

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Solution.

$$f(1.1,-2.05) = 4(1.1)(-2.05) + 3(-2.05)^2 = 3.5875$$
$$f_x = 4y$$
$$and$$
$$f_y = 4x + 6y$$
$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$L(x,y) = f(1,-2) + f_x(1,-2)(x-1) + f_y(1,-2)(y-(-2))$$
$$L(x,y) = 4(1)(-2) + 3(-2)^2 + 4(-2)(x-1) + (4(1)+6(-2))(y+2)$$

$$L(x,y) = -8 + 12 - 8x + 8 - 8y - 16$$
$$L(x,y) = -4 - 8x - 8y$$

Plug in

$$(1.1,-2.05)$$
$$L(1.1,-2.05) = -4 - 8(1.1) - 8(-2.05)$$
$$L(1.1,-2.05) = 3.6 \approx f(1.1,-2.05)$$

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14.5 The Chain Rule

Definition 12. The **gradient** of a multi-variable function is the vector containing all its partial derivatives:

$$\nabla f = \langle f_x, f_y \rangle$$
$$\nabla g = \langle g_x, g_y, g_z \rangle$$

Problem 13. Compute the gradient of the function $f(x,y,z) = 4x\cos z - y^2$. Then compute its value at the point $(1,-2,0)$.

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Solution.

$$\nabla g = \langle g_x, g_y, g_z \rangle$$
$$g_x = 4cos(z)$$
$$g_y = -2y$$
$$g_z = -4xsin(z)$$
$$\nabla g = \langle 4cos(z), -2y, -4xsin(z) \rangle$$

Plug in

$$(1,-2,0)$$
$$\nabla g = \langle 4cos(0), -2(-2), -4(1)sin(0) \rangle$$
$$\nabla g = \langle 4, -4, 0 \rangle$$

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Remark 14. If $f(P)$ is a function of multiple variables, and $\vec{\mathbf{r}}(t)$ is a vector function of t, then $f(\vec{\mathbf{r}}(t))$ is a function of t.

Problem 15. Let $f(x,y) = x^2y + 3y^2$ and $\vec{\mathbf{r}}(t) = \langle x(t), y(t)\rangle = \langle t+1, \sqrt{t}\rangle$. Write $f(\vec{\mathbf{r}}(t))$ in terms of t only, then compute $\frac{df}{dt}$.

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Solution.

$$f(\vec{\mathbf{r}}(t)) = (t+1)^2(\sqrt{t}) + 3(\sqrt{t})^2$$
$$f(\vec{\mathbf{r}}(t)) = (t^2+2t+1)\sqrt{t} + 3t$$
$$f(\vec{\mathbf{r}}(t)) = t^{\frac{5}{2}} + 2t^{\frac{3}{2}} + t^{\frac{1}{2}} + 3t$$
$$\frac{df}{dt} = \frac{5}{2}t^{\frac{3}{2}} + 3t^{\frac{1}{2}} + \frac{1}{2}t^{\frac{-1}{2}} + 3$$

$\diamondsuit$

Remark 16. The chain rule for single-variable functions may be written as

$$\frac{d}{dx}\left[f(u(x))\right] = \frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = f'(u(x))u'(x)$$

Theorem 17. Let $f(P)$ be a function of multiple variables and $\vec{\mathbf{r}}(t)$ be a function of t. Then the derivative of f with respect to t may be computed using the **Chain Rule**:

$$\frac{d}{dt}\left[f(\vec{\mathbf{r}}(t))\right] = \frac{df}{dt} = \nabla f \cdot \frac{d\vec{\mathbf{r}}}{dt}$$

Problem 18. Let f and $\vec{\mathbf{r}}$ be defined as in the previous problem. Use the Chain Rule to compute $\frac{df}{dt}$.

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Solution.

$$f(x,y) = x^2y + 3y^2$$
$$\textit{and}$$
$$\vec{\mathbf{r}}(t) = \langle x(t), y(t)\rangle = \langle t+1, \sqrt{t}\rangle$$

$$\nabla f \cdot \frac{d\vec{\mathbf{r}}}{dt} = \langle 2xy, x^2+6y\rangle \cdot \langle 1, \frac{1}{2}t^{\frac{-1}{2}}\rangle$$

$$= \langle 2(t+1)(\sqrt{t}), (t+1)^2 + 6(\sqrt{t})\rangle \cdot \langle 1, \frac{1}{2}t^{\frac{-1}{2}}\rangle$$

$$= \langle (2t+2)\sqrt{t}, t^2+2t+1+6t^{\frac{1}{2}}\rangle \cdot \langle 1, \frac{1}{2}t^{\frac{-1}{2}}\rangle$$

$$\langle 2t^{\frac{3}{2}} + 2t^{\frac{1}{2}}, t^2+2t+1+6t^{\frac{1}{2}}\rangle \cdot \langle 1, \frac{1}{2}t^{\frac{-1}{2}}\rangle$$

$$\frac{df}{dt} = 2t^{\frac{3}{2}} + 2t^{\frac{1}{2}} + \frac{1}{2}t^{\frac{3}{2}} + t^{\frac{1}{2}} + \frac{1}{2}t^{\frac{-1}{2}} + 3$$

$$\frac{df}{dt} = \frac{5}{2}t^{\frac{3}{2}} + 3t^{\frac{1}{2}} + \frac{1}{2}t^{\frac{-1}{2}} + 3$$

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Problem 19. Let $f(x, y, z) = xyz^2$, $x(t) = 2t+1$, $y(t) = t^2+1$, and $z(t) = 1-t^3$. Compute $\frac{df}{dt}$ at $t = 1$.

Solution.

$$\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$$
$$\vec{\mathbf{r}}(t) = \langle 2t+1, t^2+1, 1-t^3\rangle$$
$$\frac{d\vec{\mathbf{r}}}{dt} = \langle 2, 2t, -3t\rangle$$
$$\nabla f = \langle f_x, f_y, f_z\rangle$$
$$\nabla f = \langle yz^2, xz^2, 2xyz\rangle$$
$$\frac{d}{dt}\left[f(\vec{\mathbf{r}}(t))\right] = \frac{df}{dt} = \nabla f \cdot \frac{d\vec{\mathbf{r}}}{dt} = \langle yz^2, xz^2, 2xyz\rangle \cdot \langle 2, 2t, -3t\rangle$$
$$t = 1$$
$$\frac{d}{dt}\left[f(\vec{\mathbf{r}}(t))\right] = \frac{df}{dt} = \nabla f \cdot \frac{d\vec{\mathbf{r}}}{dt} = \langle yz^2, xz^2, 2xyz\rangle \cdot \langle 2, 2(1), -3(1)\rangle$$
$$= \langle 2yz^2 + 2xz^2 + -6xyz\rangle$$

← Also must plug in x, y, z

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Theorem 20. Suppose $f(x, y) = c$ defines y as a function of x. Then

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

Problem 21. Prove the previous theorem. (Part of the solution has been provided for you.)

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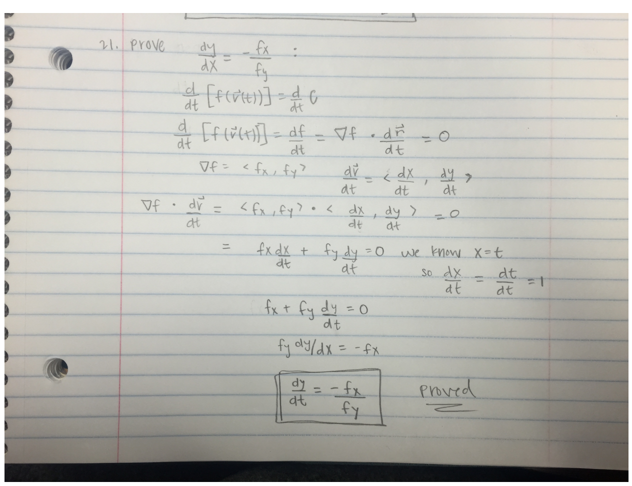
Solution. Let $y(x)$ be the function defined by $f(x, y(x)) = c$, and then let $t = x$. It follows that $f(t, y(t)) = f(\vec{\mathbf{r}}(t)) = c$, so by the Chain Rule,

$$\frac{d}{dt}[f(\vec{\mathbf{r}}(t))] = \frac{d}{dt}[c]$$



21. Prove $\frac{dy}{dx} = -\frac{f_x}{f_y}$:

$\frac{d}{dt}[f(\vec{r}(t))] = \frac{d}{dt} c$

$\frac{d}{dt}[f(\vec{r}(t))] = \frac{df}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt} = 0$

$\nabla f = \langle f_x, f_y\rangle$ $\quad \frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}\rangle$

$\nabla f \cdot \frac{d\vec{r}}{dt} = \langle f_x, f_y\rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}\rangle = 0$

$= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = 0$ we know $x = t$ so $\frac{dx}{dt} = \frac{dt}{dt} = 1$

$f_x + f_y \frac{dy}{dt} = 0$

$f_y\, dy/dx = -f_x$

$\boxed{\frac{dy}{dt} = -\frac{f_x}{f_y}}$ proved frfr Since $\frac{dy}{dt} = -\frac{f_x}{f_y}$ and $t = x$, we conclude that $\frac{dy}{dx} = -\frac{f_x}{f_y}$. ◊

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Problem 22. Find the rate of change $\frac{dy}{dx}$ for $xy^2 = 3x - 2y$ at $(-1, 3)$.

Solution.

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{-(3-y^2)}{-2-2yx} = \frac{3-y^2}{2yx+2}$$
$$at(-1,3)$$
$$\frac{3-(3)^2}{2(3)(-1)+2} = \frac{3}{2}$$
$$f(x,y) = 3x - 2y - xy^2 = 0$$

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14.6 Directional Derivatives and the Gradient Vector

Definition 23. Let $\vec{\mathbf{u}}$ be a direction. The **derivative of f in the direction $\vec{\mathbf{u}}$**, denoted $D_{\vec{\mathbf{u}}}f$, is given by $\frac{df}{ds}$ where s is the arclength parameter for the line oriented in the direction $\vec{\mathbf{u}}$.

Theorem 24. The directional derivative is the dot product of the gradient vector and $\vec{\mathbf{u}}$:

$$D_{\vec{\mathbf{u}}}f = \nabla f \cdot \vec{\mathbf{u}}$$

Remark 25. The proof of the previous theorem follows from the fact that if $\vec{\mathbf{r}}$ is the line oriented in the direction $\vec{\mathbf{u}}$, then $\frac{df}{ds} = \nabla f \cdot \frac{d\vec{\mathbf{r}}}{ds}$ and $\frac{d\vec{\mathbf{r}}}{ds} = \vec{\mathbf{u}}$.

Problem 26. Find the rate of change of $f(x,y,z) = xz^3 + 3yz$ in the direction $\vec{\mathbf{u}} = \langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \rangle$ at the point $P_0 = (-2, 0, 1)$.

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Solution.

$$D_{\vec{\mathbf{u}}}f = \nabla f \cdot \vec{\mathbf{u}}$$
$$\nabla f = \langle f_x, f_y, f_z \rangle$$
$$D_{\vec{\mathbf{u}}}f = \langle z^3, 3z, 3xz^2 + 3y \rangle \cdot ? \langle \frac{1}{3}, \frac{-2}{3}, \frac{2}{3} \rangle$$
$$D_{\vec{\mathbf{u}}}f = \langle \frac{1}{3}z^3 - 2z + 2xz^2 + 2y \rangle$$
$$P_0 = (-2, 0, 1)$$
$$\langle \frac{1}{3}(1)^3 - 2(1) + 2(-2)(1)^2 + 2(0) \rangle$$
$$\langle \frac{1}{3} - 2 - 4 \rangle = \frac{-17}{3}$$

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Problem 27. Find the rate of change of $f(x,y) = xy^2 + 3y$ in the direction of $\vec{\mathbf{A}} = \langle 2, 2 \rangle$ at the point $P_0 = (2, 0)$. (Note that $\vec{\mathbf{A}}$ isn't a unit vector, so you'll need to find its direction first.)

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Solution.

$$D_{\vec{\mathbf{u}}} f = \nabla f \cdot \vec{\mathbf{u}}$$

$$\nabla f = \langle f_x, f_y \rangle = \langle y^2, 2xy + 3 \rangle$$

We need to change $\vec{\mathbf{A}}$ to a unit vector.

$$\vec{\mathbf{u}} = \frac{\vec{[A]}}{|\vec{[A]}|} = \frac{\langle 2, 2 \rangle}{\sqrt{2^2 + 2^2}} = \frac{\langle 2, 2 \rangle}{2\sqrt{2}} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$

$$\nabla f \cdot \vec{\mathbf{u}} = \langle y^2, 2xy + 3 \rangle \cdot \langle \sqrt{2}, \sqrt{2} \rangle$$

$$= \frac{1}{\sqrt{2}} y^2 + 2\frac{1}{\sqrt{2}} xy + 3\frac{1}{\sqrt{2}}$$

$$D_{\vec{\mathbf{u}}} f = \frac{1}{\sqrt{2}} y^2 + \frac{2}{\sqrt{2}} xy + \frac{3}{\sqrt{2}}$$

$$= 0 + 0 + \frac{3}{\sqrt{2}}$$

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Problem 28. Show that the rate of change of f in the direction of $\widehat{\mathbf{j}}$ is same thing as the partial derivative of f with respect to y.

Solution.

$$D_{\widehat{\mathbf{j}}} = \nabla f \cdot \vec{j}$$

$$= \langle f_x, f_y \rangle \cdot \langle 0, 1 \rangle$$

$$= f_x(0) + f_y(1)$$

$$= f_y$$

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