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Packet 2

Part 2: Sections 14.1-14.3

14.1 Functions of Several Variables

Definition 1. A **function** f **of two variables** is a rule which assigns a real number $f(x,y)$ to each pair of real numbers (x,y) for which that rule is defined. The collection of such well-defined pairs is called the **domain** $\text{dom}(f)$ of the function, and the set of real numbers which can possibley be produced by the function is called its **range** $\text{ran}(f)$.

Definition 2. The **level curve** for each $k \in \text{ran}(f)$ is given by the equation $f(x,y) = k$. The **graph** of f is a surface in 3D space which visualizes the function, given by the equation $z = f(x,y)$.

Definition 3. A **function** f **of three variables** is a rule which assigns a real number $f(x,y,z)$ to each triple of real numbers (x,y,z) for which that rule is defined. The collection of such well-defined triples is called the **domain** $\text{dom}(f)$ of the function, and the set of real numbers which can possibley be produced by the function is called its **range** $\text{ran}(f)$.

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Problem 4. Let $f(x,y) = x\sin(x+y)$. Give the value of $f(\pi, \frac{\pi}{2})$.

Solution. $\pi \sin(\pi + \frac{\pi}{2})$

$\pi \sin(\frac{3\pi}{2})$

$\pi * (-1)$

$-\pi$

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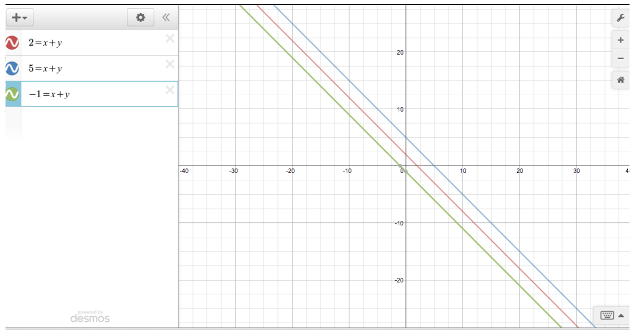
Problem 5. Let $f(x,y) = -x - y + 2$. In the xy-plane, plot the domain of f, as well as its level curves for $k = -3, 0, 3$. Then plot the graph of f in xyz space.

Solution. The domain of this function is all real numbers.

The level Curves for $k = -3, 0, 3$



For $k = -3$

$$-3 = -x - y + 2$$

$$5 = x + y$$

For $k = 0$

$$0 = -x - y + 2$$

$$2 = x + y$$

For $k = 3$

$$3 = -x - y + 2$$

$$-1 = x + y$$

$\Diamond$

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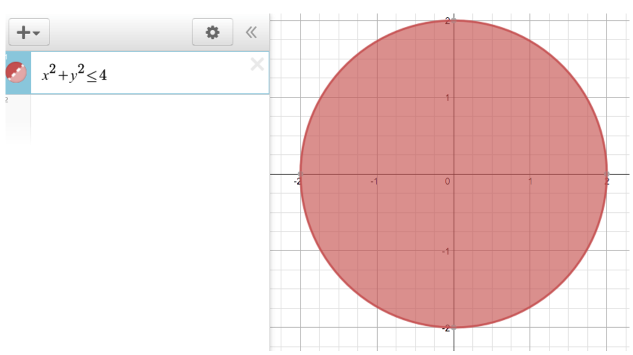
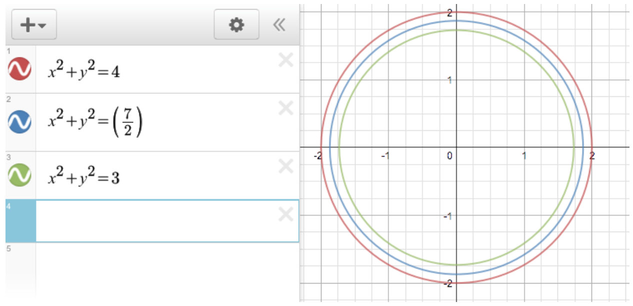
Problem 6. Let $f(x, y) = \sqrt{4 - x^2 - y^2}$. In the xy-plane, plot the domain of f, as well as its level curves for $k = 0, \frac{1}{\sqrt{2}}, 1$. Then plot the graph of f in xyz space.

Solution.

The Domain of f



The level curves for $k = 0, \frac{1}{\sqrt{2}}, 1$.



for $k = 0$

$$0 = \sqrt{4 - x^2 - y^2}$$

$$= x^2 + y^2 = 4$$

for $k = \frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}} = \sqrt{4 - x^2 - y^2}$$

$$\frac{1}{2} = 4 - x^2 - y^2$$

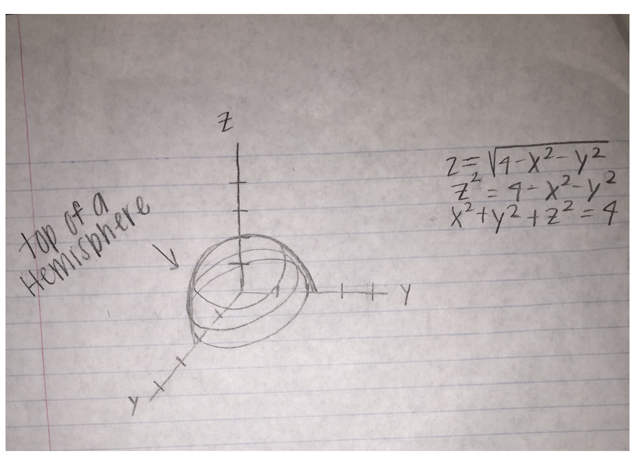
$$= x^2 + y^2 = \frac{9}{2}$$

for $k = 1$

$$1 = \sqrt{4 - x^2 - y^2}$$

$$1 = 4 - x^2 - y^2$$

$$= x^2 + y^2 = 3$$



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Definition 7. The **level surface** for each $k \in \text{ran}(f)$ is given by the equation $f(x, y, z) = k$. (Since the graph of a three variable function would require four variables and therefore is a four-dimensional object, we typically don't consider it.)

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Problem 8. Let $f(x, y, z) = \frac{x+3y^2}{z-2x}$. Give the value of $f(3, -2, 1)$.

Solution. All we have to do here is plug in (3,-2,1) into the equation

$$f(3, -2, 1) = \frac{(3) + 3(-2)^2}{(1) - 2(3)}$$

$$\frac{15}{-5} = -3$$

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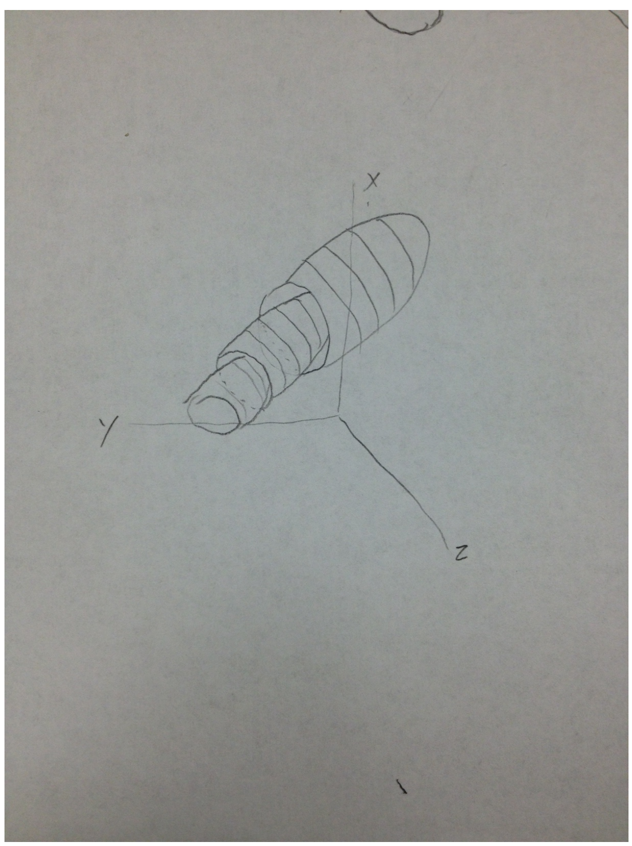
Problem 9. Let $f(x, y, z) = -x^2 + y - z^2$. In xyz space, plot the level surfaces for $k = -2, 0, 2$.

Solution.

In order to solve this problem we simply plug the values for k as the solution to the $f(x, y, z)$, and graph the equation. We find that each of these graphs are an infinite paraboloid, that are nested inside of each other.

Could use more detail



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Remark 10. If $P = (x, y)$, then we assume that $f(x, y) = f(P) = f(\vec{\mathbf{P}})$. If $P = (x, y, z)$, then we assume that $f(x, y, z) = f(P) = f(\vec{\mathbf{P}})$.

14.2 Limits and Continuity

Definition 11. If the value of the function $f(P)$ becomes arbitrarily close to the number L as points P close to P_0 are plugged into the function, then the **limit of** $f(P)$ **as** P **approaches** P_0 is L:

$$\lim_{P \to P_0} f(P) = L$$

Theorem 12. Let $f(x, y)$ be a function of two variables. If there exists a curve $y = g(x)$ passing through the point (x_0, y_0) such that $\lim_{x \to x_0} f(x, g(x))$ does not exist, then $\lim_{(x,y) \to (x_0,y_0)} f(x, y)$ does not exist.

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Problem 13. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{|x+y|}$$

does not exist by considering the function $g(x) = x$.

Solution.

$$\lim_{(x)\to(0)} \frac{x+x}{|x+x|}$$

$$\lim_{(x)\to(0)} \frac{2x}{|2x|}$$

$$\lim_{(x)\to(0^+)} \frac{2x}{|2x|} = 1$$

$$\lim_{(x)\to(0^-)} \frac{2x}{-2x} = -1$$

Don't write in math mode

Since the limit from the right does not equal the limit from the left, the limit does not exist

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Theorem 14. Let $f(x, y)$ be a function of two variables. If there exist curves $y = g(x)$ and $y = h(x)$ passing through the point (x_0, y_0) such that $\lim_{x\to x_0} f(x, g(x)) \neq \lim_{x\to x_0} f(x, h(x))$, then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

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Problem 15. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{x^6 + y^2}{x^3 y + x^6}$$

does not exist by considering the functions $g(x) = x^3$ and $h(x) = 2x^3$.

Sentence is confusing

Solution. you need to plug in x^3 for y then plug in $2x^3$ for y to prove that the two limits aren't equal to each other. After plugging in x^3, you get $\frac{2x^6}{2x^6}$, and $\frac{5x^6}{3x^6}$ after plugging in $2x^3$ the limit when $y = x^3$ is equal to 1. To find the limit of $\frac{5x^6}{3x^6}$, use L'Hopitals rule.
$\frac{5x^6}{3x^6} \to \frac{30x^5}{18x^5} \to \frac{150x^4}{90x^4} \to \frac{600x^3}{360x^3} \to \frac{1800x^2}{1080x^2} \to \frac{3600x}{2160x} \to \frac{3600}{2160} \to \frac{5}{3}$ or y'know, cancel $\frac{\cancel{x^6}}{\cancel{x^6}} = 1$
Since the two of the limits are not equal,

$$\lim_{(x,y)\to(0,0)} \frac{x^6 + y^2}{x^3y + x^6}$$

does not exist ◊

Theorem 16. The "Limit Laws" for single-variable functions also hold for multi-variable functions.

$$\lim_{P\to P_0} (f(P) \pm g(P)) = \lim_{P\to P_0} f(P) \pm \lim_{P\to P_0} g(P)$$

$$\lim_{P\to P_0} (f(P) \cdot g(P)) = \lim_{P\to P_0} f(P) \cdot \lim_{P\to P_0} g(P)$$

$$\lim_{P\to P_0} (kf(P)) = k \lim_{P\to P_0} f(P)$$

$$\lim_{P\to P_0} \frac{f(P)}{g(P)} = \frac{\lim_{P\to P_0} f(P)}{\lim_{P\to P_0} g(P)}$$

$$\lim_{P\to P_0} (f(P))^{r/s} = \left(\lim_{P\to P_0} f(P)\right)^{r/s}$$

Theorem 17. Let $P_0 = (x_0, y_0, z_0)$. Multi-variable limits which only use one variable may be reduced to a single-variable limit.

$$\lim_{P\to P_0} f(x) = \lim_{x\to x_0} f(x)$$

$$\lim_{P\to P_0} g(y) = \lim_{y\to y_0} g(y)$$

$$\lim_{P\to P_0} h(z) = \lim_{z\to z_0} h(z)$$

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Problem 18. Use the above theorems to rigorously prove that

$$\lim_{(x,y)\to(1,2)} \frac{2x+y}{y^2} = 1$$

Solution.

$$\lim_{(x,y)\to(1,2)} \frac{2x+y}{y^2} =$$

$$\frac{\lim_{(x,y)\to(1,2)} 2x+y}{\lim_{(x,y)\to(1,2)} y^2} =$$

$$\frac{\lim_{x\to 1} 2x + \lim_{y\to 2} y}{\lim_{y\to 2} y^2}$$

After completing each limit you get the equation

$$\frac{2(1)+2}{2^2} = 1$$

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Remark 19. Due to the limit laws, the "just plug it in" rule applies when plugging in does not result in an undefined operation.

Problem 20. Compute the limit

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$$\lim_{(x,y,z)\to(3,0,-1)} \frac{x\cos y}{z+x}$$

Solution.

$$\lim_{(x,y,z)\to(3,0,-1)} \frac{x\cos y}{z+x}$$

$$\lim_{(x,y,z)\to(3,0,-1)} \frac{3\cos 0}{(-1)+3}$$

$$\lim_{(x,y,z)\to(3,0,-1)} \frac{3\cos 0}{2}$$

$$= \frac{3}{2}$$

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Remark 21. There is no L'Hopital rule for multi-variable limits. However, you may still use it once the limit has been reduced to a single-variable limit.

Problem 22. Compute the limit

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$$\lim_{(x,y)\to(3,0)} \frac{xy+\sin(2y)}{y}$$

Solution. In order to solve this limit we must split the fraction into two parts and then plug in the point given into the equation

$$\lim_{(x,y)\to(3,0)} \frac{xy}{y} + \lim_{y\to 0} \frac{\sin(2y)}{y}$$

$$= \lim_{(x,y)\to(3,0)} x + \lim_{y\to 0} 2cos(2y)$$

$$3+2=5$$

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Remark 23. Factoring and canceling (including conjugation tricks) is also effective for computing multi-variable limits.

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Problem 24. Compute the limit

$$\lim_{(x,y,z)\to(1,2,4)} \frac{\sqrt{z}-xy}{z-x^2y^2}$$

Solution.

If you plug the variables in to limit as it is written you will get $\frac{0}{0}$.
Since this is not an acceptable answer we have to use the conjugate form of the equation to solve for the limit.

$$\lim_{(x,y,z)\to(1,2,4)} \frac{\sqrt{z}-xy}{z-x^2y^2} X \frac{\sqrt{z}+xy}{\sqrt{z}+xy} = \lim_{p\to(1,2,4)} \frac{1}{\sqrt{z}+xy} = \frac{1}{\sqrt{4}+(1)(2)} = \frac{1}{4}$$

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Definition 25. A function $f(P)$ is **continuous** if $\lim\limits_{P\to P_0} f(P) = f(P_0)$ for all points P_0 in its domain.

Theorem 26. If a multi-variable function is composed of continuous single-variable functions, then it is also continuous.

14.3 Partial Derivatives

Definition 27. The **partial derivative of f with respect to a variable** is the rate of change of f as that variable changes and all other variables are held constant. For example:

$$\frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$\frac{\partial g}{\partial z} = g_z(x,y,z) = \lim_{h\to 0} \frac{g(x,y,z+h)-g(x,y,z)}{h}$$

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Problem 28. Let $f(x,y,z) = xy^2 + 2z$. Use the definition of a partial derivative to prove that $\frac{\partial f}{\partial y} = 2xy$.

Solution.

$$\frac{\partial g}{\partial z} = g_z(x,y,z) = \lim_{h\to 0} \frac{g(x,y,z+h)-g(x,y,z)}{h}$$

$f(x,y,z) = xy^2 + 2z$

$$\frac{\partial \cancel{g}\, f}{\partial \cancel{z}\, y} = \lim_{h\to 0} \frac{(x(y+h)^2+2z)-(xy^2+2z)}{h}$$

$$\frac{\partial \cancel{g}\, f}{\partial \cancel{z}\, y} = \lim_{h\to 0} \frac{(x(y^2+2yh+h^2)+2z)-(xy^2+2z)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{xy^2 + 2xyh + xh^2 + 2z - xy^2 - 2z}{h}$$
$$\frac{\partial f}{\partial y} = \lim_{h\to 0} 2xy + xh$$
$$\frac{\partial f}{\partial y} = \lim_{h\to 0} 2xy + x(0)$$
$$\frac{\partial f}{\partial y} = 2xy$$

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Theorem 29. Partial derivatives may be computed in the usual way by treating all other variables as constants.

Problem 30. Compute both partial derivatives of $f(x, y) = 4x^2 - 5y^3 + xy - 1$.

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Solution.

$$\frac{\partial f}{\partial x} = f_x(x, y) = 8x + y$$
$$\frac{\partial f}{\partial y} = f_y(x, y) = -15y^2 + x$$

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Problem 31. Compute both partial derivatives of $f(x, y) = \sin(x + 3y)$.

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Solution. In order to take the partial derivative of this function one must use the chain rule and take the derivative of the function with respect to both x and y.

$$\frac{\partial f}{\partial x} = f_x(x, y) = cos(x + 3y)$$
$$\frac{\partial f}{\partial y} = f_y(x, y) = 3cos(x + 3y)$$

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Problem 32. Compute both partial derivatives of $f(x, y) = e^{xy^2}$.

Solution.

In order to compute the partial derivatives our first step is to use the chain rule. Then take the derivative of the function with respect to x and then y.

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$$\frac{\partial f}{\partial x} = f_x(x, y) = (e^{xy^2}) * (y^2)$$
$$\frac{\partial f}{\partial y} = f_x(x, y) = (e^{xy^2}) * (2xy)$$

◊

Definition 33. **Second-order partial derivatives** are the result of taking the partial derivative of a partial derivative.

Theorem 34. For sufficiently well-behaved functions, the order in which partial derivatives are taken is irrelevant. (This is sometimes called the **Mixed Derivative Theorem**.)

Problem 35. Verify the Mixed Derivative Theorem for $f(x, y) = 3x^2y^2 - x^3 + y^4 - 7$.

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Solution.

$$\frac{\partial f}{\partial x} = f_x(x, y) = 6xy^2 - 3x^2$$

$$\frac{\partial f}{\partial x} = f_{xy}(x, y) = 12xy$$

$$\frac{\partial f}{\partial y} = f_y(x, y) = 6x^2y + 4y^3$$

$$\frac{\partial f}{\partial y} = f_{yx}(x, y) = 12xy$$

$$f_{xy} = f_{yx}$$

We verified the mixed derivative theorem ◊