54

Packet 2

Part 2.4: Sections 14.7-14.8

14.7 Maximum and Minimum Values

Definition 1. Let f be a function of many variables defined near the point P_0. Then f has a **local maximum** $f(P_0)$ at P_0 if $f(P_0)$ is the largest value of f near P_0, and f has a **local minimum** $f(P_0)$ at P_0 if $f(P_0)$ is the smallest value of f near P_0. (Local maxima and minimal are also known as local extreme values or local extrema.)

Definition 2. If P_0 is a point in the domain of f and

$$\nabla f(P_0) = \vec{\mathbf{0}} \text{ or } \nabla f(P_0) \text{ DNE}$$

then P_0 is called a **critical point**.

Theorem 3. Critical points of a two-variable function occur when the tangent plane is horizontal (because $\langle 0, 0, -1\rangle$ is a normal vector) or the tangent plane does not exist.

Theorem 4. The local maximum and minimum values of a function always occur at critical points.

3/4

Problem 5. Prove that $f(x, y) = x^2 + 16y^2$ has exactly one local extreme value by showing that $(0, 0)$ is the only critical point for f, and then showing that $f(0, 0)$ is the minimum value of the function.

Solution.

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 32y$$

$$\nabla f(0, 0) = \langle 2(0), 32(0)\rangle = \langle 0, 0\rangle$$

We know that $f(0, 0)$ is a critical point from Definition 2. We also know that $f(0, 0)$ is the mininum value of the function because the range of the function is $f(x, y) \geq 0$. $\diamond$

Remark 6. By plotting the graph of f in the previous problem, you can see that $(0, 0)$ yields the lowest point on the surface.

Definition 7. The **saddle points** of f are the critical points which don't yield local extreme values.

4/4

Problem 8. Prove that $(0, 0)$ is a saddle point of the function $f(x, y) = 4x^2 - 9y^2$ by first showing that it is a critical point, and then considering the function f restricted to the curves $y = 0$ and $x = 0$ in the xy plane.

Solution.

$$\frac{\partial f}{\partial x} = 8x$$

$$\frac{\partial f}{\partial y} = -18y$$

Setting them equal to zero we get :

$$x = 0$$

and

$$y = 0$$

(0,0) is a critical point. Since it concaves up on the y axis and concaves down on the x axis, at (0,0) you will have a saddle point where they instersect. ♢

Remark 9. The term "saddle point" comes from the fact that the graph near a saddle point often looks like a saddle (such as in the previous problem).

Definition 10. The **discriminant** of a differentiable two variable function f is the function

$$f_D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

4/4

Problem 11. Compute the discriminant of the function $f(x, y) = 3x^2y - 2y^3 + 4x$.

Solution.

$$f(x, y) = 3x^2y - 2y^3 + 4x$$

$$f_x = xy + 4$$

$$f_{xx} = 6y$$

$$f_y = 3x^2 - 6y^2$$

$$f_{yy} = -12y$$

$$f_{xy} = 6x$$

$$f_D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2 = 6y(-12y) - (6x)^2 = -72y^2 - 36x^2$$

♢

Theorem 12. The **Second Derivative Test** for two-variable functions gives a way to (sometimes) determine if a critical point either yields a local maximum, a local minimum, or a saddle point. Let (a,b) be a critical point of of f where ∇f is defined.

- If $f_D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
- If $f_D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
- If $f_D(a,b) < 0$, then f has a saddle point at (a,b).
- If $f_D(a,b) = 0$, then the test is inconclusive.

2/4

Problem 13. Prove that f_{xx} could be replaced with f_{yy} in the Second Derivative Test by showing that if $f_D(a,b) > 0$, then $f_{xx}(a,b)$ and $f_{yy}(a,b)$ are either both positive or both negative.

Solution.

$$f_D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2 = (\frac{\partial^2 f}{\partial x^2})(\frac{\partial^2 f}{\partial y^2}) - (\frac{\partial^2 f}{\partial x \partial y})^2$$

if

$$f_{xx}f_{yy} - f_{xy}^2 > 0$$

$$\frac{\partial^2 f}{\partial x^2} = (\frac{\partial^2 f}{\partial x \partial y})^2(\frac{\partial y^2}{\partial^2 f}) \quad ?$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}$$

Therefore if f_{xx} is positive or negative, then f_{yy} must be the same. $\diamondsuit$

4/4

Problem 14. In an earlier problem we found that $f(x,y) = x^2 + 16y^2$ has exactly one critical point $(0,0)$. Use the Second Derivative Test to show that $f(0,0)$ is the minimum value of the function.

Solution.

$$f(x,y) = x^2 + 16y^2$$
$$f_x = 2x$$
$$f_{xx} = 2$$
$$f_y = 32y$$
$$f_{yy} = 32$$
$$f_{xy} = 0$$

$$f_D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2 = 2(32) - 0 = 64$$

If $f_D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
$64 > 0$ and $2 > 0$ so $f(0,0)$ is a local minimum.

$\diamondsuit$

4/4

Problem 15. Identify all the critical points for $f(x, y) = x^3 - 6xy + \frac{3}{2}y^2 - 1$, then use the Second Derivative Test to label each critical point as yielding a local minimum, a local maximum, or a saddle point.

Solution.

$$f(x, y) = x^3 - 6xy + \frac{3}{2}y^2 - 1$$

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x}(x^3 - 6xy + \frac{3}{2}y^2 - 1)$$

$$f_x = 3x^2 - 6y$$

$$f_x x = 6x$$

$$fy = -6x + 3y$$

$$f_y y = 3$$

$$f_x = 3x^2 - 6y$$

$$f_x y = -6$$

$$3x^2 - 6y = 0$$

$$y = \frac{x^2}{2}$$

$$-6x + 3y = 0$$

$$-6x + 3(\frac{x^2}{2}) = 0$$

$$3x(\frac{x}{2} - 2) = 0$$

$$x = 0, x = 4$$

$$y = \frac{x^2}{2}$$

$$y = 0, y = 8$$

$$f_D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2 = 6x(3) - (-6)^2 = 18x - 36$$

$$f_D(0, 0) = 18(0) - 36 = -36$$

The point at $(0, 0)$ is a Saddle Point.

$$f_D(4, 8) = 18(4) - 36 = 36$$

$$f_x x(4, 8) = 6(4) = 24$$

The point at $(4, 8)$ is a local minimum.

◊

Definition 16. Let f be a function of many variables. Then f has an **absolute maximum** $f(P_0)$ at P_0 if $f(P_0)$ is the largest value in the range of f, and f has an **absolute minimum** $f(P_0)$ at P_0 if $f(P_0)$ is the smallest value in the range of f. (Absolute maxima and minimal are also known as the absolute extreme values or absolute extrema.)

Theorem 17. A continuous function f restricted to a closed and bounded domain D always has an absolute minimum and absolute maximum value.

Theorem 18. The only possible points which can yield the absolute value of a function f of two variables x, y on a restricted domain $D \subseteq \mathbb{R}^2$ are:

- Critical points of f inside D
- Critical points of f restricted to the boundary of D
- Corners on the boundary of D

The absolute maximum and absolute minimum values may be computed by plugging in all of these candidates into f.

Remark 19. The previous theorem works because it checks all the local extreme values against each other to find the absolute largest and smallest value.

Problem 20. Find the absolute maximum and minimum value of $f(x, y) = x^2 + y^2 - 2x - 2y$ restricted to the region bounded by the triangle with vertices $(0, 0)$, $(2, 4)$, and $(2, 0)$. (Hint: this - triangle is given by the curves $y = 0$, $y = 2x$, and $x = 2$.)

4/4

Solution.

$$f(x, y) = x^2 + y^2 - 2x - 2y$$
$$f_x(x, y) = 2x - 2 = 0$$
$$x = 1$$
$$f_y(x, y) = 2y - 2 = 0$$
$$y = 1$$
$$\nabla f(1, 1) = \vec{\mathbf{0}}$$

$(1, 1)$ is a critical point.

For $y = 2x$,

$$f(x, 2x) = x^2 + (2x)^2 - 2x - 2(2x) = x^2 + 4x^2 - 2x - 4x = 5x^2 - 6x$$
$$f_x(x, 2x) = 10x - 6 = 0$$
$$x = \frac{3}{5}$$
$$y = 2(\frac{3}{5}) = \frac{6}{5}$$

$(\frac{3}{5}, \frac{6}{5})$ is a critical point on the line $y = 2x$.

For $y = 0$,

$$f(x,0) = x^2 - 2x$$
$$f_x(x,0) = 2x - 2 = 0$$
$$x = 1$$

$(1,0)$ is a critical point on the line $y = 0$.

For $x = 2$,

$$f(2,y) = 2^2 + y^2 - 2(2) - 2y$$
$$f_y(2,y) = 2y - 2 = 0$$
$$y = 1$$

$(2,1)$ is a critical point on the line x=2.

$(0,0)$,$(2,4)$, and $(2,0)$ are also critical points.

Plugging all these points back into the original equation we get:

$$f(0,0) = 0$$
$$f(\frac{3}{5}, \frac{6}{5}) = \cancel{\frac{27}{5}}$$
$$f(1,0) = -1$$
$$f(1,1) = -2$$
$$f(2,0) = 0$$
$$f(2,1) = -1$$
$$f(2,4) = 8$$

So by Theorem 18, $(2,4)$ ~~is~~ gives the absolute maximum and $(1,1)$ ~~is~~ gives the absolute minimum.

$\Diamond$

3/4

Problem 21. Find the absolute maximum and minimum value of $f(x,y) = 2xy$ restricted to the region bounded by the circle $x^2 + y^2 = 4$. (Hint: find a vector equation $\vec{\mathbf{r}}(t)$ for this circle and find the critical points of $f(\vec{\mathbf{r}}(t))$.)

Solution.

$$f(x, y) = 2xy$$

$$f_x(x, y) = 2y = 0$$

$$y = 0$$

$$f_y(x, y) = 2x = 0$$

$$x = 0$$

$$\nabla f(0, 0) = \vec{\mathbf{0}}$$

$(0, 0)$ is a critical point.

If we let $x = 2\cos t$ and $y = 2\sin t$ for the equation $x^2 + y^2 = 4$ then we have $4\cos^2 t + 4\sin^2 t = 4$.

For $x = 2\cos t$,

$$f(2\cos t, y) = (4\cos t)y$$

$$f_y(2\cos t, y) = 4\cos t = 0$$

$$t = \frac{\pi}{2}, \frac{3\pi}{2}$$

?

For $y = 2\sin t$,

$$f(x, 2\sin t) = 4x\sin t$$

$$f_x(x, 2\sin t) = 4\sin t = 0$$

$$t = 0, \pi$$

$f(0, 0) = 0$ so it is the absolute minimum. $\diamondsuit$

14.8 Lagrange Multipliers

4/4

Problem 22. A rancher wants to enclose a rectangular area using the straight edge of a cliff on one side, and barbed wire on the other three sides. If the rancher wants to maximize the area of this rectangle, what are the dimensions of the fence and the maximized area? In other words, find x, y which maximize $A(x, y) = xy$ given the constraint $2x + y = 100$, and the value of A for those values.

Solution. $y = 100 - 2x$

$A(x, y) = xy$

Substituting in for y

$A(x, y) = (100 - 2x)(x)$

$f(x) = 100x - 2x^2$

$f'(x) = 100 - 4x$

The derivative equals 0 at the point x=25. This means that this is a maximum or a minimum. To test for a maximum or minimum, plug in numbers above and below this critical point and find that 25 is a maximum of the function.

$f(0) = (100 - 2(0))(0)$

$f(0) = 0$

$f(30) = (100 - 2(30))(30)$

$f(30) = 1200$

$f(25) = (100 - 2(25))(25)$

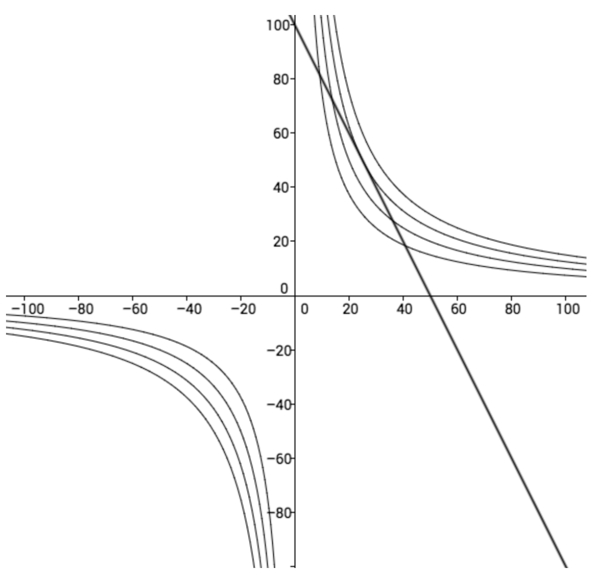
$f(25) = 1250$

Dimensions of the fence x=25 and y=50.

◊

4/4

Problem 23. Plot the constraint $2x + y = 100$ from the previous problem in the xy-plane, along with the level curves of $A(x, y) = xy$ for $k = 750, 1000, 1250, 1500$. (Make sure the image includes $x \in [-100, 100]$ and $y \in [-100, 100]$.)



Solution. ◊

Remark 24. Notice that the point (x, y) which maximizes the value of $A(x, y)$ is where the constraint $2x + y = 100$ and the level curve $xy = 1250$ share the same tangent line (and therefore normal vectors).

Theorem 25. A function $f(P)$ of many variables, constrained by the requirement $g(P) = k$ for some function g and constant k, is maximized or minimized at a point P_0 where the normal vector to the level curve/surface $f(P) = f(P_0)$ is parallel to the normal vector to the curve/surface $g(P) = k$. Put another way:

$$\nabla f = \lambda(\nabla g)$$

Theorem 26. The **Method of Lagrange Multiplers** for two-variable functions states that to maximize/minimize $f(x, y)$ on the constraint $g(x, y) = k$, you should solve the system of equations

$$f_x(x, y) = \lambda g_x(x, y)$$
$$f_y(x, y) = \lambda g_y(x, y)$$
$$g(x, y) = k$$

where λ is an unknown real number, and testing all solutions of x, y to find the maximum and minimum values of f.

4/4

Problem 27. Use the Method of Lagrange Multipliers to solve the first problem of this section. (Tip: to start, use the first two equations and eliminate the variable λ since it's not needed for the solution.)

Solution.

$$\nabla f(x, y) = \lambda(\nabla g(x, y))$$

where

$$f(x, y) = xy$$

and

$$g(x, y) = 2x + y = 100$$

To maximize $f(x, y)$ on the constraint $g(x, y)$, we need to solve the system of equations

$$f_x(x, y) = \lambda g_x(x, y)$$

$$y = 2\lambda$$

$$f_y(x, y) = \lambda g_y(x, y)$$

$$x = \lambda$$

$$g(x, y) = k$$

$$2x + y = 100$$

We get

$$y = 2x$$

Plugging y into g(x,y) we can solve for x,

$$2x + 2x = 4x = 100$$
$$x = 25$$

Plugging x in we can solve for y,

$$2(25) + y = 50 + y = 100$$
$$y = 50$$

So now we know the dimensions of the fence are x=25 and y =50. We know $A(x, y) = xy$ plugging in we find the maximum area is 25(50) = 1250.

◊

4/4

Problem 28. Find the minimum surface area of a right circular cylinder with volume equal to 432π cubic units. (Hint: $V = \pi r^2 h$ and $SA = 2\pi r(r + h)$.)

Solution.

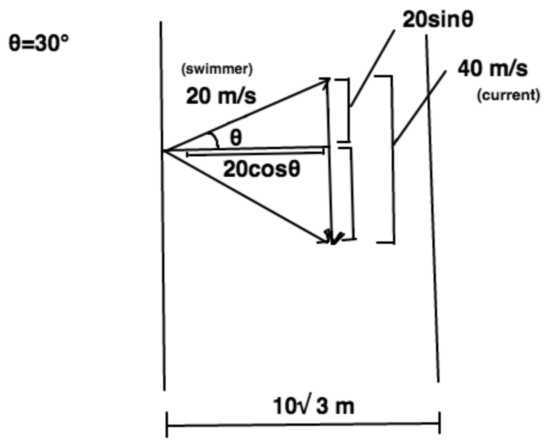
$$g(r, h) = \pi(r^2)(h) = 432(\pi)$$
$$f_r(r, h) = (\lambda)g_r(r, h)$$
$$4(\pi)(r) + 2(\pi)(h) = \lambda(2)(\pi)hr$$
$$h = \frac{2r}{\lambda(r) - 1}$$
$$f_h(r, h) = (\lambda)g_h(r, h)$$
$$2(\pi)(r) = (\lambda)(\pi)(r^2)$$
$$\lambda = \frac{2}{r}$$
$$(\pi)(r^2)(\frac{2r}{\frac{2}{r}r - 1}) = 432\pi$$
$$2r^3 = 432$$
$$r^3 = 216$$
$$r = 6 \rightarrow SA = ?$$

◊

+2

Problem 29. OPTIONAL. A river of constant width $10\sqrt{3} \approx 17.32$ meters flows 40 meters per second from north to south. A swimmer on the west side of the river can swim at a constant 20 meters per second through still waters, but since the flow of the river is faster than her top speed, this swimmer will unavoidably be pushed downstream if she tries to swim across.

Use the Method of Lagrange Multipliers to prove that if the swimmer sets an angle of $\frac{\pi}{6} = 30°$ north of east, then she will minimize the distance she is pushed downstream as she swims from the west riverbank to the east riverbank. (Hint: define $x(\theta, t)$ to be the distance she travels east after t seconds if she sets the angle θ, and define $y(\theta, t)$ to be the distance she travels south after t seconds if she sets the angle θ. Then your constraint is that $x(\theta, t)$ should be the width of the river, although that's actually not needed to solve the puzzle...)



Solution.

$$x(\theta, t) = (20\cos\theta)t$$
$$y(\theta, t) = (40 - 20\sin\theta)t$$

Which can be rewritten as:

$$f(x, y) = (40 - 20\sin x)y$$
$$g(x, y) = (20\cos x)y = 10\sqrt{3}$$

Using the Method of Lagrange Multipliers:

$$f_x(x, y) = \lambda g_x(x, y)$$
$$-20\cos x = \lambda(-20\sin x)$$
$$\cos x = \lambda(\sin x)$$
$$\lambda = \frac{\cos x}{\sin x} = \cot x$$

$$f_y(x, y) = \lambda g_y(x, y)$$
$$40 - 20\sin x = \lambda 20\cos x$$
$$40 - 20\sin x = \frac{\cos x}{\sin x} 20\cos x$$
$$40 - 20\sin x = \frac{20\cos^2 x}{\sin x}$$
$$40\sin x - 20\sin^2 x = 20\cos^2 x$$
$$40\sin x = 20\sin^2 x + 20\cos^2 x$$
$$2\sin x = \sin^2 x + \cos^2 x$$
$$2\sin x = 1$$
$$\sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6} = 30^\circ$$

So if the swimmer sets an angle of 30° north of east, she will in fact minimize the distance she is pushed downstream. ◊

Theorem 30. The **Method of Lagrange Multiplers** for three-variable functions states that to maximize/minimize $f(x, y, z)$ on the constraint $g(x, y, z) = k$, you should solve the system of equations

$$f_x(x, y, z) = \lambda g_x(x, y, z)$$
$$f_y(x, y, z) = \lambda g_y(x, y, z)$$
$$f_z(x, y, z) = \lambda g_z(x, y, z)$$
$$g(x, y, z) = k$$

where λ is an unknown real number, and testing all solutions of x, y, z to find the maximum and minimum values of f.

4/4

Problem 31. Find the maximum volume of a rectangular box without a lid which uses 48 square units of material.

Solution.

$$constraint : g(x, y, z) = xy + 2yz - 48 = 0$$
$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z), g(x, y, z) = 0$$
$$\nabla f(x, y, z) =< f_x, f_y, f_z >=< yz, zx, xy >$$
$$\nabla g(x, y, z)- < g_x, g_y, g_z >=< y + 2z, x + 2z, 2y + 2x >$$
$$(1)$$
$$yz = \lambda(y + 2z)$$
$$(2)$$
$$zx = \lambda(x + 2z)$$
$$(3)$$
$$xy = \lambda(2y + 2x)$$
$$(4)$$
$$xy + 2yz + 2zx - 48 = 0$$

multiply x through equation 1 and y through equation 2 and set them equal to each other:

$$xyz = \lambda(xy + 2zx) = \lambda(xy + 2yz) = xyz$$

reducing down we get:

$$0 = 2\lambda z(x - y)$$

since we're maximizing $\lambda \neq 0$ and $z \neq 0$ so x=y then by equation 3:

$$x^2 = 4x\lambda$$

$$x = 4\lambda$$

by equation 1:

$$4\lambda z = \lambda(4\lambda + 2z)$$

$$z = 2\lambda$$

so $x = y = 4\lambda$ and $z = 2\lambda$ pugging into equation 4:

$$16\lambda^2 + 4(2\lambda)(4\lambda) - 48 = 0$$

$$48\lambda^2 - 48 = 0$$

$$48(\lambda^2 - 1) = 0$$

$$\lambda = 1$$

So

$$x = 4$$

$$y = 4$$

$$z = 2$$

and the maximum volume is $x * y * z = 32units^2$ ◊

4/4

Problem 32. Find the highest and lowest points which lay on the curve of intersection for the cylinder $x^2 + y^2 = 8$ and the plane $2x + 2y + z = 16$.

Solution. $2x + 2y + (f(x, y)) = 16$

$f(x, y) = 2x + 2y = 16$
$f_x(x, y) = \lambda g_x(x, y)$
$2 = \lambda(2x)$
$\lambda = 1/x$
$f_y(x, y) = \lambda g_y(x, y)$
$2 = \lambda(2y)$
$\lambda = 1/y$
Therefore,
$x = y$
Plugging back in,
$x^2 + (x)^2 = 8$
$2x^2 = 8$
$x = [-2, 2]$
$y = [-2, 2]$
Plugging back into original f(x,y,z) we get z equal to
$2(2) + 2(2) + z = 16$
$4 + 4 + z = 16$
$z = 8$
$2(-2) + 2(-2) + z = 16$
$-4 - 4 + z = 16$
$z = 24$

The highest and lowest points are (2,2,8) and (-2,-2,24). We can determine which point is the highest by having the higher z value, so (-2,-2,24) is the highest point and (2,2,8) is the lowest.

$\Diamond$