Packet 2

Part 1: Sections 13.3-13.4

13.3 Arc Length and Curvature

Problem 1. Let $\vec{\mathbf{r}}(t) = \langle 6t, t^3, 3t^2 \rangle$. Use the lengths of the line segments connecting $\vec{\mathbf{r}}(0)$, $\vec{\mathbf{r}}(1)$, $\vec{\mathbf{r}}(2)$, and $\vec{\mathbf{r}}(3)$ to approximate the length of the curve from $t = 0$ to $t = 3$.

Solution. $\diamondsuit$

Definition 2. Let $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$ be a vector function. Then the **arclength** or **length** of the curve given by $\vec{\mathbf{r}}(t)$ from $t = a$ to $t = b$ is

$$L = \int_a^b \left| \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t + \Delta t) - \vec{\mathbf{r}}(t)}{\Delta t} \right| dt = \int_a^b |\vec{\mathbf{r}}'(t)|\, dt$$

Problem 3. Find the length of the curve given by $\vec{\mathbf{r}}(t) = \langle 6t, t^3, 3t^2 \rangle$ from $t = 0$ to $t = 3$.

Solution. $\diamondsuit$

Definition 4. Let $s(t)$ be the **arclength function/parameter** representing the length of a curve from the point given by $\vec{\mathbf{r}}(0)$ to the point given by $\vec{\mathbf{r}}(t)$. (Assume $s(t) < 0$ for $t < 0$.)

Theorem 5. The arclength function $s(t)$ is given by the definite integral

$$s(t) = \int_0^t |\vec{\mathbf{r}}'(\tau)|\, d\tau$$

Theorem 6. The derivative of the arclength function gives the lengths of the tangent vectors given by the derivative of the position function:

$$\frac{ds}{dt} = \left| \frac{d\vec{\mathbf{r}}}{dt} \right|$$

Problem 7. Compute $s(t)$ for $\vec{\mathbf{r}}(t) = \langle 6t, t^3, 3t^2 \rangle$, and use it to find the arclength parameter corresponding to $t = -2$.

Solution. $\diamondsuit$

Problem 8. Find the length of an arc of the circular helix with vector equation $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t \rangle$ from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

Definition 9. The **unit tangent vector** $\vec{\mathbf{T}}$ to a curve $\vec{\mathbf{r}}$ is the direction of the derivative $\vec{\mathbf{r}}'(t) = \frac{d\vec{\mathbf{r}}}{dt}$.

Theorem 10.

$$\vec{\mathbf{T}} = \frac{d\vec{\mathbf{r}}/dt}{|d\vec{\mathbf{r}}/dt|} = \frac{d\vec{\mathbf{r}}}{ds}$$

Definition 11. The **curvature** κ of a curve C at a given point is the magnitude of the rate of change of $\vec{\mathbf{T}}$ with respect to arclength s.

Theorem 12.

$$\kappa = \left|\frac{d\vec{\mathbf{T}}}{ds}\right| = \left|\frac{1}{ds/dt}\frac{d\vec{\mathbf{T}}}{dt}\right| = \frac{1}{|d\vec{\mathbf{r}}/dt|}\left|\frac{d\vec{\mathbf{T}}}{dt}\right|$$

Theorem 13. An alternate formula for curvature is given by

$$\kappa = \frac{|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)|}{|\vec{\mathbf{r}}'(t)|^3}$$

Problem 14. Prove that the helix given by the vector equation $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t \rangle$ has constant curvature.

Solution. $\diamondsuit$

Problem 15. (OPTIONAL) Prove that the alternate formula for curvature is accurate by showing

$$\frac{1}{|d\vec{\mathbf{r}}/dt|}\left|\frac{d\vec{\mathbf{T}}}{dt}\right| = \frac{|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)|}{|\vec{\mathbf{r}}'(t)|^3}$$

(Some of the solution has been provided.)

Solution. Begin by observing that $\vec{\mathbf{r}}' = \left|\frac{d\vec{\mathbf{r}}}{dt}\right|\vec{\mathbf{T}} = \frac{ds}{dt}\vec{\mathbf{T}}$, and by the product rule it follows that $\vec{\mathbf{r}}'' = \frac{d^2s}{dt^2}\vec{\mathbf{T}} + \frac{ds}{dt}\vec{\mathbf{T}}'$.

(...)

$\diamondsuit$

Definition 16. The **unit normal vector** $\vec{\mathbf{N}}$ to a curve $\vec{\mathbf{r}}$ is the direction of the derivative of the unit tangent vector $\vec{\mathbf{T}}'(t) = \frac{d\vec{\mathbf{T}}}{dt}$. (By definition, this vector points into the direction of the curve.)

Theorem 17.

$$\vec{\mathbf{N}} = \frac{\vec{\mathbf{T}}'}{|\vec{\mathbf{T}}'|}$$

Problem 18. Prove that $\vec{\mathbf{N}}$ is actually normal to the curve by using a theorem from a previous section. (Hint: $|\vec{\mathbf{T}}| = 1$.)

Solution. ◊

Definition 19. The **binormal vector** $\vec{\mathbf{B}}$ is the direction normal to both $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ according to the right-hand rule.

Theorem 20.

$$\vec{\mathbf{B}} = \vec{\mathbf{T}} \times \vec{\mathbf{N}}$$

Problem 21. Prove that $\vec{\mathbf{T}} \times \vec{\mathbf{N}}$ is a unit vector.

Solution. ◊