Calculus III - Spring 2015 - Mr. Clontz - Final Exam

Name: Solutions Class: ____________

- This exam is open-"everything", provided you do not plagiarize.
- Write your solutions so that a fellow student who has a perfect understanding of previous math classes and packets but has never seen that type of problem before could follow your work.
- You will have 150 minutes to complete this test on your own. You may not communicate with anyone during this period. Violations will be referred to the university academic honesty committee and recommended for expulsion.

1. (10 points) Find $\vec{\mathbf{r}}(t)$ given $\vec{\mathbf{r}}'(t) = \langle 2, e^t, \frac{1}{t} \rangle$ and $\vec{\mathbf{r}}(1) = \langle 3, e-3, 0 \rangle$.

$\vec{r}(t) = \langle 2t, e^t, \ln|t| \rangle + \vec{C}$

$\vec{r}(1) = \langle 2, e, \ln|1| \rangle + \vec{C} = \langle 3, e-3, 0 \rangle$

$\vec{C} = \langle 1, -3, 0 \rangle$

$\vec{r}(t) = \langle 2t, e^t, \ln|t| \rangle + \langle 1, -3, 0 \rangle$

$\vec{r}(t) = \langle 2t+1, e^t - 3, \ln|t| \rangle$

2. (10 points) Find the rate of change of $f(x, y, z) = x+2yz^2$ in the direction $\vec{\mathbf{u}} = \langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle$ at the point $P_0 = (-3, 0, 1)$.

$\nabla f = \langle 1, 2z^2, 4yz \rangle$

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$= \langle 1, 2z^2, 4yz \rangle \cdot \langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle$

$D_{\vec{u}} f \big|_{P_0} = \langle 1, 2(1)^2, 4(0)(1) \rangle \cdot \langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle$

$= \langle 1, 2, 0 \rangle \cdot \langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle$

$= \frac{1}{3} + \frac{4}{3} + 0$

$= 5/3$

3. (10 points) Prove that $f(x,y) = x^2 - 2xy - 2x + 2y - y^2$ has exactly one critical point, and that it yields a saddle point.

$\nabla f = \langle 2x - 2y - 2, -2x + 2 - 2y \rangle = \vec{0}$

$2x - 2y - 2 = 0$

$2x = 2y + 2$

$x = y + 1$

$-2x + 2 - 2y = 0$

$-2(y+1) + 2 - 2y = 0$

$-2y \cancel{-2+2} - 2y = 0$

$-4y = 0$

$y = 0$

$x = 0 + 1$

$x = 1$

$(1,0)$ is the only critical point.

$f_{xx} = 2$

$f_{xy} = -2$

$f_{yy} = -2$

$f_D = f_{xx} f_{yy} - f_{xy}^2$

$= 2(-2) - (-2)^2$

$= -4 - 4$

$= -8$

$f_D(1,0) = -8 < 0$

So $(1,0)$ is a saddle point by the 2nd Deriv. Test.

4. (10 points) Rewrite the Type II integral

$$\int_0^3 \int_y^3 \ln(1+x^2)\, dx\, dy$$

as a Type I integral by switching the order of integration. (You do not need to evaluate this integral.)

$$\int_0^3 \int_0^x \ln(1+x^2)\, dy\, dx$$

5. (10 points) Use the transformation $\vec{\mathbf{r}}(u,v) = \langle 3+u-v, 2+2u+3v\rangle$ from the unit square in the uv plane to the parallelogram R with vertices $(3,2), (4,4), (3,7), (2,5)$ in the xy plane to change

$$\iint_R y - 2x\, dA$$

to an integral of the varibles u and v. (You do not need to evaluate this integral.)

$$\vec{r}_J = \det\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = 3-(-2) = 5$$

So

$$\iint_R y-2x\, dA = \iint_G (y-2x)\,|\vec{r}_J|\, dA$$

$$= \int_0^1\int_0^1 \big((2+2u+3v)-2(3+u-v)\big)(5)\, dv\, du$$

$$= \int_0^1\int_0^1 (-4+5v)(5)\, dv\, du$$

$$= \int_0^1\int_0^1 25v-20\, dv\, du$$

6. (10 points) Prove that

$$\int_C \underbrace{\langle y^2 - 4xy, 2xy - 2x^2 + 3\rangle}_{\vec{F}} \cdot d\vec{\mathbf{r}} = -6$$

where C is the portion of the parabola $y = x^2 + x$ beginning at the point $(1, 2)$ and ending at the point $(-1, 0)$. (Hint: this *is* a conservative field.)

If $\vec{F} = \nabla f$, then

$f_x = y^2 - 4xy \Rightarrow f = xy^2 - 2x^2y + \Phi$

$f_y = 2xy - 2x^2 + 3 \Rightarrow f = xy^2 - 2x^2y + 3y + \Phi$

Satisfied by $f = xy^2 - 2x^2y + 3y$. So:

$$\int_C \vec{F} \cdot d\vec{r} = \Big[xy^2 - 2x^2y + 3y\Big]_{(1,2)}^{(-1,0)}$$

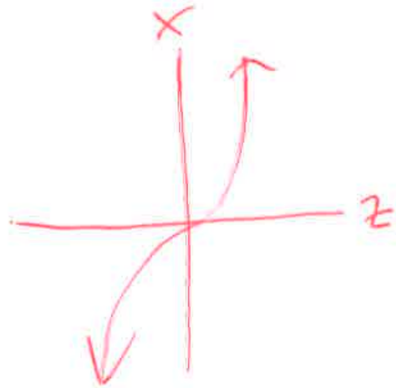
$$= \cancel{\left(-1(0)^2 - 2(-1)^2(0) + 3(0)\right)} - \left(1(2)^2 - 2(1)^2(2) + 3(2)\right)$$

$$= -(\cancel{4} - \cancel{4} + 6)$$

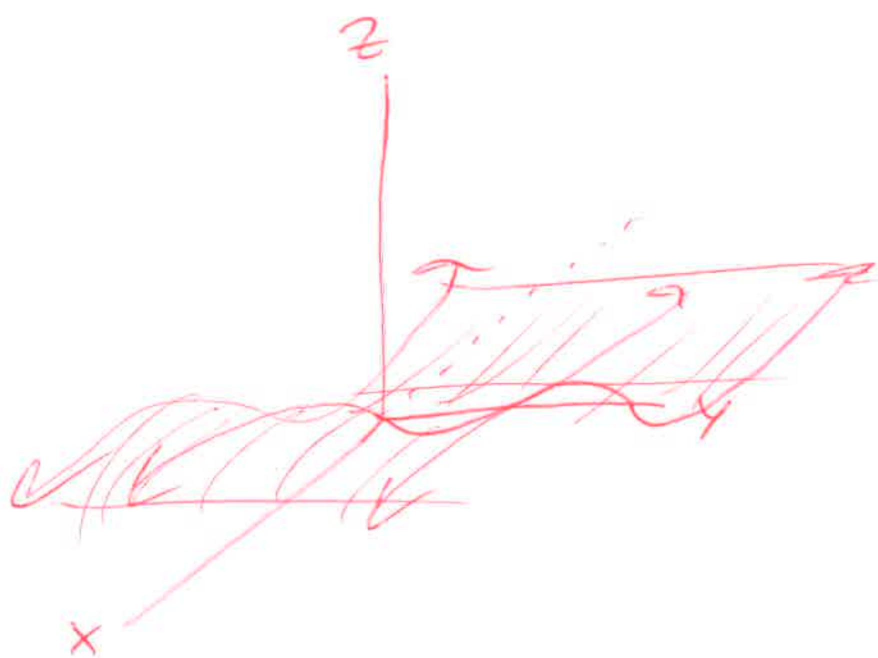
$$= -6$$

7. (10 points) Plot the curve $x = z^3$ in the xz-plane and the cylindrical surface $x = z^3$ in xyz-space.

In xz-plane:



In xyz space:



8. (10 points) Verify the Mixed Derivative Theorem ($f_{xy} = f_{yx}$) for $f(x, y) = \sin(x + 2y)$.

$f_x = \cos(x+2y)\cancel{(1+0)}$
$= \cos(x+2y)$

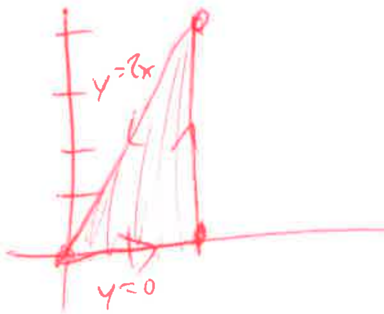
$f_y = \cos(x+2y)(0+2)$
$= 2\cos(x+2y)$

$f_{xy} = -\sin(x+2y)(0+2)$
$= -2\sin(x+2y)$

$f_{yx} = 2(-\sin(x+2y))\cancel{(1+0)}$
$= -2\sin(x+2y)$

Therefore $f_{xy} = f_{yx}$.

9. (10 points) Find the work done by a force vector field $\langle x+y, 4x-y-2\rangle$ moving an object around the boundary of the triangle with vertices $(0,0)$, $(2,0)$, and $(2,4)$ oriented counter-clockwise.



$$\int_C \vec{F}\cdot d\vec{r} = \iint_R Q_x - P_y \, dA$$

$$= \int_0^2 \int_0^{2x} 4-1 \, dA$$

$$= \int_0^2 \left[3y\right]_0^{2x} dx$$

$$= \int_0^2 6x \, dx$$

$$= 12$$

10. (10 points) Circle exactly one of the following questions, and then answer it.

- Write a sentence explaining intuitively what the curvature of $\vec{\mathbf{r}}(t) = \langle t, 1-t, 2t+5\rangle$ should be for any point on it, and then use a definition or formula from our notes to verify your intuition.
- The coordinate product of a point (x, y, z) is given by xyz. Find the point on the triangle with vertices $(6,0,0)$, $(0,6,0)$, $(0,0,6)$ which has the largest coordinate product.

Since $\vec{r}(t)$ is the vector equation for a line, curvature K should be 0. Since $K = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$, we may compute $\vec{v} = \langle 1,-1,2\rangle$, $\vec{T} = \frac{\langle 1,-1,2\rangle}{\sqrt{1+1+4}}$, and $\frac{d\vec{T}}{dt} = \langle 0,0,0\rangle$.

Thus $K = \frac{1}{\sqrt{6}}(0) = 0$.

The triangle is on the plane with equation $g = x+y+z = 6$

We want to optimize $f = xyz$. Use L. Mults:

$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z \qquad x+y+z=6$

$yz = \lambda(1) \qquad xz = \lambda(1) \qquad xy = \lambda(1)$

So $yz = xz = xy$ And $x+y+z=6$

$\Rightarrow x = y = z.$ $3x = 6$

$x = 2$
$y = 2$
$z = 2$

$(2,2,2)$

(scratch paper)

(scratch paper)