Packet 2

Part 2.2: Sections 14.1-14.3

14.1 Functions of Several Variables

Definition 1. A **function f of two variables** is a rule which assigns a real number $f(x, y)$ to each pair of real numbers (x, y) for which that rule is defined. The collection of such well-defined pairs is called the **domain** $\mathrm{dom}(f)$ of the function, and the set of real numbers which can possiblely be produced by the function is called its **range** $\mathrm{ran}(f)$.

Definition 2. The **level curve** for each $k \in \mathrm{ran}(f)$ is given by the equation $f(x, y) = k$. The **graph** of f is a surface in 3D space which visualizes the function, given by the equation $z = f(x, y)$.

Definition 3. A **function f of three variables** is a rule which assigns a real number $f(x, y, z)$ to each triple of real numbers (x, y, z) for which that rule is defined. The collection of such well-defined triples is called the **domain** $\mathrm{dom}(f)$ of the function, and the set of real numbers which can possiblely be produced by the function is called its **range** $\mathrm{ran}(f)$.

Problem 4. Let $f(x, y) = x \sin(x + y)$. Give the value of $f(\pi, \frac{\pi}{2})$.

Solution. $\Diamond$

Problem 5. Let $f(x, y) = -x - y + 2$. In the xy-plane, plot the domain of f, as well as its level curves for $k = -3, 0, 3$. Then plot the graph of f in xyz space.

Solution. $\Diamond$

Problem 6. Let $f(x, y) = \sqrt{4 - x^2 - y^2}$. In the xy-plane, plot the domain of f, as well as its level curves for $k = 0, \frac{1}{\sqrt{2}}, 1$. Then plot the graph of f in xyz space.

Solution. $\Diamond$

Definition 7. The **level surface** for each $k \in \mathrm{ran}(f)$ is given by the equation $f(x, y, z) = k$. (Since the graph of a three variable function would require four variables and therefore is a four-dimensional object, we typically don't consider it.)

Problem 8. Let $f(x, y, z) = \frac{x+3y^2}{z-2x}$. Give the value of $f(3, -2, 1)$.

Solution. $\diamond$

Problem 9. Let $f(x, y, z) = -x^2 + y - z^2$. In xyz space, plot the level surfaces for $k = -2, 0, 2$.

Solution. $\diamond$

Remark 10. If $P = (x, y)$, then we assume that $f(x, y) = f(P) = f(\vec{\mathbf{P}})$. If $P = (x, y, z)$, then we assume that $f(x, y, z) = f(P) = f(\vec{\mathbf{P}})$.

14.2 Limits and Continuity

Definition 11. If the value of the function $f(P)$ becomes arbitrarily close to the number L as points P close to P_0 are plugged into the function, then the **limit of** $f(P)$ **as** P **approaches** P_0 is L:

$$\lim_{P \to P_0} f(P) = L$$

Theorem 12. Let $f(x, y)$ be a function of two variables. If there exists a curve $y = g(x)$ passing through the point (x_0, y_0) such that $\lim_{x \to x_0} f(x, g(x))$ does not exist, then $\lim_{(x,y) \to (x_0,y_0)} f(x, y)$ does not exist.

Problem 13. Prove that

$$\lim_{(x,y) \to (0,0)} \frac{x + y}{|x + y|}$$

does not exist by considering the function $g(x) = x$.

Solution. $\diamond$

Theorem 14. Let $f(x, y)$ be a function of two variables. If there exist curves $y = g(x)$ and $y = h(x)$ passing through the point (x_0, y_0) such that $\lim_{x \to x_0} f(x, g(x)) \neq \lim_{x \to x_0} f(x, h(x))$, then $\lim_{(x,y) \to (x_0,y_0)} f(x, y)$ does not exist.

Problem 15. Prove that

$$\lim_{(x,y) \to (0,0)} \frac{x^6 + y^2}{x^3 y + x^6}$$

does not exist by considering the functions $g(x) = x^3$ and $h(x) = 2x^3$.

Solution. $\diamond$

Theorem 16. The "Limit Laws" for single-variable functions also hold for multi-variable functions.

$$\lim_{P \to P_0} (f(P) \pm g(P)) = \lim_{P \to P_0} f(P) \pm \lim_{P \to P_0} g(P)$$

$$\lim_{P \to P_0} (f(P) \cdot g(P)) = \lim_{P \to P_0} f(P) \cdot \lim_{P \to P_0} g(P)$$

$$\lim_{P \to P_0} (kf(P)) = k \lim_{P \to P_0} f(P)$$

$$\lim_{P\to P_0} \frac{f(P)}{g(P)} = \frac{\lim\limits_{P\to P_0} f(P)}{\lim\limits_{P\to P_0} g(P)}$$

$$\lim_{P\to P_0} (f(P))^{r/s} = \left(\lim_{P\to P_0} f(P)\right)^{r/s}$$

Theorem 17. Let $P_0 = (x_0, y_0, z_0)$. Multi-variable limits which only use one variable may be reduced to a single-variable limit.

$$\lim_{P\to P_0} f(x) = \lim_{x\to x_0} f(x)$$

$$\lim_{P\to P_0} g(y) = \lim_{y\to y_0} g(y)$$

$$\lim_{P\to P_0} h(z) = \lim_{z\to z_0} h(z)$$

Problem 18. Use the above theorems to rigorously prove that

$$\lim_{(x,y)\to(1,2)} \frac{2x+y}{y^2} = 1$$

Solution. ◊

Remark 19. Due to the limit laws, the "just plug it in" rule applies when plugging in does not result in an undefined operation.

Problem 20. Compute the limit

$$\lim_{(x,y,z)\to(3,0,-1)} \frac{x\cos y}{z+x}$$

Remark 21. There is no L'Hopital rule for multi-variable limits. However, you may still use it once the limit has been reduced to a single-variable limit.

Problem 22. Compute the limit

$$\lim_{(x,y)\to(3,0)} \frac{xy+\sin(2y)}{y}$$

Solution. ◊

Remark 23. Factoring and canceling (including conjugation tricks) is also effective for computing multi-variable limits.

Problem 24. Compute the limit

$$\lim_{(x,y,z)\to(1,2,4)} \frac{\sqrt{z}-xy}{z-x^2y^2}$$

Solution. ◊

Definition 25. A function $f(P)$ is **continuous** if $\lim_{P \to P_0} f(P) = f(P_0)$ for all points P_0 in its domain.

Theorem 26. If a multi-variable function is composed of continuous single-variable functions, then it is also continuous.

14.3 Partial Derivatives

Definition 27. The **partial derivative of f with respect to a variable** is the rate of change of f as that variable changes and all other variables are held constant. For example:

$$\frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial g}{\partial z} = g_z(x,y,z) = \lim_{h \to 0} \frac{g(x,y,z+h) - g(x,y,z)}{h}$$

Problem 28. Let $f(x,y,z) = xy^2 + 2z$. Use the definition of a partial derivative to prove that $\frac{\partial f}{\partial y} = 2xy$.

Solution. ◊

Theorem 29. Partial derivatives may be computed in the usual way by treating all other variables as constants.

Problem 30. Compute both partial derivatives of $f(x,y) = 4x^2 - 5y^3 + xy - 1$.

Solution. ◊

Problem 31. Compute both partial derivatives of $f(x,y) = \sin(x + 3y)$.

Solution. ◊

Problem 32. Compute both partial derivatives of $f(x,y) = e^{xy^2}$.

Solution. ◊

Definition 33. **Second-order partial derivatives** are the result of taking the partial derivative of a partial derivative. For example:

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$g_z = (g_z)_z = \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial z}\right) = \frac{\partial^2 g}{\partial z^2}$$

Theorem 34. When computing the second-order partial derivative for a sufficiently well-behaved function, the order in which the partial derivatives are taken is irrelevant. (This is sometimes called the **Mixed Derivative Theorem**.)

Problem 35. Verify the Mixed Derivative Theorem for $f(x,y) = 3x^2y^2 - x^3 + y^4 - 7$.

Solution. $\Diamond$