Packet 4

Packet 4.2: Sections 16.5-16.9

16.5 Curl and Divergence

Definition 1. The **curl** of a vector field $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is given by the expression

$$\text{curl } \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R\rangle = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$$

Problem 2. Prove that if $\vec{\mathbf{F}}$ is conservative, then curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$.

Solution. Since $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is conservative, we know from a theorem in section 16.3 from the previous packet that $Q_x = P_y$, $P_z = R_x$, and $R_y = Q_z$. Therefore:

$$\text{curl } \vec{\mathbf{F}} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \langle 0, 0, 0\rangle = \vec{\mathbf{0}}$$

$\Diamond$

Remark 3. For a vector field $\vec{\mathbf{F}}$ and direction $\vec{\mathbf{u}}$, (curl $\vec{\mathbf{F}}$) $\cdot\, \vec{\mathbf{u}}$ may be thought of as the tendency of $\vec{\mathbf{F}}$ to “spin” counter-clockwise around $\vec{\mathbf{u}}$.

Problem 4. Compute the curl of $\langle x + y, z^2 - 3, yz\rangle$ around the point $(2, 0, -1)$.

Solution. Let $\vec{\mathbf{F}} = \langle P, Q, R\rangle = \langle x + y, z^2 - 3, yz\rangle$. Then

$$\begin{aligned} \text{curl } \vec{\mathbf{F}} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle \\ &= \langle (z) - (2z), (0) - (0), (0) - (1)\rangle = \langle -z, 0, -1\rangle \end{aligned}$$

By plugging in the point $(2, 0, -1)$ we get

$$\langle -(-1), 0, -1\rangle = \langle 1, 0, -1\rangle$$

$\Diamond$

Theorem 5. Green’s Theorem may be rewritten in terms of curl as follows:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_R (\text{curl } \vec{\mathbf{F}}) \cdot \widehat{\mathbf{k}}\, dA$$

Problem 6. Prove the previous theorem.

Solution. Green's Theorem was given in section 4.1 to be

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_R Q_x - P_y \, dA$$

Since

$$\text{curl } \vec{\mathbf{F}} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

we have that

$$\text{curl } \vec{\mathbf{F}} \cdot \widehat{\mathbf{k}} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \cdot \langle 0, 0, 1 \rangle = Q_x - P_y$$

Therefore

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_R Q_x - P_y \, dA = \iint_R \text{curl } \vec{\mathbf{F}} \cdot \widehat{\mathbf{k}} \, dA$$

$\Diamond$

Definition 7. The **divergence** of a vector field $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is given by the expression

$$\text{div } \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z$$

Problem 8. Prove that the divergence of a curl vector field is always 0. Put another way, show that div (curl $\vec{\mathbf{F}}$) $= 0$.

Solution. We want to compute

$$\text{div (curl } \vec{\mathbf{F}}) = \text{div } (\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle)$$

Since divergence is the sum of the partial derivatives:

$$\text{div } (\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle) = (R_{yx} - Q_{zx}) + (P_{zy} - R_{xy}) + (Q_{xz} - P_{yz})$$

By regrouping, we find that

$$(R_{yx} - Q_{zx}) + (P_{zy} - R_{xy}) + (Q_{xz} - P_{yz}) = (R_{yx} - R_{xy}) + (P_{zy} - P_{yz}) + (Q_{xz} - Q_{zx}) = 0$$

$\Diamond$

Remark 9. Divergence measures the tendency of a vector field to diverge away from a point.

Problem 10. Compute the divergence of $\langle x + y, z^2 - 3, yz \rangle$ away from the point $(2, 0, -1)$.

Solution. Let $\vec{\mathbf{F}} = \langle P, Q, R\rangle = \langle x + y, z^2 - 3, yz\rangle$. We compute divergence as follows:

$$\text{div }\vec{\mathbf{F}} = P_x + Q_y + R_z = (1) + (0) + (y) = 1 + y$$

Plugging in the point $(2, 0, -1)$ gives

$$1 + (0) = 1$$

◊

Definition 11. The **flux** of a velocity vector field $\vec{\mathbf{F}}$ across a closed curve C is given by

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds$$

where $\vec{\mathbf{n}}$ yields outward unit normal vectors to C.

Remark 12. Flux measures the tendency of a vector field to flow outward from a closed and bounded region (or inward if the flux is negative).

Theorem 13. Green's Theorem may be rewritten in terms of divergence as follows:

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = \iint_R \text{div }\vec{\mathbf{F}}\, dA$$

Problem 14. Compute the flux of the velocity vector field $\langle x+y, x^2+y^2\rangle$ across the boundary of the unit square.

Solution. Let $\vec{\mathbf{F}} = \langle x + y, x^2 + y^2\rangle$, so then its divergence is $\text{div }\vec{\mathbf{F}} = (1) + (2y) = 1 + 2y$. By the previous theorem, the flux may be computed by

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = \iint_R \text{div }\vec{\mathbf{F}}\, dA = \iint_R 1 + 2y\, dA$$

Since R is the unit square,

$$\iint_R 1 + 2y\, dA = \int_0^1 \int_0^1 1 + 2y\, dy\, dx = \int_0^1 2\, dx = 2$$

◊

16.6 Parametric Surfaces

Remark 15. Just like a curve may be parameterized by $\vec{\mathbf{r}}(t)$ for an interval $a \leq t \leq b$, a surface may be parameterized by $\vec{\mathbf{r}}(u, v)$ for a region R in the uv plane.

Theorem 16. Following are some common surface parameterizations.

- The surface $z = f(x, y)$ may be parametrized by

$$\vec{\mathbf{r}}(x, y) = \langle x, y, f(x, y)\rangle$$

- A surface determined by a cylindrical coordinate equation may be parametrized by substituting into

$$\vec{\mathbf{r}} = \langle r\cos\theta, r\sin\theta, z\rangle$$

- A surface determined by a spherical coordinate equation may be parametrized by substituting into

$$\vec{\mathbf{r}} = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$$

Problem 17. Find a parameterization from the xy plane to the plane $2x - y + z = 7$ in xyz space.

Solution. The surface $z = f(x, y)$ may be parametrized by

$$\vec{\mathbf{r}}(x, y) = \langle x, y, f(x, y)\rangle$$

So we can rewrite the surface as $z = 7 - 2x + y = f(x, y)$, and use the parameterization

$$\vec{\mathbf{r}}(x, y) = \langle x, y, 7 - 2x + y\rangle$$

$\Diamond$

Problem 18. Find the parameterization from the rectangle $0 \leq z \leq 3$ and $0 \leq \theta \leq 2\pi$ to the conical surface $z = \sqrt{x^2 + y^2}$ below the plane $z = 3$ in xyz space. (Hint: find the cylindrical coordinate equation for the surface.)

Solution. The surface $z = \sqrt{x^2 + y^2}$ has cylindrical coordinate equation $z = \sqrt{r^2} = r$. Substituting that into the cylindrical coordinate transformation

$$\vec{\mathbf{r}}(r, \theta, z) = \langle r\cos\theta, r\sin\theta, z\rangle$$

we get

$$\vec{\mathbf{r}}(\theta, z) = \langle z\cos\theta, z\sin\theta, z\rangle$$

Since the cone revolves fully around the z-axis, $0 \leq \theta \leq 2\pi$, and we already know $0 \leq z \leq 3$. $\Diamond$

Problem 19. Find the parameterization from the rectangle $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$ to the spherical surface $x^2 + y^2 + z^2 = 9$ in xyz space. (Hint: find the spherical coordinate equation for the surface.)

Solution. The surface $x^2+y^2+z^2=9$ has spherical coordinate equation $\rho=3$. Substituting that into the spherical coordinate transformation

$$\vec{\mathbf{r}}(\rho,\phi,\theta)=\langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$$

we get

$$\vec{\mathbf{r}}(\phi,\theta)=\langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi\rangle$$

Since the sphere revolves fully around the z-axis, $0\leq\theta\leq 2\pi$, and since it includes points on the positive and negative z-axis, $0\leq\phi\leq\pi$. ◊

16.7 Surface Integrals

Definition 20. The **surface integral** of a function $f(x,y,z)$ over a surface S in xyz space is given by

$$\iint_S f(\vec{\mathbf{r}})\,d\sigma=\iint_R f(\vec{\mathbf{r}}(u,v))|\vec{\mathbf{r}}_u\times\vec{\mathbf{r}}_v|\,dA$$

where $\vec{\mathbf{r}}(u,v)$ is a parameterization from the region R in the uv plane to the surface S.

Theorem 21. The surface area of S is given by

$$\iint_S d\sigma=\iint_S 1\,d\sigma$$

Problem 22. Use the parameterization

$$\vec{\mathbf{r}}(\phi,\theta)=\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$$

from $0\leq\phi\leq\pi, 0\leq\theta\leq 2\pi$ to the unit sphere to show that the surface area of the unit sphere is 4π. (Note that this matches the formula $SA=4\pi r^2$ used in high school geometry.)

Solution. The surface area of S is given by

$$\iint_S d\sigma=\iint_R |\vec{\mathbf{r}}_\phi\times\vec{\mathbf{r}}_\theta|\,dA$$

We may compute the cross product:

$$\begin{aligned}\vec{\mathbf{r}}_\phi\times\vec{\mathbf{r}}_\theta &= \langle\cos\phi\cos\theta,\cos\phi\sin\theta,-\sin\phi\rangle\times\langle-\sin\phi\sin\theta,\sin\phi\cos\theta,0\rangle\\ &= \langle\sin^2\phi\cos\theta,-\sin^2\phi\sin\theta,\sin\phi\cos\phi\rangle\end{aligned}$$

And its magnitude is then

$$|\vec{\mathbf{r}}_\phi\times\vec{\mathbf{r}}_\theta|=|\sin\phi|$$

but we can drop the absolute values since $0\leq\phi\leq\pi$.

Using the bounds and plugging in, we get

$$\iint_R |\vec{\mathbf{r}}_\phi\times\vec{\mathbf{r}}_\theta|\,dA=\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta=\int_0^{2\pi}2\,d\theta=4\pi$$

◊

Problem 23. Show that the area of the parallelogram with vertices $(0,0,0)$, $(2,1,2)$, $(0,2,-1)$, and $(2,3,1)$ is $3\sqrt{5}$ using a surface integral. (Hint: use $\vec{\mathbf{r}}(u,v)=\langle 2u,u+2v,2u-v\rangle$.)

Solution. Note that for the parameterization $\vec{\mathbf{r}}(u,v) = \langle 2u, u+2v, 2u-v\rangle$, $\vec{\mathbf{r}}(0,0) = \langle 0,0,0\rangle$, $\vec{\mathbf{r}}(1,0) = \langle 2,1,2\rangle$, $\vec{\mathbf{r}}(0,1) = \langle 0,2,-1\rangle$, and $\vec{\mathbf{r}}(1,1) = \langle 2,3,1\rangle$. So it maps the unit square onto the given parallelogram.

So we want to compute

$$\iint_S d\sigma = \iint_R |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dA = \int_0^1 \int_0^1 |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\, dv\, du$$

We compute the cross product as follows:

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle 2,1,2\rangle \times \langle 0,2,-1\rangle = \langle -5,2,4\rangle$$

$$|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| = \sqrt{25+4+16} = \sqrt{45} = 3\sqrt{5}$$

It follows that

$$\int_0^1 \int_0^1 3\sqrt{5}\, dv\, du = \int_0^1 3\sqrt{5}\, du = 3\sqrt{5}$$

$\diamond$

Definition 24. An **orientation** of a surface is a continuous unit vector field normal to the surface.

Remark 25. Orienting a surface is akin to choosing one side or another of the surface.

Remark 26. Examples of non-orientable surfaces are the Mobiüs strip and Klein bottle.

Definition 27. The **surface integral** of a vector field $\vec{\mathbf{F}}$ over an oriented surface S in xyz space is given by

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\sigma} = \iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, d\sigma = \iint_R \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA$$

where $\vec{\mathbf{n}}$ is the orientation of the surface and giving its orientation, and $\vec{\mathbf{r}}(u,v)$ is an appropriate parameterization from the region R in the uv plane to the surface S.

Definition 28. The **flux** across a closed oriented surface (such as the boundary of a solid) is given by

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\sigma}$$

Problem 29. Use the parameterization

$$\vec{\mathbf{r}}(\phi,\theta) = \langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi\rangle$$

from $0 \le \phi \le \pi, 0 \le \theta \le 2\pi$ to the sphere $x^2+y^2+z^2 = 9$ to prove that the flux across it for the vector field $\langle x,y,z\rangle$ is

$$\int_0^{2\pi} \int_0^{\pi} 27\sin\phi\, d\phi\, d\theta$$

Solution. We want to compute

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\sigma} = \iint_R \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta)\, dA = \int_0^{2\pi} \int_0^{\pi} \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta)\, d\phi\, d\theta$$

Using the given parametrization,

$$\vec{\mathbf{F}} = \langle x, y, z \rangle = \langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi \rangle$$

and the cross product is

$$\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta = \langle 3\cos\phi\cos\theta, 3\cos\phi\sin\theta, -3\sin\phi \rangle \times \langle -3\sin\phi\sin\theta, 3\sin\phi\cos\theta, 0 \rangle$$

$$= \langle 9\sin^2\phi\cos\theta, 9\sin^2\phi\sin\theta, 9\sin\phi\cos\phi \rangle$$

So their dot product gives:

$$\int_0^{2\pi} \int_0^{\pi} \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta)\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi} 27\sin^3\phi\cos^2\theta + 27\sin^3\phi\sin^2\theta + 27\sin\phi\cos^2\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} 27\sin^3\phi + 27\sin\phi\cos^2\phi\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi} 27\sin\phi\, d\phi\, d\theta$$

$\Diamond$

16.8 Stokes' Theorem

Theorem 30. Let S be a surface with orientation $\vec{\mathbf{n}}$ and with boundary C oriented counter-clockwise with respect to $\vec{\mathbf{n}}$. Then

$$\iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot d\vec{\sigma} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

Problem 31. Let S be the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$. Use Stokes' Theorem to prove that

$$\iint_S \langle 2y, 2z, 2x \rangle \cdot d\vec{\sigma} = \int_0^{2\pi} \cos^3(t)\, dt$$

(Hint: what's the curl of $\langle z^2, x^2, y^2 \rangle$?).

Solution. Following the hint, the curl of $\langle z^2, x^2, y^2 \rangle$ may be computed to be:

$$\operatorname{curl} \langle z^2, x^2, y^2 \rangle = \langle 2y - 0, 2z - 0, 2x - 0 \rangle$$

So it follows from Stokes' Theorem that

$$\iint_S \langle 2y, 2z, 2x \rangle \cdot d\vec{\sigma} = \iint_S \operatorname{curl} \langle z^2, x^2, y^2 \rangle \cdot d\vec{\sigma} = \int_C \langle z^2, x^2, y^2 \rangle \cdot d\vec{\mathbf{r}}$$

C is the boundary of S, the circle $x^2+y^2=1$ in the plane $z=0$. It has a parametrization

$$\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, 0\rangle$$

for $0 \leq t \leq 2\pi$ and derivative

$$\frac{d\vec{\mathbf{r}}}{dt} = \langle -\sin t, \cos t, 0\rangle$$

So the line integral may be rewritten as

$$\int_C \langle z^2, x^2, y^2\rangle \cdot d\vec{\mathbf{r}} = \int_C \langle z^2, x^2, y^2\rangle \cdot \frac{d\vec{\mathbf{r}}}{dt}\,dt = \int_0^{2\pi} \langle 0^2, (\cos t)^2, (\sin t)^2\rangle \cdot \langle -\sin t, \cos t, 0\rangle\,dt$$

$$= \int_0^{2\pi} 0 + \cos^3(t) + 0\,dt = \int_0^{2\pi} \cos^3(t)\,dt$$

$\diamondsuit$

16.9 Divergence Theorem

Theorem 32. Let S be the boundary of a solid D oriented outwards. Then

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\sigma} = \iiint_D \operatorname{div} \vec{\mathbf{F}}\,dV$$

Problem 33. Let S be the boundary of the unit cube in xyz space. Use the Divergence Theorem to prove that

$$\iint_S \langle x+y, y^2+z^2, z^3+x^3\rangle \cdot d\vec{\sigma} = \int_0^1 \int_0^1 \int_0^1 1 + 2y + 3z^2\,dz\,dy\,dx$$

Solution. By the Divergence Theorem:

$$\iint_S \langle x+y, y^2+z^2, z^3+x^3\rangle \cdot d\vec{\sigma} = \iiint_D \operatorname{div} \langle x+y, y^2+z^2, z^3+x^3\rangle\,dV$$

$$= \int_0^1 \int_0^1 \int_0^1 (1) + (2y) + (3z^2)\,dz\,dy\,dx$$

$\diamondsuit$

16.10 A small remark and puzzle

Remark 34. Using derivatives, gradients, curl, and divergence, we may observe that several kinds of integrals may be evaluated by observing how the integrand behaves on the boundary of the domain of integration, and vice versa.

$$\int_{[a,b]} f'(x)\,dx = [f(x)]_a^b$$

$$\int_C \nabla f \cdot d\vec{\mathbf{r}} = [f(P)]_A^B$$

$$\iint_R \operatorname{div} \vec{\mathbf{F}}\, dA = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds$$

$$\iint_R Q_x - P_y\, dA = \int_C \langle P, Q \rangle \cdot d\vec{\mathbf{r}}$$

$$\iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot d\vec{\sigma} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

$$\iiint_D \operatorname{div} \vec{\mathbf{F}}\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\sigma}$$

Problem 35. (OPTIONAL) This has nothing to do with the above remark, but here's a puzzle for reading this far.

Wayne Brady is hosting a gameshow, and you've been called down from the audience to attempt to win fabulous prizes. Wayne gives you the choice of three doors: A, B, and C. He asks you to choose a door, explaining that only one of the three doors holds a prize behind it.

After you choose, Wayne opens one of the doors that you didn't choose to reveal nothing behind it. He then offers you the opportunity to switch your door with the other unopened door, after which you will immediately be given whatever is behind it. Should you stick with your initial guess, or should you switch, or does it even matter? Why?

Solution. Figure it out yourself. :-) ◊