# Packet 4

# Packet 4.1: Sections 16.1-16.4

## 16.1 Vector Fields

**Definition 1.** A **vector field** assigns a vector to each point in 2D or 3D space.

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\mathbf{F}}(x, y) = \langle P(x, y), Q(x, y)\rangle = \langle P(\vec{\mathbf{r}}), Q(\vec{\mathbf{r}})\rangle = \langle P, Q\rangle$$

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\mathbf{F}}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle = \langle P(\vec{\mathbf{r}}), Q(\vec{\mathbf{r}}), R(\vec{\mathbf{r}})\rangle = \langle P, Q, R\rangle$$

**Problem 2.** Sketch the vector field $\vec{\mathbf{F}} = \langle x + y, 2y\rangle$ for all $x \in \{0, 1, 2\}$ and $y \in \{0, 1, 2\}$.

**Solution.** See http://kevinmehall.net/p/equationexplorer/vectorfield.html#(x+y)i+2yj%7C%5B-1,4,-1,4%5D ◊

**Remark 3.** The gradient vector function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$$

is a vector field which yields normal vectors to the level surfaces of the function $f$.

**Problem 4.** Compute $\nabla f$ for the function $f(x, y) = x^2 - 2xy + y$, and then sketch the vector field $\nabla f$ all $x \in \{0, 1, 2\}$ and $y \in \{0, 1, 2\}$.

**Solution.** $\nabla f = \langle \frac{\partial}{\partial x}[x^2 - 2xy + y], \frac{\partial}{\partial y}[x^2 - 2xy + y]\rangle = \langle 2x - 2y, -2x + 1\rangle$

See http://kevinmehall.net/p/equationexplorer/vectorfield.html#(x+y)i+2yj%7C%5B-1,4,-1,4%5D ◊

## 16.2 Line Integrals

**Theorem 5.** Some vector functions which parameterize curves follow.

- A line segment beginning at $P_0$ and ending at $P_1$:

$$\vec{\mathbf{r}}(t) = \overrightarrow{\mathbf{P_0}} + t\overrightarrow{\mathbf{P_0P_1}}, 0 \le t \le 1$$

- A circle centered at the origin with radius $a$:

$$\vec{\mathbf{r}}(t) = \langle a\cos t, a\sin t\rangle, 0 \leq t \leq 2\pi \text{ (full counter-clockwise rotation)}$$

$$\vec{\mathbf{r}}(t) = \langle a\sin t, a\cos t\rangle, 0 \leq t \leq 2\pi \text{ (full clockwise rotation)}$$

- A planar curve given by $y = f(x)$ from $(x_0, y_0)$ to $(x_1, y_1)$

$$\vec{\mathbf{r}}(t) = \langle t, f(t)\rangle, x_0 \leq t \leq x_1 \text{ (left-to-right)}$$

$$\vec{\mathbf{r}}(t) = \langle -t, f(-t)\rangle, -x_0 \leq t \leq -x_1 \text{ (right-to-left)}$$

**Problem 6.** Give a vector function which parameterizes the line segment from the point $(0, 3, -2)$ to the point $(4, -1, 0)$.

**Solution.** Using the above theorem:

$$\vec{\mathbf{r}}(t) = \overrightarrow{\mathbf{P_0}} + t\overrightarrow{\mathbf{P_0P_1}}, 0 \leq t \leq 1$$

$$\vec{\mathbf{r}}(t) = \langle 0, 3, -2\rangle + t\langle 4-0, -1-3, 0-(-2)\rangle, 0 \leq t \leq 1$$

$$\vec{\mathbf{r}}(t) = \langle 4t, 3-4t, -2+2t\rangle, 0 \leq t \leq 1$$

◊

**Problem 7.** Give a vector function which parameterizes the curve $y = x^3 - 2x$ from the point $(1, -1)$ to the point $(-1, 1)$.

**Solution.** Using the above theorem (noting that $(1, -1)$ is to the right of $(-1, 1)$):

$$\vec{\mathbf{r}}(t) = \langle -t, f(-t)\rangle, -x_0 \leq t \leq -x_1$$

$$\vec{\mathbf{r}}(t) = \langle -t, (-t)^3 - 2(-t)\rangle, -1 \leq t \leq 1$$

$$\vec{\mathbf{r}}(t) = \langle -t, -t^3 + 2t\rangle, -1 \leq t \leq 1$$

◊

**Problem 8.** Give a vector function which parameterizes the curve $x^2 + y^2 = 9$ from the point $(3, 0)$ clockwise to the point $(0, -3)$.

**Solution.** Using the above theorem, we may get a full clockwise rotation by using:

$$\vec{\mathbf{r}}(t) = \langle a\sin t, a\cos t\rangle, 0 \leq t \leq 2\pi$$

$$\vec{\mathbf{r}}(t) = \langle 3\sin t, 3\cos t\rangle, 0 \leq t \leq 2\pi$$

To obtain the portion from $(3, 0)$ to $(0, -3)$, we note that we may plug in $\pi/2$ and $\pi$ respectively to get those points. Therefore the vector function is:

$$\vec{\mathbf{r}}(t) = \langle 3\sin t, 3\cos t\rangle, \frac{\pi}{2} \leq t \leq \pi$$

◊

**Definition 9.** The **line integral with respect to arclength** of a function of many variables $f(\vec{\mathbf{r}})$ along a curve $C$ is given by

$$\int_C f(\vec{\mathbf{r}})\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(\vec{\mathbf{r}}_{n,i})\Delta s_{n,i}$$

where for each positive integer $n$ we've defined a way to partition $C$ into $n$ pieces

$$\Delta C_{n,1}, \Delta C_{n,2}, \dots, \Delta C_{n,n}$$

where $\Delta C_{n,i}$ has length $\Delta s_{n,i}$, contains the position vector $\vec{\mathbf{r}}_{n,i}$, and

$$\lim_{n\to\infty}\max(\Delta s_{n,i}) = 0$$

**Theorem 10.** If $\vec{\mathbf{r}}(t)$ is a parametrization of $C$ for $a \leq t \leq b$, then

$$\int_C f(\vec{\mathbf{r}})\,ds = \int_{t=a}^{t=b} f(\vec{\mathbf{r}}(t))\frac{ds}{dt}\,dt$$

**Problem 11.** Evaluate $\int_C z+2xy\,ds$ where $C$ is the line segment from $(0,-1,3)$ to $(2,2,-3)$.

**Solution.** We'll use the parametrization

$$\vec{\mathbf{r}}(t) = \langle 2t, -1+3t, 3-6t\rangle, 0 \leq t \leq 1$$

for which

$$\frac{d\vec{\mathbf{r}}}{dt} = \langle 2, 3, -6\rangle$$

$$\frac{ds}{dt} = \left|\frac{d\vec{\mathbf{r}}}{dt}\right| = \sqrt{2^2+3^2+(-6)^2} = \sqrt{4+9+36} = 7$$

Therefore

$$\int_C z+2xy\,ds = \int_0^1 (z+2xy)\frac{ds}{dt}\,dt = \int_0^1 [(3-6t)+2(2t)(-1+3t)](7)\,dt$$

$$= \int_0^1 [3-10t+12t^2](7)\,dt = (7)[3t-5t^2+4t^3]_0^1 = (7)(2) = 14$$

$\diamondsuit$

**Problem 12.** Prove that $\int_C xy\,ds = \int_0^1 t^3\sqrt{1+4t^2}\,dt$ where $C$ is the parabolic arc on $y = x^2$ from $(0,0)$ to $(1,1)$.

**Solution.** We'll use the parametrization

$$\vec{\mathbf{r}}(t) = \langle t, t^2 \rangle, 0 \leq t \leq 1$$

for which

$$\frac{d\vec{\mathbf{r}}}{dt} = \langle 1, 2t \rangle$$

$$\frac{ds}{dt} = \left| \frac{d\vec{\mathbf{r}}}{dt} \right| = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2}$$

Therefore

$$\int_C xy\, ds = \int_0^1 xy \frac{ds}{dt}\, dt = \int_0^1 (t)(t^2)\sqrt{1+4t^2}\, dt = \int_0^1 t^3 \sqrt{1+4t^2}\, dt$$

◊

**Definition 13.** The **line integral of a vector field $\vec{\mathbf{F}}$** over the curve $C$ is given by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{\mathbf{F}}(\vec{\mathbf{r}}_{n,i}) \cdot \Delta\vec{\mathbf{C}}_{n,i}$$

where for each positive integer $n$ we've defined a way to approximate $C$ with $n$ vectors

$$\Delta\vec{\mathbf{C}}_{n,1}, \Delta\vec{\mathbf{C}}_{n,2}, \ldots, \Delta\vec{\mathbf{C}}_{n,n}$$

where $\vec{\mathbf{r}}_{n,i} + \Delta\vec{\mathbf{C}}_{n,i} = \vec{\mathbf{r}}_{n,i+1}$ and

$$\lim_{n\to\infty} \max(|\Delta\vec{\mathbf{C}}_{n,i}|) = 0$$

**Definition 14.** The line integral of a vector field $\vec{\mathbf{F}}$ over the curve $C$ may be computed by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$$

where $\vec{\mathbf{T}}$ yields the unit tangent vectors to the curve $C$.

**Definition 15.** If $\vec{\mathbf{r}}(t)$ is a parametrization of $C$ for $a \leq t \leq b$, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{t=a}^{t=b} \vec{\mathbf{F}} \cdot \frac{d\vec{\mathbf{r}}}{dt}\, dt$$

**Problem 16.** Prove that $\int_C \langle 2x, y - x \rangle \cdot d\vec{\mathbf{r}} = \int_0^1 23t - 7\, dt$ where $C$ is the line segment given by the vector equation $\vec{\mathbf{r}}(t) = \langle 1 - 2t, 3t \rangle$ for $0 \leq t \leq 1$.

**Solution.** We'll use the parametrization

$$\vec{\mathbf{r}}(t) = \langle 1-2t, 3t\rangle, 0 \leq t \leq 1$$

for which

$$\frac{d\vec{\mathbf{r}}}{dt} = \langle -2, 3\rangle$$

Therefore

$$\int_C \langle 2x, y-x\rangle \cdot d\vec{\mathbf{r}} = \int_0^1 \langle 2x, y-x\rangle \cdot \frac{d\vec{\mathbf{r}}}{dt}\,dt = \int_0^1 \langle 2(1-2t), 3t-(1-2t)\rangle \cdot \langle -2, 3\rangle\,dt$$

$$= \int_0^1 \langle 2-4t, 5t-1\rangle \cdot \langle -2, 3\rangle\,dt = \int_0^1 (-4+8t)+(15t-3)\,dt = \int_0^1 23t-7\,dt$$

$\Diamond$

**Remark 17.** The work done by a force vector field $\vec{\mathbf{F}}$ over the curve $C$ is given by $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

**Problem 18.** Find the work done by the force vector field $\langle -3y, 3x\rangle$ moving a particle one rotation counter-clockwise around the unit circle $x^2+y^2=1$.

**Solution.** We'll use the parametrization

$$\vec{\mathbf{r}}(t) = \langle \cos t, \sin t\rangle, 0 \leq t \leq 2\pi$$

for which

$$\frac{d\vec{\mathbf{r}}}{dt} = \langle -\sin t, \cos t\rangle$$

Therefore

$$W = \int_C \langle -3y, 3x\rangle \cdot d\vec{\mathbf{r}} = \int_0^{2\pi} \langle -3y, 3x\rangle \cdot \frac{d\vec{\mathbf{r}}}{dt}\,dt = \int_0^{2\pi} \langle -3\sin t, 3\cos t\rangle \cdot \langle -\sin t, \cos t\rangle\,dt$$

$$= \int_0^{2\pi} 3\sin^2 t + 3\cos^2 t\,dt = \int_0^{2\pi} 3\,dt = 6\pi$$

$\Diamond$

**Theorem 19.** If $C$ may be split into two curves $C_1$ and $C_2$, then

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$$

and

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

**Theorem 20.** If $-C$ is the curve $C$ oriented in the opposite direction, then

$$\int_C f\,ds = \int_{-C} f\,ds$$

and

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_{-C} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

**Problem 21.** Write a paragraph explaining why a negative appears in the previous theorem for the line integral of a vector field but not for an arclength line integral.

**Solution.** $\int_C f\,ds$ measures the area of a ribbon of height $f$ at each point above $C$, so the orientation of $C$ is irrelevant.

But $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ measures the work done by $\vec{\mathbf{F}}$ moving through the curve $C$, so if the orientation of $C$ is reversed, then the motion is in the opposite direction as before, so work is negated. $\diamond$

## 16.3 The Fundamental Theorem for Line Integrals

**Definition 22.** If $\nabla f = \vec{\mathbf{F}}$, then $f$ is a **potential function** for the **conservative field** $\vec{\mathbf{F}}$.

**Problem 23.** Prove that $\langle 2x, -3z, -3y\rangle$ is a conservative field by finding a potential function $f$ for it. Hint: such an $f$ must satisfy that $f = x^2 + \Phi_1(y,z)$, $f = -3yz + \Phi_2(x,z)$, and $f = -3yz + \Phi_3(x,y)$ for some functions $\Phi_i$. (Why?)

**Solution.** We want to solve the system

$$f_x = 2x$$
$$f_y = -3z$$
$$f_z = -3y$$

for a potential function $f$. Note $f$ must satisfy each of the anti-partial-derivatives:

$$f = x^2 + \Phi_1(y,z)$$
$$f = -3yz + \Phi_2(x,z)$$
$$f = -3yz + \Phi_3(x,y)$$

So $f = x^2 - 3yz$ satisfies all three. Therefore since $\nabla f = \vec{\mathbf{F}}$, $\vec{\mathbf{F}}$ is conservative. $\diamond$

**Theorem 24.** The Fundamental Theorem for Line Integrals: If $C$ is any smooth curve beginning at the point $A$ and ending at the point $B$, then

$$\int_C \nabla f \cdot d\vec{\mathbf{r}} = [f]_A^B = f(B) - f(A)$$

**Problem 25.** Prove that if $C$ is any smooth **closed curve** (beginning and ending at the same point), then

$$\int_C \nabla f \cdot d\vec{\mathbf{r}} = 0$$

**Solution.** Let $C$ begin and end at the point $A$.

$$\int_C \nabla f \cdot d\vec{\mathbf{r}} = [f]_A^A = f(A) - f(A) = 0$$

$\diamond$

**Problem 26.** Compute $\int_C \langle 4, z^2, 2yz\rangle \cdot d\vec{\mathbf{r}}$ where $C$ is the curve given by $\vec{\mathbf{r}}(t) = \langle 2^t, \sin(\pi t), 4t^2\rangle$ for $0 \le t \le 1$. Then compute $\int_{C'} \langle 4, z^2, 2yz\rangle \cdot d\vec{\mathbf{r}}$ where $C'$ is the line segment starting at $(1,0,0)$ and ending at $(2,0,4)$.

**Solution.** By plugging in 0 and 1 into $\vec{\mathbf{r}}(t)$, we see that both curves begin at $(1,0,0)$ and end at $(2,0,4)$.

It follows that $f = 4x + 2yz^2$ satisfies

$$f_x = 4$$

$$f_y = z^2$$

$$f_z = 2yz$$

so for both curves,

$$\int_C \langle 4, z^2, 2yz\rangle \cdot d\vec{\mathbf{r}} = \int_{C'} \langle 4, z^2, 2yz\rangle \cdot d\vec{\mathbf{r}} = [4x + 2yz^2]_{(1,0,0)}^{(2,0,4)} = [8+0] - [4+0] = 4$$

$\Diamond$

**Problem 27.** Prove that if $f$ is a potential function for the vector field $\langle P, Q, R\rangle$, then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$. (Hint: use the mixed derivative theorem.)

**Solution.** Note that $P = f_x$, $Q = f_y$, and $R = f_z$. So it follows by the mixed derivative theorem that:

$$P_y = f_{xy} = f_{yx} = Q_x$$

$$P_z = f_{xz} = f_{zx} = R_x$$

$$Q_z = f_{yz} = f_{zy} = R_y$$

$\Diamond$

**Theorem 28.** $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is a conservative vector field if and only if $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$.

**Problem 29.** Prove that $\int_C \langle ye^{xy+z}, xe^{xy+z}, e^{xy+z}\rangle \cdot d\vec{\mathbf{r}} = 0$ where $C$ is the curve given by $\vec{\mathbf{r}}(t) = \langle \frac{1}{1+t^2}, \cos t, e^{1-t^2}\rangle$ for $-1 \leq t \leq 1$.

**Solution.** By the previous theorem, we can show that $\langle ye^{xy+z}, xe^{xy+z}, e^{xy+z}\rangle$ is conservative by:

$$P_y = e^{xy+z} + xye^{xy+z} = Q_x$$

$$P_z = ye^{xy+z} = R_x$$

$$Q_z = xe^{xy+z} = R_y$$

Since $C$ starts at $\vec{\mathbf{r}}(-1) = \langle \frac{1}{2}, \cos 1, 1\rangle$ and ends at the same point $\vec{\mathbf{r}}(1) = \langle \frac{1}{2}, \cos 1, 1\rangle$, the result of Problem 25 shows that

$$\int_C \langle ye^{xy+z}, xe^{xy+z}, e^{xy+z}\rangle \cdot d\vec{\mathbf{r}} = 0$$

$\Diamond$

## 16.4 Green's Theorem

**Theorem 30.** Let $C$ be the boundary of the region $R$ in the $xy$ plane oriented counter-clockwise, and let $\vec{\mathbf{F}}$ be a two-dimensional vector field. Then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

**Problem 31.** Evaluate $\int_C \langle x^2 + y, x + y \rangle \cdot d\vec{\mathbf{r}}$ where $C$ is the boundary of the unit square oriented counter-clockwise.

**Solution.** Green's Theorem tells us that

$$\int_C \langle x^2 + y, x + y \rangle \cdot d\vec{\mathbf{r}} = \iint_R (1 - 1)\ dA = \iint_R 0\, dA = 0$$

(This also works because $C$ is a closed curve and $\langle x^2 + y, x + y \rangle$ is conservative.) ◊

**Problem 32.** Find the work done by a force vector field $\langle y, 2x \rangle$ moving an object around the boundary of the triangle with vertices $(1, 2)$, $(-1, -2)$, and $(3, -2)$ oriented clockwise.

**Solution.** Green's Theorem tells us that

$$\int_C \langle y, 2x \rangle \cdot d\vec{\mathbf{r}} = \iint_R (2 - 1)\ dA = \iint_R 1\, dA$$

Since $R$ is a triangle with base 4 and height 4, we conclude that

$$\iint_R 1\, dA = \frac{1}{2}(4)(4) = 8$$

(or we could set it up like a Type II double integral if we like doing extra work). ◊