Packet 3

Packet 3.2: Sections 15.10, 15.4, 15.8, 15.9

15.10 Change of Variables in Multiple Integrals

Remark 1. An alternate way to write the u-substitution rule from Cal I is: if $x(u)$ defines x as a function of u, and $x(u)$ transforms the interval J of u-values into the interval I of x-values, then

$$\int_I f(x)\,dx = \int_J f(x(u)) \left|\frac{dx}{du}\right| du$$

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Problem 2. Use the above alternate u-sub rule to prove that

$$\int_1^2 2xe^{x^2}\,dx = \int_1^4 e^u\,du = e^4 - e$$

Solution. $\int_1^2 2xe^{x^2}\,dx$

Let u = x^2

$du = 2x\,dx$

$u = 2^2 = 4$

$u = 1^2 = 2$

$\int_1^4 e^u\,du$

$[e^u]_1^4 = e^4 - e$

$\Diamond$

Contributors. Seay

Definition 3. A **2D transformation**

$$\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v)\rangle$$

transforms points in the uv plane to points in the xy plane.

Definition 4. The **unit square** is the square with coordinates $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

Definition 5. The **unit triangle** is the triangle with coordinates $(0,0)$, $(1,0)$, and $(1,1)$.

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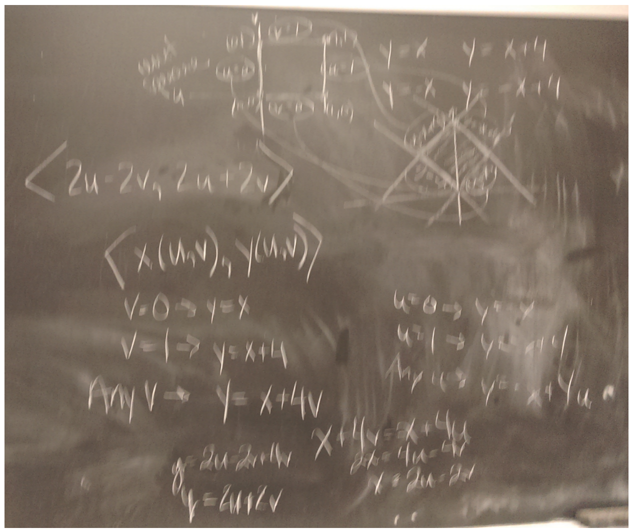
Problem 6. Show that a transformation from the unit square in the uv plane to the square with sides $y=x$, $y=x+4$, $y=-x$, and $y=-x+4$ in the xy plane could satisfy the equations $y=x+4u$ and $y=-x+4v$, and then solve this system to get the transformation $\langle x(u,v), y(u,v)\rangle$.

Solution.



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Problem 7. Find a transformation from the unit square in the uv plane to the parallelogram with vertices $(1,0)$, $(2,-1)$, $(4,0)$, and $(3,1)$ in the xy plane.

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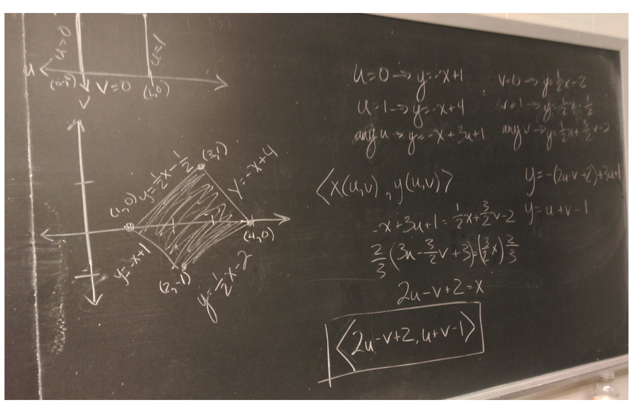
Solution.



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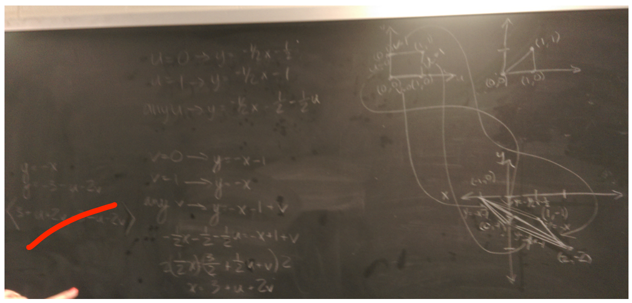
Problem 8. Find a transformation from the unit triangle in the uv plane to the triangle with vertices $(0, -1)$, $(2, -2)$, and $(-1, 0)$ in the xy plane. (Hint: complete the triangle in the xy plane to a parallelogram and then find a transformation from the unit square to that parallelogram.)

Solution.



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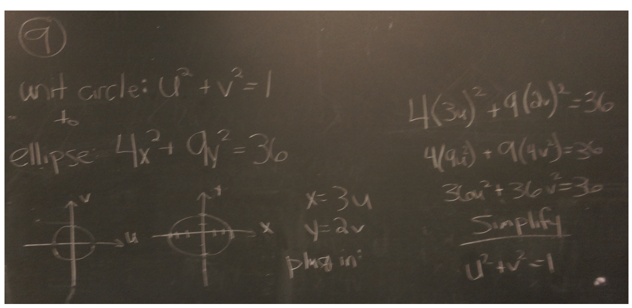
Problem 9. Find a transformation from the unit circle $u^2 + v^2 = 1$ in the uv plane to the ellipse $4x^2 + 9y^2 = 36$.

Solution.



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Contributors. Lerdo de Tejada, Willoughby, Seay, Ceasar

Definition 10. The **Jacobian** of a transformation $\vec{\mathbf{r}}(u, v) = \langle x(u, v), y(u, v)\rangle$ is given by

$$\vec{\mathbf{r}}_J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Theorem 11. If $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v)\rangle$ transforms the region G in the uv plane to the region R in the xy plane, then

$$\iint_R f(x,y)\,dA = \iint_G f(x(u,v), y(u,v))|\vec{\mathbf{r}}_J(u,v)|\,dA$$

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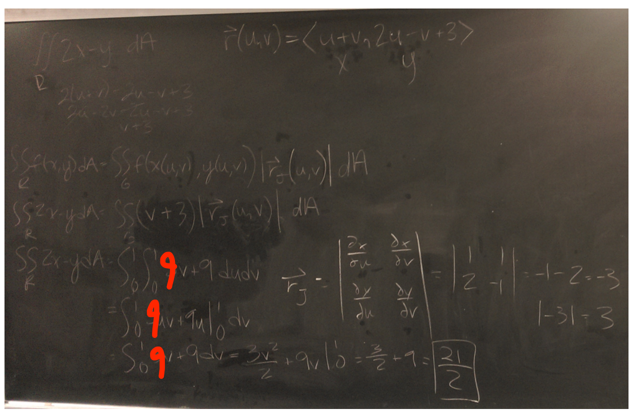
Problem 12. Evaluate $\iint_R 2x - y\,dA$ using the transformation $\vec{\mathbf{r}}(u,v) = \langle u+v, 2u-v+3\rangle$ from unit square in the uv plane into the parallelogram R with vertices $(0,3)$, $(1,5)$, $(2,4)$, and $(1,2)$ in the xy plane.

Solution.



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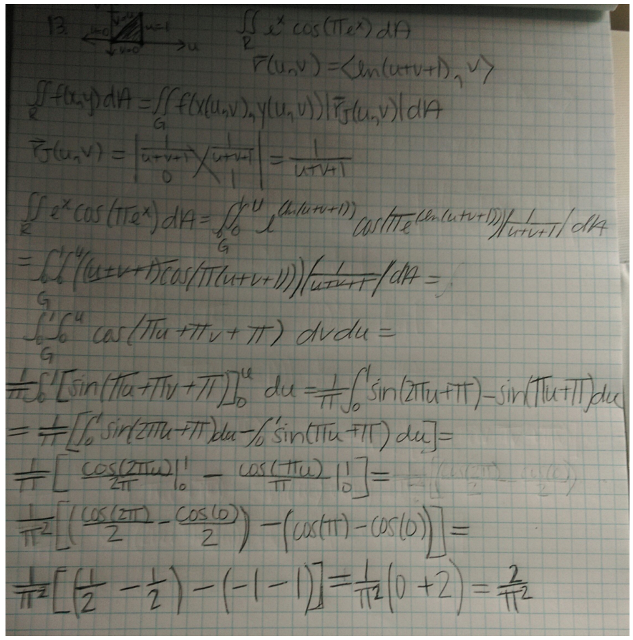
Problem 13. Evaluate $\iint_R e^x\cos(\pi e^x)\,dA$ using the transformation $\vec{\mathbf{r}}(u,v) = \langle \ln(u+v+1), v\rangle$ from the unit triangle in the uv plane into the region R bounded by $y = 0$, $y = e^x - 2$, and $y = \frac{e^x - 1}{2}$.

Solution.



13. $\iint_R e^x \cos(\pi e^x)\, dA$

$\vec{r}(u,v) = \langle \ln(u+v+1), v \rangle$

$\iint_R f(x,y)\, dA = \iint_G f(x(u,v), y(u,v)) |\vec{r}_J(u,v)|\, dA$

$\vec{r}_J(u,v) = \begin{vmatrix} \frac{1}{u+v+1} & \frac{1}{u+v+1} \\ 0 & 1 \end{vmatrix} = \frac{1}{u+v+1}$

$\iint_R e^x \cos(\pi e^x)\, dA = \iint_G{}_0^1{}_0^u e^{(\ln(u+v+1))} \cos(\pi e^{(\ln(u+v+1))}) \left|\frac{1}{u+v+1}\right| dA$

$= \iint_G{}_0^1{}_0^u (\cancel{u+v+1}) \cos(\pi(u+v+1)) \left|\frac{1}{\cancel{u+v+1}}\right| dA =$

$\iint_G{}_0^1{}_0^u \cos(\pi u + \pi v + \pi)\, dv\, du =$

$\frac{1}{\pi}\int_0^1 [\sin(\pi u + \pi v + \pi)]_0^u\, du = \frac{1}{\pi}\int_0^1 \sin(2\pi u + \pi) - \sin(\pi u + \pi)\, du$

$= \frac{1}{\pi}\left[\int_0^1 \sin(2\pi u + \pi)\, du - \int_0^1 \sin(\pi u + \pi)\, du\right] =$

$\frac{1}{\pi}\left[\frac{\cos(2\pi u)}{2\pi}\Big|_0^1 - \frac{\cos(\pi u)}{\pi}\Big|_0^1\right] =$

$\frac{1}{\pi^2}\left[\left(\frac{\cos(2\pi)}{2} - \frac{\cos(0)}{2}\right) - (\cos(\pi) - \cos(0))\right] =$

$\frac{1}{\pi^2}\left[\left(\frac{1}{2} - \frac{1}{2}\right) - (-1-1)\right] = \frac{1}{\pi^2}(0+2) = \frac{2}{\pi^2}$

$\Diamond$

Contributors. Willoughby, Lerdo de Tejada, Ceasar

Definition 14. A **3D transformation**

$$\vec{\mathbf{r}}(u,v,w) = \langle x(u,v,w), y(u,v,w), z(u,v,w) \rangle$$

transforms points in uvw space to points in xyz space.

Definition 15. The **Jacobian** of a transformation $\vec{\mathbf{r}}(u,v,w) = \langle x(u,v,w), y(u,v,w), z(u,v,w) \rangle$ is given by

$$\vec{\mathbf{r}}_J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Theorem 16. If $\vec{\mathbf{r}}(u,v,w) = \langle x(u,v,w), y(u,v,w), z(u,v,w) \rangle$ transforms the solid H in uvw space to the solid D in the xyz space, then

$$\iiint_D f(x,y,z)\, dV = \iiint_H f(x(u,v,w), y(u,v,w), z(u,v,w)) |\vec{\mathbf{r}}_J(u,v,w)|\, dV$$

15.4 Double Integrals in Polar Coordinates

Theorem 17. The polar coordinate transformation

$$\vec{\mathbf{r}}(r,\theta) = \langle r\cos\theta, r\sin\theta \rangle$$

from polar G into Cartesian R yields

$$\iint\limits_R f(x,y)\,dA = \iint\limits_G f(r\cos\theta, r\sin\theta)|r|\,dA$$

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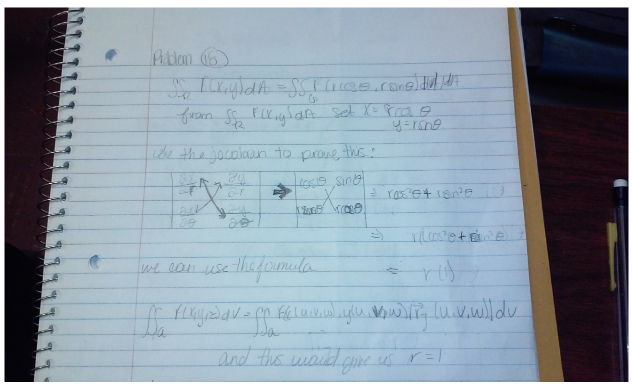
Problem 18. Prove the previous theorem.

Solution.


Problem (18)

$\iint_R f(x,y)\,dA = \iint_G f(r\cos\theta, r\sin\theta)\,dA$

from $\iint_R f(x,y)\,dA$ set $x = r\cos\theta$, $y = r\sin\theta$

use the jacobian to prove this:

$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} \Rightarrow \begin{vmatrix} \cos\theta & \sin\theta \\ r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$

$\Rightarrow r(\cos^2\theta + \sin^2\theta)$

we can use the formula $= r(1)$

$\iiint_a f(x,y,z)\,dV = \iiint_a f(x(u,v,w), y(u,v,w)) |\vec{r}_J(u,v,w)|\,dv$

and this would give us $r = 1$

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Contributors. Beverly Ceasar

Theorem 19. If the region R in the xy plane is described with polar coordinates, and is bounded by the inside/outside curves $0 \le g(\theta) \le r \le h(\theta)$ and lines $\alpha \le \theta \le \beta$, then

$$\iint\limits_R f(x,y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

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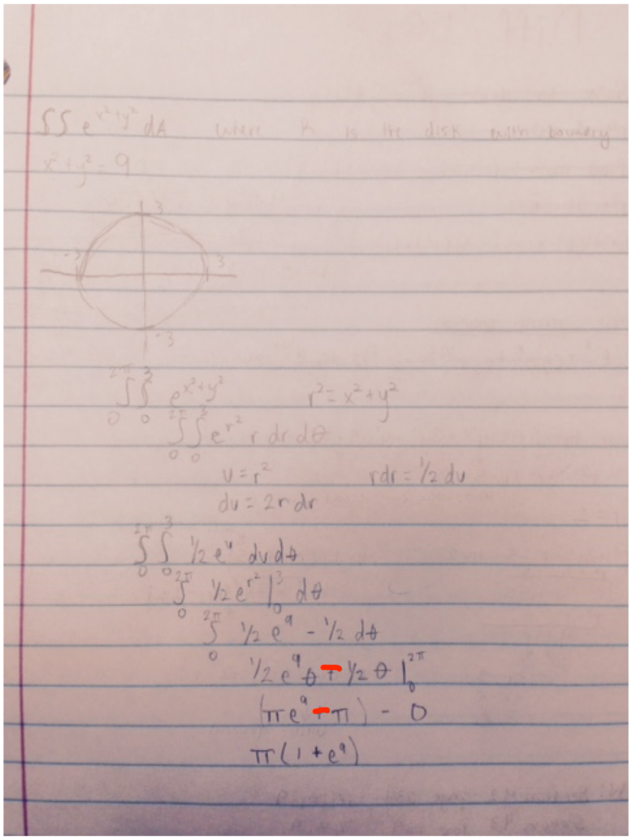
Problem 20. Evaluate $\iint_R e^{x^2+y^2}\,dA$ where R is the disk with boundary $x^2+y^2=9$.

Solution. 

$\iint e^{x^2+y^2}\,dA$ where R is the disk with boundary $x^2+y^2=9$

$\int_0^{2\pi}\int_0^3 e^{x^2+y^2}$ $\quad r^2=x^2+y^2$

$\int_0^{2\pi}\int_0^3 e^{r^2}\, r\,dr\,d\theta$

$u=r^2 \quad r\,dr = \frac{1}{2}du$

$du = 2r\,dr$

$\int_0^{2\pi}\int_0^3 \frac{1}{2}e^u\,du\,d\theta$

$\int_0^{2\pi} \frac{1}{2}e^{r^2}\Big|_0^3\,d\theta$

$\int_0^{2\pi} \frac{1}{2}e^9 - \frac{1}{2}\,d\theta$

$\frac{1}{2}e^9\theta - \frac{1}{2}\theta\Big|_0^{2\pi}$

$(\pi e^9 - \pi) - 0$

$\pi(1+e^9)$

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Contributors. Andre Samblanet

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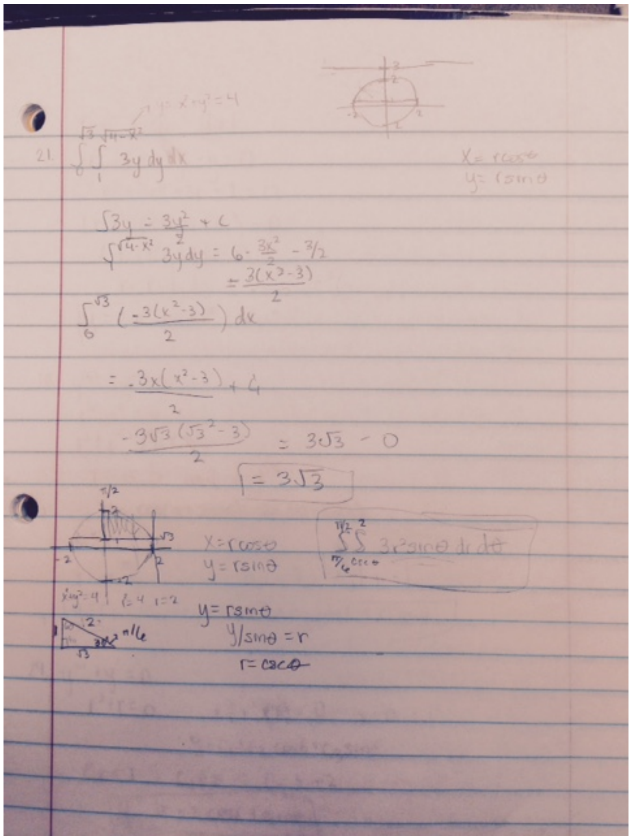
Problem 21. Prove that

$$\int_0^{\sqrt{3}}\int_1^{\sqrt{4-x^2}} 3y\,dy\,dx = \int_{\pi/6}^{\pi/2}\int_{\csc\theta}^{2} 3r^2\sin\theta\,dr\,d\theta = 3\sqrt{3}$$

Solution.  ◊

Contributors. Andre Samblanet

15.8 Triple Integrals in Cylindrical Coordinates

Theorem 22. The cylindrical coordinate transformation

$$\vec{\mathbf{r}}(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$$

from cylindrical H into Cartesian D yields

$$\iiint\limits_D f(x,y,z)\,dV = \iiint\limits_H f(r\cos\theta, r\sin\theta, z)|r|\,dV$$

Remark 23. This is equivalent to using the fact that

$$\iiint_D f(x,y,z)\,dV = \iint_R \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\right] dA$$

and then interpreting the shadow R in the xy plane with polar coordinates.

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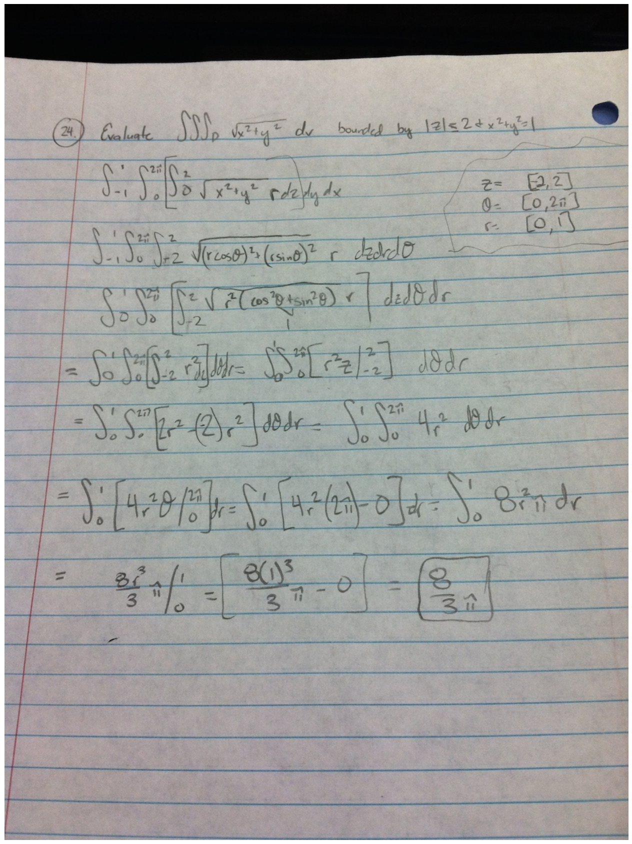
Problem 24. Evaluate $\iiint_D \sqrt{x^2+y^2}\,dV$ where D is the right circular cylinder bounded by $|z| \leq 2$ and $x^2+y^2=1$.

Solution.



24) Evaluate $\iiint_D \sqrt{x^2+y^2}\, dV$ bounded by $|z| \le 2$ & $x^2+y^2=1$

$\int_{-1}^{1}\int_0^{2\pi}\left[\int_0^2 \sqrt{x^2+y^2}\, r\,dz\right] dy\,dx$

$z = [-2, 2]$
$\theta = [0, 2\pi]$
$r = [0, 1]$

$\int_{-1}^{1}\int_0^{2\pi}\int_{-2}^{2} \sqrt{(r\cos\theta)^2+(r\sin\theta)^2}\; r\; dz\,dr\,d\theta$

$\int_0^1\int_0^{2\pi}\left[\int_{-2}^{2} \sqrt{r^2(\underbrace{\cos^2\theta+\sin^2\theta}_{1})}\; r\right] dz\,d\theta\,dr$

$= \int_0^1\int_0^{2\pi}\left[\int_{-2}^{2} r^2\,dz\right] d\theta\,dr = \int_0^1\int_0^{2\pi}\left[r^2 z\Big|_{-2}^{2}\right] d\theta\,dr$

$= \int_0^1\int_0^{2\pi}\left[2r^2-(-2)r^2\right] d\theta\,dr = \int_0^1\int_0^{2\pi} 4r^2\, d\theta\,dr$

$= \int_0^1\left[4r^2\theta\Big|_0^{2\pi}\right]dr = \int_0^1\left[4r^2(2\pi)-0\right]dr = \int_0^1 8r^2\pi\,dr$

$= \frac{8r^3}{3}\pi\Big|_0^1 = \left[\frac{8(1)^3}{3}\pi - 0\right] = \boxed{\frac{8}{3}\pi}$

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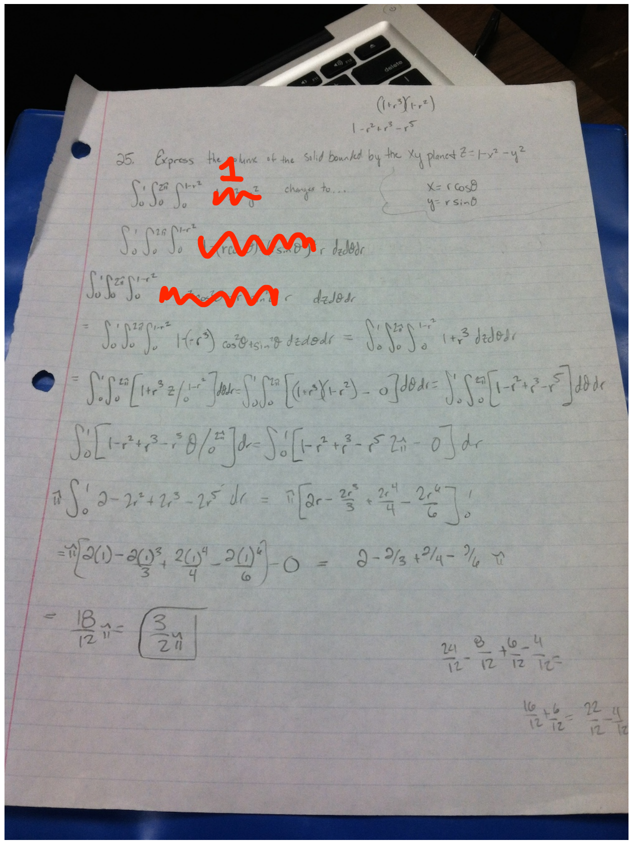
Problem 25. Express the volume of the of the solid bounded by the xy plane and $z = 1 - x^2 - y^2$ as a triple integral of the variables r, θ, z.

Solution.



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Contributors. Lerdo de Tejada

15.9 Triple Integrals in Spherical Coordinates

Theorem 26. The spherical coordinate transformation

$$\vec{\mathbf{r}}(\rho,\phi,\theta) = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$$

from spherical H into Cartesian D yields

$$\iiint_D f(x,y,z)\,dV = \iiint_H f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2|\sin\phi|\,dV$$

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Problem 27. Prove the previous theorem.

Solution.

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = -\rho^2 \sin^3\theta(\cos^2\phi + \sin^2\phi) - \rho^2 \sin^\theta \cos^2\theta(\sin^2\phi + \cos^2\phi)$$

$$= -\rho^2 \sin\theta(\sin^2\phi + \cos^2\phi) = -\rho^2 \sin\theta$$

$$dV = \rho^2 \sin\theta \, d\rho \, d\phi \, d\theta$$

Therefore the integral will become:

$$\iiint\limits_D f(x,y,z)\,dV = \iiint\limits_H f(\rho\sin\theta\cos\phi, \rho\sin\theta\sin\phi, \rho\cos\theta)\rho^2|\sin\theta|\,dV$$

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Contributors. Connor Tango

Theorem 28. If the solid D in the xyz plane is described with spherical coordinates, and is bounded by the inside/outside surfaces $h_1(\phi,\theta) \le \rho \le h_2(\phi,\theta)$, conical surfaces $0 \le g_1(\theta) \le \phi \le g_2(\theta)$, and planes $\alpha \le \theta \le \beta$, then

$$\iiint\limits_D f(x,y,z)\,dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(\phi,\theta)}^{h_2(\phi,\theta)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

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Problem 29. Prove that the volume of a sphere of radius a has volume

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx = \frac{4}{3}\pi a^3$$

Solution.

$$-a \le z \le a$$

$$-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$$

$$-\sqrt{a^2-x^2-y^2} \le x \le \sqrt{a^2-x^2-y^2}$$

Converting to spherical coordinates:

$$0 \le \rho \le a$$

$$0 \le \phi \le \pi$$

$$0 \le \theta \le 2\pi$$

$$\int_0^{2\pi} \int_0^\pi \int_0^a \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$\int_0^a \rho^2 \sin\phi \, d\rho = \frac{a^3}{3}\sin\phi$$

$$\int_0^\pi \frac{a^3}{3}\sin\phi \, d\phi = \frac{2a^3}{3}$$

$$\int_0^2 \pi\frac{2a^3}{3} = \frac{4}{3}\pi a^3$$

Therefore:

$$\int_0^{2\pi}\int_0^\pi\int_0^a \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{4}{3}\pi a^3$$

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Contributors. Connor Tango

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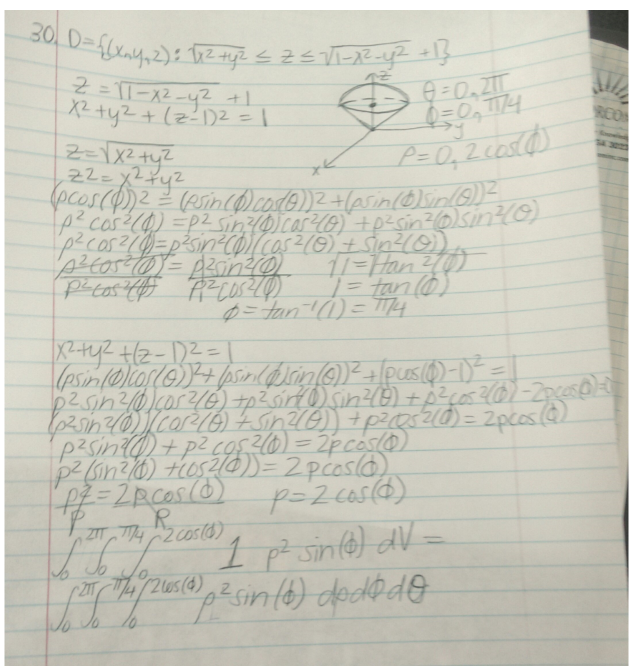
Problem 30. Express the volume of the "ice cream cone" shaped solid

$$D = \{(x,y,z) : \sqrt{x^2+y^2} \leq z \leq \sqrt{1-x^2-y^2}+1\}$$

as a triple iterated integral of the variables ρ, ϕ, θ.

Solution.



30. $D=\{(x,y,z): \sqrt{x^2+y^2} \leq z \leq \sqrt{1-x^2-y^2}+1\}$

$z = \sqrt{1-x^2-y^2}+1$
$x^2+y^2+(z-1)^2 = 1$

$\theta = 0, 2\pi$
$\phi = 0, \pi/4$
$\rho = 0, 2\cos(\phi)$

$z = \sqrt{x^2+y^2}$
$z^2 = x^2+y^2$
$(\rho\cos(\phi))^2 = (\rho\sin(\phi)\cos(\theta))^2 + (\rho\sin(\phi)\sin(\theta))^2$
$\rho^2\cos^2(\phi) = \rho^2\sin^2(\phi)\cos^2(\theta) + \rho^2\sin^2(\phi)\sin^2(\theta)$
$\rho^2\cos^2(\phi) = \rho^2\sin^2(\phi)(\cos^2(\theta)+\sin^2(\theta))$
$\frac{\rho^2\cos^2(\phi)}{\rho^2\cos^2(\phi)} = \frac{\rho^2\sin^2(\phi)}{\rho^2\cos^2(\phi)}$ $\quad \sqrt{1} = \sqrt{\tan^2(\phi)}$
$1 = \tan(\phi)$
$\phi = \tan^{-1}(1) = \pi/4$

$x^2+y^2+(z-1)^2 = 1$
$(\rho\sin(\phi)\cos(\theta))^2 + (\rho\sin(\phi)\sin(\theta))^2 + (\rho\cos(\phi)-1)^2 = 1$
$\rho^2\sin^2(\phi)\cos^2(\theta) + \rho^2\sin^2(\phi)\sin^2(\theta) + \rho^2\cos^2(\phi) - 2\rho\cos(\phi) = 0$
$(\rho^2\sin^2(\phi))(\cos^2(\theta)+\sin^2(\theta)) + \rho^2\cos^2(\phi) = 2\rho\cos(\phi)$
$\rho^2\sin^2(\phi) + \rho^2\cos^2(\phi) = 2\rho\cos(\phi)$
$\rho^2(\sin^2(\phi)+\cos^2(\phi)) = 2\rho\cos(\phi)$
$\frac{\rho^2}{\rho} = \frac{2\rho\cos(\phi)}{\rho}$ $\quad \rho = 2\cos(\phi)$

$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos(\phi)} 1 \, \rho^2\sin(\phi) \, dV =$
$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos(\phi)} \rho^2\sin(\phi) \, d\rho \, d\phi \, d\theta$

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