Packet 3

Packet 3.1: Sections 15.1-15.3 and 15.7

15.1 Double Integrals over Rectangles

Definition 1. We define the **double integral** of a function $f(x, y)$ over a region R to be

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{n,i}, y_{n,i})\Delta A_{n,i}$$

where for each positive integer n we've defined a way to partition R into n pieces

$$\Delta R_{n,1}, \Delta R_{n,2}, \ldots, \Delta R_{n,n}$$

where $\Delta R_{n,i}$ has area $\Delta A_{n,i}$, contains the point $(x_{n,i}, y_{n,i})$, and

$$\lim_{n\to\infty} \max(\Delta A_{n,i}) = 0$$

Remark 2. This basically defines the double integral to be the **Riemann sum** of a bunch of rectangular box volumes, just as the single definite integral is the Riemann sum of a bunch of rectangle areas. Therefore it represents the net volume between the curve $z = f(x, y)$ and the xy-plane above/below R.

Theorem 3. For the rectangle

$$R : a \leq x \leq b, c \leq y \leq d$$

the **Midpoint Rule** says that

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\Delta A$$

where $(\overline{x_i}, \overline{y_j})$ is the midpoint of the $i \times j$ rectangle.

Problem 4. Divide $R : 0 \leq x \leq 4, 0 \leq y \leq 2$ into four congruent pieces arranged two-by-two, and then use the midpoint rule to approximate the double integral $\iint_R 2x + 2y + 4\,dA$.

Solution. ◊

Contributors.

Problem 5. Divide $R : -2 \leq x \leq 2, 0 \leq y \leq 2$ into four congruent pieces arranged two-by-two, and then use the midpoint rule to approximate the double integral $\iint_R 12x^2y\,dA$

Solution. ◊

Contributors.

Problem 6. Divide $R : 0 \leq x \leq \pi/2, 0 \leq y \leq \pi/2$ into four congruent pieces arranged two-by-two, and then use the midpoint rule to approximate the double integral $\iint_R \cos(x+y)\,dA$

Solution. ◊

Contributors.

15.2 Iterated Integrals

Definition 7. If a solid is embedded in xyz space, and $A(x)$ is the area of that solid's cross-section for each x-value, then the solid's volume is

$$V = \int_a^b A(x)\,dx$$

Theorem 8. A double integral over a rectangle

$$R : a \leq x \leq b, c \leq y \leq d$$

can be evaluated using the **iterated integrals**:

$$\iint_R f(x,y)\,dA = \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} f(x,y)\,dy\right] dx = \int_{y=c}^{y=d} \left[\int_{x=a}^{x=b} f(x,y)\,dx\right] dy$$

Remark 9. Iterated integrals are often shortened as follows:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} f(x,y)\,dy\right] dx$$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_{y=c}^{y=d} \left[\int_{x=a}^{x=b} f(x,y)\,dx\right] dy$$

Remark 10. When evaluating iterated integrals, only the innermost d-variable acts as a variable, while other variables act as constants. Put another way, find the partial anti-derivatives.

Remark 11. The order of a double iterated integral with constant bounds may be reversed by swapping **both** the bounds of integration and the differentials dx/dy. (This will not work if there are any variables in the bounds as we'll see in the next section.)

Problem 12. Evaluate $\int_0^3 \int_2^4 xy^2 + x^3 \, dx \, dy$.

Solution. ◊

Contributors.

Problem 13. If $R : 0 \leq x \leq 4, 0 \leq y \leq 2$, then write $\iint_R 2x + 2y + 4 \, dA$ as an iterated integral. Then evaluate it, comparing its value to the approximation you found in the previous section.

Solution. ◊

Contributors.

Problem 14. If $R : -2 \leq x \leq 2, 0 \leq y \leq 2$, then write $\iint_R 12x^2 y \, dA$ as an iterated integral. Then evaluate it, comparing its value to the approximation you found in the previous section.

Solution. ◊

Contributors.

Problem 15. If $R : 0 \leq x \leq \pi/2, 0 \leq y \leq \pi/2$, then write $\iint_R \cos(x + y) \, dA$ as an iterated integral. Then evaluate it, comparing its value to the approximation you found in the previous section.

Solution. ◊

Contributors.

15.3 Double Integrals over General Regions

Theorem 16. A double integral over a **Type I** region bounded by top and bottom curves

$$R : a \leq x \leq b, g(x) \leq y \leq h(x)$$

can be evaluated using the iterated integral:

$$\iint_R f(x,y) \, dA = \int_{x=a}^{x=b} \left[\int_{y=g(x)}^{y=h(x)} f(x,y) \, dy \right] dx$$

Theorem 17. A double integral over a **Type II** region bounded by right and left curves

$$R : g(y) \leq x \leq h(y), c \leq y \leq d$$

can be evaluated using the iterated integral:

$$\iint_R f(x,y)\,dA = \int_{y=c}^{y=d} \left[\int_{x=g(y)}^{x=h(y)} f(x,y)\,dx\right] dy$$

Remark 18. Note that you *never* have variables of integration on the outside-most integral in an iterated integral.

Problem 19. Evaluate $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 6x + 30y\,dx\,dy$.

Solution. ◊

Contributors.

Problem 20. Evaluate $\iint_R 6xy + 3\,dA$ where R is the region between $x = 4 - y^2$ and $x = y^2 - 4$.

Solution. ◊

Contributors.

Problem 21. Evaluate $\iint_R 1\,dA$ where R is the triangle with vertices $(0,0)$, $(1,1)$, and $(1,2)$.

Solution. ◊

Contributors.

Remark 22. You cannot blindly switch the bounds of integration to change the order of integration for a non-rectangular region. However, if the region is both Type I and Type II, then the order of integration may be swapped by reinterpreting the region as the opposite type.

Problem 23. Evaluate the Type I iterated integral $\displaystyle\int_0^1 \int_x^1 \frac{2}{\sqrt{4+y^2}}\,dy\,dx$ by first rewriting it as a Type II iterated integral.

Solution. ◊

Contributors.

Problem 24. Evaluate the Type II iterated integral $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 3\pi\sin(\pi x^3)\,dx\,dy$ by first rewriting it as a Type I iterated integral.

Solution. ◊

Contributors.

Theorem 25. The area of a region R in the plane is

$$A = \iint\limits_R dA = \iint\limits_R 1\,dA$$

Problem 26. Express the area of the parallelogram with vertices $(-1, 2)$, $(3, 2)$, $(4, 1)$, $(0, 1)$ as a double iterated integral.

Solution. ◊

Contributors.

Definition 27. The average value of a two-variable function f over a region R is

$$\frac{1}{\text{Area of } R} \iint_R f(x, y)\,dA$$

Problem 28. Express the average value of $f(x, y) = \sin(\frac{x}{2y})$ over the triangle with vertices $(0, 1)$, $(1, 1)$, $(0, 2)$ as a double iterated integral.

Solution. ◊

Contributors.

Definition 29. The **centroid** $(\overline{x}, \overline{y})$ of a region R is the average position of all the points in R.

Problem 30. Prove that the centroid of a region R is given by the expressions:

$$\overline{x} = \frac{1}{\iint_R 1\,dA} \iint_R x\,dA$$

$$\overline{y} = \frac{1}{\iint_R 1\,dA} \iint_R y\,dA$$

(It's okay to prove one of these and say the other follows from basically the same argument.)

Solution. ◊

Contributors.

Remark 31. Not every region is Type I or Type II.

Theorem 32. If R can be split into two regions R_1, R_2, then

$$\iint\limits_{R} f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA$$

Problem 33. Express $\iint_R xe^{x+y}\,dA$ as the sum of two iterated integrals, where R is the quadrilateral with vertices at $(0,0)$, $(1,1)$, $(2,0)$, and $(1,2)$.

Solution. $\Diamond$

Contributors.

15.7 Triple Integrals

Definition 34. The **triple integral** of a function $f(x,y,z)$ over a solid D is given by

$$\iiint_D f(x,y,z)\,dV = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{n,i},y_{n,i},z_{n,i})\Delta V_{n,i}$$

where for each positive integer n we've defined a way to partition D into n pieces

$$\Delta D_{n,1}, \Delta D_{n,2}, \dots, \Delta D_{n,n}$$

where $\Delta D_{n,i}$ has volume $\Delta V_{n,i}$, contains the point $(x_{n,i},y_{n,i},z_{n,i})$, and

$$\lim_{n\to\infty}\max(\Delta V_{n,i}) = 0$$

Theorem 35. The triple integral over the rectangular box

$$D : a_1 \le x \le a_2, b_1 \le y \le b_2, c_1 \le z \le c_2$$

can be expressed as the iterated integrals:

$$\iiint_D f(x,y,z)\,dV = \int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_{c_1}^{c_2} f(x,y,z)\,dz\,dy\,dx$$

$$= \int_{b_1}^{b_2}\int_{c_1}^{c_2}\int_{a_1}^{a_2} f(x,y,z)\,dx\,dz\,dy = \int_{a_1}^{a_2}\int_{c_1}^{c_2}\int_{b_1}^{b_2} f(x,y,z)\,dy\,dz\,dx = \cdots$$

Problem 36. Evaluate $\displaystyle\iiint_D 8xz - y^2\,dV$ where D is the unit cube: $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

Solution. ◊

Contributors.

Theorem 37. If the solid D is bounded by the surfaces

$$h_1(x,y) \le z \le h_2(x,y)$$

and has shadow R in the xy-plane, then

$$\iiint_D f(x,y,z)\,dV = \iint_R \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\right] dA$$

Remark 38. z may be replaced with x or y by changing the orientation to let x or y be "up".

Problem 39. Evaluate $\displaystyle\int_{-1}^{1}\int_{1+y}^{2+y}\int_0^2 z\,dx\,dz\,dy$.

Solution. ◊

Contributors.

Problem 40. Express $\displaystyle\iiint_D xy^2z\,dV$ as a triple iterated integral, where D is the solid in the first octant bounded by the coordinate planes, $z = 1 - y^2$, and $x = 4$.

Solution. ◊

Contributors.

Theorem 41. The volume of a solid D in xyz space is

$$V = \iiint\limits_D dV = \iiint\limits_D 1\,dV$$

Problem 42. Express the volume of the pyramid with vertices $(0,0,0)$, $(3,0,0)$, $(0,2,0)$, and $(0,0,1)$ as a triple iterated integral.

Solution. ◊

Contributors.

Definition 43. The average value of a three-variable function f over a solid D is

$$\frac{1}{\text{Volume of } D}\iiint_D f(x,y,z)\,dV$$

Problem 44. Express the average value of the function $f(x,y,z) = z + xy$ over the solid bounded by the surfaces $z = 4 - x^2 - y^2$ and $z = 4x^2 + 4y^2 - 16$.

Solution. ◊

Contributors.

Theorem 45. If D can be split into two solids D_1, D_2, then

$$\iiint_D f(x,y,z)\,dV = \iiint_{D_1} f(x,y,z)\,dV + \iiint_{D_2} f(x,y,z)\,dV$$