Packet 2

Part 3: Sections 14.4-14.6

14.4 Tangent Planes and Linear Approximations

Definition 1. A **normal vector** to a surface is a vector normal to any vector tangent to a curve on the surface.

Theorem 2. Let $f(x, y)$ be a function of two variables with continuous partial derivatives, and let (a, b) be a point in the interior of f's domain. Then $\langle f_x(a, b), f_y(a, b), -1\rangle$ is normal to the surface at the point $(a, b, f(a, b))$.

Problem 3. OPTIONAL. Prove the previous theorem by using the curves $\vec{\mathbf{r}}(t) = \langle a, t, f(a, t)\rangle$ and $\vec{\mathbf{q}}(t) = \langle t, b, f(t, b)\rangle$ to yield the tangent vectors $\langle 0, 1, f_y(a, b)\rangle$ and $\langle 1, 0, f_x(a, b)\rangle$.

Solution. ◊

Definition 4. The **tangent plane** to a surface at a point is the plane passing through that point sharing the same normal vectors as the surface.

Theorem 5. The tangent plane to the surface $z = f(x, y)$ above the point (a, b) is given by the equation

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Problem 6. Prove the previous theorem.

Solution. ◊

Problem 7. Find an equation for the plane tangent to the surface $z = 4x^2 + y^2$ above the point $(1, -1)$.

Solution. ◊

Definition 8. The **linearization** $L(x, y)$ of a function $f(x, y)$ at the point (a, b) is given by the formula:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Definition 9. A function f is **differentiable** at a point if its linearization at that point approximates the value of the function nearby.

Remark 10. Basically, a differentiable function is one which looks similar to its tangent planes when zoomed in sufficiently far.

Problem 11. Approximate the value of the differentiable function $f(x, y) = 4xy + 3y^2$ at $(1.1, -2.05)$ by using its linearization at the point $(1, -2)$. Then use a calculator to approximate $f(1.1, -2.05)$.

Solution. ◊

14.5 The Chain Rule

Definition 12. The **gradient** of a multi-variable function is the vector containing all its partial derivatives:

$$\nabla f = \langle f_x, f_y \rangle$$

$$\nabla g = \langle g_x, g_y, g_z \rangle$$

Problem 13. Compute the gradient of the function $f(x, y, z) = 4x \cos z - y^2$. Then compute its value at the point $(1, -2, 0)$.

Solution. ◊

Remark 14. If $f(P)$ is a function of multiple variables, and $\vec{\mathbf{r}}(t)$ is a vector function of t, then $f(\vec{\mathbf{r}}(t))$ is a function of t.

Problem 15. Let $f(x, y) = x^2 y + 3y^2$ and $\vec{\mathbf{r}}(t) = \langle x(t), y(t) \rangle = \langle t + 1, \sqrt{t} \rangle$. Write $f(\vec{\mathbf{r}}(t))$ in terms of t only, then compute $\frac{df}{dt}$.

Solution. ◊

Remark 16. The chain rule for single-variable functions may be written as

$$\frac{d}{dx}\left[f(u(x))\right] = \frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = f'(u(x))u'(x)$$

Theorem 17. Let $f(P)$ be a function of multiple variables and $\vec{\mathbf{r}}(t)$ be a function of t. Then the derivative of f with respect to t may be computed using the **Chain Rule**:

$$\frac{d}{dt}\left[f(\vec{\mathbf{r}}(t))\right] = \frac{df}{dt} = \nabla f \cdot \frac{d\vec{\mathbf{r}}}{dt}$$

Problem 18. Let f and $\vec{\mathbf{r}}$ be defined as in the previous problem. Use the Chain Rule to compute $\frac{df}{dt}$.

Solution. $\Diamond$

Problem 19. Let $f(x, y, z) = xyz^2$, $x(t) = 2t+1$, $y(t) = t^2+1$, and $z(t) = 1-t^3$. Compute $\frac{df}{dt}$ at $t = 1$.

Solution. $\Diamond$

Theorem 20. Suppose $f(x, y) = c$ defines y as a function of x. Then

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

Problem 21. Prove the previous theorem. (Part of the solution has been provided for you.)

Solution. Let $y(x)$ be the function defined by $f(x, y(x)) = c$, and then let $t = x$. It follows that $f(t, y(t)) = f(\vec{\mathbf{r}}(t)) = c$, so by the Chain Rule,

$$\frac{d}{dt}[f(\vec{\mathbf{r}}(t))] = \frac{d}{dt}[c]$$

$$\ldots$$

Since $\frac{dy}{dt} = -\frac{f_x}{f_y}$ and $t = x$, we conclude that $\frac{dy}{dx} = -\frac{f_x}{f_y}$. $\Diamond$

Problem 22. Find the rate of change $\frac{dy}{dx}$ for $xy^2 = 3x - 2y$ at $(-1, 3)$.

Solution. $\Diamond$

14.6 Directional Derivatives and the Gradient Vector

Definition 23. Let $\vec{\mathbf{u}}$ be a direction. The **derivative of f in the direction $\vec{\mathbf{u}}$**, denoted $D_{\vec{\mathbf{u}}}f$, is given by $\frac{df}{ds}$ where s is the arclength parameter for the line oriented in the direction $\vec{\mathbf{u}}$.

Theorem 24. The directional derivative is the dot product of the gradient vector and $\vec{\mathbf{u}}$:

$$D_{\vec{\mathbf{u}}}f = \nabla f \cdot \vec{\mathbf{u}}$$

Remark 25. The proof of the previous theorem follows from the fact that if $\vec{\mathbf{r}}$ is the line oriented in the direction $\vec{\mathbf{u}}$, then $\frac{df}{ds} = \nabla f \cdot \frac{d\vec{\mathbf{r}}}{ds}$ and $\frac{d\vec{\mathbf{r}}}{ds} = \vec{\mathbf{u}}$.

Problem 26. Find the rate of change of $f(x, y, z) = xz^3+3yz$ in the direction $\vec{\mathbf{u}} = \langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \rangle$ at the point $P_0 = (-2, 0, 1)$.

Solution. $\Diamond$

Problem 27. Find the rate of change of $f(x, y) = xy^2 + 3y$ in the direction of $\vec{\mathbf{A}} = \langle 2, 2 \rangle$ at the point $P_0 = (2, 0)$. (Note that $\vec{\mathbf{A}}$ isn't a unit vector, so you'll need to find its direction first.)

Solution. ◊

Problem 28. Show that the rate of change of f in the direction of $\widehat{\mathbf{j}}$ is same thing as the partial derivative of f with respect to y.

Solution. ◊