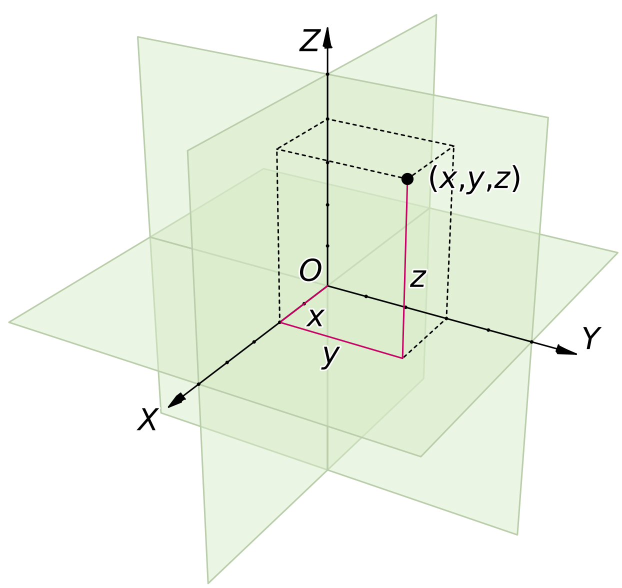
Packet 1

Sections 12.1-13.2 INSTRUCTOR SOLUTIONS

12.1 Two and Three Dimensional Space

Definition 1. Let $\mathbb{R}$ be the collection of real numbers, let $\mathbb{R}^2$ be the collection of all **ordered pairs** of real numbers, and let $\mathbb{R}^3$ be the collection of all **ordered triples** of real numbers.

$\mathbb{R}$ is known as the **real line**, $\mathbb{R}^2$ is known as the **real plane** or the xy**-plane**, and $\mathbb{R}^3$ is known as **real (3D) space** or xyz**-space**.



Definition 2. The **distance** between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in $\mathbb{R}^2$ is given by the formula

$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The **distance** between two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in $\mathbb{R}^3$ is given by the formula

$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Problem 3. Plot and find the distance between the points $(-2, 6)$ and $(3, -6)$.

Solution. The distance between $P = (-2, 6)$ and $Q = (3, -6)$ is given by the formula

$$d(P, Q) = \sqrt{(3 - (-2))^2 + (-6 - 6)^2} = 13$$

$\Diamond$

Problem 4. Plot and find the distance between the points $(0, 0, 0)$ and $(4, 2, 4)$.

Solution. The distance between $P = (0, 0, 0)$ and $Q = (4, 2, 4)$ is given by the formula

$$d(P, Q) = \sqrt{(4 - 0)^2 + (2 - 0)^2 + (4 - 0)^2} = 6$$

$\Diamond$

Problem 5. Plot and find the distance between the points $(3, 7, -2)$ and $(-1, 7, 1)$.

Solution. The distance between $P = (3, 7, -2)$ and $Q = (-1, 7, 1)$ is given by the formula

$$d(P, Q) = \sqrt{(-1 - 3)^2 + (7 - 7)^2 + (1 - (-2))^2} = 5$$

$\Diamond$

Problem 6. Plot and find the distance between the points $(8, 2, 1)$ and $(4, -2, 7)$.

Solution. The distance between $P = (8, 2, 1)$ and $Q = (4, -2, 7)$ is given by the formula

$$d(P, Q) = \sqrt{(4 - 8)^2 + (-2 - 2)^2 + (7 - 1)^2} = 2\sqrt{17}$$

$\Diamond$

Definition 7. **Simple lines** in $\mathbb{R}^2$ are given by the relations $x = a$, and $y = b$ for real numbers a, b.

Simple planes in $\mathbb{R}^3$ are given by the relations $x = a$, $y = b$, $z = c$ for real numbers a, b, c.

Definition 8. A **circle** in $\mathbb{R}^2$ is the set of all points a fixed distance (called its **radius**) from a fixed point (called its **center**). For a center (a, b) and radius r, the equation for a circle is

$$(x - a)^2 + (y - b)^2 = r^2$$

A **sphere** in $\mathbb{R}^3$ is the set of all points a fixed distance (called its **radius**) from a fixed point (called its **center**). For a center (a, b, c) and radius r, the equation for a sphere is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

Problem 9. Plot the curve $x = 3$ in the xy-plane and the surface $x = 3$ in xyz-space.

Solution. ♢

Problem 10. Plot the curve $y = -1$ in the xy-plane and and the surface $y = -1$ in xyz-space.

Solution. ♢

Problem 11. Plot the surface $z = 0$ in xyz-space.

Solution. ♢

Problem 12. Plot the curve $(x-2)^2 + (y+1)^2 = 9$ in the xy-plane.

Solution. ♢

Problem 13. Plot the surface $x^2 + y^2 + z^2 = 4$ in xyz-space.

Solution. ♢

Problem 14. Plot the curve $x^2 + (y-1)^2 + z^2 = 1$ in xyz-space.

Solution. ♢

Textbook Practice Problems: Section 12.1 *numbers* 4, 6, 7, 8, 10, 11, 12, 14, 15, 16

12.2 Vectors

Definition 15 (Vector)**.** A **vector** $\vec{\mathbf{v}}$ is a mathematical object that stores a **magnitude** (a nonnegative real number often thought of as length) and **direction**. Two vectors are **equal** if and only if they have the same magnitude and direction.

Definition 16. The **zero vector** $\vec{\mathbf{0}}$ has zero magnitude and no direction. (This is the only vector without a direction.)

Definition 17. For a given point $P = (a, b)$ in $\mathbb{R}^2$, its **position vector** is given by $\vec{\mathbf{P}} = \langle a, b\rangle$: the vector from the origin $(0, 0)$ to the point $P = (a, b)$.

For a given point $P = (a, b, c)$ in $\mathbb{R}^3$, its **position vector** is given by $\vec{\mathbf{P}} = \langle a, b, c\rangle$: the vector from the origin $(0, 0, 0)$ to the point $P = (a, b, c)$.

Theorem 18. Two vectors are equal if and only if they share the same magnitude and direction as a common position vector.

Definition 19. Since all vectors are equal to some position vector $\langle a, b\rangle$ or $\langle a, b, c\rangle$, we usually define vectors by a position vector written in this **component form**. Since the component form of a vector stores the same information as a point, we will use both interchangeably, that is, $\langle a, b\rangle = (a, b) \in \mathbb{R}^2$ and $\langle a, b, c\rangle = (a, b, c) \in \mathbb{R}^3$ (although we usually sketch them differently).

Problem 20. Plot the point $(1, 3)$ and the position vector $\langle 1, 3\rangle$ in the xy-plane.

Solution. ◊

Problem 21. Plot the point $(-2, 5)$ and the position vector $\langle -2, 5\rangle$ in the xy-plane.

Solution. ◊

Problem 22. Plot the point $(1, 1, -3)$ and the position vector $\langle 1, 1, -3\rangle$ in xyz-space.

Solution. ◊

Problem 23. Plot the point $(0, 5, 0)$ and the position vector $\langle 0, 5, 0\rangle$ in xyz-space.

Solution. ◊

Definition 24. Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$. Then the vector with initial point P and terminal point Q is defined as

$$\overrightarrow{\mathbf{PQ}} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$$

Problem 25. Plot $P = (1, 3)$ and $Q = (-3, 6)$ in the xy-plane. Then compute and plot the vector $\overrightarrow{\mathbf{PQ}}$.

Solution. The vector $\overrightarrow{\mathbf{PQ}}$ is given by

$$\overrightarrow{\mathbf{PQ}} = \langle 6 - 3, -3 - 1\rangle = \langle 3, -4\rangle$$

◊

Problem 26. Plot $P = (3, 1)$ and $Q = (0, -2)$ in the xy-plane. Then compute and plot the vector $\overrightarrow{\mathbf{PQ}}$.

Solution. The vector $\overrightarrow{\mathbf{PQ}}$ is given by

$$\overrightarrow{\mathbf{PQ}} = \langle 0 - 3, -2 - 1\rangle = \langle -3, -3\rangle$$

◊

Problem 27. Plot $P = (1, 1, 1)$ and $Q = (-3, -1, 3)$ in xyz-space. Then compute and plot the vector $\overrightarrow{\mathbf{PQ}}$.

Solution. The vector $\overrightarrow{\mathbf{PQ}}$ is given by

$$\overrightarrow{\mathbf{PQ}} = \langle -3 - 1, -1 - 1, 3 - 1\rangle = \langle -4, -2, 2\rangle$$

◊

Problem 28. Plot $P = (-2, 0, 3)$ and $Q = (1, 3, -3)$ in xyz-space. Then compute and plot the vector $\overrightarrow{\mathbf{PQ}}$.

Solution. The vector $\overrightarrow{\mathbf{PQ}}$ is given by

$$\overrightarrow{\mathbf{PQ}} = \langle 1-(-2), 3-0, -3-3\rangle = \langle -3, 3, -6\rangle$$

$\diamondsuit$

Definition 29. The magnitude $|\vec{\mathbf{v}}|$ of a vector $\vec{\mathbf{v}}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is the distance between its initial and terminal points.

Theorem 30. The magnitude of $\vec{\mathbf{v}} = \langle a, b\rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{a^2+b^2}$$

The magnitude of $\vec{\mathbf{v}} = \langle a, b, c\rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{a^2+b^2+c^2}$$

Problem 31. Evaluate the magnitude of the position vector $\langle 5, 5\rangle$.

Solution. The magnitude of $\vec{\mathbf{v}} = \langle 5, 5\rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{5^2+5^2} = 5\sqrt{2}$$

$\diamondsuit$

Problem 32. Evaluate the magnitude of the position vector $\langle -4, 3\rangle$.

Solution. The magnitude of $\vec{\mathbf{v}} = \langle -4, 3\rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{(-4)^2+3^2} = 5$$

$\diamondsuit$

Problem 33. Evaluate the magnitude of the position vector $\langle 12, -5\rangle$.

Solution. The magnitude of $\vec{\mathbf{v}} = \langle 12, -5\rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{12^2+(-5)^2} = 13$$

$\diamondsuit$

Problem 34. Evaluate the magnitude of the position vector $\langle 3, 1, -2\rangle$.

Solution. The magnitude of $\vec{\mathbf{v}} = \langle 3, 1, -2\rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{3^2+1^2+(-2)^2} = \sqrt{14}$$

$\diamondsuit$

Problem 35. Evaluate the magnitude of the position vector $\langle 4, -2, -4\rangle$.

Solution. The magnitude of $\vec{\mathbf{v}} = \langle 4, -2, -4\rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{4^2 + (-2)^2 + (-4)^2} = 6$$

$\Diamond$

Problem 36. Evaluate the magnitude of the position vector $\langle 8, 0, -6\rangle$.

Solution. The magnitude of $\vec{\mathbf{v}} = \langle 8, 0, -6\rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{8^2 + 0^2 + (-6)^2} = 10$$

$\Diamond$

Definition 37. Vector addition is defined component-wise as follows for $\mathbb{R}^2$ and $\mathbb{R}^3$

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1, u_2\rangle + \langle v_1, v_2\rangle = \langle u_1 + v_1, u_2 + v_2\rangle$$

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1, u_2, u_3\rangle + \langle v_1, v_2, v_3\rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle$$

Definition 38. A **scalar** is simply a real number by itself (as opposed to a vector of real numbers).

Definition 39. Scalar multiplication of a vector is defined component-wise as follows for $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$k\vec{\mathbf{u}} = k\langle u_1, u_2\rangle = \langle ku_1, ku_2\rangle$$

$$k\vec{\mathbf{u}} = k\langle u_1, u_2, u_3\rangle = \langle ku_1, ku_2, ku_3\rangle$$

Problem 40. Compute and plot $\vec{\mathbf{u}} = \langle 1, -3\rangle$, $\vec{\mathbf{v}} = \langle 3, 1\rangle$ and $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ in the xy-plane.

Solution. The vector $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ is given by

$$\langle 1, -3\rangle + \langle 3, 1\rangle = \langle 1 + 3, -3 + 1\rangle = \langle 4, -2\rangle$$

$\Diamond$

Problem 41. Compute and plot $\vec{\mathbf{u}} = \langle 2, 0, 1\rangle$, $\vec{\mathbf{v}} = \langle -2, 4, 2\rangle$ and $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ in xyz-space.

Solution. The vector $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ is given by

$$\langle 2, 0, 1\rangle + \langle -2, 4, 2\rangle = \langle 2 - 2, 0 + 4, 1 + 2\rangle = \langle 0, 4, 3\rangle$$

$\Diamond$

Problem 42. Compute and plot $\vec{\mathbf{u}} = \langle 8, -2\rangle$ and $\frac{1}{2}\vec{\mathbf{u}}$ in the xy-plane.

Solution. The vector $\frac{1}{2}\vec{\mathbf{u}}$ is given by

$$\frac{1}{2}\langle 8,-2\rangle = \left\langle \frac{1}{2}8, \frac{1}{2}(-2)\right\rangle = \langle 4,-1\rangle$$

$\diamond$

Problem 43. Compute and plot $\vec{\mathbf{u}} = \langle 5,3,-1\rangle$ and $3\vec{\mathbf{u}}$ in xyz-space.

Solution. The vector $3\vec{\mathbf{u}}$ is given by

$$3\langle 5,3,-1\rangle = \langle 3(5), 3(3), 3(-1)\rangle = \langle 15, 9, -3\rangle$$

$\diamond$

Definition 44. A vector $\vec{\mathbf{v}}$ is a **unit vector** if $|\vec{\mathbf{v}}| = 1$.

Theorem 45. For any non-zero vector $\vec{\mathbf{v}}$, the vector

$$\frac{1}{|\vec{\mathbf{v}}|}\vec{\mathbf{v}} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

is a unit vector.

Definition 46. The **direction** of a vector $\vec{\mathbf{v}}$ is the unit vector $\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$.

Theorem 47. Any vector $\vec{\mathbf{v}}$ is the scalar product of its magnitude and direction:

$$\vec{\mathbf{v}} = |\vec{\mathbf{v}}|\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

Problem 48. Rewrite $\langle 5,5\rangle$ as the scalar product of its magnitude and direction.

Solution. The magnitude of $\vec{\mathbf{v}} = \langle 5,5\rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{5^2+5^2} = 5\sqrt{2}$$

The direction of $\vec{\mathbf{v}} = \langle 5,5\rangle$ is then given by

$$\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = \frac{\langle 5,5\rangle}{5\sqrt{2}} = \left\langle \frac{5}{5\sqrt{2}}, \frac{5}{5\sqrt{2}}\right\rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$$

Therefore

$$\langle 5,5\rangle = 5\sqrt{2}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$$

$\diamond$

Problem 49. Rewrite $\langle -4,3\rangle$ as the scalar product of its magnitude and direction.

Solution. The magnitude of $\vec{\mathbf{v}} = \langle -4, 3\rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{(-4)^2 + 3^2} = 5$$

The direction of $\vec{\mathbf{v}} = \langle -4, 3\rangle$ is then given by

$$\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = \frac{\langle -4, 3\rangle}{5} = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle$$

Therefore

$$\langle -4, 3\rangle = 5\left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle$$

◊

Problem 50. Rewrite $\langle 12, -5\rangle$ as the scalar product of its magnitude and direction.

Solution. The magnitude of $\vec{\mathbf{v}} = \langle 12, -5\rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{12^2 + (-5)^2} = 13$$

The direction of $\vec{\mathbf{v}} = \langle 12, -5\rangle$ is then given by

$$\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = \frac{\langle 12, -5\rangle}{13} = \left\langle \frac{12}{13}, -\frac{5}{13} \right\rangle$$

Therefore

$$\langle 12, -5\rangle = 13\left\langle \frac{12}{13}, -\frac{5}{13} \right\rangle$$

◊

Problem 51. Rewrite $\langle 3, 1, -2\rangle$ as the scalar product of its magnitude and direction.

Solution. The magnitude of $\vec{\mathbf{v}} = \langle 3, 1, -2\rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14}$$

The direction of $\vec{\mathbf{v}} = \langle 12, -5\rangle$ is then given by

$$\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = \frac{\langle 3, 1, -2\rangle}{\sqrt{14}} = \left\langle \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}} \right\rangle$$

Therefore

$$\langle 3, 1, -2\rangle = \sqrt{14}\left\langle \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}} \right\rangle$$

◊

Problem 52. Rewrite $\langle 4, -2, -4\rangle$ as the scalar product of its magnitude and direction.

Solution. The magnitude of $\vec{\mathbf{v}} = \langle 4, -2, -4\rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{4^2 + (-2)^2 + (-4)^2} = 6$$

The direction of $\vec{\mathbf{v}} = \langle 4, -2, -4\rangle$ is then given by

$$\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = \frac{\langle 4, -2, -4\rangle}{6} = \left\langle \frac{4}{6}, -\frac{2}{6}, -\frac{4}{6} \right\rangle = \left\langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \right\rangle$$

Therefore

$$\langle 4, -2, -4\rangle = 6\left\langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \right\rangle$$

◊

Problem 53. Rewrite $\langle 8, 0, -6\rangle$ as the scalar product of its magnitude and direction.

Solution. The magnitude of $\vec{\mathbf{v}} = \langle 8, 0, -6\rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{8^2 + 0^2 + (-6)^2} = 10$$

The direction of $\vec{\mathbf{v}} = \langle 8, 0, -6\rangle$ is then given by

$$\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = \frac{\langle 8, 0, -6\rangle}{10} = \left\langle \frac{8}{10}, \frac{0}{10}, -\frac{6}{10} \right\rangle = \left\langle \frac{4}{5}, 0, -\frac{3}{5} \right\rangle$$

Therefore

$$\langle 8, 0, -6\rangle = 10\left\langle \frac{4}{5}, 0, -\frac{3}{5} \right\rangle$$

◊

Definition 54. The **standard unit vectors** in $\mathbb{R}^2$ are $\widehat{\mathbf{i}} = \langle 1, 0\rangle$ and $\widehat{\mathbf{j}} = \langle 0, 1\rangle$, and any vector in $\mathbb{R}^2$ can be expressed in **standard unit vector form**:

$$\langle a, b\rangle = a\widehat{\mathbf{i}} + b\widehat{\mathbf{j}}$$

The **standard unit vectors** in $\mathbb{R}^3$ are $\widehat{\mathbf{i}} = \langle 1, 0, 0\rangle$, $\widehat{\mathbf{j}} = \langle 0, 1, 0\rangle$, and $\widehat{\mathbf{k}} = \langle 0, 0, 1\rangle$, and any vector in $\mathbb{R}^3$ can be expressed in **standard unit vector form**:

$$\langle a, b, c\rangle = a\widehat{\mathbf{i}} + b\widehat{\mathbf{j}} + c\widehat{\mathbf{k}}$$

Remark 55. Since the xy-plane is the the plane $z = 0$ in xyz-space, we say the points and vectors $(a, b) = (a, b, 0) = \langle a, b\rangle = \langle a, b, 0\rangle = a\widehat{\mathbf{i}} + b\widehat{\mathbf{j}} + 0\widehat{\mathbf{k}}$ are all equal.

Problem 56. Rewrite $\langle 5, 5\rangle$ in standard unit vector form.

Solution. As $\langle a, b\rangle = a\widehat{\mathbf{i}} + b\widehat{\mathbf{j}}$:

$$\langle 5, 5\rangle = 5\widehat{\mathbf{i}} + 5\widehat{\mathbf{j}}$$

◊

Problem 57. Rewrite $\langle -4, 3\rangle$ in standard unit vector form.

Solution. As $\langle a, b\rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$:

$$\langle -4, 3\rangle = -4\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$$

$\Diamond$

Problem 58. Rewrite $\langle 3, 1, -2\rangle$ in standard unit vector form.

Solution. As $\langle a, b, c\rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$:

$$\langle 3, 1, -2\rangle = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

$\Diamond$

Problem 59. Rewrite $\langle 8, 0, -6\rangle$ in standard unit vector form.

Solution. As $\langle a, b, c\rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$:

$$\langle 8, 0, -6\rangle = 8\hat{\mathbf{i}} + 0\hat{\mathbf{j}} - 6\hat{\mathbf{k}} = 8\hat{\mathbf{i}} - 6\hat{\mathbf{k}}$$

$\Diamond$

Theorem 60. The following properties hold for any two vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and scalars a, b.

- $\vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{v}} + \vec{\mathbf{u}}$
- $(\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}} = \vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}})$
- $\vec{\mathbf{u}} + \vec{\mathbf{0}} = \vec{\mathbf{u}}$
- $\vec{\mathbf{u}} + (-\vec{\mathbf{u}}) = \vec{\mathbf{0}}$
- $0\vec{\mathbf{u}} = \vec{\mathbf{0}}$
- $1\vec{\mathbf{u}} = \vec{\mathbf{u}}$
- $a(b\vec{\mathbf{u}}) = (ab)\vec{\mathbf{u}}$
- $a(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = a\vec{\mathbf{u}} + a\vec{\mathbf{v}}$
- $(a + b)\vec{\mathbf{u}} = a\vec{\mathbf{u}} + b\vec{\mathbf{u}}$

Definition 61. Vector subtraction is defined as the addition of a negative:

$$\vec{\mathbf{u}} - \vec{\mathbf{v}} = \vec{\mathbf{u}} + (-\vec{\mathbf{v}}) = \langle u_1 - v_1, u_2 - v_2\rangle$$

$$\vec{\mathbf{u}} - \vec{\mathbf{v}} = \vec{\mathbf{u}} + (-\vec{\mathbf{v}}) = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3\rangle$$

Textbook Practice Problems: Section 12.2 *numbers* 3, 5, 13, 14, 15, 19, 21, 24, 26

12.3 The Dot Product

Definition 62. Let θ be the angle between two non-zero vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$. The **dot product** $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}$ is the product of their lengths when projected into the same direction, obtained by this formula:

$$\vec{\mathbf{u}}\cdot\vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}|\cos\theta$$

Definition 63. The dot product with a zero vector is always zero:

$$\vec{\mathbf{v}}\cdot\vec{\mathbf{0}} = \vec{\mathbf{0}}\cdot\vec{\mathbf{v}} = 0$$

Theorem 64. By the Law of Cosines:

$$\vec{\mathbf{u}}\cdot\vec{\mathbf{v}} = \langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle = u_1v_1 + u_2v_2$$

$$\vec{\mathbf{u}}\cdot\vec{\mathbf{v}} = \langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$$

Definition 65. Two vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ are **orthogonal** if $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}} = 0$.

Theorem 66. Two non-zero vectors are orthogonal if the angle θ between them is $\frac{\pi}{2}$ radians.

Theorem 67. The following properties hold for any three vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ and scalar c.

- $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}} = \vec{\mathbf{v}}\cdot\vec{\mathbf{u}}$
- $(c\vec{\mathbf{u}})\cdot\vec{\mathbf{v}} = \vec{\mathbf{u}}\cdot(c\vec{\mathbf{v}}) = c(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})$
- $\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}+\vec{\mathbf{w}}) = \vec{\mathbf{u}}\cdot\vec{\mathbf{v}} + \vec{\mathbf{u}}\cdot\vec{\mathbf{w}}$
- $\vec{\mathbf{u}}\cdot\vec{\mathbf{u}} = |\vec{\mathbf{u}}|^2$

Problem 68. Compute the angle between the vectors $\vec{\mathbf{u}} = \langle 4, -3\rangle$ and $\vec{\mathbf{v}} = \langle 5, 12\rangle$.

Solution. Note that $|\vec{\mathbf{u}}| = |\langle 4, -3\rangle| = 5$ and $|\vec{\mathbf{v}}| = |\langle 5, 12\rangle| = 13$.

By the definition of the dot product:

$$\vec{\mathbf{u}}\cdot\vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}|\cos\theta = 5(13)\cos\theta = 65\cos\theta$$

By Theorem 64:

$$\vec{\mathbf{u}}\cdot\vec{\mathbf{v}} = 4(5) + (-3)(12) = -16$$

We set these equal and solve for θ as follows:

$$65\cos\theta = -16$$

$$\cos\theta = -\frac{16}{65}$$

$$\theta = \arccos\left(-\frac{16}{65}\right) \approx 1.82 \text{ (radians)} \approx 104.3^\circ$$

$\Diamond$

Problem 69. Compute the angle between the vectors $\vec{\mathbf{u}} = \langle 1, 4, 2\rangle$ and $\vec{\mathbf{v}} = \langle 4, 1, -2\rangle$.

Solution. Note that $|\vec{\mathbf{u}}| = |\langle 1,4,2\rangle| = \sqrt{21}$ and $|\vec{\mathbf{v}}| = |\langle 4,1,-2\rangle| = \sqrt{21}$.
By the definition of the dot product:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}|\cos\theta = \sqrt{21}\sqrt{21}\cos\theta = 21\cos\theta$$

By Theorem 64:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 1(4) + 4(1) + (2)(-2) = 4$$

We set these equal and solve for θ as follows:

$$21\cos\theta = 4$$

$$\cos\theta = \frac{4}{21}$$

$$\theta = \arccos\left(\frac{4}{21}\right) \approx 1.38 \text{ (radians)} \approx 79.02^\circ$$

◊

Problem 70. Compute the angle between the vectors $\vec{\mathbf{u}} = \langle 0,5,-11\rangle$ and $\vec{\mathbf{v}} = \langle 2,0,0\rangle$.

Solution. By Theorem 64:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0(2) + 5(0) + (-11)(0) = 0$$

Therefore $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ are orthogonal, and thus $\theta = \frac{\pi}{2} = 90^\circ$ by Theorem 66.
(Note: could also solve the same way as previous problems.) ◊

Definition 71. The work W done by a force vector $\vec{\mathbf{F}}$ over a displacement vector $\vec{\mathbf{D}}$ is given by

$$W = \vec{\mathbf{F}} \cdot \vec{\mathbf{D}} = |\vec{\mathbf{F}}||\vec{\mathbf{D}}|\cos\theta$$

Textbook Practice Problems: Section 12.3 *numbers* 3, 5, 6, 7, 8, 9, 10, 11, 15, 17, 21, 27, 41, 42, 44

12.4 The Cross Product

Definition 72. For any two non-parallel vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ in $\mathbb{R}^3$, the **Right-Hand Rule** gives a specific direction orthogonal to both: position $\vec{\mathbf{u}}$ with your right thumb and $\vec{\mathbf{v}}$ with your right index finger, and let your middle finger extend orthogonal to both to give this direction.

Definition 73. Let θ be the angle between two non-zero vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ in $\mathbb{R}^3$, and let $\vec{\mathbf{n}}$ be the direction given by the Right-Hand Rule. The **cross product** $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is the vector orthogonal to both which follows the Right-Hand Rule and has magnitude equal to the area of the parallelogram formed from both.

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (|\vec{\mathbf{u}}||\vec{\mathbf{v}}|\sin\theta)\vec{\mathbf{n}}$$

$$|\vec{\mathbf{u}} \times \vec{\mathbf{v}}| = |\vec{\mathbf{u}}||\vec{\mathbf{v}}|\sin\theta$$

Definition 74. The cross product with a zero vector is always the zero vector:

$$\vec{\mathbf{v}} \times \vec{\mathbf{0}} = \vec{\mathbf{0}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$$

Theorem 75. The following properties hold for any three vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ and scalars a,b.

- $(a\vec{\mathbf{u}}) \times (b\vec{\mathbf{v}}) = (ab)(\vec{\mathbf{u}} \times \vec{\mathbf{v}})$
- $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$
- $(\vec{\mathbf{v}} + \vec{\mathbf{w}}) \times \vec{\mathbf{u}} = \vec{\mathbf{v}} \times \vec{\mathbf{u}} + \vec{\mathbf{w}} \times \vec{\mathbf{u}}$
- $\vec{\mathbf{v}} \times \vec{\mathbf{u}} = -(\vec{\mathbf{u}} \times \vec{\mathbf{v}})$

Definition 76. Two vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ are **parallel** if $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = 0$.

Theorem 77. Two non-zero vectors are parallel if the angle θ between them is 0 or π radians.

Definition 78. The cross products of the standard unit vectors are given as follows:

- $\widehat{\mathbf{i}} \times \widehat{\mathbf{j}} = \widehat{\mathbf{k}}$
- $\widehat{\mathbf{j}} \times \widehat{\mathbf{k}} = \widehat{\mathbf{i}}$
- $\widehat{\mathbf{k}} \times \widehat{\mathbf{i}} = \widehat{\mathbf{j}}$

Definition 79. A **determinant** is a short hand for writing certain commonly occuring algebraic expressions:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Theorem 80. By breaking up $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ into standard unit vectors:

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle$$

Problem 81. Compute a nonzero vector normal to both $\vec{\mathbf{u}} = \langle 4, -3, 0 \rangle$ and $\vec{\mathbf{v}} = \langle 2, 6, -3 \rangle$.

Solution. The cross-product is always normal to its factors. Therefore

$$\begin{aligned}\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ 4 & -3 & 0 \\ 2 & 6 & -3 \end{vmatrix} &= \left\langle \begin{vmatrix} -3 & 0 \\ 6 & -3 \end{vmatrix}, -\begin{vmatrix} 4 & 0 \\ 2 & -3 \end{vmatrix}, \begin{vmatrix} 4 & -3 \\ 2 & 6 \end{vmatrix} \right\rangle \\ &= \langle 9 - 0, -(-12 - 0), 24 - (-6) \rangle = \langle 9, 12, 30 \rangle \end{aligned}$$

is normal to both $\vec{\mathbf{u}}, \vec{\mathbf{v}}$. ◊

Problem 82. Compute a nonzero vector normal to both $\vec{\mathbf{u}} = \langle 1, 4, 2 \rangle$ and $\vec{\mathbf{v}} = \langle 4, 1, -2 \rangle$.

Solution. The cross-product is always normal to its factors. Therefore

$$\begin{aligned}\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ 1 & 4 & 2 \\ 4 & 1 & -2 \end{vmatrix} &= \left\langle \begin{vmatrix} 4 & 2 \\ 1 & -2 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ 4 & 1 \end{vmatrix} \right\rangle \\ &= \langle -8 - 2, -(-2 - 8), 1 - 16 \rangle = \langle -10, 10, -15 \rangle \end{aligned}$$

is normal to both $\vec{\mathbf{u}}, \vec{\mathbf{v}}$. ◊

Problem 83. Compute a nonzero vector normal to both $\vec{\mathbf{u}} = \langle 0, 5, -11 \rangle$ and $\vec{\mathbf{v}} = \langle 2, 0, 0 \rangle$.

Solution. The cross-product is always normal to its factors. Therefore

$$\begin{aligned}\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ 0 & 5 & -11 \\ 2 & 0 & 0 \end{vmatrix} &= \left\langle \begin{vmatrix} 5 & -11 \\ 0 & 0 \end{vmatrix}, -\begin{vmatrix} 0 & -11 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 5 \\ 2 & 0 \end{vmatrix} \right\rangle \\ &= \langle 0 - 0, -(0 - (-22)), 0 - 10 \rangle = \langle 0, -22, -10 \rangle \end{aligned}$$

is normal to both $\vec{\mathbf{u}}, \vec{\mathbf{v}}$. ◊

Definition 84. The torque τ done by a force vector $\vec{\mathbf{F}}$ on an arm given by $\vec{\mathbf{D}}$ is given by

$$\tau = |\vec{\mathbf{F}} \times \vec{\mathbf{D}}| = |\vec{\mathbf{F}}||\vec{\mathbf{D}}| \sin\theta$$

Theorem 85. The volume of a parallelpiped determined by the vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$, is given by the **triple scalar product**

$$(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Textbook Practice Problems: Section 12.4 *numbers* 1 – 3, 17, 19, 28, 29, 33, 35

12.5 Lines and Planes in Space

Theorem 86. Let L be the line in $\mathbb{R}^2$ normal to the vector $\overrightarrow{\mathbf{N}} = \langle A, B \rangle$ and passing through the point $P_0 = (x_0, y_0)$. Then every point $P = (x, y)$ on the line L must satisfy the following equations:

$$\overrightarrow{\mathbf{N}} \cdot \overrightarrow{\mathbf{P_0 P}} = 0$$

$$A(x - x_0) + B(y - y_0) = 0$$

Let M be the plane in $\mathbb{R}^3$ normal to the vector $\overrightarrow{\mathbf{N}} = \langle A, B, C \rangle$ and passing through the point $P_0 = (x_0, y_0, z_0)$. Then every point $P = (x, y, z)$ on the plane M must satisfy the following equations:

$$\overrightarrow{\mathbf{N}} \cdot \overrightarrow{\mathbf{P_0 P}} = 0$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Problem 87. Find an equation for the line passing through $(1, -2)$ and parallel to the line with equation $2x - y = 3$. Then plot both lines.

Solution. The line with equation $2x - y = 3$ has a normal vector $\langle 2, -1 \rangle$, so any line parallel to it would have the same normal vector.

Using the point $(1, -2)$ and the normal vector $\langle 2, -1 \rangle$, this line has the equation (by Theorem 86):

$$2(x - 1) + -1(y - (-2)) = 0$$

$$2x - y = 4$$

◊

Problem 88. Find an equation for the plane passing through $(1, 3, -2)$ and normal to the vector $\langle 3, 0, 1 \rangle$. Then plot the plane and vector.

Solution. Using the point $(1, 3, -2)$ and the normal vector $\langle 3, 0, 1 \rangle$, such a plane has the equation (by Theorem 86):

$$3(x - 1) + 0(y - 3) + 1(z - (-2)) = 0$$

$$3x + z = 1$$

◊

Problem 89. Find an equation for the plane passing through $(-2, 0, 4)$, $(1, 3, 3)$, and $(0, 0, 2)$. Then plot the plane and points.

Solution. Let $P = (-2, 0, 4)$, $Q = (1, 3, 3)$, and $R = (0, 0, 2)$ denote the given points on the plane. Then $\overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}} = \langle 3, 3, -1\rangle \times \langle 2, 0, -2\rangle$ is normal to the plane.

$$\overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 3 & -1 \\ 2 & 0 & -2 \end{vmatrix} = \left\langle \begin{vmatrix} 3 & -1 \\ 0 & -2 \end{vmatrix}, -\begin{vmatrix} 3 & -1 \\ 2 & -2 \end{vmatrix}, \begin{vmatrix} 3 & 3 \\ 2 & 0 \end{vmatrix} \right\rangle$$

$$= \langle -6 - 0, -(-6 - (-2)), 0 - 6\rangle = \langle -6, 4, -6\rangle$$

Using the point $(0, 0, 2)$ (any other point would also work) and the normal vector $\langle -6, 4, -6\rangle$, such a plane has the equation (by Theorem 86):

$$-6(x - 0) + 4(y - 0) + -6(z - 2) = 0$$

$$-6x + 4y - 6z = -12$$

$$-3x + 2y - 3z = -6$$

◊

Definition 90. Parametric equations $x(t), y(t)$ for a curve in $\mathbb{R}^2$ assign a point $(x(t), y(t))$ of the curve to each value of t.

Parametric equations $x(t), y(t), z(t)$ for a curve in $\mathbb{R}^3$ assign a point $(x(t), y(t), z(t))$ of the curve to each value of t.

Problem 91. Sketch the curve given by the parametric equations $x(t) = t$ and $y(t) = t^2$.

Solution. ◊

Problem 92. Sketch the curve given by the parametric equations $x(t) = \sin t$ and $y(t) = \frac{t}{\pi}$.

Solution. ◊

Problem 93. Sketch the curve given by the parametric equations $x(t) = 1 - t$, $y(t) = 3t$, and $z(t) = 2t - 3$.

Solution. ◊

Problem 94. Sketch the curve given by the parametric equations $x(t) = -t^2$, $y(t) = 2$, and $z(t) = t$.

Solution. ♢

Theorem 95. Let L be the line in $\mathbb{R}^2$ parallel to the vector $\vec{\mathbf{v}} = \langle a, b\rangle$ and passing through the point $P_0 = (x_0, y_0)$. Then every point $P = (x, y)$ on the line L must satisfy the following vector equation for some t:

$$\vec{\mathbf{P}} = \vec{\mathbf{v}}t + \overrightarrow{\mathbf{P_0}}$$

Thus the line is given by the parametric equations

$$x(t) = at + x_0$$

$$y(t) = bt + y_0$$

Let L be the line in $\mathbb{R}^3$ parallel to the vector $\vec{\mathbf{v}} = \langle a, b, c\rangle$ and passing through the point $P_0 = (x_0, y_0, z_0)$. Then every point $P = (x, y, z)$ on the line L must satisfy the following vector equation for some t:

$$\vec{\mathbf{P}} = \vec{\mathbf{v}}t + \overrightarrow{\mathbf{P_0}}$$

Thus the line is given by the parametric equations

$$x(t) = at + x_0$$

$$y(t) = bt + y_0$$

$$z(t) = ct + z_0$$

Problem 96. Find parametric equations for the line with equation $y = -3x + 1$ in the xy plane. Then plot the line.

Solution. The y-intercept for this line is the point $(0, 1)$.

The slope of this line is -3, so it has a rise of -3 for a run of 1. Therefore $\langle 1, -3\rangle$ is a parallel vector to this line.

Parametric equations for a line passing through $(0, 1)$ and parallel to the vector $\langle 1, -3\rangle$ are (by Theorem 95):

$$x(t) = 1t + 0 = t$$

$$y(t) = -3t + 1$$

(Alternately, we could just have let $x = t$, and deduced that $y = -3(t) + 1$ by plugging in for x.) ♢

Problem 97. Find parametric equations for the line passing through $(1, 3, -2)$ and parallel to $\langle 3, 0, 1\rangle$ in xyz space. Then plot the point, vector, and line.

Solution. Parametric equations for a line passing through $(1, 3, -2)$ and parallel to the vector $\langle 3, 0, 1\rangle$ are (by Theorem 95):

$$x(t) = 3t + 1$$

$$y(t) = 0t + 3 = 3$$

$$z(t) = 1t - 2 = t - 2$$

$\Diamond$

Problem 98. Find parametric equations for the line normal to the plane with equation $x + y + 2z = 4$ and passing through $(1, 1, 1)$ in xyz space. Then plot the point, plane, and line.

Solution. The plane with equation $x + y + 2z = 4$ must be normal to the vector $\langle 1, 1, 2\rangle$ given by its coefficients. Therefore a normal line to the vector is parallel to the vector $\langle 1, 1, 2\rangle$.

Parametric equations for a line passing through $(1, 1, 1)$ and parallel to the vector $\langle 1, 1, 2\rangle$ are (by Theorem 95):

$$x(t) = 1t + 1 = t + 1$$

$$y(t) = 1t + 1 = t + 1$$

$$z(t) = 2t + 1 = 2t + 1$$

$\Diamond$

Textbook Practice Problems: Section 12.5 *numbers* 3, 4, 6, 7, 17, 19, 24, 27, 31, 32

12.6 Cylinders and Quadratic Surfaces

Definition 99. A **cylindrical surface** is a 3D surface given by an equation of two variables.

Problem 100. Plot the curve $y = x^2$ in the xy-plane and the cylindrical surface $y = x^2$ in xyz-space.

Solution. $\Diamond$

Problem 101. Plot the curve $y = \sin z$ in the yz-plane and the cylindrical surface $y = \sin z$ in xyz-space.

Solution. $\Diamond$

Problem 102. Plot the curve $z = e^x$ in the xz-plane and the cylindrical surface $z = e^x$ in xyz-space.

Solution. ◊

Definition 103. A **trace** of an equation of x, y, z is obtained by substituting a constant for one of the variables.

Definition 104. A **quadric surface** is a surface defined by a second degree equation of x, y, z.

Remark 105. Many surfaces may be identified and sketched by using the traces $x = 0$, $y = 0$, and $z = 0$.

Definition 106. An **ellipsoid** is a quadric surface with these main traces:

- Three ellipses (with parallel ellipses)

Definition 107. An **elliptical cone** is a quadric surface with these main traces:

- Two double-lines (with parallel hyperbolas)
- One point (with parallel ellipses)

Definition 108. An **elliptical paraboloid** is a quadric surface with these main traces:

- Two parabolas (with parallel parabolas)
- One point (with parallel ellipses)

Definition 109. A **hyperbolic paraboloid** is a quadric surface with these traces:

- Two parabolas (with parallel parabolas)
- One double line (with parallel hyperbolas)

Definition 110. A **hyperboloid of one sheet** is a quadric surface with these traces:

- Two hyperbolas (with parallel hyperbolas)
- One ellipsis (with parallel ellipses)

Definition 111. A **hyperboloid of two sheets** is a quadric surface with these traces:

- Two hyperbolas (with parallel hyperbolas)
- One empty trace (with parallel ellipses)

Problem 112. Plot $x^2 - y = -z^2$ and its traces in the planes $x = 0$, $y = 0$, and $z = 0$. Name the quadric surface.

Solution. Since its main traces are two parabolas and a single point, this is an elliptical paraboloid. ◊

Problem 113. Plot $y^2 + z^2 = 4 - 4x^2$ and its traces in the planes $x = 0$, $y = 0$, and $z = 0$. Name the quadric surface.

Solution. Since its main traces are three ellipses (including one circle), this is an ellipsoid. ♢

Problem 114. Plot $z^2 - 9y^2 = x^2$ and its traces in the planes $x = 0$, $y = 0$, and $z = 0$. Name the quadric surface.

Solution. Since its main traces are two double-lines and a single point, this is an elliptical cone. ♢

Problem 115. Plot $y^2 - z^2 = 4 - 4x^2$ and its traces in the planes $x = 0$, $y = 0$, and $z = 0$. Name the quadric surface.

Solution. Since its main traces are two hyperbola and an ellipse, this is a hyperboloid of one sheet. ♢

Problem 116. Plot $4x^2 - y^2 - 4z^2 = 16$ and its traces in the planes $x = 0$, $y = 0$, and $z = 0$. Name the quadric surface.

Solution. Since its main traces are two hyperbola and one empty trace, this is a hyperboloid of two sheets. ♢

Problem 117. Plot $z = y^2 - 4x^2$ and its traces in the planes $x = 0$, $y = 0$, and $z = 0$. Name the quadric surface.

Solution. Since its main traces are two parabolas and one double-line, this is a hyperbolic paraboloid. ♢

13.1 Vector Functions and Space Curves

Definition 118. A **position function** maps a moment in time to a position in 3D (or 2D) space. It may be defined with **parametric equations**

$$x = x(t), y = y(t), z = z(t)$$

or with a **vector function**

$$\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t) \rangle$$

In either case, $x(t), y(t), z(t)$ are called the **component functions** for the position function. The **domain** of a position function is defined to be the intersection of the domains of its component functions (the values for which *every* position function is well-defined).

Problem 119. Give parametric equations and the corresponding vector function which describe motion on the curve $y = x^2$.

Solution. Suppose we let $x(t) = t$ and $y(t) = t^2$. Then for every value of t,

$$y = t^2 = (t)^2 = x^2$$

Therefore $x(t) = t$ and $y(t) = t^2$ are parametric equations for the curve.

The corresponding vector equation for the curve is $\vec{\mathbf{r}}(t) = \langle t, t^2 \rangle$. ◊

Problem 120. Give parametric equations and the corresponding vector function which describe motion on the circle $x^2 + y^2 = 9$. (Hint: $\sin^2\theta + \cos^2\theta = 1$.)

Solution. (Note that the previous solution's approach wouldn't work here: if we let $x = t$, then in order to get all possible positive and negative values for y, we must set $y = \pm\sqrt{9 - t^2}$, which isn't a function.)

Suppose we let $x(t) = 3\cos t$ and $y(t) = 3\sin t$. Then for every value of t,

$$x^2 + y^2 = (3\cos t)^2 + (3\sin t)^2 = 9\cos^2 t + 9\sin^2 t = 9(\cos^2 t + \sin^2 t) = 9$$

Therefore $x(t) = 3\cos t$ and $y(t) = 3\sin t$ are parametric equations for the curve.

The corresponding vector equation for the curve is $\vec{\mathbf{r}}(t) = \langle 3\cos t, 3\sin t \rangle$.

(Another common solution would be to let $x = 3\sin t$ and $y = 3\cos t$, which is a different correct parameterization of the curve.) ◊

Problem 121. Give parametric equations and the corresponding vector function which describe motion on the ellipse $4x^2 + 9y^2 = 36$.

Solution. Suppose we let $x(t) = 3\cos t$ and $y(t) = 2\sin t$. Then for every value of t,

$$4x^2 + 9y^2 = 4(3\cos t)^2 + 9(2\sin t)^2 = 36\cos^2 t + 36\sin^2 t = 36(\cos^2 t + \sin^2 t) = 36$$

Therefore $x(t) = 3\cos t$ and $y(t) = 2\sin t$ are parametric equations for the curve.

The corresponding vector equation for the curve is $\vec{\mathbf{r}}(t) = \langle 3\cos t, 2\sin t \rangle$. ◊

Problem 122. Describe the domain of $\vec{\mathbf{r}}(t) = \left\langle t^3, \ln\left(3 - t\right), \sqrt{t} \right\rangle$.

Solution. The domain of $x(t) = t^3$ is all real numbers.

The domain of $y(t) = \ln(3 - t)$ is all real numbers less than 3, or $(-\infty, 3) = \{t : t < 3\}$.

The domain of $z(t) = \sqrt{t}$ is all nonnegative real numbers, or $[0, \infty) = \{t : t \geq 0\}$.

Therefore the domain of $\vec{\mathbf{r}}(t)$ is the intersection of those domains: $[0, 3) = \{t : 0 \leq t < 3\}$. ◊

Definition 123. If $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$, then the **limit** of $\vec{\mathbf{r}}$ as t approaches a is defined to be the limit of its component functions:

$$\lim_{t\to a} \vec{\mathbf{r}}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

Problem 124. Compute the limit $\displaystyle\lim_{t\to -1} \left\langle \arctan t, \frac{e^{1+t}}{1-t} \right\rangle$.

Solution. The limits of the component functions are:

$$\lim_{t\to-1} \arctan t = -\frac{\pi}{4}$$

$$\lim_{t\to-1} \frac{e^{1+t}}{1-t} = \frac{1}{2}$$

Therefore the limit of the vector function is:

$$\lim_{t\to-1} \left\langle \arctan t, \frac{e^{1+t}}{1-t} \right\rangle = \left\langle \lim_{t\to-1} \arctan t, \lim_{t\to-1} \frac{e^{1+t}}{1-t} \right\rangle = \left\langle -\frac{\pi}{4}, \frac{1}{2} \right\rangle$$

$\lozenge$

Problem 125. Compute the limit $\lim_{t\to\pi/2} \langle \sin t, \cos t, \cot t\rangle$.

Solution. The limits of the component functions are:

$$\lim_{t\to\pi/2} \sin t = 1$$

$$\lim_{t\to\pi/2} \cos t = 0$$

$$\lim_{t\to\pi/2} \cot t = 0$$

Therefore the limit of the vector function is:

$$\lim_{t\to\pi/2} \langle \sin t, \cos t, \cot t\rangle = \left\langle \lim_{t\to\pi/2} \sin t, \lim_{t\to\pi/2} \cos t, \lim_{t\to\pi/2} \cot t \right\rangle = \langle 1, 0, 0\rangle$$

$\lozenge$

Problem 126. Compute the limit $\lim_{t\to1} \left\langle \frac{3t^2-3}{t+1}, \frac{\sin(2t-2)}{2t-2}, \frac{3t^2-3}{t-1} \right\rangle$.

Solution. The limits of the component functions are:

$$\lim_{t\to1} \frac{3t^2-3}{t+1} = 0$$

$$\lim_{t\to1} \frac{\sin(2t-2)}{2t-2} = 1$$

$$\lim_{t\to1} \frac{3t^2-3}{t-1} = 6$$

Therefore the limit of the vector function is:

$$\lim_{t\to1} \left\langle \frac{3t^2-3}{t+1}, \frac{\sin(2t-2)}{2t-2}, \frac{3t^2-3}{t-1} \right\rangle = \left\langle \lim_{t\to1} \frac{3t^2-3}{t+1}, \lim_{t\to1} \frac{\sin(2t-2)}{2t-2}, \lim_{t\to1} \frac{3t^2-3}{t-1} \right\rangle = \langle 0, 1, 6\rangle$$

$\lozenge$

Definition 127. The function $\vec{\mathbf{r}}(t)$ is **continuous** if

$$\lim_{t\to a}\vec{\mathbf{r}}(t)=\vec{\mathbf{r}}(a)$$

for all a in its domain.

Theorem 128. $\vec{\mathbf{r}}(t)$ is continuous exactly when all of its component functions are all continuous.

Textbook Practice Problems: Section 13.1: $7-14, 28, 30$

13.2 Derivatives and Integrals of Vector Functions

Definition 129. If $\vec{\mathbf{r}}(t)=\langle f(t),g(t),h(t)\rangle$ is a vector function where f,g,h are differentiable functions, then the **derivative** of $\vec{\mathbf{r}}(t)$ is defined to be

$$\frac{d\vec{\mathbf{r}}}{dt}=\vec{\mathbf{r}}'(t)=\langle f'(t),g'(t),h'(t)\rangle$$

Definition 130. For each real number a in the domain of $\vec{\mathbf{r}}$ and $\vec{\mathbf{r}}'$, $\vec{\mathbf{r}}'(a)$ gives a **tangent vector** to the curve for the position vector $\vec{\mathbf{r}}(a)$.

Problem 131. Compute $\vec{\mathbf{r}}'(t)$ given $\vec{\mathbf{r}}(t)=\langle t^2,3+t\rangle$. Then plot the curve corresponding to $\vec{\mathbf{r}}(t)$ and the point and tangent vectors corresponding to $t=-2$.

Solution. The derivative of a vector function is given by the derivatives of its components:

$$\vec{\mathbf{r}}'(t)=\left\langle \frac{d}{dt}[t^2],\frac{d}{dt}[3+t]\right\rangle=\langle 2t,1\rangle$$

The point on the curve corresponding to $t=-2$ is given by the position vector $\vec{\mathbf{r}}(-2)=\langle 4,1\rangle$. The tangent vector to the curve corresponding to $t=-2$ is given by the vector $\vec{\mathbf{r}}'(-2)=\langle -4,1\rangle$. ♢

Problem 132. Compute $\vec{\mathbf{r}}'(t)$ given $\vec{\mathbf{r}}(t)=\langle \sin t,t,\cos t\rangle$. Then plot the curve corresponding to $\vec{\mathbf{r}}(t)$ and the point and tangent vectors corresponding to $t=\pi$. (Hint: This curve is known as a **helix**.)

Solution. The derivative of a vector function is given by the derivatives of its components:

$$\vec{\mathbf{r}}'(t)=\left\langle \frac{d}{dt}[\sin t],\frac{d}{dt}[t],\frac{d}{dt}[\cos t]\right\rangle=\langle \cos t,1,-\sin t\rangle$$

The point on the curve corresponding to $t=\pi$ is given by the position vector $\vec{\mathbf{r}}(\pi)=\langle 0,\pi,-1\rangle$. The tangent vector to the curve corresponding to $t=\pi$ is given by the vector $\vec{\mathbf{r}}'(\pi)=\langle -1,1,0\rangle$. ♢

Problem 133. Compute $\vec{\mathbf{r}}'(t)$ given

$$\vec{\mathbf{r}}(t)=(\ln 2t)\widehat{\mathbf{i}}+(e^{2t}-2)\widehat{\mathbf{j}}+(\arcsin t)\widehat{\mathbf{k}}$$

Solution. The derivative of a vector function is given by the derivatives of its components:
$$\vec{\mathbf{r}}'(t) = \frac{d}{dt}[\ln 2t]\widehat{\mathbf{i}} + \frac{d}{dt}[e^{2t} - 2]\widehat{\mathbf{j}} + \frac{d}{dt}[\arcsin t]\widehat{\mathbf{k}} = \frac{1}{t}\widehat{\mathbf{i}} + 2e^{2t}\widehat{\mathbf{j}} + \frac{1}{\sqrt{1-t^2}}\widehat{\mathbf{k}} = \left\langle \frac{1}{t}, 2e^{2t}, \frac{1}{\sqrt{1-t^2}} \right\rangle$$

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Theorem 134. The usual differentiation rules (e.g. product rule, chain rule) for scalar functions also hold for vector functions:
$$\frac{d}{dt}[\vec{\mathbf{C}}] = \vec{\mathbf{0}}$$
$$\frac{d}{dt}[c\vec{\mathbf{u}}(t)] = c\vec{\mathbf{u}}'(t)$$
$$\frac{d}{dt}[f(t)\vec{\mathbf{C}}] = f'(t)\vec{\mathbf{C}}$$
$$\frac{d}{dt}[\vec{\mathbf{u}}(t) \pm \vec{\mathbf{v}}(t)] = \vec{\mathbf{u}}'(t) \pm \vec{\mathbf{v}}'(t)$$
$$\frac{d}{dt}[f(t)\vec{\mathbf{u}}(t)] = f(t)\vec{\mathbf{u}}'(t) + f'(t)\vec{\mathbf{u}}(t)$$
$$\frac{d}{dt}[\vec{\mathbf{u}}(t) \cdot \vec{\mathbf{v}}(t)] = \vec{\mathbf{u}}(t) \cdot \vec{\mathbf{v}}'(t) + \vec{\mathbf{u}}'(t) \cdot \vec{\mathbf{v}}(t)$$
$$\frac{d}{dt}[\vec{\mathbf{u}}(t) \times \vec{\mathbf{v}}(t)] = \vec{\mathbf{u}}(t) \times \vec{\mathbf{v}}'(t) + \vec{\mathbf{u}}'(t) \times \vec{\mathbf{v}}(t)$$
$$\frac{d\vec{\mathbf{u}}}{dt} = \frac{d}{dt}[\vec{\mathbf{u}}(f(t))] = \vec{\mathbf{u}}'(f(t))f'(t) = \frac{d\vec{\mathbf{u}}}{df}\frac{df}{dt}$$

Theorem 135. If $|\vec{\mathbf{r}}(t)| = c$ always (the curve overlays a circle centered at the origin), then
$$\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t) = 0$$

Problem 136. Prove the previous theorem. (Part of the solution has been provided.)

Solution. Since $|\vec{\mathbf{r}}(t)| = c$ and $|\vec{\mathbf{v}}|^2 = \vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$, we may differentate both sides of the following equation:
$$\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t) = c^2$$
$$\frac{d}{dt}\left[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t)\right] = \frac{d}{dt}\left[c^2\right]$$

On the left side, we may apply the dot product version of the Product Rule from Theorem 134. On the right side, note that the derivative of a constant is zero.
$$\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t) + \vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}(t) = 0$$

Then combine the two dot products together (order does not matter for dot products), then divide both sides by 2 to get the desired result.
$$2(\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t)) = 0$$
$$\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t) = 0$$

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Definition 137. If $\vec{\mathbf{R}}'(t) = \vec{\mathbf{r}}(t)$, then $\vec{\mathbf{R}}(t)$ is an **antiderivative** of $\vec{\mathbf{r}}(t)$.

Definition 138. The **indefinite integral** $\int \vec{\mathbf{r}}(t)\,dt$ is the collection of all the antiderivatives of $\vec{\mathbf{r}}(t)$.

$$\int \vec{\mathbf{r}}(t)\,dt = \vec{\mathbf{R}}(t) + \vec{\mathbf{C}}$$

$$\int \vec{\mathbf{r}}(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$$

Problem 139. Give the indefinite integral of $\vec{\mathbf{r}}(t) = \left\langle \frac{1}{t+2}, \frac{1}{(t+2)^2}, \frac{2t}{t^2+2} \right\rangle$.

Solution. The indefinite integral of a vector function is given by antiderivatives of its components, with an arbitrary constant vector of integration:

$$\int \vec{\mathbf{r}}(t)\,dt = \left\langle \int \frac{1}{t+2}\,dt, \int \frac{1}{(t+2)^2}\,dt, \int \frac{2t}{t^2+2}\,dt \right\rangle = \left\langle \ln|t+2|, -\frac{1}{t+2}, \ln|t^2+2| \right\rangle + \vec{\mathbf{C}}$$

$\Diamond$

Definition 140. The **definite integral** $\int_a^b \vec{\mathbf{r}}(t)\,dt$ is given by the definite integrals of each component function.

$$\int_a^b \vec{\mathbf{r}}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

$$\int_a^b \vec{\mathbf{r}}(t)\,dt = \vec{\mathbf{R}}(b) - \vec{\mathbf{R}}(a)$$

Theorem 141. A **differential vector equation** asks for $\vec{\mathbf{r}}(t)$ given $\vec{\mathbf{r}}'(t)$ and $\vec{\mathbf{r}}(a)$ for some value a. Such problems may either be solved by using

$$\vec{\mathbf{r}}(t) = \int_a^t \vec{\mathbf{r}}'(\tau)\,d\tau + \vec{\mathbf{r}}(a)$$

or by solving for $\vec{\mathbf{C}}$ in the indefinite integral

$$\vec{\mathbf{r}}(t) = \int \vec{\mathbf{r}}'(t)\,dt + \vec{\mathbf{C}}$$

Problem 142. Find $\vec{\mathbf{r}}(t)$ given $\vec{\mathbf{r}}'(t) = \left\langle \frac{3}{2}\sqrt{t}, 8t, 3t^2+3 \right\rangle$ and $\vec{\mathbf{r}}(1) = \langle 1, -3, 6 \rangle$.

Solution. Plugging $\vec{\mathbf{r}}'(t) = \left\langle \frac{3}{2}\sqrt{t}, 8t, 3t^2+3 \right\rangle$ and $\vec{\mathbf{r}}(1) = \langle 1, -3, 6 \rangle$ into the formula from Theorem 141:

$$\begin{aligned} \vec{\mathbf{r}}(t) &= \int_1^t \left\langle \frac{3}{2}\sqrt{\tau}, 8\tau, 3\tau^2+3 \right\rangle d\tau + \langle 1, -3, 6 \rangle \\ &= \left\langle \int_1^t \frac{3}{2}\sqrt{\tau}\, d\tau, \int_1^t 8\tau\, d\tau, \int_1^t 3\tau^2 + 3\, d\tau \right\rangle + \langle 1, -3, 6 \rangle \\ &= \left\langle t^{3/2} - 1, 4t^2 - 4, t^3 + 3t - 4 \right\rangle + \langle 1, -3, 6 \rangle \\ &= \left\langle t^{3/2}, 4t^2 - 7, t^3 + 3t + 2 \right\rangle \end{aligned}$$

Alternately, we could use the fact that $\vec{\mathbf{r}}(t)$ is an antiderivative of $\vec{\mathbf{r}}'(t)$:

$$\vec{\mathbf{r}}(t) = \int \left\langle \frac{3}{2}\sqrt{t}, 8t, 3t^2+3 \right\rangle dt = \left\langle t^{3/2}, 4t^2, t^3 + 3t \right\rangle + \vec{\mathbf{C}}$$

and plug in $t = 1$ to solve for $\vec{\mathbf{C}}$:

$$\vec{\mathbf{r}}(1) = \langle 1, -3, 6 \rangle = \left\langle 1^{3/2}, 4(1)^2, 1^3 + 3(1) \right\rangle + \vec{\mathbf{C}}$$

$$\langle 1, -3, 6 \rangle = \langle 1, 4, 4 \rangle + \vec{\mathbf{C}}$$
$$\langle 0, -7, 2 \rangle = \vec{\mathbf{C}}$$

Therefore

$$\vec{\mathbf{r}}(t) = \left\langle t^{3/2}, 4t^2, t^3 + 3t \right\rangle + \langle 0, -7, 2 \rangle = \left\langle t^{3/2}, 4t^2 - 7, t^3 + 3t + 2 \right\rangle$$

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Textbook Practice Problems: Section 13.2: $9 - 26, 35 - 40$