Packet 2

Part 2.1: Sections 13.3-13.4

13.3 Arc Length and Curvature

Problem 1. Let $\vec{\mathbf{r}}(t) = \langle 6t, t^3, 3t^2 \rangle$. Use the lengths of the line segments connecting $\vec{\mathbf{r}}(0)$, $\vec{\mathbf{r}}(1)$, $\vec{\mathbf{r}}(2)$, and $\vec{\mathbf{r}}(3)$ to approximate the length of the curve from $t = 0$ to $t = 3$.

Solution. $\Diamond$

Definition 2. Let $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$ be a vector function. Then the **arclength** or **length** of the curve given by $\vec{\mathbf{r}}(t)$ from $t = a$ to $t = b$ is

$$L = \int_a^b \left| \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t + \Delta t) - \vec{\mathbf{r}}(t)}{\Delta t} \right| dt = \int_a^b |\vec{\mathbf{r}}'(t)| \, dt$$

Problem 3. Find the length of the curve given by $\vec{\mathbf{r}}(t) = \langle 6t, t^3, 3t^2 \rangle$ from $t = 0$ to $t = 3$. (Hint: $9t^4 + 36t^2 + 36$ is a perfect square polynomial.)

Solution. $\Diamond$

Definition 4. Let $s(t)$ be the **arclength function/parameter** representing the length of a curve from the point given by $\vec{\mathbf{r}}(0)$ to the point given by $\vec{\mathbf{r}}(t)$. (Assume $s(t) < 0$ for $t < 0$.)

Theorem 5. The arclength function $s(t)$ is given by the definite integral

$$s(t) = \int_0^t |\vec{\mathbf{r}}'(\tau)| \, d\tau$$

Theorem 6. The derivative of the arclength function gives the lengths of the tangent vectors given by the derivative of the position function:

$$\frac{ds}{dt} = \left| \frac{d\vec{\mathbf{r}}}{dt} \right|$$

Problem 7. Compute $s(t)$ for $\vec{\mathbf{r}}(t) = \langle 6t, t^3, 3t^2 \rangle$, and use it to find the arclength parameter corresponding to $t = -2$.

Solution. $\Diamond$

Problem 8. Find the length of an arc of the circular helix with vector equation $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t \rangle$ from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

Definition 9. The **unit tangent vector** $\vec{\mathbf{T}}$ to a curve $\vec{\mathbf{r}}$ is the direction of the derivative $\vec{\mathbf{r}}'(t) = \frac{d\vec{\mathbf{r}}}{dt}$.

Theorem 10.

$$\vec{\mathbf{T}} = \frac{d\vec{\mathbf{r}}/dt}{|d\vec{\mathbf{r}}/dt|} = \frac{d\vec{\mathbf{r}}}{ds}$$

Problem 11. Find the unit tangent vector to the curve given by $\vec{\mathbf{r}}(t) = \langle 3t^2, 2t \rangle$ at the point where $t = -3$.

Solution. $\Diamond$

Definition 12. The **curvature** κ of a curve C at a given point is the magnitude of the rate of change of $\vec{\mathbf{T}}$ with respect to arclength s.

Theorem 13.

$$\kappa = \left| \frac{d\vec{\mathbf{T}}}{ds} \right| = \left| \frac{1}{ds/dt} \frac{d\vec{\mathbf{T}}}{dt} \right| = \frac{1}{|d\vec{\mathbf{r}}/dt|} \left| \frac{d\vec{\mathbf{T}}}{dt} \right|$$

Theorem 14. An alternate formula for curvature is given by

$$\kappa = \frac{|\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)|}{|\vec{\mathbf{r}}'(t)|^3}$$

Problem 15. Prove that the helix given by the vector equation $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t \rangle$ has constant curvature.

Solution. $\Diamond$

Problem 16. (OPTIONAL) Prove that the alternate formula for curvature is accurate by showing

$$\frac{1}{|d\vec{\mathbf{r}}/dt|} \left| \frac{d\vec{\mathbf{T}}}{dt} \right| = \frac{|\vec{\mathbf{r}}' \times \vec{\mathbf{r}}''|}{|\vec{\mathbf{r}}'|^3}$$

(Some of the solution has been provided.)

Solution. Begin by observing that $\vec{\mathbf{r}}' = \left| \frac{d\vec{\mathbf{r}}}{dt} \right| \vec{\mathbf{T}} = \frac{ds}{dt} \vec{\mathbf{T}}$, and by the product rule it follows that $\vec{\mathbf{r}}'' = \frac{d^2 s}{dt^2} \vec{\mathbf{T}} + \frac{ds}{dt} \vec{\mathbf{T}}'$.

(...)

$\Diamond$

Definition 17. The **unit normal vector** $\vec{\mathbf{N}}$ to a curve $\vec{\mathbf{r}}$ is the direction of the derivative of the unit tangent vector $\vec{\mathbf{T}}'(t) = \frac{d\vec{\mathbf{T}}}{dt}$. (By definition, this vector points into the direction of the curve.)

Theorem 18.

$$\vec{\mathbf{N}} = \frac{\vec{\mathbf{T}}'}{|\vec{\mathbf{T}}'|}$$

Problem 19. Prove that $\vec{\mathbf{N}}$ actually is normal to the curve by using a theorem from a previous section. (Hint: $|\vec{\mathbf{T}}| = 1$.)

Solution. ◊

Problem 20. Plot the curve given by $\vec{\mathbf{r}}(t) = \langle \cos(2t), \sin(2t) \rangle$, along with $\vec{\mathbf{T}}, \vec{\mathbf{N}}$ at the point where $t = \frac{\pi}{2}$.

Problem 21. Give formuals for $\vec{\mathbf{T}}, \vec{\mathbf{N}}$ in terms of t for the vector function

$$\vec{\mathbf{r}}(t) = \langle \sqrt{2} \sin t, 2 \cos t, \sqrt{2} \sin t \rangle$$

Solution. ◊

Definition 22. The **binormal vector** $\vec{\mathbf{B}}$ is the direction normal to both $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ according to the right-hand rule.

Theorem 23.

$$\vec{\mathbf{B}} = \vec{\mathbf{T}} \times \vec{\mathbf{N}}$$

Problem 24. Prove that $\vec{\mathbf{T}} \times \vec{\mathbf{N}}$ is a unit vector.

Solution. ◊

Problem 25. Given the following information about $\vec{\mathbf{r}}(t)$ at a point, evaluate the binormal vector $\vec{\mathbf{B}}$ and curvature κ at that same point:

$$\frac{d\vec{\mathbf{r}}}{dt} = \langle -3, 0, 3\sqrt{3} \rangle$$

$$\frac{d\vec{\mathbf{T}}}{dt} = \langle -\sqrt{3}, 0, -1 \rangle$$

$$\vec{\mathbf{T}} = \langle -\frac{1}{2}, 0, \frac{\sqrt{3}}{2} \rangle$$

$$\vec{\mathbf{N}} = \langle -\frac{\sqrt{3}}{2}, 0, -\frac{1}{2} \rangle$$

Solution. ◊

Definition 26. A **right-handed frame** is a group of three unit vectors which are all normal to one another and satisfy the right hand rule.

Example 27. $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ and $\vec{\mathbf{T}}, \vec{\mathbf{N}}, \vec{\mathbf{B}}$ are examples of right-handed frames.

Theorem 28. Any vector is a linear combination of the vectors in a right-handed frame.

13.4 Motion in Space, Velocity, and Acceleration

Definition 29. The **velocity** $\vec{\mathbf{v}}(t)$ of a particle at time t on a position function $\vec{\mathbf{r}}(t)$ is its rate of change with respect to t.

Definition 30. The **speed** $|\vec{\mathbf{v}}(t)|$ of a particle at time t on a position function $\vec{\mathbf{r}}(t)$ is the magnitude of its velocity.

Definition 31. The **direction** $\vec{\mathbf{T}}(t)$ of a particle at time t on a position function $\vec{\mathbf{r}}(t)$ is the direction of its velocity.

Definition 32. The **acceleration** $\vec{\mathbf{a}}(t)$ of a particle at time t on a position function $\vec{\mathbf{r}}(t)$ is the rate of change of its velocity with respect to t.

Theorem 33.

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t)$$
$$|\vec{\mathbf{v}}(t)| = |\vec{\mathbf{r}}'(t)| = \frac{ds}{dt}$$
$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$
$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = \vec{\mathbf{r}}''(t)$$

Problem 34. Given a position function $\vec{\mathbf{r}}(t) = \langle t^3, t^2 \rangle$ find its velocity, speed, and acceleration at $t = 1$.

Definition 35. Ideal projectile motion is an approximation of real-world motion assuming constant acceleration due to gravity in the y direction and no acceleration in the x direction:

$$\vec{\mathbf{a}}(t) = \langle 0, -g \rangle$$

Theorem 36. The velocity and position functions for a particle with initial velocity $\vec{\mathbf{v}}_0 = \langle v_{x,0}, v_{y,0} \rangle$ and beginning at position $P_0 = \langle x_0, y_0 \rangle$ assuming ideal projectile motion are:

$$\vec{\mathbf{v}}(t) = \langle v_{x,0}, -gt + v_{y,0} \rangle$$
$$\vec{\mathbf{r}}(t) = \left\langle v_{x,0}t + x_0, -\frac{1}{2}gt^2 + v_{y,0}t + y_0 \right\rangle$$

Problem 37. Assume ideal projectile motion and and $g = 10\frac{m}{s^2}$. What is the flight time of a projectile shot from the ground at an angle of $\pi/6$ with initial speed $100\frac{m}{s}$?

Solution. ◊

Problem 38. Assume ideal projectile motion and and $g = 10\frac{m}{s^2}$. What must have been the initial speed of a projectile shot from the ground at an angle of $\pi/3$ if it traveled 60 meters horizontally after 4 seconds?

Solution. ◊