Packet 3

Packet 3.2: Sections 15.10, 15.4, 15.8, 15.9

15.10 Change of Variables in Multiple Integrals

Remark 1. An alternate way to write the u-substitution rule from Cal I is: if $x(u)$ defines x as a function of u, and $x(u)$ transforms the interval J of u-values into the interval I of x-values, then

$$\int_I f(x)\,dx = \int_J f(x(u)) \left|\frac{dx}{du}\right| du$$

Problem 2. Use the above alternate u-sub rule to prove that

$$\int_1^2 2xe^{x^2}\,dx = \int_1^4 e^u\,du = e^4 - e$$

Solution. ◊

Contributors.

Definition 3. A **2D transformation**

$$\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v)\rangle$$

transforms points in the uv plane to points in the xy plane.

Definition 4. The **unit square** is the square with coordinates $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

Definition 5. The **unit triangle** is the triangle with coordinates $(0,0)$, $(1,0)$, and $(1,1)$.

Problem 6. Show that a transformation from the unit square in the uv plane to the square with sides $y = x$, $y = x+4$, $y = -x$, and $y = -x+4$ in the xy plane could satisfy the equations $y = x + 4u$ and $y = -x + 4v$, and then solve this system to get the transformation $\langle x(u,v), y(u,v)\rangle$.

Solution. ◊

Contributors.

Problem 7. Find a transformation from the unit square in the uv plane to the parallelogram with vertices $(1, 0)$, $(2, -1)$, $(4, 0)$, and $(3, 1)$ in the xy plane.

Solution. ◊

Contributors.

Problem 8. Find a transformation from the unit triangle in the uv plane to the triangle with vertices $(0, -1)$, $(2, -2)$, and $(-1, 0)$ in the xy plane. (Hint: complete the triangle in the xy plane to a parallelogram and then find a transformation from the unit square to that parallelogram.)

Solution. ◊

Contributors.

Problem 9. Find a transformation from the unit circle $u^2 + v^2 = 1$ in the uv plane to the ellipse $4x^2 + 9y^2 = 36$.

Solution. ◊

Contributors.

Definition 10. The **Jacobian** of a transformation $\vec{\mathbf{r}}(u, v) = \langle x(u, v), y(u, v)\rangle$ is given by

$$\vec{\mathbf{r}}_J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Theorem 11. If $\vec{\mathbf{r}}(u, v) = \langle x(u, v), y(u, v)\rangle$ transforms the region G in the uv plane to the region R in the xy plane, then

$$\iint_R f(x, y)\, dA = \iint_G f(x(u, v), y(u, v))|\vec{\mathbf{r}}_J(u, v)|\, dA$$

Problem 12. Evaluate $\iint_R 2x - y\, dA$ using the transformation $\vec{\mathbf{r}}(u, v) = \langle u + v, 2u - v + 3\rangle$ from unit square in the uv plane into the parallelogram R with vertices $(0, 3)$, $(1, 5)$, $(2, 4)$, and $(1, 2)$ in the xy plane.

Solution. ◊

Contributors.

Problem 13. Evaluate $\iint_R e^x \cos(\pi e^x)\, dA$ using the transformation $\vec{\mathbf{r}}(u,v) = \langle \ln(u+v+1), v\rangle$ from the unit triangle in the uv plane into the region R bounded by $y=0$, $y=e^x-2$, and $y=\frac{e^x-1}{2}$.

Solution. $\Diamond$

Contributors.

Definition 14. A **3D transformation**

$$\vec{\mathbf{r}}(u,v,w) = \langle x(u,v,w), y(u,v,w), z(u,v,w)\rangle$$

transforms points in uvw space to points in xyz space.

Definition 15. The **Jacobian** of a transformation $\vec{\mathbf{r}}(u,v,w) = \langle x(u,v,w), y(u,v,w), z(u,v,w)\rangle$ is given by

$$\vec{\mathbf{r}}_J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Theorem 16. If $\vec{\mathbf{r}}(u,v,w) = \langle x(u,v,w), y(u,v,w), z(u,v,w)\rangle$ transforms the solid H in uvw space to the solid D in the xyz space, then

$$\iiint_D f(x,y,z)\, dV = \iiint_H f(x(u,v,w), y(u,v,w), z(u,v,w))|\vec{\mathbf{r}}_J(u,v,w)|\, dV$$

15.4 Double Integrals in Polar Coordinates

Theorem 17. The polar coordinate transformation

$$\vec{\mathbf{r}}(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle$$

from polar G into Cartesian R yields

$$\iint\limits_R f(x,y)\, dA = \iint\limits_G f(r\cos\theta, r\sin\theta)|r|\, dA$$

Problem 18. Prove the previous theorem.

Solution. $\Diamond$

Contributors.

Theorem 19. If the region R in the xy plane is described with polar coordinates, and is bounded by the inside/outside curves $0 \leq g(\theta) \leq r \leq h(\theta)$ and lines $\alpha \leq \theta \leq \beta$, then

$$\iint\limits_R f(x,y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

Problem 20. Evaluate $\iint_R e^{x^2+y^2}\,dA$ where R is the disk with boundary $x^2+y^2=9$.

Solution. $\Diamond$

Contributors.

Problem 21. Prove that

$$\int_0^{\sqrt{3}} \int_1^{\sqrt{4-x^2}} 3y\,dy\,dx = \int_{\pi/6}^{\pi/2} \int_{\csc\theta}^{2} 3r^2 \sin\theta\,dr\,d\theta = 3\sqrt{3}$$

Solution. $\Diamond$

Contributors.

15.8 Triple Integrals in Cylindrical Coordinates

Theorem 22. The cylindrical coordinate transformation

$$\vec{\mathbf{r}}(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$$

from cylindrical H into Cartesian D yields

$$\iiint\limits_D f(x,y,z)\,dV = \iiint\limits_H f(r\cos\theta, r\sin\theta, z)|r|\,dV$$

Remark 23. This is equivalent to using the fact that

$$\iiint_D f(x,y,z)\,dV = \iint_R \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\right] dA$$

and then interpreting the shadow R in the xy plane with polar coordinates.

Problem 24. Evaluate $\iiint_D \sqrt{x^2+y^2}\,dV$ where D is the right circular cylinder bounded by $|z| \leq 2$ and $x^2+y^2=1$.

Solution. ◊

Contributors.

Problem 25. Express the volume of the of the solid bounded by the xy plane and $z = 1 - x^2 - y^2$ as a triple integral of the variables r, θ, z.

Solution. ◊

Contributors.

15.9 Triple Integrals in Spherical Coordinates

Theorem 26. The spherical coordinate transformation

$$\vec{\mathbf{r}}(\rho, \phi, \theta) = \langle \rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi \rangle$$

from spherical H into Cartesian D yields

$$\iiint\limits_D f(x, y, z)\, dV = \iiint\limits_H f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 |\sin\phi|\, dV$$

Problem 27. Prove the previous theorem.

Solution. ◊

Contributors.

Theorem 28. If the solid D in the xyz plane is described with spherical coordinates, and is bounded by the inside/outside surfaces $h_1(\phi, \theta) \leq \rho \leq h_2(\phi, \theta)$, conical surfaces $0 \leq g_1(\theta) \leq \phi \leq g_2(\theta)$, and planes $\alpha \leq \theta \leq \beta$, then

$$\iiint\limits_D f(x, y, z)\, dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(\phi,\theta)}^{h_2(\phi,\theta)} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Problem 29. Prove that the volume of a sphere of radius a has volume

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx = \frac{4}{3}\pi a^3$$

Solution. ◊

Contributors.

Problem 30. Express the volume of the "ice cream cone" shaped solid

$$D = \{(x, y, z) : \sqrt{x^2 + y^2} \leq z \leq \sqrt{1 - x^2 - y^2} + 1\}$$

as a triple iterated integral of the variables ρ, ϕ, θ.

Solution. $\Diamond$

Contributors.