Calculus III - Spring 2015 - Mr. Clontz - Test 4

Name: Solutions Class: ______ Type: ______

- This exam is open-"everything", provided you do not plagiarize. *Do not leave any unsupported answers!*
- Write your solutions so that a fellow student who has a perfect understanding of previous math classes and packets but has never seen that type of problem before could follow your work.
- Individual Test: You will have 40 minutes to complete this test on your own. You may not communicate with anyone during this period.
- Group Test: You will have 40 minutes to complete an identical test. You may collaborate with your group members during this time. All solutions must still be written by yourself, and may not be directly copied from another student.

1. (6 points) Prove that

$$\int_C x + z\, ds = \int_0^1 6 - 3t\, dt$$

where C is the line segment from $(1, -2, 1)$ to $(2, 0, -1)$.

For the line segment C, we may use the vector function

$$\vec{r}(t) = \langle 1, -2, 1 \rangle + t \langle 1, 2, -2 \rangle \qquad 0 \le t \le 1$$

$$\vec{r}(t) = \langle 1+t, -2+2t, 1-2t \rangle$$

It follows that

$$\frac{d\vec{r}}{dt} = \langle 1, 2, -2 \rangle \qquad \frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{1+4+4} = \sqrt{9} = 3$$

Therefore

$$\int_C x+z\, ds = \int_0^1 (x+z)\frac{ds}{dt}\, dt$$

$$= \int_0^1 \left((1+t)+(1-2t)\right)(3)\, dt$$

$$= \int_0^1 (2-t)(3)\, dt$$

$$= \int_0^1 6-3t\, dt$$

$$\left(= \left[6t - \frac{3}{2}t^2\right]_0^1 = 9/2 \right)$$

Typo!

2. (6 points) Prove that

$$\int_C \langle -y^2, x\rangle \cdot d\vec{\mathbf{r}} = \int_0^{2\pi} 8\sin^3 t + 4\cos^2 t\, dt$$

where C is the full counter-clockwise rotation of the circle $x^2 + y^2 = 4$. (Hint: this is *not* a conservative field.)

For the circle C, we may use the vector function

$$\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle \qquad 0 \leq t \leq 2\pi$$

It follows that

$$\frac{d\vec{r}}{dt} = \langle -2\sin t, 2\cos t\rangle$$

Therefore

$$\int_C \langle -y^2, x\rangle \cdot d\vec{r} = \int_0^{2\pi} \langle -y^2, x\rangle \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_0^{2\pi} \langle -4\sin^2 t, 2\cos t\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt$$

$$= \int_0^{2\pi} 8\sin^3 t + 4\cos^2 t\, dt$$

Typo!

3. (6 points) Prove that

$$\int_C \underbrace{\langle 2xy, x^2 - 6yz, -3y^2\rangle}_{\vec{F}} \cdot d\vec{\mathbf{r}} = +34$$

where C is a curve beginning at $(1, 2, 3)$ and ending at $(4, 0, 1)$. (Hint: this *is* a conservative field.)

Since $\vec{F}$ is conservative, it must have a potential function f satisfying

$$f_x = 2xy \rightarrow f = x^2y + \Phi$$
$$f_y = x^2 - 6yz \rightarrow f = x^2y - 3y^2z + \Phi$$
$$f_z = -3y^2 \rightarrow f = -3y^2z + \Phi$$

Since $f = x^2y - 3y^2z$ satisfies all three,

$$\int_C \vec{F} \cdot d\vec{r} = \left[x^2y - 3y^2z\right]_{(1,2,3)}^{(4,0,1)}$$

$$= (0 - 0) - (2 - 36)$$

$$= 34$$

4. (6 points) Let C be the boundary of a surface S. Use Stokes' Theorem (section 16.8) to prove that

$$\iint_S \langle 2z, 2x, 2y\rangle \cdot d\vec{\sigma} = \int_C \underbrace{\langle \overset{P}{x-y^2}, \overset{Q}{y-z^2}, \overset{R}{z-x^2}\rangle}_{\vec{F}} \cdot d\vec{\mathbf{r}}$$

We may compute

$$\begin{aligned}\operatorname{curl} \vec{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle \\ &= \langle 0-(-2z), 0-(-2x), 0-(-2y)\rangle \\ &= \langle 2z, 2x, 2y\rangle\end{aligned}$$

Therefore by Stokes's Theorem:

$$\begin{aligned}\iint_S \langle 2z, 2x, 2y\rangle \cdot d\vec{\sigma} &= \iint_S \operatorname{curl} \vec{F} \cdot d\vec{\sigma} \\ &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_C \langle x-y^2, y-z^2, z-x^2\rangle \cdot d\vec{r}\end{aligned}$$

Typo

5. (6 points) Let $\vec{\mathbf{E}} = \langle E_1, E_2, E_3 \rangle$ be an electric vector field defined for each point in the interior D of a closed surface S. By Gauss's Law, if there is a total charge of Q contained by S, then the total electric flux across the surface S is given by

$$\iint_S \vec{\mathbf{E}} \cdot d\vec{\sigma} = \frac{Q}{\epsilon_0}$$

where ϵ_0 is known as the electric constant (roughly 8.85×10^{-12}, not that it matters). Prove that the total charge Q may be computed as

$$\epsilon_0 \iiint_D \frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}\, dV$$

(Of course, no knowledge of physics is required to answer this question.)

Solving for Q we find

$$Q = \varepsilon_0 \iint_S \vec{E} \cdot d\vec{\sigma}$$

By the divergence theorem:

$$Q = \varepsilon_0 \iiint_D \operatorname{div} \vec{E}\, dV$$

$$= \varepsilon_0 \iiint_D \frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}\, dV$$