10.4 The Cross Product

Definition 1. For any two non-parallel and non-zero vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ in $\mathbb{R}^3$, the **Right-Hand Rule** gives a specific direction orthogonal to both: position both vectors at the origin, and draw a line orthogonal to the plane containing both vectors. Then place your right thumb near $\vec{\mathbf{u}}$ and your right index finger near $\vec{\mathbf{v}}$. The direction on the orthogonal line given by extending your middle finger is the direction given by the RHR.

Definition 2. Let $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ be vectors. Their **cross product** $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is the vector constructed as follows:

1. If either of $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ is the zero vector $\vec{\mathbf{0}}$, then $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$.
2. If $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ are parallel, then $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$.
3. Otherwise, let $\vec{\mathbf{n}}$ be the unit vector given by the vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ and the RHR, and let a be the area of the parallelogram determined by the vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$. Then $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = a\vec{\mathbf{n}}$.

Theorem 3. The cross products of the standard unit vectors are given as follows:

- $\widehat{\mathbf{i}} \times \widehat{\mathbf{j}} = \widehat{\mathbf{k}}$
- $\widehat{\mathbf{j}} \times \widehat{\mathbf{i}} = -\widehat{\mathbf{k}}$
- $\widehat{\mathbf{j}} \times \widehat{\mathbf{k}} = \widehat{\mathbf{i}}$
- $\widehat{\mathbf{k}} \times \widehat{\mathbf{j}} = -\widehat{\mathbf{i}}$
- $\widehat{\mathbf{k}} \times \widehat{\mathbf{i}} = \widehat{\mathbf{j}}$
- $\widehat{\mathbf{i}} \times \widehat{\mathbf{k}} = -\widehat{\mathbf{j}}$
- $\widehat{\mathbf{i}} \times \widehat{\mathbf{i}} = \vec{\mathbf{0}}$
- $\widehat{\mathbf{j}} \times \widehat{\mathbf{j}} = \vec{\mathbf{0}}$
- $\widehat{\mathbf{k}} \times \widehat{\mathbf{k}} = \vec{\mathbf{0}}$

Theorem 4. The following properties hold for any three vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ and scalars a,b.

- $\vec{\mathbf{v}} \times \vec{\mathbf{u}} = -(\vec{\mathbf{u}} \times \vec{\mathbf{v}})$
- $(a\vec{\mathbf{u}}) \times (b\vec{\mathbf{v}}) = (ab)(\vec{\mathbf{u}} \times \vec{\mathbf{v}})$
- $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$
- $(\vec{\mathbf{v}} + \vec{\mathbf{w}}) \times \vec{\mathbf{u}} = \vec{\mathbf{v}} \times \vec{\mathbf{u}} + \vec{\mathbf{w}} \times \vec{\mathbf{u}}$

Problem 5. Compute $(3\widehat{\mathbf{i}} - 4\widehat{\mathbf{j}}) \times (\widehat{\mathbf{j}} + 2\widehat{\mathbf{k}})$.

Definition 6. A **determinant** is shorthand for writing the following algebraic expressions:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Theorem 7. The area of a parallelogram determined by two vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ with angle θ is given by $\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \sin\theta$.

Problem 8. Find the area of the parallelogram determined by the vectors $\langle 0, 3\rangle$ and $\langle 2, 2\rangle$

Theorem 9. The area of a parallelogram determined by two $2D$ vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ with angle θ is given by the absolute value of the determinant $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$.

Problem 10. Use this to resolve the previous problem.

Problem 11. Find the area of the triangle with vertices at $(2, 3)$, $(-1, 4)$, and $(1, 1)$

Theorem 12. The volume of a parallelepiped determined by three three-dimensional vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ is given by the absolute value of their **triple scalar product**, the determinant $\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$.

Problem 13. Find the volume of the parallelepiped determined by the vectors $\langle 1, 2, 3\rangle$, $\langle 0, -1, 4\rangle$, and $\langle 2, 2, 0\rangle$.

Theorem 14. By breaking up $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ into standard unit vectors:

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\widehat{\mathbf{i}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\widehat{\mathbf{j}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\widehat{\mathbf{k}}$$

Problem 15. Recompute $(3\widehat{\mathbf{i}} - 4\widehat{\mathbf{j}}) \times (\widehat{\mathbf{j}} + 2\widehat{\mathbf{k}})$.

Problem 16. Find the area of the parallelogram determined by $\vec{\mathbf{u}} = \langle 4, -3, 0\rangle$ and $\vec{\mathbf{v}} = \langle 2, 6, -3\rangle$.

Problem 17. Find a unit vector orthogonal to both $\vec{\mathbf{u}} = \langle 4, -3, 0\rangle$ and $\vec{\mathbf{v}} = \langle 2, 6, -3\rangle$.

Theorem 18. The triple scalar product of three vectors is also given by

$$\vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Definition 19. The torque τ done by a force vector $\vec{\mathbf{F}}$ on an arm given by $\vec{\mathbf{D}}$ is given by

$$\tau = |\vec{\mathbf{F}} \times \vec{\mathbf{D}}| = |\vec{\mathbf{F}}||\vec{\mathbf{D}}| \sin\theta$$

Problem 20. Find the torque enacted by the force $\langle 2, 2, -2 \rangle$ on a wrench at the point $(4, 3, 2)$ and bolt centered at the point $(1, 0, -2)$.