Packet 4

Packet 4.1: Sections 16.1-16.4

16.1 Vector Fields

Definition 1. A **vector field** assigns a vector to each point in 2D or 3D space.

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\mathbf{F}}(x, y) = \langle P(x, y), Q(x, y)\rangle = \langle P(\vec{\mathbf{r}}), Q(\vec{\mathbf{r}})\rangle = \langle P, Q\rangle$$

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\mathbf{F}}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle = \langle P(\vec{\mathbf{r}}), Q(\vec{\mathbf{r}}), R(\vec{\mathbf{r}})\rangle = \langle P, Q, R\rangle$$

Problem 2. Sketch the vector field $\vec{\mathbf{F}} = \langle x + y, 2y\rangle$ for all $x \in \{0, 1, 2\}$ and $y \in \{0, 1, 2\}$.

Solution. $\Diamond$

Contributors.

Remark 3. The gradient vector function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$$

is a vector field which yields normal vectors to the level surfaces of the function f.

Problem 4. Compute ∇f for the function $f(x, y) = x^2 - 2xy + y$, and then sketch the vector field ∇f all $x \in \{0, 1, 2\}$ and $y \in \{0, 1, 2\}$.

Solution. $\Diamond$

Contributors.

16.2 Line Integrals

Theorem 5. Some vector functions which parameterize curves follow.

- A line segment beginning at P_0 and ending at P_1:

$$\vec{\mathbf{r}}(t) = \overrightharpoon{\mathbf{P_0}} + t\overrightharpoon{\mathbf{P_0P_1}}, 0 \leq t \leq 1$$

- A circle centered at the origin with radius a:

$$\vec{\mathbf{r}}(t) = \langle a\cos t, a\sin t\rangle, 0 \leq t \leq 2\pi \text{ (full counter-clockwise rotation)}$$

$$\vec{\mathbf{r}}(t) = \langle a\sin t, a\cos t\rangle, 0 \leq t \leq 2\pi \text{ (full clockwise rotation)}$$

- A planar curve given by $y = f(x)$ from (x_0, y_0) to (x_1, y_1)

$$\vec{\mathbf{r}}(t) = \langle t, f(t)\rangle, x_0 \leq t \leq x_1 \text{ (left-to-right)}$$

$$\vec{\mathbf{r}}(t) = \langle -t, f(-t)\rangle, -x_0 \leq t \leq -x_1 \text{ (right-to-left)}$$

Problem 6. Give a vector function which parameterizes the line segment from the point $(0, 3, -2)$ to the point $(4, -1, 0)$.

Solution. $\Diamond$

Contributors.

Problem 7. Give a vector function which parameterizes the curve $y = x^3 - 2x$ from the point $(1, -1)$ to the point $(-1, 1)$.

Solution. $\Diamond$

Contributors.

Problem 8. Give a vector function which parameterizes the curve $x^2 + y^2 = 9$ from the point $(3, 0)$ clockwise to the point $(0, -3)$.

Solution. $\Diamond$

Contributors.

Definition 9. The **line integral with respect to arclength** of a function of many variables $f(\vec{\mathbf{r}})$ along a curve C is given by

$$\int_C f(\vec{\mathbf{r}})\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(\vec{\mathbf{r}}_{n,i})\Delta s_{n,i}$$

where for each positive integer n we've defined a way to partition C into n pieces

$$\Delta C_{n,1}, \Delta C_{n,2}, \dots, \Delta C_{n,n}$$

where $\Delta C_{n,i}$ has length $\Delta s_{n,i}$, contains the position vector $\vec{\mathbf{r}}_{n,i}$, and

$$\lim_{n\to\infty}\max(\Delta s_{n,i}) = 0$$

Theorem 10. If $\vec{\mathbf{r}}(t)$ is a parametrization of C for $a \leq t \leq b$, then

$$\int_C f(\vec{\mathbf{r}})\,ds = \int_{t=a}^{t=b} f(\vec{\mathbf{r}}(t))\frac{ds}{dt}\,dt$$

Problem 11. Evaluate $\int_C z+2xy\,ds$ where C is the line segment from $(0,-1,3)$ to $(2,2,-3)$.

Solution. ◊

Contributors.

Problem 12. Prove that $\int_C xy\,ds = \int_0^1 t^3\sqrt{1+2t}\,dt$ where C is the parabolic arc on $y = x^2$ from $(0,0)$ to $(1,1)$.

Solution. ◊

Contributors.

Definition 13. The **line integral of a vector field $\vec{\mathbf{F}}$** over the curve C is given by

$$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds$$

where $\vec{\mathbf{T}}$ is the unit tangent vector to the curve C at the position vector $\vec{\mathbf{r}}_0$.

Definition 14. If $\vec{\mathbf{r}}(t)$ is a parametrization of C for $a \leq t \leq b$, then

$$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_{t=a}^{t=b} \vec{\mathbf{F}}\cdot\frac{d\vec{\mathbf{r}}}{dt}\,dt$$