Packet 4

Packet 4.1: Sections 16.1-16.4

16.1 Vector Fields

Definition 1. A **vector field** assigns a vector to each point in 2D or 3D space.

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\mathbf{F}}(x, y) = \langle P(x, y), Q(x, y)\rangle = \langle P(\vec{\mathbf{r}}), Q(\vec{\mathbf{r}})\rangle = \langle P, Q\rangle$$

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\mathbf{F}}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle = \langle P(\vec{\mathbf{r}}), Q(\vec{\mathbf{r}}), R(\vec{\mathbf{r}})\rangle = \langle P, Q, R\rangle$$

Problem 2. Sketch the vector field $\vec{\mathbf{F}} = \langle x + y, 2y\rangle$ for all $x \in \{0, 1, 2\}$ and $y \in \{0, 1, 2\}$.

Solution. $\Diamond$

Contributors.

Remark 3. The gradient vector function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$$

is a vector field which yields normal vectors to the level surfaces of the function f.

Problem 4. Compute ∇f for the function $f(x, y) = x^2 - 2xy + y$, and then sketch the vector field ∇f all $x \in \{0, 1, 2\}$ and $y \in \{0, 1, 2\}$.

Solution. $\Diamond$

Contributors.

16.2 Line Integrals

Theorem 5. Some vector functions which parameterize curves follow.

- A line segment beginning at P_0 and ending at P_1:

$$\vec{\mathbf{r}}(t) = \overrightharpoon{\mathbf{P_0}} + t\overrightharpoon{\mathbf{P_0P_1}}, 0 \leq t \leq 1$$

- A circle centered at the origin with radius a:

$$\vec{\mathbf{r}}(t) = \langle a\cos t, a\sin t\rangle, 0 \leq t \leq 2\pi \text{ (full counter-clockwise rotation)}$$

$$\vec{\mathbf{r}}(t) = \langle a\sin t, a\cos t\rangle, 0 \leq t \leq 2\pi \text{ (full clockwise rotation)}$$

- A planar curve given by $y = f(x)$ from (x_0, y_0) to (x_1, y_1)

$$\vec{\mathbf{r}}(t) = \langle t, f(t)\rangle, x_0 \leq t \leq x_1 \text{ (left-to-right)}$$

$$\vec{\mathbf{r}}(t) = \langle -t, f(-t)\rangle, -x_0 \leq t \leq -x_1 \text{ (right-to-left)}$$

Problem 6. Give a vector function which parameterizes the line segment from the point $(0, 3, -2)$ to the point $(4, -1, 0)$.

Solution. ◊

Contributors.

Problem 7. Give a vector function which parameterizes the curve $y = x^3 - 2x$ from the point $(1, -1)$ to the point $(-1, 1)$.

Solution. ◊

Contributors.

Problem 8. Give a vector function which parameterizes the curve $x^2 + y^2 = 9$ from the point $(3, 0)$ clockwise to the point $(0, -3)$.

Solution. ◊

Contributors.

Definition 9. The **line integral with respect to arclength** of a function of many variables $f(\vec{\mathbf{r}})$ along a curve C is given by

$$\int_C f(\vec{\mathbf{r}})\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(\vec{\mathbf{r}}_{n,i})\Delta s_{n,i}$$

where for each positive integer n we've defined a way to partition C into n pieces

$$\Delta C_{n,1}, \Delta C_{n,2}, \dots, \Delta C_{n,n}$$

where $\Delta C_{n,i}$ has length $\Delta s_{n,i}$, contains the position vector $\vec{\mathbf{r}}_{n,i}$, and

$$\lim_{n\to\infty} \max(\Delta s_{n,i}) = 0$$

Theorem 10. If $\vec{\mathbf{r}}(t)$ is a parametrization of C for $a \leq t \leq b$, then

$$\int_C f(\vec{\mathbf{r}})\,ds = \int_{t=a}^{t=b} f(\vec{\mathbf{r}}(t))\frac{ds}{dt}\,dt$$

Problem 11. Evaluate $\int_C z+2xy\,ds$ where C is the line segment from $(0,-1,3)$ to $(2,2,-3)$.

Solution. ♢

Contributors.

Problem 12. Prove that $\int_C xy\,ds = \int_0^1 t^3\sqrt{1+4t^2}\,dt$ where C is the parabolic arc on $y = x^2$ from $(0,0)$ to $(1,1)$.

Solution. ♢

Contributors.

Definition 13. The **line integral of a vector field $\vec{\mathbf{F}}$** over the curve C is given by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{\mathbf{F}}(\vec{\mathbf{r}}_{n,i}) \cdot \Delta\vec{\mathbf{C}}_{n,i}$$

where for each positive integer n we've defined a way to approximate C with n vectors

$$\Delta\vec{\mathbf{C}}_{n,1}, \Delta\vec{\mathbf{C}}_{n,2}, \dots, \Delta\vec{\mathbf{C}}_{n,n}$$

where $\vec{\mathbf{r}}_{n,i} + \Delta\vec{\mathbf{C}}_{n,i} = \vec{\mathbf{r}}_{n,i+1}$ and

$$\lim_{n\to\infty} \max(|\Delta\vec{\mathbf{C}}_{n,i}|) = 0$$

Definition 14. The line integral of a vector field $\vec{\mathbf{F}}$ over the curve C may be computed by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$$

where $\vec{\mathbf{T}}$ yields the unit tangent vectors to the curve C.

Definition 15. If $\vec{\mathbf{r}}(t)$ is a parametrization of C for $a \leq t \leq b$, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{t=a}^{t=b} \vec{\mathbf{F}} \cdot \frac{d\vec{\mathbf{r}}}{dt}\, dt$$

Problem 16. Prove that $\int_C \langle 2x, y-x\rangle \cdot d\vec{\mathbf{r}} = \int_0^1 23t - 7\, dt$ where C is the line segment given by the vector equation $\vec{\mathbf{r}}(t) = \langle 1-2t, 3t\rangle$ for $0 \leq t \leq 1$.

Solution. ♢

Contributors.

Remark 17. The work done by a force vector field $\vec{\mathbf{F}}$ over the curve C is given by $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

Problem 18. Find the work done by the force vector field $\langle -3y, 3x\rangle$ moving a particle one rotation counter-clockwise around the unit circle $x^2 + y^2 = 1$.

Solution. ♢

Contributors.

Theorem 19. If C may be split into two curves C_1 and C_2, then

$$\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds$$

and

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

Theorem 20. If $-C$ is the curve C oriented in the opposite direction, then

$$\int_C f\, ds = \int_{-C} f\, ds$$

and

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_{-C} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

Problem 21. Write a paragraph explaining why a negative appears in the previous theorem for the line integral of a vector field but not for an arclength line integral.

Solution. $\Diamond$

Contributors.

16.3 The Fundamental Theorem for Line Integrals

Definition 22. If $\nabla f = \vec{\mathbf{F}}$, then f is a **potential function** for the **conservative field** $\vec{\mathbf{F}}$.

Problem 23. Prove that $\langle 2x, -3z, -3y\rangle$ is a conservative field by finding a potential function f for it. Hint: such an f must satisfy that $f = x^2 + \Phi_1(y,z)$, $f = -3yz + \Phi_2(x,z)$, and $f = -3yz + \Phi_3(x,y)$ for some functions Φ_i. (Why?)

Solution. $\Diamond$

Contributors.

Theorem 24. The Fundamental Theorem for Line Integrals: If C is any smooth curve beginning at the point A and ending at the point B, then

$$\int_C \nabla f \cdot d\vec{\mathbf{r}} = [f]_A^B = f(B) - f(A)$$

Problem 25. Prove that if C is any smooth **closed curve** (beginning and ending at the same point), then

$$\int_C \nabla f \cdot d\vec{\mathbf{r}} = 0$$

Solution. $\Diamond$

Contributors.

Problem 26. Compute $\int_C \langle 4, z^2, 2yz\rangle \cdot d\vec{\mathbf{r}}$ where C is the curve given by $\vec{\mathbf{r}}(t) = \langle 2^t, \sin(\pi t), 4t^2\rangle$ for $0 \leq t \leq 1$. Then compute $\int_{C'} \langle 4, z^2, 2yz\rangle \cdot d\vec{\mathbf{r}}$ where C' is the line segment starting at $(1,0,0)$ and ending at $(2,0,4)$.

Solution. $\Diamond$

Contributors.

Problem 27. Prove that if f is a potential function for the vector field $\langle P, Q, R\rangle$, then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$. (Hint: use the mixed derivative theorem.)

Solution. $\Diamond$

Contributors.

Theorem 28. $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is a conservative vector field if and only if $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$.

Problem 29. Prove that $\int_C \langle ye^{xy+z}, xe^{xy+z}, e^{xy+z}\rangle \cdot d\vec{\mathbf{r}} = 0$ where C is the curve given by $\vec{\mathbf{r}}(t) = \langle \frac{1}{1+t^2}, \cos t, e^{1-t^2}\rangle$ for $-1 \leq t \leq 1$.

Solution. $\diamondsuit$

Contributors.

16.4 Green's Theorem

Theorem 30. Let C be the boundary of the region R in the xy plane oriented counter-clockwise, and let $\vec{\mathbf{F}}$ be a two-dimensional vector field. Then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Problem 31. Evaluate $\int_C \langle x^2 - y, x + y\rangle \cdot d\vec{\mathbf{r}}$ where C is the boundary of the unit square oriented counter-clockwise.

Solution. $\diamondsuit$

Contributors.

Problem 32. Find the work done by a force vector field $\langle y, 2x\rangle$ moving an object around the boundary of the triangle with vertices $(1, 2)$, $(-1, -2)$, and $(3, -2)$ oriented clockwise.

Solution. $\diamondsuit$

Contributors.