Packet 4

Packet 4.2: Sections 16.5-16.9

16.5   Curl and Divergence

Definition 1. The **curl** of a vector field $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is given by the expression

$$\operatorname{curl} \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

Problem 2. Prove that if $\vec{\mathbf{F}}$ is conservative, then $\operatorname{curl} \vec{\mathbf{F}} = \vec{\mathbf{0}}$.

Solution. $\diamondsuit$

Contributors.

Remark 3. For a vector field $\vec{\mathbf{F}}$ and direction $\vec{\mathbf{u}}$, $(\operatorname{curl} \vec{\mathbf{F}}) \cdot \vec{\mathbf{u}}$ may be thought of as the tendency of $\vec{\mathbf{F}}$ to "spin" counter-clockwise around $\vec{\mathbf{u}}$.

Problem 4. Compute the curl of $\langle x + y, z^2 - 3, yz \rangle$ around the point $(2, 0, -1)$.

Solution. $\diamondsuit$

Contributors.

Theorem 5. Green's Theorem may be rewritten in terms of curl as follows:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_R (\operatorname{curl} \vec{\mathbf{F}}) \cdot \widehat{\mathbf{k}} \, dA$$

Problem 6. Prove the previous theorem.

Solution. $\diamondsuit$

Contributors.

Definition 7. The **divergence** of a vector field $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is given by the expression

$$\text{div } \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z$$

Problem 8. Prove that the divergence of a curl vector field is always 0. Put another way, show that $\text{div}\,(\text{curl } \vec{\mathbf{F}}) = 0$.

Solution.   ◇

Contributors.

Remark 9. Divergence measures the tendency of a vector field to diverge away from a point.

Problem 10. Compute the divergence of $\langle x + y, z^2 - 3, yz \rangle$ away from the point $(2, 0, -1)$.

Solution.   ◇

Contributors.

Definition 11. The **flux** of a velocity vector field $\vec{\mathbf{F}}$ across a closed curve C is given by

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} \, ds$$

where $\vec{\mathbf{n}}$ yields outward unit normal vectors to C.

Remark 12. Flux measures the tendency of a vector field to flow outward from a closed and bounded region (or inward if the flux is negative).

Theorem 13. Green's Theorem may be rewritten in terms of divergence as follows:

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} \, ds = \iint_R \text{div } \vec{\mathbf{F}} \, dA$$

Problem 14. Compute the flux of the velocity vector field $\langle x + y, x^2 + y^2 \rangle$ across the boundary of the unit square.

Solution.   ◇

Contributors.

16.6   Parametric Surfaces

Remark 15. Just like a curve may be parameterized by $\vec{r}(t)$ for an interval $a \leq t \leq b$, a surface may be parameterized by $\vec{r}(u, v)$ for a region R in the uv plane.

Theorem 16. Following are some common surface parameterizations.

- The surface $z = f(x, y)$ may be parametrized by

$$\vec{r}(x, y) = \langle x, y, f(x, y) \rangle$$

- A surface determined by a cylindrical coordinate equation may be parametrized by substituting into

$$\vec{r} = \langle r \cos \theta, r \sin \theta, z \rangle$$

- A surface determined by a spherical coordinate equation may be parametrized by substituting into

$$\vec{r} = \langle \rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi \rangle$$

Problem 17. Find a parameterization from the xy plane to the plane $2x - y + z = 7$ in xyz space.

Solution.   ◇

Contributors.

Problem 18. Find the parameterization from the rectangle $0 \leq z \leq 3$ and $0 \leq \theta \leq 2\pi$ to the conical surface $z = \sqrt{x^2 + y^2}$ below the plane $z = 3$ in xyz space. (Hint: find the cylindrical coordinate equation for the surface.)

Solution.   ◇

Contributors.

Problem 19. Find the parameterization from the rectangle $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$ to the spherical surface $x^2 + y^2 + z^2 = 9$ in xyz space. (Hint: find the spherical coordinate equation for the surface.)

Solution.   ◇

Contributors.

16.7   Surface Integrals

Definition 20. The **surface integral** of a function $f(x, y, z)$ over a surface S in xyz space is given by

$$\iint_S f(\vec{\mathbf{r}}) \, d\sigma = \iint_R f(\vec{\mathbf{r}}(u, v))|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| \, dA$$

where $\vec{\mathbf{r}}(u, v)$ is a parameterization from the region R in the uv plane to the surface S.

Theorem 21. The surface area of S is given by

$$\iint_S d\sigma = \iint_S 1 \, d\sigma$$

Problem 22. Use the parameterization

$$\vec{\mathbf{r}}(\phi, \theta) = \langle \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \rangle$$

from $0 \le \phi \le \pi, 0 \le \theta \le 2\pi$ to the unit sphere to show that the surface area of the unit sphere is 4π. (Note that this matches the formula $SA = 4\pi r^2$ used in high school geometry.)

Solution. $\diamond$

Contributors.

Problem 23. Show that the area of the parallelogram with vertices $(0, 0, 0)$, $(2, 1, 2)$, $(0, 2, -1)$, and $(2, 3, 1)$ is $3\sqrt{5}$ using a surface integral. (Hint: use $\vec{\mathbf{r}}(u, v) = \langle 2u, u + 2v, 2u - v \rangle$.)

Solution. $\diamond$

Contributors.

Definition 24. An **orientation** of a surface is a continuous unit vector field normal to the surface.

Remark 25. Orienting a surface is akin to choosing one side or another of the surface.

Remark 26. Examples of non-orientable surfaces are the Mobiüs strip and Klein bottle.

Definition 27. The **surface integral** of a vector field $\vec{\mathbf{F}}$ over an oriented surface S in xyz space is given by

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\sigma} = \iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} \, d\sigma = \iint_R \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) \, dA$$

where $\vec{\mathbf{n}}$ is the orientation of the surface and giving its orientation, and $\vec{\mathbf{r}}(u, v)$ is an appropriate parameterization from the region R in the uv plane to the surface S.

Definition 28. The **flux** across a closed oriented surface (such as the boundary of a solid) is given by

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\sigma}$$

Problem 29. Use the parameterization

$$\vec{\mathbf{r}}(\phi, \theta) = \langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi \rangle$$

from $0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$ to the sphere $x^2 + y^2 + z^2 = 9$ to prove that the flux across it for the vector field $\langle x, y, z \rangle$ is

$$\int_0^{2\pi} \int_0^{\pi} 27\sin\phi \, d\phi \, d\theta$$

Solution. ◇

Contributors.

16.8 Stokes' Theorem

Theorem 30. Let S be a surface with orientation $\vec{\mathbf{n}}$ and with boundary C oriented counterclockwise with respect to $\vec{\mathbf{n}}$. Then

$$\iint_S \text{curl } \vec{\mathbf{F}} \cdot d\vec{\sigma} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

Problem 31. Let S be the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$. Use Stokes' Theorem to prove that

$$\iint_S \langle 2y, 2z, 2x \rangle \cdot d\vec{\sigma} = \int_0^{2\pi} \cos^3(t) \, dt$$

(Hint: what's the curl of $\langle z^2, x^2, y^2 \rangle$?).

Solution. ◇

Contributors.

16.9 Divergence Theorem

Theorem 32. Let S be the boundary of a solid D oriented outwards. Then

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\sigma} = \iiint_D \text{div } \vec{\mathbf{F}} \, dV$$

Problem 33. Let S be the boundary of the unit cube in xyz space. Use the Divergence Theorem to prove that

$$\iint_S \langle x + y, y^2 + z^2, z^3 + x^3 \rangle \cdot d\vec{\sigma} = \int_0^1 \int_0^1 \int_0^1 1 + 2y + 3z^2 \, dz \, dy \, dx$$

Solution. ◇

Contributors.

16.10   A small remark and puzzle

Remark 34. Using derivatives, gradients, curl, and divergence, we may observe that several kinds of integrals may be evaluated by observing how the integrand behaves on the boundary of the domain of integration, and vice versa.

$$\int_{[a,b]} f'(x)\, dx = [f(x)]_a^b$$

$$\int_C \nabla f \cdot d\vec{\mathbf{r}} = [f(P)]_A^B$$

$$\iint_R \text{div}\, \vec{\mathbf{F}}\, dA = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds$$

$$\iint_R Q_x - P_y\, dA = \int_C \langle P, Q \rangle \cdot d\vec{\mathbf{r}}$$

$$\iint_S \text{curl}\, \vec{\mathbf{F}} \cdot d\vec{\sigma} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

$$\iiint_D \text{div}\, \vec{\mathbf{F}}\, dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\sigma}$$

Problem 35. (OPTIONAL) This has nothing to do with the above remark, but here's a puzzle for reading this far.

Wayne Brady is hosting a gameshow, and you've been called down from the audience to attempt to win fabulous prizes. Wayne gives you the choice of three doors: A, B, and C. He asks you to choose a door, explaining that only one of the three doors holds a prize behind it.

After you choose, Wayne opens one of the doors that you didn't choose to reveal nothing behind it. He then offers you the opportunity to switch your door with the other unopened door, after which you will immediately be given whatever is behind it. Should you stick with your initial guess, or should you switch, or does it even matter? Why?

Solution. ◇

Contributors.