Packet 3

Packet 3.1: Sections 15.1-15.3

15.1 Double Integrals over Rectangles

Definition 1. We define the **double integral** of a function $f(x, y)$ over a region R to be

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{n,i}, y_{n,i})\Delta A_{n,i}$$

where for each positive integer n we've defined a way to partition R into n pieces

$$\Delta R_{n,1}, \Delta R_{n,2}, \ldots, \Delta R_{n,n}$$

where $\Delta R_{n,i}$ has area $\Delta A_{n,i}$, contains the point $(x_{n,i}, y_{n,i})$, and

$$\lim_{n\to\infty} \max(\Delta A_{n,i}) = 0$$

Remark 2. This basically defines the double integral to be the **Riemann sum** of a bunch of rectangular box volumes, just as the single definite integral is the Riemann sum of a bunch of rectangle areas. Therefore it represents the net volume between the curve $z = f(x, y)$ and the xy-plane above/below R.

Theorem 3. For the rectangle

$$R : a \le x \le b, c \le y \le d$$

the **Midpoint Rule** says that

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\Delta A$$

where $(\overline{x_i}, \overline{y_j})$ is the midpoint of the $i \times j$ rectangle.

Problem 4. Divide $R : 0 \le x \le 4, 0 \le y \le 2$ into four congruent pieces arranged two-by-two, and then use the midpoint rule to approximate the double integral $\displaystyle\iint_R 2x + 2y + 4\,dA$.

Solution. $\Diamond$

Contributors.

Problem 5. Divide $R : -2 \leq x \leq 2, 0 \leq y \leq 2$ into four congruent pieces arranged two-by-two, and then use the midpoint rule to approximate the double integral $\iint_R 12x^2y\,dA$

Solution. $\Diamond$

Contributors.

Problem 6. Divide $R : 0 \leq x \leq \pi/2, 0 \leq y \leq \pi/2$ into four congruent pieces arranged two-by-two, and then use the midpoint rule to approximate the double integral $\iint_R \cos(x+y)\,dA$

Solution. $\Diamond$

Contributors.

15.2 Iterated Integrals

Definition 7. If a solid is embedded in xyz space, and $A(x)$ is the area of that solid's cross-section for each x-value, then the solid's volume is

$$V = \int_a^b A(x)\,dx$$

Theorem 8. A double integral over a rectangle

$$R : a \leq x \leq b, c \leq y \leq d$$

can be evaluated using the **iterated integrals**:

$$\iint_R f(x,y)\,dA = \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} f(x,y)\,dy\right]\,dx = \int_{y=c}^{y=d} \left[\int_{x=a}^{x=b} f(x,y)\,dx\right]\,dy$$

Remark 9. Iterated integrals are often shortened as follows:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} f(x,y)\,dy\right]\,dx$$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_{y=c}^{y=d} \left[\int_{x=a}^{x=b} f(x,y)\,dx\right]\,dy$$

Remark 10. When evaluating iterated integrals, only the innermost d-variable acts as a variable, while other variables act as constants. Put another way, find the partial anti-derivatives.

Problem 11. If $R : 0 \leq x \leq 4, 0 \leq y \leq 2$, then write $\iint_R 2x + 2y + 4\,dA$ as an iterated integral. Then evaluate it, comparing its value to the approximation you found in the previous section.

Solution. $\Diamond$

Contributors.

Problem 12. If $R : -2 \leq x \leq 2, 0 \leq y \leq 2$, then write $\iint_R 12x^2y\,dA$ as an iterated integral. Then evaluate it, comparing its value to the approximation you found in the previous section.

Solution. $\Diamond$

Contributors.

Problem 13. If $R : 0 \leq x \leq \pi/2, 0 \leq y \leq \pi/2$, then write $\iint_R \cos(x + y)\,dA$ as an iterated integral. Then evaluate it, comparing its value to the approximation you found in the previous section.

Solution. $\Diamond$

Contributors.