Packet 3

Packet 3.1: Sections 15.1-15.3 and 15.7

15.1 Double Integrals over Rectangles

Definition 1. We define the **double integral** of a function $f(x, y)$ over a region R to be

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{n,i}, y_{n,i})\Delta A_{n,i}$$

where for each positive integer n we've defined a way to partition R into n pieces

$$\Delta R_{n,1}, \Delta R_{n,2}, \ldots, \Delta R_{n,n}$$

where $\Delta R_{n,i}$ has area $\Delta A_{n,i}$, contains the point $(x_{n,i}, y_{n,i})$, and

$$\lim_{n\to\infty} \max(\Delta A_{n,i}) = 0$$

Remark 2. This basically defines the double integral to be the **Riemann sum** of a bunch of rectangular box volumes, just as the single definite integral is the Riemann sum of a bunch of rectangle areas. Therefore it represents the net volume between the curve $z = f(x, y)$ and the xy-plane above/below R.

Theorem 3. For the rectangle

$$R : a \le x \le b, c \le y \le d$$

the **Midpoint Rule** says that

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\Delta A$$

where $(\overline{x_i}, \overline{y_j})$ is the midpoint of the $i \times j$ rectangle.

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Problem 4. Divide $R : 0 \le x \le 4, 0 \le y \le 2$ into four congruent pieces arranged two-by-two, and then use the midpoint rule to approximate the double integral $\iint_R 2x + 2y + 4\,dA$.

Solution.

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$$\sum f = f(1,\frac{1}{2})(2) + f(1,\frac{3}{2})(2) + f(3,\frac{1}{2})(2) + f(3,\frac{3}{2})(2)$$

$$= (2(1) + 2(\frac{1}{2}) + 4)(2) + (2(1) + 2(\frac{3}{2}) + 4)(2) + (2(3) + 2(\frac{1}{2}) + 4)(2) + (2(3) + 2(\frac{3}{2})4)(2)$$

$$= 14 + 18 + 22 + 26 = 80$$

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Contributors. Blake Wade, Connor Dayton

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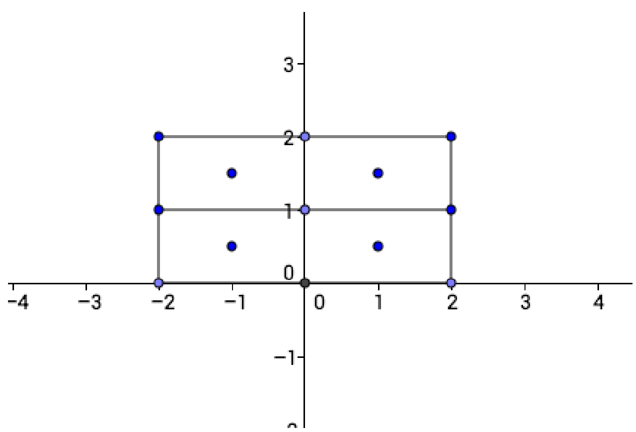
Problem 5. Divide $R : -2 \leq x \leq 2, 0 \leq y \leq 2$ into four congruent pieces arranged two-by-two, and then use the midpoint rule to approximate the double integral $\iint_R 12x^2y\,dA$



Solution.

$$\sum f = f(-1, 1.5)(2) + f(-1, .5)(2) + f(1, 1.5)(2) + f(1, .5)(2)$$

$$= 2(18) + 2(6) + 2(18) + 2(6)$$

$$= 96$$

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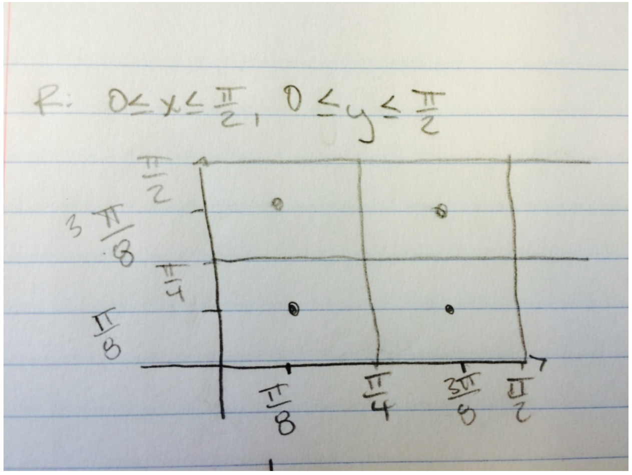
Problem 6. Divide $R : 0 \leq x \leq \pi/2, 0 \leq y \leq \pi/2$ into four congruent pieces arranged two-by-two, and then use the midpoint rule to approximate the double integral $\iint_R \cos(x+y)\,dA$

Solution.



$$\sum f = f(\frac{\pi}{8},\frac{\pi}{8})(\frac{\pi^2}{16}) + f(\frac{\pi}{8},\frac{3\pi}{8})(\frac{\pi^2}{16}) + f(\frac{3\pi}{8},\frac{\pi}{8})(\frac{\pi^2}{16}) + f(\frac{3\pi}{8},\frac{3\pi}{8})(\frac{\pi^2}{16})$$

$$= cos(\frac{\pi}{8}+\frac{\pi}{8})(\frac{\pi^2}{16}) + cos(\frac{\pi}{8}+\frac{3\pi}{8})(\frac{\pi^2}{16}) + cos(\frac{3\pi}{8}+\frac{\pi}{8})(\frac{\pi^2}{16}) + cos(\frac{3\pi}{8}+\frac{3\pi}{8})(\frac{\pi^2}{16})$$

$$= cos(\frac{\pi}{4})(\frac{\pi^2}{16}) + cos(\frac{\pi}{2})(\frac{\pi^2}{16}) + cos(\frac{\pi}{2})(\frac{\pi^2}{16}) + cos(\frac{3\pi}{4})(\frac{\pi^2}{16})$$

$$= (\frac{\sqrt{2}}{2})(\frac{\pi^2}{16}) + (0)(\frac{\pi^2}{16}) + (0)(\frac{\pi^2}{16}) + (-\frac{\sqrt{2}}{2})(\frac{\pi^2}{16})$$

$$= 0$$

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Contributors. Blake Diggs, Blake Wade

15.2 Iterated Integrals

Definition 7. If a solid is embedded in xyz space, and $A(x)$ is the area of that solid's cross-section for each x-value, then the solid's volume is

$$V = \int_a^b A(x)\,dx$$

Theorem 8. A double integral over a rectangle

$$R : a \leq x \leq b, c \leq y \leq d$$

can be evaluated using the **iterated integrals**:

$$\iint_R f(x,y)\,dA = \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} f(x,y)\,dy\right] dx = \int_{y=c}^{y=d} \left[\int_{x=a}^{x=b} f(x,y)\,dx\right] dy$$

Remark 9. Iterated integrals are often shortened as follows:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} f(x,y)\,dy\right] dx$$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_{y=c}^{y=d} \left[\int_{x=a}^{x=b} f(x,y)\,dx\right] dy$$

Remark 10. When evaluating iterated integrals, only the innermost d-variable acts as a variable, while other variables act as constants. Put another way, find the partial anti-derivatives.

Remark 11. The order of a double iterated integral with constant bounds may be reversed by swapping **both** the bounds of integration and the differentials dx/dy. (This will not work if there are any variables in the bounds as we'll see in the next section.)

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Problem 12. Evaluate $\displaystyle\int_0^3 \int_2^4 xy^2 + x^3\,dx\,dy$.

Solution. $\displaystyle\int_0^3 \left[\int_2^4 xy^2 + x^3\,dx\right] dy$
$\int_2^4 xy^2 + x^3\,dx = [(x^2y^2)/2 + x^4/4]_2^4 = 6y^2 + 60$
$\displaystyle\int_0^3 6y^2 + 60\,dy$
$= [2y^3 + 60y]_0^3 = 234$

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Problem 13. If $R : 0 \leq x \leq 4, 0 \leq y \leq 2$, then write $\displaystyle\iint_R 2x + 2y + 4\,dA$ as an iterated integral. Then evaluate it, comparing its value to the approximation you found in the previous section.

Solution. $\int_{x=0}^{x=4} \left[\int_{y=0}^{y=2} 2x + 2y + 4dy\right] dx$
$\int_{y=0}^{y=2} 2x + 2y + 4dy = (2xy + y^2 + 4y)_0^2 = 4x + 4 + 8 - 0 - 0 - 0 = 4x + 12$
$\int_{x=0}^{x=4} (4x + 12) = (2x^2 + 12x)_0^4 = 32 + 48 - 0 - 0 = 80$ ◊

Contributors. Will ParKer, Connor Dayton

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Problem 14. If $R : -2 \leq x \leq 2, 0 \leq y \leq 2$, then write $\displaystyle\iint_R 12x^2y\,dA$ as an iterated integral. Then evaluate it, comparing its value to the approximation you found in the previous section.

Solution.

$$\int_{x=-2}^{x=2}\left[\int_{y=0}^{y=2} 12x^2 y dy\right] dx$$

$$\int_{x=-2}^{x=2}[12x^2\frac{y^2}{2}]_0^2 dx = \int_{x=-2}^{x=2}[6x^2y^2]_0^2 dx = \int_{x=-2}^{x=2}[6x^2(2)^2]dx = \int_{x=-2}^{x=2}[24x^2]dx$$

$$= [24\frac{x^3}{3}]_{x=-2}^{x=2} = [8x^3]_{x=-2}^{x=2} = 8(8) - 8(-8) = 128$$

$\Diamond$

Contributors. BlaKe Wade

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Problem 15. If $R : 0 \leq x \leq \pi/2, 0 \leq y \leq \pi/2$, then write $\int_R \cos(x+y)\, dA$ as an iterated integral. Then evaluate it, comparing its value to the approximation you found in the previous section.

Solution.

$$\int_{y=0}^{y=\pi/2}\left[\int_{x=0}^{x=\pi/2} \cos(x+y)dx\right] dy$$

$$\int_{x=0}^{x=\pi/2} \cos(x+y)dx = \sin(x+y)|_{x=0}^{x=\pi/2} = \sin(\pi/2+y) - \sin(y)$$

$$\int_{y=0}^{y=\pi/2}(\sin(\pi/2+y) - \sin(y))dy = (-\cos(\pi/2+y) + \cos(y))|_{y=0}^{y=\pi/2}$$

$$= -\cos(\pi/2+\pi/2) + \cos(\pi/2) + \cos(\pi/2+0) - \cos(0)$$

$$-(-1) + (0) + (0) - (1) = 0$$

$\Diamond$

Contributors. Will Parker, Natalie Patton

15.3 Double Integrals over General Regions

Theorem 16. A double integral over a **Type I** region bounded by top and bottom curves

$$R : a \leq x \leq b, g(x) \leq y \leq h(x)$$

can be evaluated using the iterated integral:

$$\iint_R f(x,y)\, dA = \int_{x=a}^{x=b}\left[\int_{y=g(x)}^{y=h(x)} f(x,y)\, dy\right] dx$$

Theorem 17. A double integral over a **Type II** region bounded by right and left curves

$$R : g(y) \leq x \leq h(y), c \leq y \leq d$$

can be evaluated using the iterated integral:

$$\iint_R f(x,y)\,dA = \int_{y=c}^{y=d} \left[\int_{x=g(y)}^{x=h(y)} f(x,y)\,dx \right] dy$$

Remark 18. Note that you *never* have variables of integration on the outside-most integral in an iterated integral.

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Problem 19. Evaluate $\int_0^4 \int_{\sqrt{y}}^2 6x + 30y\,dx\,dy$.

Solution.

$$\int_0^4 \left[\int_{\sqrt{y}}^2 6x + 30y dx \right] dy$$

$$\int_{\sqrt{y}}^2 (6x + 30y)dx = (3x^2 + 30xy)|_{\sqrt{y}}^2$$

$$= 3(2)^2 + 30(2)y - 3(\sqrt{y})^2 - 30(\sqrt{y})$$

$$= 12 + 60y - 3y - 30y^{3/2}$$

$$= 12 + 57y - 30y^{3/2}$$

$$\int_0^4 (12 + 57y - 30y^{3/2})dy$$

$$= (12y + 28.5y^2 - 12y^{5/2})|_0^4$$

$$= 12(4) + 28.5(4)^2 - 12(4)^{5/2} - 0 - 0 - 0$$

$$= 48 + 456 - 384$$

$$= 120$$

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Contributors. Will Parker, Natalie Patton

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Problem 20. Evaluate $\iint_R 6xy + 3\,dA$ where R is the region between $x = 4 - y^2$ and $x = y^2 - 4$.

?

Solution. $\int_{-2}^{2}\int_{y^2-4}^{4-y^2} 6xy + 3dxdy$
$= \int_{y^2-4}^{4-y^2} 6xy + 3dx = [3x^2y + 3x]$ from $y^2 - 4$ to $4 - y^2$
$= 3(4-y^2)^2 2y + 3(4-y^2) - 3(y^2-4)^2 2y + 3(y^2-4)$
$= 2y(3y^4 - 24y^2 + 48) + 12 - 3y^2 - 2y(3y^4 - 24y^2 + 48) + 3y^2 - 12$
$= \int_{-2}^{2} -96y^3 + 192ydy$
$= [-24y^4 + 96y^2]$ from -2 to 2
$= -24 * (2)^4 + 96 * (2)^2 - -24 * (-2)^4 + 96 * (-2)^2$
$= -384 + 384 - (-384 + 384)$
$= 0$

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Contributors. Connor Dayton

Problem 21. Evaluate $\iint_R 6xy + 3\,dA$ where R is the region between $x = 4 - y^2$ and $x = y^2 - 4$.

Solution. ◊

Contributors.

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Problem 22. Evaluate $\iint_R 1\,dA$ where R is the triangle with vertices $(0,0)$, $(1,1)$, and $(1,2)$.

Solution.

how?

$$\int_0^1 \int_x^{2x} 1\,dy\,dx$$

$$\int_0^1 [\int_x^{2x} 1\,dy]\,dx = \int_0^1 [y]_x^{2x}\,dx = \int_0^1 [2x - x]\,dx$$

$$\int_0^1 x\,dx = [\frac{x^2}{2}]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

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Contributors. Blake Wade

Remark 23. You cannot blindly switch the bounds of integration to change the order of integration for a non-rectangular region. However, if the region is both Type I and Type II, then the order of integration may be swapped by reinterpreting the region as the opposite type.

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Problem 24. Evaluate the Type I iterated integral $\displaystyle\int_0^1 \int_x^1 \frac{2}{\sqrt{4+y^2}}\,dy\,dx$ by first rewriting it as a Type II iterated integral.

Solution.

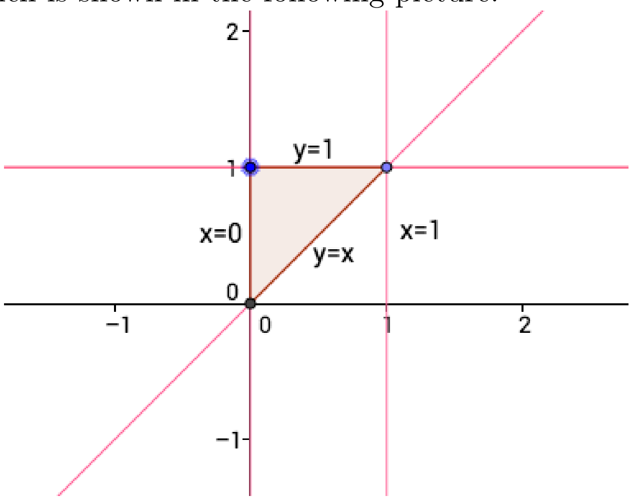
$$\int_{x=0}^{x=1}\left[\int_{y=x}^{y=1}\frac{2}{\sqrt{4+y^2}}\,dy\right]dx$$

According to the limits of integration of the given integral, the region of integration is

$$0 \le x \le 1$$

$$x \le y \le 1$$

which is shown in the following picture.



Since we can also describe the region by

$$0 \le y \le 1$$

$$0 \le x \le y$$

, the integral with the order changed is

$$\int_0^1\int_x^1\frac{2}{\sqrt{4+y^2}}\,dy\,dx=\int_0^1\int_0^y\frac{2}{\sqrt{4+y^2}}\,dx\,dy$$

With this new $dx\,dy$ order, we first integrate with respect to x:

$$\int_{y=0}^{y=1}\left[\int_{x=0}^{x=y}\frac{2}{\sqrt{4+y^2}}\,dx\right]dy$$

$$\int_0^1\frac{2x}{\sqrt{4+y^2}}\Big|_0^y\,dy$$

$$\int_0^1\frac{2y}{\sqrt{4+y^2}}\,dy$$

$$u=y^2+4$$

$$du=2y\,dy$$

$$\int u^{-\frac{1}{2}}\,du$$

$$= 2u^{\frac{1}{2}}] = 2\sqrt{4+y^2}$$

$$2\sqrt{4+y^2}|_0^1$$

$$= 4 - 2\sqrt{5}$$

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Contributors. Natalie Patton

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Problem 25. Evaluate the Type II iterated integral $\int_0^1 \int_{\sqrt{y}}^1 3\pi \sin(\pi x^3)\,dx\,dy$ by first rewriting it as a Type I iterated integral.

Solution. Changing the bounds to evaluate it as a Type I integral

$\int_0^{x^2} \int_0^1 3\pi \sin(\pi x^3)\,dx\,dy$ ◊

Contributors. Blake Diggs

Theorem 26. The area of a region R in the plane is

$$A = \iint_R dA = \iint_R 1\,dA$$

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Problem 27. Express the area of the parallelogram with vertices $(-1, 2)$, $(3, 2)$, $(4, 1)$, $(0, 1)$ as a double iterated integral.

Solution. Area of a parrallelogram = base*height

from graphing the verticies, I was able to get the height and the base

$\int_1^2 \int_0^4 dxdy$

$\int_0^4 dx = 4 - 0 = 4$

$\int_1^2 4dy = 4(2) - 4(1) = 4$

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Contributors. Connor Dayton

Definition 28. The average value of a two-variable function f over a region R is

$$\frac{1}{\text{Area of } R} \iint_R f(x, y)\,dA$$

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Problem 29. Express the average value of $f(x, y) = \sin(\frac{x}{2y})$ over the triangle with vertices $(0, 1)$, $(1, 1)$, $(0, 2)$ as a double iterated integral.

Solution. Area of R $= (1/2)b * h = (1/2)(1-0)(2-1) = 1/2$
average value$= 2 * \int_1^2 \int_0^{-y+2} sin(\frac{x}{2y})dxdy$

how?

$\diamond$

Contributors. Will Parker

Definition 30. The **centroid** $(\overline{x}, \overline{y})$ of a region R is the average position of all the points in R.

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Problem 31. Prove that the centroid of a region R is given by the expressions:

$$\overline{x} = \frac{1}{\iint_R 1\,dR} \iint_R x\,dR$$

$$\overline{y} = \frac{1}{\iint_R 1\,dR} \iint_R y\,dR$$

(It's okay to prove one of these and say the other follows from basically the same argument.)

Solution.

$$\frac{1}{\text{Area of } R} \iint_R f(x,y)\,dA = \frac{1}{\iint_R 1dR} \iint_R f(x,y)\,dA$$

$$\overline{x} = \frac{1}{\iint_R 1\,dR} \iint_R x\,dR$$

$$\overline{y} = \frac{1}{\iint_R 1\,dR} \iint_R y\,dR$$

I replaced Area of R with $\iint_R 1dR$ from theorem 26, to get this expression. Then i pulled out x because im looking for the average value of x. and same with y.

$\diamond$

Contributors. Blake Wade

Remark 32. Not every region is Type I or Type II.

Theorem 33. If R can be split into two regions R_1, R_2, then

$$\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA$$

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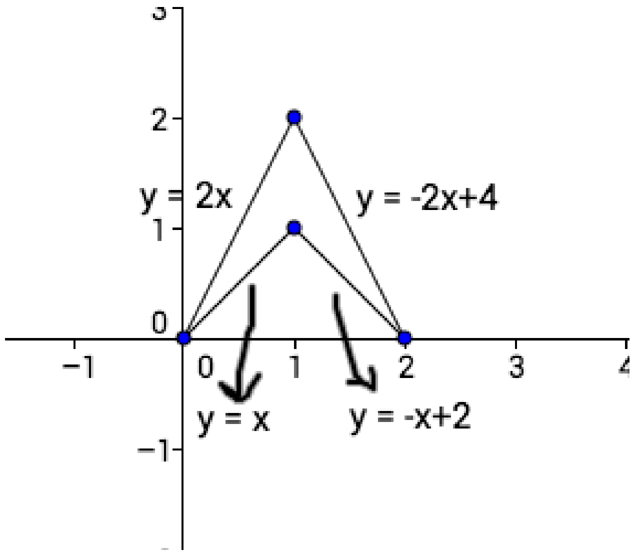
Problem 34. Express $\iint_R xe^{x+y}\,dA$ as the sum of two iterated integrals, where R is the quadrilateral with vertices at $(0,0)$, $(1,1)$, $(2,0)$, and $(1,2)$.



Solution.

$$\int_0^1 \left[\int_x^{2x} xe^{x+y}\,dy\right]\,dx + \int_1^2 \left[\int_{-x+2}^{-2x+4} xe^{x+y}\,dy\right]\,dx$$

$\diamond$

Contributors. Natalie Patton

15.7 Triple Integrals

Definition 35. The **triple integral** of a function $f(x, y, z)$ over a solid D is given by

$$\iiint_D f(x, y, z)\,dV = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{n,i}, y_{n,i}, z_{n,i})\Delta V_{n,i}$$

where for each positive integer n we've defined a way to partition D into n pieces

$$\Delta D_{n,1}, \Delta D_{n,2}, \ldots, \Delta D_{n,n}$$

where $\Delta D_{n,i}$ has volume $\Delta V_{n,i}$, contains the point $(x_{n,i}, y_{n,i}, z_{n,i})$, and

$$\lim_{n\to\infty} \max(\Delta V_{n,i}) = 0$$

Theorem 36. The triple integral over the rectangular box

$$D : a_1 \le x \le a_2, b_1 \le y \le b_2, c_1 \le z \le c_2$$

can be expressed as the iterated integrals:

$$\iiint_D f(x, y, z)\,dV = \int_{a_1}^{a_2} \int_{b_1}^{b_2} \int_{c_1}^{c_2} f(x, y, z)\,dz\,dy\,dx$$

$$= \int_{b_1}^{b_2} \int_{c_1}^{c_2} \int_{a_1}^{a_2} f(x, y, z)\,dx\,dz\,dy = \int_{a_1}^{a_2} \int_{c_1}^{c_2} \int_{b_1}^{b_2} f(x, y, z)\,dy\,dz\,dx = \cdots$$

Problem 37. Evaluate $\iiint_D 8xz - y^2\,dV$ where D is the unit cube: $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

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Solution.

$$\iint_D \left[\int 8xz - y^2\,dx\right] dy\,dz$$

$$\int_D \left[\int 4x^2z - y^2x\,dy\right] dz$$

$$\int_D 4x^2zy - (y^3)/(3x)\,dz$$

$$2x^2z^2y - (y^3xz)/3 + C$$

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Contributors. Blake Diggs

Theorem 38. If the solid D is bounded by the surfaces

$$h_1(x,y) \leq z \leq h_2(x,y)$$

and has shadow R in the xy-plane, then

$$\iiint_D f(x,y,z)\,dV = \iint_R \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\right] dA$$

Remark 39. z may be replaced with x or y by changing the orientation to let x or y be "up".

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Problem 40. Evaluate $\int_{-1}^{1}\int_{1+y}^{2+y}\int_{0}^{2} z\,dx\,dz\,dy$.

Solution. $\int_{-1}^{1}\int_{1+y}^{2+y}\int_{0}^{2} z\,dx\,dz\,dy$
$\int_0^2 z\,dx = 2z$
$\int_{1+y}^{2+y} 2z\,dz = (2+y)^2 - (1+y)^2 = y^2 + 4y + 4 - (y^2 + 2y + 1) = 2y + 3$
$\int_{-1}^{1} 2y + 3\,dy = y^2 + 3y = 1 + 3 - (\cancel{-}+1 - 3) = \cancel{10}\ 6$

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Contributors. Connor Dayton

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Problem 41. Express $\iiint_D xy^2z\,dV$ as a triple iterated integral, where D is the solid in the first octant bounded by the coordinate planes, $z = 1 - y^2$, and $x = 4$.

?

Solution. $\iiint_D xy^2z\,dV = \iint_R (\int_{z=0}^{z=1-y^2} xd^2zdz)dA =$

$\int_{x=0}^{x=4}\int_{y=0}^{y=1}\int_{z=0}^{z=1-y^2} xy^2zdzdydx$ ✓ ◊

why?

Contributors. Will Parker

Theorem 42. The volume of a solid D in xyz space is

$$V = \iiint_D dV = \iiint_D 1\,dV$$

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Problem 43. Express the volume of the pyramid with vertices $(0,0,0)$, $(3,0,0)$, $(0,2,0)$, and $(0,0,1)$ as a triple iterated integral.

? why

Solution.

$$\int_{x=0}^{x=3}\int_{y=0}^{y=-\frac{2}{3}x+2}\int_{z=0}^{z=-\frac{2}{3}x-\frac{1}{2}y+1} 1dzdydx$$

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Contributors. Blake Wade

Definition 44. The average value of a three-variable function f over a solid D is

$$\frac{1}{\text{Volume of } D}\iiint_D f(x,y,z)\,dV$$

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Problem 45. Express the average value of the function $f(x,y,z) = z + xy$ over the solid bounded by the surfaces $z = 4 - x^2 - y^2$ and $z = 4x^2 + 4y^2 - 16$ as triple integrals.

Solution. Volume of the Domain: $V = \iiint_D 1\,dV$

$= \int_{4x^2+4y^2-16}^{4-x^2-y^2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-2}^{2} 1dzdydx$

This is found by taking the bounds set by the higher and lower functions, then the top and bottom of the circle made by the functions meeting, and the points on the x axis where the circle touches.

$\iiint_D f(x,y,z)dV =$

$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{4x^2+4y^2-16}^{4-x^2-y^2} z + xydzdydx$ ✓

Average value $= \frac{1}{\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{4x^2+4y^2-16}^{4-x^2-y^2} 1dzdydx} * \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{4x^2+4y^2-16}^{4-x^2-y^2} z + xydzdydx$ ✓

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Contributors. Will Parker

Theorem 46. If D can be split into two solids D_1, D_2, then

$$\iiint_D f(x,y,z)\,dV = \iiint_{D_1} f(x,y,z)\,dV + \iiint_{D_2} f(x,y,z)\,dV$$