Calculus III - Spring 2015 - Mr. Clontz - Test 4 Makeup

Name: Solutions Class: ______ Type: ______

- This exam is open-"everything", provided you do not plagiarize. *Do not leave any unsupported answers!*
- Write your solutions so that a fellow student who has a perfect understanding of previous math classes and packets but has never seen that type of problem before could follow your work.
- Individual Test: You will have 40 minutes to complete this test on your own. You may not communicate with anyone during this period.
- Group Test: You will have 40 minutes to complete an identical test. You may collaborate with your group members during this time. All solutions must still be written by yourself, and may not be directly copied from another student.

1. (8 points) Prove that

$$\int_C xy\,ds = \int_0^{2\pi} 8\sin t\cos t\,dt$$

where C is the full counter-clockwise rotation of the circle $x^2 + y^2 = 4$.

Let C be modeled by the vector equation

$\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle \qquad 0 \le t \le 2\pi$

$\Rightarrow \frac{d\vec{r}}{dt} = \langle -2\sin t, 2\cos t\rangle \quad \Rightarrow \frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$

Then

$$\int_C xy\,ds = \int_0^{2\pi} (2\cos t)(2\sin t)(2)\,dt$$

$$= \int_0^{2\pi} 8\sin t\cos t\,dt$$

2. (7 points) Prove that

$$\int_C \langle yz, -x, x+y \rangle \cdot d\vec{\mathbf{r}} = \int_0^1 4t^2 + 6t - 6\,dt$$

where C is the line segment from $(1, 0, 1)$ to $(0, 2, -1)$. (Hint: this is *not* a conservative field.)

Let C be modeled by

$\vec{r}(t) = \langle 1-t, 2t, 1-2t \rangle \quad 0 \le t \le 1$

$d\vec{r}/dt = \langle -1, 2, -2 \rangle$

Then

$$\int_C \langle yz, -x, x+y \rangle \cdot d\vec{r} = \int_0^1 \langle 2t(1-2t), -(1-t), 1-t+1-2t \rangle \cdot \langle -1, 2, -2 \rangle\,dt$$

$$= \int_0^1 (4t^2 - \cancel{2t}) + (\cancel{2t} - 2) + (6t - 4)\,dt$$

$$= \int_0^1 4t^2 + 6t - 6\,dt$$

3. (8 points) Prove that

$$\int_C \langle 2 + 3yz, 3xz, 3xy \rangle \cdot d\vec{\mathbf{r}} = 2$$

where C is a curve beginning at $(1, 2, 0)$ and ending at $(2, 0, -1)$. (Hint: this *is* a conservative field.)

Assume $\nabla f = \langle 2+3yz, 3xz, 3xy \rangle$. Then

$$f_x = 2+3yz \Rightarrow f = 2x + 3xyz + \Phi$$

$$f_y = 3xz \Rightarrow f = 3xyz + \Phi$$

$$f_z = 3xy \Rightarrow f = 3xyz + \Phi$$

which is satisfied by $f = 2x + 3xyz$.

So

$$\int_C \nabla f \cdot d\vec{r} = \Big[2x + 3xyz\Big]_{(1,2,0)}^{(2,0,-1)}$$

$$= (4+0) - (2+0)$$

$$= 2$$

4. (7 points) Let D be the solid bounded by the a surface S. Use the Divergence Theorem (section 16.9) to prove that

$$\iint_S \langle x^2+1, y^2-xz, 1+z^2\rangle \cdot d\vec{\sigma} = 2\iiint_D x+y+z\,dV$$

Let $\vec{F} = \langle x^2+1, y^2-xz, 1+z^2\rangle$, so

$\operatorname{div}\vec{F} = 2x+2y+2z$.

Then

$$\iint_S \vec{F}\cdot d\vec{s} = \iiint_D \operatorname{div}\vec{F}\,dV$$

$$= \iiint_D 2x+2y+2z\,dV$$

$$= 2\iiint_D x+y+z\,dV$$