The *recursion principle* is the rule that states that $\mathsf{rec}_{A\times B}(f) : A\times B \to C$ is well defined, and it is taken as a primitive notion. We are asked to instead take the well-definedness of pr_1 and pr_2 as primitive instead and to derive the recursion principle.

Definitions of $\mathsf{rec}_{A\times B}$, pr_1, and pr_2 are given below for convenience.

$$\mathsf{rec}_{A\times B} : (A \to B \to C) \to A\times B \to C$$
$$\mathsf{rec}_{A\times B}(f)((a,b)) :\equiv f(a)(b)$$

$$\mathsf{pr}_1 : A\times B \to A$$
$$\mathsf{pr}_1 :\equiv \mathsf{rec}_{A\times B}(\lambda\, a\, b \mapsto a)$$

$$\mathsf{pr}_2 : A\times B \to B$$
$$\mathsf{pr}_2 :\equiv \mathsf{rec}_{A\times B}(\lambda\, a\, b \mapsto b)$$

(In agreement with the notation used in the text, $\lambda\, a\, b \mapsto \Phi$ is merely shorthand for $\lambda\, a \mapsto (\lambda\, b \mapsto \Phi)$. We can think of such an expression as a "two-variable" curried function [mmmm, curry].)

We need primitive definitions of pr_1 and pr_2.

$$\mathsf{pr}_1 : A\times B \to A$$
$$\mathsf{pr}_1(a,b) :\equiv a$$

$$\mathsf{pr}_2 : A\times B \to B$$
$$\mathsf{pr}_2(a,b) :\equiv b$$

We define

$$\mathsf{rec}'_{A\times B} : (A \to B \to C) \to A\times B \to C$$
$$\mathsf{rec}'_{A\times B}(f)((a,b)) :\equiv f(\mathsf{pr}_1(a,b))(\mathsf{pr}_2(a,b))$$

We have

$$\begin{aligned}\mathsf{rec}_{A\times B}(f)(a,b) &= f(a)(b)\\ &= f(\mathsf{pr}_1(a,b))(\mathsf{pr}_2(a,b))\\ &= \mathsf{rec}'_{A\times B}(f)(a,b)\end{aligned}$$

So $\mathsf{rec}_{A\times B} = \mathsf{rec}'_{A\times B}$.