

Intro to Topology

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Abstract

These are course notes for an introductory undergraduate or dual-listed course for general topology, to be used in an inquiry-based learning classroom.

*Motivation: Curves and Surfaces In this section, we will develop an intuition for a topological space and the purpose of topology by investigating two natural examples of topological spaces: curves and surfaces.

Unlike the rest of these notes, we will not rigorously define these concepts. (For example, what do I mean by “locally looks like” in [Definition 0.1](#)?) However, many of these ideas will return later in the course and be handled more carefully.

Definition 0.1 A **curve** is a set of points such that for every point in the set, the set locally looks like a (possibly bent or curved) copy of the real line \mathbb{R} or the half line $\mathbb{R}^* = \{x \in \mathbb{R} : x \geq 0\}$. ◇

For example, [Figure 0.2](#), [Figure 0.3](#), and [Figure 0.4](#) are all examples of curves in two or three-dimensional Euclidean space.

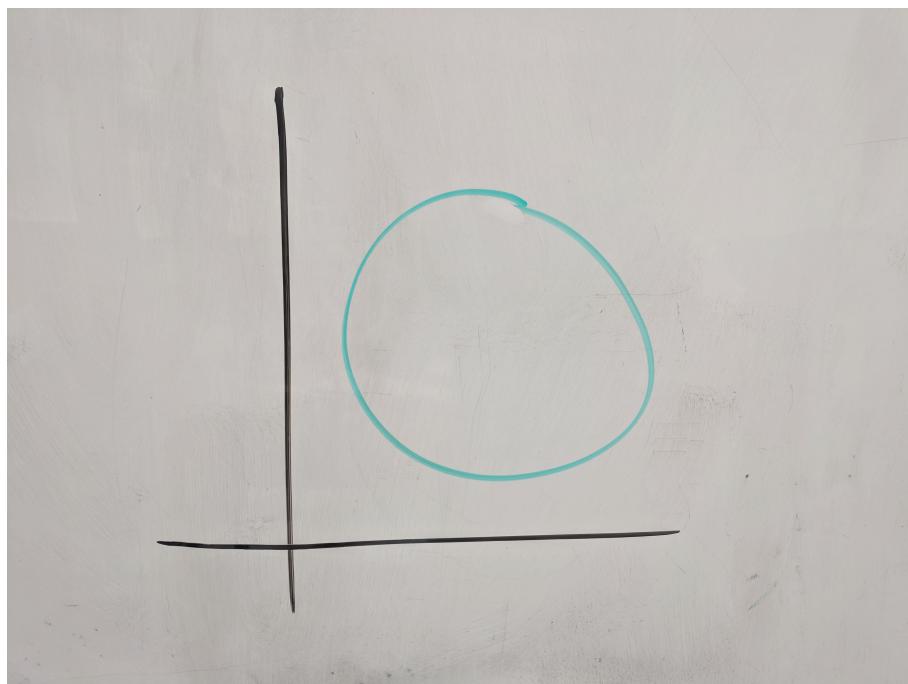


Figure 0.2 A circle

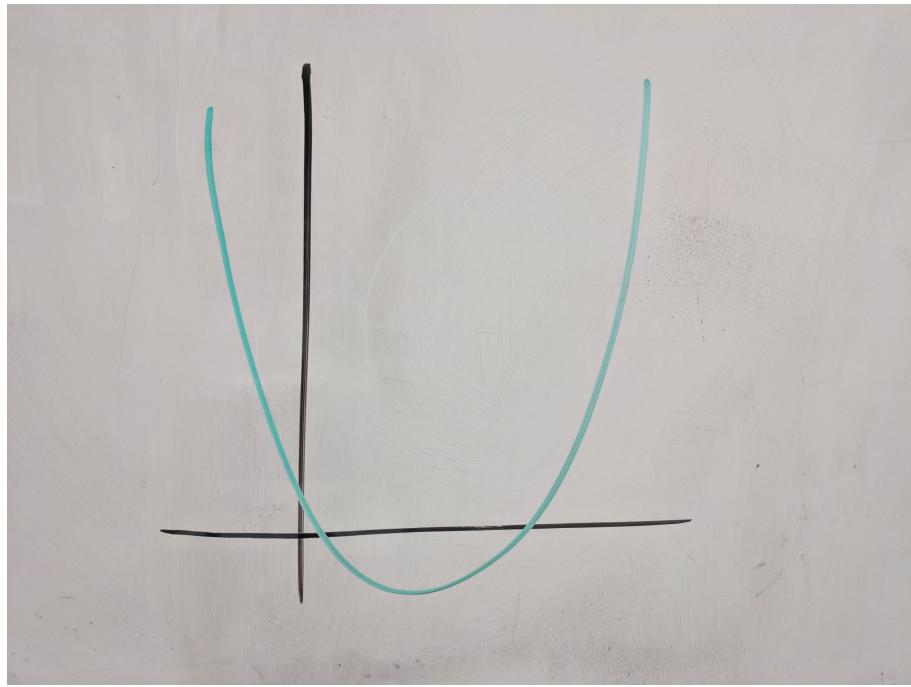


Figure 0.3 A parabola

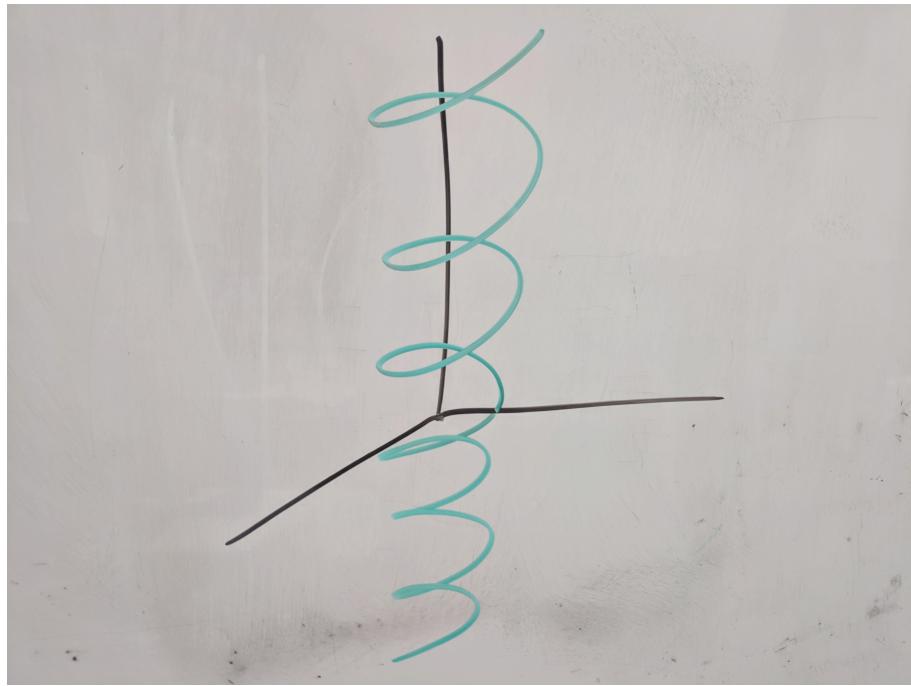


Figure 0.4 A helix

Note the following differences between [Figure 0.2](#) and [Figure 0.3](#):

- Removing a point from [Figure 0.3](#) would split it into two disconnected parts, but [Figure 0.2](#) would remain connected after a point is removed.
- [Figure 0.2](#) is bounded while [Figure 0.3](#) extends unboundedly.¹

¹This topological distinction makes sense as both are closed subsets of \mathbb{R}^2 ; see [Section 6](#) for more info.

These differences would remain no matter how the curves were stretched or bent. However, while there are certainly geometrical differences between [Figure 0.3](#) and [Figure 0.4](#), they are in a certain sense the same object that has been bent or stretched into a different shape.

Definition 0.5 Two objects are said to be **topologically equivalent** or **homeomorphic** if one may be bent or stretched into the shape of the other. \diamond

So this means that all geometrically similar shapes are homeomorphic (as in [Figure 0.6](#)), but we also use the idea of homeomorphism to compare other objects in our daily lives.

For example, while many of them are not curves by our definition, the letters of the alphabet may be considered as topological objects. [Figure 0.7](#) illustrates several homeomorphic expressions of the letter “A”.

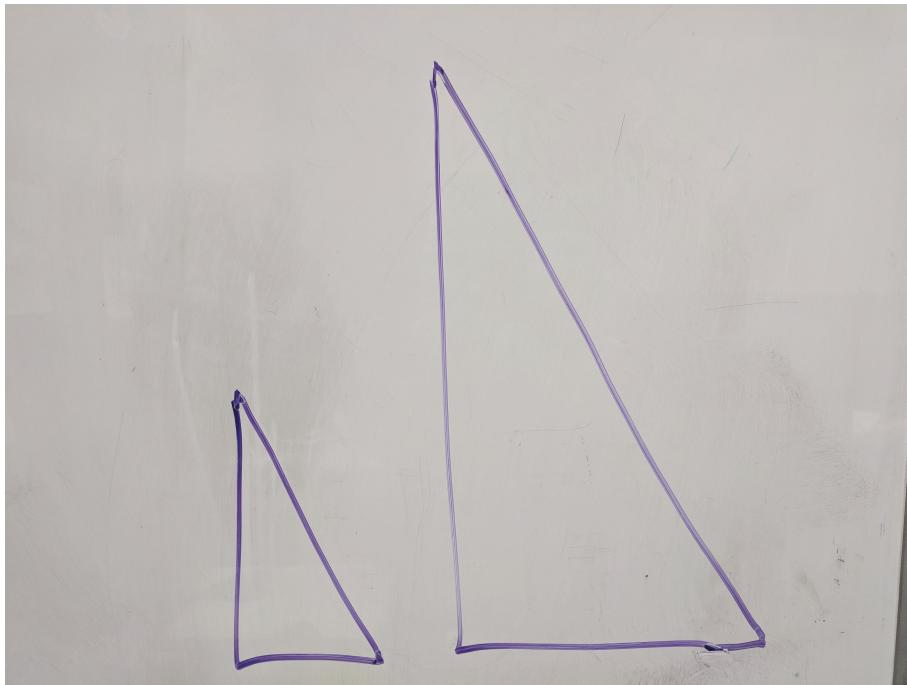


Figure 0.6 Two similar triangles

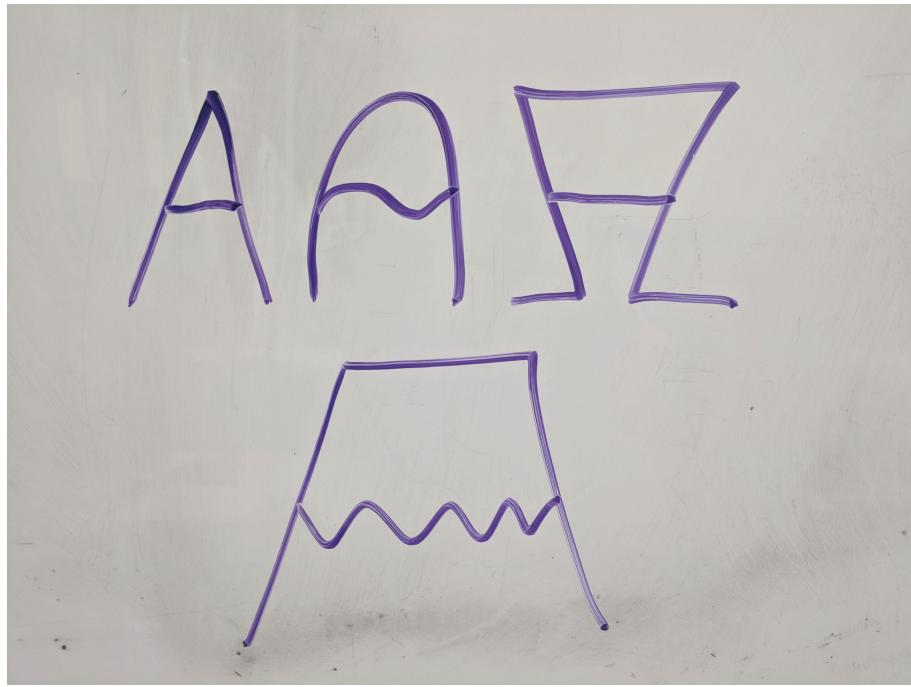


Figure 0.7 The letter “A” in several fonts.

A homeomorphism is more carefully defined in [Section 3](#), but the central idea is that of “neighborhoods”. For each of the letters “A” in [Figure 0.7](#), note that there are two endpoints and two triad intersections whose neighborhoods look different from the other neighborhoods within the letter; see [Figure 0.8](#).

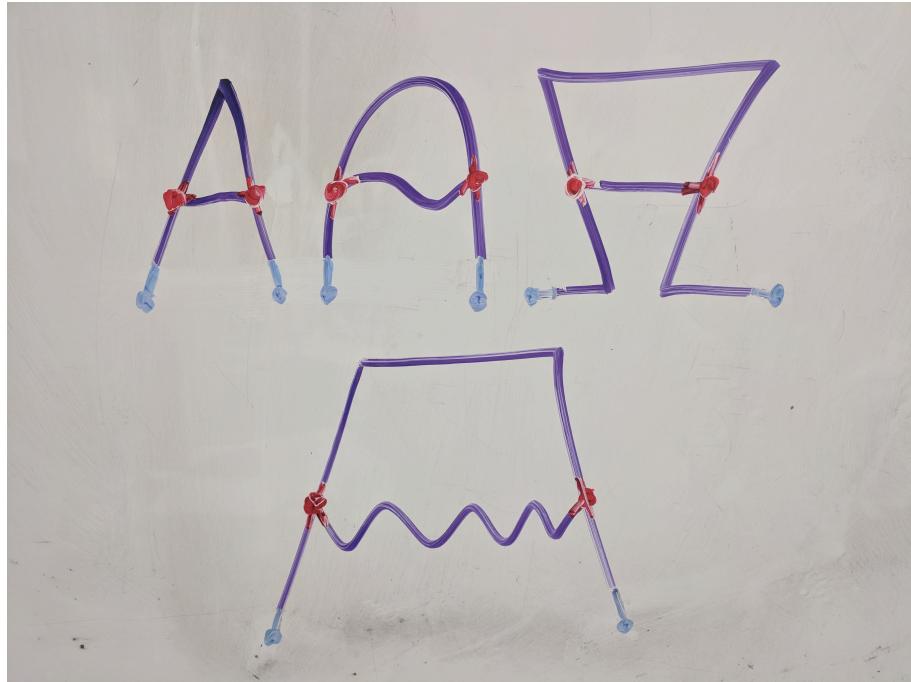


Figure 0.8 Neighborhoods within the letter “A”.

Definition 0.9 A **surface** is a set of points such that for every point in the set, the set locally looks like a (possibly bent or curved) copy of the plane \mathbb{R}^2

or the half-plane $\mathbb{R}^{2*} = \{\langle x, y \rangle \in \mathbb{R}^2 : x \geq 0\}$. ◊

A classic example of the topology of surfaces is the following joke: “A topologist is a mathematician who cannot tell the difference between his doughnut and coffee cup.” The joke is a lot funnier² once you’ve seen [this animated GIF on Wikipedia³](#).

The “doughnut”’s surface is known mathematically as a “torus”, shown in [Figure 0.10](#). A sphere is shown in [Figure 0.11](#), and a surface that cannot be embedded in \mathbb{R}^3 , the Klein bottle, is shown in [Figure 0.12](#).

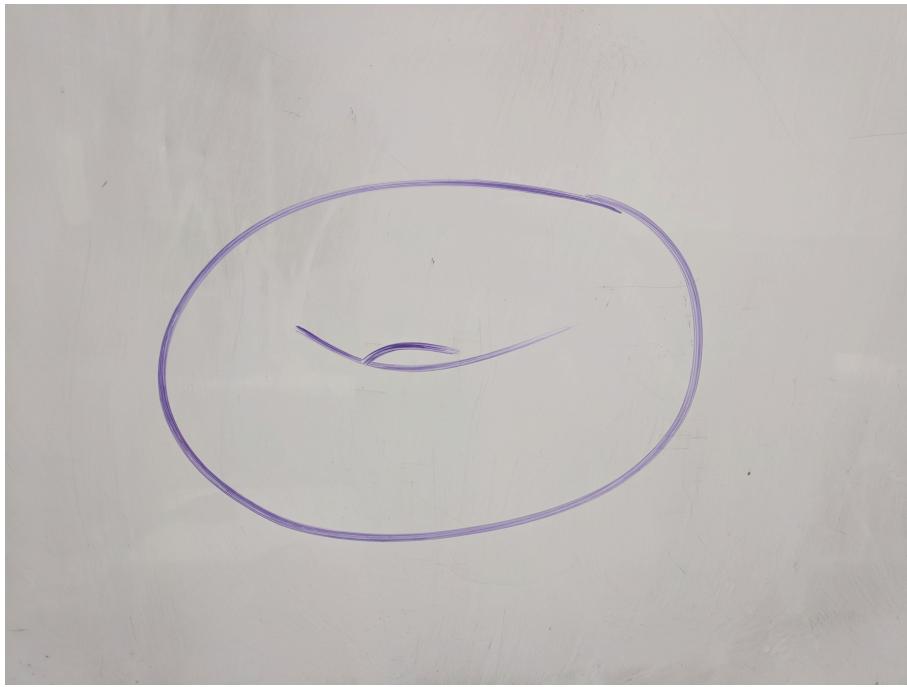


Figure 0.10 A torus.

²Eh, maybe.

³https://en.wikipedia.org/wiki/File:Mug_and_Torus_morph.gif

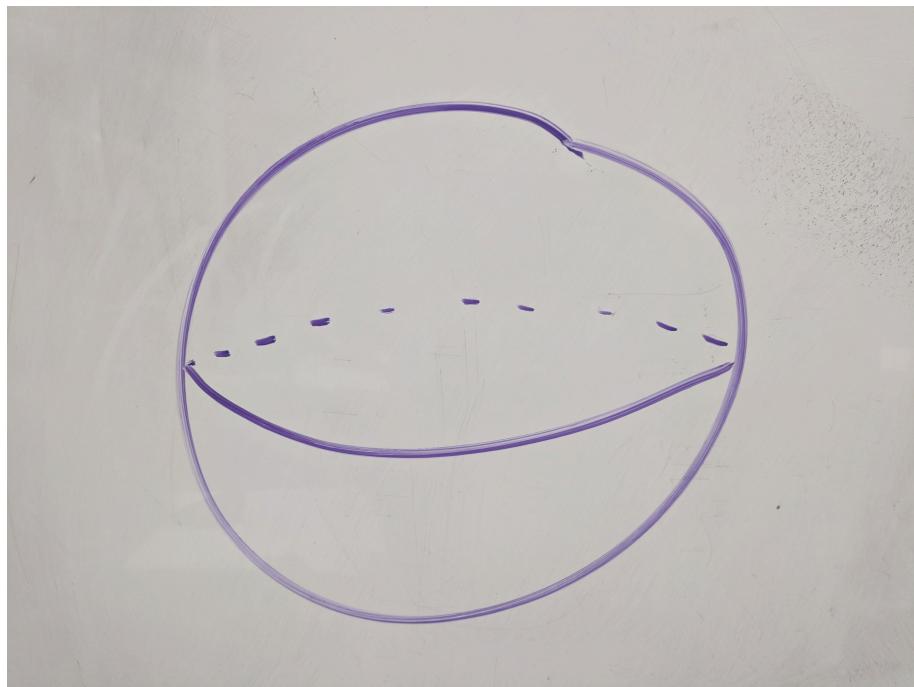


Figure 0.11 A sphere.

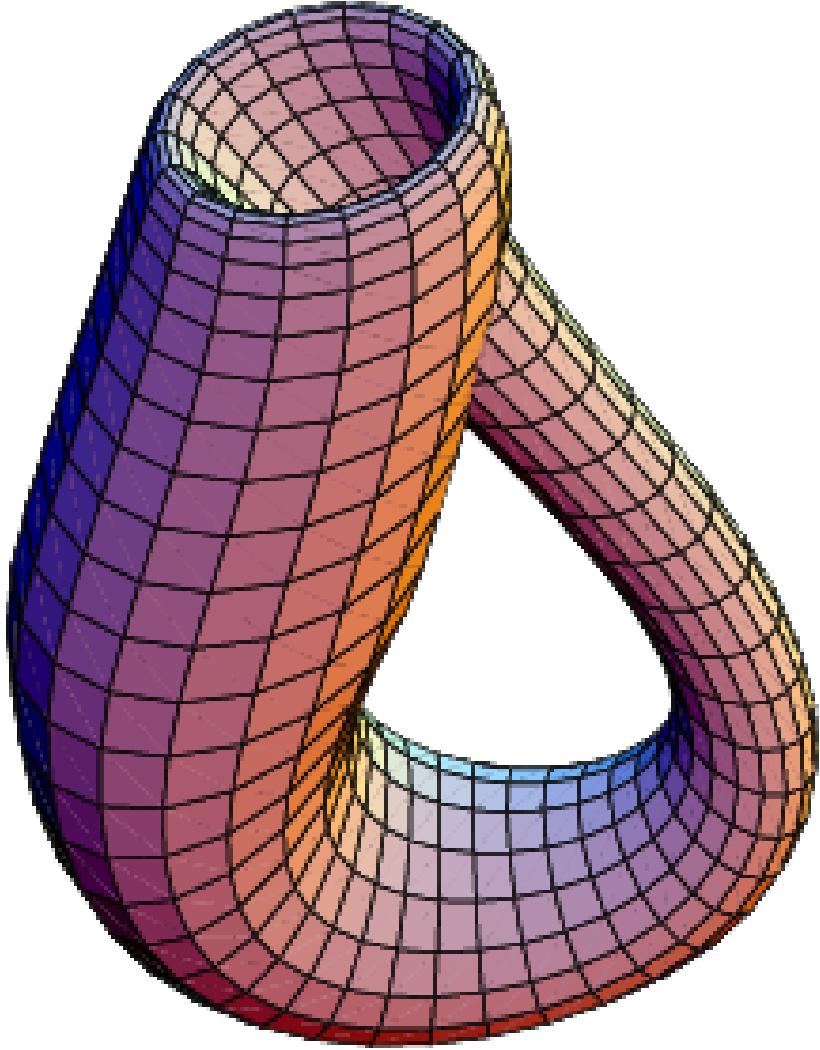


Figure 0.12 A Klein bottle.

While these shapes appear very different, they can all be defined as a “quotient space” ([Section 9](#)) of the unit square in \mathbb{R}^2 .

In order to study so-called “topological spaces” such as these, we will begin by distilling down the notion of a “neighborhood” for an arbitrary set.

1 Topological Spaces

Definition 1.1 Let $a, b \in \mathbb{R}$. The **open interval** from a to b is the set

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

◇

Definition 1.2 Let $x \in \mathbb{R}$ and $S \subseteq \mathbb{R}$. The point x is a **limit point** of the set S if and only if for every open interval (a, b) containing x , there is a point $y \in S$ such that $x \neq y$ and $y \in (a, b)$. ◇

Checkpoint 1.3 Determine if each set has the number 0 as a limit point.

- (a) \mathbb{Z}
- (b) $\mathbb{R} \setminus \mathbb{Z}$
- (c) $\left\{ \frac{1}{n+1} : n \in \omega \right\}$
- (d) \mathbb{Q}
- (e) Any finite set $F \subseteq \mathbb{R}$

Definition 1.4 A subset $U \subseteq \mathbb{R}$ is called **open** if and only if for every point $x \in U$, there exists an open interval (a, b) such that $x \in (a, b) \subseteq U$. \diamond

Checkpoint 1.5 Determine if each set is open or not open.

1. $[\pi, 42)$
2. $(-3, -1) \cup (4, 5.5)$
3. $\{x : 2x + 1 > 5\}$
4. \mathbb{Z}
5. $\mathbb{R} \setminus \mathbb{Z}$
6. \mathbb{Q}
7. A finite set $F \subseteq \mathbb{R}$

Theorem 1.6 A subset $U \subseteq \mathbb{R}$ is open if and only if there exists a collection of open intervals \mathcal{U} such that $U = \bigcup \mathcal{U}$.

Proposition 1.7 Let $x \in \mathbb{R}$ and $S \subseteq \mathbb{R}$. The point x is a limit point of the set S if and only if for every open set U containing x , there is a point $y \in S$ such that $x \neq y$ and $y \in U$.

Theorem 1.8 The open subsets of \mathbb{R} satisfy the following properties.

1. \emptyset and \mathbb{R} are open sets.
2. If \mathcal{U} is a collection of open sets, then $\bigcup \mathcal{U}$ is also an open set.
3. If U, V are open sets, then $U \cap V$ is an open set.

Definition 1.9 Let X be a set, and let $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfy the following properties.

1. $\emptyset, X \in \mathcal{T}$.
2. If $\mathcal{U} \subseteq \mathcal{T}$, then $\bigcup \mathcal{U} \in \mathcal{T}$.
3. If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

Then \mathcal{T} is called a **topology** on X , the pair $\langle X, \mathcal{T} \rangle$ is called a **topological space**, and elements $U \in \mathcal{T}$ are called **open sets** of the space. (Usually $\langle X, \mathcal{T} \rangle$ is abbreviated to just X when the topology is known from context.) \diamond

Definition 1.10 Let $\mathcal{T} \subseteq \mathcal{P}(\mathbb{R})$ be the collection of open subsets of \mathbb{R} defined by [Definition 1.4](#). Then by [Theorem 1.8](#), \mathcal{T} is a valid topology for \mathbb{R} called the **Euclidean topology**. \diamond

Theorem 1.11 Let X be any set. Then the following sets are topologies on

X .

1. $\mathcal{T} = \mathcal{P}(X)$ is called the **discrete topology**.
2. $\mathcal{T} = \{\emptyset, X\}$ is called the **indiscrete topology**.

Proposition 1.12 Let \mathcal{T} be a topology, and let $\mathcal{U} \subseteq \mathcal{T}$ be finite. Then $\bigcap \mathcal{U} \in \mathcal{T}$.

Proposition 1.13 Let \mathcal{T} be the Euclidean topology. There exists a collection $\mathcal{U} = \{U_n : n \in \omega\}$ such that $\bigcap \mathcal{U} \notin \mathcal{T}$.

Definition 1.14 Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$. The following are called **intervals** of real numbers.

$$\begin{aligned}(a, b) &= \{x \in \mathbb{R} : a < x < b\} \\ [a, b) &= \{x \in \mathbb{R} : a \leq x < b\} \\ (a, b] &= \{x \in \mathbb{R} : a < x \leq b\} \\ [a, b] &= \{x \in \mathbb{R} : a \leq x \leq b\}\end{aligned}$$

◊

Checkpoint 1.15 Show that each of the following is an example of a topological space $\langle X, \mathcal{T} \rangle$.

1. Let $X = \mathbb{R}$ and $\mathcal{T} = \{(x, \infty) : x \in \mathbb{R}\} \cup \{[x, \infty) : x \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.
2. Let $X = \mathbb{R}$ and $\mathcal{T} = \{(x, y) : x, y \in \mathbb{R} \cup \{-\infty, \infty\}$ and $x < 0 < y\} \cup \{\emptyset\}$.
3. Let $X = \mathbb{R}$ and $U \in \mathcal{T}$ if for each $x \in U$, there exists $a, b \in \mathbb{R}$ such that $x \in [a, b] \subseteq U$.
4. Let $X = \{0, 1\}$ and $\mathcal{T} = \{\emptyset, \{0\}, X\}$.
5. Let $X = \mathbb{Z}$, $E = \{n \in \mathbb{Z} : n \text{ is even}\}$, $D = \{n \in \mathbb{Z} : n \text{ is odd}\}$, and $\mathcal{T} = \{\emptyset, E, D, X\}$.

Definition 1.16 Let $\langle X, \mathcal{T} \rangle$ be a topological space and let $x \in X$. The set $N \subseteq X$ is called a **neighborhood** of x if and only if there exists an open set $U \in \mathcal{T}$ such that $x \in U \subseteq N$.

◊

Proposition 1.17 A subset U of a topological space X is open if and only if U is a neighborhood of every point it contains.

Definition 1.18 Let $S \subseteq X$ be a subset of a topological space. The point x is a **limit point** of the set S if and only if for every neighborhood of U of x , there is a point $y \in S$ such that $x \neq y$ and $y \in U$.

◊

Proposition 1.19 The point $x \in \mathbb{R}$ is a limit point of $S \subseteq \mathbb{R}$ according to [Definition 1.2](#) if and only if it is a limit point according to [Definition 1.18](#) (where \mathbb{R} is assumed to have the Euclidean topology).

Definition 1.20 Let $S \subseteq X$ be a subset of a topological space. Then S' is the set of all limit points of S , called the **derived set** of S .

◊

Definition 1.21 Let $S \subseteq X$ be a subset of a topological space. Then $\text{cl } S = S \cup S'$ is called the **closure** of S .

◊

Checkpoint 1.22 Calculate $\text{cl } S$ for each of the following examples.

1. $S = (-1, 1) \subseteq \mathbb{R}$ where \mathbb{R} has the Euclidean topology.
2. $S = (-1, 1) \subseteq \mathbb{R}$ where \mathbb{R} has the discrete topology.
3. $S = (-1, 1) \subseteq \mathbb{R}$ where \mathbb{R} has the indiscrete topology.

4. $S = \mathbb{Z} \subseteq \mathbb{R}$ where \mathbb{R} has the Euclidean topology.
5. $S = \mathbb{Q} \subseteq \mathbb{R}$ where \mathbb{R} has the Euclidean topology.

Definition 1.23 Let $H \subseteq X$ be a subset of a topological space. Then H is called **closed** if and only if $H = \text{cl } S$. \diamond

Theorem 1.24 Let $H \subseteq X$ be a subset of a topological space. Then H is closed if and only if the set $X \setminus H$ is open.

Proposition 1.25 The closed subsets of a topological space X satisfy the following properties.

1. \emptyset and X are closed sets.
2. If \mathcal{H} is a collection of closed sets, then $\bigcap \mathcal{H}$ is also a closed set.
3. If H, L are closed sets, then $H \cup L$ is a closed set.

Theorem 1.26 A topological space X is T_1 if and only if every finite subset of X is closed.

Definition 1.27 Let $S \subseteq X$ be a subset of a topological space. The point x is a **boundary point** of the set S if and only if for every neighborhood of U of x , both $U \cap S$ and $U \setminus S$ are non-empty.

Let $\text{bd } S$ collect all the boundary points of S . \diamond

Proposition 1.28 Let $a, b \in \mathbb{R}$. Then $\text{bd}(a, b) = \text{bd}(a, b] = \text{bd}[a, b) = \text{bd}[a, b] = \{a, b\}$ with respect to the Euclidean topology.

Definition 1.29 Let $S \subseteq X$ be a subset of a topological space. The point x is a **interior point** of the set S if and only if there exists a neighborhood U of x such that $x \in U \subseteq S$.

Let $\text{int } S$ collect all the interior points of S . \diamond

Proposition 1.30 Let $U \subseteq X$ be a subset of a topological space. Then U is open if and only if $U = \text{int } U$.

Definition 1.31 Let $S \subseteq X$ be a subset of a topological space. The point x is a **exterior point** of the set S if and only if there exists a neighborhood U of x such that $x \in U \subseteq X \setminus S$.

Let $\text{ext } S$ collect all the exterior points of S . \diamond

Definition 1.32 A **partition** of a set X is a collection \mathcal{P} such that $X = \bigcup \mathcal{P}$ and $A \cap B = \emptyset$ for all $A, B \in \mathcal{P}$ where $A \neq B$. \diamond

Proposition 1.33 Let $S \subseteq X$ be a subset of a topological space. Then $\{\text{int } S, \text{bd } S, \text{ext } S\}$ is a partition of X .

Proposition 1.34 Let $S \subseteq X$ be a subset of a topological space. Then $\text{cl } S = \text{int } S \cup \text{bd } S = S \cup \text{bd } S$.

Checkpoint 1.35 Let A be a subset of a topological space X . Prove or disprove the following.

1. $\text{int int } A = \text{int } A$
2. $\text{int cl } A = \text{int } A$
3. $\text{bd bd } A = \text{bd } A$
4. $\text{ext ext } A = \text{int } A$
5. $\text{int ext } A = \text{ext } A$
6. $\text{int bd } A = \emptyset$

$$7. \text{ cext } A = X \setminus \text{int } A$$

Checkpoint 1.36 Let A, B be subsets of a topological space X . Prove or disprove the following.

1. $\text{int}(A \cap B) = \text{int } A \cap \text{int } B$
2. $\text{int}(A \cup B) = \text{int } A \cup \text{int } B$
3. $\text{bd}(A \cap B) = \text{bd } A \cap \text{bd } B$
4. $\text{bd}(A \cup B) = \text{bd } A \cup \text{bd } B$
5. $\text{cl}(A \cap B) = \text{cl } A \cap \text{cl } B$
6. $\text{cl}(A \cup B) = \text{cl } A \cup \text{cl } B$

Definition 1.37 Let $Y \subseteq X$ for a topological space $\langle X, \mathcal{T} \rangle$. Then the **subspace topology** for Y is given by $\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}$. \diamond

Proposition 1.38 *The subspace topology is a valid topology.*

Definition 1.39 The **Cantor set** is the subset $C \subseteq \mathbb{R}$ defined by $C = \bigcap_{n \in \omega} C_n$, where $C_0 = [0, 1]$ and

$$C_{n+1} = C_n \setminus \bigcup_{0 \leq k < 3^n} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right).$$

This set is usually considered as a closed subset of the Euclidean line, or as a subspace of the Euclidean line. \diamond

2 Basis, Density, and Size

Definition 2.1 Let $\langle X, \mathcal{T} \rangle$ be a topological space. A subset $\mathcal{B} \subseteq \mathcal{T}$ is called a **basis** for the topology if for every $x \in X$ and neighborhood U of x , there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. \diamond

Proposition 2.2 $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$ is a basis for the Euclidean topology.

Theorem 2.3 Let $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfy the following properties:

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. If $x \in A \in \mathcal{B}$ and $x \in B \in \mathcal{B}$, there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

Then $\mathcal{T} = \{\bigcup \mathcal{U} : \mathcal{U} \subseteq \mathcal{B}\}$ is a topology, and \mathcal{B} is a basis for that topology. We call this the **topology generated by the basis**.

Proposition 2.4 $\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}\}$ is a basis for a topology different from the Euclidean topology, called the **Sorgenfrey topology**.

Checkpoint 2.5 Examples of bases. Calculate the topology generated by each basis on \mathbb{R} .

1. $\mathcal{B} = \{(a, b) : a, b \in \mathbb{Q}\}$
2. $\mathcal{B} = \{(a, \infty) : a \in \mathbb{R}\}$
3. $\mathcal{B} = \{\{x\} : x \in \mathbb{R}\}$
4. $\mathcal{B} = \{[a, b] : a, b \in \mathbb{R}\}$
5. $\mathcal{B} = \{[a, b] : a, b \in \mathbb{R}, a < 0 < b\}$

Theorem 2.6 Two bases $\mathcal{B}_0, \mathcal{B}_1$ generate the same topology if and only if for all $x \in B_0 \in \mathcal{B}_0$ there exists $B_1 \in \mathcal{B}_1$ such that $x \in B_1 \subseteq B_0$, and for all $x \in B_1 \in \mathcal{B}_1$ there exists $B_0 \in \mathcal{B}_0$ such that $x \in B_0 \subseteq B_1$.

Theorem 2.7 Let $\mathcal{S} \subseteq \mathcal{P}(X)$ and

$$\mathcal{T} = \bigcap \{\mathcal{T}^* \subseteq \mathcal{P}(X) : \mathcal{S} \subseteq \mathcal{T}^* \text{ and } \mathcal{T}^* \text{ is a topology on } X\}.$$

Then \mathcal{T} is a topology.

Definition 2.8 The set $\mathcal{S} \subseteq \mathcal{P}(X)$ in [Theorem 2.7](#) is called a **subbasis** generating the topology

$$\mathcal{T} = \bigcap \{\mathcal{T}^* \subseteq \mathcal{P}(X) : \mathcal{S} \subseteq \mathcal{T}^* \text{ and } \mathcal{T}^* \text{ is a topology on } X\}.$$

◊

Checkpoint 2.9 Topologies generated from subbases. Calculate the topology on \mathbb{R} generated by each subbasis.

1. $\{(-\infty, x) : x \in \mathbb{R}\} \cup \{(y, \infty) : y \in \mathbb{R}\}$
2. $\{(-\infty, x] : x \in \mathbb{R}\} \cup \{[y, \infty) : y \in \mathbb{R}\}$
3. $\{\{0\}\}$
4. $\mathcal{T} \cup \left\{ \mathbb{R} \setminus \left\{ \frac{1}{2^n} : n \in \omega \right\} \right\}$

Theorem 2.10 Let $\mathcal{S} \subseteq \mathcal{P}(X)$ and

$$\mathcal{B} = \{X\} \cup \bigcap \{\mathcal{B}^* \subseteq \mathcal{P}(X) : \mathcal{S} \subseteq \mathcal{B}^* \text{ and } B_1, B_2 \in \mathcal{B}^* \Rightarrow B_1 \cap B_2 \in \mathcal{B}^*\}.$$

Then \mathcal{B} is a basis for a topology on X , and the topology generated by the basis \mathcal{B} is same as the topology generated by the subbasis \mathcal{S} .

Definition 2.11 A subset $D \subseteq X$ of a topological space is called **dense** if and only if $\text{cl } D = X$. ◊

Checkpoint 2.12 Determine which of these are dense subsets of \mathbb{R} .

1. \mathbb{Q}
2. \mathbb{Z}
3. $\mathbb{R} \setminus \mathbb{Q}$
4. $\mathbb{R} \setminus \mathbb{Z}$

Theorem 2.13 A subset D of a topological space is dense if and only if every nonempty open set of the space contains a point of D .

Proposition 2.14 Let X be a topological space, and let $D \subseteq E \subseteq X$. If D is dense, then E is also dense.

Definition 2.15 A space is said to have **countable weight** (or be **second-countable**) if it has a basis whose cardinality is countable. ◊

Definition 2.16 A space is said to have **countable density** (or be **separable**, no relation to [Section 4](#)) if it has a dense subset whose cardinality is countable. ◊

Proposition 2.17 Every space with countable cardinality has countable weight.

Example 2.18 The reals with the Euclidean topology are a space with uncountable cardinality, but countable weight and countable density. \square

Theorem 2.19 Every space with countable weight has countable density.

Example 2.20 The reals with the Sorgenfrey topology are a space with uncountable cardinality and weight, but countable density. \square

Proposition 2.21 The reals with the discrete topology are a space with uncountable cardinality weight, and density.

Checkpoint 2.22 Find or define a non-discrete topological space which has uncountable weight and density.

3 Continuity & Homeomorphisms

Definition 3.1 Let $f : X \rightarrow Y$ be a function. For $A \subseteq X$, let $f[A] = \{f(x) : x \in A\}$. For $y \in Y$, let $f^{-1}(y) = \{x \in X : f(x) = y\}$. For $B \subseteq Y$, let $f^{-1}[B] = \{x \in X : f(x) \in B\}$. \diamond

Definition 3.2 Let X, Y be topological spaces with $x \in X$, and let $f : X \rightarrow Y$ be a function such that for every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f[U] \subseteq V$. Then f is said to be **continuous at the point x** .

A function that is continuous at every point of its domain is called **continuous**. \diamond

Proposition 3.3 A function $f : X \rightarrow Y$ is continuous if and only if $f^{-1}[V]$ is an open subset of X for every open $V \subseteq Y$.

Proposition 3.4 Let X, Y be topological spaces.

1. The identity function $\iota : X \rightarrow X$ defined by $\iota(x) = x$ is continuous.
2. Let $y \in Y$. The constant function $c_y : X \rightarrow Y$ defined by $c_y(x) = y$ is continuous.
3. Every function whose domain is a discrete space is continuous.
4. Every function whose range is an indiscrete space is continuous.

Checkpoint 3.5 Continuous functions $\mathbb{R} \rightarrow \mathbb{R}$. Verify that each of the following functions $f : \mathbb{R} \rightarrow \mathbb{R}$ are continuous.

1. $f(x) = |x|$
2. $f(x) = x^2$
3. $f(x) = g(x) + h(x)$ for $g, h : \mathbb{R} \rightarrow \mathbb{R}$ continuous.
4. $f(x) = g(x)h(x)$ for $g, h : \mathbb{R} \rightarrow \mathbb{R}$ continuous.

Theorem 3.6 If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both continuous, then $g \circ f : X \rightarrow Z$ is continuous.

Definition 3.7 Let $f : X \rightarrow Y$ be a bijection such that both f and its inverse f^{-1} are continuous. Then f is called a **homeomorphism** and X, Y are said to be **homeomorphic**. \diamond

Checkpoint 3.8 Properties preserved by continuous functions. Determine if the following hold if $f : X \rightarrow Y$ is a continuous surjection. If not, determine if they hold if f is a continuous bijection. If not, show that they

hold if f is a homeomorphism.

1. If $U \subseteq X$ is open, then $f[U] \subseteq Y$ is open.
2. If $H \subseteq X$ is closed, then $f[H] \subseteq Y$ is closed.
3. If x is a limit point of $A \subseteq X$, then $f(x)$ is a limit point of $f[A] \subseteq Y$.

Proposition 3.9 *Every topological space is homeomorphic to itself.*

Proposition 3.10 *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both homeomorphisms, then $g \circ f : X \rightarrow Z$ is a homeomorphism.*

Theorem 3.11 *Let $a < b$ and $c < d$ be real numbers. Then (a, b) and (c, d) are homeomorphic subspaces of the Euclidean line.*

Theorem 3.12 *\mathbb{R} with the Euclidean topology is homeomorphic to its subspace $(0, 1)$.*

Theorem 3.13 *Let \mathcal{B} be a basis for the Euclidean topology on \mathbb{R} . Give $K = \mathbb{R} \cup \{-\infty, \infty\}$ the topology generated by the basis $\{[-\infty, x) : x \in \mathbb{Q}\} \cup \{(x, \infty] : x \in \mathbb{Q}\} \cup \mathcal{B}$. Then K is homeomorphic to the subspace $[0, 1]$ of the Euclidean line.*

Proposition 3.14 *The real line with the Sorgenfrey topology generated by the basis $\{[a, b) : a, b \in \mathbb{R}\}$ is homeomorphic to the real line with the reverse Sorgenfrey topology generated by the basis $\{(a, b] : a, b \in \mathbb{R}\}$.*

4 Separation

Definition 4.1 The following are known as **separation axioms** for a topological space $\langle X, \mathcal{T} \rangle$.

1. \mathcal{T} is said to be T_0 if and only if for all points $x, y \in X$ such that $x \neq y$, there either exists a neighborhood U of x such that $y \notin U$, or there exists a neighborhood V of y such that $x \notin V$.
2. \mathcal{T} is said to be T_1 if and only if for all points $x, y \in X$ such that $x \neq y$, there exists a neighborhood U of x such that $y \notin U$.
3. \mathcal{T} is said to be T_2 (also known as **Hausdorff**) if and only if for all points $x, y \in X$ such that $x \neq y$, there exist neighborhoods U, V of x, y (respectively) such that $U \cap V = \emptyset$.

◊

Proposition 4.2 $T_2 \Rightarrow T_1 \Rightarrow T_0$.

Checkpoint 4.3 Find or create an example of a topological space $\langle X, \mathcal{T} \rangle$ that is:

1. Not T_0 .
2. T_0 but not T_1 .
3. T_1 but not T_2 .

Theorem 4.4 *Let X be a finite topological space. Then X is T_1 if and only if X has the discrete topology.*

Proposition 4.5 *The Euclidean real line is a non-discrete Hausdorff topological space.*

Definition 4.6 The following are also known as **separation axioms** for a topological space $\langle X, \mathcal{T} \rangle$.

1. \mathcal{T} is said to be **regular** if and only if for all points $x \in X$ and closed subsets $H \subseteq X$ such that $x \notin H$, there exist open sets $U, V \in \mathcal{T}$ such that $x \in U, H \subseteq V, U \cap V = \emptyset$.
2. \mathcal{T} is said to be T_3 if and only if it is both regular and T_1
3. \mathcal{T} is said to be **normal** if and only if for all closed subsets $H, L \subseteq X$ such that $H \cap L = \emptyset$, there exist open sets $U, V \in \mathcal{T}$ such that $H \subseteq U, L \subseteq V, U \cap V = \emptyset$.
4. \mathcal{T} is said to be T_4 if and only if it is both normal and T_1

◊

Proposition 4.7 $T_{n+1} \Rightarrow T_n$ for $n \in \{0, 1, 2, 3\}$.

Theorem 4.8 The real line \mathbb{R} equipped with the Euclidean topology is T_4 .

Checkpoint 4.9 Find or create an example of a topological space that is:

1. T_2 but not regular.
2. T_3 but not T_4
3. Regular but not T_3 .
4. Normal but not T_4 .
5. Regular but not normal.
6. Normal but not regular.

Proposition 4.10 A space is regular if and only if for every point x and neighborhood U , there exists a neighborhood V of x such that $x \in V \subseteq \text{cl } V \subseteq U$.

Proposition 4.11 A space is normal if and only if for closed set H and open set $U \supseteq H$, there exists an open set $V \supseteq H$ such that $H \subseteq V \subseteq \text{cl } V \subseteq U$.

Theorem 4.12 A topological space is T_3 if and only if it is regular and T_0 .

Proposition 4.13 Let $n \in \{0, 1, 2, 3\}$. A subspace of a T_n space is also T_n .

5 Metric Spaces

Definition 5.1 Let $d : X^2 \rightarrow [0, \infty)$ be a function satisfying the following for all $x, y, z \in X$.

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Then d is said to be a **metric** on the set X , and

$$B_r(x) = \{y \in X : d(x, y) < r\}$$

is said to be a **metric ball around x** . ◊

Checkpoint 5.2 Examples of metrics. Verify that each of the following

is a metric.

1. $d(x, y) = 1$ for all distinct $x, y \in X$, and $d(x, x) = 0$
2. $d(x, y) = |y - x|$ for all $x, y \in \mathbb{R}$
3. $d(\langle x_0, x_1 \rangle, \langle y_0, y_1 \rangle) = \sqrt{(y_1 - y_0)^2 + (x_1 - x_0)^2}$ for all $\langle x_0, x_1 \rangle, \langle y_0, y_1 \rangle \in \mathbb{R}^2$.
4. $d(\langle x_0, x_1 \rangle, \langle y_0, y_1 \rangle) = |y_1 - y_0| + |x_1 - x_0|$ for all $\langle x_0, x_1 \rangle, \langle y_0, y_1 \rangle \in \mathbb{R}^2$.
5. $d(\langle x_0, x_1 \rangle, \langle y_0, y_1 \rangle) = \max\{|y_1 - y_0|, |x_1 - x_0|\}$ for all $\langle x_0, x_1 \rangle, \langle y_0, y_1 \rangle \in \mathbb{R}^2$.

Theorem 5.3 Let d be a metric on a set X . Then

$$\mathcal{B} = \{B_r(x) : x \in X, r > 0\}$$

is a basis for a topology on X .

Definition 5.4 The topology generated by the basis given in [Theorem 5.3](#) is called the **topology generated by the metric**.

A given topology is said to be **metrizable** if there exists some metric that generates it. Two metrics are said to be **topologically equivalent** if they generate the same topology. \diamond

Proposition 5.5 Every discrete space is metrizable.

Theorem 5.6 Every metrizable space is T_4 .

Theorem 5.7 Let $B_r(x), B'_r(x)$ be the metric balls around x given by two metrics d, d' respectively. Then d, d' are topologically equivalent if and only if for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_\delta(x) \subseteq B'_\epsilon(x)$ and $B_\delta(x)' \subseteq B_\epsilon(x)$.

Definition 5.8 For $\vec{x} = \langle x_0, \dots, x_{n-1} \rangle \in \mathbb{R}^n$, let $\vec{x}(i) = x_i$. \diamond

Theorem 5.9 The following metrics on \mathbb{R}^n are topologically equivalent.

1. $d(\vec{x}, \vec{y}) = \sqrt{\sum_{0 \leq i < n} (\vec{y}(i) - \vec{x}(i))^2}$
2. $d(\vec{x}, \vec{y}) = \sum_{0 \leq i < n} |\vec{y}(i) - \vec{x}(i)|$
3. $d(\vec{x}, \vec{y}) = \max \{|\vec{y}(i) - \vec{x}(i)| : 0 \leq i < n\}$

Definition 5.10 The topology generated by the metrics given in [Theorem 5.9](#) is called the **Euclidean topology** on \mathbb{R}^n . \diamond

Definition 5.11 A **local basis at a point x** is a collection of open sets \mathcal{B}_x such that for every neighborhood U of x , there exists $B \in \mathcal{B}_x$ such that $x \in B \subseteq U$. \diamond

Definition 5.12 A space is said to have **local countable weight** (or be **first-countable**) if there exists a countable local basis at every point of the space. \diamond

Proposition 5.13 Every space of countable weight has local countable weight.

Proposition 5.14 Every metrizable space has local countable weight

Theorem 5.15 Let X be metrizable. Then X has countable weight if and only if it has countable density.

Proposition 5.16 Every space \mathbb{R}^n with the Euclidean topology has countable weight and density.

Theorem 5.17 For $\vec{x}, \vec{y} \in \mathbb{R}^2$, let $d(\vec{x}, \vec{y}) = 1$ if $\vec{x}(1) \neq \vec{y}(1)$, and $d(\vec{x}, \vec{y}) = |\vec{y}(0) - \vec{x}(0)|$ otherwise. Then d is a metric generating a non-discrete topology on \mathbb{R}^2 with uncountable weight and uncountable density.

Theorem 5.18 The subspace $\{\vec{x} : \vec{x}(1) \in \{0, 1\}\}$ of the space defined in [Theorem 5.17](#) is homeomorphic to the subspace $(0, 1) \cup (2, 3)$ of the Euclidean line.

Definition 5.19 A point x is called a **sequential limit point** of a set A iff there exists a countable subset $B \subseteq A \setminus \{x\}$ such that every neighborhood of x contains all but finitely many points of B . \diamond

Proposition 5.20 Every sequential limit point of a set is a limit point of that set.

Theorem 5.21 Let X have local countable weight. Then x is a limit point of a set if and only if x is a sequential limit point of that set.

Definition 5.22 A **Cauchy sequence** is a countably infinite set A such that for all $\epsilon > 0$, the set $\{x \in A : \exists y \in A (d(x, y) \geq \epsilon)\}$ is finite. \diamond

Definition 5.23 A **complete metric** is a metric such that every Cauchy sequence has a sequential limit point.

A topology that can be generated by a complete metric is said to be **completely metrizable**. \diamond

Proposition 5.24 Every Euclidean space is completely metrizable.

Proposition 5.25 Let $d : X \rightarrow [0, \infty)$ be a metric and $Y \subseteq X$. Then d restricted to Y generates the subspace topology on Y . (Therefore, every subspace of a metrizable space is metrizable.)

Theorem 5.26 The subspace $(0, 1)$ of the Euclidean line is completely metrizable, but not by the topology inherited from \mathbb{R} .

Theorem 5.27 The subspace \mathbb{Q} of the Euclidean line is metrizable, but not completely metrizable.

Theorem 5.28 The subspace $\mathbb{R} \setminus \mathbb{Q}$ of the Euclidean line is completely metrizable, but not by the topology inherited from \mathbb{R} .

Theorem 5.29 Metrizable and completely metrizable are topological properties. That is, if X and Y are homeomorphic, then X is (completely) metrizable if and only if Y is too.

6 Compactness

Definition 6.1 A collection $\mathcal{A} \subseteq \mathcal{P}(X)$ is said to **cover** a subset $Y \subseteq X$ iff $Y \subseteq \bigcup \mathcal{A}$. \diamond

Definition 6.2 A subset $K \subseteq X$ of a topological space is said to be **compact** iff for every collection \mathcal{U} of open sets covering K , there exists a finite subcollection $\mathcal{F} \subseteq \mathcal{U}$ that also covers K . \diamond

Checkpoint 6.3 Determine if each of the following subsets of the Euclidean line \mathbb{R} is compact.

1. \mathbb{R}

2. \mathbb{Z}

3. $\{2^{-n} : n \in \omega\}$
4. $\{0\} \cup \{2^{-n} : n \in \omega\}$
5. $(0, 1)$
6. $[0, 1]$

Definition 6.4 A subset $R \subseteq X$ of a topological space is said to be **relatively compact** iff for every collection of open sets \mathcal{U} covering X , there exists a finite subcollection $\mathcal{F} \subseteq \mathcal{U}$ that covers R . \diamond

Theorem 6.5 Let X be regular. A subset $R \subseteq X$ is relatively compact if and only if $\text{cl } R$ is compact.

Theorem 6.6 Let X be compact and K be a closed subset of X . Then K is compact.

Proposition 6.7 Every finite subset of a space is compact.

Proposition 6.8 Every finite union of compact subsets is compact.

Theorem 6.9 Every compact subset of a Hausdorff space is closed.

Proposition 6.10 Let $\mathcal{T} = \{\emptyset\} \cup \{\omega \setminus F : F \text{ is finite}\}$ be the cofinite topology on ω . Every subset of ω is compact under this topology.

Theorem 6.11 Let $f : X \rightarrow Y$ be continuous and $K \subseteq X$ be compact. Then $f[K]$ is compact.

Corollary 6.12 Compactness is a topological property.

Theorem 6.13 Every infinite subset of a compact set has a limit point.

Theorem 6.14 Let $\mathcal{K} = \{K_n : n \in \omega\}$ be a collection of non-empty compact subsets of a topological space such that $K_{n+1} \subseteq K_n$ for all $n \in \omega$. Then $\bigcap \mathcal{K}$ is a non-empty compact set.

Theorem 6.15 Let X be metrizable. Then the following are equivalent for $K \subseteq X$.

1. K is compact
2. Every infinite subset of K has a limit point.
3. Every infinite subset of K has a sequential limit point.

Lemma 6.16 A topological space X is Hausdorff if and only if for every pair of disjoint compact subsets H, K there exist disjoint open sets U, V such that $H \subseteq U$ and $K \subseteq V$.

Theorem 6.17 Every compact Hausdorff space is T_4 .

7 Connectedness

Definition 7.1 A pair of open sets $\{A, B\}$ satisfying $A \cap Y \neq \emptyset, B \cap Y \neq \emptyset$, $Y \subseteq A \cup B$, and $A \cap B \cap Y = \emptyset$ for a subset Y of a topological space X is called a **disconnection** of Y .

A subset of a space for which a disconnection exists is called **disconnected**; otherwise, the space is called **connected**. \diamond

Definition 7.2 A set which is both closed and open is said to be **clopen**. \diamond

Proposition 7.3 A pair $\{A, B\}$ of open sets is a disconnection of $Y \subseteq X$ if and only if $\{A \cap Y, B \cap Y\}$ is a partition of Y by non-empty clopen sets in the subspace topology.

Corollary 7.4 A space itself is disconnected if and only if it is the union of two disjoint non-empty clopen subsets.

Proposition 7.5 The Euclidean line with a point removed $\mathbb{R} \setminus \{0\}$ is disconnected.

Lemma 7.6 Let $\mathbb{R} = U \cup V$ for open sets U, V and let $x \in U, y \in V$ with $x \leq y$. Then $\inf\{z \in [x, y] : z \in V\} \in U \cap V$.

Corollary 7.7 The Euclidean line is connected.

Proposition 7.8 The Sorgenfrey topology on \mathbb{R} is disconnected.

Theorem 7.9 If a subset A of a topological space is connected, then $\text{cl } A$ is connected.

Proposition 7.10 If a subset A of a topological space X is connected and $f : X \rightarrow Y$ is continuous, then $f[A]$ is connected.

Corollary 7.11 Connectedness is a topological property.

Proposition 7.12 Let $\{0, 1\}$ have the discrete topology. Then a topological space X is connected if and only if every continuous function $f : X \rightarrow \{0, 1\}$ is constant.

Theorem 7.13 If \mathcal{A} is a collection of connected subsets of a topological space with $\bigcap \mathcal{A} \neq \emptyset$, then $\bigcup \mathcal{A}$ is connected.

Definition 7.14 Suppose for every two points $x, y \in A \subseteq X$, there exists a continuous function $f : [0, 1] \rightarrow A$ such that $f(0) = x$ and $f(1) = y$. Such a space is said to be **path connected**. \diamond

Proposition 7.15 Every path connected space is connected.

Checkpoint 7.16 Find or create an example of a connected space that's not path connected.

Theorem 7.17 Let

$$S = \left\{ \langle x, y \rangle : x \in (0, 1] \text{ and } y = \sin\left(\frac{1}{x}\right) \right\}$$

(the **topologist's sine curve**). Then $\text{cl } S$ is a subset of the Euclidean space \mathbb{R}^2 that is connected but not path connected.

8 Product Spaces

Definition 8.1 Let X, Y be topological spaces, generated respectively by the bases $\mathcal{B}_X, \mathcal{B}_Y$. Then the **product space** is given by the set $X \times Y = \{\langle x, y \rangle : x \in X, y \in Y\}$ with the topology generated by the basis $\mathcal{B} = \{U \times V : U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$. \diamond

Proposition 8.2 The product spaces $X \times Y$ and $Y \times X$ are homeomorphic.

Proposition 8.3 Let $p \in Y$. The subspace $X \times \{p\} = \{\langle x, p \rangle : x \in X\}$ of $X \times Y$ is homeomorphic to X .

Theorem 8.4 The Euclidean space \mathbb{R}^{n+1} is homeomorphic to the product space $\mathbb{R}^n \times \mathbb{R}$.

Proposition 8.5 *The product $X \times Y$ is Hausdorff if and only if X, Y are each Hausdorff.*

Theorem 8.6 *The product $X \times Y$ is regular if and only if X, Y are each regular.*

Lemma 8.7 *Let $S = \mathbb{R}$ equipped with the Sorgenfrey topology. Then the product space $S \times S$ contains two disjoint closed subsets $H = \{\langle x, -x \rangle : x \in \mathbb{Q}\}$ and $L = \{\langle x, -x \rangle : x \in \mathbb{R} \setminus \mathbb{Q}\}$ that cannot be separated by a pair of open sets.*

Theorem 8.8 *Let $S = \mathbb{R}$ equipped with the Sorgenfrey topology. Then S is normal, but $S \times S$ is not normal.*

Proposition 8.9 *The diagonal $\Delta = \{\langle x, x \rangle : x \in X\}$ of $X \times X$ is homeomorphic to X .*

Definition 8.10 For a product space $X \times Y$, its **projection maps** $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ are defined by $\pi_X(\langle x, y \rangle) = x$ and $\pi_Y(\langle x, y \rangle) = y$. \diamond

Checkpoint 8.11 Properties of projection maps. Verify the following properties of projection maps.

1. Every projection map is continuous.
2. The projection of an open set is always open.
3. The projection of a closed set is not always closed.

Theorem 8.12 *The product $X \times Y$ is metrizable if and only if X, Y are each metrizable.*

Lemma 8.13 *Let Y be compact. If \mathcal{U} is an open cover of $X \times Y$, then for each $x \in X$ there exists a finite subcollection $\mathcal{F}_x \subseteq \mathcal{U}$ and an open neighborhood U_x of x such that $U_x \times Y \subseteq \bigcup \mathcal{F}_x$.*

Theorem 8.14 *The product $X \times Y$ is compact if and only if X, Y are each compact.*

Theorem 8.15 *The product $X \times Y$ is connected if and only if X, Y are each connected.*

9 Quotient Spaces

Theorem 9.1 *Let X^* be a partition of a space X with topology \mathcal{T} . Then the set $\mathcal{T}^* = \{U^* \subseteq X^* : \bigcup U^* \in \mathcal{T}\}$ defines a topology on X^* .*

Definition 9.2 The topology defined in [Theorem 9.1](#) is called the **quotient topology**, and X^* is called a **quotient space** or **identification space**. \diamond

Theorem 9.3 *Let $(X \times Y)^* = \{\{x\} \times Y : x \in X\}$ partition the product $X \times Y$. Then the quotient space $(X \times Y)^*$ is homeomorphic to X .*

Definition 9.4 A subset $R \subseteq X^2$ is called a **relation** on X , where the notation xRy is equivalent to writing $\langle x, y \rangle \in R$.

A relation \sim on X is called an **equivalence relation** if it satisfies the following for all $x, y, z \in X$.

1. $x \sim x$. (Reflexivity)
2. $x \sim y$ implies $y \sim x$. (Symmetry)
3. $x \sim y$ and $y \sim z$ implies $x \sim z$. (Transitivity)

◊

Proposition 9.5 Let X^* be a partition of X and define the relation \sim on X such that $x \sim y$ if and only if $\{x, y\} \subseteq A$ for some $A \in X^*$. Then \sim is an equivalence relation.

Theorem 9.6 Let \sim be an equivalence relation on X , and let $[x] = \{y \in X : x \sim y\}$. Then $X^* = \{[x] : x \in X\}$ is a partition of X .

Definition 9.7 Let \sim be an equivalence relation on a topological space X . Then X/\sim denotes the quotient space defined by the partition X^* given in [Theorem 9.6](#). ◊

Proposition 9.8 Let R be a relation on X . Then

$$\sim = \bigcap \{\sim^* \subseteq X^2 : R \subseteq \sim^* \text{ and } \sim^* \text{ is an equivalence relation on } X\}$$

is an equivalence relation on X .

(Therefore, an equivalence relation may be defined as the minimal equivalence relation satisfying a list of relationships.)

Proposition 9.9 Define the equivalence relation \sim_Y on $X \times Y$ by $\langle x, a \rangle \sim_Y \langle x, b \rangle$. Then the quotient space $(X \times Y)/\sim_Y$ is homeomorphic to X .

Checkpoint 9.10 Curves and surfaces defined as quotients. Show that each of the following Euclidean subspaces and quotients of Euclidean subspaces are homeomorphic.

1. $X = [0, 1]/\sim$ where $0 \sim 1$, and $Y = \{\langle x, y \rangle \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
2. $X = [0, 2]/\sim$ where $0 \sim 1 \sim 2$, and $Y = \{\langle x, y \rangle \in \mathbb{R}^2 : (x - 1)^2 + y^2 = 1\} \cup \{\langle x, y \rangle \in \mathbb{R}^2 : (x + 1)^2 + y^2 = 1\}$.
3. $X = [0, 1]^2/\sim$ where $\langle 0, y \rangle \sim \langle 1, y \rangle$, and $Y = \{\langle x, y \rangle \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2\}$.
4. $X = [0, 1]^2/\sim$ where $\langle x, y \rangle \sim \langle z, w \rangle$ whenever at least one of x, y and at least one of z, w is in $\{0, 1\}$, and $Y = \{\langle x, y, z \rangle \in \mathbb{R}^2 : 1 \leq x^2 + y^2 + z^2 = 1\}$.

Definition 9.11 The **hypersphere** of dimension n is the quotient space $S^n = [0, 1]^n/\sim$ given by $\vec{x} \sim \vec{y}$ whenever there exist $i, j \in \{0, \dots, n\}$ such that $\vec{x}(i), \vec{y}(j) \in \{0, 1\}$. ◊

Definition 9.12 The **Möbius strip** is the quotient space $M = [0, 1]^2/\sim$ given by $\langle 0, y \rangle \sim \langle 1, 1 - y \rangle$. ◊

Definition 9.13 The **torus** is the quotient space $T = [0, 1]^2/\sim$ given by $\langle 0, y \rangle \sim \langle 1, y \rangle$ and $\langle x, 0 \rangle \sim \langle x, 1 \rangle$. ◊

Definition 9.14 The **Klein bottle** is the quotient space $K = [0, 1]^2/\sim$ given by $\langle 0, y \rangle \sim \langle 1, 1 - y \rangle$ and $\langle x, 0 \rangle \sim \langle x, 1 \rangle$. ◊

A Assumptions

Here is a brief overview of basic results and definitions concerning sets and the reals that should be assumed for this course.

Definition A.1

- \mathbb{R} is the set of real numbers.
- \mathbb{Z} is the set of integers.

- $\omega = \{z \in \mathbb{Z} : z \geq 0\} = \{0, 1, 2, \dots\}$ is the set of natural numbers, which includes zero.
- $\mathbb{Q} = \left\{ \frac{z}{n+1} : z \in \mathbb{Z}, n \in \omega \right\}$ is the set of rational numbers.

◊

Theorem A.2 De Morgan's Laws: Let \mathcal{A} be a collection of subsets of X .

$$X \setminus \bigcup_{A \in \mathcal{A}} A = \bigcap_{A \in \mathcal{A}} (X \setminus A)$$

$$X \setminus \bigcap_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} (X \setminus A)$$

Theorem A.3 The Archimedean Property of the real numbers guarantees that for each positive real number $x > 0$, there exists a natural number $n \in \omega$ such that $\frac{1}{n} < x$.

Theorem A.4 Let $S \subseteq \mathbb{R}$ be a set of real numbers with a lower bound. Then there exists a **greatest lower bound** (a.k.a. **infimum**) $\text{glb } S = \inf S$.

Let $S \subseteq \mathbb{R}$ be a set of real numbers with a lower bound. Then there exists a **least upper bound** (a.k.a. **supremum**) $\text{lub } S = \sup S$.

Definition A.5 The **cardinality** of a finite set is the natural number that is equal to the number of elements it contains. For example, $S = \{-1, \{5, -e\}, \pi\}$ has cardinality $|S| = 3$.

A set S that has a bijection with ω is said to be **countable** and have cardinality $|S| = \aleph_0$.

A set S that has no injection with ω is said to be **uncountable** and have cardinality $|S| > \aleph_0$.

A set S that has a bijection with \mathbb{R} is said to be **continuum-sized** and have cardinality $|S| = \mathfrak{c}$. ◊