Intro to Topology

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These are course notes for an introductory undergraduate or dual-listed course for general topology, to be used in an inquiry-based learning classroom.

Preface

These notes have been written to serve as an introduction to topology for undergraduate or beginning graduate students of mathematics. Unlike a more comprehensive textbook, these notes are written to be used in an inquiry-based learning classroom. As such, no proofs have been provided. Instead, the results of the course have been carefully scaffolded to allow students to prove most of the theorems on their own, with the support of the instructor.

The exception to this format is Section 1.1, which provides a short narrative to movitave the study of topology by discussing the neighborhoods of points in subspaces of $\mathbb{R}^3$, particularly unions of curves and surfaces.

The only prerequisite for this course is an undergraduate-level introduction to proofs course, which should include all the naive set theory necessary to study topology at the introductory level. Appendix A provides a few definitions and theorems that should be assumed for this course (e.g. properties of the reals, $0 \in \mathbb{N}$).

Contents

Chapter 1

The only chapter

1.1 Curves and Surfaces

In this section, we will develop an intuition for a topological space and the purpose of topology by investigating two natural examples of topological spaces: curves and surfaces.

Unlike the rest of these notes, we will not rigorously define these concepts. (For example, what do I mean by "locally looks like" in Definition 1.1.1?) However, many of these ideas will return later in the course and be handled more carefully.

Definition 1.1.1 A **curve** is a set of points such that for every point in the set, the set locally looks like a (possibly bent or curved) copy of the real line $\mathbb{R}$ or the half line $\mathbb{R}^* = \{x \in \mathbb{R} : x \geq 0\}$. $\Diamond$

For example, Figure 1.1.2, Figure 1.1.3, and Figure 1.1.4 are all examples of curves in two or three-dimensional Euclidean space.

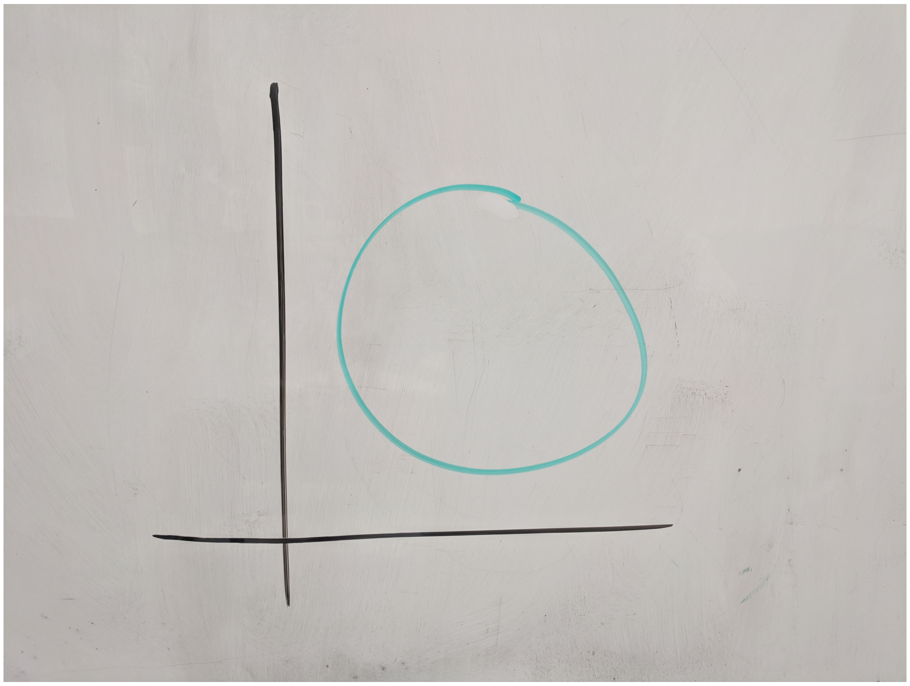

Figure 1.1.2: A circle

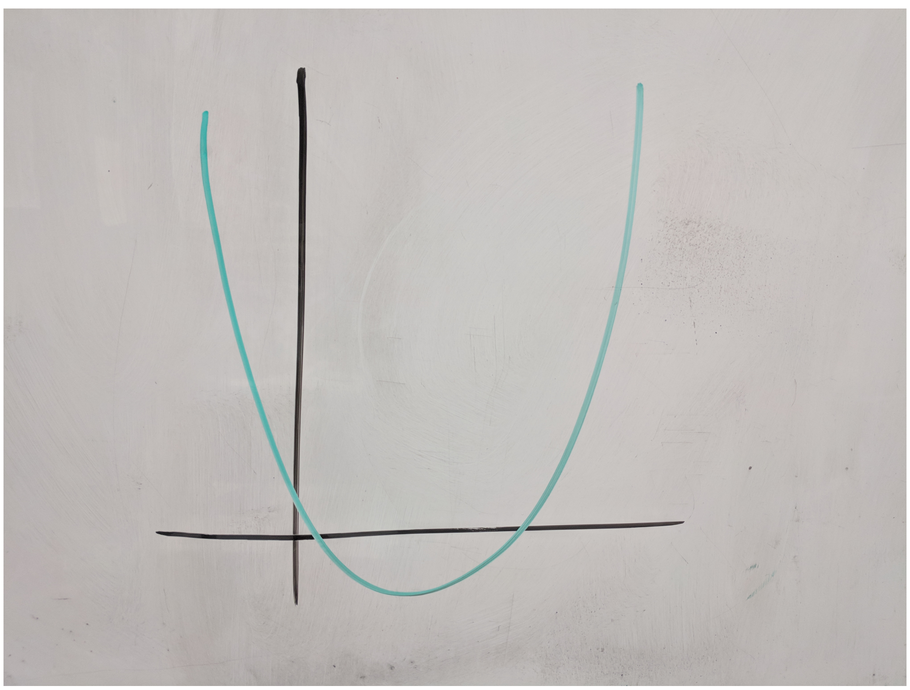

Figure 1.1.3: A parabola

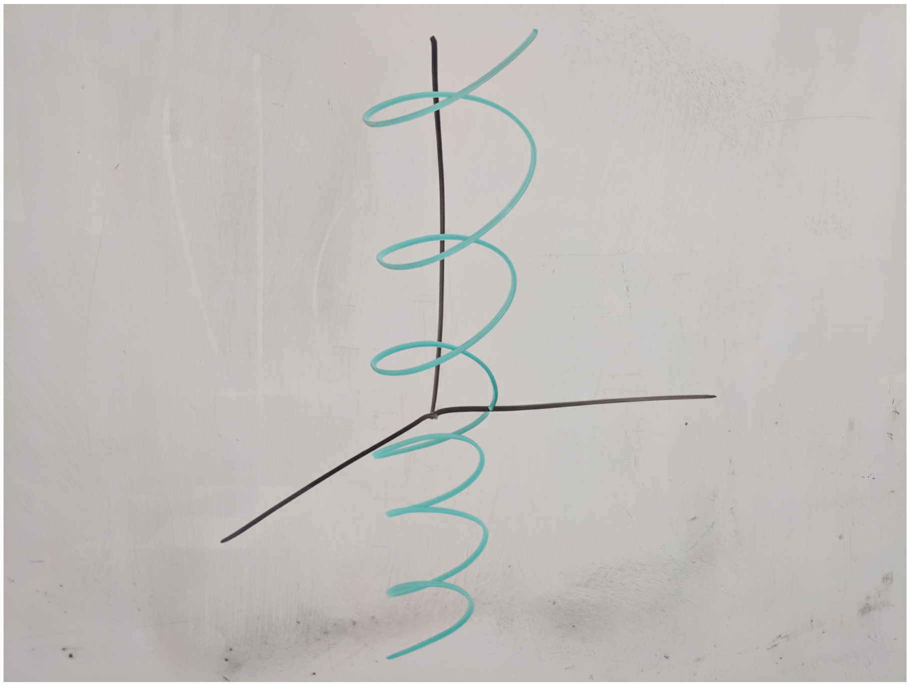

Figure 1.1.4: A helix

Note the following differences between Figure 1.1.2 and Figure 1.1.3:

- Removing a point from Figure 1.1.3 would split it into two disconnected parts, but Figure 1.1.2 would remain connected after a point is removed.
- Figure 1.1.2 is bounded while Figure 1.1.3 extends unboundedly. [1]

These differences would remain no matter how the curves were stretched or bent. However, while there are certainly geometrical differences beteween Figure 1.1.3 and Figure 1.1.4, they are in a certain sense the same object that has been bent or stretched into a different shape.

Definition 1.1.5 Two objects are said to be **topologically equivalent** or **homeomorphic** if one may be bent or stretched into the shape of the other. $\Diamond$

So this means that all geometrically similar shapes are homeomorphic (as in Figure 1.1.6), but we also use the idea of homeomorphism to compare other objects in our daily lives.

For example, while many of them are not curves by our definition, the letters of the alphabet may be considered as topological objects. Figure 1.1.7 illustrates several homeomorphic expressions of the letter "A".

[1] This topological distinction makes sense as both are closed subsets of $\mathbb{R}^2$; see Section 1.5 for more info.

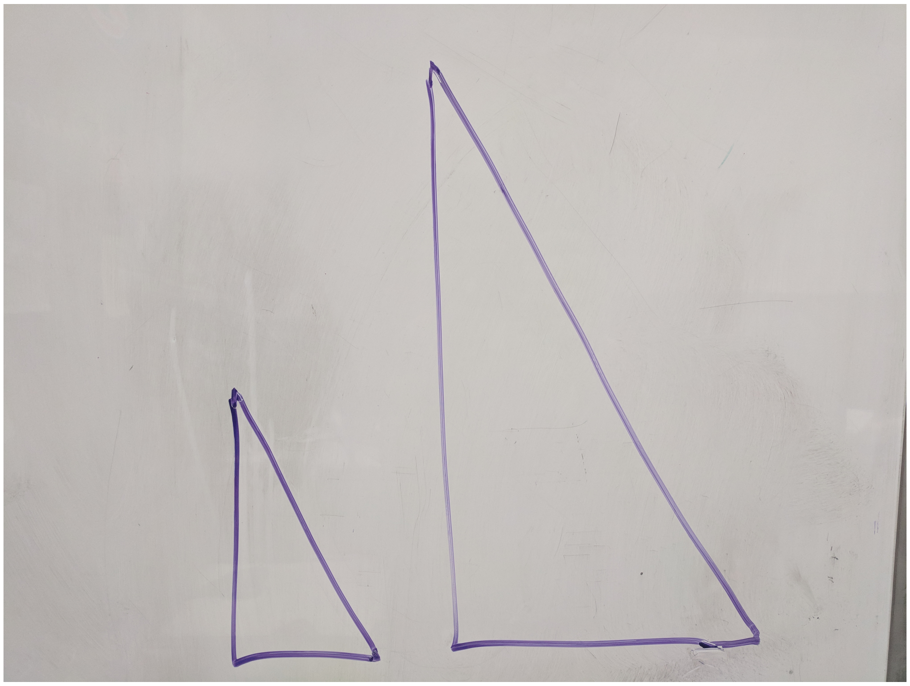

Figure 1.1.6: Two similar triangles

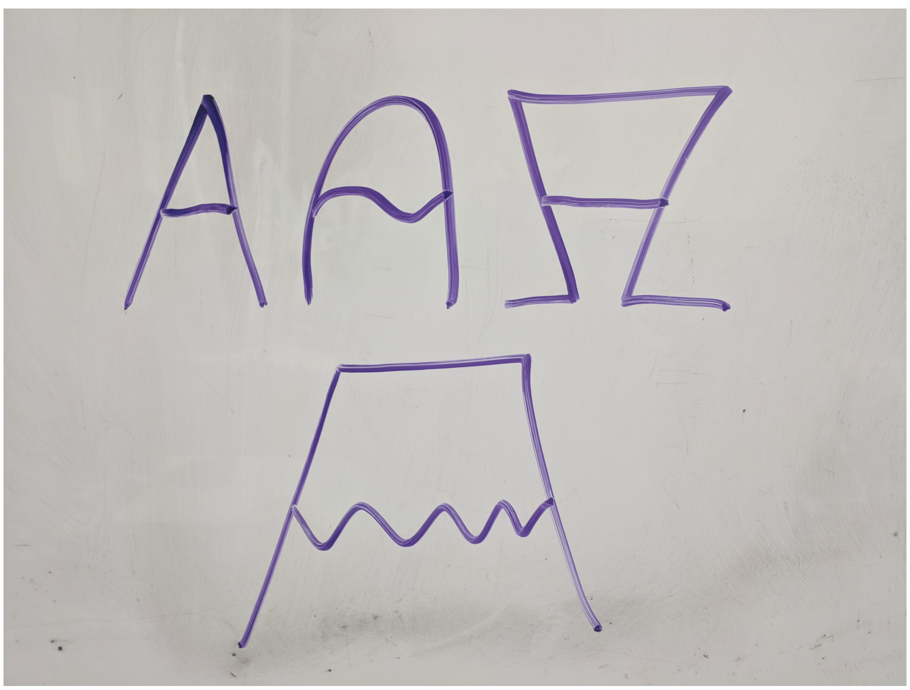

Figure 1.1.7: The letter "A" in several fonts.

A homeomorphism is more carefully defined in Section 1.3, but the central idea is that of "neighborhoods". For each of the letters "A" in Figure 1.1.7, note that there are two endpoints and two triad intersections whose neighborhoods look different from the other neighborhoods within the letter; see Figure 1.1.8.

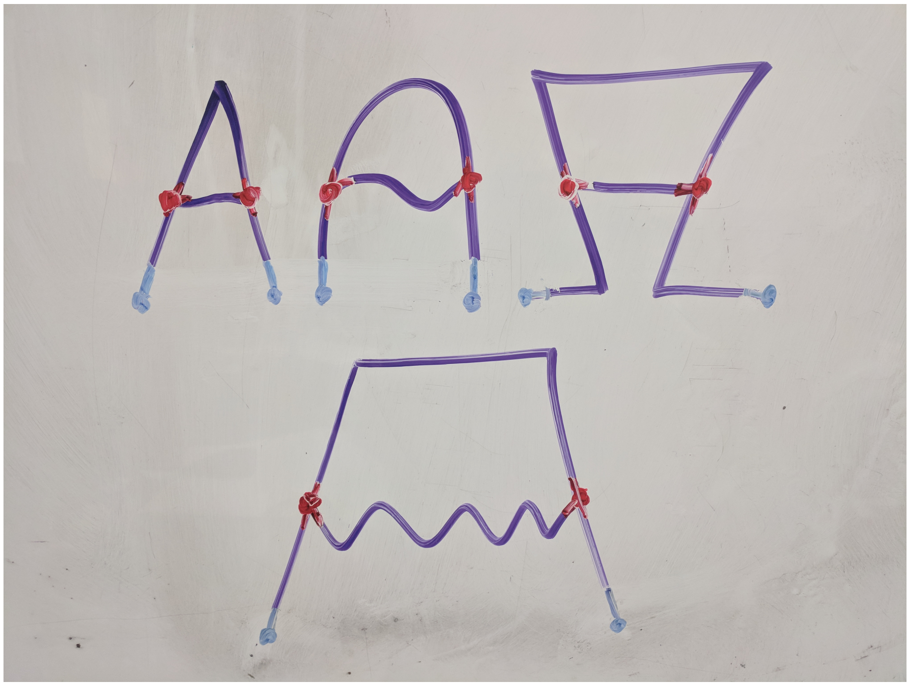

Figure 1.1.8: Neighborhoods within the letter "A".

Definition 1.1.9 A **surface** is a set of points such that for every point in the set, the set locally looks like a (possibly bent or curved) copy of the plane $\mathbb{R}^2$ or the half-plane $\mathbb{R}^{2*} = \{\langle x, y\rangle \in \mathbb{R}^2 : x \geq 0\}$. $\Diamond$

A classic example of the topology of surfaces is the following joke: "A topologist is a mathematician who cannot tell the difference between his doughnut and coffee cup." The joke is a lot funnier[2] once you've seen this animated GIF on Wikipedia.

The "doughnut"'s surface is known mathematically as a "torus", shown in Figure 1.1.10. A sphere is shown in Figure 1.1.11, and a surface that cannot be cannot be embedded in $\mathbb{R}^3$, the Klein bottle, is shown in Figure 1.1.12.

[2] Eh, maybe.

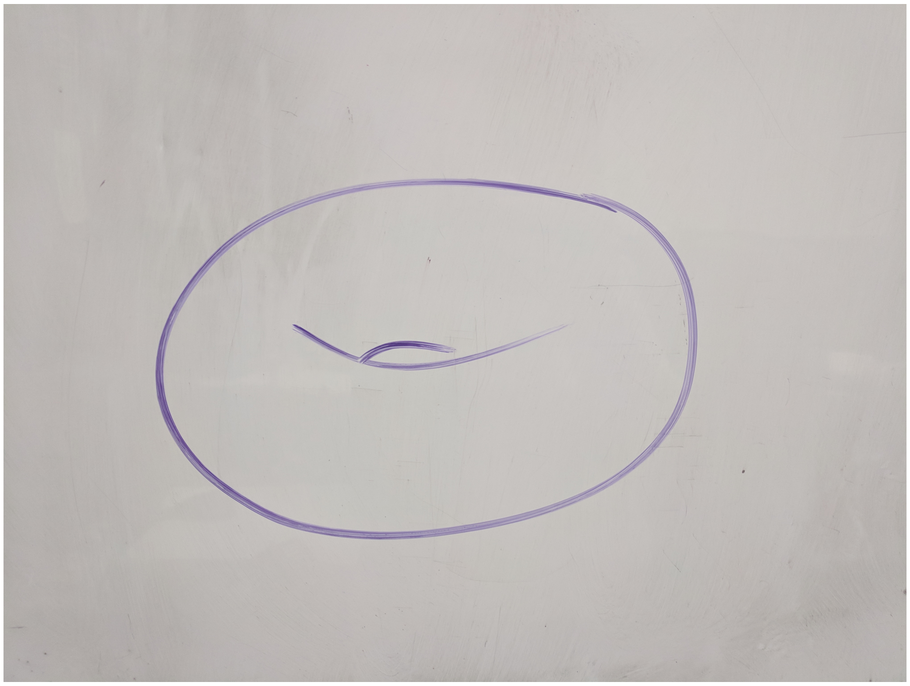

Figure 1.1.10: A torus.

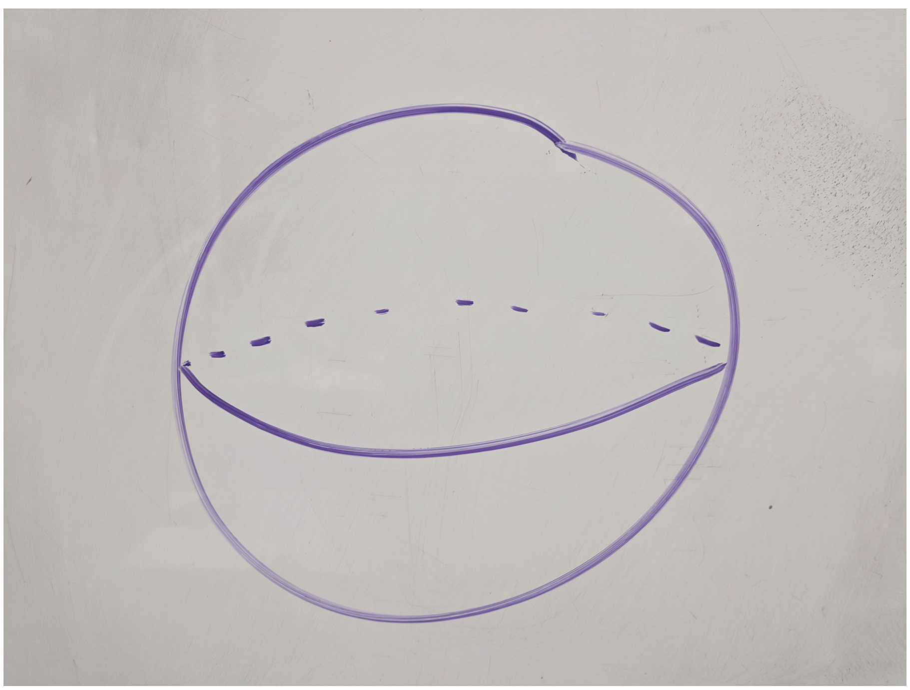

Figure 1.1.11: A sphere.

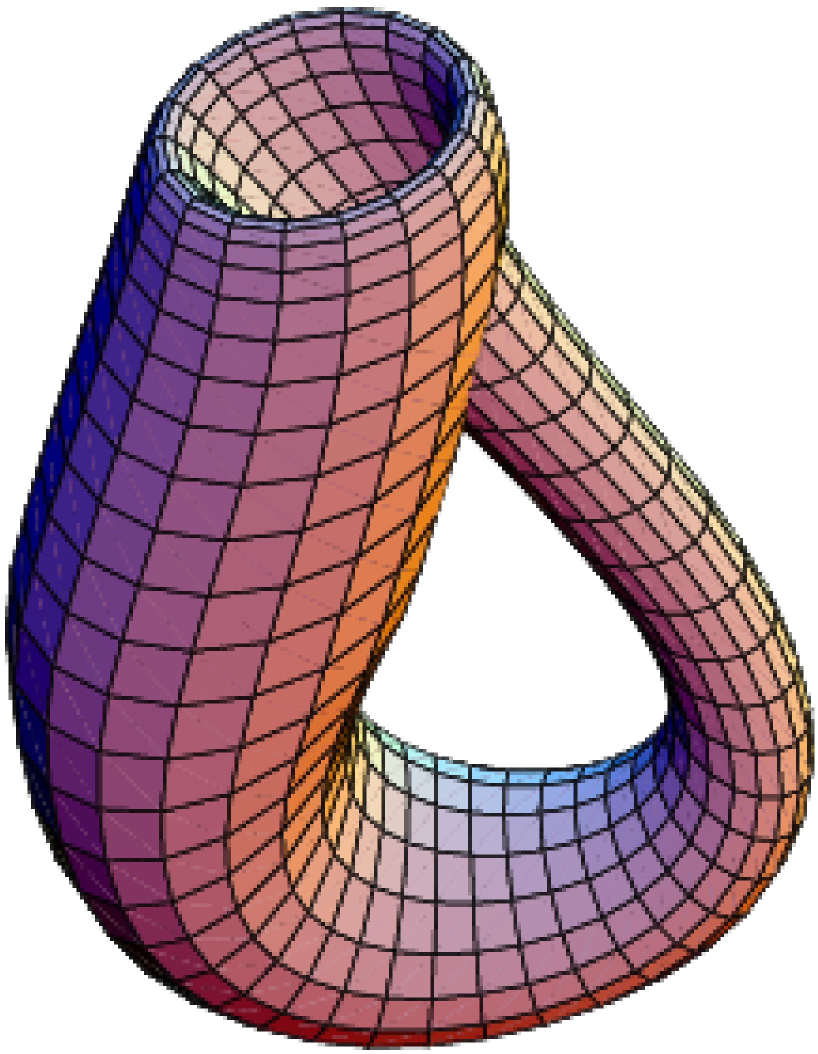

Figure 1.1.12: A Klein bottle.

While these shapes appear very different, they can all be defined as a "quotient space" (Section 1.8) of the unit square in $\mathbb{R}^2$.

In order to study so-called "topological spaces" such as these, we will begin by distilling down the notion of a "neighborhood" for an arbitrary set.

1.2 Topological Spaces

Definition 1.2.1 Let $a, b \in \mathbb{R}$. The **open interval** from a to b is the set

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

$\Diamond$

Definition 1.2.2 Let $x \in \mathbb{R}$ and $S \subseteq \mathbb{R}$. The point x is a **limit point** of the set S if and only if for every open interval (a, b) containing x, there is a point $y \in S$ such that $x \neq y$ and $y \in (a, b)$. $\Diamond$

Example 1.2.3 Determine if each set has the number 0 as a limit point.

1. $\mathbb{Z}$
2. $\mathbb{R} \setminus \mathbb{Z}$
3. $\left\{\frac{1}{n+1} : n \in \mathbb{N}\right\}$
4. $\mathbb{Q}$
5. A finite set $F \subseteq \mathbb{R}$

$\square$

Definition 1.2.4 A subset $U \subseteq \mathbb{R}$ is called **open** if and only if for every point $x \in U$, there exists an open interval (a, b) such that $x \in (a, b) \subseteq U$. $\Diamond$

Example 1.2.5 Determine if each set is open or not open.

1. $[\pi, 42)$
2. $(-3, -1) \cup (4, 5.5)$
3. $\{x : 2x + 1 > 5\}$
4. $\mathbb{Z}$
5. $\mathbb{R} \setminus \mathbb{Z}$
6. $\mathbb{Q}$
7. A finite set $F \subseteq \mathbb{R}$

$\square$

Theorem 1.2.6 *A subset $U \subseteq \mathbb{R}$ is open if and only if there exists a collection of open intervals $\mathcal{U}$ such that $U = \bigcup \mathcal{U}$.*

Proposition 1.2.7 *Let $x \in \mathbb{R}$ and $S \subseteq \mathbb{R}$. The point x is a limit point of the set S if and only if for every open set U containing x, there is a point $y \in S$ such that $x \neq y$ and $y \in U$.*

Theorem 1.2.8 *The open subsets of $\mathbb{R}$ satisfy the following properties.*

1. *$\emptyset$ and $\mathbb{R}$ are open sets.*
2. *If $\mathcal{U}$ is a collection of open sets, then $\bigcup \mathcal{U}$ is also an open set.*
3. *If U, V are open sets, then $U \cap V$ is an open set.*

Definition 1.2.9 Let X be a set, and let $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfy the following properties.

1. $\emptyset, X \in \mathcal{T}$.
2. If $\mathcal{U} \subseteq \mathcal{T}$, then $\bigcup \mathcal{U} \in \mathcal{T}$.
3. If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

Then $\mathcal{T}$ is called a **topology** on X, the pair $\langle X, \mathcal{T} \rangle$ is called a **topological space**, and elements $U \in \mathcal{T}$ are called **open sets** of the space. (Usually $\langle X, \mathcal{T} \rangle$ is abbreviated to just X when the topology is known from context.) $\Diamond$

Definition 1.2.10 Let $\mathcal{T} \subseteq \mathcal{P}(\mathbb{R})$ be the collection of open subsets of $\mathbb{R}$ defined by Definition 1.2.4. Then by Theorem 1.2.8, $\mathcal{T}$ is a valid topology for $\mathbb{R}$ called the **Euclidean topology**. $\Diamond$

Theorem 1.2.11 *Let X be any set. Then the following sets are topologies on X.*

1. *$\mathcal{T} = \mathcal{P}(X)$ is called the **discrete topology**.*

2. *$\mathcal{T} = \{\emptyset, X\}$ is called the **indiscrete topology**.*

Proposition 1.2.12 *Let $\mathcal{T}$ be a topology, and let $\mathcal{U} \subseteq \mathcal{T}$ be finite. Then $\bigcap \mathcal{U} \in \mathcal{T}$.*

Proposition 1.2.13 *Let $\mathcal{T}$ be the Euclidean topology. There exists a collection $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ such that $\bigcap \mathcal{U} \notin \mathcal{T}$.*

Definition 1.2.14 Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$. The following are called **intervals** of real numbers.

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

$\Diamond$

Example 1.2.15 Show that each of the following is an example of a topological space $\langle X, \mathcal{T} \rangle$.

1. Let $X = \mathbb{R}$ and $\mathcal{T} = \{(x, \infty) : x \in \mathbb{R}\} \cup \{[x, \infty) : x \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.

2. Let $X = \mathbb{R}$ and $\mathcal{T} = \{(x, y) : x, y \in \mathbb{R} \cup \{-\infty, \infty\} \text{ and } x < 0 < y\} \cup \{\emptyset\}$.

3. Let $X = \mathbb{R}$ and $U \in \mathcal{T}$ if for each $x \in U$, there exists $a, b \in \mathbb{R}$ such that $x \in [a, b) \subseteq U$.

4. Let $X = \{0, 1\}$ and $\mathcal{T} = \{\emptyset, \{0\}, X\}$.

5. Let $X = \mathbb{Z}$, $E = \{n \in \mathbb{Z} : n \text{ is even}\}$, $D = \{n \in \mathbb{Z} : n \text{ is odd}\}$, and $\mathcal{T} = \{\emptyset, E, D, X\}$.

$\square$

Definition 1.2.16 Let $\langle X, \mathcal{T} \rangle$ be a topological space and let $x \in X$. The set $N \subseteq X$ is called a **neighborhood** of x if and only if there exists an open set $U \in \mathcal{T}$ such that $x \in U \subseteq N$. $\Diamond$

Proposition 1.2.17 *A subset U of a topological space X is open if and only if U is a neighborhood of every point it contains.*

Definition 1.2.18 The following are known as **separation axioms** for a topological space $\langle X, \mathcal{T} \rangle$.

1. $\mathcal{T}$ is said to be T_0 if and only if for all points $x, y \in X$ such that $x \neq y$, there either exists a neighborhood U of x such that $y \notin U$, or there exists a neighborhood V of y such that $x \notin V$.

2. $\mathcal{T}$ is said to be T_1 if and only if for all points $x, y \in X$ such that $x \neq y$, there exists a neighborhood U of x such that $y \notin U$.

3. $\mathcal{T}$ is said to be T_2 (also known as **Hausdorff**) if and only if for all points $x, y \in X$ such that $x \neq y$, there exist neighborhoods U, V of x, y (respectively) such that $U \cap V = \emptyset$.

$\Diamond$

Proposition 1.2.19 $T_2 \Rightarrow T_1 \Rightarrow T_0$.

Example 1.2.20 Find or create an example of a topological space $\langle X, \mathcal{T} \rangle$ that is:

1. Not T_0.

2. T_0 but not T_1.

3. T_1 but not T_2.

$\square$

Theorem 1.2.21 *Let X be a finite topological space. Then X is T_1 if and only if X has the discrete topology.*

Proposition 1.2.22 *The Euclidean real line is a non-discrete Hausdorff topological space.*

Definition 1.2.23 Let $S \subseteq X$ be a subset of a topological space. The point x is a **limit point** of the set S if and only if for every neighborhood of U of x, there is a point $y \in S$ such that $x \neq y$ and $y \in U$. $\Diamond$

Proposition 1.2.24 *The point $x \in \mathbb{R}$ is a limit point of $S \subseteq \mathbb{R}$ according to Definition 1.2.2 if and only if it is a limit point according to Definition 1.2.23 (where $\mathbb{R}$ is assumed to have the Euclidean topology).*

Definition 1.2.25 Let $S \subseteq X$ be a subset of a topological space. Then S' is the set of all limit points of S, called the **derived set** of S. $\Diamond$

Definition 1.2.26 Let $S \subseteq X$ be a subset of a topological space. Then $\operatorname{cl} S = S \cup S'$ is called the **closure** of S. $\Diamond$

Example 1.2.27 Calculate $\operatorname{cl} S$ for each of the following examples.

1. $S = (-1, 1) \subseteq \mathbb{R}$ where $\mathbb{R}$ has the Euclidean topology.

2. $S = (-1, 1) \subseteq \mathbb{R}$ where $\mathbb{R}$ has the discrete topology.

3. $S = (-1, 1) \subseteq \mathbb{R}$ where $\mathbb{R}$ has the indiscrete topology.

4. $S = \mathbb{Z} \subseteq \mathbb{R}$ where $\mathbb{R}$ has the Euclidean topology.

5. $S = \mathbb{Q} \subseteq \mathbb{R}$ where $\mathbb{R}$ has the Euclidean topology.

$\square$

Definition 1.2.28 Let $H \subseteq X$ be a subset of a topological space. Then H is called **closed** if and only if $H = \operatorname{cl} S$. $\Diamond$

Theorem 1.2.29 *Let $H \subseteq X$ be a subset of a topological space. Then H is closed if and only if there exists an open set U such that $H = X \setminus U$.*

Proposition 1.2.30 *The closed subsets of a topological space X satisfy the following properties.*

1. *$\emptyset$ and X are closed sets.*

2. *If $\mathcal{H}$ is a collection of closed sets, then $\bigcap \mathcal{H}$ is also a closed set.*

3. *If H, L are closed sets, then $H \cup L$ is a closed set.*

Theorem 1.2.31 *A topological space X is T_1 if and only if every finite subset of X is closed.*

Definition 1.2.32 Let $S \subseteq X$ be a subset of a topological space. The point x is a **boundary point** of the set S if and only if for every neighborhood of U of x, both $U \cap S$ and $U \setminus S$ are non-empty.

Let $\operatorname{bd} S$ collect all the boundary points of S. ◊

Proposition 1.2.33 *Let $a, b \in \mathbb{R}$. Then $\operatorname{bd}(a, b) = \operatorname{bd}(a, b] = \operatorname{bd}[a, b) = \operatorname{bd}[a, b] = \{a, b\}$ with respect to the Eudlidean topology.*

Definition 1.2.34 Let $S \subseteq X$ be a subset of a topological space. The point x is a **interior point** of the set S if and only if there exists a neighborhood U of x such that $x \in U \subseteq S$.

Let $\operatorname{int} S$ collect all the interior points of S. ◊

Proposition 1.2.35 *Let $U \subseteq X$ be a subset of a topological space. Then U is open if and only if $U = \operatorname{int} U$.*

Definition 1.2.36 Let $S \subseteq X$ be a subset of a topological space. The point x is a **exterior point** of the set S if and only if there exists a neighborhood U of x such that $x \in U \subseteq X \setminus S$.

Let $\operatorname{ext} S$ collect all the exterior points of S. ◊

Definition 1.2.37 A **partition** of a set X is a collection $\mathcal{P}$ such that $X = \bigcup \mathcal{P}$ and $A \cap B = \emptyset$ for all $A, B \in \mathcal{P}$ where $A \neq B$. ◊

Proposition 1.2.38 *Let $S \subseteq X$ be a subset of a topological space. Then $\{\operatorname{int} S, \operatorname{bd} S, \operatorname{ext} S\}$ is a partition of X.*

Proposition 1.2.39 *Let $S \subseteq X$ be a subset of a topological space. Then $\operatorname{cl} S = \operatorname{int} S \cup \operatorname{bd} S = S \cup \operatorname{bd} S$.*

Example 1.2.40 Let A be a subset of a topological space X. Prove or disprove the following.

1. $\operatorname{int} \operatorname{int} A = \operatorname{int} A$
2. $\operatorname{int} \operatorname{cl} A = \operatorname{int} A$
3. $\operatorname{bd} \operatorname{bd} A = \operatorname{bd} A$
4. $\operatorname{ext} \operatorname{ext} A = \operatorname{int} A$
5. $\operatorname{int} \operatorname{ext} A = \operatorname{ext} A$
6. $\operatorname{int} \operatorname{bd} A = \emptyset$
7. $\operatorname{cl} \operatorname{ext} A = X \setminus \operatorname{int} A$

□

Example 1.2.41 Let A, B be subsets of a topological space X. Prove or disprove the following.

1. $\operatorname{int}(A \cap B) = \operatorname{int} A \cap \operatorname{int} B$
2. $\operatorname{int}(A \cup B) = \operatorname{int} A \cup \operatorname{int} B$
3. $\operatorname{bd}(A \cap B) = \operatorname{bd} A \cap \operatorname{bd} B$
4. $\operatorname{bd}(A \cup B) = \operatorname{bd} A \cup \operatorname{bd} B$
5. $\operatorname{cl}(A \cap B) = \operatorname{cl} A \cap \operatorname{cl} B$

6. $\operatorname{cl}(A \cup B) = \operatorname{cl} A \cup \operatorname{cl} B$

□

Definition 1.2.42 A subset $D \subseteq X$ of a topological space is called **dense** if and only if $\operatorname{cl} D = X$. ◊

Example 1.2.43 Determine which of these are dense subsets of $\mathbb{R}$.

1. $\mathbb{Q}$
2. $\mathbb{Z}$
3. $\mathbb{R} \setminus \mathbb{Q}$
4. $\mathbb{R} \setminus \mathbb{Z}$

□

Theorem 1.2.44 *A subset D of a topological space is dense if and only if every nonempty open set of the space contains a point of D.*

Proposition 1.2.45 *Let X be a topological space, and let $D \subseteq E \subseteq X$. If D is dense, then E is also dense.*

Definition 1.2.46 Let $Y \subseteq X$ for a topological space $\langle X, \mathcal{T} \rangle$. Then the **subspace topology** for Y is given by $\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}$. ◊

Proposition 1.2.47 *The subspace topology is a valid topology.*

Proposition 1.2.48 *Let $n \in \{0, 1, 2\}$. A subspace of a T_n space is also T_n.*

Definition 1.2.49 The **Cantor set** is the subset $C \subseteq \mathbb{R}$ defined by $C = \bigcap_{n \in \mathbb{N}} C_n$, where $C_0 = [0, 1]$ and

$$C_{n+1} = C_n \setminus \bigcup_{0 \leq k < 3^n} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right).$$

This set is usually considered as a closed subset of the Euclidean line, or as a subspace of the Euclidean line. ◊

Definition 1.2.50 Let $\langle X, \mathcal{T} \rangle$ be a topological space. A subset $\mathcal{B} \subseteq \mathcal{T}$ is called a **basis** for the topology if for every $x \in X$ and neighborhood U of x, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. ◊

Proposition 1.2.51 *$\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$ is a basis for the Euclidean topology.*

Theorem 1.2.52 *Let $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfy the following properties:*

1. *For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.*
2. *If $x \in A \in \mathcal{B}$ and $x \in B \in \mathcal{B}$, there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.*

*Then $\mathcal{T} = \{\bigcup \mathcal{U} : \mathcal{U} \subseteq \mathcal{B}\}$ is a topology, and $\mathcal{B}$ is a basis for that topology. We call this the **topology generated by the basis**.*

Proposition 1.2.53 *$\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}\}$ is a basis for a topology different from the Euclidean topology, called the **Sorgenfrey topology**.*

Example 1.2.54 Examples of bases. Calculate the topology generated by each basis on $\mathbb{R}$.

1. $\mathcal{B} = \{(a, b) : a, b \in \mathbb{Q}\}$

2. $\mathcal{B} = \{(a, \infty) : a \in \mathbb{R}\}$

3. $\mathcal{B} = \{\{x\} : x \in \mathbb{R}\}$

4. $\mathcal{B} = \{[a, b] : a, b \in \mathbb{R}\}$

5. $\mathcal{B} = \{[a, b] : a, b \in \mathbb{R}, a < 0 < b\}$

□

Theorem 1.2.55 *Let $\mathcal{S} \subseteq \mathcal{P}(X)$ and*

$$\mathcal{T} = \bigcap \{\mathcal{T}^\star \subseteq \mathcal{P}(X) : \mathcal{S} \subseteq \mathcal{T}^\star \textit{ and } \mathcal{T}^\star \textit{ is a topology on } X\}.$$

Then $\mathcal{T}$ is a topology.

Definition 1.2.56 The set $\mathcal{S} \subseteq \mathcal{P}(X)$ in Theorem 1.2.55 is called a **subbasis** generating the topology

$$\mathcal{T} = \bigcap \{\mathcal{T}^\star \subseteq \mathcal{P}(X) : \mathcal{S} \subseteq \mathcal{T}^\star \text{ and } \mathcal{T}^\star \text{ is a topology on } X\}.$$

◊

Example 1.2.57 Topologies generated from subbases. Calculate the topology on $\mathbb{R}$ generated by each subbasis.

1. $\{(-\infty, x) : x \in \mathbb{R}\} \cup \{(y, \infty) : y \in \mathbb{R}\}$

2. $\{(-\infty, x] : x \in \mathbb{R}\} \cup \{[y, \infty) : y \in \mathbb{R}\}$

3. $\{\{0\}\}$

4. $\mathcal{T} \cup \left\{\mathbb{R} \setminus \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}\right\}$

□

Theorem 1.2.58 *Let $\mathcal{S} \subseteq \mathcal{P}(X)$ and*

$$\mathcal{B} = \{X\} \cup \bigcap \{\mathcal{B}^\star \subseteq \mathcal{P}(X) : \mathcal{S} \subseteq \mathcal{B}^\star \textit{ and } B_1, B_2 \in \mathcal{B}^\star \Rightarrow B_1 \cap B_2 \in \mathcal{B}^\star\}.$$

Then $\mathcal{B}$ is a basis for a topology on X, and the topology generated by the basis $\mathcal{B}$ is same as the topology generated by the subbasis $\mathcal{S}$.

Definition 1.2.59 The following are also known as **separation axioms** for a topological space $\langle X, \mathcal{T}\rangle$.

1. $\mathcal{T}$ is said to be **regular** if and only if for all points $x \in X$ and closed subsets $H \subseteq X$ such that $x \notin H$, there exist open sets $U, V \in \mathcal{T}$ such that $x \in U, H \subseteq V, U \cap V = \emptyset$.

2. $\mathcal{T}$ is said to be T_3 if and only if it is both regular and T_1

3. $\mathcal{T}$ is said to be **normal** if and only if for all closed subsets $H, L \subseteq X$ such that $H \cap L = \emptyset$, there exist open sets $U, V \in \mathcal{T}$ such that $H \subseteq U, L \subseteq V, U \cap V = \emptyset$.

4. $\mathcal{T}$ is said to be T_4 if and only if it is both normal and T_1

◊

Proposition 1.2.60 $T_{n+1} \Rightarrow T_n$ *for* $n \in \{0, 1, 2, 3\}$.

Theorem 1.2.61 *The real line $\mathbb{R}$ equipped with the Euclidean topology is T_4.*

Example 1.2.62 Find or create an example of a topological space that is:

1. T_2 but not regular.
2. T_3 but not T_4
3. Regular but not T_3.
4. Normal but not T_4.
5. Regular but not normal.
6. Normal but not regular.

$\square$

Theorem 1.2.63 *A topological space is T_3 if and only if it is regular and T_0.*

1.3 Continuity & Homeomorphisms

Definition 1.3.1 Let $f : X \to Y$ be a function. For $A \subseteq X$, let $f[A] = \{f(x) : x \in A\}$. For $y \in Y$, let $f^{\leftarrow}(y) = \{x \in X : f(x) = y\}$. For $B \subseteq Y$, let $f^{\leftarrow}[B] = \{x \in X : f(x) \in B\}$. $\diamondsuit$

Definition 1.3.2 Let X, Y be topological spaces with $x \in X$, and let $f : X \to Y$ be a function such that for every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f[U] \subseteq V$. Then f is said to be **continuous at the point** x.

A function that is continuous at every point of its domain is called **continuous**. $\diamondsuit$

Proposition 1.3.3 *A function $f : X \to Y$ is continuous if and only if $f^{\leftarrow}[V]$ is an open subset of X for every open $V \subseteq Y$.*

Proposition 1.3.4 *Let X, Y be topological spaces.*

1. *The identity function $\iota : X \to X$ defined by $\iota(x) = x$ is continuous.*
2. *Let $y \in Y$. The constant function $c_y : X \to Y$ defined by $c_y(x) = y$ is continuous.*
3. *Every function whose domain is a discrete space is continuous.*
4. *Every function whose range is an indiscrete space is continuous.*

Example 1.3.5 Continuous functions $\mathbb{R} \to \mathbb{R}$. Verify that each of the following functions $f : \mathbb{R} \to \mathbb{R}$ are continuous.

1. $f(x) = |x|$
2. $f(x) = x^2$
3. $f(x) = g(x) + h(x)$ for $g, h : \mathbb{R} \to \mathbb{R}$ continuous.
4. $f(x) = g(x)h(x)$ for $g, h : \mathbb{R} \to \mathbb{R}$ continuous.

$\square$

Theorem 1.3.6 *If $f : X \to Y$ and $g : Y \to Z$ are both continuous, then $g \circ f : X \to Z$ is continuous.*

Definition 1.3.7 Let $f : X \to Y$ be a bijection such that both f and its inverse f^{-1} are continuous. Then f is called a **homeomorphism** and X, Y are said to be **homeomorphic**. $\Diamond$

Example 1.3.8 Properties preserved by continuous functions. Determine if the following hold if $f : X \to Y$ is a continous surjection. If not, determine if they hold if f is a continuous bijection. If not, show that they hold if f is a homeomorphism.

1. If X is Hausdorff, then Y is Hausdorff.
2. If Y is Hausdorff, then X is Hausdorff.
3. If $U \subseteq X$ is open, then $f[U] \subseteq Y$ is open.
4. If $H \subseteq X$ is closed, then $f[H] \subseteq Y$ is closed.
5. If x is a limit point of $A \subseteq X$, then $f(x)$ is a limit point of $f[A] \subseteq Y$.

$\square$

Proposition 1.3.9 *Every topological space is homeomorphic to itself.*

Proposition 1.3.10 *If $f : X \to Y$ and $g : Y \to Z$ are both homeomorphisms, then $g \circ f : X \to Z$ is a homeomorphism.*

Theorem 1.3.11 *Let $a < b$ and $c < d$ be real numbers. Then (a, b) and (c, d) are homeomorphic subspaces of the Euclidean line.*

Theorem 1.3.12 *$\mathbb{R}$ with the Euclidean topology is homeomorphic to its subspace $(0, 1)$.*

Theorem 1.3.13 *Let $\mathcal{B}$ be a basis for the Euclidean topology on $\mathbb{R}$. Give $K = \mathbb{R} \cup \{-\infty, \infty\}$ the topology generated by the basis $\{[-\infty, x) : x \in \mathbb{Q}\} \cup \{(x, \infty] : x \in \mathbb{Q}\} \cup \mathcal{B}$. Then K is homeomorphic to the subspace $[0, 1]$ of the Euclidean line.*

Proposition 1.3.14 *The real line with the Sorgenfrey topology generated by the basis $\{[a, b) : a, b \in \mathbb{R}\}$ is homeomorphic to the real line with the reverse Sorgenfrey topology generated by the basis $\{(a, b] : a, b \in \mathbb{R}\}$.*

1.4 Metric Spaces

Definition 1.4.1 Let $d : X^2 \to [0, \infty)$ be a function satisfying the following for all $x, y, z \in X$.

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Then d is said to be a **metric** on the set X, and

$$B_r(x) = \{y \in X : d(x, y) < r\}$$

is said to be a **metric ball around** x. $\Diamond$

Example 1.4.2 Examples of metrics. Verify that each of the following is a metric.

1. $d(x, y) = 1$ for all distinct $x, y \in X$, and $d(x, x) = 0$

2. $d(x, y) = |y - x|$ for all $x, y \in \mathbb{R}$
3. $d(\langle x_0, x_1\rangle, \langle y_0, y_1\rangle) = \sqrt{(y_1 - y_0)^2 + (x_1 - x_0)^2}$ for all $\langle x_0, x_1\rangle, \langle y_0, y_1\rangle \in \mathbb{R}^2$.
4. $d(\langle x_0, x_1\rangle, \langle y_0, y_1\rangle) = |y_1 - y_0| + |x_1 - x_0|$ for all $\langle x_0, x_1\rangle, \langle y_0, y_1\rangle \in \mathbb{R}^2$.
5. $d(\langle x_0, x_1\rangle, \langle y_0, y_1\rangle) = \max\{|y_1 - y_0|, |x_1 - x_0|\}$ for all $\langle x_0, x_1\rangle, \langle y_0, y_1\rangle \in \mathbb{R}^2$.

□

Theorem 1.4.3 *Let d be a metric on a set X. Then*

$$\mathcal{B} = \{B_r(x) : x \in X, r > 0\}$$

is a basis for a topology on X.

Definition 1.4.4 The topology generated by the basis given in Theorem 1.4.3 is called the **topology generated by the metric**.

A given topology is said to be **metrizable** if there exists some metric that generates it. Two metrics are said to be **topologically equivalent** if they generate the same topology. ◊

Proposition 1.4.5 *Every discrete space is metrizable.*

Theorem 1.4.6 *Every metrizable space is T_4.*

Theorem 1.4.7 *Two bases $\mathcal{B}_0, \mathcal{B}_1$ generate the same topology if and only if for all $x \in B_0 \in \mathcal{B}_0$ there exists $B_1 \in \mathcal{B}_1$ such that $x \in B_1 \subseteq B_0$, and for all $x \in B_1 \in \mathcal{B}_1$ there exists $B_0 \in \mathcal{B}_0$ such that $x \in B_0 \subseteq B_1$.*

Theorem 1.4.8 *Let $B_{0,r}(x), B_{1,r}(x)$ be the metric balls around x given by two metrics d_0, d_1 respectively. Then d_0, d_1 are topologically equivalent if and only if for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{0,\delta}(x) \subseteq B_{1,\epsilon}$ and $B_{1,\delta}(x) \subseteq B_{0,\epsilon}(x)$.*

Definition 1.4.9 For $\vec{x} = \langle x_0, \ldots, x_{n-1}\rangle \in \mathbb{R}^n$, let $\vec{x}(i) = x_i$. ◊

Theorem 1.4.10 *The following metrics on $\mathbb{R}^n$ are topologically equivalent.*

1. $d(\vec{x}, \vec{y}) = \sqrt{\sum_{0 \le i < n} (\vec{y}(i) - \vec{x}(i))^2}$
2. $d(\vec{x}, \vec{y}) = \sum_{0 \le i < n} |\vec{y}(i) - \vec{x}(i)|$
3. $d(\vec{x}, \vec{y}) = \max\{|\vec{y}(i) - \vec{x}(i)| : 0 \le i < n\}$

Definition 1.4.11 The topology generated by the metrics given in Theorem 1.4.10 is called the **Euclidean topology** on $\mathbb{R}^n$. ◊

Definition 1.4.12 A **local basis at a point** x is a collection of open sets $\mathcal{B}_x$ such that for every neighborhood U of x, there exists $B \in \mathcal{B}_x$ such that $x \in B_x \subseteq U$. ◊

Definition 1.4.13 A space is said to be **first-countable** if there exists a countable local basis at every point of the space.

A space is said to be **second-countable** if there exists a countable basis for the space. ◊

Proposition 1.4.14 *Every second-countable space is first-countable.*

Proposition 1.4.15 *Every metrizable space is first-countable*

Definition 1.4.16 A space is said to be **separable** if there exists a countable dense subset of the space. ♢

Theorem 1.4.17 *Let X be metrizable. Then X is second-countable if and only if it is separable.*

Proposition 1.4.18 *Every Euclidean space is separable and second-countable.*

Theorem 1.4.19 *For $\vec{x}, \vec{y} \in \mathbb{R}^2$, let $d(\vec{x}, \vec{y}) = 1$ if $\vec{x}(1) \neq \vec{y}(1)$, and $d(\vec{x}, \vec{y}) = |\vec{y}(0) - \vec{x}(0)|$ otherwise. Then d is a metric generating a non-separable, non-discrete topology on $\mathbb{R}^2$.*

Theorem 1.4.20 *The subspace $\{\vec{x} : \vec{x}(1) \in \{0, 1\}\}$ of the space defined in Theorem 1.4.19 is homeomorphic to the subspace $(0, 1) \cup (2, 3)$ of the Euclidean line.*

Definition 1.4.21 A point x is called a **sequential limit point** of a set A iff there exists a countable subset $B \subseteq A \setminus \{x\}$ such that every neighborhood of x contains all but finitely many points of B. ♢

Proposition 1.4.22 *Every sequential limit point of a set is a limit point of that set.*

Theorem 1.4.23 *Let X be first-countable. Then x is a limit point of a set if and only if x is a sequental limit point of that set.*

Definition 1.4.24 A **Cauchy sequence** is a countably infinte set A such that for all $\epsilon > 0$, the set $\{x \in A : \exists y \in A(d(x, y) \geq \epsilon)\}$ is finite. ♢

Definition 1.4.25 A **complete metric** is a metric such that every Cauchy sequence has a sequential limit point.

A topology that can be generated by a complete metric is said to be **completely metrizable**. ♢

Proposition 1.4.26 *Every Euclidean space is completely metrizable.*

Proposition 1.4.27 *Let $d : X \to [0, \infty)$ be a metric and $Y \subseteq X$. Then d restricted to Y generates the subspace topology on Y. (Therefore, every subspace of a metrizable space is metrizable.)*

Theorem 1.4.28 *The subspace $(0, 1)$ of the Euclidean line is completely metrizable, but not by the topology inherited from $\mathbb{R}$.*

Theorem 1.4.29 *The subspace $\mathbb{Q}$ of the Euclidean line is metrizable, but not completely metrizable.*

Theorem 1.4.30 *The subspace $\mathbb{R} \setminus \mathbb{Q}$ of the Euclidean line is completely metrizable, but not by the topology inherited from $\mathbb{R}$.*

Theorem 1.4.31 *Metrizable and completely metrizable are topological properties. That is, if X and Y are homeomorphic, then X is (completely) metrizable if and only if Y is too.*

1.5 Compactness

Definition 1.5.1 A collection $\mathcal{A} \subseteq \mathcal{P}(X)$ is said to **cover** a subset $Y \subseteq X$ iff $Y \subseteq \bigcup \mathcal{A}$. ♢

Definition 1.5.2 A subset $K \subseteq X$ of a topological space is said to be **compact** iff for every collection of open sets $\mathcal{U}$ covering K, there exists a finite subcollection $\mathcal{F} \subseteq \mathcal{U}$ that also covers K. ♢

Example 1.5.3 Determine if each of the following subsets of the Euclidean line is compact.

1. $\mathbb{R}$
2. $\mathbb{Z}$
3. $\{2^{-n} : n \in \mathbb{N}\}$
4. $\{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$
5. $(0, 1)$
6. $[0, 1]$

□

Definition 1.5.4 A subset $R \subseteq X$ of a topological space is said to be **relatively compact** iff for every collection of open sets $\mathcal{U}$ covering X, there exists a finite subcollection $\mathcal{F} \subseteq \mathcal{U}$ that covers K. ♢

Theorem 1.5.5 *A space is regular if and only if for every point x and neighborhood U, there exists a neighborhood V of x such that $x \in V \subseteq \operatorname{cl} V \subseteq U$.*

Theorem 1.5.6 *Let X be regular. A subset $R \subseteq X$ is relatively compact if and only if $\operatorname{cl} R$ is compact.*

Theorem 1.5.7 *Let X be compact and K be a closed subset of X. Then K is compact.*

Proposition 1.5.8 *Every finite subset of a space is compact.*

Proposition 1.5.9 *Every finite union of compact subsets is compact.*

Theorem 1.5.10 *Every compact subset of a Hausdorff space is closed.*

Proposition 1.5.11 *Let $\mathcal{T} = \{\emptyset\} \cup \{\mathbb{N} \setminus F : F \text{ is finite}\}$ be the cofinite topology on $\mathbb{N}$. Every subset of $\mathbb{N}$ is compact under this topology.*

Theorem 1.5.12 *Let $f : X \to Y$ be continuous and $K \subseteq X$ be compact. Then $f[K]$ is compact.*

Corollary 1.5.13 *Compactness is a topological property.*

Theorem 1.5.14 *Every infinite subset of a compact set has a limit point.*

Theorem 1.5.15 *Let $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$ be a collection of non-empty compact subsets of a topological space such that $K_{n+1} \subseteq K_n$ for all $n \in \mathbb{N}$. Then $\bigcap \mathcal{K}$ is a non-empty compact set.*

Theorem 1.5.16 *Let X be metrizable. Then the following are equivalent for $K \subseteq X$.*

1. *K is compact*

2. *Every infinite subset of K has a limit point.*

3. *Every infinite subset of K has a sequential limit point.*

Lemma 1.5.17 *A topological space X is Hausdorff if and only if for every pair of disjoint compact subsets H, K there exist disjoint open sets U, V such that $H \subseteq U$ and $K \subseteq V$.*

Theorem 1.5.18 *Every compact Hausdorff space is T_4.*

1.6 Connectedness

Definition 1.6.1 A pair of open sets $\{A, B\}$ satisfying $A \cap Y \neq \emptyset$, $B \cap Y \neq \emptyset$, $Y \subseteq A \cup B$, and $A \cap B \cap Y = \emptyset$ for a subset Y of a topological space X is called a **disconnection** of Y.

A space for which a diconnection exists is called **disconnected**; otherwise, the space is called **connected**. $\Diamond$

Proposition 1.6.2 *A pair $\{A, B\}$ of open sets is a disconnection of $Y \subseteq X$ if and only if $\{A \cap Y, B \cap Y\}$ is a partition of Y by non-empty clopen (both closed and open) sets in the subspace topology.*

(Corollary: A space itself is disconnected iff it is the union of two disjoint non-empty clopen subsets.)

Proposition 1.6.3 *The Euclidean line with a point removed $\mathbb{R} \setminus \{0\}$ is disconnected.*

Lemma 1.6.4 *Let $\mathbb{R} = U \cup V$ for open sets U, V and let $x \in U, y \in V$ with $x \leq y$. Then $\inf\{z \in [x, y] : z \in V\} \in U \cap V$.*

Corollary 1.6.5 *The Euclidean line is connected.*

Theorem 1.6.6 *The Sorgenfrey topology on $\mathbb{R}$ is disconnected.*

Theorem 1.6.7 *If a subset A of a topological space is connected, then $\operatorname{cl} A$ is connected.*

Proposition 1.6.8 *If a subset A of a topological space is connected and $f : X \to Y$ is continuous, then $f[A]$ is connected.*

Corollary 1.6.9 *Connectedness is a topological property.*

Proposition 1.6.10 *Let $\{0, 1\}$ have the discrete topology. Then a topological space X is connected if and only if every continuous function $f : X \to \{0, 1\}$ is constant.*

Theorem 1.6.11 *If $\mathcal{A}$ is a collection of connected subsets of a topological space with $\bigcap \mathcal{A} \neq \emptyset$, then $\bigcup \mathcal{A}$ is connected.*

Definition 1.6.12 Suppose for every two points $x, y \in A \subseteq X$, there exists a continuous function $f : [0, 1] \to A$ such that $f(0) = x$ and $f(1) = y$. Such a space is said to be **path connected**. $\Diamond$

Proposition 1.6.13 *Every path connected space is connected.*

Theorem 1.6.14 *For $\vec{x}, \vec{y} \in \mathbb{R}^2$, let*

$$B(\vec{x}, \vec{y}) = \{\vec{z} : \vec{z}(0) \in (\vec{x}(0), \vec{y}(0)) \text{ or } (\vec{x}(0) = \vec{y}(0) = \vec{z}(0) \text{ and } \vec{z}(1) \in (\vec{x}(1), \vec{y}(1))\}.$$

Then $\mathcal{B} = \left\{B(\vec{x}, \vec{y}) : \vec{x}, \vec{y} \in \mathbb{R}^2\right\}$ is a basis for a topology on $\mathbb{R}^2$ that is connected *but not path connected.*

Theorem 1.6.15 *Let*

$$S = \left\{ \langle x, y \rangle : x \in (0, 1] \text{ and } y = \sin\left(\frac{1}{x}\right) \right\}$$

(the ***topologist's sine curve****). Then* $\operatorname{cl} S$ *is a subset of the Euclidean space* $\mathbb{R}^2$ *that is connected but not path connected.*

1.7 Product Spaces

Definition 1.7.1 Let X, Y be topological spaces, generated respectively by the bases $\mathcal{B}_X, \mathcal{B}_Y$. Then the **product space** is given by the set $X \times Y = \{\langle x, y \rangle : x \in X, y \in Y\}$ with the topology generated by the basis $\mathcal{B} = \{U \times V : U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$. $\diamondsuit$

Theorem 1.7.2 *The Euclidean space* $\mathbb{R}^{n+1}$ *is homeomorphic to the product space* $\mathbb{R}^n \times \mathbb{R}$.

Proposition 1.7.3 *The product* $X \times Y$ *is Hausdorff if and only if* X, Y *are each Hausdorff.*

Theorem 1.7.4 *The product* $X \times Y$ *is regular if and only if* X, Y *are each regular.*

Lemma 1.7.5 *Let* $S = \mathbb{R}$ *equipped with the Sorgenfrey topology. Then the product space* $S \times S$ *contains two disjoint closed subsets* $H = \{\langle x, -x \rangle : x \in \mathbb{Q}\}$ *and* $L = \{\langle x, -x \rangle : x \in \mathbb{R} \setminus \mathbb{Q}\}$ *that cannot be separated by a pair of open sets.*

Theorem 1.7.6 *Let* $S = \mathbb{R}$ *equipped with the Sorgenfrey topology. Then* S *is normal, but* $S \times S$ *is not normal.*

Proposition 1.7.7 *Let* $p \in Y$. *The subspace* $\{\langle x, p \rangle : x \in X\}$ *of* $X \times Y$ *is homeomorphic to* X.

Proposition 1.7.8 *The* ***diagonal*** $\Delta = \{\langle x, x \rangle : x \in X\}$ *of* $X \times X$ *is homeomorphic to* X.

Proposition 1.7.9 *The product spaces* $X \times Y$ *and* $Y \times X$ *are homeomorphic.*

Definition 1.7.10 For a product space $X \times Y$, its **projection maps** $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are defined by $\pi_X(\langle x, y \rangle) = x$ and $\pi_Y(\langle x, y \rangle) = y$. $\diamondsuit$

Example 1.7.11 Properties of projection maps. Verify the following properties of projection maps.

1. Every projection map is continuous.
2. The projection of an open set is always open.
3. The projection of a closed set is not always closed.

$\square$

Theorem 1.7.12 *The product* $X \times Y$ *is metrizable if and only if* X, Y *are each metrizable.*

Lemma 1.7.13 *Let Y be compact. If $\mathcal{U}$ is an open cover of $X \times Y$, then for each $x \in X$ there exists a finite subcollection $\mathcal{F}_x \subseteq \mathcal{U}$ and an open neighborhood U_x of x such that $U_x \times Y \subseteq \bigcup \mathcal{F}_x$.*

Theorem 1.7.14 *The product $X \times Y$ is compact if and only if X, Y are each compact.*

Theorem 1.7.15 *The product $X \times Y$ is connected if and only if X, Y are each connected.*

1.8 Quotient Spaces

Definition 1.8.1 A **quotient map** is a surjection $f : X \to Y$ such that $V \subseteq Y$ is open if and only if $f^{\leftarrow}[V] \subseteq X$ is open. $\Diamond$

Proposition 1.8.2 *Let $\langle X, \mathcal{T}_X \rangle$ be a topological space, and let $f : X \to Y$ be a surjection. Then $\mathcal{T}_Y = \{V \subseteq Y : f^{\leftarrow}[V] \in \mathcal{T}_X\}$ is a topology on Y such that f is a quotient map.*

Definition 1.8.3 The topology defined in Proposition 1.8.2 is known as the **quotient topology** induced by f. $\Diamond$

Definition 1.8.4 Let X^* be a partition of a topological space X, and let $f : X \to X^*$ be the surjection given by letting $f(x) = A$ iff $x \in A$. Then X^* paired with the quotient topology induced by f is called a **quotient space** or **identification space**. $\Diamond$

Theorem 1.8.5 *Let X^* be a quotient space of X and $V \subseteq X^*$. Then V is open in X^* if and only if $\bigcup V \subseteq X$ is open in X.*

Theorem 1.8.6 *Let $(X \times Y)^* = \{\{x\} \times Y : x \in X\}$ partition the product $X \times Y$. Then the quotient space $(X \times Y)^*$ is homeomorphic to X.*

Definition 1.8.7 A subset $R \subseteq X^2$ is called a **relation** on X, where the notation xRy is equivalent to writing $\langle x, y \rangle \in R$.

A relation $\sim$ on X is called an **equivalence relation** if it satisfies the following for all $x, y, z \in X$.

1. $x \sim x$. (Reflexivity)
2. $x \sim y$ implies $y \sim x$. (Symmetry)
3. $x \sim y$ and $y \sim z$ implies $y \sim z$. (Transitivity)

$\Diamond$

Theorem 1.8.8 *Let X^* be a partition of X and define the relation $\sim$ on X such that $x \sim y$ if and only if $\{x, y\} \subseteq A$ for some $A \in X^*$. Then $\sim$ is an equivalence relation.*

Theorem 1.8.9 *Let $\sim$ be an equivalence relation on X, and let $[x] = \{y \in X : x \sim y\}$. Then $X^* = \{[x] : x \in X\}$ is a partition of X.*

Definition 1.8.10 Let $\sim$ be an equivalence relation on a topological space X. Then $X/\sim$ denotes the quotient space defined by the partition X^* given in Theorem 1.8.9. $\Diamond$

Proposition 1.8.11 *Let R be a relation on X. Then*

$$\sim = \bigcap \Big\{ \sim^\star \subseteq X^2 : R \subseteq \sim^\star \ \ \textit{and} \ \ \sim^\star \ \ \textit{is an equivalence relation on } X \Big\}$$

is an equivalence relation on X.

(Therefore, an equivalence relation may be defined as the minimal equivalence relation satisfying a list of relationships.)

Example 1.8.12 Curves and surfaces defined as quotients. Show that each of the following Euclidean subspaces and quotients of Euclidean subspaces are homeomorphic.

1. $[0,1]/\sim$ where $0 \sim 1$, and $\big\{\langle x,y\rangle \in \mathbb{R}^2 : x^2+y^2 = 1\big\}$.
2. $[0,2]/\sim$ where $0 \sim 1 \sim 2$, and $\big\{\langle x,y\rangle \in \mathbb{R}^2 : (x-1)^2+y^2 = 1\big\} \cup \big\{\langle x,y\rangle \in \mathbb{R}^2 : (x+1)^2+y^2 = 1\big\}$.
3. $[0,1]^2/\sim$ where $\langle 0,y\rangle \sim \langle 1,y\rangle$, and $\big\{\langle x,y\rangle \in \mathbb{R}^2 : 1 \leq x^2+y^2 \leq 2\big\}$.
4. $[0,1]^2/\sim$ where $\langle x,y\rangle \sim \langle z,w\rangle$ whenever at least one of x,y and at least one of z,w is in $\{0,1\}$, and $\big\{\langle x,y,z\rangle \in \mathbb{R}^2 : 1 \leq x^2+y^2+z^2 = 1\big\}$.

□

Definition 1.8.13 The **hypersphere** of dimension n is the quotient space $S^n = [0,1]^n/\sim$ given by $\vec{x} \sim \vec{y}$ whenever there exist $i,j \in \{0,\ldots,n\}$ such that $\vec{x}(i), \vec{y}(j) \in \{0,1\}$. ◊

Definition 1.8.14 The **Möbius strip** is the quotient space $M = [0,1]^2/\sim$ given by $\langle 0,y\rangle \sim \langle 1,1-y\rangle$. ◊

Definition 1.8.15 The **torus** is the quotient space $T = [0,1]^2/\sim$ given by $\langle 0,y\rangle \sim \langle 1,y\rangle$ and $\langle x,0\rangle \sim \langle x,1\rangle$. ◊

Definition 1.8.16 The **Klein bottle** is the quotient space $K = [0,1]^2/\sim$ given by $\langle 0,y\rangle \sim \langle 1,1-y\rangle$ and $\langle x,0\rangle \sim \langle x,1\rangle$. ◊

Appendix A

Assumptions

Here is a brief overview of basic results and definitions concerning sets and the reals that should be assumed for this course.

Definition A.0.1

- $\mathbb{R}$ is the set of real numbers.
- $\mathbb{Z}$ is the set of integers.
- $\mathbb{N} = \{z \in \mathbb{Z} : z \geq 0\} = \{0, 1, 2, \ldots\}$ is the set of natural numbers, which includes zero.
- $\mathbb{Q} = \left\{\frac{z}{n+1} : z \in \mathbb{Z}, n \in \mathbb{N}\right\}$ is the set of rational numbers.

$\Diamond$

Theorem A.0.2 ***De Morgan's Laws****: Let $\mathcal{A}$ be a collection of subsets of X.*

$$X \setminus \bigcup_{A \in \mathcal{A}} A = \bigcap_{A \in \mathcal{A}} (X \setminus A)$$

$$X \setminus \bigcap_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} (X \setminus A)$$

Theorem A.0.3 *The* ***Archemedian Property*** *of the real numbers guarantees that for each positive real number $x > 0$, there exists a natural number $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.*

Theorem A.0.4 *Let $S \subseteq \mathbb{R}$ be a set of real numbers with a lower bound. Then there exists a* ***greatest lower bound*** *(a.k.a.* ***infimum****)* $\operatorname{glb} S = \inf S$.

Let $S \subseteq \mathbb{R}$ be a set of real numbers with a lower bound. Then there exists a ***least upper bound*** *(a.k.a.* ***supremum****)* $\operatorname{lub} S = \sup S$.