Linear Algebra for Team-Based Inquiry Learning

Spring 2024 Math 0520 Edition

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Website: Linear Algebra for Team-Based Inquiry Learning[1]

[1] linear.tbil.org

[2] github.com/TeamBasedInquiryLearning/linear-algebra/blob/main/LICENSE.md

Contents

Chapter 1: Systems of Linear Equations (LE)

1.1 Linear Systems, Vector Equations, and Augmented Matrices (LE1)

Activity 1.1.10 All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is the empty set.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity 1.1.11 Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

(a) Find three different solutions for this system.

(b) Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to write the solution set $\left\{ \left[\begin{array}{c} ? \\ a \end{array}\right] \middle| a \in \mathbb{R} \right\}$ for the linear system.

Activity 1.1.12 Consider the following linear system.

$$\begin{alignedat}{5} x_1 &+ 2x_2 & & & &- x_4 &&= && 3 \\ & & & x_3 &&+ 4x_4 &&= &&-2 \end{alignedat}$$

Describe the solution set

$$\left\{ \left[\begin{array}{c} ? \\ a \\ ? \\ b \end{array}\right] \middle| a, b \in \mathbb{R} \right\}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

1.2 Row Reduction of Matrices (LE2)

Activity 1.2.2 Consider whether these matrix manipulations (A) *must keep* or (B) *could change* the solution set for the corresponding linear system.

(a) Swapping two rows, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 5 & 6 \\ 1 & 2 & 3 \end{array}\right] \qquad \begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned} \qquad \begin{aligned} 4x+5y&=6 \\ x+2y&=3 \end{aligned}$$

(b) Swapping two columns, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 5 & 4 & 6 \end{array}\right] \qquad \begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned} \qquad \begin{aligned} 2x+y&=6 \\ 5x+4y&=3 \end{aligned}$$

(c) Add a constant to every term of a row, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1+6 & 2+6 & 3+6 \\ 4 & 5 & 6 \end{array}\right] \qquad \begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned} \qquad \begin{aligned} 7x+8y&=9 \\ 4x+5y&=3 \end{aligned}$$

(d) Multiply a row by a nonzero constant, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 6 & 9 \\ 4 & 5 & 6 \end{array}\right] \qquad \begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned} \qquad \begin{aligned} 3x+6y&=9 \\ 4x+5y&=3 \end{aligned}$$

(e) Add a constant multiple of one row to another row, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4+3 & 5+6 & 6+9 \end{array}\right] \qquad \begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned} \qquad \begin{aligned} ?\,x+\,?\,y&=\,? \\ ?\,x+\,?\,y&=\,? \end{aligned}$$

(f) Replace a column with zeros, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 4 & 0 & 6 \end{array}\right] \qquad \begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned} \qquad \begin{aligned} ?\,x+\,?\,y&=\,? \\ ?\,x+\,?\,y&=\,? \end{aligned}$$

(g) Replace a row with zeros, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x+2y&=3 \\ 4x+5y&=6 \end{aligned} \qquad \begin{aligned} ?\,x+\,?\,y&=\,? \\ ?\,x+\,?\,y&=\,? \end{aligned}$$

Row Reduction of Matrices (LE2)

Activity 1.2.4 Each of the following linear systems has the same solution set.

A)

$$\begin{aligned} x+2y+z&=3\\ -x-y+z&=1\\ 2x+5y+3z&=7 \end{aligned}$$

B)

$$\begin{aligned} 2x+5y+3z&=7\\ -x-y+z&=1\\ x+2y+z&=3 \end{aligned}$$

C)

$$\begin{aligned} x-z&=1\\ y+2z&=4\\ y+z&=1 \end{aligned}$$

D)

$$\begin{aligned} x+2y+z&=3\\ y+2z&=4\\ 2x+5y+3z&=7 \end{aligned}$$

E)

$$\begin{aligned} x-z&=1\\ y+z&=1\\ z&=3 \end{aligned}$$

F)

$$\begin{aligned} x+2y+z&=3\\ y+2z&=4\\ y+z&=1 \end{aligned}$$

Sort these six equivalent linear systems from most complicated to simplest (in your opinion).

Activity 1.2.5 Here we've written the sorted linear systems from Activity 1.2.4 as augmented matrices.

$$\left[\begin{array}{ccc|c} 2 & 5 & 3 & 7\\ -1 & -1 & 1 & 1\\ 1 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3\\ -1 & -1 & 1 & 1\\ 2 & 5 & 3 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3\\ 0 & 1 & 2 & 4\\ 2 & 5 & 3 & 7 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3\\ 0 & \boxed{1} & 2 & 4\\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 1\\ 0 & \boxed{1} & 2 & 4\\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 1\\ 0 & \boxed{1} & 1 & 1\\ 0 & 0 & -1 & -3 \end{array}\right]$$

Assign the following row operations to each step used to manipulate each matrix to the next:

$R_3 - 1R_2 \to R_3$ $\qquad$ $R_2 + 1R_1 \to R_2$ $\qquad$ $R_1 \leftrightarrow R_3$

$R_3 - 2R_1 \to R_3$ $\qquad$ $R_1 - 2R_3 \to R_1$

Activity 1.2.7 Recall that a matrix is in **reduced row echelon form** (**RREF**) if

1. The leftmost nonzero term of each row is 1. We call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term that is either above or below a pivot is 0.
4. All zero rows (rows whose terms are all 0) are at the bottom of the matrix.

For each matrix, mark the leading terms, and label it as RREF or not RREF. For the ones not in RREF, determine which rule is violated and how it might be fixed.

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 \end{array}\right] \qquad C = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0\\ 1 & 2 & 0 & 3\\ 0 & 0 & 1 & -1 \end{array}\right]$$

Row Reduction of Matrices (LE2)

Activity 1.2.8 Recall that a matrix is in **reduced row echelon form** (**RREF**) if

1. The leftmost nonzero term of each row is 1. We call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term that is either above or below a pivot is 0.
4. All zero rows (rows whose terms are all 0) are at the bottom of the matrix.

For each matrix, mark the leading terms, and label it as RREF or not RREF. For the ones not in RREF, determine which rule is violated and how it might be fixed.

$$D = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad E = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad F = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Activity 1.2.10 Consider the matrix

$$\begin{bmatrix} 2 & 6 & -1 & 6 \\ 1 & 3 & -1 & 2 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the first move in converting to RREF?

A. Add row 3 to row 2 ($R_2 + R_3 \to R_2$)

B. Add row 2 to row 3 ($R_3 + R_2 \to R_3$)

C. Swap row 1 to row 2 ($R_1 \leftrightarrow R_2$)

D. Add -2 row 2 to row 1 ($R_1 - 2R_2 \to R_1$)

Activity 1.2.11 Consider the matrix

$$\begin{bmatrix} \boxed{1} & 3 & -1 & 2 \\ 2 & 6 & -1 & 6 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

A. Add row 1 to row 3 ($R_3 + R_1 \to R_3$)

B. Add -2 row 1 to row 2 ($R_2 - 2R_1 \to R_2$)

C. Add 2 row 2 to row 3 ($R_3 + 2R_2 \to R_3$)

D. Add 2 row 3 to row 2 ($R_2 + 2R_3 \to R_2$)

Row Reduction of Matrices (LE2)

Activity 1.2.12 Consider the matrix

$$\begin{bmatrix} \boxed{1} & 3 & -1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

A. Add row 1 to row 2 ($R_2 + R_1 \to R_2$)

B. Add -1 row 3 to row 2 ($R_2 - R_3 \to R_2$)

C. Add -1 row 2 to row 3 ($R_3 - R_2 \to R_3$)

D. Add row 2 to row 1 ($R_1 + R_2 \to R_1$)

Activity 1.2.14 Complete the following RREF calculation (multiple row operations may be needed for certain steps):

$$A = \begin{bmatrix} 2 & 3 & 2 & 3 \\ -2 & 1 & 6 & 1 \\ -1 & -3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ -2 & 1 & 6 & 1 \\ -1 & -3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ 0 & \boxed{1} & ? & ? \\ 0 & ? & ? & ? \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & ? & ? \\ 0 & \boxed{1} & ? & ? \\ 0 & 0 & ? & ? \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 0 & -2 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Activity 1.2.15 Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{bmatrix}.$$

Compute RREF(A).

Activity 1.2.16 Consider the non-augmented and augmented matrices

$$A = \begin{bmatrix} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{bmatrix} \qquad B = \left[\begin{array}{ccc|c} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{array}\right].$$

Can RREF(A) be used to find RREF(B)?

A. Yes, RREF(A) and RREF(B) are exactly the same.

B. Yes, RREF(A) may be slightly modified to find RREF(B).

C. No, a new calculuation is required.

Row Reduction of Matrices (LE2)

Activity 1.2.17 Free browser-based technologies for mathematical computation are available online.

- Go to `https://sagecell.sagemath.org/`.
- In the dropdown on the right, you can select a number of different languages. Select "Octave" for the Matlab-compatible syntax used by this text.
- Type `rref([1,3,2;2,5,7])` and then press the Evaluate button to compute the RREF of $\left[\begin{array}{ccc} 1 & 3 & 2 \\ 2 & 5 & 7 \end{array}\right]$.

Activity 1.2.18 In the HTML version of this text, code cells are often embedded for your convenience when RREFs need to be computed.

Try this out to compute RREF $\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 3 & 0 & 6 \end{array}\right]$.

1.3 Counting Solutions for Linear Systems (LE3)

Activity 1.3.2 Consider the following system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11. \end{aligned}$$

(a) Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF}\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

(b) Use the RREF matrix to write a linear system equivalent to the original system.

(c) How many solutions must this system have?

A. Zero B. Only one C. Infinitely-many

Activity 1.3.3 Consider the vector equation

$$x_1 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 13 \\ 10 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

(a) Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF}\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

(b) Use the RREF matrix to write a linear system equivalent to the original system.

(c) How many solutions must this system have?

A. Zero B. Only one C. Infinitely-many

Activity 1.3.4 What contradictory equations besides $0 = 1$ may be obtained from the RREF of an augmented matrix?

A. $x = 0$ is an obtainable contradiction

B. $x = y$ is an obtainable contradiction

C. $0 = 17$ is an obtainable contradiction

D. $0 = 1$ is the only obtainable contradiction

Counting Solutions for Linear Systems (LE3)

Activity 1.3.5 Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

(a) Find its corresponding augmented matrix A and find $\mathrm{RREF}(A)$.

(b) Use the RREF matrix to write a linear system equivalent to the original system.

(c) How many solutions must this system have?

A. Zero B. One C. Infinitely-many

Activity 1.3.8 For each vector equation, write an explanation for whether each solution set has no solutions, one solution, or infinitely-many solutions. If the set is finite, describe it using set notation.

(a)

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -6 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \\ 4 \end{bmatrix}$$

(b)

$$x_1 \begin{bmatrix} -2 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 13 \end{bmatrix}$$

(c)

$$x_1 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ -5 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} -7 \\ -9 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

1.4 Linear Systems with Infinitely-Many Solutions (LE4)

Activity 1.4.1 Consider this simplified linear system found to be equivalent to the system from Activity 1.3.5:

$$\begin{aligned} x_1 + 2x_2 \quad &= 4 \\ x_3 &= -1 \end{aligned}$$

Earlier, we determined this system has infinitely-many solutions.

(a) Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$.

(b) Let $x_2 = b$ and write the solution set in the form $\left\{ \begin{bmatrix} ? \\ b \\ ? \end{bmatrix} \middle| \, b \in \mathbb{R} \right\}$.

(c) Which of these was easier? What features of the RREF matrix $\left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right]$ caused this?

Activity 1.4.3 Find the solution set for the system

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 - \ x_5 &= \ \ 3 \\ -x_1 + \ x_2 + 3x_3 - x_4 + 2x_5 &= -3 \\ x_1 - 2x_2 - \ x_3 + x_4 + \ x_5 &= \ \ 2 \end{aligned}$$

by doing the following.

(a) Row-reduce its augmented matrix.

(b) Assign letters to the free variables (given by the non-pivot columns):

$$? = a$$

$$? = b$$

(c) Solve for the bound variables (given by the pivot columns) to show that

$$? = 1 + 5a + 2b$$

$$? = 1 + 2a + 3b$$

$$? = 3 + 3b$$

(d) Replace x_1 through x_5 with the appropriate expressions of a, b in the following set-builder notation.

$$\left\{ \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array}\right] \middle| \, a, b \in \mathbb{R} \right\}$$

Activity 1.4.5 Consider the following system of linear equations.

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 5 \\ -5 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 13 \\ -13 \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \\ -12 \end{bmatrix}.$$

(a) Explain how to find a simpler system or vector equation that has the same solution set.

(b) Explain how to describe this solution set using set notation.

Activity 1.4.6 Consider the following system of linear equations.

$$\begin{array}{rcrcrcr} x_1 & & & - & 2\,x_3 & = & -3 \\ 5\,x_1 & + & x_2 & - & 7\,x_3 & = & -18 \\ 5\,x_1 & - & x_2 & - & 13\,x_3 & = & -12 \\ x_1 & + & 3\,x_2 & + & 7\,x_3 & = & -12 \end{array}$$

(a) Explain how to find a simpler system or vector equation that has the same solution set.

(b) Explain how to describe this solution set using set notation.

Chapter 2: Euclidean Vectors (EV)

2.1 Linear Combinations (EV1)

Activity 2.1.4 Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

(a) Sketch the four Euclidean vectors

$$1\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix},\quad 3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix},\quad 0\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\quad -2\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-2\\-4\end{bmatrix}$$

in the xy plane by placing a dot at the (x, y) coordinate associated with each vector.

(b) Sketch a representation of all the vectors belonging to

$$\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\2\end{bmatrix}\middle|\, a\in\mathbb{R}\right\}$$

in the xy plane by plotting their (x, y) coordinates as dots. What best describes this sketch?

A. A line B. A plane C. A parabola D. A circle

Activity 2.1.6 Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

(a) Sketch the following five Euclidean vectors in the xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix}=?\qquad 0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=?\qquad 1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=?$$

$$-2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=?\qquad -1\begin{bmatrix}1\\2\end{bmatrix}+-2\begin{bmatrix}-1\\1\end{bmatrix}=?$$

(b) Sketch a representation of all the vectors belonging to

$$\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\2\end{bmatrix}+b\begin{bmatrix}-1\\1\end{bmatrix}\middle|\, a,b\in\mathbb{R}\right\}$$

in the xy plane. What best describes this sketch?

A. A line B. A plane C. A parabola D. A circle

Activity 2.1.7 Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}6\\-4\end{bmatrix},\begin{bmatrix}-3\\2\end{bmatrix}\right\}$ in the xy plane. What best describes this sketch?

A. A line

B. A plane

C. A parabola

D. A cube

Activity 2.1.8 Consider the following questions to discover whether a Euclidean vector belongs to a span.

(a) The Euclidean vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when there exists a solution to which of these vector equations?

A. $x_1 \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$

B. $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$

C. $x_1 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = 0$

(b) Use technology to find RREF of the corresponding augmented matrix, and then use that matrix to find the solution set of the vector equation.

(c) Given this solution set, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

Activity 2.1.10 Consider this claim about a vector equation:
$\begin{bmatrix} -6 \\ 2 \\ -6 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}$.

(a) Write a statement involving the solutions of a vector equation that's equivalent to this claim.

(b) Explain why the statement you wrote is true.

(c) Since your statement was true, use the solution set to describe a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}$ that equals $\begin{bmatrix} -6 \\ 2 \\ -6 \end{bmatrix}$.

Activity 2.1.11 Consider this claim about a vector equation:
$\begin{bmatrix} -5 \\ -1 \\ -7 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix} \right\}$.

Linear Combinations (EV1)

(a) Write a statement involving the solutions of a vector equation that's equivalent to this claim.

(b) Explain why the statement you wrote is false, to conclude that the vector does not belong to the span.

2.2 Spanning Sets (EV2)

Activity 2.2.2 How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the xy plane to support your answer.

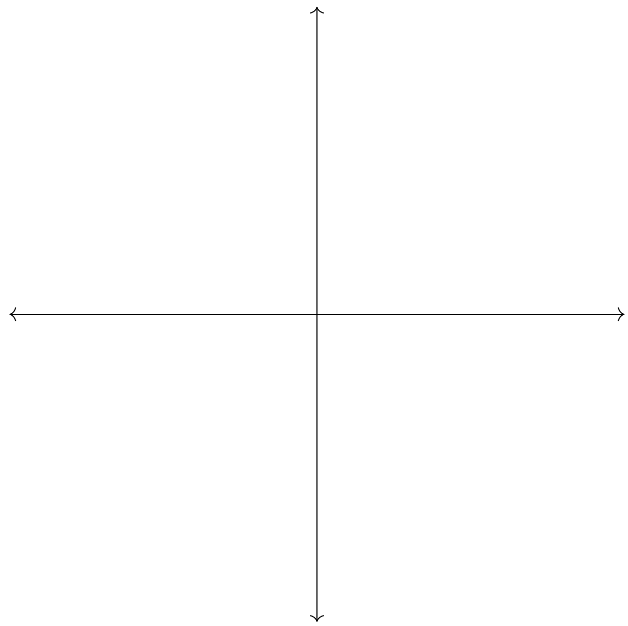


Figure 7 The xy plane $\mathbb{R}^2$

A. 1

B. 2

C. 3

D. 4

E. Infinitely Many

Activity 2.2.3 How many vectors are required to span $\mathbb{R}^3$?

Spanning Sets (EV2)

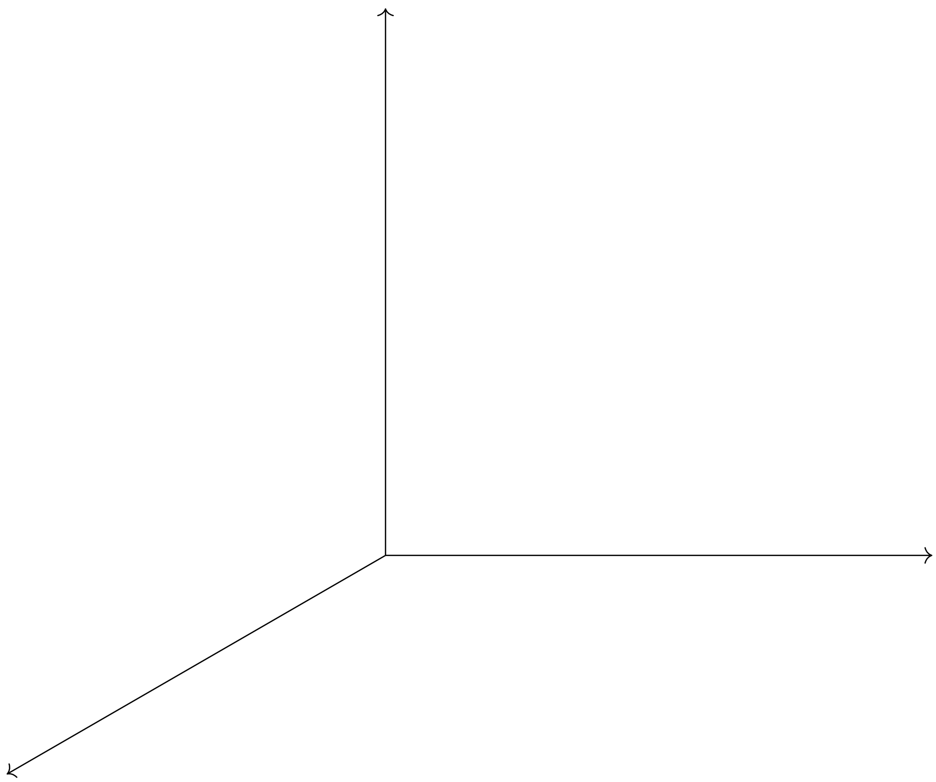


Figure 8 $\mathbb{R}^3$ space

A. 1

B. 2

C. 3

D. 4

E. Infinitely Many

Activity 2.2.5 Consider the question: Does every vector in $\mathbb{R}^3$ belong to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$?

(a) Determine if $\begin{bmatrix}7\\-3\\-2\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$.

(b) Determine if $\begin{bmatrix}2\\5\\7\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$.

(c) An arbitrary vector $\begin{bmatrix}?\\?\\?\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$ provided the equation

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

has...

A. no solutions.

B. exactly one solution.

C. at least one solution.

D. infinitely-many solutions.

(d) We're guaranteed at least one solution if the RREF of the corresponding augmented matrix has no contradictions; likewise, we have no solutions if the RREF corresponds to the contradiction $0 = 1$. Given

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & ? \\ -1 & 0 & -2 & ? \\ 0 & 1 & 2 & ? \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & ? \\ 0 & 1 & 2 & ? \\ 0 & 0 & 0 & ? \end{array}\right]$$

we may conclude that the set does not span all of $\mathbb{R}^3$ because...

A. the row $[0\,1\,2\,|\,?]$ prevents a contradiction.

B. the row $[0\,1\,2\,|\,?]$ allows a contradiction.

C. the row $[0\,0\,0\,|\,?]$ prevents a contradiction.

D. the row $[0\,0\,0\,|\,?]$ allows a contradiction.

Activity 2.2.7 Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$ and the question "Does $\mathbb{R}^4 = \operatorname{span} S$?"

(a) Rewrite this question in terms of the solutions to a vector equation.

(b) Answer your new question, and use this to answer the original question.

Activity 2.2.8 Let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^7$ be three Euclidean vectors, and suppose $\vec{w}$ is another vector with $\vec{w} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. What can you conclude about $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

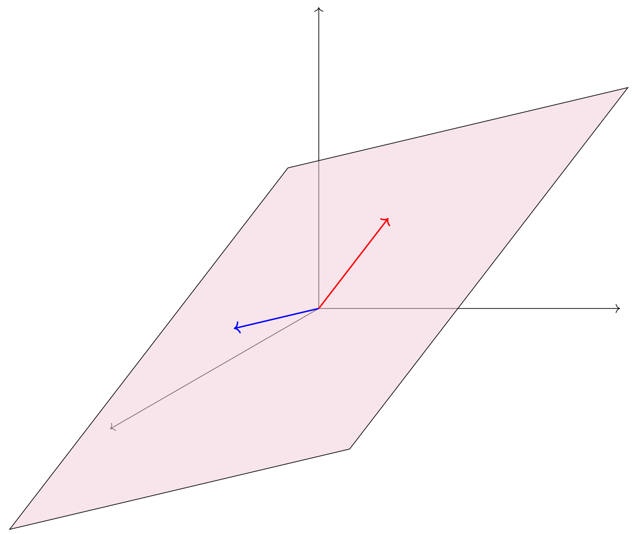
A. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is larger than $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

B. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is the same as $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

C. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is smaller than $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

2.3 Subspaces (EV3)

Activity 2.3.2 Consider two non-colinear vectors in $\mathbb{R}^3$. If we look at all linear combinations of those two vectors (that is, their span), we end up with a planar subspace within $\mathbb{R}^3$. Call this plane S.



(a) For any unspecified $\vec{u}, \vec{v} \in S$, is it the case that $\vec{u} + \vec{v} \in S$?

A. Yes.

B. No.

(b) For any unspecified $\vec{u} \in S$ and $c \in \mathbb{R}$, is it the case that $\vec{u} + \begin{bmatrix} c \\ c \\ c \end{bmatrix} \in S$?

A. Yes.

B. No.

(c) For any unspecified $\vec{u} \in S$ and $c \in \mathbb{R}$, is it the case that $c\vec{u} \in S$?

A. Yes.

B. No.

Activity 2.3.5 Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + 2y + z = 0 \right\}$.

(a) Let's assume that $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ are in S. What are we allowed to assume?

A. $x + 2y + z = 0$.

B. $a + 2b + c = 0$.

C. Both of these.

D. Neither of these.

(b) Which equation must be verified to show that $\vec{v}+\vec{w}=\begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$ also belongs to S?

A. $(x+a)+2(y+b)+(z+c)=0$.

B. $x+a+2y+b+z+c=0$.

C. $x+2y+z=a+2b+c$.

(c) Use the assumptions from (a) to verify the equation from (b).

(d) Is S is a subspace of $\mathbb{R}^3$?

A. Yes

B. No

C. Not enough information

(e) Show that $k\vec{v}=\begin{bmatrix} kx \\ ky \\ kz \end{bmatrix}$ also belongs to S for any $k\in\mathbb{R}$ by verifying $(kx)+2(ky)+(kz)=0$ under these assumptions.

(f) Is S is a subspace of $\mathbb{R}^3$?

A. Yes

B. No

C. Not enough information

Activity 2.3.6 Let $S=\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, x+2y+z=4\right\}$.

(a) Which of these statements is valid?

A. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\in S$, *and* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}\in S$, so S *is* a subspace.

B. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\in S$, *and* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}\in S$, so S *is not* a subspace.

C. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\in S$, *but* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}\notin S$, so S *is* a subspace.

D. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\in S$, *but* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}\notin S$, so S *is not* a subspace.

(b) Which of these statements is valid?

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in S$, *and* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S$, so S *is* a subspace.

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in S$, *and* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S$, so S *is not* a subspace.

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in S$, *but* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin S$, so S *is* a subspace.

(d) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in S$, *but* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin S$, so S *is not* a subspace.

Activity 2.3.8 Consider these subsets of $\mathbb{R}^3$:

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| y = z + 1 \right\} \qquad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| y = |z| \right\} \qquad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| z = xy \right\}.$$

(a) Show R isn't a subspace by showing that $\vec{0} \notin R$.

(b) Show S isn't a subspace by finding two vectors $\vec{u}, \vec{v} \in S$ such that $\vec{u} + \vec{v} \notin S$.

(c) Show T isn't a subspace by finding a vector $\vec{v} \in T$ such that $2\vec{v} \notin T$.

Activity 2.3.9 Consider the following two sets of Euclidean vectors:

$$U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| 7\,x + 4\,y = 0 \right\} \qquad W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| 3\,xy^2 = 0 \right\}$$

Explain why one of these sets is a subspace of $\mathbb{R}^2$ and one is not.

Activity 2.3.10 Consider the following attempted proof that

$$U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x + y = xy \right\}$$

is closed under scalar multiplication.

Let $\begin{bmatrix} x \\ y \end{bmatrix} \in U$, so we know that $x + y = xy$. We want to show $k \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix} \in U$, that is, $(kx) + (ky) = (kx)(ky)$. This is verified by the following calculation:

$$\begin{aligned} (kx) + (ky) &= (kx)(ky) \\ k(x + y) &= k^2 xy \end{aligned}$$

Subspaces (EV3)

$$0[k(x+y)] = 0[k^2xy]$$
$$0 = 0$$

Is this reasoning valid?

A. Yes

B. No

2.4 Linear Independence (EV4)

Activity 2.4.1 Consider the two sets

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}.$$

Which of the following is true?

A. $\operatorname{span} S$ is bigger than $\operatorname{span} T$.

B. $\operatorname{span} S$ and $\operatorname{span} T$ are the same size.

C. $\operatorname{span} S$ is smaller than $\operatorname{span} T$.

Activity 2.4.3 Consider the following three vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}.$$

(a) Let $\vec{w} = 3\vec{v}_1 - \vec{v}_2 - 5\vec{v}_3 = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$. The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}\}$ is...

A. linearly dependent: at least one vector is a linear combination of others

B. linearly independent: no vector is a linear combination of others

(b) Find

$$\operatorname{RREF} \left[\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{w} \end{array} \right] = \operatorname{RREF} \begin{bmatrix} -2 & 1 & -2 & ? \\ 0 & 3 & 5 & ? \\ 0 & 0 & 4 & ? \end{bmatrix} = ?.$$

What does this tell you about solution set for the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{w} = \vec{0}$?

A. It is inconsistent.

B. It is consistent with one solution.

C. It is consistent with infinitely many solutions.

(c) Which of these might explain the connection?

A. A pivot column establishes linear independence and creates a contradiction.

B. A non-pivot column both describes a linear combination and reveals the number of solutions.

C. A pivot row describes the bound variables and prevents a contradiction.

D. A non-pivot row prevents contradictions and makes the vector equation solvable.

Linear Independence (EV4)

Activity 2.4.5 Find

$$\mathrm{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 1 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly dependent (the part that shows its linear system has infinitely many solutions).

Activity 2.4.7

(a) Write a statement involving the solutions of a vector equation that's equivalent to each claim:

(i) "The set of vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 11 \\ 6 \\ 3 \end{bmatrix} \right\}$ is linearly *independent*."

(ii) "The set of vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 11 \\ 6 \\ 3 \end{bmatrix} \right\}$ is linearly *dependent*."

(b) Explain how to determine which of these statements is true.

Activity 2.4.8 What is the largest number of $\mathbb{R}^4$ vectors that can form a linearly independent set?

A. 3

B. 4

C. 5

D. You can have infinitely many vectors and still be linearly independent.

Activity 2.4.9 Is is possible for the set of Euclidean vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{0}\}$ to be linearly independent?

A. Yes

B. No

2.5 Identifying a Basis (EV5)

Activity 2.5.1 Consider the set of vectors

$$S = \left\{ \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -16 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(a) Express the vector $\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in S, i.e. find scalars such that

$$\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix} = ? \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} + ? \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} + ? \begin{bmatrix} 0 \\ -16 \\ -5 \\ -3 \end{bmatrix} + ? \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + ? \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

(b) Find a *different* way to express the vector $\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in S.

(c) Consider another vector $\begin{bmatrix} 8 \\ 6 \\ 7 \\ 5 \end{bmatrix}$. Without computing the RREF of another matrix, how many ways can this vector be written as a linear combination of the vectors in S?

A. Zero.

B. One.

C. Infinitely-many.

D. Computing a new matrix RREF is necessary.

Activity 2.5.2 Let's review some of the terminology we've been dealing with...

(a) If every vector in a vector space can be constructed as one or more linear combination of vectors in a set S, we can say...

A. the set S spans the vector space.

B. the set S fails to span the vector space.

C. the set S is linearly independent.

D. the set S is linearly dependent.

(b) If the zero vector $\vec{0}$ can be constructed as a *unique* linear combination of vectors in a set S (the combination multiplying every vector by the scalar value 0), we can say...

A. the set S spans the vector space.

B. the set S fails to span the vector space.

C. the set S is linearly independent.

D. the set S is linearly dependent.

(c) If every vector of a vector space can either be constructed as a *unique* linear combination of vectors in a set S, or not at all, we can say...

A. the set S spans the vector space.

B. the set S fails to span the vector space.

C. the set S is linearly independent.

D. the set S is linearly dependent.

Activity 2.5.5 Let S be a basis (Definition 2.5.3) for a vector space. Then...

A. the set S must both span the vector space and be linearly independent.

B. the set S must span the vector space but could be linearly dependent.

C. the set S must be linearly independent but could fail to span the vector space.

D. the set S could fail to span the vector space and could be linearly dependent.

Activity 2.5.6 The vectors

$$\hat{\imath} = (1,0,0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{\jmath} = (0,1,0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat{k} = (0,0,1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ used frequently in multivariable calculus.

Find the unique linear combination of these vectors

$$?\,\hat{\imath} + \,?\,\hat{\jmath} + \,?\,\hat{k}$$

that equals the vector

$$(3,-2,4) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

in xyz space.

Activity 2.5.8 Take the RREF of an appropriate matrix to determine if each of the following sets is a basis for $\mathbb{R}^4$.

Identifying a Basis (EV5)

(a)

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(b)

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(c)

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}3\\13\\7\\16\end{bmatrix},\begin{bmatrix}-1\\10\\7\\14\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(d)

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix},\begin{bmatrix}3\\6\\1\\5\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

(e)

$$\left\{\begin{bmatrix}5\\3\\0\\-1\end{bmatrix},\begin{bmatrix}-2\\1\\0\\3\end{bmatrix},\begin{bmatrix}4\\5\\1\\3\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

Activity 2.5.9 If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$, that means RREF$[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4]$ has a pivot in every row (because it spans), and has a pivot in every column (because it's linearly independent).

What is RREF$[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4]$?

$$\mathrm{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4] = \begin{bmatrix}?&?&?&?\\?&?&?&?\\?&?&?&?\\?&?&?&?\end{bmatrix}$$

2.6 Subspace Basis and Dimension (EV6)

Activity 2.6.2 Consider the subspace of $\mathbb{R}^4$ given by $W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}$.

(a) Mark the column of $\operatorname{RREF}\begin{bmatrix}2 & 2 & 2 & 1\\3 & 0 & -3 & 5\\0 & 1 & 2 & -1\\1 & -1 & -3 & 0\end{bmatrix}$ that shows that W's spanning set is linearly dependent.

(b) What would be the result of removing the vector that gave us this column?

A. The set still spans W, and remains linearly dependent.

B. The set still spans W, but is now also linearly independent.

C. The set no longer spans W, and remains linearly dependent.

D. The set no longer spans W, but is now linearly independent.

Activity 2.6.5

(a) Find a basis for $\operatorname{span} S$ where

$$S = \left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}.$$

(b) Find a basis for $\operatorname{span} T$ where

$$T = \left\{\begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}.$$

Activity 2.6.9 Consider the following subspace W of $\mathbb{R}^4$:

$$W = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}-2\\0\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\1\\-5\\5\end{bmatrix}, \begin{bmatrix}12\\-3\\15\\-18\end{bmatrix}\right\}.$$

(a) Explain and demonstrate how to find a basis of W.

(b) Explain and demonstrate how to find the dimension of W.

Subspace Basis and Dimension (EV6)

Activity 2.6.10 The dimension of a subspace may be found by doing what with an appropriate RREF matrix?

A. Count the rows.

B. Count the non-pivot columns.

C. Count the pivots.

D. Add the number of pivot rows and pivot columns.

2.7 Homogeneous Linear Systems (EV7)

Activity 2.7.2 Consider the homogeneous vector equation $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$.

(a) Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are both solutions, we know that

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0} \text{ and } b_1\vec{v}_1 + \cdots + b_n\vec{v}_n = \vec{0}.$$

Therefore by adding these equations,

$$(a_1 + b_1)\vec{v}_1 + \cdots + (a_n + b_n)\vec{v}_n = \vec{0}$$

shows that $\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$ is also a solution. Thus the solution set of a homogeneous system is...

A. Closed under addition.

B. Not closed under addition.

C. Linearly dependent.

D. Linearly independent.

(b) Similarly, if $c \in \mathbb{R}$, $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is a solution. Thus the solution set of a homogeneous system is also closed under scalar multiplication, and therefore...

A. A basis for $\mathbb{R}^n$.

B. A subspace of $\mathbb{R}^n$.

C. All of $\mathbb{R}^n$.

D. The empty set.

Activity 2.7.3 Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \phantom{{}- x_3} + x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - x_4 &= 0 \end{aligned}$$

(a) Find its solution set (a subspace of $\mathbb{R}^4$).

(b) Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}.$$

(c) Rewrite this solution space in the form

$$\text{span} \left\{ \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \right\}.$$

(d) Which of these choices best describes the set of two vectors $\left\{ \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \right\}$ used in this span?

A. The set is linearly dependent.

B. The set is linearly independent.

C. The set spans all of $\mathbb{R}^4$.

D. The set fails to span the solution space.

Activity 2.7.5 Consider the homogeneous system of equations

$$\begin{aligned} 2x_1 + 4x_2 + 2x_3 - 4x_4 &= 0 \\ -2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - 4x_4 &= 0 \end{aligned}$$

Find a basis for its solution space.

Activity 2.7.6 Consider the homogeneous vector equation

$$x_1 \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -4 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Find a basis for its solution space.

Activity 2.7.7 Consider the homogeneous system of equations

$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= 0 \\ 2x_1 + 6x_2 + 4x_3 &= 0 \\ x_1 + 6x_2 - 4x_3 &= 0 \end{aligned}$$

(a) Find its solution space.

(b) Which of these is the best choice of basis for this solution space?

A $\{\}$

B $\{\vec{0}\}$

C The basis does not exist

Activity 2.7.8 To create a computer-animated film, an animator first models a scene as a subset of $\mathbb{R}^3$. Then to transform this three-dimensional visual data for display on a two-dimensional movie screen or television set, the computer could apply a linear tranformation that maps visual information at the point $(x, y, z) \in \mathbb{R}^3$ onto the pixel located at $(x + y, y - z) \in \mathbb{R}^2$.

(a) What homoegeneous linear system describes the positions (x, y, z) within the original scene that would be aligned with the pixel $(0, 0)$ on the screen?

(b) Solve this system to describe these locations.

Chapter 3: Algebraic Properties of Linear Maps (AT)

3.1 Linear Transformations (AT1)

Activity 3.1.4 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}.$$

(a) Compute the result of adding vectors before a T transformation:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix}\right)$$

A. $\begin{bmatrix} x - u + z - w \\ 3y - 3v \end{bmatrix}$

B. $\begin{bmatrix} x + u - z - w \\ 3y + 3v \end{bmatrix}$

C. $\begin{bmatrix} x + u \\ 3y + 3v \\ z + w \end{bmatrix}$

D. $\begin{bmatrix} x - u \\ 3y - 3v \\ z - w \end{bmatrix}$

(b) Compute the result of adding vectors after a T transformation:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix}$$

A. $\begin{bmatrix} x - u + z - w \\ 3y - 3v \end{bmatrix}$

B. $\begin{bmatrix} x + u - z - w \\ 3y + 3v \end{bmatrix}$

C. $\begin{bmatrix} x + u \\ 3y + 3v \\ z + w \end{bmatrix}$

D. $\begin{bmatrix} x - u \\ 3y - 3v \\ z - w \end{bmatrix}$

(c) Is T a linear transformation?

A. Yes.

B. No.

C. More work is necessary to know.

(d) Compute the result of scalar multiplcation before a T transformation:

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right)$$

A. $\left[\begin{array}{c} cx - cz \\ 3cy \end{array}\right]$

B. $\left[\begin{array}{c} cx + cz \\ -3cy \end{array}\right]$

C. $\left[\begin{array}{c} x + c \\ 3y + c \\ z + c \end{array}\right]$

D. $\left[\begin{array}{c} x - c \\ 3y - c \\ z - c \end{array}\right]$

(e) Compute the result of scalar multiplcation after a T transformation:

$$cT\left(\left[\begin{array}{c} x \\ y \\ z \end{array}\right]\right) = c\left[\begin{array}{c} x - z \\ 3y \end{array}\right]$$

A. $\left[\begin{array}{c} cx - cz \\ 3cy \end{array}\right]$

B. $\left[\begin{array}{c} cx + cz \\ -3cy \end{array}\right]$

C. $\left[\begin{array}{c} x + c \\ 3y + c \\ z + c \end{array}\right]$

D. $\left[\begin{array}{c} x - c \\ 3y - c \\ z - c \end{array}\right]$

(f) Is T a linear transformation?

A. Yes.

B. No.

C. More work is necessary to know.

Activity 3.1.5 Let $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$S\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left[\begin{array}{c} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{array}\right]$$

(a) Compute

$$S\left(\left[\begin{array}{c} 0 \\ 1 \end{array}\right] + \left[\begin{array}{c} 2 \\ 3 \end{array}\right]\right) = S\left(\left[\begin{array}{c} 2 \\ 4 \end{array}\right]\right)$$

A. $\left[\begin{array}{c} 6 \\ 4 \\ 7 \\ 0 \end{array}\right]$ B. $\left[\begin{array}{c} -3 \\ 0 \\ 1 \\ 5 \end{array}\right]$ C. $\left[\begin{array}{c} -3 \\ -1 \\ 7 \\ 5 \end{array}\right]$ D. $\left[\begin{array}{c} 6 \\ 4 \\ 10 \\ -1 \end{array}\right]$

(b) Compute

$$S\left(\left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right) + S\left(\left[\begin{array}{c} 2 \\ 3 \end{array}\right]\right) = \left[\begin{array}{c} 0 + 1 \\ 0^2 \\ 1 + 3 \\ 1 - 2^0 \end{array}\right] + \left[\begin{array}{c} 2 + 3 \\ 2^2 \\ 3 + 3 \\ 3 - 2^2 \end{array}\right]$$

A. $\begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} -3 \\ 0 \\ 1 \\ 5 \end{bmatrix}$ C. $\begin{bmatrix} -3 \\ -1 \\ 7 \\ 5 \end{bmatrix}$ D. $\begin{bmatrix} 6 \\ 4 \\ 10 \\ -1 \end{bmatrix}$

(c) Is T a linear transformation?

A. Yes.

B. No.

C. More work is necessary to know.

Activity 3.1.6 Fill in the $?$s, assuming $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear:

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = T\left(?\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = ?T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

Activity 3.1.8

(a) Consider the following maps of Euclidean vectors $P : \mathbb{R}^3 \to \mathbb{R}^3$ and $Q : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$P\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -2\,x - 3\,y - 3\,z \\ 3\,x + 4\,y + 4\,z \\ 3\,x + 4\,y + 5\,z \end{bmatrix} \quad \text{and} \quad Q\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 4\,y + 9\,z \\ y - 2\,z \\ 8\,y^2 - 3\,xz \end{bmatrix}.$$

Which do you *suspect*?

A. P is linear, but Q is not.

B. Q is linear, but P is not.

C. Both maps are linear.

D. Neither map is linear.

(b) Consider the following map of Euclidean vectors $S : \mathbb{R}^2 \to \mathbb{R}^2$

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2\,y \\ 9\,xy \end{bmatrix}.$$

Prove that S *is not* a linear transformation.

(c) Consider the following map of Euclidean vectors $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 8\,x - 6\,y \\ 6\,x - 4\,y \end{bmatrix}.$$

Prove that T *is* a linear transformation.

3.2 Standard Matrices (AT2)

Activity 3.2.2 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What is $T\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right)$?

A. $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

B. $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

C. $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

D. $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

Activity 3.2.3 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What is $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$?

A. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

C. $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

D. $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

Activity 3.2.4 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What is $T\left(\begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix}\right)$?

A. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

C. $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

D. $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

Activity 3.2.5 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. What piece of information would help you compute

$T\left(\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}\right)$?

A. The value of $T\left(\begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}\right)$.

B. The value of $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$.

C. The value of $T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

D. Any of the above.

Activity 3.2.8 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T(\vec{e}_1) = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \qquad T(\vec{e}_2) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \qquad T(\vec{e}_3) = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \qquad T(\vec{e}_4) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Write the standard matrix $[T(\vec{e}_1) \cdots T(\vec{e}_n)]$ for T.

Activity 3.2.9 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

(a) Compute $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.

(b) Find the standard matrix for T.

Activity 3.2.11 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.

(b) Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Activity 3.2.12 Compute the following linear transformations of vectors given their standard matrices.

Standard Matrices (AT2)

(a)

$$T_1\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \text{ for the standard matrix } A_1 = \begin{bmatrix} 4 & 3 \\ 0 & -1 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$$

(b)

$$T_2\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix}\right) \text{ for the standard matrix } A_2 = \begin{bmatrix} 4 & 3 & 0 & -1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

(c)

$$T_3\left(\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}\right) \text{ for the standard matrix } A_3 = \begin{bmatrix} 4 & 3 & 0 \\ 0 & -1 & 3 \\ 5 & 1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

3.3 Image and Kernel (AT3)

Activity 3.3.1 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes the set of all vectors that transform into $\vec{0}$?

A. $\left\{\begin{bmatrix} a \\ a \end{bmatrix} \middle| a \in \mathbb{R}\right\}$

B. $\left\{\begin{bmatrix} a \\ 0 \end{bmatrix} \middle| a \in \mathbb{R}\right\}$

C. $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

D. $\left\{\begin{bmatrix} a \\ b \end{bmatrix} \middle| a, b \in \mathbb{R}\right\}$

Activity 3.3.3 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\ker T$, the set of all vectors that transform into $\vec{0}$?

A. $\left\{\begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| a \in \mathbb{R}\right\}$

B. $\left\{\begin{bmatrix} a \\ a \\ 0 \end{bmatrix} \middle| a \in \mathbb{R}\right\}$

C. $\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

D. $\left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| a, b, c \in \mathbb{R}\right\}$

Activity 3.3.4 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x + 4y - z \\ x + 2y + z \end{bmatrix}$$

(a) Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

(b) Use RREF(A) to solve this homogeneous system of equations and find a basis for the kernel of T.

Image and Kernel (AT3)

Activity 3.3.5 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 2x + 4y + 2z - 4w \\ -2x - 4y + z + w \\ 3x + 6y - z - 4w \end{bmatrix}.$$

Find a basis for the kernel of T.

Activity 3.3.6 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes the set of all vectors that are the result of using T to transform $\mathbb{R}^2$ vectors?

A. $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

D. $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$

Activity 3.3.8 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^3$ vectors?

A. $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

D. $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

Activity 3.3.9 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) & T(\vec{e}_4) \end{bmatrix}.$$

Consider the question: Which vectors $\vec{w}$ in $\mathbb{R}^3$ belong to $\operatorname{Im} T$?

Image and Kernel (AT3)

(a) Determine if $\begin{bmatrix} 12 \\ 3 \\ 3 \end{bmatrix}$ belongs to $\operatorname{Im} T$.

(b) Determine if $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ belongs to $\operatorname{Im} T$.

(c) An arbitrary vector $\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ belongs to $\operatorname{Im} T$ provided the equation

$$x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + x_3 T(\vec{e}_3) + x_4 T(\vec{e}_4) = \vec{w}$$

has...

A. no solutions.

B. exactly one solution.

C. at least one solution.

D. infinitely-many solutions.

(d) Based on this, how do $\operatorname{Im} T$ and $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ relate to each other?

A. The set $\operatorname{Im} T$ contains $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ but is not equal to it.

B. The set $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ contains $\operatorname{Im} T$ but is not equal to it.

C. The set $\operatorname{Im} T$ and $\operatorname{span}\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4)\}$ are equal to each other.

D. There is no relation between these two sets.

Activity 3.3.12 Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of T.

Activity 3.3.13 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Which of the following is equal to the dimension of the kernel of T?

A. The number of pivot columns

B. The number of non-pivot columns

C. The number of pivot rows

D. The number of non-pivot rows

Image and Kernel (AT3)

Activity 3.3.14 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Which of the following is equal to the dimension of the image of T?

A. The number of pivot columns

B. The number of non-pivot columns

C. The number of pivot rows

D. The number of non-pivot rows

Activity 3.3.16 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right]\right) = \left[\begin{array}{c} x - y + 5\,z + 3\,w \\ -x - 4\,z - 2\,w \\ y - 2\,z - w \end{array}\right].$$

(a) Explain and demonstrate how to find the image of T and a basis for that image.

(b) Explain and demonstrate how to find the kernel of T and a basis for that kernel.

(c) Explain and demonstrate how to find the rank and nullity of T, and why the rank-nullity theorem holds for T.

3.4 Injective and Surjective Linear Maps (AT4)

Activity 3.4.2 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T injective?

A. Yes, because $T(\vec{v}) = T(\vec{w})$ whenever $\vec{v} = \vec{w}$.

B. Yes, because $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.

C. No, because $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$.

D. No, because $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$.

Activity 3.4.3 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T injective?

A. Yes, because $T(\vec{v}) = T(\vec{w})$ whenever $\vec{v} = \vec{w}$.

B. Yes, because $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.

C. No, because $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$.

D. No, because $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$.

Activity 3.4.5 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T surjective?

A. Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$, there exists $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ such that $T(\vec{v}) = \vec{w}$.

B. No, because $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

C. No, because $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Activity 3.4.6 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T surjective?

A. Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{v} = \begin{bmatrix} x \\ y \\ 42 \end{bmatrix} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{w}$.

B. Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{w}$.

C. No, because $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Activity 3.4.7 Let $T : V \to W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

A. T is injective

B. T is not injective

C. T is surjective

D. T is not surjective

Activity 3.4.9 Let $T : V \to \mathbb{R}^3$ be a linear transformation where $\operatorname{Im} T$ may be spanned by only two vectors. What can you conclude?

A. T is injective

B. T is not injective

C. T is surjective

D. T is not surjective

Injective and Surjective Linear Maps (AT4)

Activity 3.4.12 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. The kernel of T is trivial, i.e. $\ker T = \{\vec{0}\}$.
2. The image of T equals its codomain, i.e. $\operatorname{Im} T = \mathbb{R}^m$.
3. For every $\vec{w} \in \mathbb{R}^m$, the set $\{\vec{v} \in \mathbb{R}^n | T(\vec{v}) = \vec{w}\}$ contains exactly one vector.

Activity 3.4.13 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. The columns of A span $\mathbb{R}^m$.
2. The columns of A form a basis for $\mathbb{R}^m$.
3. The columns of A are linearly independent.

Activity 3.4.14 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. $\mathrm{RREF}(A)$ is the identity matrix.
2. Every column of $\mathrm{RREF}(A)$ has a pivot.
3. Every row of $\mathrm{RREF}(A)$ has a pivot.

Activity 3.4.15 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Determine whether each of the following statements means T is (A) *injective*, (B) *surjective*, or (C) *bijective* (both).

1. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has a solution for all $\vec{b} \in \mathbb{R}^m$.
2. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has exactly one solution for all $\vec{b} \in \mathbb{R}^m$.
3. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$ has exactly one solution.

Activity 3.4.17 What can you conclude about the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$?

A. Its standard matrix has more columns than rows, so T is not injective.

B. Its standard matrix has more columns than rows, so T is injective.

C. Its standard matrix has more rows than columns, so T is not surjective.

D. Its standard matrix has more rows than columns, so T is surjective.

Activity 3.4.18 What can you conclude about the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$?

A. Its standard matrix has more columns than rows, so T is not injective.

B. Its standard matrix has more columns than rows, so T is injective.

C. Its standard matrix has more rows than columns, so T is not surjective.

D. Its standard matrix has more rows than columns, so T is surjective.

Activity 3.4.20 Suppose $T : \mathbb{R}^n \to \mathbb{R}^4$ with standard matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is bijective.

(a) How many pivot rows must RREF A have?

(b) How many pivot columns must RREF A have?

(c) What is RREF A?

Activity 3.4.21 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

A. RREF(A) is the identity matrix.

B. The columns of A form a basis for $\mathbb{R}^n$

C. The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has exactly one solution for each $\vec{b} \in \mathbb{R}^n$.

Activity 3.4.23 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the standard matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & 1 \\ 6 & 2 & 1 \end{bmatrix}.$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

Injective and Surjective Linear Maps (AT4)

Activity 3.4.24 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

Activity 3.4.25 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x - y \\ x + 3y \end{bmatrix}.$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

Activity 3.4.26 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

A. T is neither injective nor surjective

B. T is injective but not surjective

C. T is surjective but not injective

D. T is bijective.

3.5 Vector Spaces (AT5)

Activity 3.5.2 Which of the following properites of $\mathbb{R}^2$ Euclidean vectors is NOT true?

A. $\left[\begin{array}{c}x_1\\x_2\end{array}\right]+\left(\left[\begin{array}{c}y_1\\y_2\end{array}\right]+\left[\begin{array}{c}z_1\\z_2\end{array}\right]\right)=\left(\left[\begin{array}{c}x_1\\x_2\end{array}\right]+\left[\begin{array}{c}y_1\\y_2\end{array}\right]\right)+\left[\begin{array}{c}z_1\\z_2\end{array}\right].$

B. $\left[\begin{array}{c}x_1\\x_2\end{array}\right]+\left[\begin{array}{c}y_1\\y_2\end{array}\right]=\left[\begin{array}{c}y_1\\y_2\end{array}\right]+\left[\begin{array}{c}x_1\\x_2\end{array}\right].$

C. There exists some $\left[\begin{array}{c}?\\?\end{array}\right]$ where $\left[\begin{array}{c}x_1\\x_2\end{array}\right]+\left[\begin{array}{c}?\\?\end{array}\right]=\left[\begin{array}{c}x_1\\x_2\end{array}\right].$

D. There exists some $\left[\begin{array}{c}?\\?\end{array}\right]$ where $\left[\begin{array}{c}x_1\\x_2\end{array}\right]+\left[\begin{array}{c}?\\?\end{array}\right]=\left[\begin{array}{c}0\\0\end{array}\right].$

E. $\frac{1}{2}\left(\left[\begin{array}{c}x_1\\x_2\end{array}\right]+\left[\begin{array}{c}y_1\\y_2\end{array}\right]\right)$ is the only vector whose endpoint is equally distant from the endpoints of $\left[\begin{array}{c}x_1\\x_2\end{array}\right]$ and $\left[\begin{array}{c}y_1\\y_2\end{array}\right].$

Activity 3.5.4 Which of the following properites of $\mathbb{R}^2$ Euclidean vectors is NOT true?

A. $a\left(b\left[\begin{array}{c}x_1\\x_2\end{array}\right]\right)=ab\left[\begin{array}{c}x_1\\x_2\end{array}\right].$

B. $1\left[\begin{array}{c}x_1\\x_2\end{array}\right]=\left[\begin{array}{c}x_1\\x_2\end{array}\right].$

C. There exists some $?$ such that $?\left[\begin{array}{c}x_1\\x_2\end{array}\right]=\left[\begin{array}{c}y_1\\y_2\end{array}\right].$

D. $a(\vec{u}+\vec{v})=a\vec{u}+a\vec{v}.$

E. $(a+b)\vec{v}=a\vec{v}+b\vec{v}.$

Activity 3.5.9 Consider the set $V=\{(x,y)\,|\,y=2^x\}$.
Which of the following vectors is not in V?

A. $(0,0)$

B. $(1,2)$

C. $(2,4)$

D. $(3,8)$

Activity 3.5.10 Consider the set $V=\{(x,y)\,|\,y=2^x\}$ with the operation $\oplus$ defined by

$$(x_1,y_1)\oplus(x_2,y_2)=(x_1+x_2,y_1y_2).$$

Let $\vec{u},\vec{v}$ be in V with $\vec{u}=(1,2)$ and $\vec{v}=(2,4)$. Using the operations defined for V, which of the following is $\vec{u}\oplus\vec{v}$?

A. $(2,6)$

B. $(2,8)$

C. $(3,6)$

D. $(3,8)$

Activity 3.5.11 Consider the set $V = \{(x,y) \,|\, y = 2^x\}$ with operations $\oplus, \odot$ defined by

$$(x_1,y_1) \oplus (x_2,y_2) = (x_1+x_2, y_1y_2) \qquad c \odot (x,y) = (cx, y^c).$$

Let $a=2, b=-3$ be scalars and $\vec{u} = (1,2) \in V$.

(a) Verify that

$$(a+b) \odot \vec{u} = \left(-1, \frac{1}{2}\right).$$

(b) Compute the value of

$$(a \odot \vec{u}) \oplus (b \odot \vec{u}).$$

Activity 3.5.12 Consider the set $V = \{(x,y) \,|\, y = 2^x\}$ with operations $\oplus, \odot$ defined by

$$(x_1,y_1) \oplus (x_2,y_2) = (x_1+x_2, y_1y_2) \qquad c \odot (x,y) = (cx, y^c).$$

Let a, b be unspecified scalars in $\mathbb{R}$ and $\vec{u} = (x,y)$ be an unspecified vector in V.

(a) Show that both sides of the equation

$$(a+b) \odot (x,y) = (a \odot (x,y)) \oplus (b \odot (x,y))$$

simplify to the expression $(ax+bx, y^a y^b)$.

(b) Show that V contains an additive identity element $\vec{z} = (\,?\,,\,?\,)$ satisfying

$$(x,y) \oplus (\,?\,,\,?\,) = (x,y)$$

for all $(x,y) \in V$.

That is, pick appropriate values for $\vec{z} = (\,?\,,\,?\,)$ and then simplify $(x,y) \oplus (\,?\,,\,?\,)$ into just (x,y).

(c) Is V a vector space?

A. Yes

B. No

C. More work is required

Activity 3.5.14 Let $V = \{(x,y) \,|\, x,y \in \mathbb{R}\}$ have operations defined by

$$(x_1,y_1) \oplus (x_2,y_2) = (x_1+y_1+x_2+y_2, x_1^2+x_2^2)$$
$$c \odot (x,y) = (x^c, y+c-1).$$

(a) Show that 1 is the scalar multiplication identity element by simplifying $1 \odot (x,y)$ to (x,y).

(b) Show that V does not have an additive identity element $\vec{z} = (z, w)$ by showing that $(0, -1) \oplus (z, w) \neq (0, -1)$ no matter what the values of z, w are.

(c) Is V a vector space?

A. Yes

B. No

C. More work is required

Activity 3.5.15 Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + 3y_2) \qquad c \odot (x, y) = (cx, cy).$$

(a) Show that scalar multiplication distributes over vector addition, i.e.

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$$

for *all* $c \in \mathbb{R}$, $(x_1, y_1), (x_2, y_2) \in V$.

(b) Show that vector addition is not associative, i.e.

$$(x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)) \neq ((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3)$$

for *some* vectors $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$.

(c) Is V a vector space?

A. Yes

B. No

C. More work is required

3.6 Polynomial and Matrix Spaces (AT6)

Activity 3.6.2 Let V be a vector space with the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Which of these completes the following definition for a bijective linear map $T : V \to \mathbb{R}^3$?

$$T(\vec{v}) = T(a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3) = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

A. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} a+b+c \\ 0 \\ 0 \end{bmatrix}$ C. $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Activity 3.6.4 The matrix space $M_{2,2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$ has the basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(a) What is the dimension of $M_{2,2}$?

A. 2

B. 3

C. 4

D. 5

(b) Which Euclidean space is $M_{2,2}$ isomorphic to?

A. $\mathbb{R}^2$

B. $\mathbb{R}^3$

C. $\mathbb{R}^4$

D. $\mathbb{R}^5$

(c) Describe an isomorphism $T : M_{2,2} \to \mathbb{R}^?$:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} ? \\ \\ \vdots \\ \\ ? \end{bmatrix}$$

Activity 3.6.5 The polynomial space $\mathcal{P}^4 = \{a + bx + cx^2 + dx^3 + ex^4 | a, b, c, d, e \in \mathbb{R}\}$ has the basis

$$\left\{1, x, x^2, x^3, x^4\right\}.$$

(a) What is the dimension of $\mathcal{P}^4$?

A. 2

B. 3

C. 4

D. 5

(b) Which Euclidean space is $\mathcal{P}^4$ isomorphic to?

A. $\mathbb{R}^2$

B. $\mathbb{R}^3$

C. $\mathbb{R}^4$

D. $\mathbb{R}^5$

(c) Describe an isomorphism $T : \mathcal{P}^4 \to \mathbb{R}^{?}$:

$$T\left(a + bx + cx^2 + dx^3 + ex^4\right) = \begin{bmatrix} ? \\ \\ \vdots \\ \\ ? \end{bmatrix}$$

Activity 3.6.7 Consider how to construct the polynomial $x^3 + x^2 + 5x + 1$ as a linear combination of polynomials from the set

$$\left\{x^3 - 2\,x^2 + x + 2, 2\,x^2 - 1, -x^3 + 3\,x^2 + 3\,x - 2, x^3 - 6\,x^2 + 9\,x + 5\right\}.$$

(a) Describe the vector space involved in this problem, and an isomorphic Euclidean space and relevant Eucldean vectors that can be used to solve this problem.

(b) Show how to construct an appropriate Euclidean vector from an approriate set of Euclidean vectors.

(c) Use this result to answer the original question.

Chapter 4: Matrices (MX)

4.1 Matrices and Multiplication (MX1)

Activity 4.1.2 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by the 2×3 standard matrix B and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by the 4×2 standard matrix A:

$$B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}.$$

(a) What are the domain and codomain of the composition map $S \circ T$?

A. The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^2$

B. The domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^4$

C. The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^4$

D. The domain is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^3$

(b) What size will the standard matrix of $S \circ T$ be?

A. 4 (rows) $\times$ 3 (columns)

B. 3 (rows) $\times$ 4 (columns)

C. 3 (rows) $\times$ 2 (columns)

D. 2 (rows) $\times$ 4 (columns)

(c) Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

(d) Compute $(S \circ T)(\vec{e}_2)$.

(e) Compute $(S \circ T)(\vec{e}_3)$.

(f) Use $(S \circ T)(\vec{e}_1), (S \circ T)(\vec{e}_2), (S \circ T)(\vec{e}_3)$ to write the standard matrix for $S \circ T$.

Activity 4.1.4 Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

(a) Write the dimensions (rows $\times$ columns) for A, B, AB, and BA.

(b) Find the standard matrix AB of $S \circ T$.

(c) Find the standard matrix BA of $T \circ S$.

Matrices and Multiplication (MX1)

Activity 4.1.5 Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

(a) Find the domain and codomain of each of the three linear maps corresponding to A, B, and C.

(b) Only one of the matrix products AB, AC, BA, BC, CA, CB can actually be computed. Compute it.

Activity 4.1.6 Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

(a) Compute the product BA by hand.

(b) Check your work using technology. Using Octave:

```
B = [3 -4 0 ; 2 0 -1 ; 0 -3 3]
A = [2 7 -1 ; 0 3 2  ; 1 1 -1]
B*A
```

Activity 4.1.7 Of the following three matrices, only two may be multiplied.

$$A = \begin{bmatrix} -1 & 3 & -2 & -3 \\ 1 & -4 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -6 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ -2 & 4 & -1 \\ -2 & 3 & -1 \end{bmatrix}$$

Explain which two can be multiplied and why. Then show how to find their product.

Activity 4.1.8 Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y \\ 3x+5y \\ -x-2y \end{bmatrix}$ In Fact 3.2.10 we adopted the notation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y \\ 3x+5y \\ -x-2y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Matrices and Multiplication (MX1)

Verify that $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ y \\ 3x+5y \\ -x-2y \end{bmatrix}$ in terms of matrix multiplication.

4.2 The Inverse of a Matrix (MX2)

Activity 4.2.1 Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix B such that $BA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Check your guess using technology.

Activity 4.2.4 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following items into three groups of statements: a group that means T is *injective*, a group that means T is *surjective*, and a group that means T is *bijective*.

A. $T(\vec{x}) = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^m$

B. $T(\vec{x}) = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^m$

C. $T(\vec{x}) = \vec{0}$ has a unique solution.

D. The columns of A span $\mathbb{R}^m$

E. The columns of A are linearly independent

F. The columns of A are a basis of $\mathbb{R}^m$

G. Every column of $\text{RREF}(A)$ has a pivot

H. Every row of $\text{RREF}(A)$ has a pivot

I. $m = n$ and $\text{RREF}(A) = I$

Activity 4.2.6 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear bijection given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

(a) To find $\vec{x} = T^{-1}(\vec{e}_1)$, we need to find the unique solution for $T(\vec{x}) = \vec{e}_1$. Which of these linear systems can be used to find this solution?

A. $\begin{alignedat}{4} 2x_1 & -1x_2 & -6x_3 & = x_1 \\ 2x_1 & +1x_2 & +3x_3 & = 0 \\ 1x_1 & +1x_2 & +4x_3 & = 0 \end{alignedat}$

B. $\begin{alignedat}{4} 2x_1 & -1x_2 & -6x_3 & = x_1 \\ 2x_1 & +1x_2 & +3x_3 & = x_2 \\ 1x_1 & +1x_2 & +4x_3 & = x_3 \end{alignedat}$

C. $\begin{alignedat}{4} 2x_1 & -1x_2 & -6x_3 & = 1 \\ 2x_1 & +1x_2 & +3x_3 & = 0 \\ 1x_1 & +1x_2 & +4x_3 & = 0 \end{alignedat}$

D. $\begin{alignedat}{4} 2x_1 & -1x_2 & -6x_3 & = 1 \\ 2x_1 & +1x_2 & +3x_3 & = 1 \\ 1x_1 & +1x_2 & +4x_3 & = 1 \end{alignedat}$

(b) Use that system to find the solution $\vec{x} = T^{-1}(\vec{e}_1)$ for $T(\vec{x}) = \vec{e}_1$.

(c) Similarly, solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$, and solve $T(\vec{x}) = \vec{e}_3$ to find $T^{-1}(\vec{e}_3)$.

(d) Use these to write
$$A^{-1} = [T^{-1}(\vec{e}_1) \quad T^{-1}(\vec{e}_2) \quad T^{-1}(\vec{e}_3)],$$
the standard matrix for T^{-1}.

Activity 4.2.7 Find the inverse A^{-1} of the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & -4 \\ 1 & 1 & 0 & -4 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

by computing how it transforms each of the standard basis vectors for $\mathbb{R}^4$: $T^{-1}(\vec{e}_1)$, $T^{-1}(\vec{e}_2)$, $T^{-1}(\vec{e}_3)$, and $T^{-1}(\vec{e}_4)$.

Activity 4.2.8 Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible?

A. Yes, because its transformation is a bijection.

B. Yes, because its transformation is not a bijection.

C. No, because its transformation is a bijection.

D. No, because its transformation is not a bijection.

Activity 4.2.10 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

(a) Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

(b) If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1}, find the 2×2 matrix
$$A^{-1}A = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}.$$

4.3 Solving Systems with Matrix Inverses (MX3)

Activity 4.3.1 Consider the following linear system with a unique solution:

$$\begin{alignedat}{5} 3x_1 &\; - \;& 2x_2 &\; - \;& 2x_3 &\; - \;& 4x_4 &\; = \;& -7 \\ 2x_1 &\; - \;& x_2 &\; - \;& x_3 &\; - \;& x_4 &\; = \;& -1 \\ -x_1 && &\; + \;& x_3 && &\; = \;& -1 \\ &\; - \;& x_2 && &\; - \;& 2x_4 &\; = \;& -5 \end{alignedat}$$

(a) Suppose we let

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - 2x_2 - 2x_3 - 4x_4 \\ 2x_1 - x_2 - x_3 - x_4 \\ -x_1 + x_3 \\ - x_2 - 2x_4 \end{bmatrix}.$$

Which of these choices would help us solve the given system?

A. Compute $T\left(\begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}\right)$

B. Find $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}$

(b) How can we express this in terms of matrix multiplication?

A. $\begin{bmatrix} 3 & -2 & -2 & -4 \\ 2 & -1 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}$

B. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -2 & -4 \\ 2 & -1 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}$

C. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{bmatrix} 3 & -2 & -2 & -4 \\ 2 & -1 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix}$

D. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ -1 \\ -5 \end{bmatrix} \begin{bmatrix} 3 & -2 & -2 & -4 \\ 2 & -1 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix}$

(c) How could a matrix equation of the form $A\vec{x} = \vec{b}$ be solved for $\vec{x}$?

A. Multiply: $(\text{RREF } A)(A\vec{x}) = (\text{RREF } A)\vec{b}$

B. Add: $(\text{RREF } A) + A\vec{x} = (\text{RREF } A) + \vec{b}$

C. Multiply: $(A^{-1})(A\vec{x}) = (A^{-1})\vec{b}$

D. Add: $(A^{-1}) + A\vec{x} = (A^{-1}) + \vec{b}$

(d) Find $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ using the method you chose in (c).

Activity 4.3.3 Consider the vector equation

$$x_1 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ -1 \end{bmatrix}$$

with a unique solution.

(a) Explain and demonstrate how this problem can be restated using matrix multiplication.

(b) Use the properties of matrix multiplication to find the unique solution.

4.4 Row Operations as Matrix Multiplication (MX4)

Row Operations as Matrix Multiplication (MX4)

Activity 4.4.1 Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

(a) Which of these tweaks of the identity matrix yields a matrix that doubles the third row of A when left-multiplying? $(2R_3 \to R_3)$

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

A. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

D. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(b) Which of these tweaks of the identity matrix yields a matrix that swaps the first and third rows of A when left-multiplying? $(R_1 \leftrightarrow R_3)$

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

A. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) Which of these tweaks of the identity matrix yields a matrix that adds 5 times the third row of A to the first row when left-multiplying? $(R_1 + 5R_3 \to R_1)$

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5(1) & 7+5(1) & -1+5(-1) \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

A. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 5 & 5 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Row Operations as Matrix Multiplication (MX4)

Activity 4.4.3 What would happen if you *right*-multiplied by the tweaked identity matrix rather than left-multiplied?

A. The manipulated rows would be reversed.

B. Columns would be manipulated instead of rows.

C. The entries of the resulting matrix would be rotated 180 degrees.

Activity 4.4.4 Consider the two row operations $R_2 \leftrightarrow R_3$ and $R_1 + R_2 \rightarrow R_1$ applied as follows to show $A \sim B$:

$$\begin{aligned} A = \begin{bmatrix} -1 & 4 & 5 \\ 0 & 3 & -1 \\ 1 & 2 & 3 \end{bmatrix} &\sim \begin{bmatrix} -1 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} \\ &\sim \begin{bmatrix} -1+1 & 4+2 & 5+3 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = B \end{aligned}$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A:

$$B = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} A$$

Check your work using technology.

Activity 4.4.5 Let A be *any* 4×4 matrix.

(a) Give a 4×4 matrix M that may be used to perform the row operation $-5R_2 \rightarrow R_2$.

(b) Give a 4×4 matrix Y that may be used to perform the row operation $R_2 \leftrightarrow R_3$.

(c) Use matrix multiplication to describe the matrix obtained by applying $-5R_2 \rightarrow R_2$ and then $R_2 \leftrightarrow R_3$ to A (note the order).

Chapter 5: Geometric Properties of Linear Maps (GT)

5.1 Row Operations and Determinants (GT1)

Activity 5.1.1 The image in Figure 46 illustrates how the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ transforms the unit square.

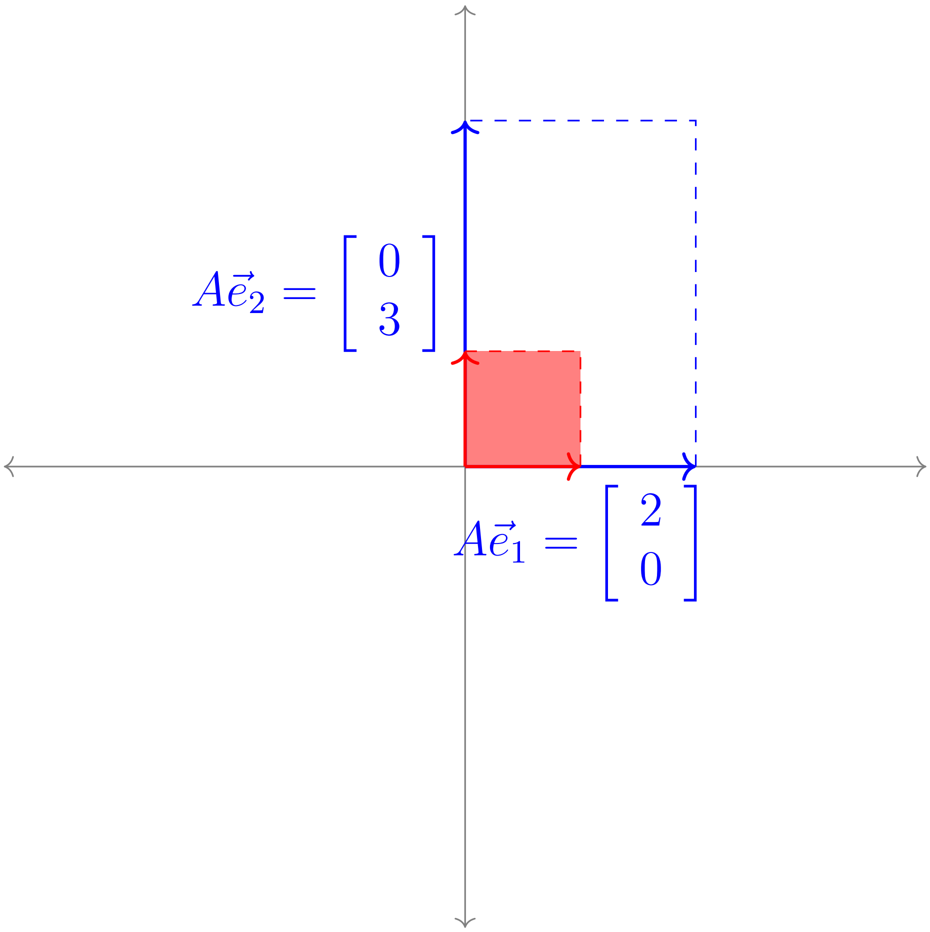


Figure 46 Transformation of the unit square by the matrix A.

(a) What are the lengths of $A\vec{e}_1$ and $A\vec{e}_2$?

(b) What is the area of the transformed unit square?

Activity 5.1.2 The image below illustrates how the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$ transforms the unit square.

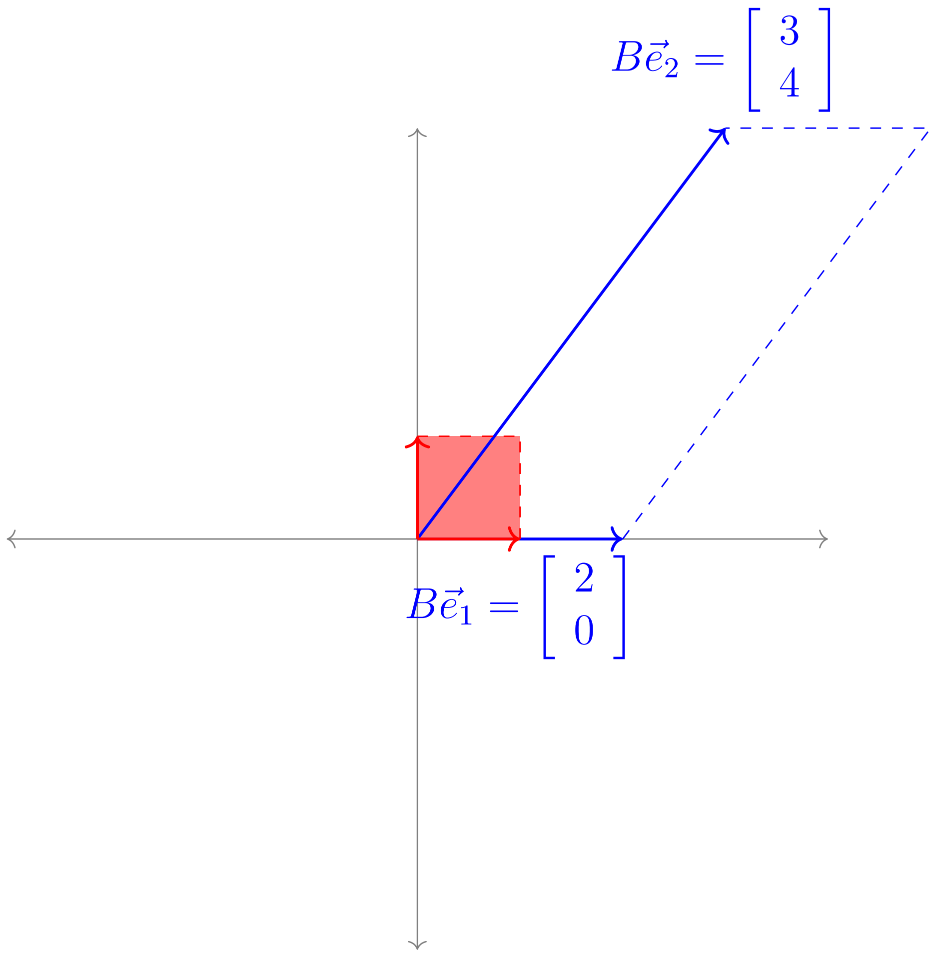


Figure 47 Transformation of the unit square by the matrix B

(a) What are the lengths of $B\vec{e}_1$ and $B\vec{e}_2$?

(b) What is the area of the transformed unit square?

Activity 5.1.6 The transformation of the unit square by the standard matrix $[\vec{e}_1 \ \ \vec{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ is illustrated below. If $\det([\vec{e}_1 \ \ \vec{e}_2]) = \det(I)$ is the area of resulting parallelogram, what is the value of $\det([\vec{e}_1 \ \ \vec{e}_2]) = \det(I)$?

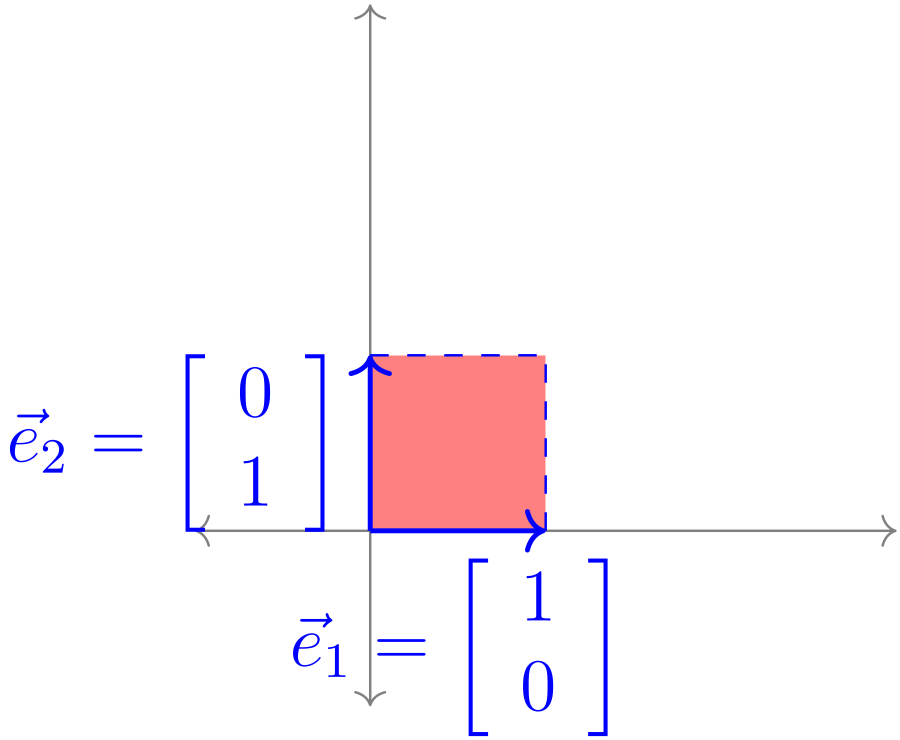


Figure 51 The transformation of the unit square by the identity matrix.

The value for $\det([\vec{e}_1\ \ \vec{e}_2]) = \det(I)$ is:

A. 0

B. 1

C. 2

D. 4

Activity 5.1.7 The transformation of the unit square by the standard matrix $[\vec{v}\ \ \vec{v}]$ is illustrated below: both $T(\vec{e}_1) = T(\vec{e}_2) = \vec{v}$. If $\det([\vec{v}\ \ \vec{v}])$ is the area of the generated parallelogram, what is the value of $\det([\vec{v}\ \ \vec{v}])$?

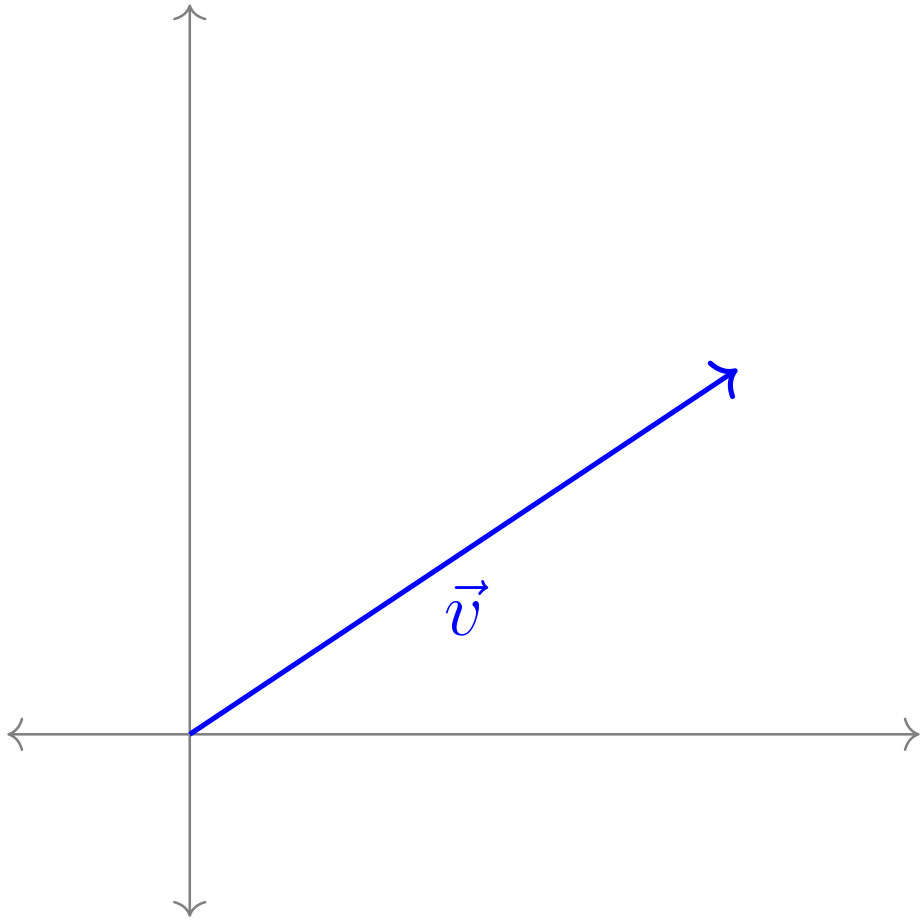


Figure 52 Transformation of the unit square by a matrix with identical columns.

The value of $\det([\vec{v}\ \ \vec{v}])$ is:

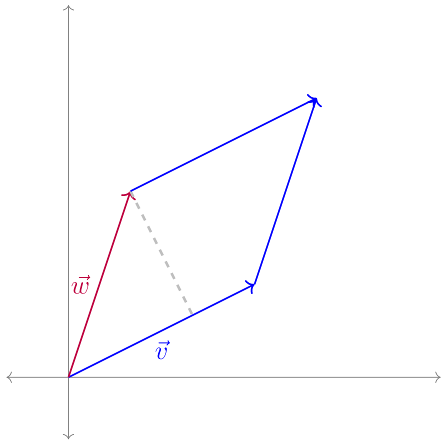
A. 0

B. 1

C. 2

D. 4

Activity 5.1.8 The transformations of the unit square by the standard matrices $[\vec{v}\ \ \vec{w}]$ and $[c\vec{v}\ \ \vec{w}]$ are illustrated below. Describe the value of $\det([c\vec{v}\ \ \vec{w}])$.



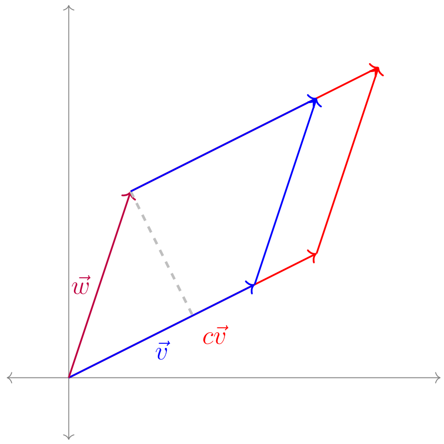


Figure 53 The parallelograms generated by $\vec{v}$ and $\vec{w}/c\vec{w}$

Describe the value of $\det([c\vec{v}\ \ \vec{w}])$:

A. $\det([\vec{v}\ \ \vec{w}])$

B. $c\det([\vec{v}\ \ \vec{w}])$

C. $c^2\det([\vec{v}\ \ \vec{w}])$

D. Cannot be determined from this information.

Activity 5.1.12 The parallelograms generated by the standard matrices $[\vec{u}\ \ \vec{w}]$, $[\vec{v}\ \ \vec{w}]$ and $[\vec{u}+\vec{v}\ \ \vec{w}]$ are illustrated below.

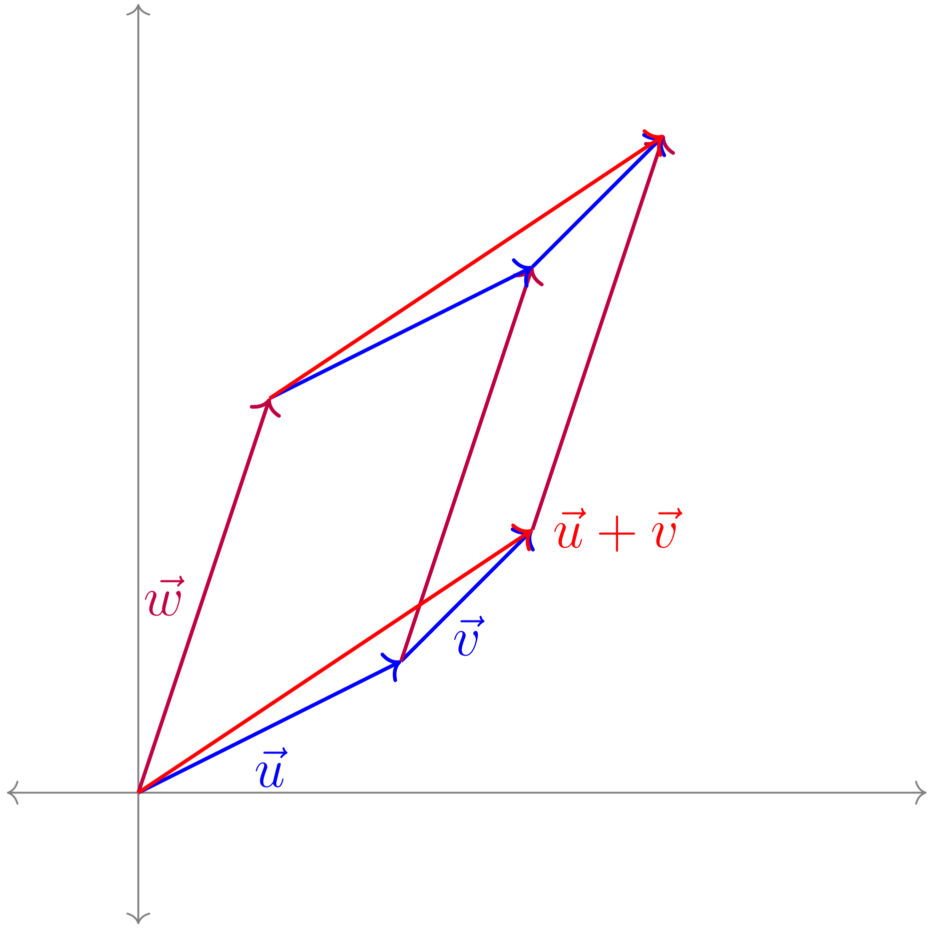


Figure 57 Parallelogram generated by $\vec{u}+\vec{v}$ and $\vec{w}$

Describe the value of $\det([\vec{u}+\vec{v} \;\; \vec{w}])$.

A. $\det([\vec{u} \;\; \vec{w}]) = \det([\vec{v} \;\; \vec{w}])$

B. $\det([\vec{u} \;\; \vec{w}]) + \det([\vec{v} \;\; \vec{w}])$

C. $\det([\vec{u} \;\; \vec{w}]) \det([\vec{v} \;\; \vec{w}])$

D. Cannot be determined from this information.

Activity 5.1.18 The transformation given by the standard matrix A scales areas by 4, and the transformation given by the standard matrix B scales areas by 3. By what factor does the transformation given by the standard matrix AB scale areas?

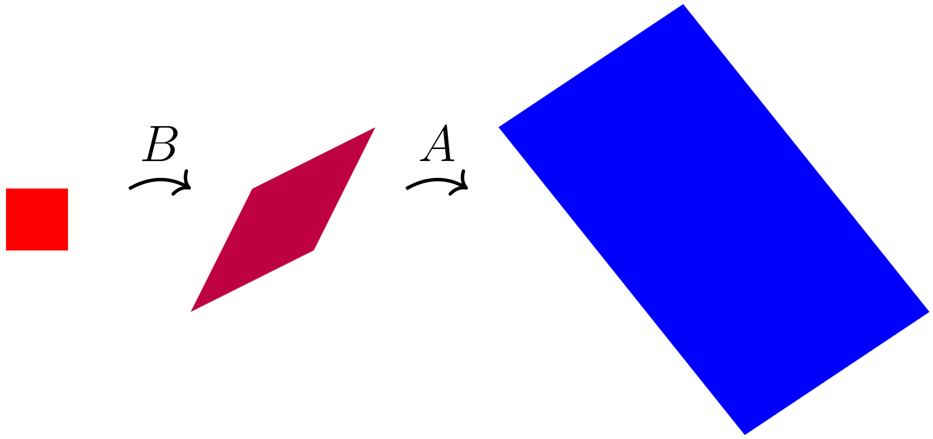


Figure 60 Area changing under the composition of two linear maps

A. 1

B. 7

C. 12

D. Cannot be determined

Activity 5.1.22 Consider the row operation $R_1 + 4R_3 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 1+4(9) & 2+4(10) & 3+4(11) & 4+4(12) \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) Find $\det R$ by comparing with the previous slide.

(c) If $C \in M_{4,4}$ is a matrix with $\det(C) = -3$, find

$$\det(RC) = \det(R)\det(C).$$

Activity 5.1.23 Consider the row operation $R_1 \leftrightarrow R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to I.

(b) If $C \in M_{4,4}$ is a matrix with $\det(C) = 5$, find $\det(RC)$.

Row Operations and Determinants (GT1)

Activity 5.1.24 Consider the row operation $3R_2 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3(5) & 3(6) & 3(7) & 3(8) \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{4,4}$ is a matrix with $\det(C) = -7$, find $\det(RC)$.

Activity 5.1.25 Let A be *any* 4×4 matrix with determinant 2.

(a) Let B be the matrix obtained from A by applying the row operation $R_1 - 5R_3 \to R_1$. What is $\det B$?

A -4 | B -2 | C 2 | D 10

(b) Let M be the matrix obtained from A by applying the row operation $R_3 \leftrightarrow R_1$. What is $\det M$?

A -4 | B -2 | C 2 | D 10

(c) Let P be the matrix obtained from A by applying the row operation $2R_4 \to R_4$. What is $\det P$?

A -4 | B -2 | C 2 | D 10

Activity 5.1.29 Complete the following derivation for a formula calculating 2×2 determinants:

$$\begin{aligned} \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= ? \det \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \\ &= ? \det \begin{bmatrix} 1 & b/a \\ c - c & d - bc/a \end{bmatrix} \\ &= ? \det \begin{bmatrix} 1 & b/a \\ 0 & d - bc/a \end{bmatrix} \\ &= ? \det \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \\ &= ? \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= ? \det I \\ &= ? \end{aligned}$$

5.2 Computing Determinants (GT2)

Activity 5.2.2 The following image illustrates the transformation of the unit cube by the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

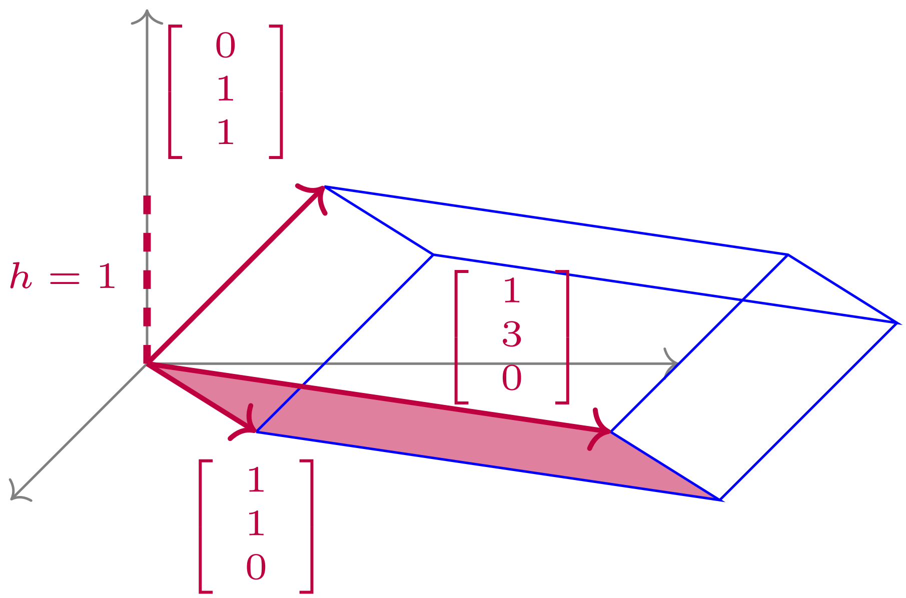


Figure 62 Transformation of the unit cube by the linear transformation.

Recall that for this solid $V = Bh$, where h is the height of the solid and B is the area of its parallelogram base. So what must its volume be?

A. $\det \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

B. $\det \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

C. $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

D. $\det \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

Activity 5.2.5 Remove an appropriate row and column of $\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 12 \\ 3 & 2 & -1 \end{bmatrix}$ to simplify the determinant to a 2×2 determinant.

Activity 5.2.6 Simplify $\det \begin{bmatrix} 0 & 3 & -2 \\ 2 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

(a) Factor out a 2 from a column.

(b) Swap rows or columns to put a 1 on the main diagonal.

Computing Determinants (GT2)

Activity 5.2.7 Simplify $\det \begin{bmatrix} 4 & -2 & 2 \\ 3 & 1 & 4 \\ 1 & -1 & 3 \end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

(a) Use row/column operations to create two zeroes in the same row or column.

(b) Factor/swap as needed to get a row/column of all zeroes except a 1 on the main diagonal.

Activity 5.2.9 Rewrite

$$\det \begin{bmatrix} 2 & 1 & -2 & 1 \\ 3 & 0 & 1 & 4 \\ -2 & 2 & 3 & 0 \\ -2 & 0 & -3 & -3 \end{bmatrix}$$

as a multiple of a determinant of a 3×3 matrix.

Activity 5.2.10 Compute $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$ by using any combination of row/column operations.

Activity 5.2.13 Based on the previous activities, which technique is easier for computing determinants?

A. Memorizing formulas.

B. Using row/column operations.

C. Laplace expansion.

D. Some other technique.

Activity 5.2.14 Use your preferred technique to compute $\det \begin{bmatrix} 4 & -3 & 0 & 0 \\ 1 & -3 & 2 & -1 \\ 3 & 2 & 0 & 3 \\ 0 & -3 & 2 & -2 \end{bmatrix}$.

5.3 Eigenvalues and Characteristic Polynomials (GT3)

Activity 5.3.1 An invertible matrix M and its inverse M^{-1} are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Which of the following is equal to $\det(M)\det(M^{-1})$?

A. -1

B. 0

C. 1

D. 4

Activity 5.3.5 Finding the eigenvalues λ that satisfy

$$A\vec{x} = \lambda\vec{x} = \lambda(I\vec{x}) = (\lambda I)\vec{x}$$

for some nontrivial eigenvector $\vec{x}$ is equivalent to finding nonzero solutions for the matrix equation

$$(A - \lambda I)\vec{x} = \vec{0}.$$

(a) If λ is an eigenvalue, and T is the transformation with standard matrix $A - \lambda I$, which of these must contain a non-zero vector?

A. The kernel of T

B. The image of T

C. The domain of T

D. The codomain of T

(b) Therefore, what can we conclude?

A. A is invertible

B. A is not invertible

C. $A - \lambda I$ is invertible

D. $A - \lambda I$ is not invertible

(c) And what else?

A. $\det A = 0$

B. $\det A = 1$

C. $\det(A - \lambda I) = 0$

D. $\det(A - \lambda I) = 1$

Activity 5.3.8 Let $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$.

(a) Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

(b) Set this characteristic polynomial equal to zero and factor to determine the eigenvalues of A.

Eigenvalues and Characteristic Polynomials (GT3)

Activity 5.3.9 Find all the eigenvalues for the matrix $A = \begin{bmatrix} 3 & -3 \\ 2 & -4 \end{bmatrix}$.

Activity 5.3.10 Find all the eigenvalues for the matrix $A = \begin{bmatrix} 1 & -4 \\ 0 & 5 \end{bmatrix}$.

Activity 5.3.11 Find all the eigenvalues for the matrix $A = \begin{bmatrix} 3 & -3 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$.

5.4 Eigenvectors and Eigenspaces (GT4)

Activity 5.4.1 It's possible to show that -2 is an eigenvalue for $\begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 9 \\ 3 & 0 & 4 \end{bmatrix}$.

Compute the kernel of the transformation with standard matrix

$$A - (-2)I = \begin{bmatrix} ? & 4 & -2 \\ 2 & ? & 9 \\ 3 & 0 & ? \end{bmatrix}$$

to find all the eigenvectors $\vec{x}$ such that $A\vec{x} = -2\vec{x}$.

Activity 5.4.3 Find a basis for the eigenspace for the matrix $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ associated with the eigenvalue 3.

Activity 5.4.4 Find a basis for the eigenspace for the matrix $\begin{bmatrix} 5 & -2 & 0 & 4 \\ 6 & -2 & 1 & 5 \\ -2 & 1 & 2 & -3 \\ 4 & 5 & -3 & 6 \end{bmatrix}$ associated with the eigenvalue 1.

Activity 5.4.5 Find a basis for the eigenspace for the matrix $\begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ associated with the eigenvalue 2.

Chapter A: Applications

A.1 Civil Engineering: Trusses and Struts

Activity A.1.2 Consider the representation of a simple truss pictured below. All of the seven struts are of equal length, affixed to two anchor points applying a normal force to nodes C and E, and with a $10000N$ load applied to the node given by D.

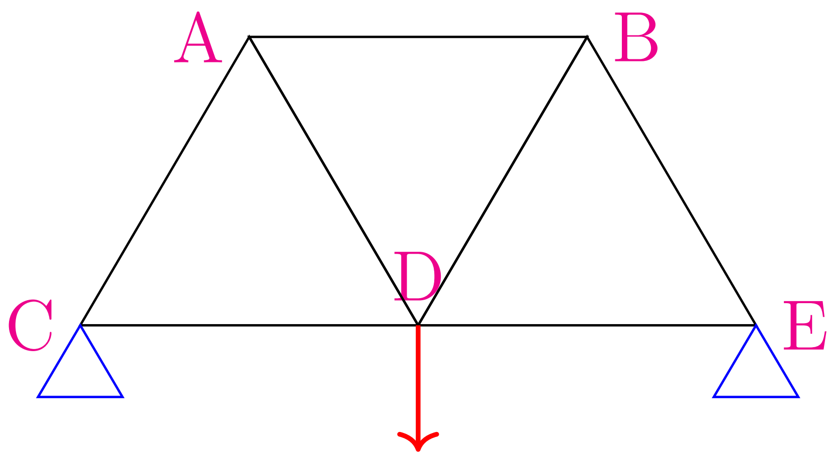


Figure 71 A simple truss

Which of the following must hold for the truss to be stable?

1. All of the struts will experience compression.
2. All of the struts will experience tension.
3. Some of the struts will be compressed, but others will be tensioned.

Activity A.1.5 Using the conventions of the previous remark, and where $\vec{L}$ represents the load vector on node D, find four more vector equations that must be satisfied for each of the other four nodes of the truss.

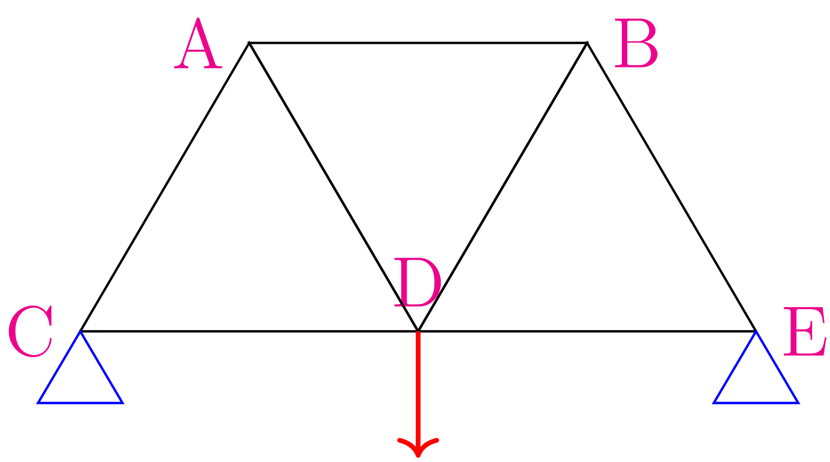


Figure 74 A simple truss

$$A : ?$$

$$B : ?$$

$$C : \vec{F}_{CA} + \vec{F}_{CD} + \vec{N}_C = \vec{0}$$

$$D : ?$$

$$E : ?$$

Activity A.1.8 To write a linear system that models the truss under consideration with constant load 10000 newtons, how many scalar variables will be required?

- 7: 5 from the nodes, 2 from the anchors
- 9: 7 from the struts, 2 from the anchors
- 11: 7 from the struts, 4 from the anchors
- 12: 7 from the struts, 4 from the anchors, 1 from the load
- 13: 5 from the nodes, 7 from the struts, 1 from the load

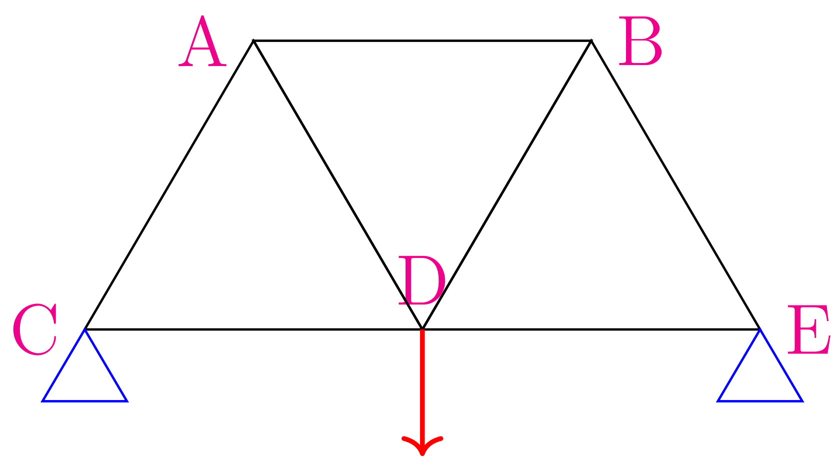


Figure 75 A simple truss

Activity A.1.12 Expand the vector equation given below using sine and cosine of appropriate angles, then compute each component (approximating $\sqrt{3}/2 \approx 0.866$).

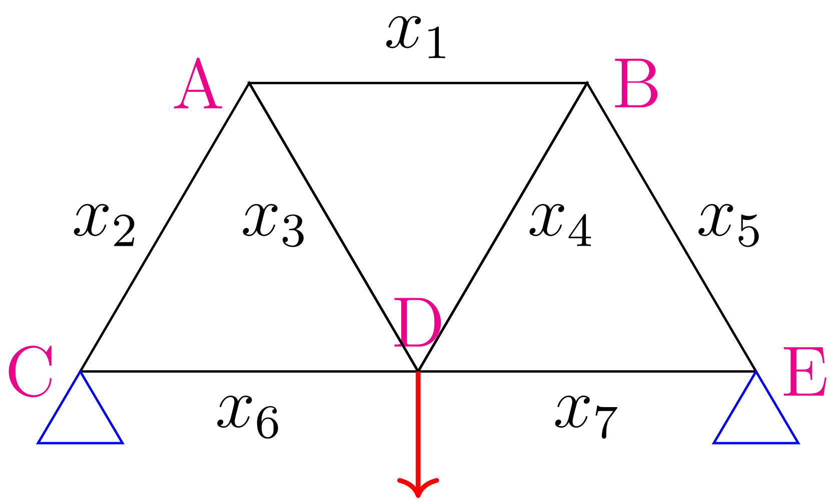


Figure 79 Variables for the truss

$$D : \vec{F}_{DA} + \vec{F}_{DB} + \vec{F}_{DC} + \vec{F}_{DE} = -\vec{L}$$

$$\Leftrightarrow x_3 \begin{bmatrix} \cos(\ ?\ ) \\ \sin(\ ?\ ) \end{bmatrix} + x_4 \begin{bmatrix} \cos(\ ?\ ) \\ \sin(\ ?\ ) \end{bmatrix} + x_6 \begin{bmatrix} \cos(\ ?\ ) \\ \sin(\ ?\ ) \end{bmatrix} + x_7 \begin{bmatrix} \cos(\ ?\ ) \\ \sin(\ ?\ ) \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$\Leftrightarrow x_3 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_4 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_6 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_7 \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

A.2 Computer Science: PageRank

Activity A.2.1

In the picture below, each circle represents a webpage, and each arrow represents a link from one page to another.

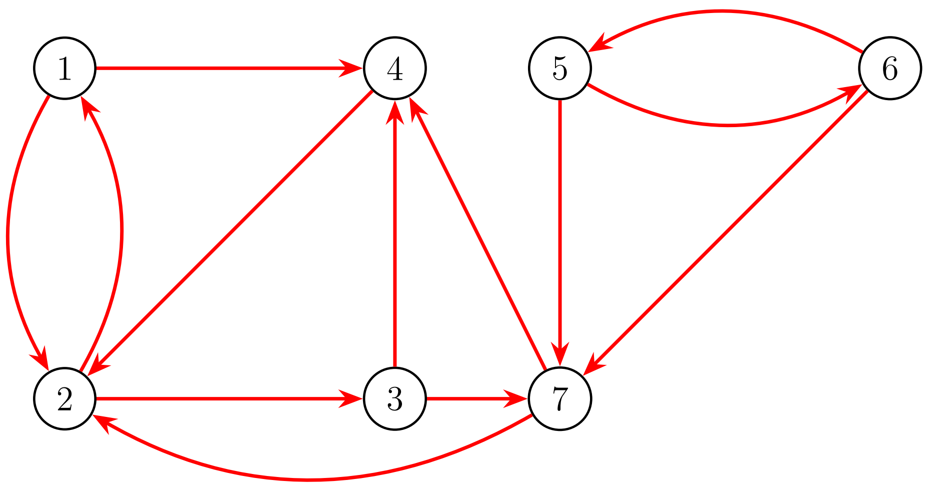


Figure 81 A seven-webpage network

Based on how these pages link to each other, write a list of the 7 webpages in order from most important to least important.

Activity A.2.5 Thus, our \$978,000,000,000 problem is what kind of problem?

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A. An antiderivative problem

B. A bijection problem

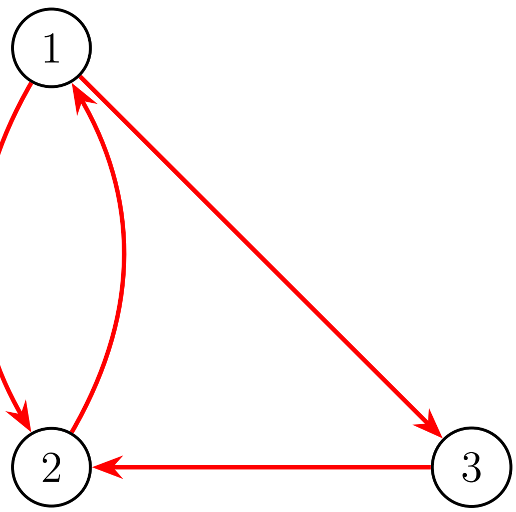
C. A cofactoring problem

D. A determinant problem

E. An eigenvector problem

Activity A.2.6 Find a page rank vector $\vec{x}$ satisfying $A\vec{x} = 1\vec{x}$ for the following network's page rank matrix A.

That is, find the eigenspace associated with $\lambda = 1$ for the matrix A, and choose a vector from that eigenspace.



$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Figure 84 A three-webpage network

Activity A.2.8 Compute the 7×7 page rank matrix for the following network.

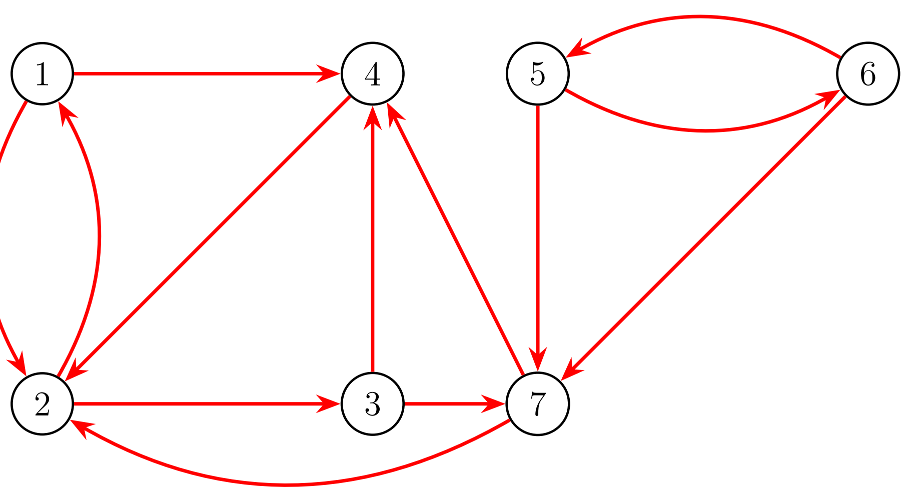


Figure 85 A seven-webpage network

For example, since website 1 distributes its endorsement equally between 2 and 4, the first column is $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Activity A.2.9 Find a page rank vector for the given page rank matrix.

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

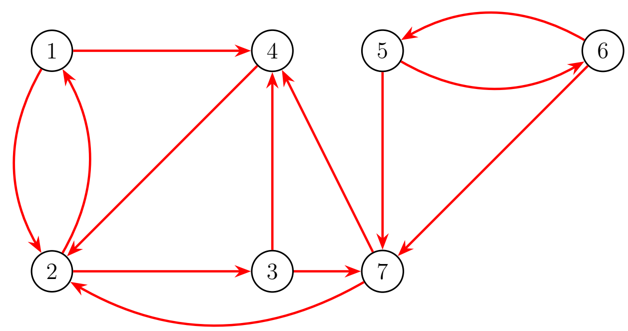


Figure 86 A seven-webpage network

Which webpage is most important?

Activity A.2.11 Given the following diagram, use a page rank vector to rank the pages 1 through 7 in order from most important to least important.

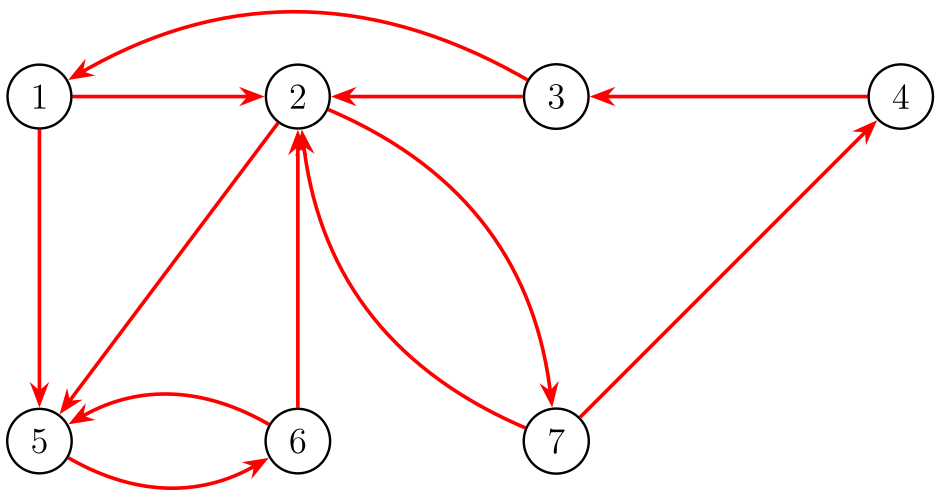


Figure 88 Another seven-webpage network

A.3 Geology: Phases and Components

Activity A.3.3 To study this vector space, each of the three components $\vec{c}_1, \vec{c}_2, \vec{c}_3$ may be considered as the three components of a Euclidean vector.

$$\vec{p}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{p}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{p}_5 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Determine if the set of phases is linearly dependent or linearly independent.

Activity A.3.4 Geologists are interested in knowing all the possible chemical reactions among the 5 phases:

$$\vec{p}_1 = \mathrm{Ca_3MgSi_2O_8} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \vec{p}_2 = \mathrm{CaMgSiO_4} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{p}_3 = \mathrm{CaSiO_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{p}_4 = \mathrm{CaMgSi_2O_6} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \vec{p}_5 = \mathrm{Ca_2MgSi_2O_7} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

That is, they want to find numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\vec{p}_1 + x_2\vec{p}_2 + x_3\vec{p}_3 + x_4\vec{p}_4 + x_5\vec{p}_5 = 0.$$

(a) Set up a system of equations equivalent to this vector equation.

(b) Find a basis for its solution space.

(c) Interpret each basis vector as a vector equation and a chemical equation.

Activity A.3.5 We found two basis vectors $\begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, corresponding to the vector and chemical equations

$$\begin{aligned} 2\vec{p}_2 + 2\vec{p}_3 &= \vec{p}_1 + \vec{p}_4 & \qquad 2\mathrm{CaMgSiO_4} + 2\mathrm{CaSiO_3} &= \mathrm{Ca_3MgSi_2O_8} + \mathrm{CaMgSi_2O_6} \\ \vec{p}_2 + \vec{p}_3 &= \vec{p}_5 & \mathrm{CaMgSiO_4} + \mathrm{CaSiO_3} &= \mathrm{Ca_2MgSi_2O_7} \end{aligned}$$

Combine the basis vectors to produce a chemical equation among the five phases that does not involve $\vec{p}_2 = \mathrm{CaMgSiO_4}$.