Proposition 1. *Puzzle 1: Find an infinite collection of open intervals in $\mathbb{R}$ whose intersection is not an open interval.*

Proof. $X = \bigcap\{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{Z}^+\}$ ☐

Proposition 2. *Any finite union of closed sets is closed, and any arbitrary intersection of closed sets is closed.*

Proof. Proposition 1 - Any finite union of closed sets is closed.
Let C, D be closed sets.
Let A, B be compliments of C, D so A, B are open.
Then, A $\cap$ B is also open.
Thus, X$\setminus$(A $\cap$ B) is closed.
X$\setminus$(A $\cap$ B) = X$\setminus$A $\cup$ X$\setminus$B = C $\cup$ D

Inductive Proof:
Base Case: C $\cup$ D is closed.
Inductive Hypothesis:
Assume $C_1 \cup C_2 \cup ... \cup C_n$ is closed for closed sets C_i.
Then $C_1 \cup C_2 \cup ... \cup C_n$ = K, K $\cup$ C_{n+1} is closed.

Proposition 1 - Any arbitrary intersection of closed sets is closed.
Let $\mathcal{C}$ be a collection of closed sets.
Let $\mathcal{U}$ = {X$\setminus$C : C $\in \mathcal{C}$}, so $\mathcal{U}$ is a collection of open sets.
Then, $\bigcup\mathcal{U}$ is also open.
Thus, X$\setminus\bigcup\mathcal{U}$ is closed.
X$\setminus\bigcup\mathcal{U} = \bigcap\mathcal{C}$
Therefore, $\bigcap\mathcal{C}$ is closed. ☐

Lemma 3. *A set U is open if and only if for every point $x \in U$, there exists an open set U_x where $x \in U_x \subseteq U$*

Proof. Suppose U is open
Then for all x $\in$ U there exists an open subset of U, U_x such that x $\in$ U_x ☐

Proof. Suppose there is an open set {U_x : x $\in$ U_x} $\subseteq$ U.
We want to show that U is open.
If $U_x \subseteq$ U, then for all $x \in U_x$, $x \in$ U
$U_x \subseteq$ U is open if $U_x \subseteq \tau$
If $U_x \subseteq \tau$, then $\bigcup U_x \in \tau$
$\bigcup U_x$ = U $\in \tau$, so U is open. ☐

Proposition 4. *A set K in a topological space X is closed if and only if K contains all its limit points.*

Proof. Suppose K contains all its limit points.
This is what you had already given
If K = X, then K is closed because $\emptyset$ is open.

Otherwise let x $\in$ X\K, so x is not a limit point of K.
Then, $\exists\ x \in \mathrm{U}_x \in \tau$ such that $\mathrm{U}_x \cap$ K $= \emptyset$.
So, $x \in \mathrm{U}_x \subseteq$ X\K.
Thus, $\bigcup\{\mathrm{U}_x : x \in \mathrm{X}\backslash\mathrm{K}\} \supseteq \mathrm{X}\backslash\mathrm{K}$ and $\bigcup\{\mathrm{U}_x : x \in \mathrm{X}\backslash\mathrm{K}\} \subseteq \mathrm{X}\backslash\mathrm{K}$ since $\mathrm{U}_x \subseteq \mathrm{X}\backslash\mathrm{K}$.
Therefore, $\bigcup\{\mathrm{U}_x : x \in \mathrm{X}\backslash\mathrm{K}\} = \mathrm{X}\backslash\mathrm{K}$, so X\K is open and K is closed. □

Proof. Suppose K is closed.
Still unsure how to apply the lemma to the double implication...also unsure if I have the lemma correctly in order to even use it in the first place... □

Proposition 5. *Verify the discrete and indiscrete topologies are topologies*

Proof. Prove the discrete topology is an actual topology.

The discrete topology on a set X is $\tau = \mathcal{P}(\mathrm{X})$
$\emptyset$, X $\in \mathcal{P}(\mathrm{X})$, So $\emptyset$, X $\in \tau$
Let U $\in \tau$, or U $\subseteq \mathcal{P}(\mathrm{X})$, then the $\bigcup \mathrm{U} \in \mathcal{P}(\mathrm{X})$ or $\bigcup \mathrm{U} \in \tau$
Let U, V $\in \tau$, then the intersection of U, V is also in τ. □

Proof. Prove the indiscrete topology is an actual topology.

The indiscrete topology on a set X is given as $\tau = \{\emptyset, \mathrm{X}\}$.
Clearly, $\emptyset$, X $\in \tau$.
Let $\mathcal{U}$ be a collection of all open sets in X, then the $\bigcup \mathcal{U}$ is also $\in \tau$.
Let U, V be open sets in X, then U, V $\in \tau$.
Thus the intersection f open sets U, V is also $\in \tau$ □

Proposition 6. *Find a basis for the indiscrete topology on* $\mathbb{R}$.

Proof. Find a basis for the indiscrete topology on $\mathbb{R}$.

The indiscrete topology on $\mathbb{R}$ is given as $\tau = \{\emptyset, \mathbb{R}\}$
A basis on $\mathbb{R}$ is a collection $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$ s.t. for each $x \in \mathbb{R}\ \exists$ B $\in \mathcal{B} : x \in \mathcal{B}$.
Let $\mathcal{B} = \mathbb{R}$. Since $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$, for each $x \in \mathbb{R}$, there is a subset B $\in \mathcal{B} : x \in \mathcal{B}$.
Thus, $x \in \mathcal{B}$ and B is a basis on $\mathbb{R}$.
Furthermore, if two sets B_1, $\mathrm{B}_2 \in \mathcal{B}$, then their intersection is also in $\mathcal{B}$. Let $x \in \mathrm{B}_1$, B_2. Then $x \in \mathrm{B}_1 \cap \mathrm{B}_2$.
$\mathrm{B}_1 \cap \mathrm{B}_2 = \mathrm{B}_3$, so $\mathrm{B}_3 \in \mathcal{B}$, and $x \in \mathrm{B}_3 \subseteq \mathrm{B}_1 \cap \mathrm{B}_2$.
Thus B_3 is also a basis on $\mathbb{R}$. □