
Proposition 1. Find an infinite collection of open intervals in \mathbb{R} whose intersection is not an open interval.

Proof. $\bigcap \{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{Z}^+\} = \{0\}$ which is not an open interval or even open. \square

Proposition 2. Any finite union of closed sets is closed, and any arbitrary intersection of closed sets is closed.

Proof. We proceed by showing that any finite union of closed sets is closed:

Let C, D be closed sets.

Let A, B be compliments of C, D so A, B are open.

Then, $A \cap B$ is also open.

Thus, $X \setminus (A \cap B)$ is closed.

By Demorgan's Law, $X \setminus (A \cap B) = X \setminus A \cup X \setminus B = C \cup D$ which is closed.

Now that we've shown that $C \cup D$ is closed for all C, D ; Assume $C_1 \cup C_2 \cup \dots \cup C_n$ is closed for closed sets C_i . Then for $C_1 \cup \dots \cup C_n \cup C_{n+1}$, let $K = C_1 \cup \dots \cup C_n$.

Thus $K \cup C_{n+1} = C_1 \cup \dots \cup C_n \cup C_{n+1}$ is closed.

Now we show that any arbitrary intersection of closed sets is closed.

Let \mathcal{C} be a collection of closed sets.

Let $\mathcal{U} = \{X \setminus C : C \in \mathcal{C}\}$, so \mathcal{U} is a collection of open sets.

Then, $\bigcup \mathcal{U}$ is also open.

Thus, $X \setminus \bigcup \mathcal{U}$ is closed.

By Demorgan's Law, $X \setminus \bigcup \mathcal{U} = \bigcap \mathcal{C}$

Therefore, $\bigcap \mathcal{C}$ is closed. \square

Lemma 3. A set U is open if and only if for every point $x \in U$, there exists an open set U_x where $x \in U_x \subseteq U$

Proof. Suppose U is open, then for all $x \in U$ there exists an open $U_x = U$, such that $x \in U_x \subseteq U$. To show the converse, suppose that for each $x \in U$ there is an open set U_x where $x \in U_x \subseteq U$. For $x \in U, x \in U_x$ so $x \in \bigcup \{U_x : x \in U\}$. Thus $U \subseteq \bigcup \{U_x : x \in U\}$. Now let $y \in U_x$ for some $x \in U$, then $U_x \subseteq U$. Thus $U \supseteq \bigcup \{U_x : x \in U\}$. Therefore $U = \bigcup \{U_x : x \in U\}$. \square

Proposition 4. A set K in a topological space X is closed if and only if K contains all its limit points.

Proof. Suppose K contains all its limit points. If $K = X$, then K is closed because \emptyset is open. Otherwise let $x \in X \setminus K$, so x is not a limit point of K . Then, $\exists x \in U_x \in \tau$ such that $U_x \cap K = \emptyset$ since $x \notin K$. So, $x \in U_x \subseteq X \setminus K$. By the lemma, $X \setminus K$ is open so K is closed.

To show the converse is true as well, let $x \in X \setminus K$, which is an open set by the lemma. Since $(X \setminus K) \cap K$ is the empty set, x is not a limit point of K . So, if ℓ is any limit point of K , then $\ell \notin X \setminus K$, so $\ell \in K$ and K contains all of its limit points. \square

Proposition 5. Verify the discrete and indiscrete topologies are topologies

Proof. To prove the discrete topology is an actual topology, first we define that the discrete topology on a set X is $\tau = \mathcal{P}(X)$. $\emptyset, X \in \mathcal{P}(X)$, So $\emptyset, X \in \tau$. Now, let $\mathcal{U} \subseteq \tau = \mathcal{P}(X)$, then the $\bigcup \mathcal{U} \in \mathcal{P}(X)$. Let $U, V \in \tau = \mathcal{P}(X)$ then the intersection $U \cap V \in \mathcal{P}(X) = \tau$. Now to show that the indiscrete topology is an actual topology, we define that the indiscrete topology on a set X is given as $\tau = \{\emptyset, X\}$. Clearly, $\emptyset, X \in \tau$. Let \mathcal{U} be a collection of open sets in X , then $\bigcup \mathcal{U} = X$, thus $\bigcup \mathcal{U} \in \tau$. \square

Definition 6. A collection of sets $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a basis if:

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

The set $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$ is called the topology generated by \mathcal{B} .

Theorem 7. The “topology generated by \mathcal{B} ” is actually a topology.

Proof. τ is $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$. So $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} \subseteq \mathcal{P}(X)$. $\emptyset, X \in \tau$ because for $\mathcal{B}' = \emptyset, \bigcup \mathcal{B}' = \emptyset$ and for $\mathcal{B}' = X, \bigcup \mathcal{B}' = X$. Let $\mathcal{U} \subseteq \tau$ and $\bigcup \mathcal{U} \in \tau$ because for each $U \in \mathcal{U}, U = \bigcup \mathcal{B}'_U$ for some $\mathcal{B}'_U \subseteq \mathcal{B}$. So, $\bigcup \mathcal{U} = \bigcup \{\bigcup \mathcal{B}'_U : U \in \mathcal{U}\}$. Let $\mathcal{B}' = \bigcup \{\mathcal{B}'_U : U \in \mathcal{U}\}$. So, $\bigcup \mathcal{U} = \bigcup \mathcal{B}'$. For $U, V \in \{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}, U = B_1$ and $V = B_2, B_1, B_2 \in \mathcal{B}$ such that $B_1 = \mathcal{B}$ and $B_2 = \mathcal{B}$. Then $U \cap V \in \{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} \supseteq W$ so there exists $x \in W, U : x \in U \cap V \supseteq W$. Therefore $x \in W \subseteq U \cap V \in \tau$. \square

Theorem 8. Let τ be a topology on X . Then $\mathcal{B} \subseteq \tau$ generates τ if:

1. For all $x \in U \in \tau$, there exists $B \in \mathcal{B}$ where $x \in B \subseteq U \in \tau$
2. For all $B_1, B_2 \in \mathcal{B}$ with $x \in (B_1 \cap B_2)$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

Theorem 9. $\{\mathbb{X}\}$ is a basis for $\tau = \{\emptyset, \mathbb{X}\}$. (indiscrete)

Proof. :

1. Let $\mathcal{B} = \{\mathbb{X}\}$ and let $x \in \mathbb{X} = B \in \mathcal{B}$.
2. Now consider $B_1, B_2 \in \mathcal{B}$. $B_1 = \mathbb{X}$ and $B_2 = \mathbb{X}$ Let $x \in B_1 \cap B_2$, then $x \in B_3 = \mathbb{X} : B_1 \cap B_2 = \mathbb{X}$.

\square

Theorem 10. $\{\{x\} : x \in X\}$ is a basis for $\tau = \mathcal{P}(X) = \{U : U \subseteq X\}$. (discrete)

Proof. :

1. Let $x \in U \in \tau = \mathcal{P}(X)$. Then for $B = \{x\} \in \mathcal{B}, x \in B \subseteq U$.
2. Let $B_1, B_2 \in \mathcal{B}$. Let $y \in B_1 \cap B_2$ so $B_1 = B_2 = \{y\}$. Let $B_3 = \{y\}$ so $y \in B_3 = \{y\} \subseteq B_1 \cap B_2 = \{y\}$.

□

Definition 11. The Euclidean topology on \mathbb{R} is the topology generated by the basis $\{(a, b) : a < b \in \mathbb{R}\}$.

Theorem 12. $\{(a, b) : a < b \in \mathbb{R}\}$ is a basis.

Proof. :

1. Let $\mathcal{B} = \{(a, b) : a < b \in \mathbb{R}\}$. Let $x \in \mathbb{R}$ and let $B = (x - 1, x + 1)$, then $B \subseteq \mathbb{R}$ and $B \in \mathcal{B}$.
2. For $B_1 = (a_1, b_1)$, $B_2 = (a_2, b_2)$, $a_1 < b_1$, $a_2 < b_2 \in \mathbb{R}$ with $x \in B_1 \cap B_2$, then there is $B_3 = (\max(a_1, a_2), \min(b_1, b_2)) \subseteq B_1 \cap B_2$ such that $a_1, a_2 < x < b_1, b_2 \in B_3$.

□

Theorem 13. $\{(a, b) : a < b \in \mathbb{Q}\}$ is a basis for the Euclidean topology.

Proof. :

1. Let $\mathcal{B} = \{(a, b) : a < b \in \mathbb{Q}\}$. Let $x, y \in \mathbb{Q}$ and let $B = (x - y, x + y)$, so $B \subseteq \mathbb{Q}$ and $x \in B \in \mathcal{B}$.
2. Now, for $B_1, B_2 \in \mathcal{B}$, $\exists x \in B_1 \cap B_2 : x \in B_1$ and $x \in B_2$. $a_1, a_2 < b_1, b_2 \in \mathbb{Q}$. Let $B_1 = (a_1, b_1)$ and $B_2 = (a_2, b_2)$. Now let $B_3 = (\max(a_1, a_2), \min(b_1, b_2))$ with $a_1, a_2 < x < b_1, b_2$ so $x \in B_3$. So, $x \in B_3 \subseteq B_1 \cap B_2$.

□

Definition 14. A subset K of a topological space X is said to be **compact** if for every open cover \mathcal{U} of K (every collection \mathcal{U} of open sets such that $\bigcup \mathcal{U} = K$) there exists a finite subcollection $\mathcal{F} \subseteq \mathcal{U}$ that also covers K (that is, $\bigcup \mathcal{F} = K$).

Theorem 15. Any finite union of compact subsets is compact.

Proof. By definition if K_1 and K_2 are compact then for each \mathcal{U}_1 and \mathcal{U}_2 of K_1 and K_2 such that $\bigcup \mathcal{U}_1 = K_1$ and $\bigcup \mathcal{U}_2 = K_2$. So, $\bigcup (\mathcal{U}_1 \cup \mathcal{U}_2) = K_1 \cup K_2$. Let $K_3 = K_1 \cup K_2$ and $\mathcal{U}_3 = \mathcal{U}_1 \cup \mathcal{U}_2$. Then for K_3 there exists \mathcal{U}_3 such that $\bigcup \mathcal{U}_3 = K_3$.

Also by definition there must exist \mathcal{F}_1 and \mathcal{F}_2 such that $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{U}_1, \mathcal{U}_2$ respectively. Then, \mathcal{F}_1 and $\mathcal{F}_2 \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$. Let $\mathcal{F}_3 = \mathcal{F}_1 \cup \mathcal{F}_2$, then $\mathcal{F}_3 \subseteq \mathcal{U}_3$. Now observe that $\bigcup \mathcal{F}_1 = K_1$ and $\bigcup \mathcal{F}_2 = K_2$. The $\bigcup (\mathcal{F}_1 \cup \mathcal{F}_2) = K_1 \cup K_2$ and $\bigcup \mathcal{F}_3 = K_1 \cup K_2$ so $K_1 \cup K_2$ must be compact. □

Proposition 16. Any finite subset of a topological space is compact.

Proof. A subset K of space \mathbb{X} is compact if for every open cover \mathcal{U} of K there exists $\mathcal{F} \subseteq \mathcal{U}$ such that $\bigcup \mathcal{F} = K$.

Let $K \subseteq \mathbb{X}$, let $\mathcal{U} = \{(a, b) : a < b \in K\}$ then $\bigcup \mathcal{U} = K$. Let $\mathcal{F} = \{F : F = U \in \mathcal{U}\}$, then $\bigcup \mathcal{F} = \bigcup \mathcal{U} = K$, so $\bigcup \mathcal{F} = K$.

□

Proposition 17. *The subset $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ of \mathbb{R} is compact.*

Proof. Assume K is compact, then for all \mathcal{U} of K , $\bigcup \mathcal{U} = K$. Now we find $\mathcal{F} \subseteq \mathcal{U}$ that also covers K . For $\bigcup \mathcal{U} = K$, then $0, 1 \in \bigcup \mathcal{U}$. Let $\mathbb{A} = \bigcap \{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{Z}^+\}$ and let $\mathbb{C} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. So $\mathbb{A} \subseteq \mathcal{U}$ and $\mathbb{C} \subseteq \mathcal{U}$ so $\mathbb{A} \cup \mathbb{C} \subseteq \mathcal{U}$, let $\mathcal{F} = \mathbb{A} \cup \mathbb{C}$, then $\bigcup \mathcal{F} = \bigcup (\{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\})$ so $\bigcup \mathcal{F} = K$.

□

Proposition 18. *Any unbounded subset of \mathbb{R} is not compact.*

Proposition 19. *Any open interval $(a, b) \subseteq \mathbb{R}$ is not compact.*

Definition 20. A subset K of a topological space X is said to be **Lindelöf** if for every open cover \mathcal{U} of K (every collection \mathcal{U} of open sets such that $\bigcup \mathcal{U} = K$) there exists a countable subcollection $\mathcal{F} \subseteq \mathcal{U}$ that also covers K (that is, $\bigcup \mathcal{F} = K$).

Definition 21. A subset K of a topological space X is said to be **σ -compact** if there exist compact subspaces K_n of X for $n \in \mathbb{N}$ such that $K = \bigcup_{n \in \mathbb{N}} K_n$.

Definition 22. A subset K of a topological space X is said to be **hemicompact** if there exist compact subspaces K_n of X for $n \in \mathbb{N}$ such that $K = \bigcup_{n \in \mathbb{N}} K_n$, and for every compact subset $H \subseteq K$, $H \subseteq K_n$ for some $n \in \mathbb{N}$.