Proposition 1. *Find an infinite collection of open intervals in $\mathbb{R}$ whose intersection is not an open interval.*

Proof. $\bigcap\{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{Z}^+\} = \{0\}$ which is not an open interval or even open. $\square$

Proposition 2. *Any finite union of closed sets is closed, and any arbitrary intersection of closed sets is closed.*

Proof. We proceed by showing that any finite union of closed sets is closed:
Let C, D be closed sets.
Let A, B be compliments of C, D so A, B are open.
Then, A $\cap$ B is also open.
Thus, X\(A $\cap$ B) is closed.
By Demorgan's Law, X\(A $\cap$ B) = X\A $\cup$ X\B = C $\cup$ D which is closed.

Now that we've shown that C $\cup$ D is closed for all C, D; Assume $C_1 \cup C_2 \cup ... \cup C_n$ is closed for closed sets C_i. Then for $C_1 \cup ... \cup C_n \cup C_{n+1}$, let $K = C_1 \cup ... \cup C_n$.
Thus $K \cup C_{n+1} = C_1 \cup ... \cup C_n \cup C_{n+1}$ is closed.
Now we show that any arbitrary intersection of closed sets is closed.
Let $\mathcal{C}$ be a collection of closed sets.
Let $\mathcal{U} = \{X\backslash C : C \in \mathcal{C}\}$, so $\mathcal{U}$ is a collection of open sets.
Then, $\bigcup \mathcal{U}$ is also open.
Thus, $X\backslash \bigcup \mathcal{U}$ is closed.
By Demorgan's Law, $X\backslash \bigcup \mathcal{U} = \bigcap \mathcal{C}$
Therefore, $\bigcap \mathcal{C}$ is closed. $\square$

Lemma 3. *A set U is open if and only if for every point $x \in U$, there exists an open set U_x where $x \in U_x \subseteq U$*

Proof. Suppose U is open, then for all x $\in$ U there exists an open $U_x = U$, such that x $\in U_x \subseteq$ U. To show the converse, suppose that for each x $\in$ U there is an open set U_x where x $\in U_x \subseteq$ U. For $x \in U, x \in U_x$ so $x \in \bigcup\{U_x : x \in U\}$. Thus $U \subseteq \bigcup\{U_x : x \in U_x\}$. Now let y $\in U_x$ for some x $\in$ U, then $U_x \subseteq$ U. Thus $U \supseteq \bigcup\{ U_x :$ x $\in$ U $\}$. Therefore U $= \bigcup\{ U_x :$ x $\in$ U $\}$. $\square$

Proposition 4. *A set K in a topological space X is closed if and only if K contains all its limit points.*

Proof. Suppose K contains all its limit points. If K $=$ X, then K is closed because $\emptyset$ is open. Otherwise let x $\in$ X\K, so x is not a limit point of K. Then, $\exists \; x \in U_x \in \tau$ such that $U_x \cap$ K $= \emptyset$ since $x \notin K$. So, $x \in U_x \subseteq$ X\K. By the lemma, X\K is open so K is closed.

To show the converse is true as well, let x $\in$ X\K, which is an open set by the lemma. Since (X\K)$\cap$K is the empty set, x is not a limit point of K. So, if ℓ is any limit point of K, then $\ell \notin$ X\K, so $\ell \in$ K and K contains all of its limit points. $\square$

Proposition 5. *Verify the discrete and indiscrete topologies are topologies*

Proof. To prove the discrete topology is an actual topology, first we define that the discrete topology on a set X is $\tau = \mathcal{P}(X)$. $\emptyset, X \in \mathcal{P}(X)$, So $\emptyset, X \in \tau$. Now, let $\mathcal{U} \subseteq \tau = \mathcal{P}(X)$, then the $\bigcup U \in \mathcal{P}(X)$. Let $U, V \in \tau = \mathcal{P}(X)$ then the intersection $U \cap V \in \mathcal{P}(X) = \tau$. Now to show that the indiscrete topology is an actual topology, we define that the indiscrete topology on a set X is given as $\tau = \{\emptyset, X\}$. Clearly, $\emptyset, X \in \tau$. Let $\mathcal{U}$ be a collection of open sets in X, then $\bigcup \mathcal{U} = X$, thus $\bigcup \mathcal{U} \in \tau$. $\square$

Definition 6. A collection of sets $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a basis if:

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.

2. For all $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

The set $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$ is called the topology generated by $\mathcal{B}$.

Theorem 7. *The "topology generated by $\mathcal{B}$" is actually a topology.*

Proof. NEEDS WORK
τ is $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$. So $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} \subseteq \mathcal{P}(X)$. $\emptyset, X \in \tau$ because for $\mathcal{B}' = \emptyset, \bigcup \mathcal{B}' = \emptyset$ and for $\mathcal{B}' = X, \bigcup \mathcal{B}' = X$. Let $\mathcal{U} \subseteq \tau$ and $\bigcup \mathcal{U} \in \tau$ because for each $U \in \mathcal{U}, U = \bigcup \mathcal{B}'_U$ for some $\mathcal{B}'_U \subseteq \mathcal{B}$ So, $\bigcup \mathcal{U} = \bigcup \{\bigcup \mathcal{B}'_U : U \in \mathcal{U}\}$. Let $\mathcal{B}' = \bigcup \{\mathcal{B}'_U : U \in \mathcal{U}\}$. So, $\bigcup \mathcal{U} = \bigcup \mathcal{B}'$.
For $U, V \in \{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}, U = B_1$ and $V = B_2, B_1, B_2 \in \mathcal{B}$ such that $B_1 = \mathcal{B}$ and $B_2 = \mathcal{B}$. Then $U \cap V \in \{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} \supseteq W$ so there exists $x \in V, U : x \in U \cap V \supseteq W$. Therefore $x \in W \subseteq U \cap V \in \tau$. $\square$

Theorem 8. *Let τ be a topology on X. Then $\mathcal{B} \subseteq \tau$ generates τ if:*
1. For all $x \in U \in \tau$, there exists $B \in \mathcal{B}$ where $x \in B \subseteq U \in \tau$
2. For all $B_1, B_2 \in \mathcal{B}$ with $x \in (B_1 \cap B_2)$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

Proof. :

1. For $x \in \mathbb{X} \in \tau$, select $x \in B \in \mathcal{B}$, so $x \in \mathbb{X}$ and $x \in B$, then $x \in B \subseteq \mathbb{X} \in \tau$.

2. See definition 6-2.

3. Let $\mathcal{B}' \subseteq \mathcal{B}$ and $\mathcal{B}' = \mathcal{B}$,so $\bigcup \mathcal{B}' = \bigcup \mathcal{B} = \tau$.

$\square$

Theorem 9. $\{\mathbb{X}\}$ *is a basis for $\tau = \{\emptyset, \mathbb{X}\}$. (indiscrete)*

Proof. :

1. Let $\mathcal{B} = \{\mathbb{X}\}$ and let $x \in \mathbb{X} = B \in \mathcal{B}$.

2. Now consider $B_1, B_2 \in \mathcal{B}$. $B_1 = \mathbb{X}$ and $B_2 = \mathbb{X}$ Let $x \in B_1 \cap B_2$, then $x \in B_3 = \mathbb{X} : B_1 \cap B_2 = \mathbb{X}$.

$\square$

Theorem 10. $\{\{x\} : x \in \mathbb{X}\}$ *is a basis for* $\tau = \mathcal{P}(\mathbb{X}) = \{ U : U \subseteq \mathbb{X}\}$. *(discrete)*

Proof. :

1. Let $x \in U \in \tau = \mathcal{P}(\mathbb{X})$. Then for $B = \{x\} \in \mathcal{B}, x \in B \subseteq U$.

2. Let $B_1, B_2 \in \mathcal{B}$. Let $y \in B_1 \cap B_2$. Let $B_3 = \{y\}$ $y \in B_3 = \{y\} \subseteq B_1 \cap B_2$.

$\square$

Definition 11. The Euclidean topology on $\mathbb{R}$ is the topology generated by the basis $\{(a,b) : a < b \in \mathbb{R}\}$.

Theorem 12. $\{(a, b) : a < b \in \mathbb{R}\}$ *is a basis.*

Proof. :

1. Let $\mathcal{B} = \{(a,b) : a < b \in \mathbb{R}\}$. Let $x \in \mathbb{R}$ and let $B = (x-1, x+1)$, then $x \in B \subseteq \mathbb{R}$ and $B \in \mathcal{B}$.

2. For $B_1 = (a_1, b_1)$, $B_2 = (a_2, b_2)$, $a_1 < b_1$, $a_2 < b_2 \in \mathbb{R}$ with $x \in B_1 \cap B_2$, then there is $B_3 = (max(a_1, a_2), min(b_1, b_2)) \subseteq B_1 \cap B_2$ such that $a_1, a_2 < x < b_1, b_2$ so $x \in B_3$.

$\square$

Theorem 13. $\{(a, b) : a < b \in \mathbb{Q}\}$ *is a basis for the Euclidean topology.*

Proof. :

1. Let $\mathcal{B} = \{(a,b) : a < b \in \mathbb{Q}\}$ Let $x \in \mathbb{R}$ then, $\llcorner x \lrcorner - 1 < x < \ulcorner x \urcorner + 1$. Let B $= (\llcorner x \lrcorner - 1, \ulcorner x \urcorner + 1)$, so $x \in B \in \mathcal{B}$ and $B \subseteq \mathbb{Q}$.

2. Now, for $B_1, B_2 \in \mathcal{B}, \exists x \in B_1 \cap B_2 : x \in B_1$ and $x \in B_2$. $a_1, a_2 < b_1, b_2 \in \mathbb{Q}$. Let $B_1 = (a_1, b_1)$ and $B_2 = (a_2, b_2)$. Now let $B_3 = (max(a_1, a_2), min(b_1, b_2))$ with $a_1, a_2 < x < b_1, b_2$ so $x \in B_3$. So, $x \in B_3 \subseteq B_1 \cap B_2$.

$\square$

Definition 14. A subset K of a topological space X is said to be **compact** if for every open cover $\mathcal{U}$ of K (every collection $\mathcal{U}$ of open sets such that $\bigcup \mathcal{U} \supseteq K$) there exists a finite subcollection $\mathcal{F} \subseteq \mathcal{U}$ that also covers K (that is, $\bigcup \mathcal{F} \supseteq K$).

Theorem 15. *Any finite union of compact subsets is compact.*

Proof. K_1 and K_2 are compact, so let $\mathcal{U}_1$ and $\mathcal{U}_2$ be open covers of K_1 and K_2 respectively. So $\bigcup \mathcal{U}_1 \supseteq K_1$, and $\bigcup \mathcal{U}_2 \supseteq K_2$ and $\bigcup(\mathcal{U}_1 \cup \mathcal{U}_2) \supseteq K_1 \cup K_2$. $K_3 = K_1 \cup K_2$. There is a $\mathcal{U}_3 \supseteq K_3$. So let $\mathcal{U}_1 = \mathcal{U}_3$ and $\mathcal{U}_2 = \mathcal{U}_3$. For finite $\mathcal{F}_3 \subseteq \mathcal{U}_3, \mathcal{F}_3 \supseteq K_3$, so $\mathcal{F}_3 \supseteq K_1 \cup K_2$. Let $\mathcal{F}_1 = \mathcal{F}_3$ and $\mathcal{F}_2 = \mathcal{F}_3$, so $K_1 \cup K_2$ is compact. $\square$

Proposition 16. *Any finite subset of a topological space is compact.*

Proof. For $F = \{x_i : 0 \leq x_i < n\} \subseteq \mathbb{X}$, let $\mathcal{U}$ be an open covering on F so $\bigcup \mathcal{U} \supseteq F$. Then we can find a finite $\mathcal{F} \subseteq \mathcal{U}$ such that $\bigcup \mathcal{F} \supseteq F$. Suppose $\mathcal{U}_0 = \{x_0 : -1 \leq x_0 < n+1\} \subseteq \mathcal{U}$ then let $\mathcal{F} = \mathcal{U}_0$, so $\bigcup \mathcal{F} \supseteq F$. $\square$

Proposition 17. *The subset $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ of $\mathbb{R}$ is compact.*

Proof. Let $K = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\} \subseteq \mathbb{R}$ and let $\mathcal{U}$ be an open covering on K such that $\bigcup \mathcal{U} \supseteq K$. So for $\mathcal{U}_0 \subseteq \mathcal{U}, \mathcal{U}_0 = \{n_0 : -1 \leq n_0 \leq 2, n_0 \in \mathbb{Z}\}$. $\bigcup \mathcal{U}_0 \supseteq K$. So, let finite $\mathcal{F} = \mathcal{U}_0$, so $\mathcal{F} \subseteq \mathcal{U}$ and $\mathcal{F} \supseteq K$. $\square$

Proposition 18. *Any unbounded subset of $\mathbb{R}$ is not compact.*

Proof. For unbounded $K \subseteq \mathbb{R}$ suppose there is an open covering $\mathcal{U}$ on K. Then there exists a finite $\mathcal{F} \subseteq \mathcal{U}$ such that $\bigcup \mathcal{F} \supseteq K$. For $\mathcal{F}$ to be finite it must be bounded. So, there is no $\bigcup \mathcal{F} \supseteq K$. Essentially we would need $\mathcal{F} = \{(-\infty, n) : n \in \mathbb{Z}^-\} \cup \{(n-3, \infty) : n \in \mathbb{Z}^+\}$ which is not finite. $\square$

Proposition 19. *Any open interval $(a, b) \subseteq \mathbb{R}$ is not compact.*

Proof. $\square$

Definition 20. A subset K of a topological space X is said to be **Lindelöf** if for every open cover $\mathcal{U}$ of K (every collection $\mathcal{U}$ of open sets such that $\bigcup \mathcal{U} \supseteq K$) there exists a countable subcollection $\mathcal{F} \subseteq \mathcal{U}$ that also covers K (that is, $\bigcup \mathcal{F} \supseteq K$).

Definition 21. A subset K of a topological space X is said to be σ**-compact** if there exist compact subspaces K_n of X for $n \in \mathbb{N}$ such that $K = \bigcup_{n \in \mathbb{N}} K_n$.

Theorem 22. *Every σ-compact subset of a topological space is Lindelöf.*

Proof. $\square$

Definition 23. A subset K of a topological space X is said to be **hemicompact** if there exist compact subspaces K_n of X for $n \in \mathbb{N}$ such that $K = \bigcup_{n \in \mathbb{N}} K_n$, and for every compact subset $H \subseteq K$, $H \subseteq K_n$ for some $n \in \mathbb{N}$.

Theorem 24. *Every hemicompact subset of a topological space is σ-compact.*

Proof. This proof is trivial as it is defined as stated here in definition 21. $\square$