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**Proposition 1.** Find an infinite collection of open intervals in  $\mathbb{R}$  whose intersection is not an open interval.

*Proof.*  $\bigcap\{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{Z}^+\} = \{0\}$  which is not an open interval or even open.  $\square$

**Proposition 2.** Any finite union of closed sets is closed, and any arbitrary intersection of closed sets is closed.

*Proof.* We proceed by showing that any finite union of closed sets is closed:

Let  $C, D$  be closed sets.

Let  $A, B$  be compliments of  $C, D$  so  $A, B$  are open.

Then,  $A \cap B$  is also open.

Thus,  $X \setminus (A \cap B)$  is closed.

By Demorgan's Law,  $X \setminus (A \cap B) = X \setminus A \cup X \setminus B = C \cup D$  which is closed.

Now that we've shown that  $C \cup D$  is closed for all  $C, D$ ; Assume  $C_1 \cup C_2 \cup \dots \cup C_n$  is closed for closed sets  $C_i$ . Then for  $C_1 \cup \dots \cup C_n \cup C_{n+1}$ , let  $K = C_1 \cup \dots \cup C_n$ .

Thus  $K \cup C_{n+1} = C_1 \cup \dots \cup C_n \cup C_{n+1}$  is closed.

Now we show that any arbitrary intersection of closed sets is closed.

Let  $\mathcal{C}$  be a collection of closed sets.

Let  $\mathcal{U} = \{X \setminus C : C \in \mathcal{C}\}$ , so  $\mathcal{U}$  is a collection of open sets.

Then,  $\bigcup \mathcal{U}$  is also open.

Thus,  $X \setminus \bigcup \mathcal{U}$  is closed.

By Demorgan's Law,  $X \setminus \bigcup \mathcal{U} = \bigcap \mathcal{C}$

Therefore,  $\bigcap \mathcal{C}$  is closed.  $\square$

**Lemma 3.** A set  $U$  is open if and only if for every point  $x \in U$ , there exists an open set  $U_x$  where  $x \in U_x \subseteq U$

*Proof.* Suppose  $U$  is open, then for all  $x \in U$  there exists an open  $U_x = U$ , such that  $x \in U_x \subseteq U$ . To show the converse, suppose that for each  $x \in U$  there is an open set  $U_x$  where  $x \in U_x \subseteq U$ . For  $x \in U, x \in U_x$  so  $x \in \bigcup\{U_x : x \in U\}$ . Thus  $U \subseteq \bigcup\{U_x : x \in U\}$ . Now let  $y \in U_x$  for some  $x \in U$ , then  $U_x \subseteq U$ . Thus  $U \supseteq \bigcup\{U_x : x \in U\}$ . Therefore  $U = \bigcup\{U_x : x \in U\}$ .  $\square$

**Proposition 4.** A set  $K$  in a topological space  $X$  is closed if and only if  $K$  contains all its limit points.

*Proof.* Suppose  $K$  contains all its limit points. If  $K = X$ , then  $K$  is closed because  $\emptyset$  is open. Otherwise let  $x \in X \setminus K$ , so  $x$  is not a limit point of  $K$ . Then,  $\exists x \in U_x \in \tau$  such that  $U_x \cap K = \emptyset$  since  $x \notin K$ . So,  $x \in U_x \subseteq X \setminus K$ . By the lemma,  $X \setminus K$  is open so  $K$  is closed.

To show the converse is true as well, let  $x \in X \setminus K$ , which is an open set by the lemma. Since  $(X \setminus K) \cap K$  is the empty set,  $x$  is not a limit point of  $K$ . So, if  $\ell$  is any limit point of  $K$ , then  $\ell \notin X \setminus K$ , so  $\ell \in K$  and  $K$  contains all of its limit points.  $\square$

**Proposition 5.** Verify the discrete and indiscrete topologies are topologies

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*Proof.* To prove the discrete topology is an actual topology, first we define that the discrete topology on a set  $X$  is  $\tau = \mathcal{P}(X)$ .  $\emptyset, X \in \mathcal{P}(X)$ , So  $\emptyset, X \in \tau$ . Now, let  $\mathcal{U} \subseteq \tau = \mathcal{P}(X)$ , then the  $\bigcup \mathcal{U} \in \mathcal{P}(X)$ . Let  $U, V \in \tau = \mathcal{P}(X)$  then the intersection  $U \cap V \in \mathcal{P}(X) = \tau$ . Now to show that the indiscrete topology is an actual topology, we define that the indiscrete topology on a set  $X$  is given as  $\tau = \{\emptyset, X\}$ . Clearly,  $\emptyset, X \in \tau$ . Let  $\mathcal{U}$  be a collection of open sets in  $X$ , then  $\bigcup \mathcal{U} = X$ , thus  $\bigcup \mathcal{U} \in \tau$ .  $\square$

**Definition 6.** A collection of sets  $\mathcal{B} \subseteq \mathcal{P}(X)$  is called a basis if:

1. For all  $x \in X$ , there exists  $B \in \mathcal{B}$  such that  $x \in B$ .
2. For all  $B_1, B_2 \in \mathcal{B}$  with  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  with  $x \in B_3 \subseteq B_1 \cap B_2$ .

The set  $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$  is called the topology generated by  $\mathcal{B}$ .

**Theorem 7.** The “topology generated by  $\mathcal{B}$ ” is actually a topology.

*Proof.*  $\tau$  is  $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$ . So  $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} \subseteq \mathcal{P}(X)$ .  $\emptyset, X \in \tau$  because for  $\mathcal{B}' = \emptyset, \bigcup \mathcal{B}' = \emptyset$  and for  $\mathcal{B}' = X, \bigcup \mathcal{B}' = X$ . Let  $\mathcal{U} \subseteq \tau$  and  $\bigcup \mathcal{U} \in \tau$  because for each  $U \in \mathcal{U}, U = \bigcup \mathcal{B}'_U$  for some  $\mathcal{B}'_U \subseteq \mathcal{B}$ . So,  $\bigcup \mathcal{U} = \bigcup \{\bigcup \mathcal{B}'_U : U \in \mathcal{U}\}$ . Let  $\mathcal{B}' = \bigcup \{\mathcal{B}'_U : U \in \mathcal{U}\}$ . So,  $\bigcup \mathcal{U} = \bigcup \mathcal{B}'$ . For  $U, V \in \{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}, U = B_1$  and  $V = B_2, B_1, B_2 \in \mathcal{B}$  such that  $B_1 = \bigcup \mathcal{B}'$  and  $B_2 = \bigcup \mathcal{B}''$ . Then  $U \cap V \in \{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} \supseteq W$  so there exists  $x \in W, U : x \in U \cap V \supseteq W$ . Therefore  $x \in W \subseteq U \cap V \in \tau$ .  $\square$

**Theorem 8.** Let  $\tau$  be a topology on  $X$ . Then  $\mathcal{B} \subseteq \tau$  generates  $\tau$  if:

1. For all  $x \in U \in \tau$ , there exists  $B \in \mathcal{B}$  where  $x \in B \subseteq U \in \tau$
2. For all  $B_1, B_2 \in \mathcal{B}$  with  $x \in (B_1 \cap B_2)$ , there exists  $B_3 \in \mathcal{B}$  with  $x \in B_3 \subseteq B_1 \cap B_2$ .

**Theorem 9.**  $\{\mathbb{X}\}$  is a basis for  $\tau = \{\emptyset, \mathbb{X}\}$ . (indiscrete)

*Proof.* Let  $\mathcal{B} = \{\mathbb{X}\}$  and let  $x \in \mathbb{X} = B \in \mathcal{B}$ . Now consider  $B_1, B_2 \in \mathcal{B}$ .  $B_1 = \mathbb{X}$  and  $B_2 = \mathbb{X}$  Let  $x \in B_1 \cap B_2$ , then  $x \in B_3 = \mathbb{X} : B_1 \cap B_2 = \mathbb{X}$ .  $\square$

**Theorem 10.**  $\{\{x\} : x \in X\}$  is a basis for  $\tau = \mathcal{P}(X) = \{U : U \subseteq X\}$ . (discrete)

*Proof.* Let  $x \in U \in \tau = \mathcal{P}(X)$ . Then for  $B = \{x\} \in \mathcal{B}, x \in B \subseteq U$ . Let  $B_1, B_2 \in \mathcal{B}$ . Let  $y \in B_1 \cap B_2$  so  $B_1 = B_2 = \{y\}$ . Let  $B_3 = \{y\}$  so  $y \in B_3 = \{y\} \subseteq B_1 \cap B_2 = \{y\}$ .  $\square$

**Definition 11.** The Euclidean topology on  $\mathbb{R}$  is the topology generated by the basis  $\{(a, b) : a < b \in \mathbb{R}\}$ .

**Theorem 12.**  $\{(a, b) : a < b \in \mathbb{R}\}$  is a basis.

*Proof.* Let  $\mathcal{B} = \{(a, b) : a < b \in \mathbb{R}\}$ . Let  $x \in \mathbb{R}$  and let  $B = (x - 1, x + 1)$ , then  $B \subseteq \mathbb{R}$  and  $B \in \mathcal{B}$ . For  $B_1 = (a_1, b_1), B_2 = (a_2, b_2), a_1 < b_1, a_2 < b_2 \in \mathbb{R}$  with  $x \in B_1 \cap B_2$ , then there is  $B_3 = (\max(a_1, a_2), \min(b_1, b_2)) \subseteq B_1 \cap B_2$  such that  $a_1, a_2 < x < b_1, b_2 \in B_3$ .  $\square$

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**Theorem 13.**  $\{(a, b) : a < b \in \mathbb{Q}\}$  is a basis for the Euclidean topology.

*Proof.* Let  $\mathcal{B} = \{(a, b) : a < b \in \mathbb{Q}\}$ . Let  $x, y \in \mathbb{Q}$  and let  $B = (x-y, x+y)$ , so  $B \subseteq \mathbb{Q}$  and  $x \in B \in \mathcal{B}$ . Now, for  $B_1, B_2 \in \mathcal{B}$ ,  $\exists x \in B_1 \cap B_2 : x \in B_1$  and  $x \in B_2$ .  $a_1, a_2 < b_1, b_2 \in \mathbb{Q}$ . Let  $B_1 = (a_1, b_1)$  and  $B_2 = (a_2, b_2)$ . Now let  $B_3 = (\max(a_1, a_2), \min(b_1, b_2))$  with  $a_1, a_2 < x < b_1, b_2 \in \mathcal{B}$ . So,  $x \in B_3 \subseteq B_1 \cap B_2$ .  $\square$

**Definition 14.** A subset  $K$  of a topological space  $X$  is said to be **compact** if for every open cover  $\mathcal{U}$  of  $K$  (every collection  $\mathcal{U}$  of open sets such that  $\bigcup \mathcal{U} = K$ ) there exists a finite subcollection  $\mathcal{F} \subseteq \mathcal{U}$  that also covers  $K$  (that is,  $\bigcup \mathcal{F} = K$ ).

**Theorem 15.** Any finite union of compact subsets is compact.

*Proof.* By definition if  $K_1$  and  $K_2$  are compact then for each  $\mathcal{U}_1$  and  $\mathcal{U}_2$  of  $K_1$  and  $K_2$  such that  $\bigcup \mathcal{U}_1 = K_1$  and  $\bigcup \mathcal{U}_2 = K_2$ . So,  $\bigcup (\mathcal{U}_1 \cup \mathcal{U}_2) = K_1 \cup K_2$ . Let  $K_3 = K_1 \cup K_2$  and  $\mathcal{U}_3 = \mathcal{U}_1 \cup \mathcal{U}_2$ . Then for  $K_3$  there exists  $\mathcal{U}_3$  such that  $\bigcup \mathcal{U}_3 = K_3$ .

Also by definition there must exist  $\mathcal{F}_1$  and  $\mathcal{F}_2$  such that  $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{U}_1, \mathcal{U}_2$  respectively. Then,  $\mathcal{F}_1$  and  $\mathcal{F}_2 \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$ . Let  $\mathcal{F}_3 = \mathcal{F}_1 \cup \mathcal{F}_2$ , then  $\mathcal{F}_3 \subseteq \mathcal{U}_3$ . Now observe that  $\bigcup \mathcal{F}_1 = K_1$  and  $\bigcup \mathcal{F}_2 = K_2$ . The  $\bigcup (\mathcal{F}_1 \cup \mathcal{F}_2) = K_1 \cup K_2$  and  $\bigcup \mathcal{F}_3 = K_1 \cup K_2$  so  $K_1 \cup K_2$  must be compact.  $\square$

**Proposition 16.** Any finite subset of a topological space is compact.

*Proof.* A subset  $K$  of space  $\mathbb{X}$  is compact if for every open cover  $\mathcal{U}$  of  $K$  there exists  $\mathcal{F} \subseteq \mathcal{U}$  such that  $\bigcup \mathcal{F} = K$ .

Let  $K \subseteq \mathbb{X}$ , let  $\mathcal{U} = \{(a, b) : a < b \in K\}$  then  $\bigcup \mathcal{U} = K$ . Let  $\mathcal{F} = \{F : F = U \in \mathcal{U}\}$ , then  $\bigcup \mathcal{F} = \bigcup \mathcal{U} = K$ , so  $\bigcup \mathcal{F} = K$ .  $\square$

**Proposition 17.** The subset  $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\}$  of  $\mathbb{R}$  is compact.

*Proof.* Assume  $K$  is compact, then for all  $\mathcal{U}$  of  $K$ ,  $\bigcup \mathcal{U} = K$ . Now we find  $\mathcal{F} \subseteq \mathcal{U}$  that also covers  $K$ . For  $\bigcup \mathcal{U} = K$ , then  $0, 1 \in \bigcup \mathcal{U}$ . Let  $\mathbb{A} = \bigcap \{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{Z}^+\}$  and let  $\mathbb{C} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ . So  $\mathbb{A} \subseteq \mathcal{U}$  and  $\mathbb{C} \subseteq \mathcal{U}$  so  $\mathbb{A} \cup \mathbb{C} \subseteq \mathcal{U}$ , let  $\mathcal{F} = \mathbb{A} \cup \mathbb{C}$ , then  $\bigcup \mathcal{F} = \bigcup (\{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\})$  so  $\bigcup \mathcal{F} = K$ .  $\square$

**Proposition 18.** Any unbounded subset of  $\mathbb{R}$  is not compact.

**Proposition 19.** Any open interval  $(a, b) \subseteq \mathbb{R}$  is not compact.

**Definition 20.** A subset  $K$  of a topological space  $X$  is said to be **Lindelöf** if for every open cover  $\mathcal{U}$  of  $K$  (every collection  $\mathcal{U}$  of open sets such that  $\bigcup \mathcal{U} = K$ ) there exists a countable subcollection  $\mathcal{F} \subseteq \mathcal{U}$  that also covers  $K$  (that is,  $\bigcup \mathcal{F} = K$ ).

**Definition 21.** A subset  $K$  of a topological space  $X$  is said to be  **$\sigma$ -compact** if there exist compact subspaces  $K_n$  of  $X$  for  $n \in \mathbb{N}$  such that  $K = \bigcup_{n \in \mathbb{N}} K_n$ .

**Definition 22.** A subset  $K$  of a topological space  $X$  is said to be **hemicompact** if there exist compact subspaces  $K_n$  of  $X$  for  $n \in \mathbb{N}$  such that  $K = \bigcup_{n \in \mathbb{N}} K_n$ , and for every compact subset  $H \subseteq K$ ,  $H \subseteq K_n$  for some  $n \in \mathbb{N}$ .