
Proposition 1. Find an infinite collection of open intervals in \mathbb{R} whose intersection is not an open interval.

Proof. $\bigcap\{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{Z}^+\} = \{0\}$ which is not an open interval or even open. \square

Proposition 2. Any finite union of closed sets is closed, and any arbitrary intersection of closed sets is closed.

Proof. We proceed by showing that any finite union of closed sets is closed:

Let C, D be closed sets.

Let A, B be compliments of C, D so A, B are open.

Then, $A \cap B$ is also open.

Thus, $X \setminus (A \cap B)$ is closed.

By Demorgan's Law, $X \setminus (A \cap B) = X \setminus A \cup X \setminus B = C \cup D$ which is closed.

Now that we've shown that $C \cup D$ is closed for all C, D ; Assume $C_1 \cup C_2 \cup \dots \cup C_n$ is closed for closed sets C_i . Then for $C_1 \cup \dots \cup C_n \cup C_{n+1}$, let $K = C_1 \cup \dots \cup C_n$.

Thus $K \cup C_{n+1} = C_1 \cup \dots \cup C_n \cup C_{n+1}$ is closed.

Now we show that any arbitrary intersection of closed sets is closed.

Let \mathcal{C} be a collection of closed sets.

Let $\mathcal{U} = \{X \setminus C : C \in \mathcal{C}\}$, so \mathcal{U} is a collection of open sets.

Then, $\bigcup \mathcal{U}$ is also open.

Thus, $X \setminus \bigcup \mathcal{U}$ is closed.

By Demorgan's Law, $X \setminus \bigcup \mathcal{U} = \bigcap \mathcal{C}$

Therefore, $\bigcap \mathcal{C}$ is closed. \square

Lemma 3. A set U is open if and only if for every point $x \in U$, there exists an open set U_x where $x \in U_x \subseteq U$

Proof. Suppose U is open, then for all $x \in U$ there exists an open $U_x = U$, such that $x \in U_x \subseteq U$. To show the converse, suppose that for each $x \in U$ there is an open set U_x where $x \in U_x \subseteq U$. For $x \in U, x \in U_x$ so $x \in \bigcup \{U_x : x \in U\}$. Thus $U \subseteq \bigcup \{U_x : x \in U\}$. Now let $y \in U_x$ for some $x \in U$, then $U_x \subseteq U$. Thus $U \supseteq \bigcup \{U_x : x \in U\}$. Therefore $U = \bigcup \{U_x : x \in U\}$. \square

Proposition 4. A set K in a topological space X is closed if and only if K contains all its limit points.

Proof. Suppose K contains all its limit points. If $K = X$, then K is closed because \emptyset is open. Otherwise let $x \in X \setminus K$, so x is not a limit point of K . Then, $\exists x \in U_x \in \tau$ such that $U_x \cap K = \emptyset$ since $x \notin K$. So, $x \in U_x \subseteq X \setminus K$. By the lemma, $X \setminus K$ is open so K is closed.

To show the converse is true as well, let $x \in X \setminus K$, which is an open set by the lemma. Since $(X \setminus K) \cap K$ is the empty set, x is not a limit point of K . So, if ℓ is any limit point of K , then $\ell \notin X \setminus K$, so $\ell \in K$ and K contains all of its limit points. \square

Proposition 5. Verify the discrete and indiscrete topologies are topologies

Proof. To prove the discrete topology is an actual topology, first we define that the discrete topology on a set X is $\tau = \mathcal{P}(X)$. $\emptyset, X \in \mathcal{P}(X)$, So $\emptyset, X \in \tau$. Now, let $\mathcal{U} \subseteq \tau = \mathcal{P}(X)$, then the $\bigcup \mathcal{U} \in \mathcal{P}(X)$. Let $U, V \in \tau = \mathcal{P}(X)$ then the intersection $U \cap V \in \mathcal{P}(X) = \tau$. Now to show that the indiscrete topology is an actual topology, we define that the indiscrete topology on a set X is given as $\tau = \{\emptyset, X\}$. Clearly, $\emptyset, X \in \tau$. Let \mathcal{U} be a collection of open sets in X , then $\bigcup \mathcal{U} = X$, thus $\bigcup \mathcal{U} \in \tau$. \square

Definition 6. A collection of sets $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a basis if:

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

The set $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$ is called the topology generated by \mathcal{B} .

Theorem 7. The “topology generated by \mathcal{B} ” is actually a topology.

Theorem 8. Let τ be a topology on X . Then $\mathcal{B} \subseteq \tau$ generates τ if:

1. For all $x \in U \in \tau$, there exists $B \in \mathcal{B}$ where $x \in B \subseteq U \in \tau$
2. For all $B_1, B_2 \in \mathcal{B}$ with $x \in (B_1 \cap B_2)$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

Theorem 9. If \mathcal{B} is a basis for τ , then $\tau = \{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$. (Every open set is a union of basic sets)

Proof. Let $\mathcal{B} \subseteq \tau$, then for any $\mathcal{B}' \subseteq \mathcal{B}$, $\mathcal{B}' \in \tau$. Thus $\tau = \{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$. \square

Theorem 10. $\{\mathbb{X}\}$ is a basis for $\tau = \{\emptyset, \mathbb{X}\}$. (indiscrete)

Proof. Let $\mathcal{B} = \mathbb{X}$ and let $x \in \mathbb{X}$ so $x \in \mathcal{B}$. Then there is $B \subseteq \mathcal{B}$ such that $x \in B$. Now consider $B_1, B_2 \in \mathcal{B}$. $B_1 = \mathbb{X}$ and $B_2 = \mathbb{X}$ Let $x \in B_1 \cap B_2$ and let $B_3 \subseteq B_1 \cap B_2$, then $x \in B_3 = \mathbb{X}$ \square

Theorem 11. $\{\{x\} : x \in X\}$ is a basis for $\tau = \mathcal{P}(X) = \{U : U \subseteq X\}$. (discrete)

Proof. Let $x \in U \in \tau = \mathcal{P}(X)$. Then for $B = \{x\} \in \mathcal{B}$, $x \in B \subseteq U$. Let $B_1, B_2 \in \mathcal{B}$. Then $B_1 = \{x\}$ and $B_2 = \{x\}$. Let $y \in B_1 \cap B_2$ and let $B_3 \subseteq B_1 \cap B_2$, then $y \in B_3 = \{x\}$ so $\{y\} = \{x\}$. \square

Definition 12. The Euclidean topology on \mathbb{R} is the topology generated by the basis $\{(a, b) : a < b \in \mathbb{R}\}$.

Theorem 13. $\{(a, b) : a < b \in \mathbb{R}\}$ is a basis.

Proof. Let $\mathcal{B} = \{(a, b) : a < b \in \mathbb{R}\}$ and let $(a, b) \in \mathbb{R}$ so $(a, b) \in \mathcal{B}$. Then there is $B \subseteq \mathcal{B}$ such that $(a, b) \in B$. Now consider $B_1, B_2 \in \mathcal{B}$. $B_1 = \mathcal{B}$ and $B_2 = \mathcal{B}$. Let $(a, b) \in B_1 \cap B_2$ and let $B_3 \subseteq B_1 \cap B_2$, then $(a, b) \in B_3 = \mathcal{B}$. \square

Theorem 14. $\{(a, b) : a < b \in \mathbb{Q}\}$ is a basis for the Euclidean topology.

Proof. By the same argument as above this also holds true because $\{(a, b) : a < b \in \mathbb{Q}\}$ also contains all of the $(a', b') \in \mathbb{R}$. \square