**Proposition 1.** *Puzzle 1: Find an infinite collection of open intervals in $\mathbb{R}$ whose intersection is not an open interval.*

*Proof.* $\bigcap\{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{Z}^+\} = \{0\}$ which is not an open interval or open. □

**Proposition 2.** *Any finite union of closed sets is closed, and any arbitrary intersection of closed sets is closed.*

*Proof.* Proposition 1 - We proceed by showing that any finite union of closed sets is closed:
Let C, D be closed sets.
Let A, B be compliments of C, D so A, B are open.
Then, A ∩ B is also open.
Thus, X\(A ∩ B) is closed.
By Demorgan's Law, X\(A ∩ B) = X\A ∪ X\B = C ∪ D which is closed.

Now that we've shown that C ∪ D is closed for all C, D; Assume $C_1 \cup C_2 \cup ... \cup C_n$ is closed for closed sets $C_i$. Then for $C_1 \cup ... \cup C_n \cup C_{n+1}$, let $K = C_1 \cup ... \cup C_n$. Thus $K \cup C_{n+1} = C_1 \cup ... \cup C_n \cup C_{n+1}$ is closed.
Now we show that any arbitrary intersection of closed sets is closed.
Let $\mathcal{C}$ be a collection of closed sets.
Let $\mathcal{U} = \{X \backslash C : C \in \mathcal{C}\}$, so $\mathcal{U}$ is a collection of open sets.
Then, $\bigcup \mathcal{U}$ is also open.
Thus, $X \backslash \bigcup \mathcal{U}$ is closed.
By Demorgan's Law, $X \backslash \bigcup \mathcal{U} = \bigcap \mathcal{C}$
Therefore, $\bigcap \mathcal{C}$ is closed. □

**Lemma 3.** *A set U is open if and only if for every point $x \in U$, there exists an open set $U_x$ where $x \in U_x \subseteq U$*

*Proof.* Suppose U is open, then for all x ∈ U there exists an open $U_x = U$, such that $x \in U_x \subseteq U$. To show the converse, for each x ∈ U there is an open set $U_x$ where $x \in U_x \subseteq U$. To show that $U = \bigcup\{ U_x : x \in U \}$, let x ∈ U, then for each $U_x$, $U \subseteq U_x$. Thus $U \subseteq \bigcup\{ U_x : x \in U \}$. Now let $y \in U_x$ for some x ∈ U, then $U_x \subseteq U$. Thus $U \supseteq \bigcup\{ U_x : x \in U \}$. Therefore $U = \bigcup\{ U_x : x \in U \}$. □

**Proposition 4.** *A set K in a topological space X is closed if and only if K contains all its limit points.*

*Proof.* Suppose K contains all its limit points. If K = X, then K is closed because ∅ is open. Otherwise let x ∈ X\K, so x is not a limit point of K. Then, $\exists\ x \in U_x \in \tau$ such that $U_x \cap K = \emptyset$. So, $x \in U_x \subseteq X \backslash K$. By the lemma, X\K is open so K is closed.

To show the converse is true as well, let x ∈ X\K, which is an open set by the lemma. Since (X\K)∩K is the empty set, x is not a limit point of K. So, if $\ell$ is any limit point of K, then $\ell \notin X \backslash K$, so $\ell \in K$ and K contains all of its limit points. □

**Proposition 5.** *Verify the discrete and indiscrete topologies are topologies*

*Proof.* To prove the discrete topology is an actual topology, first we define that the discrete topology on a set X is $\tau = \mathcal{P}(\mathrm{X})$. $\emptyset$, X $\in \mathcal{P}(\mathrm{X})$, So $\emptyset$, X $\in \tau$. Now, let $\mathcal{U} \subseteq \tau = \mathcal{P}(\mathrm{X})$, then the $\bigcup \mathrm{U} \in \mathcal{P}(\mathrm{X})$. Let U, V $\in \tau = \mathcal{P}(\mathrm{X})$ then the intersection U $\cap$ V $\in \mathcal{P}(\mathrm{X}) = \tau$. Now to show that the indiscrete topology is an actual topology, we define that the indiscrete topology on a set X is given as $\tau = \{\emptyset$, X$\}$. Clearly, $\emptyset$, X $\in \tau$. Let $\mathcal{U}$ be a collection of open sets in X, then $\bigcup \mathcal{U}$ = X, thus $\bigcup \mathcal{U} \in \tau$. □

**Definition 6.** Let $\tau$ be a topology on X. Then $\mathcal{B} \subseteq \tau$ is a basis for $\tau$ if:
1. For all x $\in$ U $\in \tau$, there exists B $\in \mathcal{B}$ where x $\in$ B $\subseteq$ U $\in \tau$
2. For all $\mathrm{B}_1$, $\mathrm{B}_2 \in \mathcal{B}$ with x $\in$ ($\mathrm{B}_1 \cap$ $\mathrm{B}_2$), there exists $\mathrm{B}_3 \in \mathcal{B}$ with x $\in \mathrm{B}_3 \subseteq \mathrm{B}_1 \cap$ $\mathrm{B}_2$.

**Theorem 7.** *If $\mathcal{B}$ is a basis for $\tau$, then $\tau = \{\bigcup \mathcal{B}' : \mathcal{B}' \in \mathcal{B}\}$. (Every open set is a union of basic sets)*

*Proof.* Let $\mathcal{B} \subseteq \tau$, then for any $\mathcal{B}' \in \mathcal{B}$, $\mathcal{B}' \in \tau$. Furthermore the $\bigcup \mathcal{B}' = \mathcal{B}$, then $\bigcup \mathcal{B}' \subseteq \tau$ and $\bigcup \mathcal{B}'$ $= \tau$. □

**Theorem 8.** $\{\mathbb{X}\}$ *is a basis for* $\tau = \{\emptyset, \mathbb{X}\}$. *(indiscrete)*

*Proof.* $\mathbb{X} \in \{\emptyset, \mathbb{X}\} = \tau$. First, let U $= \mathbb{X} \in \tau$, and let x $\in$ U $\in \tau$, then for all x $\in$ U, x $\in \mathbb{X}$, and for some $\mathrm{X}_n \in \mathbb{X}$, x $\in \mathrm{X}_n$. Thus, x $\in \mathrm{X}_n \subseteq$ U $\in \tau$. Now, we let x $\in \mathrm{X}_1$, $\mathrm{X}_2$. Then x $\in$ ($\mathrm{X}_1 \cap \mathrm{X}_2$). Let $\mathrm{X}_3 \subseteq (\mathrm{X}_1 \cap \mathrm{X}_2)$ such that $\mathrm{X}_3 = (\mathrm{X}_1 \cap \mathrm{X}_2)$ then x $\in \mathrm{X}_3$. Thus, x $\in \mathrm{X}_3 \subseteq (\mathrm{X}_1 \cap \mathrm{X}_2) \in \mathbb{X}$. Then $\{\mathbb{X}\}$ is a basis for $\tau = \{\emptyset, \mathbb{X}\}$. □

**Theorem 9.** $\{\{X\} : x \in X\}$ *is a basis for* $\tau = \mathcal{P}(X) = \{\ U : U \subseteq X\}$. *(discrete)*

*Proof.* The $\bigcup\{$ U : U $\subseteq \mathbb{X}\} = \mathbb{X}$, so for all x $\in$ U, x $\in \mathbb{X} \in \tau$. So, there exists X $\subseteq \mathbb{X}$ such that x $\in$ X, thus x $\in$ X $\subseteq \mathbb{X} \in \tau$. Again, the $\bigcup\{$ U : U $\subseteq \mathbb{X}\} = \mathbb{X}$, so let x $\in \mathrm{X}_1$, $\mathrm{X}_2 \subseteq \mathbb{X}$ then x $\in \mathbb{X}$, and x $\in (\mathrm{X}_1 \cap \mathrm{X}_2) = \mathrm{X}_3$. $\mathrm{X}_3 = (\mathrm{X}_1 \cap \mathrm{X}_2) \subseteq \mathbb{X}$, so $\mathrm{X}_3 \in \mathbb{X}$ with x $\in \mathrm{X}_3 \subseteq (\mathrm{X}_1 \cap \mathrm{X}_2) \subseteq \mathbb{X} \in \tau$. □

**Definition 10.** The Euclidean topology on $\mathbb{R}$ is the topology generated by the basis $\{$(a, b) : a $<$ b are in $\mathbb{R}\}$.

**Theorem 11.** $\{(a,\ b) : a < b \in \mathbb{R}\}$ *is a basis.*

*Proof.* □

**Theorem 12.** $\{(a,\ b) : a < b \in \mathbb{Q}\}$ *is a basis for the Euclidean topology.*

*Proof.* □