Proposition 1. *Find an infinite collection of open intervals in $\mathbb{R}$ whose intersection is not an open interval.*

Proof. $\bigcap\{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{Z}^+\} = \{0\}$ which is not an open interval or even open. $\square$

Proposition 2. *Any finite union of closed sets is closed, and any arbitrary intersection of closed sets is closed.*

Proof. We proceed by showing that any finite union of closed sets is closed:
Let C, D be closed sets.
Let A, B be compliments of C, D so A, B are open.
Then, A $\cap$ B is also open.
Thus, X$\backslash$(A $\cap$ B) is closed.
By Demorgan's Law, X$\backslash$(A $\cap$ B) = X$\backslash$A $\cup$ X$\backslash$B = C $\cup$ D which is closed.

Now that we've shown that C $\cup$ D is closed for all C, D; Assume $C_1 \cup C_2 \cup ... \cup C_n$ is closed for closed sets C_i. Then for $C_1 \cup ... \cup C_n \cup C_{n+1}$, let $K = C_1 \cup ... \cup C_n$.
Thus $K \cup C_{n+1} = C_1 \cup ... \cup C_n \cup C_{n+1}$ is closed.
Now we show that any arbitrary intersection of closed sets is closed.
Let $\mathcal{C}$ be a collection of closed sets.
Let $\mathcal{U} = \{X\backslash C : C \in \mathcal{C}\}$, so $\mathcal{U}$ is a collection of open sets.
Then, $\bigcup\mathcal{U}$ is also open.
Thus, $X\backslash\bigcup\mathcal{U}$ is closed.
By Demorgan's Law, $X\backslash\bigcup\mathcal{U} = \bigcap\mathcal{C}$
Therefore, $\bigcap\mathcal{C}$ is closed. $\square$

Lemma 3. *A set U is open if and only if for every point $x \in U$, there exists an open set U_x where $x \in U_x \subseteq U$*

Proof. Suppose U is open, then for all x $\in$ U there exists an open U_x = U, such that x $\in U_x \subseteq$ U. To show the converse, suppose that for each x $\in$ U there is an open set U_x where x $\in U_x \subseteq$ U. For $x \in U, x \in U_x$ so $x \in \bigcup\{U_x : x \in U\}$. Thus $U \subseteq \bigcup\{U_x : x \in U_x\}$. Now let y $\in U_x$ for some x $\in$ U, then $U_x \subseteq$ U. Thus $U \supseteq \bigcup\{ U_x : x \in U \}$. Therefore $U = \bigcup\{ U_x : x \in U \}$. $\square$

Proposition 4. *A set K in a topological space X is closed if and only if K contains all its limit points.*

Proof. Suppose K contains all its limit points. If K = X, then K is closed because $\emptyset$ is open. Otherwise let x $\in$ X$\backslash$K, so x is not a limit point of K. Then, $\exists\ x \in U_x \in \tau$ such that $U_x \cap$ K = $\emptyset$ since $x \notin K$. So, $x \in U_x \subseteq$ X$\backslash$K. By the lemma, X$\backslash$K is open so K is closed.

To show the converse is true as well, let x $\in$ X$\backslash$K, which is an open set by the lemma. Since (X$\backslash$K)$\cap$K is the empty set, x is not a limit point of K. So, if ℓ is any limit point of K, then $\ell \notin$ X$\backslash$K, so $\ell \in$ K and K contains all of its limit points. $\square$

Proposition 5. *Verify the discrete and indiscrete topologies are topologies*

Proof. To prove the discrete topology is an actual topology, first we define that the discrete topology on a set X is $\tau = \mathcal{P}$(X). $\emptyset$, X $\in \mathcal{P}$(X), So $\emptyset$, X $\in \tau$. Now, let $\mathcal{U} \subseteq \tau = \mathcal{P}$(X), then the $\bigcup$U $\in \mathcal{P}$(X). Let U, V $\in \tau = \mathcal{P}$(X) then the intersection U $\cap$ V $\in \mathcal{P}$(X) $= \tau$. Now to show that the indiscrete topology is an actual topology, we define that the indiscrete topology on a set X is given as $\tau = \{\emptyset$, X$\}$. Clearly, $\emptyset$, X $\in \tau$. Let $\mathcal{U}$ be a collection of open sets in X, then $\bigcup\mathcal{U}$ = X, thus $\bigcup\mathcal{U} \in \tau$. □

Definition 6. A collection of sets $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a basis if:

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

The set $\{\bigcup\mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$ is called the topology generated by $\mathcal{B}$.

Theorem 7. *The "topology generated by $\mathcal{B}$" is actually a topology.*

Proof. NEEDS WORK
τ is $\{\bigcup\mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$. So $\{\bigcup\mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} \subseteq \mathcal{P}(X)$. $\emptyset, X \in \tau$ because for $\mathcal{B}' = \emptyset, \bigcup\mathcal{B}' = \emptyset$ and for $\mathcal{B}' = X, \bigcup\mathcal{B}' = X$. Let $\mathcal{U} \subseteq \tau$ and $\bigcup\mathcal{U} \in \tau$ because for each $U \in \mathcal{U}, U = \bigcup\mathcal{B}'_U$ for some $\mathcal{B}'_U \subseteq \mathcal{B}$ So, $\bigcup\mathcal{U} = \bigcup\{\bigcup\mathcal{B}'_U : U \in \mathcal{U}\}$. Let $\mathcal{B}' = \bigcup\{\mathcal{B}'_U : U \in \mathcal{U}\}$. So, $\bigcup\mathcal{U} = \bigcup\mathcal{B}'$.
For $U, V \in \{\bigcup\mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}, U = B_1$ and $V = B_2, B_1, B_2 \in \mathcal{B}$ such that $B_1 = \mathcal{B}$ and $B_2 = \mathcal{B}$. Then $U \cap V \in \{\bigcup\mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} \supseteq W$ so there exists $x \in V, U : x \in U \cap V \supseteq W$. Therefore $x \in W \subseteq U \cap V \in \tau$. □

Theorem 8. *Let τ be a topology on X. Then $\mathcal{B} \subseteq \tau$ generates τ if:*
1. For all $x \in U \in \tau$, there exists $B \in \mathcal{B}$ where $x \in B \subseteq U \in \tau$
2. For all B_1, $B_2 \in \mathcal{B}$ with $x \in (B_1 \cap B_2)$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

Proof. :

1. For $x \in \mathbb{X} \in \tau$, select $x \in B \subseteq \mathbb{X} \in \tau$, so 1 of definition 6 holds.
2. See definition 6-2.
3. Let $U \in \tau$, then for every $x \in U$, there exists $B_x \in \mathcal{B}$ where $x \in B_x \subseteq U \in \tau$. Let $\mathcal{B}' = \{B_x : B_x \in \mathcal{B}\}$ so $U = \bigcup\mathcal{B}'$. Now let $\mathcal{B}' \subseteq \mathcal{B} \subseteq \tau$, and since $\mathcal{B}' \subseteq \tau, \bigcup\mathcal{B}' \in \tau$.

□

Theorem 9. $\{\mathbb{X}\}$ *is a basis for $\tau = \{\emptyset, \mathbb{X}\}$. (indiscrete)*

Proof. :

1. Let $\mathcal{B} = \{\mathbb{X}\}$ and let $x \in \mathbb{X} = B \in \mathcal{B}$.
2. Now consider $B_1, B_2 \in \mathcal{B}$. $B_1 = \mathbb{X}$ and $B_2 = \mathbb{X}$ Let $x \in B_1 \cap B_2$, then $x \in B_3 \subseteq B_1 \cap B_2$.

3. Choose $\mathcal{B}_\emptyset = \emptyset \subseteq \mathcal{B}$ and $\mathcal{B}_\mathbb{X} = \{\mathbb{X}\} \subseteq \mathcal{B}$ where $\bigcup \mathcal{B}_\emptyset = \emptyset$ and $\bigcup \mathcal{B}_\mathbb{X} = \mathbb{X}$, so since $\mathcal{P}(\mathcal{B}) = \{\emptyset, \{\mathbb{X}\}\}$, $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} = \{\bigcup \mathcal{B}_\emptyset, \bigcup \mathcal{B}_\mathbb{X}\} = \tau = \{\emptyset, \{\mathbb{X}\}\}$.

□

Theorem 10. $\mathcal{B} = \{\{x\} : x \in \mathbb{X}\}$ *is a basis for* $\tau = \mathcal{P}(\mathbb{X}) = \{\ U : U \subseteq \mathbb{X}\}$. *(discrete)*

Proof. :

1. Let $x \in U \in \tau = \mathcal{P}(\mathbb{X})$. Then for $B = \{x\} \in \mathcal{B}, x \in B \subseteq U$.
2. Let $B_1, B_2 \in \mathcal{B}$. Let $y \in B_1 \cap B_2$. Let $B_3 = \{y\}$ $y \in B_3 = \{y\} \subseteq B_1 \cap B_2$.
3. Let $U \in \tau = \mathcal{P}(\mathbb{X})$. Then $U = \{x : x \in U\} = \bigcup\{x : x \in U\} = \bigcup \mathcal{B}_U$ for $\mathcal{B}_U = \{\{x\} : x \in U\} \subseteq \mathcal{B}$.
   Let $\mathcal{B}' \subseteq \mathcal{B}$. Then $\bigcup \mathcal{B}' \in \mathcal{P}(\mathbb{X}) \in \tau$.

□

Definition 11. The Euclidean topology on $\mathbb{R}$ is the topology generated by the basis $\{(a, b) : a < b \in \mathbb{R}\}$.

Theorem 12. $\{(a,\ b)\ :\ a <\ b \in \mathbb{R}\}$ *is a basis.*

Proof. :

1. Let $\mathcal{B} = \{(a, b) : a < b \in \mathbb{R}\}$. Let $x \in \mathbb{R}$ and let $B = (x-1, x+1)$, then $x \in B \subseteq \mathbb{R}$ and $B \in \mathcal{B}$.
2. For $B_1 = (a_1, b_1)$, $B_2 = (a_2, b_2)$, $a_1 < b_1$, $a_2 < b_2 \in \mathbb{R}$ with $x \in B_1 \cap B_2$, then there is $B_3 = (max(a_1, a_2), min(b_1, b_2)) \subseteq B_1 \cap B_2$ such that $a_1, a_2 < x < b_1, b_2$ so $x \in B_3$.

□

Theorem 13. $\{(a,\ b)\ :\ a <\ b \in \mathbb{Q}\}$ *is a basis for the Euclidean topology.*

Proof. :

1. Let $\mathcal{B} = \{(a, b) : a < b \in \mathbb{Q}\}$ Let $x \in \mathbb{R}$ then, $\lfloor x \rfloor - 1 < x < \lceil x \rceil + 1$. Let B = $(\lfloor x \rfloor - 1, \lceil x \rceil + 1)$, so $x \in B \in \mathcal{B}$ and $B \subseteq \mathbb{Q}$.
2. Now, for $B_1, B_2 \in \mathcal{B}, \exists x \in B_1 \cap B_2 : x \in B_1$ and $x \in B_2$. $a_1, a_2 < b_1, b_2 \in \mathbb{Q}$. Let $B_1 = (a_1, b_1)$ and $B_2 = (a_2, b_2)$. Now let $B_3 = (max(a_1, a_2), min(b_1, b_2))$ with $a_1, a_2 < x < b_1, b_2$ so $x \in B_3$. So, $x \in B_3 \subseteq B_1 \cap B_2$.
3. Now show it generates τ...

□

Definition 14. A subset K of a topological space X is said to be **compact** if for every open cover $\mathcal{U}$ of K (every collection $\mathcal{U}$ of open sets such that $\bigcup\mathcal{U} \supseteq K$) there exists a finite subcollection $\mathcal{F} \subseteq \mathcal{U}$ that also covers K (that is, $\bigcup\mathcal{F} \supseteq K$).

Theorem 15. *Any finite union of compact subsets is compact.*

Proof. K_1 and K_2 are compact, so let $\mathcal{U}_3$ be an open cover of $K_3 = K_1 \cup K_2$. So let $\mathcal{U}_1 = \mathcal{U}_3, \bigcup\mathcal{U}_3 \supseteq K_1$ and $\mathcal{U}_2 = \mathcal{U}_3, \bigcup\mathcal{U}_3 \supseteq K_2$. Since K_1, K_2 are compact choose finite $\mathcal{F}_1 \subseteq \mathcal{U}_1 = \mathcal{U}_3$ and $\mathcal{F}_2 \subseteq \mathcal{U}_2 = \mathcal{U}_3$, so $\bigcup\mathcal{F}_1 \supseteq K_1$ and $\bigcup\mathcal{F}_2 \supseteq K_2$ and $\bigcup(\mathcal{F}_1 \cup \mathcal{F}_2) \supseteq K_3$. Since $\mathcal{F}_1 \cup \mathcal{F}_2$ is a finite subset of $\mathcal{U}_3, K_3$ is compact. □

Proposition 16. *Any finite subset of a topological space is compact.*

Proof. For $F = \{x_i : 0 \leq i < n\} \subseteq \mathbb{X}$, let $\mathcal{U}$ be an open covering of F so $\bigcup\mathcal{U} \supseteq F$. For each $0 \leq i < n$, let $U_i \in \mathcal{U}$ satisfy $x_i \in U_i$. Let finite $\mathcal{F} = \{U_i : 0 \leq i < n\} \subseteq \mathcal{U}$, so $\bigcup\mathcal{F} \supseteq F$ and F is compact. □

Proposition 17. *The subset $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ of $\mathbb{R}$ is compact.*

Proof. Let $K = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\} \subseteq \mathbb{R}$ and let $\mathcal{U}$ be an open covering of K. We may choose $U_0 \in \mathcal{U}$ where $0 \in U_0$. So pick $a < b \in \mathbb{R}$ where $0 \in (a, b) \subseteq U_0$. Let $N \in \mathbb{Z}^+$ where $0 < \frac{1}{N} < b$. Now let finite $F = \{\frac{1}{m} : 1 \leq m < N\}$ so it is compact, and choose $\mathcal{F} \subseteq \mathcal{U}$ with $\bigcup\mathcal{F} \supseteq F$. So, $\{\bigcup\mathcal{F} \cup (a, b) \cup U_0\} \supseteq K$. □

Proposition 18. *Any unbounded subset of $\mathbb{R}$ is not compact.*

Proof. Let K be an unbounded subset of $\mathbb{R}$. Let $\mathcal{U} = \{(n-2, n) : n \in \mathbb{Z}\}$. Let $n_i \in \mathbb{Z}$ be arbitrary for $0 \leq i < n$. Then $\mathcal{F} = \{(n_i - 2, n_i) : 0 \leq i < n\}$ is an arbitrary finite sub collection of $\mathcal{U}$. However, $\bigcup\mathcal{F} \not\supseteq K$, for $min(n_i - 2 : 0 \leq i < n)$ and $max(n_i : 0 \leq i < n)$ we can find a $k \in K : k < min(n_i - 2 : 0 \leq i < n)$ or $k > max(n_i : 0 \leq i < n)$, so $k \notin \bigcup\mathcal{F}$ and K is not compact. □

Proposition 19. *Any open interval $(a, b) \subseteq \mathbb{R}$ is not compact.*

Proof. Let $K = (a, b) \subseteq \mathbb{R}$, Let $\mathcal{U} = \{(x, y) : x, y \in \mathbb{R}, a < x < y < b\}$. So, $\bigcup\mathcal{U} \supseteq K$, and $\bigcup\mathcal{U} = K$. Let $n \in \mathbb{Z}$, let $x_i, y_i \in \mathbb{R}$ be arbitrary with $0 \leq i < n$. Then, $\mathcal{F} = \{(x_i, y_i) : 0 \leq i < n\}$ is an arbitrary finite subset of $\mathcal{U}$. However $\bigcup\mathcal{F} \not\supseteq K$, for $min(x_i : 0 \leq i < n)$ we can find a $k \in K : k < min(x_i : 0 \leq i < n)$ which is greater than a so $k \notin \bigcup\mathcal{F}$ and K is not compact. □

Definition 20. A subset K of a topological space X is said to be **Lindelöf** if for every open cover $\mathcal{U}$ of K (every collection $\mathcal{U}$ of open sets such that $\bigcup\mathcal{U} \supseteq K$) there exists a countable subcollection $\mathcal{F} \subseteq \mathcal{U}$ that also covers K (that is, $\bigcup\mathcal{F} \supseteq K$).

Definition 21. A subset K of a topological space X is said to be ***σ*-compact** if there exist compact subspaces K_n of X for $n \in \mathbb{N}$ such that $K = \bigcup_{n \in \mathbb{N}} K_n$.

Theorem 22. *Every* $\boldsymbol{\sigma}$***-compact*** *subset of a topological space is* ***Lindelöf****.*

Proof. Let K be σ-compact. Let $\mathcal{F}_n = \{U_{n0}, U_{n1}, ..., U_{nm}\}$ be a finite subcover of $\mathcal{U}$ for K_n. So, $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is a countable subcover of $\mathcal{U}$ for K. Therefore K is Lindelöf.

□

Definition 23. A subset K of a topological space X is said to be **hemicompact** if for every compact subset $H \subseteq K$, $H \subseteq K_n$ for some $n \in \mathbb{N}$.

Theorem 24. *Every* ***hemicompact*** *subset of a topological space is* $\boldsymbol{\sigma}$***-compact****.*

Proof. For all compact $C \subseteq K \subseteq X$, $C \subseteq K_{n\in\mathbb{N}} \subseteq K$.
$K_{n\in\mathbb{N}} \subseteq X$. Since all compact $C_{i\in\mathbb{N}}$ are subsets of some $K_{n\in\mathbb{N}}$, $\bigcup\limits_{n\in\mathbb{N}} K_n \supseteq \bigcup\limits_{i\in\mathbb{N}} C_i$ so $\bigcup\limits_{n\in\mathbb{N}} K_n = K$. □

Definition 25. X is **locally compact** if for every $x \in X$ there exists an open set U and compact K with $x \in U \subseteq K$.

Definition 26. x is a **cluster point** of $\langle p_n \rangle_{n<\omega}$ if and only if for every neighborhood U of x, U hits infinitely many p_n.

Definition 27. $\boldsymbol{Gru_{\mathcal{K},\mathcal{P}}(X)}$ is a game defined as... $\mathcal{K}$ wins if and only if $\langle p_n \rangle_{n<\omega}$ lacks a **cluster point**.

Definition 28. $\boldsymbol{\mathcal{K} \underset{pre}{\uparrow} Gru_{\mathcal{K},\mathcal{P}}(X)}$ is the statement that $\mathcal{K}$ has a winning strategy $\sigma : \omega \to \mathcal{K}(X)$ that depends only on the round number (ignores the moves of $\mathcal{P}$).

Theorem 29. *If X is* ***Lindelöf****, then* $\boldsymbol{\mathcal{K} \underset{pre}{\uparrow} Gru_{\mathcal{K},\mathcal{P}}(X)}$

Proof. Let X be Lindelöf, so there is a countable subcollection $\mathcal{C} \subseteq \mathcal{U}$, where $\mathcal{U}$ is an arbitrary covering of X. So, $\mathcal{C} = \{U_n : U_n \in \mathcal{U}, n < \omega\}$ and $\bigcup \mathcal{C} \supseteq X$.

For $Gru_{\mathcal{K},\mathcal{P}}(X)$, $\mathcal{K}$ chooses compact $K_{i\leq n<\omega} \supseteq U_{n<\omega} \in \mathcal{C}$ and $\mathcal{P}$ chooses $p \in X$ such that $p \notin K_{i\leq n<\omega}$. Since $\{K_i : i \leq n < \omega\}$ is countable and $\bigcup\{K_i : i \leq n < \omega\} \supseteq X$, $\mathcal{P}$ has countably many $p_{m\leq n<\omega}$. Therefore $\langle p_m \rangle$ is finite and X has no cluster points. Thus, $\mathcal{K} \underset{pre}{\uparrow} Gru_{\mathcal{K},\mathcal{P}}(X)$. □

Lemma 30. *Let K be a compact subset of X. If $\langle p_n \rangle$ is a sequence of points in K, then $\langle p_n \rangle$ has a cluster point.*

Proof. This is obviously true if the sequence only uses finitely many different points (one point must be repeated infinitely often, so it is its own cluster point). So assume $\langle p_n \rangle$ uses infinitely many different points. If $\langle p_n \rangle$ didn't have a cluster point, then for each $x \in K$, let U_x contain only finitely many p_n. $\{U_x : x \in K\}$ covers K, so choose x_i for $i \leq n : \{U_{x_i} : i \leq n\}$ covers K. But each U_{x_i} contains only finitely many p_n, so $\langle p_n \rangle$ cannot have infinitely many different points to choose from and $\langle p_n \rangle$ does not have a cluster point. □

Theorem 31. $\boldsymbol{\mathcal{K} \underset{pre}{\uparrow} Gru_{\mathcal{K},\mathcal{P}}(X)} \Rightarrow X$ *is* ***hemicompact****.*

Proof. We prove the contrapositive. Let σ be a predetermined strategy, and assume X isn't hemicompact. So there exists a compact K that isn't a union of finite $\sigma(n)$, so player $\mathcal{P}$ may always legally choose a point $p_n \in K$. Then by the lemma, $\langle p_n \rangle$ has a cluster point. Therefore σ isn't a winning strategy. $\square$