Proposition 1. *Find an infinite collection of open intervals in* $\mathbb{R}$ *whose intersection is not an open interval.*

Proof. $\bigcap\{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{Z}^+\} = \{0\}$ which is not an open interval or even open. $\square$

Proposition 2. *Any finite union of closed sets is closed, and any arbitrary intersection of closed sets is closed.*

Proof. We proceed by showing that any finite union of closed sets is closed:
Let C, D be closed sets.
Let A, B be compliments of C, D so A, B are open.
Then, A $\cap$ B is also open.
Thus, X$\backslash$(A $\cap$ B) is closed.
By Demorgan's Law, X$\backslash$(A $\cap$ B) = X$\backslash$A $\cup$ X$\backslash$B = C $\cup$ D which is closed.

Now that we've shown that C $\cup$ D is closed for all C, D; Assume $\mathrm{C}_1 \cup \mathrm{C}_2 \cup ... \cup \mathrm{C}_n$ is closed for closed sets C_i. Then for $\mathrm{C}_1 \cup ... \cup \mathrm{C}_n \cup \mathrm{C}_{n+1}$, let $\mathrm{K} = \mathrm{C}_1 \cup ... \cup \mathrm{C}_n$.
Thus $\mathrm{K} \cup \mathrm{C}_{n+1} = \mathrm{C}_1 \cup ... \cup \mathrm{C}_n \cup \mathrm{C}_{n+1}$ is closed.
Now we show that any arbitrary intersection of closed sets is closed.
Let $\mathcal{C}$ be a collection of closed sets.
Let $\mathcal{U} = \{\mathrm{X}\backslash\mathrm{C} : \mathrm{C} \in \mathcal{C}\}$, so $\mathcal{U}$ is a collection of open sets.
Then, $\bigcup\mathcal{U}$ is also open.
Thus, X$\backslash\bigcup\mathcal{U}$ is closed.
By Demorgan's Law, X$\backslash\bigcup\mathcal{U} = \bigcap\mathcal{C}$
Therefore, $\bigcap\mathcal{C}$ is closed. $\square$

Lemma 3. *A set U is open if and only if for every point* $x \in U$*, there exists an open set* U_x *where* $x \in U_x \subseteq U$

Proof. Suppose U is open, then for all x $\in$ U there exists an open U_x = U, such that x $\in \mathrm{U}_x \subseteq$ U. To show the converse, suppose that for each x $\in$ U there is an open set U_x where x $\in \mathrm{U}_x \subseteq$ U. For $x \in U, x \in U_x$ so $x \in \bigcup\{U_x : x \in U\}$. Thus $U \subseteq \bigcup\{U_x : x \in U_x\}$. Now let y $\in \mathrm{U}_x$ for some x $\in$ U, then $\mathrm{U}_x \subseteq$ U. Thus U $\supseteq \bigcup\{$ U_x : x $\in$ U $\}$. Therefore U = $\bigcup\{$ U_x : x $\in$ U $\}$. $\square$

Proposition 4. *A set K in a topological space X is closed if and only if K contains all its limit points.*

Proof. Suppose K contains all its limit points. If K = X, then K is closed because $\emptyset$ is open. Otherwise let x $\in$ X$\backslash$K, so x is not a limit point of K. Then, $\exists$ $x \in \mathrm{U}_x \in \tau$ such that $\mathrm{U}_x \cap$ K = $\emptyset$ since $x \notin K$. So, $x \in \mathrm{U}_x \subseteq$ X$\backslash$K. By the lemma, X$\backslash$K is open so K is closed.

To show the converse is true as well, let x $\in$ X$\backslash$K, which is an open set by the lemma. Since (X$\backslash$K)$\cap$K is the empty set, x is not a limit point of K. So, if ℓ is any limit point of K, then $\ell \notin$ X$\backslash$K, so $\ell \in$ K and K contains all of its limit points. $\square$

Proposition 5. *Verify the discrete and indiscrete topologies are topologies*

Proof. To prove the discrete topology is an actual topology, first we define that the discrete topology on a set X is $\tau = \mathcal{P}(\mathrm{X})$. $\emptyset$, X $\in \mathcal{P}(\mathrm{X})$, So $\emptyset$, X $\in \tau$. Now, let $\mathcal{U} \subseteq \tau = \mathcal{P}(\mathrm{X})$, then the $\bigcup \mathrm{U} \in \mathcal{P}(\mathrm{X})$. Let U, V $\in \tau = \mathcal{P}(\mathrm{X})$ then the intersection U $\cap$ V $\in \mathcal{P}(\mathrm{X}) = \tau$. Now to show that the indiscrete topology is an actual topology, we define that the indiscrete topology on a set X is given as $\tau = \{\emptyset$, X$\}$. Clearly, $\emptyset$, X $\in \tau$. Let $\mathcal{U}$ be a collection of open sets in X, then $\bigcup \mathcal{U} =$ X, thus $\bigcup \mathcal{U} \in \tau$. □

Definition 6. A collection of sets $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a basis if:

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

The set $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$ is called the topology generated by $\mathcal{B}$.

Theorem 7. *The "topology generated by $\mathcal{B}$" is actually a topology.*

Proof. NEEDS WORK
τ is $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$. So $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} \subseteq \mathcal{P}(X)$. $\emptyset, X \in \tau$ because for $\mathcal{B}' = \emptyset, \bigcup \mathcal{B}' = \emptyset$ and for $\mathcal{B}' = X, \bigcup \mathcal{B}' = X$. Let $\mathcal{U} \subseteq \tau$ and $\bigcup \mathcal{U} \in \tau$ because for each $U \in \mathcal{U}, U = \bigcup \mathcal{B}'_U$ for some $\mathcal{B}'_U \subseteq \mathcal{B}$ So, $\bigcup \mathcal{U} = \bigcup \{\bigcup \mathcal{B}'_U : U \in \mathcal{U}\}$. Let $\mathcal{B}' = \bigcup \{\mathcal{B}'_U : U \in \mathcal{U}\}$. So, $\bigcup \mathcal{U} = \bigcup \mathcal{B}'$.
For $U, V \in \{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}, U = B_1$ and $V = B_2, B_1, B_2 \in \mathcal{B}$ such that $B_1 = \mathcal{B}$ and $B_2 = \mathcal{B}$. Then $U \cap V \in \{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} \supseteq W$ so there exists $x \in V, U : x \in U \cap V \supseteq W$. Therefore $x \in W \subseteq U \cap V \in \tau$. □

Theorem 8. *Let τ be a topology on X. Then $\mathcal{B} \subseteq \tau$ generates τ if:*
1. For all $x \in U \in \tau$, there exists $B \in \mathcal{B}$ where $x \in B \subseteq U \in \tau$
2. For all B_1, $B_2 \in \mathcal{B}$ with $x \in (B_1 \cap B_2)$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

Proof. :

1. For $x \in \mathbb{X} \in \tau$, select $x \in B \subseteq \mathbb{X} \in \tau$, so 1 of definition 6 holds.
2. See definition 6-2.
3. Let $U \in \tau$, then for every $x \in U$, there exists $B_x \in \mathcal{B}$ where $x \in B_x \subseteq U \in \tau$. Let $\mathcal{B}' = \{B_x : B_x \in \mathcal{B}\}$ so $U = \bigcup \mathcal{B}'$. Now let $\mathcal{B}' \subseteq \mathcal{B} \subseteq \tau$, and since $\mathcal{B}' \subseteq \tau, \bigcup \mathcal{B}' \in \tau$.

□

Theorem 9. $\{\mathbb{X}\}$ *is a basis for $\tau = \{\emptyset, \mathbb{X}\}$. (indiscrete)*

Proof. :

1. Let $\mathcal{B} = \{\mathbb{X}\}$ and let $x \in \mathbb{X} = B \in \mathcal{B}$.
2. Now consider $B_1, B_2 \in \mathcal{B}$. $B_1 = \mathbb{X}$ and $B_2 = \mathbb{X}$ Let $x \in B_1 \cap B_2$, then $x \in B_3 \subseteq B_1 \cap B_2$.

3. Choose $\mathcal{B}_\emptyset = \emptyset \subseteq \mathcal{B}$ and $\mathcal{B}_\mathbb{X} = \{\mathbb{X}\} \subseteq \mathcal{B}$ where $\bigcup \mathcal{B}_\emptyset = \emptyset$ and $\bigcup \mathcal{B}_\mathbb{X} = \mathbb{X}$, so since $\mathcal{P}(\mathcal{B}) = \{\emptyset, \{\mathbb{X}\}\}$, $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} = \{\bigcup \mathcal{B}_\emptyset, \bigcup \mathcal{B}_\mathbb{X}\} = \tau = \{\emptyset, \{\mathbb{X}\}\}$.

□

Theorem 10. $\mathcal{B} = \{\{x\} : x \in \mathbb{X}\}$ *is a basis for* $\tau = \mathcal{P}(\mathbb{X}) = \{\ U : U \subseteq \mathbb{X}\}$. *(discrete)*

Proof. :

1. Let $x \in U \in \tau = \mathcal{P}(\mathbb{X})$. Then for $B = \{x\} \in \mathcal{B}, x \in B \subseteq U$.
2. Let $B_1, B_2 \in \mathcal{B}$. Let $y \in B_1 \cap B_2$. Let $B_3 = \{y\}$ $y \in B_3 = \{y\} \subseteq B_1 \cap B_2$.
3. Let $U \in \tau = \mathcal{P}(\mathbb{X})$. Then $U = \{x : x \in U\} = \bigcup\{x : x \in U\} = \bigcup \mathcal{B}_U$ for $\mathcal{B}_U = \{\{x\} : x \in U\} \subseteq \mathcal{B}$.
   Let $\mathcal{B}' \subseteq \mathcal{B}$. Then $\bigcup \mathcal{B}' \in \mathcal{P}(\mathbb{X}) \in \tau$.

□

Definition 11. The Euclidean topology on $\mathbb{R}$ is the topology generated by the basis $\{(a, b) : a < b \in \mathbb{R}\}$.

Theorem 12. $\{(a,\ b)\ :\ a < b \in \mathbb{R}\}$ *is a basis.*

Proof. :

1. Let $\mathcal{B} = \{(a, b) : a < b \in \mathbb{R}\}$. Let $x \in \mathbb{R}$ and let $B = (x-1, x+1)$, then $x \in B \subseteq \mathbb{R}$ and $B \in \mathcal{B}$.
2. For $B_1 = (a_1, b_1)$, $B_2 = (a_2, b_2)$, $a_1 < b_1$, $a_2 < b_2 \in \mathbb{R}$ with $x \in B_1 \cap B_2$, then there is $B_3 = (max(a_1, a_2), min(b_1, b_2)) \subseteq B_1 \cap B_2$ such that $a_1, a_2 < x < b_1, b_2$ so $x \in B_3$.

□

Theorem 13. $\{(a,\ b)\ :\ a < b \in \mathbb{Q}\}$ *is a basis for the Euclidean topology.*

Proof. :

1. Let $\mathcal{B} = \{(a, b) : a < b \in \mathbb{Q}\}$ Let $x \in \mathbb{R}$ then, $\lfloor x \rfloor - 1 < x < \lceil x \rceil + 1$. Let B $= (\lfloor x \rfloor - 1, \lceil x \rceil + 1)$, so $x \in B \in \mathcal{B}$ and $B \subseteq \mathbb{Q}$.
2. Now, for $B_1, B_2 \in \mathcal{B}, \exists x \in B_1 \cap B_2 : x \in B_1$ and $x \in B_2$. $a_1, a_2 < b_1, b_2 \in \mathbb{Q}$. Let $B_1 = (a_1, b_1)$ and $B_2 = (a_2, b_2)$. Now let $B_3 = (max(a_1, a_2), min(b_1, b_2))$ with $a_1, a_2 < x < b_1, b_2$ so $x \in B_3$. So, $x \in B_3 \subseteq B_1 \cap B_2$.
3. Now show it generates τ...

$\square$

Definition 14. A subset K of a topological space X is said to be **compact** if for every open cover $\mathcal{U}$ of K (every collection $\mathcal{U}$ of open sets such that $\bigcup \mathcal{U} \supseteq K$) there exists a finite subcollection $\mathcal{F} \subseteq \mathcal{U}$ that also covers K (that is, $\bigcup \mathcal{F} \supseteq K$).

Theorem 15. *Any finite union of compact subsets is compact.*

Proof. K_1 and K_2 are compact, so let $\mathcal{U}_3$ be an open cover of $K_3 = K_1 \cup K_2$. So let $\mathcal{U}_1 = \mathcal{U}_3, \bigcup \mathcal{U}_3 \supseteq K_1$ and $\mathcal{U}_2 = \mathcal{U}_3, \bigcup \mathcal{U}_3 \supseteq K_2$. Since K_1, K_2 are compact choose finite $\mathcal{F}_1 \subseteq \mathcal{U}_1 = \mathcal{U}_3$ and $\mathcal{F}_2 \subseteq \mathcal{U}_2 = \mathcal{U}_3$, so $\bigcup \mathcal{F}_1 \supseteq K_1$ and $\bigcup \mathcal{F}_2 \supseteq K_2$ and $\bigcup (\mathcal{F}_1 \cup \mathcal{F}_2) \supseteq K_3$. Since $\mathcal{F}_1 \cup \mathcal{F}_2$ is a finite subset of $\mathcal{U}_3, K_3$ is compact. $\square$

Proposition 16. *Any finite subset of a topological space is compact.*

Proof. For $F = \{x_i : 0 \leq i < n\} \subseteq \mathbb{X}$, let $\mathcal{U}$ be an open covering of F so $\bigcup \mathcal{U} \supseteq F$. For each $0 \leq i < n$, let $U_i \in \mathcal{U}$ satisfy $x_i \in U_i$. Let finite $\mathcal{F} = \{U_i : 0 \leq i < n\} \subseteq \mathcal{U}$, so $\bigcup \mathcal{F} \supseteq F$ and F is compact. $\square$

Proposition 17. *The subset $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ of $\mathbb{R}$ is compact.*

Proof. Let $K = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\} \subseteq \mathbb{R}$ and let $\mathcal{U}$ be an open covering of K. We may choose $U_0 \in \mathcal{U}$ where $0 \in U_0$. So pick $a < b \in \mathbb{R}$ where $0 \in (a,b) \subseteq U_0$. Let $N \in \mathbb{Z}^+$ where $0 < \frac{1}{N} < b$. Now let finite $F = \{\frac{1}{m} : 1 \leq m < N\}$ so it is compact, and choose $\mathcal{F} \subseteq \mathcal{U}$ with $\bigcup \mathcal{F} \supseteq F$. So, $\{\bigcup \mathcal{F} \cup (a,b) \cup U_0\} \supseteq K$. $\square$

Proposition 18. *Any unbounded subset of $\mathbb{R}$ is not compact.*

Proof. Let K be an unbounded subset of $\mathbb{R}$. Let $\mathcal{U} = \{(n-2, n) : n \in \mathbb{Z}\}$. Let $n_i \in \mathbb{Z}$ be arbitrary for $0 \leq i < n$. Then $\mathcal{F} = \{(n_i - 2, n_i) : 0 \leq i < n\}$ is an arbitrary finite sub collection of $\mathcal{U}$. However, $\bigcup \mathcal{F} \not\supseteq K$, for $min(n_i - 2 : 0 \leq i < n)$ and $max(n_i : 0 \leq i < n)$ we can find a $k \in K : k < min(n_i - 2 : 0 \leq i < n)$ or $k > max(n_i : 0 \leq i < n)$, so $k \notin \bigcup \mathcal{F}$ and K is not compact. $\square$

Proposition 19. *Any open interval $(a,b) \subseteq \mathbb{R}$ is not compact.*

Proof. Let $K = (a,b) \subseteq \mathbb{R}$, Let $\mathcal{U} = \{(x,y) : x, y \in \mathbb{R}, a < x < y < b\}$. So, $\bigcup \mathcal{U} \supseteq K$, and $\bigcup \mathcal{U} = K$. Let $n \in \mathbb{Z}$, let $x_i, y_i \in \mathbb{R}$ be arbitrary with $0 \leq i < n$. Then, $\mathcal{F} = \{(x_i, y_i) : 0 \leq i < n\}$ is an arbitrary finite subset of $\mathcal{U}$. However $\bigcup \mathcal{F} \not\supseteq K$, for $min(x_i : 0 \leq i < n)$ we can find a $k \in K : k < min(x_i : 0 \leq i < n)$ which is greater than a so $k \notin \bigcup \mathcal{F}$ and K is not compact. $\square$

Definition 20. A subset K of a topological space X is said to be **Lindelöf** if for every open cover $\mathcal{U}$ of K (every collection $\mathcal{U}$ of open sets such that $\bigcup \mathcal{U} \supseteq K$) there exists a countable subcollection $\mathcal{F} \subseteq \mathcal{U}$ that also covers K (that is, $\bigcup \mathcal{F} \supseteq K$).

Definition 21. A subset K of a topological space X is said to be **$\boldsymbol{\sigma}$-compact** if there exist compact subspaces K_n of X for $n \in \mathbb{N}$ such that $K = \bigcup_{n \in \mathbb{N}} K_n$.

Theorem 22. *Every* $\boldsymbol{\sigma}$***-compact*** *subset of a topological space is* ***Lindelöf****.*

Proof. Let K be σ-compact. Let $\mathcal{F}_n = \{U_{n0}, U_{n1}, ..., U_{nm}\}$ be a finite subcover of $\mathcal{U}$ for K_n. So, $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is a countable subcover of $\mathcal{U}$ for K. Therefore K is Lindelöf.

□

Definition 23. A subset K of a topological space X is said to be **hemicompact** if there exists compact sets $K_n \subseteq K$ for $n \in \mathbb{N}$ such that for every compact subset $H \subseteq K$, $H \subseteq K_n$ for some $n \in \mathbb{N}$.

Theorem 24. *Every* ***hemicompact*** *subset of a topological space is* $\boldsymbol{\sigma}$***-compact****.*

Proof. Let $K_n \subseteq K$ witness hemicompactness, of course $\bigcup\limits_{n\in\mathbb{N}} K_n \subseteq K$, since $K_n \subseteq K$. Let $k \in K$, so $\{k\}$ is compact and $\bigcup\{k\} = K$. For $k_n \in K$, choose $K_n : k_n \in K_n \subseteq K$, so $K = \bigcup\limits_{n\in\mathbb{N}} K_n$ □

Definition 25. X is **locally compact** if for every $x \in X$ there exists an open set U and compact K with $x \in U \subseteq K$.

Definition 26. x is a **cluster point** of $\langle p_n \rangle_{n<\omega}$ if and only if for every neighborhood U of x, U hits infinitely many p_n.

Definition 27. $\boldsymbol{Gru_{\mathcal{K},\mathcal{P}}(X)}$ is a game defined as the *Gruenhagecompact/pointgame* with players $\mathcal{K}$, *mathcalP* played on a topological space X. During round n, $\mathcal{K}$ chooses a compact subset K_n of X, followed by $\mathcal{P}$ choosing a point $p_n \in X$ such that $p_n \notin \bigcup_{m\leq n} K_m$. $\mathcal{K}$ wins if and only if $\langle p_n \rangle_{n<\omega}$ lacks a **cluster point**.

Definition 28. $\boldsymbol{\mathcal{K} \underset{pre}{\uparrow} Gru_{\mathcal{K},\mathcal{P}}(X)}$ is the statement that $\mathcal{K}$ has a winning strategy $\sigma : \omega \to \mathcal{K}(X)$ that depends only on the round number (ignores the moves of $\mathcal{P}$).

Theorem 29. *If X is* ***Lindelöf*** *and locally compact, then* $\boldsymbol{\mathcal{K} \underset{pre}{\uparrow} Gru_{\mathcal{K},\mathcal{P}}(X)}$

Proof. For $x \in X$, let $x \in U_x \subseteq K_x$ for U_x open and K_x compact. Then by definition of Lindelöf choose $U_n \in \mathcal{U} = \{U_x : x \in X\}$ such that $\bigcup\limits_{n\in\mathbb{N}} U_n = X$. Then for each U_n, there exists compact K_n where $U_n \subseteq K_n$.
For $Gru_{\mathcal{K},\mathcal{P}}(X)$, let $\sigma(n) = K_n$ denote a strategy for $\mathcal{K}$, $\mathcal{K}$ chooses compact $K_n \supseteq U_n \in \mathcal{C}$ and $\mathcal{P}$ chooses $p \in X$ such that $p \notin K_n$. $\{K_i : i < \omega\}$ is countable and $\bigcup\{K_i : i < \omega\} \supseteq X$. For $\mathcal{P}$ to win, there must be $U_n \in \mathcal{C}$ such that $\sigma(n) \not\supseteq U_n$, but we've already established that $\bigcup\{K_i : i < \omega\} \supseteq X$ so contradiction, thus $\mathcal{K} \underset{pre}{\uparrow} Gru_{\mathcal{K},\mathcal{P}}(X)$. □

Lemma 30. *Let K be a compact subset of X. If $\langle p_n \rangle$ is a sequence of points in K, then $\langle p_n \rangle$ has a cluster point.*

Proof. This is obviously true if the sequence only uses finitely many different points (one point must be repeated infinitely often, so it is its own cluster point). So without loss of generality $\langle p_n \rangle$ uses infinitely many different points. If $\langle p_n \rangle$ didn't have a cluster point, then for each $x \in K$, let U_x contain only finitely many p_n. $\{U_x : x \in K\}$ covers K, so choose x_i for $i \leq n : \{U_{x_i} : i \leq n\}$ covers

K. But each U_{x_i} contains only finitely many p_n, so $\langle p_n \rangle$ cannot have infinitely many different points to choose, contradiction. So $\langle p_n \rangle$ does have a cluster point. $\square$

Theorem 31. $\boldsymbol{\mathcal{K} \underset{pre}{\uparrow} Gru_{\mathcal{K},\mathcal{P}}(X)} \Rightarrow X$ *is* ***hemicompact.***

Proof. We prove the contrapositive. Let σ be a predetermined strategy, and assume X isn't hemicompact. So there exists a compact K that isn't a union of finite $\sigma(n)$, so player $\mathcal{P}$ may always legally choose a point $p_n \in K$. Then by the lemma, $\langle p_n \rangle$ has a cluster point. Therefore σ isn't a winning strategy. $\square$