
Proposition 1. *Puzzle 1: Find an infinite collection of open intervals in \mathbb{R} whose intersection is not an open interval.*

Proof. $\bigcap\{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{Z}^+\} = \{0\}$ which is not an open interval or open. \square

Proposition 2. *Any finite union of closed sets is closed, and any arbitrary intersection of closed sets is closed.*

Proof. Proposition 1 - We proceed by showing that any finite union of closed sets is closed:

Let C, D be closed sets.

Let A, B be compliments of C, D so A, B are open.

Then, $A \cap B$ is also open.

Thus, $X \setminus (A \cap B)$ is closed.

By Demorgan's Law, $X \setminus (A \cap B) = X \setminus A \cup X \setminus B = C \cup D$ which is closed.

Now that we've shown that $C \cup D$ is closed for all C, D ; Assume $C_1 \cup C_2 \cup \dots \cup C_n$ is closed for closed sets C_i . Then for $C_1 \cup \dots \cup C_n \cup C_{n+1}$, let $K = C_1 \cup \dots \cup C_n$. Thus $K \cup C_{n+1} = C_1 \cup \dots \cup C_n \cup C_{n+1}$ is closed.

Now we show that any arbitrary intersection of closed sets is closed.

Let \mathcal{C} be a collection of closed sets.

Let $\mathcal{U} = \{X \setminus C : C \in \mathcal{C}\}$, so \mathcal{U} is a collection of open sets.

Then, $\bigcup \mathcal{U}$ is also open.

Thus, $X \setminus \bigcup \mathcal{U}$ is closed.

By Demorgan's Law, $X \setminus \bigcup \mathcal{U} = \bigcap \mathcal{C}$

Therefore, $\bigcap \mathcal{C}$ is closed. \square

Lemma 3. *A set U is open if and only if for every point $x \in U$, there exists an open set U_x where $x \in U_x \subseteq U$*

Proof. Suppose U is open, then for all $x \in U$ there exists an open $U_x = U$, such that $x \in U_x \subseteq U$. To show the converse, for each $x \in U$ there is an open set U_x where $x \in U_x \subseteq U$. To show that $U = \bigcup \{U_x : x \in U\}$, let $x \in U$, then for each U_x , $U \subseteq U_x$. Thus $U \subseteq \bigcup \{U_x : x \in U\}$. Now let $y \in U_x$ for some $x \in U$, then $U_x \subseteq U$. Thus $U \supseteq \bigcup \{U_x : x \in U\}$. Therefore $U = \bigcup \{U_x : x \in U\}$. \square

Proposition 4. *A set K in a topological space X is closed if and only if K contains all its limit points.*

Proof. Suppose K contains all its limit points. If $K = X$, then K is closed because \emptyset is open. Otherwise let $x \in X \setminus K$, so x is not a limit point of K . Then, $\exists x \in U_x \in \tau$ such that $U_x \cap K = \emptyset$. So, $x \in U_x \subseteq X \setminus K$. By the lemma, $X \setminus K$ is open so K is closed.

To show the converse is true as well, let $x \in X \setminus K$, which is an open set by the lemma. Since $(X \setminus K) \cap K$ is the empty set, x is not a limit point of K . So, if ℓ is any limit point of K , then $\ell \notin X \setminus K$, so $\ell \in K$ and K contains all of its limit points. \square

Proposition 5. *Verify the discrete and indiscrete topologies are topologies*

Proof. To prove the discrete topology is an actual topology, first we define that the discrete topology on a set X is $\tau = \mathcal{P}(X)$. $\emptyset, X \in \mathcal{P}(X)$, So $\emptyset, X \in \tau$. Now, let $\mathcal{U} \subseteq \tau = \mathcal{P}(X)$, then the $\bigcup \mathcal{U} \in \mathcal{P}(X)$. Let $U, V \in \tau = \mathcal{P}(X)$ then the intersection $U \cap V \in \mathcal{P}(X) = \tau$. Now to show that the indiscrete topology is an actual topology, we define that the indiscrete topology on a set X is given as $\tau = \{\emptyset, X\}$. Clearly, $\emptyset, X \in \tau$. Let \mathcal{U} be a collection of open sets in X , then $\bigcup \mathcal{U} = X$, thus $\bigcup \mathcal{U} \in \tau$. \square

Definition 6. A collection of sets $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a basis if:

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

The set $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$ is called the topology generated by \mathcal{B} .

Theorem 7. The “topology generated by \mathcal{B} ” is actually a topology.

Theorem 8. Let τ be a topology on X . Then $\mathcal{B} \subseteq \tau$ generates τ if:

1. For all $x \in U \in \tau$, there exists $B \in \mathcal{B}$ where $x \in B \subseteq U \in \tau$
2. For all $B_1, B_2 \in \mathcal{B}$ with $x \in (B_1 \cap B_2)$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

Theorem 9. If \mathcal{B} is a basis for τ , then $\tau = \{\bigcup \mathcal{B}' : \mathcal{B}' \in \mathcal{B}\}$. (Every open set is a union of basic sets)

Proof. Let $\mathcal{B} \subseteq \tau$, then for any $\mathcal{B}' \in \mathcal{B}$, $\mathcal{B}' \in \tau$. Furthermore the $\bigcup \mathcal{B}' = \mathcal{B}$, then $\bigcup \mathcal{B}' \subseteq \tau$ and $\bigcup \mathcal{B}' = \tau$. \square

Theorem 10. $\{\mathbb{X}\}$ is a basis for $\tau = \{\emptyset, \mathbb{X}\}$. (indiscrete)

Proof. $\mathbb{X} \in \{\emptyset, \mathbb{X}\} = \tau$. First, let $U = \mathbb{X} \in \tau$, and let $x \in U \in \tau$, then for all $x \in U$, $x \in \mathbb{X}$, and for some $X_n \in \mathbb{X}$, $x \in X_n$. Thus, $x \in X_n \subseteq U \in \tau$. Now, we let $x \in X_1, X_2$. Then $x \in (X_1 \cap X_2)$. Let $X_3 \subseteq (X_1 \cap X_2)$ such that $X_3 = (X_1 \cap X_2)$ then $x \in X_3$. Thus, $x \in X_3 \subseteq (X_1 \cap X_2) \in \mathbb{X}$. Then $\{\mathbb{X}\}$ is a basis for $\tau = \{\emptyset, \mathbb{X}\}$. \square

Theorem 11. $\{\{X\} : x \in X\}$ is a basis for $\tau = \mathcal{P}(X) = \{U : U \subseteq X\}$. (discrete)

Proof. The $\bigcup \{U : U \subseteq \mathbb{X}\} = \mathbb{X}$, so for all $x \in U$, $x \in \mathbb{X} \in \tau$. So, there exists $X \subseteq \mathbb{X}$ such that $x \in X$, thus $x \in X \subseteq \mathbb{X} \in \tau$. Again, the $\bigcup \{U : U \subseteq \mathbb{X}\} = \mathbb{X}$, so let $x \in X_1, X_2 \subseteq \mathbb{X}$ then $x \in \mathbb{X}$, and $x \in (X_1 \cap X_2) = X_3$. $X_3 = (X_1 \cap X_2) \subseteq \mathbb{X}$, so $X_3 \in \mathbb{X}$ with $x \in X_3 \subseteq (X_1 \cap X_2) \subseteq \mathbb{X} \in \tau$. \square

Definition 12. The Euclidean topology on \mathbb{R} is the topology generated by the basis $\{(a, b) : a < b \text{ are in } \mathbb{R}\}$.

Theorem 13. $\{(a, b) : a < b \in \mathbb{R}\}$ is a basis.

Proof. \square

Theorem 14. $\{(a, b) : a < b \in \mathbb{Q}\}$ is a basis for the Euclidean topology.

Proof. \square