Proposition 1. *Find an infinite collection of open intervals in $\mathbb{R}$ whose intersection is not an open interval.*

Proof. $\bigcap\{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{Z}^+\} = \{0\}$ which is not an open interval or even open. $\square$

Proposition 2. *Any finite union of closed sets is closed, and any arbitrary intersection of closed sets is closed.*

Proof. We proceed by showing that any finite union of closed sets is closed:
Let C, D be closed sets.
Let A, B be compliments of C, D so A, B are open.
Then, A $\cap$ B is also open.
Thus, X$\backslash$(A $\cap$ B) is closed.
By Demorgan's Law, X$\backslash$(A $\cap$ B) = X$\backslash$A $\cup$ X$\backslash$B = C $\cup$ D which is closed.

Now that we've shown that C $\cup$ D is closed for all C, D; Assume $C_1 \cup C_2 \cup ... \cup C_n$ is closed for closed sets C_i. Then for $C_1 \cup ... \cup C_n \cup C_{n+1}$, let $K = C_1 \cup ... \cup C_n$.
Thus $K \cup C_{n+1} = C_1 \cup ... \cup C_n \cup C_{n+1}$ is closed.
Now we show that any arbitrary intersection of closed sets is closed.
Let $\mathcal{C}$ be a collection of closed sets.
Let $\mathcal{U} = \{X\backslash C : C \in \mathcal{C}\}$, so $\mathcal{U}$ is a collection of open sets.
Then, $\bigcup\mathcal{U}$ is also open.
Thus, $X\backslash\bigcup\mathcal{U}$ is closed.
By Demorgan's Law, $X\backslash\bigcup\mathcal{U} = \bigcap\mathcal{C}$
Therefore, $\bigcap\mathcal{C}$ is closed. $\square$

Lemma 3. *A set U is open if and only if for every point $x \in U$, there exists an open set U_x where $x \in U_x \subseteq U$*

Proof. Suppose U is open, then for all x $\in$ U there exists an open $U_x = U$, such that $x \in U_x \subseteq U$. To show the converse, suppose that for each x $\in$ U there is an open set U_x where $x \in U_x \subseteq U$. For $x \in U, x \in U_x$ so $x \in \bigcup\{U_x : x \in U\}$. Thus $U \subseteq \bigcup\{U_x : x \in U_x\}$. Now let $y \in U_x$ for some $x \in U$, then $U_x \subseteq U$. Thus $U \supseteq \bigcup\{ U_x : x \in U \}$. Therefore $U = \bigcup\{ U_x : x \in U \}$. $\square$

Proposition 4. *A set K in a topological space X is closed if and only if K contains all its limit points.*

Proof. Suppose K contains all its limit points. If K = X, then K is closed because $\emptyset$ is open. Otherwise let x $\in$ X$\backslash$K, so x is not a limit point of K. Then, $\exists\ x \in U_x \in \tau$ such that $U_x \cap K = \emptyset$ since $x \notin K$. So, $x \in U_x \subseteq X\backslash K$. By the lemma, X$\backslash$K is open so K is closed.

To show the converse is true as well, let x $\in$ X$\backslash$K, which is an open set by the lemma. Since (X$\backslash$K)$\cap$K is the empty set, x is not a limit point of K. So, if ℓ is any limit point of K, then $\ell \notin X\backslash K$, so $\ell \in K$ and K contains all of its limit points. $\square$

Proposition 5. *Verify the discrete and indiscrete topologies are topologies*

Proof. To prove the discrete topology is an actual topology, first we define that the discrete topology on a set X is $\tau = \mathcal{P}$(X). $\emptyset$, X $\in \mathcal{P}$(X), So $\emptyset$, X $\in \tau$. Now, let $\mathcal{U} \subseteq \tau = \mathcal{P}$(X), then the $\bigcup$U $\in \mathcal{P}$(X). Let U, V $\in \tau = \mathcal{P}$(X) then the intersection U $\cap$ V $\in \mathcal{P}$(X) $= \tau$. Now to show that the indiscrete topology is an actual topology, we define that the indiscrete topology on a set X is given as $\tau = \{\emptyset$, X$\}$. Clearly, $\emptyset$, X $\in \tau$. Let $\mathcal{U}$ be a collection of open sets in X, then $\bigcup\mathcal{U}$ = X, thus $\bigcup\mathcal{U} \in \tau$. □

Definition 6. A collection of sets $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a basis if:

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

The set $\{\bigcup\mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$ is called the topology generated by $\mathcal{B}$.

Theorem 7. *The "topology generated by $\mathcal{B}$" is actually a topology.*

Proof. τ is $\{\bigcup\mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$. So $\{\bigcup\mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} \subseteq \mathcal{P}(X)$. $\emptyset, X \in \tau$ because for $\mathcal{B}' = \emptyset, \bigcup\mathcal{B}' = \emptyset$ and for $\mathcal{B}' = X, \bigcup\mathcal{B}' = X$. Let $\mathcal{U} \subseteq \tau$ and $\bigcup\mathcal{U} \in \tau$ because for each $U \in \mathcal{U}, U = \bigcup\mathcal{B}'_U$ for some $\mathcal{B}'_U \subseteq \mathcal{B}$ So, $\bigcup\mathcal{U} = \bigcup\{\bigcup\mathcal{B}'_U : U \in \mathcal{U}\}$. Let $\mathcal{B}' = \bigcup\{\mathcal{B}'_U : U \in \mathcal{U}\}$. So, $\bigcup\mathcal{U} = \bigcup\mathcal{B}'$.
For $U, V \in \{\bigcup\mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}, U = B_1$ and $V = B_2, B_1, B_2 \in \mathcal{B}$ such that $B_1 = \mathcal{B}$ and $B_2 = \mathcal{B}$. Then $U \cap V \in \{\bigcup\mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} \supseteq W$ so there exists $x \in V, U : x \in U \cap V \supseteq W$. Therefore $x \in W \subseteq U \cap V \in \tau$. □

Theorem 8. *Let τ be a topology on X. Then $\mathcal{B} \subseteq \tau$ generates τ if:*
1. For all $x \in U \in \tau$, there exists $B \in \mathcal{B}$ where $x \in B \subseteq U \in \tau$
2. For all B_1, $B_2 \in \mathcal{B}$ with $x \in (B_1 \cap B_2)$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

Proof. :

1. For $x \in U \in \tau$, let $B = \{x\}$, so $x \in B \subseteq U \in \tau$.
2. $B_1, B_2 \in \mathcal{B}$. Let $B_3 = (B_1 \cap B_2)$, so $B_3 \subseteq (B_1 \cap B_2)$. Since $x \in (B_1 \cap B_2)$, $x \in B_3$. Thus, $x \in B_3 \subseteq (B_1 \cap B_2)$.

□

Theorem 9. $\{\mathbb{X}\}$ *is a basis for $\tau = \{\emptyset, \mathbb{X}\}$. (indiscrete)*

Proof. :

1. Let $\mathcal{B} = \{\mathbb{X}\}$ and let $x \in \mathbb{X} = B \in \mathcal{B}$.
2. Now consider $B_1, B_2 \in \mathcal{B}$. $B_1 = \mathbb{X}$ and $B_2 = \mathbb{X}$ Let $x \in B_1 \cap B_2$, then $x \in B_3 = \mathbb{X} : B_1 \cap B_2 = X$.

□

Theorem 10. $\{\{x\} : x \in \mathbb{X}\}$ *is a basis for* $\tau = \mathcal{P}(\mathbb{X}) = \{\ U : U \subseteq \mathbb{X}\}$. *(discrete)*

Proof. :

1. Let $x \in U \in \tau = \mathcal{P}(\mathbb{X})$. Then for $B = \{x\} \in \mathcal{B}, x \in B \subseteq U$.
2. Let $B_1, B_2 \in \mathcal{B}$. Let $y \in B_1 \cap B_2$ so $B_1 = B_2 = \{y\}$. Let $B_3 = \{y\}$ so $y \in B_3 = \{y\} \subseteq B_1 \cap B_2 = \{y\}$.

□

Definition 11. The Euclidean topology on $\mathbb{R}$ is the topology generated by the basis $\{(a, b) : a < b \in \mathbb{R}\}$.

Theorem 12. $\{(a,\ b)\ :\ a < b \in \mathbb{R}\}$ *is a basis.*

Proof. :

1. Let $\mathcal{B} = \{(a, b) : a < b \in \mathbb{R}\}$. Let $x \in \mathbb{R}$ and let $B = (x - 1, x + 1)$, then $B \subseteq \mathbb{R}$ and $B \in \mathcal{B}$.
2. For $B_1 = (a_1, b_1)$, $B_2 = (a_2, b_2)$, $a_1 < b_1$, $a_2 < b_2 \in \mathbb{R}$ with $x \in B_1 \cap B_2$, then there is $B_3 = (max(a_1, a_2), min(b_1, b_2)) \subseteq B_1 \cap B_2$ such that $a_1, a_2 < x < b_1, b_2$ so $x \in B_3$.

□

Theorem 13. $\{(a,\ b)\ :\ a < b \in \mathbb{Q}\}$ *is a basis for the Euclidean topology.*

Proof. :

1. Let $\mathcal{B} = \{(a, b) : a < b \in \mathbb{Q}\}$. Let $x, y \in \mathbb{Q}$ and let $B = (x-y, x+y)$, so $B \subseteq \mathbb{Q}$ and $x \in B \in \mathcal{B}$.
2. Now, for $B_1, B_2 \in \mathcal{B}, \exists x \in B_1 \cap B_2 : x \in B_1$ and $x \in B_2$. $a_1, a_2 < b_1, b_2 \in \mathbb{Q}$. Let $B_1 = (a_1, b_1)$ and $B_2 = (a_2, b_2)$. Now let $B_3 = (max(a_1, a_2), min(b_1, b_2))$ with $a_1, a_2 < x < b_1, b_2$ so $x \in B_3$. So, $x \in B_3 \subseteq B_1 \cap B_2$.

□

Definition 14. A subset K of a topological space X is said to be **compact** if for every open cover $\mathcal{U}$ of K (every collection $\mathcal{U}$ of open sets such that $\bigcup \mathcal{U} \supseteq K$) there exists a finite subcollection $\mathcal{F} \subseteq \mathcal{U}$ that also covers K (that is, $\bigcup \mathcal{F} \supseteq K$).

Theorem 15. *Any finite union of compact subsets is compact.*

Proof. K_1 and K_2 are compact so $\bigcup \mathcal{U}_1 \supseteq K_1$ and $\mathcal{U}_2 \supseteq K_2$. $\bigcup(\mathcal{U}_1 \cup \mathcal{U}_2) \supseteq K_1 \cup K_2$. $K_3 = K_1 \cup K_2$. Also, for finite $\mathcal{F}_1 \subseteq \mathcal{U}_1$ and $\mathcal{F}_2 \subseteq \mathcal{U}_2, \bigcup \mathcal{F}_1 \supseteq K_1 and \bigcup \mathcal{F}_2 \supseteq K_2$. So, $\bigcup(\mathcal{F}_1 \cup \mathcal{F}_2) \supseteq K_1 \cup K_2$. Thus, $K_1 \cup K_2$ is compact. □

Proposition 16. *Any finite subset of a topological space is compact.*

Proof. As shown previously, when K is compact there is $\mathcal{U}$ such that $\bigcup \mathcal{U} \supseteq K$. Also, there is a finite $\mathcal{F}$ such that $\bigcup \mathcal{F} \supseteq K$. For $F \in \mathcal{F}$, we can find an open covering $\mathcal{U}' \subseteq \mathcal{U}$ such that $\mathcal{U}' \supseteq F$. Also, we can find a finite $\mathcal{F}' \subseteq \mathcal{U}'$ such that $\bigcup \mathcal{F}' \supseteq F$. $\square$

Proposition 17. *The subset $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ of $\mathbb{R}$ is compact.*

Proof. $K = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. Let $\mathcal{U} = \{(n, n+1) : -1 < n < 2\}$. Then $\bigcup \mathcal{U} \supseteq K$. Now, there are clearly finitely many $F \in \mathcal{F} \subseteq \mathcal{U}$ such that $\bigcup \mathcal{F} \supseteq K$. $\square$

Proposition 18. *Any unbounded subset of $\mathbb{R}$ is not compact.*

Proof. For a subset to be compact it must be bounded by definition. An unbounded $K \subseteq \mathbb{R}$ cannot be compact because there is no finite $\mathcal{F}$ such that $\bigcup \mathcal{F} \supseteq K$ because K itself is not bound so it is impossible for $\bigcup \mathcal{F} \supseteq K$. Thus K cannot be compact. $\square$

Proposition 19. *Any open interval $(a, b) \subseteq \mathbb{R}$ is not compact.*

Proof. By the same argument, an open interval (a, b) is not bounded, and therefore not compact. For every open cover $\mathcal{U}$ of (a, b) has infinitely many $\mathcal{F} \subseteq \mathcal{U}$ such that $\bigcup \mathcal{F} \supseteq (a, b)$. So there are not finitely many $F \in \mathcal{F}$ and (a, b) is not compact. $\square$

Definition 20. A subset K of a topological space X is said to be **Lindelöf** if for every open cover $\mathcal{U}$ of K (every collection $\mathcal{U}$ of open sets such that $\bigcup \mathcal{U} \supseteq K$) there exists a countable subcollection $\mathcal{F} \subseteq \mathcal{U}$ that also covers K (that is, $\bigcup \mathcal{F} \supseteq K$).

Definition 21. A subset K of a topological space X is said to be **σ-compact** if there exist compact subspaces K_n of X for $n \in \mathbb{N}$ such that $K = \bigcup_{n \in \mathbb{N}} K_n$.

Theorem 22. *Every σ-compact subset of a topological space is Lindelöf.*

Proof. $K_{n \in \mathbb{N}}$ are countable since $\mathbb{N}$ is countable. If $K = \bigcup_{n \in \mathbb{N}} K_n$, then $\bigcup_{n \in \mathbb{N}} K_n \supseteq K$. $\square$

Definition 23. A subset K of a topological space X is said to be **hemicompact** if there exist compact subspaces K_n of X for $n \in \mathbb{N}$ such that $K = \bigcup_{n \in \mathbb{N}} K_n$, and for every compact subset $H \subseteq K$, $H \subseteq K_n$ for some $n \in \mathbb{N}$.

Theorem 24. *Every hemicompact subset of a topological space is σ-compact.*

Proof. By definition of σ-compact, we have compact subspaces K_n such that for σ-compact K', $K' = \bigcup_{n \in \mathbb{N}} K_n$. Obviously, for any subset H of the hemicompact set K, $H \subseteq K_n \subseteq K$ and since $K' = \bigcup_{n \in \mathbb{N}} K_n, H \subseteq K'$. $\square$