Project Summary

Intellectual Merit

Broader Impacts

Project Description

1 Overview

In the summers of 1967 and 1968, Lynn Arthur Steen and J. Arthur Seebach, Jr. coordinated two NSF-funded undergraduate research experiences at Saint Olaf College. Each summer, five undergraduate students assisted them in investigating and cataloging known and novel results in general topology. Specifically, their work was a treatment of topological spaces, properties preserved by homeomorphism, and theorems connecting these properties (e.g. all compact Hausdorff spaces are normal). In 1970, Steen and Seebach published a handbook covering this work, titled *Counterexamples in Topology* [5].

Mary Ellen Rudin wrote the following in her review [4] of the original text. "*Counterexamples in Topology* is a valuable addition to the small collection of books I keep on the shelf in my office." "The book is completely unique; no other book now in print serves its purpose." Recognizing the maze of counterexamples littered throughout the field of set-theoretic topology, Rudin suggested that students could benefit from the text as a guidebook. "Even those of us who work exactly in the area will profit from its organization." Several other "*Counterexamples*" texts in the tradition of Stein and Seebach have been published in several fields, including (but not limited to) real analysis [9], differential equations [3], probability [6], and graph theory [1].

Of the students involved in these topology REUs, several moved on to careers in mathematical research. These include John Feroe, Professor of Mathematics and Statistics at Vassar College; Gary Gruenhage, Professor Emeritus of Mathematics at Auburn University, specializing in set-theoretic topology; and Linda Ness, Chief Research Scientist: Applied Research, Vencore Labs. Stein and Seebach write the following in their introduction to *Counterexamples*, "We acknowledge that theirs was a twofold contribution: not only did they explore and develop many examples, but they proved by their own example the efficacy of examples for the undergraduate study of topology."

The handbook stands as a useful resource, even today. However, the examples in this original work are relatively elementary, and the research community has shifted focus away from some topics and onto others over the last four decades. To this end, the proposed REU program is a next-generation approach to extending *Counterexamples*, keeping in mind the needs of modern students and researchers in topology, and taking advantage of modern technology.

The PiBase, currently hosted at `https://topology.jdabbs.com`, is an open, web-based application for exploring and collaboratively contributing to the corpus of topological theorems and examples. It provides a rich search interface that allows both students and researchers to find examples relevant to the properties they are studying, and to do so without a protracted literature search. It includes an automated deduction engine, so that as new properties and theorems are added, existing examples are automatically updated, ensuring that new discoveries are shared publicly and visibly. Finally, the deduction engine can provide links back to all supporting assertions - eventually producing either a list of citations to the original research or a formally verifiable proof of each fact in the database. This application has already been used on a small scale for both research and educational purposes, and holds potential for drastically increasing the efficiency of topological research and instruction.

Participants in the proposed REU will contribute to the PiBase database, auditing its current entries as well as adding new content from more recent publications such as Watson's survey of topological planks and resolutions [7]. Participants will also be given mentorship as they choose a problem based on this work for the purpose of original research.

As a result of the proposed program, all students of set-theoretic topology will benefit, especially the program's participants. General topology is the backbone of many mathematical fields, and participants will be given the opportunity to develop their knowledge of this core. Generally, it is difficult to develop novel open questions in set-theoretic topology which are accessible by an undergraduate researcher. However, the PiBase application automatically detects which properties of a space cannot be deduced from the other properties and theorems entered into its database from the literature, providing a plethora of material on which the REU participants, as well as any undergraduate student, may work as original research. By uncovering the questions which researchers have not yet thought to ask or rigorously pursue, a robust PiBase will allow students to obtain valuable experience working on truely open problems, while contributing to the collective knowledge of the set-theoretic topology research community.

The proposers' aim is to simultaneously provide an authentic research experience for the undergraduate participants while developing the PiBase research assistant for use by the entire research community. Students are not all expected to pursue mathematical research as a career; however, the problem-solving skills developed during the program, and the exposure to software development and research cyberinfrastructure, will certainly benefit all participants regardless of their eventual careers. However, it is the investigators' hope that many participants will have their interest in mathematical research solidified by this experience, or even have it germinated for the first time.

Researchers of set-theoretic topology will also appreciate the product of this REU. Several major open questions in topology simply ask for the existence of, or counterexample to, a topological space satisfying certain properties (perhaps under various set-theoretic axioms). Additionally, it is not uncommon for seminar talks to be derailed by pondering the existence of one counterexample or another. So much of the community's knowledge is scattered across a diaspora of peer-reviwed papers in numerous journals, meaning many "open" questions may actually be a simple corollary of results from two or more heretofore unconnected articles, which would be automatically connected by a robust PiBase. Likewise, several properties have been studied under various names, whether for historical reasons, or because these properties were later shown to be equivalent; other properties share the same names, while actually being distinct (at least in a sufficiently general setting). Spaces and properties in the PiBase are tagged with unique IDs, preventing any ambiguity, and providing researchers a common language when referencing existing spaces and properties from the literature.

The benefits of this program will not be restricted to only students of topology, or even researchers in set-theoretic topology. After the PiBase database has been updated to reflect the modern status of topological research, data on its utility as a tool for students and researchers may be collected. At its core, PiBase is a tool which may be generalized to relate the objects, categorical invariants, and theorems relating those invariants within any given mathematical category. Once the PiBase is battled-tested within one field, it will serve as a proof of concept for researchers of different categories, and can be adapted to serve those communities as well.

As described in more detail below, the proposed program will take place over two summers, each summer lasting eight weeks with a team of eight talented undergraduate students. The sponsoring institution is The University of South Alabama, located in Mobile, Alabama. The program will be run virtually, in order to reach out to all United States citizens, nationals, and permanent residents, regardless of their geographical location, as well as allow for the seamless interaction of guest speakers and mentors from around the world. The resources saved by not funding housing or travel for the REU will be repurposed for supporting stipends, conference travel, and conference lodging for students.

Leadership of the proposed program is divided between Dr. Steven Clontz, handling the logistics of running the program and the combined research and cyberinfrastructure expertise required for nurturing a next-generation *Counterexamples*, and Dr. Ziqin Feng, bringing his research expertise in the fields of set-theroetic topology and set theory. They will also be supported by James Dabbs, the developer and maintainer of the PiBase project, who received his Masters in mathematics studing set-theoretic topology before entering industry as a software engineer.

{TODO Add participating organizations' commitment to the REU activity.}

2 Nature of Student Activities

Despite the differing nature of onsite and virtual undergraduate research experiences, student activities for the proposed program will be very similar to those in an onsite mathematics REU. All meetings will take place between organizers and participants using the low-cost persistent chatroom and videoconferencing service Slack for Education, utilizing features such as screensharing, virtual whiteboarding, and mathematical typesetting. Virtual meetings will be scheduled between 8am and 5pm for all participating timezones.

The first two weeks of the program will serve to orient participants to the field, building upon their existing knowledge of the topology of Euclidean space $\mathbb{R}^n$ and any previous topology courses they have taken. This orientation will begin by covering selected spaces and their properites from the PiBase database, and later expanding to theorems in the database. Group investigations will take place during morning meetings, led by one of the organizers, followed by assigned work to be done in pairs by the participants in the afternoons. The length of this orientation period may be adjusted based upon the competency of the participants, but particularly in the first summer this will also serve to audit the existing contents of the PiBase database. Organizers will serve as mentors during this period, and will check in with the participants during the afternoon to provide advice or assistance as required.

Following this introductory period, the remainder of the summer will consist of each student being assigned several spaces and properties from the literature, to be studied and catalogued in the PiBase database. These assignments will be chosen by the organizers based upon the competencies and interests of the participants, and some will be assigned to multiple participants for the purposes of peer review and collaboration. Once or twice each day, a student will lead a presentation with an organizer and the other participants based upon the material they have investigated since their last presentation. Occassionally, these presentations will be substituted with talks from active researchers in set-theoretic topology or other fields of interest to the participants. These speakers will be brought in virtually through free videoconferencing solutions from anywhere in the world.

As participants work through their assigned material, several questions will arise from blanks in the database which cannot be directly answered from the literature. Under the organizers' mentorship, participants will choose several questions for their original research. Generally, these questions will be of one of the following forms:

- "Does space S have property P?"
- "Do properties $\{P_i : 0 \leq i < n\}$ imply properties $\{Q_i : 0 \leq i < m\}$?"
- "How can a space be constructed such that it satisifes properties $\{P_i : 0 \leq i < n\}$?"

Such questions can be automatically generated by PiBase; however, it is up to the participants and their mentors to choose questions appropriate for their competencies and interests. The expectation for each participant is that his or her work will result in one or more publications within journals appropriate for the significance of the obtained results. Each participant is also expected to give a talk on their research at either a meeting of the AMS/MAA, a conference on undergraduate research, or a topology conference, following the conclusion of the program.

Knowledge of elementary topological theory has become fundamentally important in a wide range of fields in mathematics, including but not limited to analysis, algebra, descriptive set theory, and functional analysis. The construction of topological examples plays an essential rule in illustrating this theory. However, it often requires great effort to construct needed examples as the theory of topological constructions is immensely deep. Watson's survey [7], the organizers' primary source for published counterexamples to be entered into PiBase, showcases two powerful processes of building new topological spaces.

2.1 Planks

A plank is a product of two topological spaces X_0 and X_1, each with some distinguished points, and with a particular subset removed. This process provides a powerful method to construct interesting topological spaces with desired topological properties. Many important examples in the field were constructed as planks. Two classical examples of planks are the Tychonoff plank and the Dieudonne plank. The constructions of both planks are based on two simple topological spaces $\omega + 1$ and $\omega_1 + 1$ equipped with the order topology, where ω is the first infinite ordinal and ω_1 is the first uncountable ordinal. The Tychonoff plank [8, pp. 122-123] {TODO cite Alexandroff and Urysohn; Tychonoff} is the topological space $(\omega + 1) \times (\omega_1 + 1) \setminus \{(\omega, \omega_1)\}$, which is the simplest example of a pseudocompact space which is not countably compact. The Dieudonne plank [2] [5, Ex38] is the topological space $(\omega_1 + 1)^2 \setminus (\omega_1 + 1) \times \omega_1$ which demonstrates that countable compact spaces need not be normal.

The construction of plank-type examples also appears in analysis. A Dowker-style plank is obtained by taking a subspace of the products of Tychonoff spaces Z_0, Z_1 to be determined by a function $f : Z_0 \to Z_1$. Using this technique, Haydon (also, Gardner and Pfeffer) {TODO cite} constructed a normal realcompact space which admits a locally trivial yet non-trivial regular Borel measure. Let E be a subset of $[0, 1]$ with positive outer measure of minimal cardinality κ, and let

$f : E \to \kappa$ be a bijection. Then the Haydon plank is the topological space $\{(\alpha, e) : e \in E, \alpha \leq f(e)\}$, a subset of $\kappa \times [0, 1]$. A Borel measure on a space X is called locally trivial if X can be covered by open sets of measure zero. A Borel measure on X is called trivial if the measure of X is zero.

2.2 Resolutions

Many of the classical examples in topology are constructed through the same fundamental operation in the construction of topological spaces - the operation of resolutions. Such examples include the double arrow space, the Alexandroff duplicate, the butterfly space, and the Michael line. All these examples mentioned here play important roles in understanding non-trivial topological concepts.

Let $[0, 1] \times 2$ be the product of the unit interval and the two-point discrete space. Define a topology on $[0, 1] \times 2$ {TODO cite (Alexandroff and Urysohn)} by declaring the singletons $\{(x, 1)\}$ to be open, and for any open set $U \subset [0, 1]$, declare $(U \times 2) \setminus \{(x, 1)\}$ to be open as well. The space constructed is a hereditarily paracompact first countable compact Hausdorff space which has an uncountable disjoint family of open sets. Furthermore, let $\mathbb{P}$ be the set of irrationals and $\mathbb{Q}$ be the set of rationals. Take the subspace $(\mathbb{P} \times \{1\}) \cup (\mathbb{Q} \times \{0\})$ of $[0, 1] \times 2$ under the topology defined above. We get the Michael line which is zero-dimensional and hereditarily paracompact, but not metrizable.

Fedorcuk {TODO cite} gives the following definition of resolution which generalized this elegant and fundamental process of resolution.

Definition. Suppose X is a topological space and $\{Y_x : x \in\}$ are topological spaces, and for each $x \in X$, $f_x : X \to Y_x$ is a continuous mapping. We topologize $Z = \bigcup\{\{x\} \times Y_x : x \in X\}$ by defining an open set $U \otimes W$ for each $x \in X$, each open set $U \subset X$ such that $x \in U$ and each open set $W \subset Y_x$. We define $U \otimes W = (\{x\} \times W) \cup \bigcup\{\{x'\} \times Y_{x'} : x' \in U \cap f_x^{-1}(W)\}$. We call Z the *resolution* of X at each point $x \in X$ into Y_x by the mapping f_x.

So, resolve each element of $[0, 1]$ at each interior point $x \in [0, 1]$ into 2 by the mapping $f_x(x') = 0$ for $x' < x$ and $f_x(x') = 1$ for $x' > x$. This produces the double arrow space which is hereditarily separable and hereditarily Lindelof, but not metrizable. The Sorgenfrey line is the subspace $[0, 1] \times \{0\}$ of the double arrow space, a normal space whose square is not normal.

Another important example obtained by resolution is the topological sine curve. Resolve $[0, 1]$ at 0 in to $[0, 1]$ by the function $f : (0, 1] \to [0, 1]$ such that for each $\epsilon > 0$, $[0, 1] \subset f((0, \epsilon))$. The topological sine curve is an important example of a continuum (a connected compact metric space) which is not locally connected.

2.3 Other benefits of participation

In addition to the mathematical training and experience the participants will receive, they will also be exposed to several technologies and topics outside set-theoretic topology. These topics include, but are not limited to, the cyberinfrastructure of mathematical research, mathematical typesetting, web application development, teamwork/leadership skills, pedagogy, and presentation skills. These skills will benefit the student participants in nearly any chosen career.

While the development of PiBase as a tool for modern students and researchers of topology is an important goal, the primary focus of the proposed program is the academic and professional development of the participating students. These participants will develop relationships not only with their peers and the program organizers, but also with various invited speakers and the greater community developing the PiBase source and contributing to its database. Participants will be required to show evidence of collaboration with each other and their mentors on their assigned projects in order to develop these important collegial relationships. Participants will be encouraged to stay involved in the PiBase community after the completion of the REU, perhaps even returning as a virtual speaker during a future year, as long as their research interests overlap with the content of the program.

3 The Research Environment

{TODO add explanation of the benefits of a virtual research environment}

{TODO Steven's research bio}

{TODO Ziqin's research bio}

4 Student Recruitment and Selection

Recruitment for this program will be centralized on a webpage hosted by the University of South Alabama. Participants will be able to apply through a responsive online form a desktop computer, tablet, or mobile device. The site will include a brief description of the program, describing the PiBase application and the mathematical concepts involved. Emails advertising this site will be disseminated to the organizers' network of colleagues in various areas of topology, attracting promising candidates from across the country with an interest in topology and the cyberinfrastructre of mathematical research. The program will also be announced on popular social media outlets such as Reddit. Demographics underrepresented in mathematics will be especially encouraged to apply; as such, email advertisements will be also be distributed to organizations which represent such groups.

Each application will outline the candidate's academic experience and ability, as well as his or her background and interests. Only U.S. citizens, U.S. nationals, and permanent residents of the United States will be eligible to apply. After an evaluation of applications, the organizers will select roughly twice as many candidates as there are spaces for the REU as a shortlist. These candidates will be given a short interview over Skype or Google Hangouts to assess their ability to communicate and collaborate virtually, and confirm the academic experience and ability represented by their application. These candidates will be ranked based on their applications and interviews, and invitations will be given to the top half of candidates, with the other candidates put on a waitlist in case a selected candidate declines to accept his or her invitation. Candidates who can contribute to the diversity of the REU program will be given preference when selecting candidates for interviews and invitations, as will candidates who represent academic institutions where STEM research opportunities are limited. Preference will also be given to candidates who demonstrate a need for a virtual research experience over a traditional onsite REU. Students from the organizers' home institutions will be given no preferential treatment during this process, and no more than one-quarter of the invited participants will come from those institutions.

5 Project Evaluation and Reporting

The two main metrics of success for this project will be the academic success of its participants, followed by the utility of the PiBase database they help develop during the program. Participants will be encouraged to stay involved in the PiBase community after the completion of the program, and to maintain correspondence with the program organizers. By the nature of the Slack service, most written communictions are catalogued and may be searched for evaluation purposes.

The project directors will collect evaluations from the particpants at the end of each summer, and track their publications, talks, and accomplishments following the program. Once PiBase is ready for production, researchers associated with the Spring Topology and Dynamics Conference and the Summer Conference on Topology and its Applications will be encouraged to try it and provide feedback on its utility. From this combined data, the project directors will write a final evaluation of the program for the NSF.

6 Broader Impacts

The broader goal of the proposed program is to develop next-generation techniques in the cyber-infrastructure of mathematical research, not only within participating students but for the entire research community. Generally, researchers in set-theoretic topology and its related fields have not taken much advantage of technology beyond mathematical typesetting tools and email. In contrast, this program relies heavily on vairous internet-based communication tools, which if adopted by the wider reserach community would enhance productivity in producing new results by connecting researchers and facilitating collaboration. It also serves to further develop the PiBase topology research assistant, which will help all reserachers of topology be more productive in their work by faciliting communications, centralizing topological results, and providing a common language for otherwise ambiguous terminology or notation.

By virtue of being a virtual REU, this program also has the potential to bring a rigorous research experience to students who otherwise may not be able to participate in a traditional onsite REU. The only requirement for the participants is consistent access to high-speed interent on a laptop or desktop computer; all required software and services will be freely available to the participants. Furthermore, because students will never "leave" the REU host, the research experience can extend well beyond the eight weeks for an interested participant, and the relationships forged during the REU experience will not waver due to geographical distance after the end of the program.

7 Results From Prior NSF Support

No prior NSF support has been given for this project.

References Cited

[1] Michael Capobianco and John C. Molluzzo. *Examples and counterexamples in graph theory.* North-Holland, New York-Amsterdam-Oxford, 1978. Foreword by Gary Chartrand.

[2] Jean Dieudonné. Une généralisation des espaces compacts. *J. Math. Pures Appl. (9)*, 23:65–76, 1944.

[3] John M. Rassias. *Counterexamples in differential equations and related topics.* World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.

[4] Mary Ellen Rudin. Reviews: Counterexamples in Topology. *Amer. Math. Monthly*, 78(7):803–804, 1971.

[5] Lynn Arthur Steen and J. Arthur Seebach, Jr. *Counterexamples in topology.* Dover Publications, Inc., Mineola, NY, 1995. Reprint of the second (1978) edition.

[6] Jordan M. Stoyanov. *Counterexamples in probability.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Ltd., Chichester, 1987.

[7] Stephen Watson. The construction of topological spaces: planks and resolutions. In *Recent progress in general topology (Prague, 1991)*, pages 673–757. North-Holland, Amsterdam, 1992.

[8] Stephen Willard. *General topology.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.

[9] Gary L. Wise and Eric B. Hall. *Counterexamples in probability and real analysis.* The Clarendon Press, Oxford University Press, New York, 1993.

Biographical Sketch: Steven Clontz

1 Professional Preparation

• Auburn University	Auburn, AL	Mathematics	B.S., 2008
• Auburn University	Auburn, AL	Mathematics	M.S., 2010
• Auburn University	Auburn, AL	Mathematics	Ph.D., 2015

2 Appointments

- Assistant Professor, The Univerity of South Alabama, 2016-present.
- Visiting Assistant Professor, The University of North Carolina at Charlotte, 2015-2016.

3 Publications

3.1 Related to Proposal

- Clontz, S. and Gruenhage, G. *Proximal compact spaces are Corson compact.* Topology Appl. 173 (2014), 1-8.
- Clontz, S. *Zero-Markov information in topological games.* Ala. J. Math. 39 (2015).
- Clontz, S. *Tactic-proximal compact spaces are strong Eberlein compact.* Topology Appl. 204 (2016), 306-317.
- Clontz, S. *On k-tactics for Gruenhage's compact-point game.* Q&A in Gen. Topology 34 (2016), 1-10.
- Clontz, S. and Varagona, S. *Destruction of metrizability in generalized inverse limits.* Top. Proc. 48 (2016), 289-297.

4 Synergistic Activities

- Director of Mathematical Puzzle Programs, an organization which organizes mathematical puzzle events to encourage young people to engage in mathematical problem-solving.
- Maintainer of the Online Streams in Mathematics project, which provides support for streaming mathematical talks and lectures online using free open-source software.
- Reviewer for Mathematical Reviews / MathSciNet.

- Founder and software engineer for Teloga.com, customer relationship management software-as-a-service for collegiate music organizations from 2011-2015.
- Served the National Youth Leadership Training program at Camp Westmoreland, Florence, AL from 2002-2011 as director, quartermaster, and numerous other roles.

5 Collaborators and Other Affiliations

5.1 Collaborators and Co-Editors

- Gary Gruenhage (Auburn University)
- Scott Varagona (University of Montevallo)

5.2 Graduate Advisors and Postdoctoral Sponsors

- Gary Gruenhage (PhD Advisor, Auburn University)
- Alan Dow (Mentor, UNC Charlotte)

5.3 Thesis Advisor and Postgraduate-Scholar Sponsor

- Undergraduate Project: Tanyce James (UNC Charlotte)