MA 126 — Spring 2017 — Prof. Clontz — Final Exam

Name: ______________________________

- Each question corresponds to a Standard for this course.
- Four questions are asked of all students, attached to this cover sheet.
- Each student may choose up to 12 additional questions from the provided booklet. Each choice must be clearly marked at the top of a provided answer sheet and stapled to this cover sheet upon submission.
- When grading, each response will be marked as follows:
  - $\checkmark$: The response is demonstrates complete understanding of the Standard.
  - $\star$: The response may indicate full understanding of the Standard, but clarification or minor corrections are required.
  - $\times$: The response does not demonstrate complete understanding of the Standard.
- Up to three $\star$ marks will be converted to $\checkmark$ marks automatically.
- Only responses marked as $\checkmark$ count toward your grade for the semester.
- This Assessment is due after 120 minutes.

C16a: This student is able to... Approximate series and power series within appropriate margins of error.	Mark: (Instructor Use Only)

Recall that the Maclaurin series generated by e^x is $\sum_{k=0}^{\infty} \frac{x^k}{k!}$. Use Taylor's Formula for error

$$R_n(x) = \frac{f^{n+1}(x_n)}{(n+1)!}(x-a)^{n+1}$$

to show that the value of $\frac{1}{e}$ is within 0.01 of $\frac{3}{8} = 0.375$.

C16b: This student is able to... Approximate series and power series within appropriate margins of error.	Mark: (Instructor Use Only)

Recall that the Maclaurin series generated by $\cos(x)$ is $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{2k!}$. Use Taylor's Formula for error

$$R_n(x) = \frac{f^{n+1}(x_n)}{(n+1)!}(x-a)^{n+1}$$

to show that the value of $\cos(0.5)$ is within 0.01 of $\frac{7}{8} = 0.875$.

S16: This student is able to... Find a power series converging to a function.	Mark: (Instructor Use Only)

Recall that the Maclaurin series generated by e^x is $\sum_{k=0}^{\infty} \frac{x^k}{k!}$. Prove that $3x^2e^{x^3} = \sum_{k=0}^{\infty} \frac{3x^{3k+2}}{k!}$.

S17: This student is able to... Prove the convergence of a Taylor or Maclaurin Series using Taylors Formula.	Mark: (Instructor Use Only)

Recall that the Maclaurin series generated by e^x is $\sum_{k=0}^{\infty} \frac{x^k}{k!}$. Use Taylor's Formula for error

$$R_n(x) = \frac{f^{n+1}(x_n)}{(n+1)!}(x-a)^{n+1}$$

to prove that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Write the code (e.g. C09b) for the question you are attempting to answer:	Mark: (Instructor Use Only)

C01a: This student is able to...
Derive properties of the logarithmic and exponential functions from their definitions.

Use the definition $\ln x = \int_1^x \frac{1}{t}\,dt$ to prove that $\ln(4x) = \ln x + \ln 4$ for all positive real numbers x.

C01b: This student is able to...
Derive properties of the logarithmic and exponential functions from their definitions.

Let $f^{\leftarrow}$ denote the inverse function of an invertable function f; in particular, if $f(x) = \exp(x)$, then $f^{\leftarrow}(x) = \ln(x)$.
Use the theorems $\frac{d}{dx}[f^{\leftarrow}(x)] = \frac{1}{f'(f^{\leftarrow}(x))}$ and $\frac{d}{dx}[\exp x] = \exp x$ to prove that $\frac{d}{dx}[\ln x] = \frac{1}{x}$.

C02a: This student is able to...
Prove hyperbolic function identities.

Use the definitions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}$$

to prove the following identity.

$$\cosh^2(x) = 1 + \sinh^2(x)$$

C02b: This student is able to...
Prove hyperbolic function identities.

Use the definitions

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}$$

to prove the following identity.

$$\operatorname{sech}^2(x) + \tanh^2(x) = 1$$

C03a: This student is able to...
Use integration by substitution.

Find $\displaystyle\int 2x^3\sqrt{x^2+1}\,dx$.

C03b: This student is able to...
Use integration by substitution.

Find $\displaystyle\int \frac{e^{2x}}{e^{2x}+1}\,dx$.

C04a: This student is able to...
Use integration by parts.

Find $\int 3y^2 e^y \, dy$.

C04b: This student is able to...
Use integration by parts.

assessmentTitle Find $\int \sin(x) \cosh(x) \, dx$.

C05a: This student is able to...
Identify and use appropriate integration techniques.

Match each of the five integrals on the left with the most appropriate integration technique listed on the right. Multiple techniques may be technically possible, but choose the technique most useful to begin integration. Every integral and technique is used exactly once in the correct answer.

a) $\displaystyle\int \sin^3(x) \cos^4(x) \, dx$	1) Integration by Substiution
b) $\displaystyle\int \frac{x^2+x+1}{x^3+x} \, dx$	2) Method of Partial Fractions
c) $\displaystyle\int e^x \cos(1+e^x) \, dx$	3) Trigonometric Identities
d) $\displaystyle\int x \sin(x) \, dx$	4) Trigonometric Substitution
e) $\displaystyle\int \frac{1}{4+x^2} \, dx$	5) Integration by Parts

C05b: This student is able to...
Identify and use appropriate integration techniques.

(See C05a for instructions.)

a) $\displaystyle\int \frac{3x+4}{(x-1)(x+2)^2}\,dx$

b) $\displaystyle\int x^2\sqrt{3x^3+4}\,dx$

c) $\displaystyle\int \frac{1}{4+x^2}\,dx$

d) $\displaystyle\int e^x \sin(x)\,dx$

e) $\displaystyle\int \sec^4(x)\tan^2(x)\,dx$

1) Integration by Substiution

2) Method of Partial Fractions

3) Trigonometric Identities

4) Trigonometric Substitution

5) Integration by Parts

C06a: This student is able to...
Express an area between curves as a definite integral.

Find a definite integral equal to the area between the curves $y = 2x$ and $y = x^2$. (Do not solve your integral.)

C06b: This student is able to...
Express an area between curves as a definite integral.

Find a definite integral equal to the area between the curves $y = \sqrt{4-x^2}$ and $y = 1$. (Do not solve your integral.)

C07a: This student is able to...
Use the washer or cylindrical shell method to express a volume of revolution as a definite integral.

Find a definite integral equal to the volume of the solid of revolution obtained by rotating the triangle with vertices $(1,1)$, $(2,1)$, and $(1,2)$ around the axis $x = 0$. (Do not solve your integral.)

C07b: This student is able to...
Use the washer or cylindrical shell method to express a volume of revolution as a definite integral.

Find a definite integral equal to the volume of the solid of revolution obtained by rotating the region bounded by $y = x$ and $y = x^2$ around the axis $y = 0$. (Do not solve your integral.)

C08a: This student is able to...
Express the work done in a system as a definite integral.

Find the work required to pull up a fully extended 50 foot cable that weighs 200 pounds. (Do not solve your integral.)

C08b: This student is able to...
Express the work done in a system as a definite integral.

Hooke's Law states that the force required to stretch a spring x units from its natural length requires $F(x) = kx$ units of force for some constant k (depending on the spring). Suppose a spring satisfies $k = 5$ and is naturally length 9. Find a definite integral equal to the work required to compress this spring from length 11 to length 14. (Do not solve your integral.)

C09a: This student is able to...
Parametrize a curve to express an arclength or area as a definite integral.

Find the arclength of $x = 3y^2$ between $(3, -1)$ and $(12, 2)$.
(Do not solve your integral.)

C09b: This student is able to...
Parametrize a curve to express an arclength or area as a definite integral.

Recall the following. A smooth curve parametrized by one-to-one functions $x(t), y(t)$ on $a \leq t \leq b$ where $y(t) \geq 0$ may be rotated around the x-axis to yield a surface of revolution. Its area is given by $2\pi \int_a^b y(t)\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$.
Use this to find a definite integral equal to the conical surface area obtained by rotating the line segment connecting $(1, 0)$ and $(5, 2)$ around the line $y = 0$.
(Do not solve your integral.)

C10a: This student is able to...
Use polar coordinates to express an arclength or area as a definite integral.

Find a definite integral equal to the circumference of the cardioid $r = 3 + 3\cos\theta$.
(Do not solve your integral.)

C10b: This student is able to...
Use polar coordinates to express an arclength or area as a definite integral.

Find a definite integral equal to the area inside the cardioid $r = 2 - 2\sin\theta$.
(Do not solve your integral.)

C11a: This student is able to...
Compute the limit of a convergent sequence.

Find $\lim_{n\to\infty} \dfrac{4n+n^4}{5n^4+n^2-3}$.

C11b: This student is able to...
Compute the limit of a convergent sequence.

Find $\lim_{n\to\infty} \dfrac{\sqrt{e^{2n}+1}}{e^{n+1}}$.

C12a: This student is able to...
Express as a limit and find the value of a convergent geometric or telescoping series.

Find the value of the convergent series $\sum_{k=2}^{\infty} \frac{2}{6^{k-1}}$.

C12b: This student is able to...
Express as a limit and find the value of a convergent geometric or telescoping series.

Find the value of the convergent series $\sum_{k=1}^{\infty} (\frac{2k+3}{k} - \frac{2k+5}{k+1})$.

C13a: This student is able to...
Identify and use appropriate techniques for determining the convergence or divergence of a series.

Recall the following types of series and techniques for determining series converence.

- Telescoping Series
- Geometric Series
- Alternating Series Test
- Integral Test
- p-Series Test
- Ratio Test
- Root Test
- Comparison Test (Direct/Limit)

Label the following four series with an appropriate type of series or technique for determining series convergence. Then label whether each series converges or diverges (you do not need to show any work).

1) $\displaystyle\sum_{k=0}^{\infty} \frac{k}{\sqrt{k^3+1}}$ 2) $\displaystyle\sum_{m=1}^{\infty} 2m^{-1/2}$ 3) $\displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{n}{n^2+1}$

C13b: This student is able to...
Identify and use appropriate techniques for determining the convergence or divergence of a series.

(See C13a for instructions.)

1) $\displaystyle\sum_{k=0}^{\infty} \frac{k^2+3k+2}{2^k}$ 2) $\displaystyle\sum_{m=1}^{\infty} \frac{4^m}{m!}$ 3) $\displaystyle\sum_{n=3}^{\infty} \frac{2^{2n}}{5^n}$

C14a: This student is able to...
Identify the domain of a function defined as a power series.

Prove that $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} = 1 + (x-2) + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{6} + \ldots$ is defined for all real numbers x.

C14b: This student is able to...
Identify the domain of a function defined as a power series.

Prove that the domain of $g(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n2^n} = \frac{(x-2)}{2} + \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} + \ldots$ is $0 \leq x < 4$.

C15a: This student is able to...
Generate a Taylor or Maclaurin Series from a function.

Generate the Maclaurin Series for $e^{x/2}$.

C15b: This student is able to...
Generate a Taylor or Maclaurin Series from a function.

Generate the Taylor Series for $3x^2 + 4x + 7$ at $x = 1$. Write your answer in the form $c_0 + c_1(x-1) + c_2(x-1)^2$.

S01: This student is able to...
Find derivatives and integrals involving logrithmic and exponential functions.

a) Find $\frac{d}{dz}[e^{2\ln(z)}]$.
b) Find $\displaystyle\int \left(e^y - \frac{2}{y}\right) dy$.

S02: This student is able to...
Find derivatives and integrals involving hypberbolic functions.

a) Find $\frac{d}{dv}[4\sinh(3v) - \operatorname{sech}(v^2)]$.
b) Find $\displaystyle\int (\sinh(x) - 2\operatorname{sech}^2(x))\, dx$.

S03: This student is able to...
Integrate products of trigonometric functions.

Find $\int \sin^3(y)\cos^3(y)\, dy$.

S04: This student is able to...
Use trigonometric substitution.

Find $\displaystyle\int \frac{4}{4+z^2}\, dz$.

S05: This student is able to...
Use partial fractions to integrate rational functions.

Find $\displaystyle\int \frac{3x^2+2x+4}{(x^2+4)(x+1)}\, dx$.

S06: This student is able to...
Use cross-sectioning to express a volume as a definite integral.

Find a definite integral that equals the volume of a solid whose base is the triangle with vertices $(0,0)$, $(2,2)$, and $(2,0)$, and whose cross-sections perpindicular to the x-axis are squares with bases on the xy plane. (Do not solve your integral.)

S07: This student is able to...
Derive a formula for the volume of a three dimensional solid.

Prove that the volume of a sphere with radius a is $V = \frac{4}{3}\pi a^3$.

S08: This student is able to...
Parametrize planar curves and sketch parametrized curves.

a) Give a parameterization of the curve $xy = 9$ from $(1,9)$ to $(27, \frac{1}{3})$.
b) Sketch the curve parameterized by $x = 4+t$, $y = 5-2t$ for $-1 \leq t \leq 2$.

S09: This student is able to...
Use parametric equations to find and use tangent slopes.

Find the slope of the tangent line to the curve defined parametrically by $x = 4 + \cos(t)$, $y = 5 + \sin(t)$ for $0 \leq t \leq 2\pi$ at the point $(4, 6)$.

S10: This student is able to...
Convert and sketch polar and Cartesian coordinates and equations.

a) Find a Cartesian coordinate equal to the polar coordinate $p(4, -2\pi/3)$.
b) Sketch the cardioid $r = 5 + 5\sin\theta$ in the xy plane.

S11: This student is able to...
Define and use explicit and recursive formulas for sequences.

Give an explicit or recursive formula matching the sequence $\langle t_n \rangle_{n=0}^{\infty} = \langle 0, 1, 3, 6, 10, 15, 21, 28, \dots \rangle$.

S12: This student is able to...
Use the alternating series test to determine series convergence.

Does $\sum_{m=0}^{\infty} (-1)^m \frac{3}{(\ln(m+5))^2}$ converge or diverge?

S13: This student is able to...
Use the integral test to determine series convergence.

a) Does $\int_1^{\infty} \frac{2}{\sqrt{x}}\,dx$ converge or diverge?
b) Based on (a), does $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$ converge or diverge?

S14: This student is able to...
Use the ratio and root tests to determine series convergence.

Does $\sum_{n=0}^{\infty} \frac{2^n}{(n+2)!}$ converge or diverge?

S15: This student is able to...
Use the comparison tests to determine series convergence.

Does $\sum_{n=0}^{\infty} \frac{4n+1}{\sqrt{n^4+4}}$ converge or diverge?