# Module S: Structure of vector spaces

# What structure do vector spaces have?

At the end of this module, students will be able to...

**S1.** **Linear independence.** ... determine if a set of Euclidean vectors is linearly dependent or independent.

**S2.** **Basis verification.** ... determine if a set of Euclidean vectors is a basis of $\mathbb{R}^n$.

**S3.** **Basis computation.** ... compute a basis for the subspace spanned by a given set of Euclidean vectors.

**S4.** **Dimension.** ... compute the dimension of a subspace of $\mathbb{R}^n$.

**S5.** **Abstract vector spaces.** ... solve exercises related to standards V3-S4 when posed in terms of polynomials or matrices.

**S6.** **Basis of solution space.** ... find a basis for the solution set of a homogeneous system of equations.

**Readiness Assurance Outcomes**

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.
- Apply linear combinations and spanning sets **V3,V4**.

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting complex numbers (Khan Academy): `http://bit.ly/1PE3ZMQ`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

# Module S Section 1

**Activity S.1.1** *(∼10 min)*

Consider the two sets

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}$$

Which of the following is true?

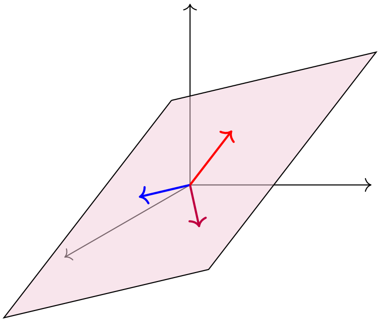
(A) span $S$ is bigger than span $T$.

(B) span $S$ and span $T$ are the same size.

(C) span $S$ is smaller than span $T$.

**Definition S.1.2**

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.



You can think of linearly dependent sets as containing a redundant vector, in the sense that you can drop a vector out without reducing the span of the set. In the above image, all three vectors lay on the same planar subspace, but only two vectors are needed to span the plane, so the set is linearly dependent.

**Activity S.1.3** *(∼10 min)*

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$. Suppose $3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$, so the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. Which of the following is true of the vector equation $x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = \mathbf{0}$ ?

(A) It is consistent with one solution

(B) It is consistent with infinitely many solutions

(C) It is inconsistent.

**Fact S.1.4**

For any vector space, the set $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is linearly dependent if and only if $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{z}$ is consistent with infinitely many solutions.

**Activity S.1.5** *(∼10 min)*
Find

$$\operatorname{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent.

**Fact S.1.6**

A set of Euclidean vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is linearly dependent if and only if RREF $\begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}$ has a column without a pivot position.

**Activity S.1.7** *(∼5 min)*

Is the set of Euclidean vectors $\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$ linearly dependent or linearly independent?

**Activity S.1.8** *(∼10 min)*

Is the set of polynomials $\left\{x^3+1, x^2+2x, x^2+7x+4\right\}$ linearly dependent or linearly independent?

**Activity S.1.9** *(∼5 min)*

What is the largest number of vectors in $\mathbb{R}^4$ that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

**Activity S.1.10** *(~5 min)*

What is the largest number of vectors in

$$\mathcal{P}^4 = \left\{ ax^4 + bx^3 + cx^2 + dx + e \,\middle|\, a, b, c, d, e \in \mathbb{R} \right\}$$

that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

**Activity S.1.11** *(~5 min)*

What is the largest number of vectors in

$$\mathcal{P} = \{f(x) \mid f(x) \text{ is any polynomial}\}$$

that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

# Module S Section 2

## Definition S.2.1

A **basis** is a linearly independent set that spans a vector space.

The **standard basis** of $\mathbb{R}^n$ is the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \cdots \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For $\mathbb{R}^3$, these are the vectors $\mathbf{e}_1 = \hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{e}_3 = \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

**Observation S.2.2**

A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

For example, in many Calculus 3 courses, vectors often expressed in their component form

$$(3, -2, 4) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

or in their standard basic vector form

$$3\mathbf{e}_1 - 2\mathbf{e}_2 + 4\mathbf{e}_3 = 3\hat{\imath} - 2\hat{\jmath} + 4\hat{k}.$$

**Activity S.2.3** *(~15 min)*

Label each of the sets $A, B, C, D, E$ as:

- SPANS $\mathbb{R}^4$ or DOES NOT SPAN $\mathbb{R}^4$
- LINEARLY INDEPENDENT or LINEARLY DEPENDENT
- BASIS FOR $\mathbb{R}^4$ or NOT A BASIS FOR $\mathbb{R}^4$

$$A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\} \qquad D = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\}$$

$$E = \left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

**Activity S.2.4** *(∼10 min)*
If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$, that means RREF$[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$ doesn't have a column without a pivot position, and doesn't have a row of zeros. What is RREF$[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$?

$$\text{RREF}[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4] = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

**Fact S.2.5**

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis for $\mathbb{R}^n$ if and only if $m = n$ and

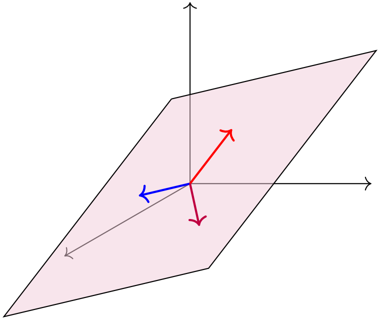
$$\mathrm{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

That is, a basis for $\mathbb{R}^n$ must have exactly $n$ vectors and its square matrix must row-reduce to the so-called **identity matrix** containing all zeros except for a downward diagonal of ones. (We will learn where the identity matrix gets its name in a later module.)

## Observation S.2.6

Recall that a **subspace** of a vector space is a subset that is itself a vector space.

One easy way to construct a subspace is to take the span of set, but a linearly dependent set contains "redundant" vectors. For example, only two of the three vectors in the following image are needed to span the planar subspace.



**Activity S.2.7** *(~10 min)*

Consider the subspace $W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}$ of $\mathbb{R}^4$.

**Activity S.2.7** *(~10 min)*

Consider the subspace $W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}$ of $\mathbb{R}^4$.

*Part 1:* Use RREF $\begin{bmatrix}2 & 2 & 2 & 1\\3 & 0 & -3 & 5\\0 & 1 & 2 & -1\\1 & -1 & -3 & 0\end{bmatrix}$ to identify a vector that causes the spanning set for $W$ to be linearly dependent.

## Activity S.2.7 *(~10 min)*

Consider the subspace $W = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$ of $\mathbb{R}^4$.

*Part 1:* Use RREF $\begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$ to identify a vector that causes the spanning set for $W$ to be linearly dependent.

*Part 2:* Find a basis for $W$ by removing a vector from the spanning set to make it linearly independent.

**Fact S.2.8**
Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$. The easiest basis describing span $S$ is the set of vectors in $S$ given by the pivot columns of $\mathrm{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$.

Put another way, to compute a basis for the subspace span $S$, simply remove the vectors corresponding to the non-pivot columns of $\mathrm{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$.

**Activity S.2.9** *(~10 min)*

Let $W$ be the subspace of $\mathbb{R}^4$ given by

$$W = \operatorname{span}\left\{\begin{bmatrix}1\\3\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\2\end{bmatrix}, \begin{bmatrix}4\\5\\3\\0\end{bmatrix}, \begin{bmatrix}3\\2\\2\\1\end{bmatrix}\right\}$$

Find a basis for $W$.

**Activity S.2.10** *(∼10 min)*
Let $W$ be the subspace of $\mathcal{P}^3$ given by

$$W = \operatorname{span}\left\{x^3+3x^2+x-1, 2x^3-x^2+x+2, 4x^3+5x^3+3x, 3x^3+2x^2+2x+1\right\}$$

Find a basis for $W$.

# Module S Section 3

**Observation S.3.1**

Recall from last class: to compute a basis for the subspace $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, simply remove the vectors corresponding to the non-pivot columns of $\operatorname{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$.

**Activity S.3.2** *(∼10 min)*
Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

**Activity S.3.2** *(∼10 min)*
Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

*Part 1:* Find a basis for span $S$

**Activity S.3.2** *(∼10 min)*
Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

*Part 1:* Find a basis for span $S$
*Part 2:* Find a basis for span $T$

**Fact S.3.3**

A vector space has a lot of bases, but all bases for a given vector space must be the same size.

**Definition S.3.4**

The **dimension** of a vector space is given by the cardinality/size of any basis for the vector space.

## Activity S.3.5 *(∼15 min)*

Find the dimension of each subspace of $\mathbb{R}^4$.

$$\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

$$\operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix}\right\}$$

$$\operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}3\\13\\7\\16\end{bmatrix},\begin{bmatrix}-1\\10\\7\\14\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix}\right\}$$

$$\operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix},\begin{bmatrix}3\\6\\1\\5\end{bmatrix}\right\}$$

$$\operatorname{span}\left\{\begin{bmatrix}5\\3\\0\\-1\end{bmatrix},\begin{bmatrix}-2\\1\\0\\3\end{bmatrix},\begin{bmatrix}4\\5\\1\\3\end{bmatrix}\right\}$$

## Fact S.3.6

Every vector space with finite dimension, that is, every vector space with a basis of the form $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is **isomorphic** to a Euclidean space $\mathbb{R}^n$:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

**Observation S.3.7**

Several interesting vector spaces are infinite-dimensional:

- The space of polynomials $\mathcal{P}$ (consider the set $\{1, x, x^2, x^3, \dots\}$).
- The space of continuous functions $C(\mathbb{R})$ (which contains all polynomials, in addition to other functions like $e^x$).
- The space of real number sequences $\mathbb{R}^\infty$ (consider the set $\{(1, 0, 0, \dots), (0, 1, 0, \dots), (0, 0, 1, \dots), \dots\}$).

**Definition S.3.8**
A **homogeneous** system of linear equations is one of the form

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}.$$

Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are solutions, so is $\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$ i.e. if

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$$

and

$$b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n = \mathbf{0}$$

then

$$(a_1 + b_1)\mathbf{v}_1 + \cdots + (a_n + b_n)\mathbf{v}_n = \mathbf{0}.$$

Similarly, if $c \in \mathbb{R}$, $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is a solution. Thus the solution set of a homogeneous system is a subspace.

**Activity S.3.9** *(∼10 min)*

Consider the homogeneous system of equations

$$\begin{alignedat}{5}
x_1 &{}+{}& 2x_2 & & &{}+{}& x_4 &= 0\\
2x_1 &{}+{}& 4x_2 &{}-{}& x_3 &{}-{}& 2x_4 &= 0\\
3x_1 &{}+{}& 6x_2 &{}-{}& x_3 &{}-{}& x_4 &= 0
\end{alignedat}$$

**Activity S.3.9** *(∼10 min)*
Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \phantom{{}- x_3} + x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - x_4 &= 0 \end{aligned}$$

*Part 1:* Find the solution set.

**Activity S.3.9** *(∼10 min)*
Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \phantom{{}- x_3} + x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - x_4 &= 0 \end{aligned}$$

*Part 1:* Find the solution set.
*Part 2:* Rewrite the solution set in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

.

**Activity S.3.9** *(∼10 min)*

Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \phantom{{}-x_3} + \phantom{2}x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - \phantom{2}x_4 &= 0 \end{aligned}$$

*Part 1:* Find the solution set.

*Part 2:* Rewrite the solution set in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

.

*Part 3:* Find a basis for the solution set.

**Activity S.3.10** *(∼10 min)*
Consider the homogeneous system of equations

$$\begin{aligned} x_1 - 3x_2 + 2x_3 \phantom{+ 3x_4} &= 0 \\ 2x_1 - 6x_2 + 4x_3 + 3x_4 &= 0 \\ -2x_1 + 6x_2 - 4x_3 - 4x_4 &= 0 \end{aligned}$$

Find a basis for the solution set.

**Activity S.3.11** *(∼5 min)*

Suppose $W$ is a subspace of $\mathcal{P}^8$, and you know that the set $\{x^3 + x, x^2 + 1, x^4 - x\}$ is a linearly independent subset of $W$. What can you conclude about $W$?

(a) The dimension of $W$ is no more than 3

(b) The dimension of $W$ is 3

(c) The dimension of $W$ is at least 3

**Activity S.3.12** *(∼5 min)*

Suppose $W$ is a subspace of $\mathcal{P}^8$, and you know that $W$ is spanned by the six vectors

$$\{x^4 - x, x^3 + x, x^3 + x + 1, x^4 + 2x, x^3, 2x + 1\}$$

Without doing any calculation, what can you conclude about $W$?

(a) The dimension of $W$ is no more than 6

(b) The dimension of $W$ is 6

(c) The dimension of $W$ is at least 6