Module G: Geometry of Linear Maps

How can we understand linear maps geometrically?

At the end of this module, students will be able to...

G1. **Row operations.** ... represent a row operation as matrix multiplication, and compute how the operation affects the determinant.

G2. **Determinants.** ... compute the determinant of a square matrix.

G3. **Eigenvalues.** ... find the eigenvalues of a 2×2 matrix.

G4. **Eigenvectors.** ... find a basis for the eigenspace of a square matrix associated with a given eigenvalue.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **A1**.
- Recall and use the definition of a linear transformation **A2**.
- Find all roots of quadratic polynomials (including complex ones).
- Interpret the statement "A is an invertible matrix" in many equivalent ways in different contexts.

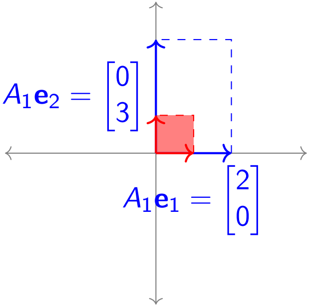
The following resources will help you prepare for this module.

- Finding the area of a parallelogram (Khan Academy): `http://bit.ly/2BO5iWx`
- Factoring quadratics (Khan Academy): `http://bit.ly/1XjfbV2`
- Factoring quadratics using area models (Youtube): `https://youtu.be/Aa-v1EK7DR4`
- Finding complex roots of quadratics (Youtube): `https://www.youtube.com/watch?v=2yBhDsNE0wg`

Module G Section 1

Activity G.1.1 (*∼5 min*)

The image below illustrates how the linear transformation $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ transforms the unit square.

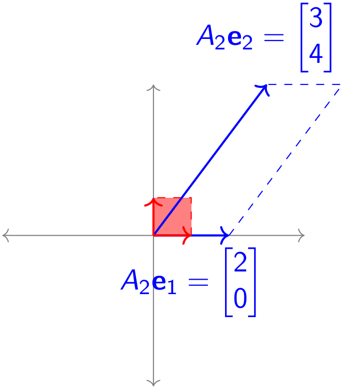


(a) What are the lengths of $A_1\mathbf{e}_1$ and $A_1\mathbf{e}_2$?

(b) What is the area of the transformed unit square?

Activity G.1.2 *(∼5 min)*

The image below illustrates how the linear transformation $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A_2 = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$. transforms the unit square.



(a) What are the lengths of $A_2\mathbf{e}_1$ and $A_2\mathbf{e}_2$?

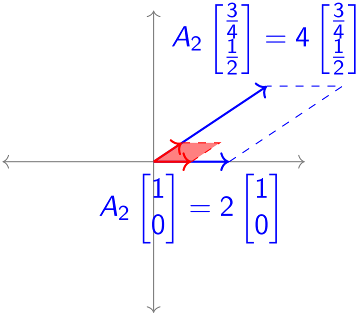
(b) What is the area of the transformed unit square?

Observation G.1.3

It is possible to find two nonparallel vectors that are scaled but not rotated by the linear map given by A_2.

$$A_2\mathbf{e}_1 = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\mathbf{e}_1$$

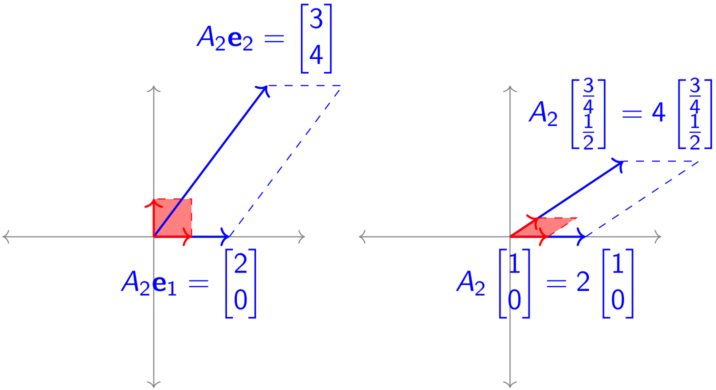
$$A_2\begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 4\begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$



The process for finding such vectors will be covered later in this module.

Observation G.1.4

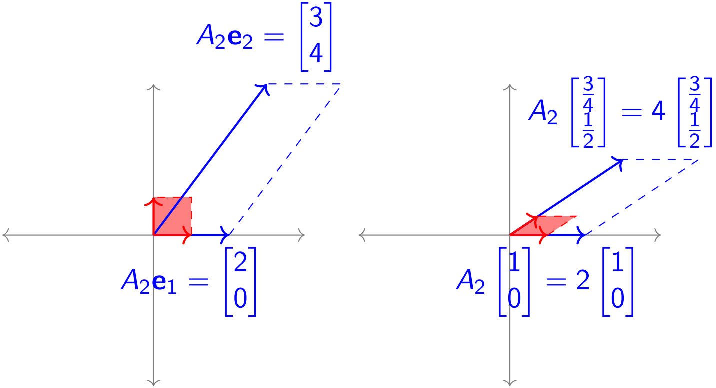
Notice that while a linear map can transform vectors in various ways, linear maps always transform parallelograms into parallelograms, and these areas are always transformed by the same factor: in the case of $A_2 = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, this factor is 8.



Since this change in area is always the same for a given linear map, it will be equal to the value of the transformed unit square (which begins with area 1).

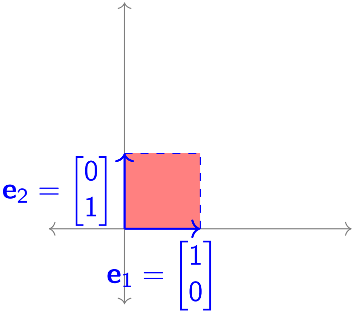
Remark G.1.5

We will define the **determinant** of a square matrix A, or $\det(A)$ for short, to be the factor by which A scales areas, but we first need to figure out the properties it must satisfy.



Activity G.1.6 (*∼2 min*)

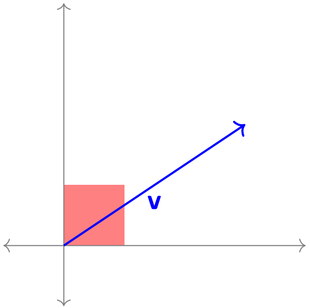
The transformation of the unit square by the standard matrix $[\mathbf{e}_1 \ \mathbf{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ is illustrated below. What is $\det([\mathbf{e}_1 \ \mathbf{e}_2]) = \det(I)$, the area of the transformed unit square shown here?



a) 0
b) 1
c) 2
d) Cannot be determined

Activity G.1.7 *(~2 min)*

The transformation of the unit square by the standard matrix $[\mathbf{v}\ \ \mathbf{v}]$ is illustrated below: both $T(\mathbf{e}_1) = T(\mathbf{e}_2) = \mathbf{v}$. What is $\det([\mathbf{v}\ \ \mathbf{v}])$, the area of the transformed unit square shown here?



a) 0

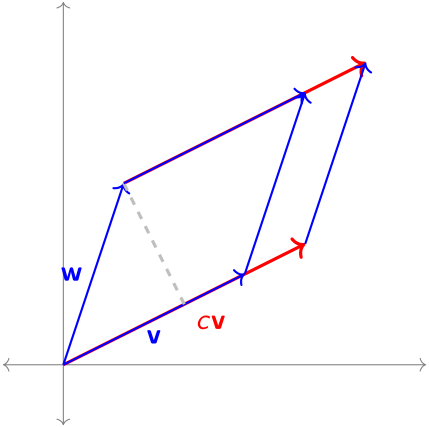
b) 1

c) 2

d) Cannot be determined

Activity G.1.8 *(∼5 min)*

The transformations of the unit square by the standard matrices $[\mathbf{v}\ \ \mathbf{w}]$ and $[c\mathbf{v}\ \ \mathbf{w}]$ are illustrated below. How are $\det([\mathbf{v}\ \ \mathbf{w}])$ and $\det([c\mathbf{v}\ \ \mathbf{w}])$ related?



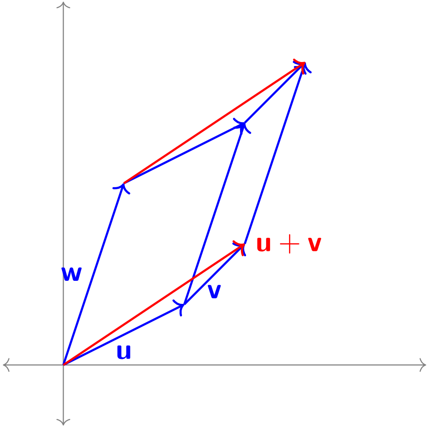
a) $\det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

b) $c + \det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

c) $c\det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

Activity G.1.9 *(∼5 min)*

The transformations of unit squares by the standard matrices $[\mathbf{u}\ \ \mathbf{w}]$, $[\mathbf{v}\ \ \mathbf{w}]$ and $[\mathbf{u}+\mathbf{v}\ \ \mathbf{w}]$ are illustrated below. How is $\det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$ related to $\det([\mathbf{u}\ \ \mathbf{w}])$ and $\det([\mathbf{v}\ \ \mathbf{w}])$?



a) $\det([\mathbf{u}\ \ \mathbf{w}]) = \det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

b) $\det([\mathbf{u}\ \ \mathbf{w}]) + \det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

c) $\det([\mathbf{u}\ \ \mathbf{w}])\det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

Definition G.1.10

The **determinant** is the unique function $\det : M_{n,n} \to \mathbb{R}$ satisfying these properties:

P1: $\det(I) = 1$

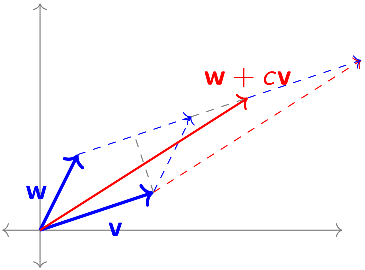
P2: $\det(A) = 0$ whenever two columns of the matrix are identical.

P3: $\det[\cdots \ c\mathbf{v} \ \cdots] = c\det[\cdots \ \mathbf{v} \ \cdots]$, assuming no other columns change.

P4: $\det[\cdots \ \mathbf{v}+\mathbf{w} \ \cdots] = \det[\cdots \ \mathbf{v} \ \cdots] + \det[\cdots \ \mathbf{w} \ \cdots]$, assuming no other columns change.

Observation G.1.11

The determinant must also satisfy other properties. Consider $\det([\mathbf{v} + c\mathbf{w} \quad \mathbf{w}])$ and $\det([\mathbf{v} \quad \mathbf{w}])$.



The base of both parallelograms is $\mathbf{v}$, while the height has not changed, so the determinant does not change either. This can be proven using the other properties of the determinant:

$$\begin{aligned}
\det([\mathbf{v} + c\mathbf{w} \quad \mathbf{w}]) &= \det([\mathbf{v} \quad \mathbf{w}]) + \det([c\mathbf{w} \quad \mathbf{w}]) \\
&= \det([\mathbf{v} \quad \mathbf{w}]) + c\det([\mathbf{w} \quad \mathbf{w}]) \\
&= \det([\mathbf{v} \quad \mathbf{w}]) + c \cdot 0 \\
&= \det([\mathbf{v} \quad \mathbf{w}])
\end{aligned}$$

Observation G.1.12

Columns may be swapped by adding/subtracting columns from one another, which we've just seen doesn't change the determinant.

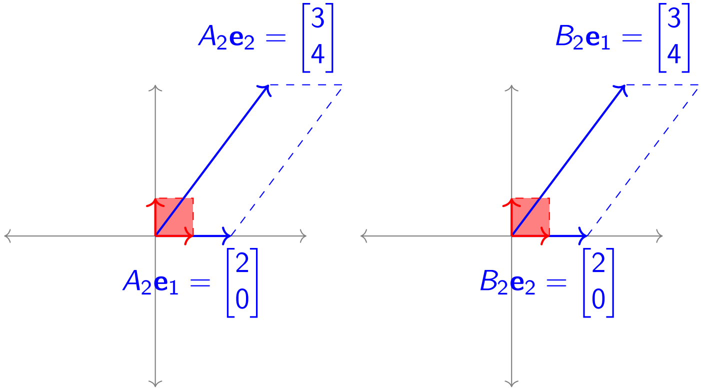
$$\begin{aligned}
\det([\mathbf{v} \quad \mathbf{w}]) &= \det([\mathbf{v}+\mathbf{w} \quad \mathbf{w}]) \\
&= \det([\mathbf{v}+\mathbf{w} \quad \mathbf{w}-(\mathbf{v}+\mathbf{w})]) \\
&= \det([\mathbf{v}+\mathbf{w} \quad -\mathbf{v}]) \\
&= \det([\mathbf{v}+\mathbf{w}-\mathbf{v} \quad -\mathbf{v}]) \\
&= \det([\mathbf{w} \quad -\mathbf{v}]) \\
&= -\det([\mathbf{w} \quad \mathbf{v}])
\end{aligned}$$

So swapping two columns results in a negation of the determinant. Therefore, determinants represent a *signed* area, since they are not always positive.

Remark G.1.13

Swapping columns may be thought of as a reflection, which is represented by a negative determinant. For example, the following matrices transform the unit square into the same parallelogram, but the second matrix reflects its orientation.

$$A_2 = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix}$$



Fact G.1.14

We've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

(a) Multiplying a column by a scalar multiplies the determinant by that scalar:

$$c\det([\cdots \quad \mathbf{v} \quad \cdots]) = \det([\cdots \quad c\mathbf{v} \quad \cdots])$$

(b) Swapping two columns changes the sign of the determinant:

$$\det([\cdots \quad \mathbf{v} \quad \cdots \quad \mathbf{w} \quad \cdots]) = -\det([\cdots \quad \mathbf{w} \quad \cdots \quad \mathbf{v} \quad \cdots])$$

(c) Adding a multiple of a column to another column does not change the determinant:

$$\det([\cdots \quad \mathbf{v} \quad \cdots \quad \mathbf{w} \quad \cdots]) = \det([\cdots \quad \mathbf{v}+c\mathbf{w} \quad \cdots \quad \mathbf{w} \quad \cdots])$$

Activity G.1.15 *(∼5 min)*

The transformation given by the standard matrix A scales areas by 4, and the transformation given by the standard matrix B scales areas by 3. How must the transformation given by the standard matrix AB scale areas?

(a) 1

(b) 7

(c) 12

(d) Cannot be determined

Fact G.1.16

Since the transformation given by the standard matrix AB is obtained by applying the transformations given by A and B, it follows that

$$\det(AB) = \det(A)\det(B)$$

Remark G.1.17

Recall that row operations may be produced by matrix multiplication.

- Multiply the first row of A by c: $\begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$
- Swap the first and second row of A: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$
- Add c times the third row to the first row of A: $\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Fact G.1.18

The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:

- Scaling a row: $\det \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c$
- Swapping rows: $\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1$
- Adding a row multiple to another row:

  $$\det \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & c - 1c \\ 0 & 1 & 0 - 0c \\ 0 & 0 & 1 - 0c \end{bmatrix} = \det(I) = 1$$

Activity G.1.19 *(~5 min)*

Consider the row operation $R_1 + 4R_3 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1+4(7) & 2+4(8) & 3+4(9) \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Find $\det R$ by comparing with the previous slide.

(c) If $C \in M_{3,3}$ is a matrix with $\det(C) = -3$, find

$$\det(RC) = \det(R)\det(C).$$

Activity G.1.20 *(∼5 min)*

Consider the row operation $R_1 \leftrightarrow R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to I.

(b) If $C \in M_{3,3}$ is a matrix with $\det(C) = 5$, find $\det(RC)$.

Activity G.1.21 *(∼5 min)*

Consider the row operation $3R_2 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 3(4) & 3(5) & 3(6) \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det(C) = -7$, find $\det(RC)$.

Module G Section 2

Fact G.2.1

We've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

(a) Multiplying a column by a scalar multiplies the determinant by that scalar:

$$c\det(\begin{bmatrix}\cdots & \mathbf{v} & \cdots\end{bmatrix}) = \det(\begin{bmatrix}\cdots & c\mathbf{v} & \cdots\end{bmatrix})$$

(b) Swapping two columns changes the sign of the determinant:

$$\det(\begin{bmatrix}\cdots & \mathbf{v} & \cdots & \mathbf{w} & \cdots\end{bmatrix}) = -\det(\begin{bmatrix}\cdots & \mathbf{w} & \cdots & \mathbf{v} & \cdots\end{bmatrix})$$

(c) Adding a multiple of a column to another column does not change the determinant:

$$\det(\begin{bmatrix}\cdots & \mathbf{v} & \cdots & \mathbf{w} & \cdots\end{bmatrix}) = \det(\begin{bmatrix}\cdots & \mathbf{v}+c\mathbf{w} & \cdots & \mathbf{w} & \cdots\end{bmatrix})$$

Fact G.2.2

Since the row operation matrices may be obtained with column operations, we can also use row operations to simplify determinants.

1. Multiplying rows by scalars: $\det\begin{bmatrix}\vdots\\ cR\\ \vdots\end{bmatrix} = c\det\begin{bmatrix}\vdots\\ R\\ \vdots\end{bmatrix}$
2. Swapping two rows: $\det\begin{bmatrix}\vdots\\ R\\ \vdots\\ S\\ \vdots\end{bmatrix} = -\det\begin{bmatrix}\vdots\\ S\\ \vdots\\ R\\ \vdots\end{bmatrix}$
3. Adding multiples of rows to other rows: $\det\begin{bmatrix}\vdots\\ R\\ \vdots\\ S\\ \vdots\end{bmatrix} = \det\begin{bmatrix}\vdots\\ R+cS\\ \vdots\\ S\\ \vdots\end{bmatrix}$

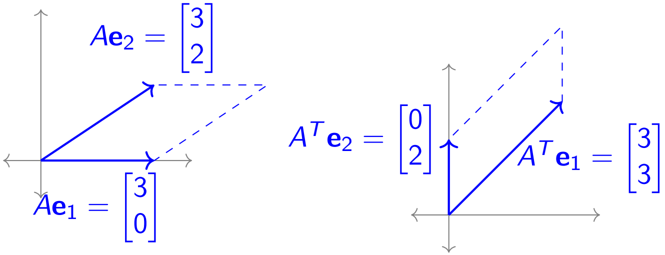
Definition G.2.3

The **transpose** of a matrix is given by rewriting its columns as rows and vice versa:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Fact G.2.4

Since row and column operations both affect determinants the same way, the determinant of a matrix and its transpose are the same. For example, let $A = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$, so $A^T = \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix}$; both matrices scale the unit square by 6, even though the parallelograms are not congruent.



Activity G.2.5 *(∼10 min)*

Compute the determinant of $\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$ by row reducing it to a nicer matrix.

For example, $\det \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

Fact G.2.6

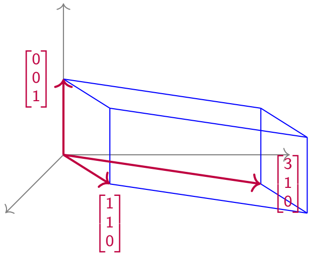
This same process allows us to prove a more convenient formula:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

In higher dimensions, the formulas become unreasonable. For example, the formula for 4×4 matrices has 24 terms!

Activity G.2.7 *(∼5 min)*

The following image illustrates the transformation of the unit cube by the matrix $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.



This volume is equal to which of the following areas?

(a) $\det \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\det \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ (c) $\det \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$ (d) $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Fact G.2.8

If column i of a matrix is $\mathbf{e}_i$, then both column and row i may be removed without changing the value of the determinant. For example, the second column of the following matrix is $\mathbf{e}_2$, so:

$$\det\begin{bmatrix} 3 & 0 & -1 & 5 \\ 2 & 1 & 4 & 0 \\ -1 & 0 & 1 & 11 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \det\begin{bmatrix} 3 & -1 & 5 \\ -1 & 1 & 11 \\ 3 & 0 & 1 \end{bmatrix}$$

Therefore the same holds for the transpose:

$$\det\begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 5 & 0 & 11 & 1 \end{bmatrix} = \det\begin{bmatrix} 3 & -1 & 3 \\ -1 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$$

Geometrically, this is the fact that if the height is 1, the base $\times$ height formula reduces to the area/volume/etc. of the $n-1$ dimensional base.

Activity G.2.9 *($\sim$5 min)*

Compute $\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 12 \\ 3 & 2 & -1 \end{bmatrix}$.

Activity G.2.10 *(∼5 min)*

Compute $\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$.

(a) -1

(b) 0

(c) 1

Activity G.2.11 *(∼10 min)*

Compute $\det \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -5 \\ 0 & 3 & 3 \end{bmatrix}$

Hint: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(a) 3 (b) 6 (c) 9 (d) 12

Activity G.2.12 *(∼15 min)*

Compute $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$.

Observation G.2.13

$$\det\begin{bmatrix}2&3&5&0\\0&3&2&0\\1&2&0&3\\-1&-1&2&2\end{bmatrix}=(-1)(0)\det\begin{bmatrix}2&3&5&0\\0&3&2&0\\1&2&0&3\\-1&-1&2&2\end{bmatrix}+(1)(3)\det\begin{bmatrix}2&3&5&0\\0&3&2&0\\1&2&0&3\\-1&-1&2&2\end{bmatrix}+$$

$$(-1)(2)\det\begin{bmatrix}2&3&5&0\\0&3&2&0\\1&2&0&3\\-1&-1&2&2\end{bmatrix}+(1)(0)\det\begin{bmatrix}2&3&5&0\\0&3&2&0\\1&2&0&3\\-1&-1&2&2\end{bmatrix}$$

$$=3\det\begin{bmatrix}2&5&0\\1&0&3\\-1&2&2\end{bmatrix}+(-1)(2)\det\begin{bmatrix}2&3&0\\1&2&3\\-1&-1&2\end{bmatrix}$$

This technique is called **Laplace expansion** or **cofactor expansion**.

Activity G.2.14 *(∼10 min)*

Compute $\det \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & -1 \\ 1 & 2 & 0 & 3 \\ -1 & -3 & 2 & -2 \end{bmatrix}$.

Module G Section 3

Activity G.3.1 *(∼5 min)*

An invertible matrix M and its inverse M^{-1} are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Compute $\det(M)$ and $\det(M^{-1})$.

Activity G.3.2 *(∼5 min)*
Suppose the matrix M is invertible, so there exists M^{-1} with $MM^{-1} = I$. It follows that $\det(M)\det(M^{-1}) = \det(I)$.

What is the only number that $\det(M)$ cannot equal?

(a) -1

(b) 0

(c) 1

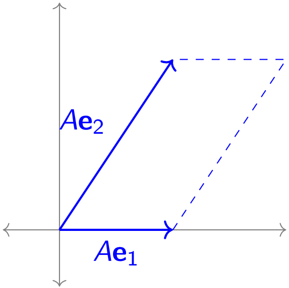
(d) $\frac{1}{\det(M^{-1})}$

Fact G.3.3

- For every invertible matrix M, $\det(M^{-1}) = \frac{1}{\det(M)}$.
- Furthermore, a square matrix M is invertible if and only if $\det(M) \neq 0$.

Observation G.3.4

Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$



It is easy to see geometrically that

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is less obvious (but easily verified by computation) that

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Definition G.3.5
Let $A \in \mathbb{R}^{n\times n}$. An **eigenvector** is a vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x}$ is parallel to $\mathbf{x}$. In other words, $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.

We call this λ an **eigenvalue** of A.

Observation G.3.6

Since $\lambda\mathbf{x} = \lambda(I\mathbf{x})$, we can find the eigenvalues and eigenvectors satisfying $A\mathbf{x} = \lambda\mathbf{x}$ by inspecting $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

- Since we already know that $(A - \lambda I)\mathbf{0} = \mathbf{0}$ for any value of λ, we are more interested in finding values of λ such that $A - \lambda I$ has a nontrivial kernel.
- Thus $\operatorname{RREF}(A - \lambda I)$ must have a non-pivot column, and therefore $A - \lambda I$ cannot be invertible.
- Since $A - \lambda I$ cannot be invertible, our eigenvalues must satisfy $\det(A - \lambda I) = 0$.

Definition G.3.7
Computing $\det(A - \lambda I)$ results in the **characteristic polynomial** of A.

For example, when $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, we have

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$$

Thus the characteristic polynomial of A is

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2$$

Activity G.3.8 *(∼15 min)*

Compute $\det(A - \lambda I)$ to find the characteristic polynomial of $A = \begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}$.

Activity G.3.9 *(∼15 min)*

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Activity G.3.9 *(∼15 min)*

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Activity G.3.9 *(∼15 min)*

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Part 2: Find the roots of the characteristic polynomial to determine the eigenvalues of A.

Activity G.3.9 *($\sim$15 min)*

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Part 2: Find the roots of the characteristic polynomial to determine the eigenvalues of A.

Part 3: Compute the kernel of the transformation given by

$$A - 2I = \begin{bmatrix} 2-2 & 2 \\ 0 & 3-2 \end{bmatrix}$$

to determine all the eigenvectors associated to the eigenvalue 2.

Activity G.3.9 *(∼15 min)*

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Part 2: Find the roots of the characteristic polynomial to determine the eigenvalues of A.

Part 3: Compute the kernel of the transformation given by

$$A - 2I = \begin{bmatrix} 2-2 & 2 \\ 0 & 3-2 \end{bmatrix}$$

to determine all the eigenvectors associated to the eigenvalue 2.

Part 4: Compute the kernel of the transformation given by $A - 3I$ to determine all the eigenvectors associated to the eigenvalue 3.

Definition G.3.10

The kernel of the transformation given by $A - \lambda I$ contains all the eigenvectors associated with λ. Since kernel is a subspace of $\mathbb{R}^n$, we call this kernel the **eigenspace** associated with the eigenvalue λ.

Activity G.3.11 *(~15 min)*

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

Activity G.3.11 *(∼15 min)*

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Activity G.3.11 *(∼15 min)*

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Part 2: Find the roots of the characteristic polynomial $(3 - \lambda)(\lambda^2 - 4\lambda - 12)$ to determine the eigenvalues of A.

Activity G.3.11 *(∼15 min)*

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Part 2: Find the roots of the characteristic polynomial $(3-\lambda)(\lambda^2 - 4\lambda - 12)$ to determine the eigenvalues of A.

Part 3: Compute the kernels of $A - \lambda I$ for each eigenvalue $\lambda \in \{-2, 3, 6\}$ to determine the respective eigenspaces.

Observation G.3.12

Recall that if a is a root of the polynomial $p(\lambda)$, the **multiplicity** of a is the largest number k such that $p(\lambda) = q(\lambda)(\lambda - a)^k$ for some polynomial $q(\lambda)$.

For this reason, the **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

Example G.3.13

If $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 3 & 3 \\ 0 & 0 & -1 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = (\lambda - 3)^2(\lambda + 1)$.

The eigenvalues are 3 (with algebraic multiplicity 2) and -1 (with algebraic multiplicity 1).

Module G Section 4

Observation G.4.1

Recall from last class:

- To find the eigenvalues of a matrix A, we need to find values of λ such that $A - \lambda I$ has a nontrivial kernel. Equivalently, we want values where $A - \lambda I$ is not invertible, so we want to know the values of λ where $\det(A - \lambda I) = 0$.
- $\det(A - \lambda I)$ is a polynomial with variable λ, called the **characteristic polynomial** of A. Thus the roots of the characteristic polynomial of A are exactly the eigenvalues of A.
- Once an eigenvalue λ is found, the **eigenspace** containing all **eigenvectors** $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is given by $\ker(A - \lambda I)$.

Activity G.4.2 *(∼5 min)*

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Activity G.4.2 *(∼5 min)*

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Part 1: Compute the eigenvalues of A.

Activity G.4.2 *(∼5 min)*

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Part 1: Compute the eigenvalues of A.

Part 2: Sketch a picture of the transformation of the unit square. What about this picture reveals that A has no real eigenvectors?

Activity G.4.3 *(∼5 min)*

If A is a 4×4 matrix, what is the largest number of eigenvalues A can have?

(a) 3

(b) 4

(c) 5

(d) 6

(e) It can have infinitely many

Observation G.4.4

An $n \times n$ matrix may have between 0 and n real-valued eigenvalues. But the Fundamental Theorem of Algebra implies that if complex eigenvalues are included, then every $n \times n$ matrix has exactly n eigenvalues (counting algebraic multiplicites).

Activity G.4.5 *(∼5 min)*

The matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$ has characteristic polynomial $-\lambda(\lambda - 2)^2$.

Find the dimension of the eigenspace of A associated to the eigenvalue 2 (the dimension of the kernel of $A - 2I$).

Activity G.4.6 *(∼5 min)*

The matrix $B = \begin{bmatrix} -3 & -9 & 5 \\ -2 & -2 & 2 \\ -7 & -13 & 9 \end{bmatrix}$ has characteristic polynomial $-\lambda(\lambda-2)^2$.

Find the dimension of the eigenspace of B associated to the eigenvalue 2 (the dimension of the kernel of $B - 2I$).

Observation G.4.7

In the first example, the (2 dimensional) plane spanned by $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}4\\-2\\0\end{bmatrix}$ was preserved. In the second example, only the (one dimensional) line spanned by $\begin{bmatrix}1\\0\\1\end{bmatrix}$ is preserved.

Definition G.4.8

While the **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial, the **geometric multiplicity** of an eigenvalue is the dimension of its eigenspace.

Fact G.4.9

As we've seen, the geometric multiplicity may be different than its algebraic multiplicity, but it cannot exceed it.

This fact is explored deeper and explained in Math 316, Linear Algebra II

Activity G.4.10 *(∼20 min)*

Consider the 4×4 matrix

$$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$$

Activity G.4.10 *(∼20 min)*
Consider the 4×4 matrix

$$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$$

Part 1: Use technology (e.g. Wolfram Alpha) to find its characteristic polynomial.

Activity G.4.10 *(∼20 min)*
Consider the 4×4 matrix

$$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$$

Part 1: Use technology (e.g. Wolfram Alpha) to find its characteristic polynomial.
Part 2: Find the algebraic and geometric multiplicities for both eigenvalues.