

Module P: Applications of Linear Algebra

Module P Section 1

Definition P.1.1

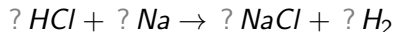
In chemistry, we learn that when the two substances

- Hydrochloric acid HCl (formed from 1 H and 1 Cl atom)
- Sodium Na (formed from 1 Na atom)

react, their atoms rearrange to form the substances

- Salt $NaCl$ (formed from 1 Na and 1 Cl atom)
- Hydrogen gas H_2 (formed from 2 H atoms).

This may be represented by the **chemical equation**

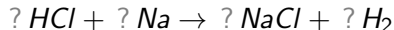


where each $?$ represents the amount of that substance before/after the reaction.

Activity P.1.2 (*~5 min*)

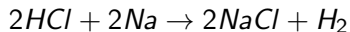
The **law of conservation of mass** states that the quantity of atoms before and after a chemical reaction must remain the same.

Find positive integers so that both sides of the chemical equation represent the same amount of matter:



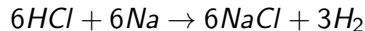
Definition P.1.3

A chemical equation is **balanced** if the given quantities of each substance before and after the reaction are equal and minimal positive integers:



Observation P.1.4

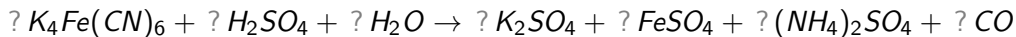
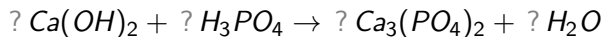
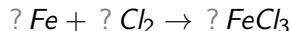
For example, the following equation isn't balanced because all the integers may be divided by three:



Therefore if a chemical equation can be balanced, there is exactly one correct solution.

Activity P.1.5 (*~15 min*)

Balance the following chemical equations:



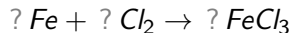
(Note that $(NH_4)_2SO_4$ represents 2 *N*, 8 *H*, 1 *S*, and 4 *O*.)

Observation P.1.6

For the purposes of balancing chemical equations, the set

$$L = \{\mathbf{A} \mid \mathbf{A} \text{ is combination of elements}\}$$

may be treated as a kind of **vector space**. This means that balancing the chemical equation



may be achieved by finding a solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to the vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl}).$$

Activity P.1.7 (*~5 min*)

To solve the vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl})$$

we are only concerned with the subspace $W = \text{span}\{\mathbf{Cl}, \mathbf{Fe}\}$ of L . Since the element \mathbf{Fe} cannot be created from the element \mathbf{Cl} in a chemical reaction and vice versa, the set $\{\mathbf{Cl}, \mathbf{Fe}\}$:

- a) spans W , but is linearly dependent.
- b) is linearly independent, but does not span W .
- c) is a basis for W .

Observation P.1.8

$W = \text{span}\{\mathbf{Cl}, \mathbf{Fe}\}$ is a two-dimensional subspace of L , so as usual we'd rather work with its isomorphic Euclidean space \mathbb{R}^2 .

Thus we should assign a transformation of bases such as:

$$\mathbf{Cl} \leftrightarrow \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{Fe} \leftrightarrow \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Activity P.1.9 (*~10 min*)

Rewrite the $W = \text{span}\{\mathbf{Cl}, \mathbf{Fe}\}$ vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl})$$

using the transformation of bases

$$\mathbf{Cl} \leftrightarrow \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{Fe} \leftrightarrow \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and show how it may be simplified to

$$x \begin{bmatrix} 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \end{bmatrix} - z \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Activity P.1.10 (*~10 min*)

Consider the Euclidean vector equation

$$x \begin{bmatrix} 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \end{bmatrix} - z \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Activity P.1.10 (~ 10 min)

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Part 1: Find its solution set.

Activity P.1.10 (~ 10 min)

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Part 1: Find its solution set.

Part 2: Find a vector in the solution space that consists of minimal positive integers.

Activity P.1.10 (~ 10 min)

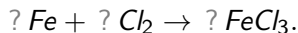
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Part 1: Find its solution set.

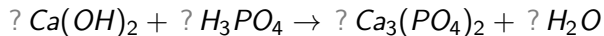
Part 2: Find a vector in the solution space that consists of minimal positive integers.

Part 3: Balance the chemical equation



Activity P.1.11 (*~10 min*)

Balance the chemical equation



by first converting it into an \mathbb{R}^4 vector equation and finding its solution set.