UNCOMPLETABLE MOORE SPACES

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INTRODUCTION

In [2] and [3], Mary Ellen Rudin described two fantastic Moore spaces each having the following seemingly contradictory properties:

1. non-separable,
2. every collection of mutually exclusive open sets is countable,
3. cannot be imbedded in any complete Moore space,
4. every metric subspace is nowhere dense.

One of the spaces has, in addition, the following properties:

5. arc-wise connected,
6. locally arc-wise connected.

These spaces occupy key positions in the theory of Moore spaces. They are also probably the most complicated examples of Moore spaces known. In this paper, with the full advantage of hindsight at our disposal, we shall describe two very simple Moore spaces - the simpler one having properties 1 - 4 and the other one having properties 1 - 6.

TERMINOLOGY

We recall here some definitions and notations which are by now more or less standard.

For each positive integer n, E^n will denote the n-dimensional euclidean space with its usual topology. If K is a collection of sets, K^* will denote the union of all members of K. Moreover if $p \in K^*$, then $K^*(p)$ will denote the union of all members of K containing p.

Suppose that (X,T) is a topological space. G is a *development* for (X,T) means that G is a sequence of open covers of X and for each point p and each open set U with $p \in U$, there is a positive integer n such that $G_n^*(p) \subset U$. A space which admits a development is called a *developable* space. A T_1, regular, developable space is called a *Moore space*. G is a *complete development* for (X,T) means that G is a development for (X,T) with the additional properties that (i) $G_n \supset G_{n+1}$ for each positive integer n, and (ii) if M is a decreasing sequence of closed sets and for each positive integer i there is a member U of G_i with $M_i \subset U$, then $\bigcap_{i=1}^{\infty} M_i \neq \phi$. A *complete Moore space* is a Moore space which admits a complete development.

SOME PROPOSITIONS CONCERNING MOORE SPACES.

The first two properties of Moore spaces listed below are used to rephrase the problem of constructing spaces with properties 1 - 4. Proposition 3 gives a way of obtaining some other solutions to the problem once one solution is given.

Proposition 1. If a Moore space has properties 1 and 2 then it also has property 3.

Proof. The proposition follows from [1] and Theorem 3 of [3].

Proposition 2. If S is a Moore space with property 2 and no open set in S is separable then S has property 4.

Proof. Let X be a subspace of S such that $\overline{X}$ contains an open set U of S. Then $X \cap U$ has property 2. To see this, let K be a collection of mutually exclusive subsets of $X \cap U$ each open in $X \cap U$. For each $V \in K$ let V' be an open subset of U with

$V' \cap X = V$. Clearly $V' \cap W' = \phi$ for any $V, W \in K$ with $V \neq W$. So K is countable became $\{V' : V \quad K\}$ is countable. Now if in addition X is metric, then it follows that $X \cap U$ is separable which implies that U is separable - a contradiction.

Proposition 1 and the argument for Proposition 2 easily gives the following fact.

Proposition 3. If S is a Moore space with properties 1-4 then every dense subspace of S also has properties 1-4.

A MOORE SPACE Λ WITH PROPERTIES 1-4.

Points of Λ. p is a point of Λ if and only if p is a non-empty finite subset of the real numbers.

Regions of Λ. R is a region of Λ if and only if there is a $p \in \Lambda$ and an open subset U of the real line with $p \subset U$ such that

$$R = \{q \in \Lambda \mid p \subseteq q \subset U\}.$$

Let $R(p,U)$ denote such a region R.

I. The regions are well-defined. That is, the collection

$$\{R(p,U) : p \in \Lambda,\ U \text{ open in } E',\ p \subset U\}$$

is a topological base for Λ. To see this, let $z \in R(p,U) \cap R(q,V)$ and note that $R(z, U \cap V) \subseteq R(p,U) \cap R(q,V)$. Further note from the argument above, that if X is an open set in Λ and $p \in X$ then there is an open subset U of E' with $p \subset U$ and $R(p,U) \subset X$.

II. The space Λ is Hausdorff. For suppose that $x, y \in \Lambda$ and $x \neq y$. Without loss of generality, assume that $t \in x - y$. Choose an open set U in E^1 such that $y \subseteq U$ and $t \notin U$. Then, $R(y,U) \cap R(x,E^1) = \phi$.

III. Each region is closed. To see this, let $R(p,U)$ be a region and let $z \in \Lambda - R(p,U)$. We shall show that z is not a limit point of $R(p,U)$. From the definition of $R(p,U)$, either $p \not\subseteq z$ or $z \not\subseteq U$. In case $p \not\subseteq z$, let $t \in p - z$ and let V be an open set in E^1 such that $z \subset V$ and $t \notin V$. Then $R(z,V) \cap R(p,U) = \phi$. Hence z is not a limit point of $R(p,U)$. In case $z \not\subseteq U$, $R(z,E^1) \cap R(p,U) = \phi$. Hence again z is not a limit point of $R(p,U)$.

IV. The space Λ is developable. For each $p \in \Lambda$ and each positive integer n, let $S_{\frac{1}{n}}(p) = R(p,U)$ where U is the union of the open segments of radius $\frac{1}{n}$ with the points in p as centers. Clearly, $S_{\frac{1}{n}}(p) \supseteq S_{\frac{1}{n+1}}(p)$. Also observe that for each $R(p,V)$ there is a positive integer n such that $S_{\frac{1}{n}}(p) \subset R(p,V)$. Now let G be the sequence such that for each positive integer n, $G_n = \{S_{\frac{1}{n}}(p) : p \in \Lambda\}$. We shall show that G is a development for Λ. Clearly, for each n, G_n is an open cover of Λ. Suppose now that X is an open set in Λ and $p \subseteq X$. In view of the last sentence of I and the second two sentences in IV, choose a positive integer n such that $S_{\frac{1}{n}}(p) \subset X$ and $\frac{1}{n} < |a-b|$ for each $a,b \in p$ with $a \neq b$. Note that if $p \in S_{\frac{1}{n}}(q)$ for some $q \in \Lambda$ then not only $q \subseteq p$ but also $p \subseteq q$, for otherwise, if $a \in p - q$, then, by the definition of $S_{\frac{1}{n}}(q)$, there is a b in q and hence in p such that $|a-b| < \frac{1}{n}$ – a contradiction to the choice of n. So, $S_{\frac{1}{n}}(p)$ is the only member of G_n containing p. Therefore, $G_n{}^*(p) = S_{\frac{1}{n}}(p) \subset X$.

V. No region is separable. Suppose that C is a countable subset of some region $R(p,U)$. Let $x \in U - C^*$ and denote by q the set $p \cup \{x\}$. Then $q \in R(p,U)$ but $R(q,U)$ contains no point of C.

VI. Every collection of mutually exclusive open set of Λ is countable. Suppose that C is an uncountable collection of mutually exclusive open sets. We can easily get an uncountable subcollection C' of G^* - that is sets of the form $S_{\frac{1}{n}}(p)$ - which are mutually exclusive. It follows that there are N and M, each a positive integer, and an uncountable subcollection C'' of C' such that if $X \in C''$ then $X = S_{\frac{1}{N}}(p)$ for some p with exactly M elements. To each p with $S_{\frac{1}{n}}(p) \in C''$, assign the point $\tilde{p} = (p_1, p_2, \cdots, p_M)$ in E^M where $\{p_1, p_2, \cdots, p_M\} = p$ and $p_1 < p_2 < \cdots < p_M$. The set $\{\tilde{p}\}$ of all such points is uncountable and consequently contains a limit point. Therefore let $\tilde{p}$ and $\tilde{q}$ be such that $|\tilde{p}_i - \tilde{q}_i| < \frac{1}{N}$ for $i = 1, \cdots, M$. Then

$$p \cup q = \{p_1, p_2, \cdots, p_M, q_1, q_2, \cdots, q_M\} \in S_{\frac{1}{N}}(p) \cap S_{\frac{1}{N}}(q),$$

a contradiction to C'' being a collection of mutually exclusive sets.

VII. The space Λ is a Moore space with properties 1 - 4. That Λ is regular is a consequence of III. So, I, II, III, and IV say that Λ is a Moore space. Property 1 follows from V. Property 2 is VI. So property 3 follows from Proposition 1. In view of Proposition 2, V implies property 4.

A MOORE SPACE ψ WITH PROPERTIES 1 - 6.

Points of ψ. p is a point of ψ if and only if p is a non-negative function from E' to E' such that $\{x : p(x) > 0\}$ is finite and non-empty.

Regions of ψ. R is a region if and only if there are $p \in \psi$ and $\varepsilon > 0$ such that $q \in R$ if and only if

(i) for each $x \in E^1$ with $p(x) = 0$, $q(x) < p(y) + \varepsilon$ for some y with $|y - x| < \varepsilon$, and

(ii) $|q(x) - p(x)| < \varepsilon$ for each x with $p(x) > 0$.

We denote such a region by $R(p,\varepsilon)$.

Notation. For the convenience in proving some of the properties claimed for ψ, we introduce some notation.

For each $p \in \psi$, $\varepsilon > 0$, and $x \in E^1$, let $f(p,\varepsilon,x)$ denote $\sup.\{p(y) : |x - y| < \varepsilon\}$, and let $g(p,\varepsilon,x)$ denote the number y nearest to x such that $p(y) = f(p,\varepsilon,x)$.

For each $p \in \psi$, let $\eta(p)$ denote the number

$$\min.\{\{p(x) : p(x) > 0\} \cup \{|x - y| : x \neq y,\ p(x) > 0,\ p(y) > 0\}\}.$$

See page 7 for illustration of $f(p,\varepsilon,x)$.

Lemma 1. $q \in R(p,\varepsilon)$ if and only if

$$q(x) < f(p,\varepsilon,x) + \varepsilon \quad \text{for each } x \in E^1 \quad \text{and}$$

$$|q(x) - p(x)| < \varepsilon \quad \text{for each } x \text{ with } p(x) > 0.$$

Lemma 2. If $0 < \varepsilon' < \varepsilon$, then $R(p,\varepsilon') \subset R(p,\varepsilon)$ for each $p \in \psi$.

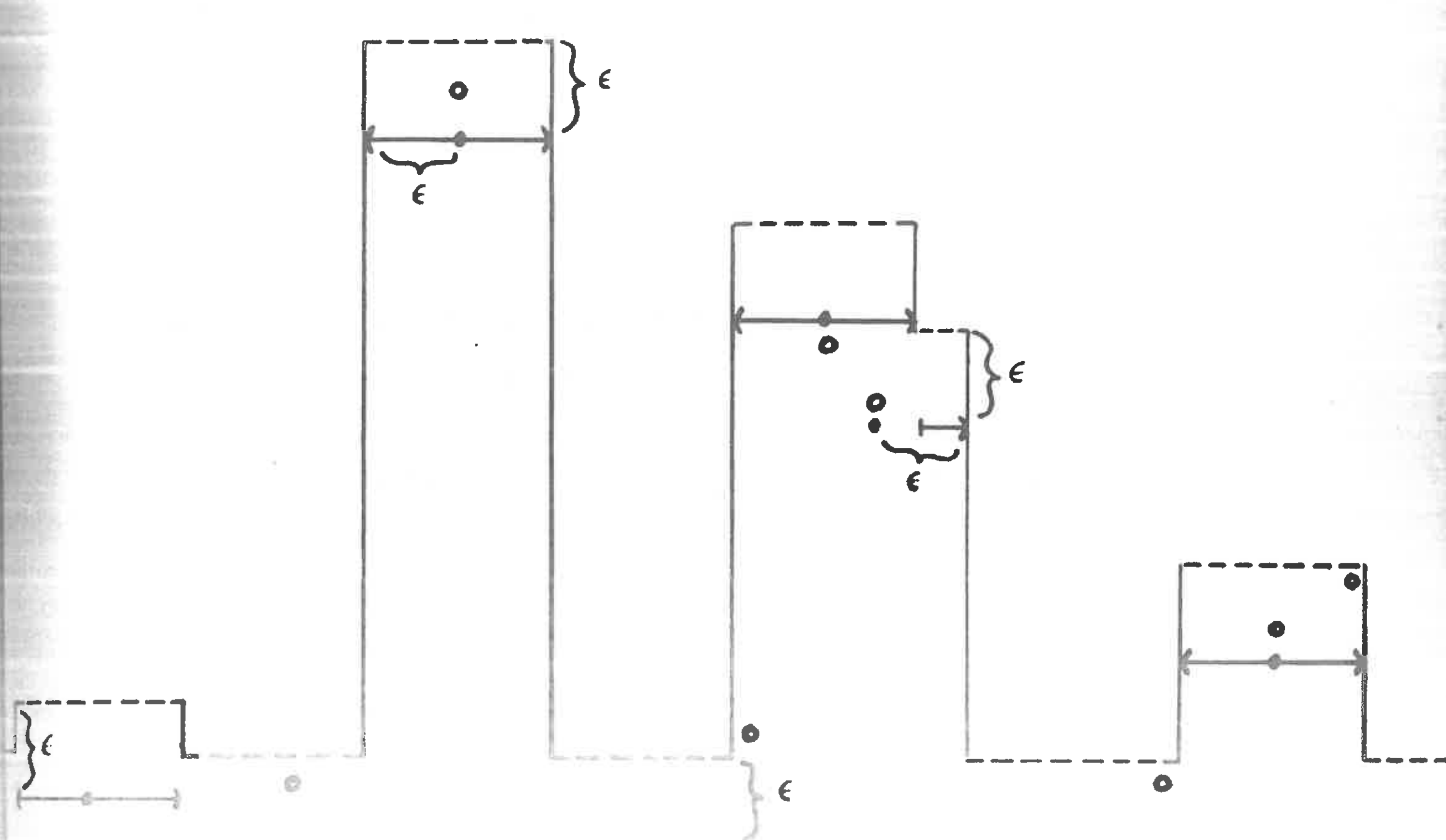
Lemma 3. If $p,q \in \psi$, $\varepsilon < \eta(p)$ and $p \in R(q,\frac{\varepsilon}{2})$ then $R(q,\frac{\varepsilon}{2}) \subset R(p,\varepsilon)$.

Proof. Suppose that $z \in R(q,\frac{\varepsilon}{2})$ and let $x \in E^1$. Let $y = g(q,\frac{\varepsilon}{2},x)$. Clearly $|y - x| < \frac{\varepsilon}{2} < \frac{\eta(p)}{2}$. It follows that,

$$z(x) < q(y) + \frac{\varepsilon}{2} < p(y) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < f(p,\frac{\varepsilon}{2},x) + \varepsilon < f(p,\varepsilon,x) + \varepsilon.$$

Also, if $p(x) > 0$, then $q(y) > p(x) - \frac{\varepsilon}{2} > \eta(p) - \frac{\varepsilon}{2} > \frac{\varepsilon}{2}$. So,

<u>ILLUSTRATION</u>



represents the non-zero values of p .

represents the graph of $f(p,\epsilon,\ )$.

represents the graph of $f(p,\epsilon,\ ) +\ $.

represents the non-zero values of a typical point $g\ \ R(p,\ )$.

$p(y) > q(y) - \frac{\varepsilon}{2} > 0$. Hence $x = y$. Consequently, $|z(x) - p(x)| \leq |z(x) - q(x)| + |q(x) - p(x)| < \varepsilon$. Hence, by Lemma 1, $z \in R(p,\varepsilon)$. Thus $R(q,\frac{\varepsilon}{2}) \subset R(p,\varepsilon)$.

Lemma 4. If $p,q \in \psi$, $\varepsilon < \eta(p)$, $0 < \varepsilon' \leq \varepsilon$, $\varepsilon' < \min.\{|x-y| : x \neq y,\ p(x) > 0,\ q(y) > 0\}$, and $R(p,\frac{\varepsilon}{2}) \cap R(q,\frac{\varepsilon'}{2}) \neq \phi$, then $q \in R(p,\varepsilon)$.

Proof. Let $z \in R(p,\frac{\varepsilon}{2}) \cap R(q,\frac{\varepsilon'}{2})$ and suppose that $x \in E^1$. Let $y = g(q,\frac{\varepsilon'}{2},x)$. Clearly $|x-y| < \frac{\varepsilon'}{2}$. It follows that

$$q(x) < z(x) + \frac{\varepsilon'}{2} < f(p,\frac{\varepsilon}{2},x) + \frac{\varepsilon}{2} + \frac{\varepsilon'}{2} < f(p,\varepsilon,x) + \varepsilon .$$

Moreover, if $p(x) > 0$, $z(x) > p(x) - \frac{\varepsilon}{2} > \eta(p) - \frac{\varepsilon}{2} > \frac{\varepsilon}{2}$. And so $q(y) > z(x) - \frac{\varepsilon'}{2} > 0$. Hence, $x = y$. Then, $|q(x) - p(x)| \leq |z(x) - q(x)| + |z(x) - p(x)| < \varepsilon$. Therefore, by Lemma 1, $q \in R(p,\varepsilon)$.

Lemma 5. If $q \in R(p,\varepsilon)$ then there is an $\varepsilon' > 0$ such that $R(q,\varepsilon') \subset R(p,\varepsilon)$.

Proof. Choose $0 < \varepsilon' < \min.\{\{\varepsilon - |x - g(p,\varepsilon,x)| : q(x) > 0\} \cup \{\varepsilon + f(p,\varepsilon,x) - q(x) : x \in E^1\} \cup \{\varepsilon - |p(x) - q(x)| : p(x) > 0\} \cup \{|x-y| : x \neq y,\ p(x) > 0,\ q(y) > 0\}\}$. Let $z \in R(q,\varepsilon')$ and $x \in E^1$. Put $y = g(q,\varepsilon',x)$ if $z(x) < \varepsilon$, then $z(x) < p(x) + \varepsilon \leq f(p,\varepsilon,x) + \varepsilon$. In case $z(x) \geq \varepsilon$, we have $q(y) > z(x) - \varepsilon' \geq \varepsilon - \varepsilon' > 0$. So,

$$|x - g(p,\varepsilon,y)| \leq |x - y| + |y - g(p,\varepsilon,y)| < \varepsilon' + |y - g(p,\varepsilon,y)| < \varepsilon.$$

Hence,

$$z(x) < q(y) + \varepsilon' < q(y) + \varepsilon + f(p,\varepsilon,y) - q(y) = p(g(p,\varepsilon,y)) + \varepsilon \leq f(p,\varepsilon,x) + \varepsilon.$$

If $p(x) > 0$ then $x = y$. So, in case $q(x) > 0$, $|z(x) - q(x)| < \varepsilon'$, and *in* case $q(x) = 0$, $z(x) < q(x) + \varepsilon'$, hence $|z(x) - q(x)| < \varepsilon'$. Therefore, $|z(x) - p(x)| \leq |z(x) - q(x)| + |q(x) - p(x)| < \varepsilon' + |q(x) - p(x)| < \varepsilon$. Hence, by Lemma 1, $z \in R(p,\varepsilon)$. So we have that $R(q,\varepsilon') \subset R(p,\varepsilon)$.

I. The regions are well-defined, that is, the set $\{R(p,\varepsilon):p \in \psi, \varepsilon > 0\}$ is a topological base for ψ. This is an immediate consequence of Lemmas 5 and 2.

II. The space ψ is a Hausdorff space. To see this, suppose that p and q are two points in ψ. Without loss of generality assume that $t \in E^1$ with $p(t) > q(t)$. Let $\varepsilon = \min. \{\frac{\eta(p)}{2}, \frac{p(t) - q(t)}{2}\}$. Clearly $q \notin R(p,\varepsilon)$. Now choose ε' as in Lemma 4. Then by Lemma 4, $R(p,\frac{\varepsilon}{2}) \cap R(q,\frac{\varepsilon'}{2}) = \phi$.

III. The space ψ is regular. In view of Lemmas 2 and 5, it suffices to show that for each $p \in \psi$ there is an $\varepsilon > 0$ such that $\overline{R(p,\frac{\varepsilon'}{2})} \subset R(p,\varepsilon')$ for each $\varepsilon' < \varepsilon$. According to Lemma 4, $\eta(p)$ is precisely such an ε.

IV. The space ψ is developable. For each positive integer n, let $G_n = \{R(p,\frac{1}{n}):p \in \psi\}$. It is clear that $\{G_n\}$ is a sequence of open covers of ψ. In view of Lemmas 5 and 2, the fact that $\{G_n\}$ is a development for ψ follows from Lemma 3.

V. No region is separable. To see this, take a region $R(p,\varepsilon)$ and a countable subset of it, say, C. Let $S = \bigcup_{q \in C} \{x:q(x) > 0\}$. S is countable. Let $t \in E^1 - S$. Choose $z \in \psi$ such that $z(x) = p(x)$ for each $x \neq t$, and $z(t) = \frac{\varepsilon}{2}$. Then $z \in R(p,\varepsilon)$ but $R(z,\frac{\varepsilon}{4}) \cap C = \phi$.

VI. Every collection of pairwise disjoint open sets is countable. Assume, on the contrary, that there is an uncountable such collection. In *the* presence of Lemmas 5 and 2, usual counting arguments give M and N, each a positive integer, and an uncountable subcollection K of G_N of pairwise disjoint sets such that for each $R(p,\frac{1}{N}) \in K$, the set $\{x:p(x) > 0\}$ has exactly M elements. Assign to each p with $R(p,\frac{1}{N}) \in K$ the point $\tilde{p} \in E^{2M}$ such that $\tilde{p} = (x_1,x_2,\cdots,x_M,p(x_1),p(x_2),\cdots,p(x_M))$ where $x_1 < x_2 < \cdots < x_M$ and $\{x_1,x_2,\cdots,x_M\} = \{x:p(x) > 0\}$. Since the set $\{\tilde{p}\}$ of all such points in E^{2M} is uncountable, it contains a limit point. So let $\tilde{p}$ and $\tilde{q}$ be two points with $|\tilde{p}_j - \tilde{q}_j| < \frac{1}{N}$ for $j = 1,\cdots,2M$. Then $\sup.\{p,q\} = \{(u,v):u \in E', \ v = \max\{p(u),q(u)\}\} \in R(p,\frac{1}{N}) \cap R(q,\frac{1}{N})$ - a contradiction to K being a collection of pairwise disjoint sets.

VII. Each region is pathwise connected. To see this, let $q \in R(p,\varepsilon)$ and define $\pi:[0,1] \to R(p,\varepsilon)$ as follows:

$$\pi(t)(x) = tq(x) \quad \text{if} \quad q(x) > 0 \quad \text{but} \quad p(x) = 0$$

$$\pi(t)(x) = tq(x) + (1-t)p(x) \quad \text{if} \quad p(x) > 0.$$

$$\pi(t)(x) = 0 \quad \text{otherwise}.$$

That π is a continuous function and $\pi(0) = p$ and $\pi(1) = q$ can be easily verified.

VIII. The space ψ is pathwise connected. The same sort path π as in VII will join any two points.

IX. The space ψ is a Moore space with properties 1-6. I, II, III and IV say that ψ is a Moore space. From V we get property 1. VI is property 2. Property 3 now follows from Proposition 6. V also gives property 4. Properties 5 and 6 follow from VII and VIII respectively.

References

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