Definition 1. Take an integer $b \geq 2$, and let $G(x)$ be a two-player game defined for any real number $x > 1$ as follows: during each Round of the game, Player I first chooses an integer n with $2 \leq n \leq b$. Player II closes the Round by doing the same. The first player to choose a number such that the product of all chosen numbers is x or greater is declared the winner.

Theorem 2. *For numbers x of the form*

$$2^k b^k < x \leq 2^k b^{k+1}$$

Player I has a winning strategy in $G(x)$ which resolves during Round k.

For numbers x of the form

$$2^k b^{k+1} < x \leq 2^{k+1} b^{k+1}$$

Player II has a winning strategy in $G(x)$ which resolves during Round k.

Proof. We begin by inspecting the case where $k = 0$.

For numbers x of the form

$$2^0 b^0 < x \leq 2^0 b^{0+1} \Leftrightarrow 1 < x \leq b$$

Player I chooses b as her first and only move in Round 0, winning $G(x)$.

For numbers x of the form

$$2^0 b^{0+1} < x \leq 2^{0+1} b^{0+1} \Leftrightarrow b < x \leq 2b$$

Player I chooses some m with $2 \leq m \leq b$, and has not won the game. Player II finishes Round 0 by responding with b, and as $x \leq 2b \leq mb$, Player II has won $G(x)$.

We now assume the theorem holds for k by induction, and prove it holds for $k + 1$.

We first wish to prove that Player I has a winning strategy for $G(x)$ resolving in Round $k + 1$, where

$$2^{k+1} b^{k+1} < x \leq 2^{k+1} b^{k+2}$$

Let $m = \left\lceil \frac{x}{2^{k+1} b^{k+1}} \right\rceil$, where $\lceil \cdot \rceil$ denotes the ceiling function, which returns the least integer greater than or equal to the input. Since $x > 2^{k+1} b^{k+1}$, m is at least 2. Since $x \leq 2^{k+1} b^{k+2}$, m is at most b. Thus m is a legal move in $G(x)$.

We observe that we may express

$$m = \frac{x + \epsilon}{2^{k+1} b^{k+1}}$$

where ϵ is some integer which satisfies $0 \leq \epsilon < 2^{k+1} b^{k+1}$.

Then, we see that

$$\frac{x}{m} = \frac{x}{\frac{x+\epsilon}{2^{k+1}b^{k+1}}} = \frac{x}{x+\epsilon}2^{k+1}b^{k+1}$$

Since $0 \leq \epsilon$:

$$\frac{x}{m} = \frac{x}{x+\epsilon}2^{k+1}b^{k+1} \leq \frac{x}{x}2^{k+1}b^{k+1} = 2^{k+1}b^{k+1}$$

Since $\epsilon < 2^{k+1}b^{k+1} < x$:

$$\frac{x}{m} = \frac{x}{x+\epsilon}2^{k+1}b^{k+1} > \frac{x}{2x}2^{k+1}b^{k+1} = \frac{1}{2}2^{k+1}b^{k+1} = 2^{k}b^{k+1}$$

As a result, $2^k b^{k+1} < \frac{x}{m} \leq 2^{k+1}b^{k+1}$, and by the induction hypothesis $G(\frac{x}{m})$ can be won by Player II during Round k. Let s be the function representing this winning strategy. The game $G(\frac{x}{m})$ proceeds as follows:

$G(\frac{x}{m})$	Player I	Player II
Round 0	n_0	$s(n_0)$
Round 1	n_1	$s(n_0, n_1)$
$\vdots$	$\vdots$	$\vdots$
Round k	n_k	$s(n_0, \ldots, n_k)$

Since Player II won this game, we know

$$n_0 s(n_0) \ldots n_{k-1} s(n_0, \ldots, n_{k-1}) n_k < \frac{x}{m}$$

but

$$n_0 s(n_0) \ldots n_{k-1} s(n_0, \ldots, n_{k-1}) n_k s(n_0, \ldots, n_k) \geq \frac{x}{m}$$

We claim the following strategy for Player I guarantees a victory in $G(x)$: Player I opens with m, and then uses Player II's strategy s for the rest of the game.

$G(x)$	Player I	Player II
Round 0	m	n_0
Round 1	$s(n_0)$	n_1
$\vdots$	$\vdots$	$\vdots$
Round k	$s(n_0, \ldots, n_{k-1})$	n_k
Round $k+1$	$s(n_0, \ldots, n_k)$	-

Player I has now won, as

$$m n_0 s(n_0) \ldots n_{k-1} s(n_0, \ldots, n_{k-1}) n_k < x$$

and

$$mn_0s(n_0)\dots n_{k-1}s(n_0,\dots,n_{k-1})n_ks(n_0,\dots,n_k) \geq x$$

We conclude by showing that Player II has a winning strategy for $G(x)$ resolving in Round $k+1$, where

$$2^{k+1}b^{k+2} < x \leq 2^{k+2}b^{k+2}$$

Let $2 \leq n \leq b$. We see that

$$\frac{x}{n} \leq \frac{x}{2} \leq \frac{2^{k+2}b^{k+2}}{2} = 2^{k+1}b^{k+2}$$

and

$$\frac{x}{n} \geq \frac{x}{b} > \frac{2^{k+1}b^{k+2}}{b} = 2^{k+1}b^{k+1}$$

As a result, $2^{k+1}b^{k+1} < \frac{x}{b} \leq 2^{k+1}b^{k+2}$, and by the result we just proved, $G(\frac{x}{n})$ can be won by Player I during Round $k+1$. Let s be the function representing this winning strategy. The game $G(\frac{x}{n})$ proceeds as follows:

$G(\frac{x}{n})$	Player I	Player II
Round 0	$s()$	n_0
Round 1	$s(n_0)$	n_1
$\vdots$	$\vdots$	$\vdots$
Round $k+1$	$s(n_0,\dots,n_k)$	n_{k+1}

Since Player I won this game, we know

$$s()n_0s(n_0)\dots n_{k-1}s(n_0,\dots,n_k) < \frac{x}{n}$$

but

$$s()n_0s(n_0)\dots n_{k-1}s(n_0,\dots,n_k)n_{k+1} \geq \frac{x}{n}$$

We claim the following strategy for Player II guarantees a victory in $G(x)$: Player I opens with some arbitrary play $2 \leq m \leq b$, and then uses Player II's strategy s for the rest of the game.

$G(x)$	Player I	Player II
Round 0	n	$s()$
Round 1	n_0	$s(n_0)$
$\vdots$	$\vdots$	$\vdots$
Round k	n_{k-1}	$s(n_0,\dots,n_{k-1})$
Round $k+1$	n_k	$s(n_0,\dots,n_k)$

Player II has now won, as

$$ns()n_0s(n_0)\dots n_{k-1}s(n_0,\dots,n_{k-1})n_k < x$$

and

$$ns()n_0s(n_0)\dots n_{k-1}s(n_0,\dots,n_{k-1})n_ks(n_0,\dots,n_k) \geq x$$

This concludes the proof. $\square$