Problem 26. Write the following vectors as the scalar product of their magnitude and direction:

- $\langle 5,5\rangle$ $\to |\langle 5,5\rangle| = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2} \to \langle 5,5\rangle = \boxed{5\sqrt{2}\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle}$
- $\langle -4,3\rangle$ $\to |\langle -4,3\rangle| = \sqrt{16+9} = \sqrt{25} = 5 \to \langle -4,3\rangle = \boxed{5\langle -4/5, 3/5\rangle}$
- $\langle 12,-5\rangle$ $\to |\langle 12,-5\rangle| = \sqrt{144+25} = \sqrt{169} = 13 \to \langle 12,-5\rangle = \boxed{13\langle 12/13, -5/13\rangle}$
- $\langle 3,1,-2\rangle$ $\to |\langle 3,1,-2\rangle| = \sqrt{9+1+4} = \sqrt{14} \to \langle 3,1,-2\rangle = \boxed{\sqrt{14}\langle \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}\rangle}$
- $\langle 4,-2,-4\rangle$ $\to |\langle 4,-2,-4\rangle| = \sqrt{16+4+16} = \sqrt{36} = 6 \to \langle 4,-2,-4\rangle = \boxed{6\langle 2/3, -1/3, -2/3\rangle}$
- $\langle 8,0,-6\rangle$ $\to |\langle 8,0,-6\rangle| = \sqrt{64+0+36} = \sqrt{100} = 10 \to \langle 8,0,-6\rangle = \boxed{10\langle 4/5, 0, -3/5\rangle}$

Definition 27. The **standard unit vectors** in $\mathbb{R}^2$ are $\hat{\mathbf{i}} = \langle 1,0\rangle$ and $\hat{\mathbf{j}} = \langle 0,1\rangle$, and any vector in $\mathbb{R}^2$ can be expressed in **standard unit vector form**:

$$\langle a,b\rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$$

The **standard unit vectors** in $\mathbb{R}^3$ are $\hat{\mathbf{i}} = \langle 1,0,0\rangle$, $\hat{\mathbf{j}} = \langle 0,1,0\rangle$, and $\hat{\mathbf{k}} = \langle 0,0,1\rangle$, and any vector in $\mathbb{R}^3$ can be expressed in **standard unit vector form**:

$$\langle a,b,c\rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$$

Note 28. Since the xy-plane is the the plane $z = 0$ in xyz-space, we say the points $(a,b) = (a,b,0)$ and vectors $\langle a,b\rangle = \langle a,b,0\rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$ are equal.

Problem 29. Write the following vectors in standard unit vector form.

- $\langle 5, 5\rangle$ $= 5\hat{i} + 5\hat{j}$
- $\langle -4, 3\rangle$ $= -4\hat{i} + 3\hat{j}$
- $\langle 12, -5\rangle$ $= 12\hat{i} - 5\hat{j}$
- $\langle 3, 1, -2\rangle$ $= 3\hat{i} + \hat{j} - 2\hat{k}$
- $\langle 4, -2, -4\rangle$ $= 4\hat{i} - 2\hat{j} - 4\hat{k}$
- $\langle 8, 0, -6\rangle$ $= 8\hat{i} \cancel{+0\hat{j}} - 6\hat{k}$

Theorem 30. The following properties hold for any two vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and scalars a, b.

- $\vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{v}} + \vec{\mathbf{u}}$
- $(\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}} = \vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}})$
- $\vec{\mathbf{u}} + \vec{\mathbf{0}} = \vec{\mathbf{u}}$
- $\vec{\mathbf{u}} + (-\vec{\mathbf{u}}) = \vec{\mathbf{0}}$
- $0\vec{\mathbf{u}} = \vec{\mathbf{0}}$
- $1\vec{\mathbf{u}} = \vec{\mathbf{u}}$
- $a(b\vec{\mathbf{u}}) = (ab)\vec{\mathbf{u}}$
- $a(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = a\vec{\mathbf{u}} + a\vec{\mathbf{v}}$
- $(a + b)\vec{\mathbf{u}} = a\vec{\mathbf{u}} + b\vec{\mathbf{u}}$

Definition 31. Vector subtraction is defined as the addition of a negative:

$$\vec{\mathbf{u}} - \vec{\mathbf{v}} = \vec{\mathbf{u}} + (-\vec{\mathbf{v}}) = \langle u_1 - v_1, u_2 - v_2\rangle$$

$$\vec{\mathbf{u}} - \vec{\mathbf{v}} = \vec{\mathbf{u}} + (-\vec{\mathbf{v}}) = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3\rangle$$

Suggested Homework: Section 12.2 numbers 3, 5, 13, 14, 15, 19, 21, 24, 26

12.3 The Dot Product

Definition 32. Let θ be the angle between two non-zero vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$. The **dot product** $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$ is the product of their lengths when projected into the same direction, obtained by this formula:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

Definition 33. The dot product with a zero vector is always zero:

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{0}} = \vec{\mathbf{0}} \cdot \vec{\mathbf{v}} = 0$$

Theorem 34. By the Law of Cosines:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle = u_1 v_1 + u_2 v_2$$

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Definition 35. Two vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ are **orthogonal** if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.

Theorem 36. Two non-zero vectors are orthogonal if the angle θ between them is $\frac{\pi}{2}$ radians.

Theorem 37. The following properties hold for any three vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ and scalar c.

- $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$
- $(c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}}) = c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})$
- $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$
- $\vec{\mathbf{u}} \cdot \vec{\mathbf{u}} = |\vec{\mathbf{u}}|^2$

Problem 38. Solve for $\cos\theta$ for the following pairs of vectors.

- $\vec{\mathbf{u}} = \langle 4, -3 \rangle$
  $\vec{\mathbf{v}} = \langle 5, 12 \rangle$

  $\vec{u} \cdot \vec{v} = (4)(5) + (-3)(12)$
  $= 20 - 36$
  $= -16$

  $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$
  $-16 = (5)(13) \cos\theta$
  $\cos\theta = \boxed{-\frac{16}{65}}$

- $\vec{\mathbf{u}} = \langle 1, 4, 2 \rangle$
  $\vec{\mathbf{v}} = \langle 4, 1, -2 \rangle$

  $\vec{u} \cdot \vec{v} = (1)(4) + (4)(1) + (2)(-2)$
  $= 4$

  $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$
  $4 = \sqrt{21}\sqrt{21} \cos\theta$
  $\cos\theta = \boxed{\frac{4}{21}}$

- $\vec{\mathbf{u}} = \langle 0, 5, -11 \rangle$
  $\vec{\mathbf{v}} = \langle 2, 0, 0 \rangle$

  $\vec{u} \cdot \vec{v} = (0)(2) + (5)(0) + (-11)(0)$
  $= 0$

  $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$
  $0 = \cancel{|\vec{u}||\vec{v}|} \cos\theta$
  $\cos\theta = \boxed{0}$

Definition 39. The work W done by a force vector $\vec{\mathbf{F}}$ over a displacement vector $\vec{\mathbf{D}}$ is given by

$$W = \vec{\mathbf{F}} \cdot \vec{\mathbf{D}} = |\vec{\mathbf{F}}||\vec{\mathbf{D}}| \cos\theta$$

Suggested Homework: Section 12.3 numbers 3, 5, 6, 7, 8, 9, 10, 11, 15, 17, 21, 27, 41, 42, 44

12.4 The Cross Product

Definition 40. For any two non-parallel vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ in $\mathbb{R}^3$, the **Right-Hand Rule** gives a specific direction orthogonal to both: position $\vec{\mathbf{u}}$ with your right thumb and $\vec{\mathbf{v}}$ with your right index finger, and let your middle finger extend orthogonal to both to give this direction.

Definition 41. Let θ be the angle between two non-zero vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ in $\mathbb{R}^3$, and let $\vec{\mathbf{n}}$ be the direction given by the Right-Hand Rule. The **cross product** $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is the vector orthogonal to both which follows the Right-Hand Rule and has magnitude equal to the area of the parallelogram formed from both.

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (|\vec{\mathbf{u}}||\vec{\mathbf{v}}| \sin\theta)\vec{\mathbf{n}}$$

$$|\vec{\mathbf{u}} \times \vec{\mathbf{v}}| = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \sin\theta$$

Definition 42. The cross product with a zero vector is always the zero vector:

$$\vec{\mathbf{v}} \times \vec{\mathbf{0}} = \vec{\mathbf{0}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$$

Theorem 43. The following properties hold for any three vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ and scalars a,b.

- $(a\vec{\mathbf{u}}) \times (b\vec{\mathbf{v}}) = (ab)(\vec{\mathbf{u}} \times \vec{\mathbf{v}})$
- $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$
- $(\vec{\mathbf{v}} + \vec{\mathbf{w}}) \times \vec{\mathbf{u}} = \vec{\mathbf{v}} \times \vec{\mathbf{u}} + \vec{\mathbf{w}} \times \vec{\mathbf{u}}$
- $\vec{\mathbf{v}} \times \vec{\mathbf{u}} = -(\vec{\mathbf{u}} \times \vec{\mathbf{v}})$

Definition 44. Two vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ are **parallel** if $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = 0$.

Theorem 45. Two non-zero vectors are parallel if the angle θ between them is 0 or π radians.

Definition 46. The cross products of the standard unit vectors are given as follows:

- $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$
- $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$
- $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$

Definition 47. A **determinant** is a short hand for writing certain commonly occuring algebraic expressions:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Theorem 48. By breaking up $\vec{u}$, $\vec{v}$ into standard unit vectors:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle$$

Problem 49. Use the cross product to find a vector normal to both $\vec{u}$ and $\vec{v}$.

- $\vec{u} = \langle 4, -3, 0 \rangle$
  $\vec{v} = \langle 2, 6, -3 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -3 & 0 \\ 2 & 6 & -3 \end{vmatrix} = \hat{i}\begin{vmatrix} -3 & 0 \\ 6 & -3 \end{vmatrix} - \hat{j}\begin{vmatrix} 4 & 0 \\ 2 & -3 \end{vmatrix} + \hat{k}\begin{vmatrix} 4 & -3 \\ 2 & 6 \end{vmatrix}$$
$$= \langle 9, 12, 30 \rangle$$

- $\vec{u} = \langle 1, 4, 2 \rangle$
  $\vec{v} = \langle 4, 1, -2 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 4 & 1 & -2 \end{vmatrix} = \hat{i}\begin{vmatrix} 4 & 2 \\ 1 & -2 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 4 \\ 4 & 1 \end{vmatrix}$$
$$= \langle -10, 10, -14 \rangle$$

- $\vec{u} = \langle 0, 5, -11 \rangle$
  $\vec{v} = \langle 2, 0, 0 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 5 & -11 \\ 2 & 0 & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} 5 & -11 \\ 0 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 0 & -11 \\ 2 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 0 & 5 \\ 2 & 0 \end{vmatrix}$$
$$= \langle 0, -22, -10 \rangle$$

Definition 50. The torque τ done by a force vector $\vec{F}$ on an arm given by $\vec{D}$ is given by

$$\tau = |\vec{F} \times \vec{D}| = |\vec{F}||\vec{D}| \sin\theta$$

Theorem 51. The volume of a parallelpiped determined by the vectors $\vec{u}$, $\vec{v}$, $\vec{w}$, is given by the **triple scalar product**

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Suggested Homework: Section 12.4 numbers 1 – 3, 17, 19, 28, 29, 33, 35

12.5 Lines and Planes in Space

Theorem 52. Let L be the line in $\mathbb{R}^2$ normal to the vector $\vec{\mathbf{N}} = \langle A, B\rangle$ and passing through the point $P_0 = (x_0, y_0)$. Then every point $P = (x, y)$ on the line L must satisfy the following equations:

$$\vec{\mathbf{N}} \cdot \overrightarrow{\mathbf{P_0P}} = 0$$

$$A(x - x_0) + B(y - y_0) = 0$$

Let M be the plane in $\mathbb{R}^3$ normal to the vector $\vec{\mathbf{N}} = \langle A, B, C\rangle$ and passing through the point $P_0 = (x_0, y_0, z_0)$. Then every point $P = (x, y, z)$ on the plane M must satisfy the following equations:

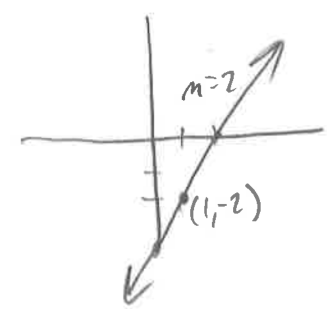
$$\vec{\mathbf{N}} \cdot \overrightarrow{\mathbf{P_0P}} = 0$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

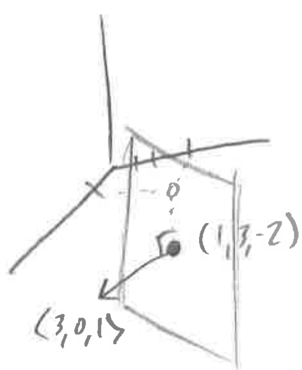
Problem 53. Sketch and find equations for the following lines and planes:

- The line passing through $(1, -2)$ and parallel to the line with equation $2x - y = 3$.
- The plane passing through $(1, 3, -2)$ and normal to the vector $\langle 3, 0, 1\rangle$.
- The plane passing through $(-2, 0, 4)$, $(1, 3, 3)$, and $(0, 0, 2)$.

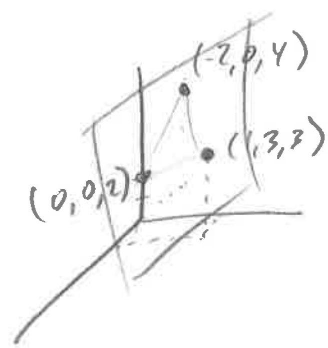
(Work omitted)



$2x - y = 4$



$3x + z = -4$



$6x - 4y + 6z = 12$

OR

$3x - 2y + 3z = 6$

etc.

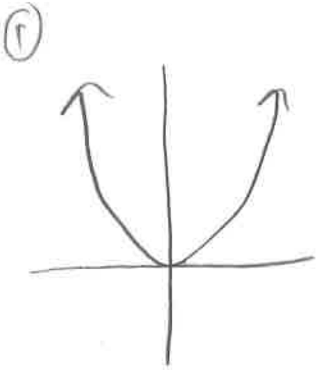
Definition 54. Parametric equations $x(t), y(t)$ for a curve in $\mathbb{R}^2$ assign a point $(x(t), y(t))$ of the curve to each value of t.

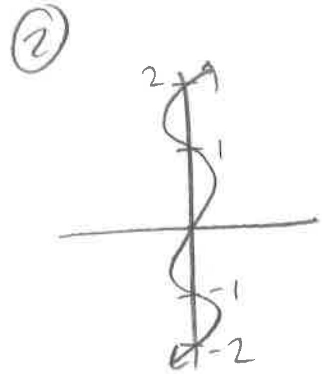
Parametric equations $x(t), y(t), z(t)$ for a curve in $\mathbb{R}^3$ assign a point $(x(t), y(t), z(t))$ of the curve to each value of t.

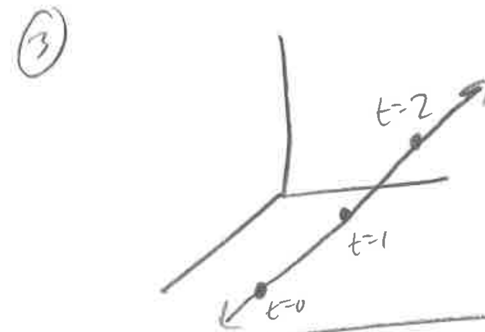
Problem 55. Sketch the curves given by the following parametric equations.

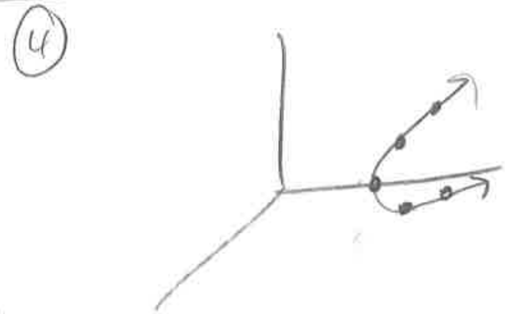
- 1 $x(t) = t$, $y(t) = t^2$
- 2 $x(t) = \sin t$, $y(t) = \frac{t}{\pi}$
- 3 $x(t) = 1 - t$, $y(t) = 3t$, $z(t) = 2t - 3$
- 4 $x(t) = -t^2$, $y(t) = 2$, $z(t) = t$

((Charts omitted))









Theorem 56. Let L be the line in $\mathbb{R}^2$ parallel to the vector $\vec{\mathbf{v}} = \langle a, b\rangle$ and passing through the point $P_0 = (x_0, y_0)$. Then every point $P = (x, y)$ on the line L must satisfy the following vector equation for some t:

$$\vec{\mathbf{P}} = \vec{\mathbf{v}}t + \vec{\mathbf{P_0}}$$

Thus the line is given by the parametric equations

$$x(t) = at + x_0$$

$$y(t) = bt + y_0$$

Let L be the line in $\mathbb{R}^3$ parallel to the vector $\vec{\mathbf{v}} = \langle a, b, c\rangle$ and passing through the point $P_0 = (x_0, y_0, z_0)$. Then every point $P = (x, y, z)$ on the line L must satisfy the following vector equation for some t:

$$\vec{\mathbf{P}} = \vec{\mathbf{v}}t + \vec{\mathbf{P_0}}$$

Thus the line is given by the parametric equations

$$x(t) = at + x_0$$

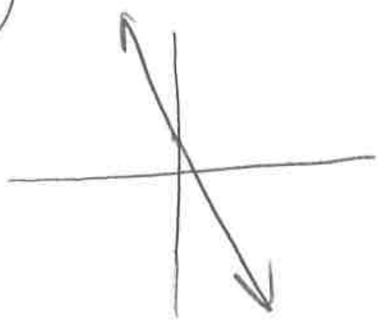
$$y(t) = bt + y_0$$

$$z(t) = ct + z_0$$

Problem 57. Sketch and give parametric equations for the following lines.

1. • The line with equation $y = -3x + 1$ in the xy plane.
2. • The line passing through $(1, 3, -2)$ and parallel to $\langle 3, 0, 1\rangle$.
3. • The line normal to the plane with equation $x+y+2z = 4$ and passing through $(1, 1, 1)$.

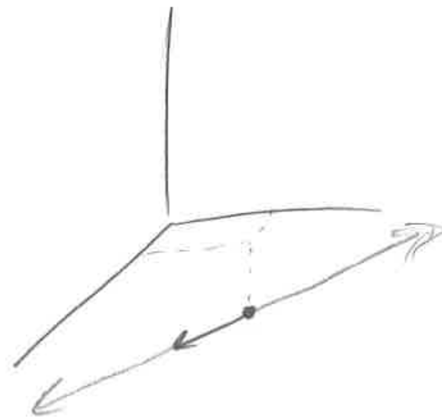
1



$x(t) = t$
$y(t) = -3t + 1$

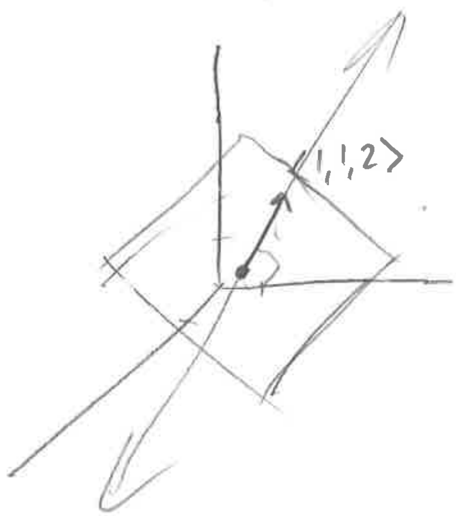
2

$x(t) = 3t + 1$
$y(t) = 0t + 3$
$z(t) = t - 2$



3

$x(t) = 1t + 1$
$y(t) = 1t + 1$
$z(t) = 2t + 1$



Suggested Homework: Section 12.5 numbers 3, 4, 6, 7, 17, 19, 24, 27, 31, 32