11.2 Series

Observation 45. What do we mean when we write $\pi = 3.1415926535\ldots$? It is a convenient way to write the following:

$$\pi = \frac{3}{10^0} + \frac{1}{10^1} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \frac{3}{10^9} + \frac{5}{10^{10}} + \cdots .$$

Definition 46 (Series). Adding up the terms in an infinite sequence is a **series**. That is to say, given a sequence $(a_n)_{n=1}^{\infty}$, the series would be denoted as

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \ldots$$

Definition 47 (Partial Sum). Let (a_n) be a sequence. The **partial sums** of the sequence are

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.
\end{aligned}$$

Definition 48 (Definition of Series). If $s_n = \sum_{i=1}^{n} a_i$ is the n^{th} partial sum of the sequence $(a_i)_{i=1}^{\infty}$, then the **value** or **sum** of the series $\sum_{i=1}^{\infty} a_i$ is defined to be the limit of its sequence of partial sums:

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} s_n$$

whenever the limit exists.

Definition 49 (Series Convergence & Divergence). The series $\sum_{i=1}^{\infty} a_i$ **converges** or **diverges** based on whether its sequence of partial sums

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} s_n$$

converges or diverges.

Problem 50. Determine whether or not the series $\sum_{i=1}^{\infty} a_i$ converges or diverges, given its n^{th} partial sum $s_n = a_1 + a_2 + \cdots + a_n = \dfrac{2n^2}{3n^2+5}$. 9/10

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n^2}{3n^2+5} = \boxed{\frac{2}{3}}$$

$\boxed{\text{Converges}}$

Note The above problem does not say anything about the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2}{3n^2+5}$.

Theorem 51 (Geometric Series). The **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

The geometric series is divergent if $|r| \geq 1$.

Problem 52. Show that the geometric series converges when $|r| < 1$ and diverges otherwise. 9/10

Hint Show that its n^{th} partial sum is $s_n = a\frac{1-r^n}{1-r}$.

$s_n = ar^0 + \cancel{\cdots} + \cancel{ar^{n-1}}$

$rs_n = \cancel{ar^1} + \cancel{\cdots} + ar^n$

$s_n - rs_n = ar^0 - ar^n$

$(1-r)s_n = a(1-r^n)$

$s_n = a\dfrac{1-r^n}{1-r}$ $(r \neq 1)$

So $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} a\frac{1-r^n}{1-r}$

$\displaystyle = a\frac{1-\lim_{n\to\infty} r^n}{1-r}$

When $|r|<1$:

$\displaystyle = a\frac{1-0}{1-r}$

$\displaystyle = \boxed{\frac{a}{1-r}}$

$\boxed{\text{Converges}}$

When $r=-1$ or $|r|>1$:

$\displaystyle\lim_{n\to\infty} r^n$

$\boxed{\text{diverges}}$

Problem 53. Find the sum of the series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

~~9~~ ~~10~~

$$= \sum_{n=1}^{\infty} \underset{a}{(5)} \underset{r}{\left(-\frac{2}{3}\right)}^{n-1} \qquad (|r|<1)$$

$$= \frac{5}{1-\left(-\frac{2}{3}\right)} = \frac{5}{5/3} = \boxed{3}$$

$\boxed{\text{Converges}}$

Problem 54. Is $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

~~4~~ ~~10~~

$$= \sum_{n=1}^{\infty} 4^n 3^{-(n-1)} \qquad = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}}$$

$\boxed{\text{diverges}}$

since $\left|\frac{4}{3}\right| = \frac{4}{3} \geq 1$

$$= \sum_{n=1}^{\infty} 4\,\frac{4^{n-1}}{3^{n-1}}$$

Problem 55. Compute the sum of the series $\sum_{n=0}^{\infty} ar^n$ for $|r| < 1$.

~~0~~ ~~10~~

$$\sum_{n=0}^{\infty} ar^n = ar^0 + ar^1 + ar^2 + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \boxed{\frac{a}{1-r}}$$

Theorem 56 (Harmonic Series). The **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

Problem 57. Show that the harmonic series diverges.

$\cancel{9}$ $\cancel{10}$

$$= (1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

$\boxed{\text{diverges}}$

Problem 58. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and find its sum.

$\cancel{9}/10$

Hint Show that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

$$\frac{1}{n} - \frac{1}{n+1} = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} = \frac{1}{n(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty} \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \cancel{\left(\frac{1}{2} - \frac{1}{3}\right)} + \cdots + \cancel{\left(\frac{1}{n}\right.} - \left.\frac{1}{n+1}\right)$$

$$= \lim_{n\to\infty} \frac{1}{1} - \cancel{\frac{1}{n+1}}$$

$$= \boxed{1}$$

$\boxed{\text{Converges}}$

Theorem 59 (Divergence Test). If $\underbrace{\lim_{n\to\infty} a_n \neq 0}_{P}$, then $\underbrace{\sum_{n=1}^{\infty} a_n \text{ is divergent.}}_{Q}$

Problem 60. Show that $\sum_{n=1}^{\infty} \frac{2n^2}{3n^2+5}$ diverges. 9/10

$$\lim_{n\to\infty} \frac{2n^2}{3n^2+5} = \frac{2}{3} \neq 0$$

so $\sum_{n=1}^{\infty} \frac{2n^2}{3n^2+5}$ $\boxed{\text{diverges}}$

Problem 61. Write the contrapositive of the Divergence Test. 10/10

If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

Problem 62. Use the contrapositive of the Divergence Test to show that the sequence $\langle (0.6)^n \rangle$ converges. 10/10

$\sum_{n=1}^{\infty} (0.6)^n$ is geo. series,

$|0.6| < 1$, so it converges.

Thus $\langle (0.6)^n \rangle$ $\boxed{\text{converges}}$ to 0 as a sequence.

Problem 63. Find the sum of $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$ $= \sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n}$ 10/10

$$= \left(\lim_{n\to\infty} \frac{3}{1} - \cancel{\frac{3}{2}} + \cancel{\frac{3}{2}} - \cancel{\frac{3}{3}} + \cdots + \cancel{\frac{3}{n}} - \frac{3}{n+1} \right) + \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$$

$$= \lim_{n\to\infty} \left(3 - \frac{3}{n+1}\right) + \frac{1/2}{1-1/2}$$

$$= 3 + 1 = \boxed{4}$$

Suggested Problems: Section 11.2 numbers 3, 15, 17, 18, 21, 27, 29, 30, 31, 33, 38, 45