Chapter 6

Applications of Integrals

6.1 Area Between Curves

6.1.1 Area with Respect to the x-axis

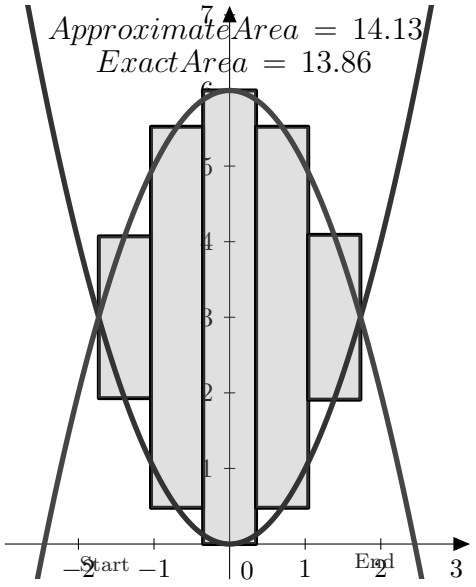

Figure 6.1: Area Between Two Curves

Goal 1. To be able to find the area between two curves.

Motivation 2. Say we want to find the area enclosed by two curves. Over a given area, there is a top function and a bottom function. Let's call the top function $f(x)$ and the bottom function $g(x)$. Just as we did with area under the curve, we can split this region into n rectangles and add up all of their areas to approximate the area between these curves. Each rectangle has width Δx. To find the length of each rectangle, let x_i^* denote a point within the i^{th} rectangle. Then the length (or height) of rectangle i is $f(x_i^*) - g(x_i^*)$.

Theorem 3. The area, A, between two curves can be found by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f\left(x_i^*\right) - g\left(x_i^*\right)\right) \Delta x.$$

Recall 4. From Chapter 5.3, we have seen that

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x = \int_a^b f(x)\,dx.$$

Following the same line of reasoning, we have

Theorem 5. The area A of the region bounded by the curves $y = f(x), y = g(x), x = a$ and $x = b$, where f and g are continous functions such that $f(x) \geq g(x)$ for all $x \in [a, b]$ is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f\left(x_i^*\right) - g\left(x_i^*\right)\right) \Delta x = \int_a^b f(x) - g(x)\,dx.$$

(Think: "Top minus Bottom")

Problem 6. Find the area of the region bounded above by $y = e^x$, below by $y = x$, and on the sides by $x = 0$ and $x = 1$.

Problem 7. Find the area bounded by $y = x^2$ and $y = 2x - x^2$.

Problem 8. Find the area bounded by $y = \sin(x)$ and $y = \cos(x)$ from $x = 0$ to $x = \frac{\pi}{2}$.

6.1.2 Area with Respect to the y-axis

Finding the area with respect to the y-axis is very similar. One simply has to tilt his or her head to the right. That is to say that instead of thinking "Top Minus Bottom," think "Right minus Left."

Problem 9. Find the area enclosed by $y = x - 1$ and $y^2 = 2x + 6$.

Problem 10. Find the area enclosed by $x = 2y - y^2$ and $x = y^2 - 4y$.

Suggested Homework: Section 6.1 numbers 1, 4, 12, 24, 27, 44, 50