

# Chapter 12

## Vectors and the Geometry of Space

### 12.1 Two and Three Dimensional Space

**Definition 1.** Let  $\mathbb{R}$  be the collection of real numbers, let  $\mathbb{R}^2$  be the collection of all **ordered pairs** of real numbers, and let  $\mathbb{R}^3$  be the collection of all **ordered triples** of real numbers.

$\mathbb{R}$  is known as the **real line**,  $\mathbb{R}^2$  is known as the **real plane** or the  **$xy$ -plane**, and  $\mathbb{R}^3$  is known as **real (3D) space** or  **$xyz$ -space**.

**Definition 2.** The **distance** between two points  $P = (x_1, y_1)$  and  $Q = (x_2, y_2)$  in  $\mathbb{R}^2$  is given by the formula

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The **distance** between two points  $P = (x_1, y_1, z_1)$  and  $Q = (x_2, y_2, z_2)$  in  $\mathbb{R}^3$  is given by the formula

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

**Problem 3.** Plot and find the distance between the following pairs of points:

- $(-2, 6)$  and  $(3, -6)$
- $(0, 0, 0)$  and  $(4, 2, 4)$
- $(3, 7, -2)$  and  $(-1, 7, 1)$
- $(8, 2, 1)$  and  $(4, -2, 7)$

**Definition 4.** **Simple lines** in  $\mathbb{R}^2$  are given by the relations  $x = a$ , and  $y = b$  for real numbers  $a, b$ .

**Simple planes** in  $\mathbb{R}^3$  are given by the relations  $x = a$ ,  $y = b$ ,  $z = c$  for real numbers  $a, b, c$ .

**Definition 5.** A **circle** in  $\mathbb{R}^2$  is the set of all points a fixed distance (called its **radius**) from a fixed point (called its **center**). For a center  $(a, b)$  and radius  $r$ , the equation for a circle is

$$(x - a)^2 + (y - b)^2 = r^2$$

A **sphere** in  $\mathbb{R}^3$  is the set of all points a fixed distance (called its **radius**) from a fixed point (called its **center**). For a center  $(a, b, c)$  and radius  $r$ , the equation for a sphere is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

**Question 6.** Sketch the following curves and surfaces.

- $x = 3$  in the  $xy$ -plane and  $xyz$ -space.
- $y = -1$  in the  $xy$ -plane and  $xyz$ -space.
- $z = 0$  in  $xyz$ -space.
- $(x - 2)^2 + (y + 1)^2 = 9$  in the  $xy$ -plane.
- $x^2 + y^2 + z^2 = 4$  in  $xyz$ -space.
- $x^2 + (y - 1)^2 + z^2 = 1$  in  $xyz$ -space.

Suggested Homework: Section 12.1 numbers 4, 6, 7, 8, 10, 11, 12, 14, 15, 16

## 12.2 Vectors

**Definition 7** (Vector). A **vector**  $\vec{v}$  is a mathematical object that stores a **magnitude** (a nonnegative real number often thought of as length) and **direction**. Two vectors are **equal** if and only if they have the same magnitude and direction.

**Definition 8.** The **zero vector**  $\vec{0}$  has zero magnitude and no direction. (This is the only vector without a direction.)

**Definition 9.** For a given point  $P = (a, b)$  in  $\mathbb{R}^2$ , its **position vector** is given by  $\vec{P} = \langle a, b \rangle$ : the vector from the origin  $(0, 0)$  to the point  $P = (a, b)$ .

For a given point  $P = (a, b, c)$  in  $\mathbb{R}^3$ , its **position vector** is given by  $\vec{P} = \langle a, b, c \rangle$ : the vector from the origin  $(0, 0, 0)$  to the point  $P = (a, b, c)$ .

**Theorem 10.** Two vectors are equal if and only if they share the same magnitude and direction as a common position vector.

**Definition 11.** Since all vectors are equal to some position vector  $\langle a, b \rangle$  or  $\langle a, b, c \rangle$ , we usually define vectors by a position vector written in this **component form**. Since the component form of a vector stores the same information as a point, we will use both interchangeably, that is,  $\langle a, b \rangle = (a, b) \in \mathbb{R}^2$  and  $\langle a, b, c \rangle = (a, b, c) \in \mathbb{R}^3$  (although we usually sketch them differently).

**Problem 12.** Plot the following points and position vectors.

- $(1, 3)$  and  $\langle 1, 3 \rangle$  in the  $xy$ -plane.
- $(-2, 5)$  and  $\langle -2, 5 \rangle$  in the  $xy$ -plane.
- $(1, 1, -3)$  and  $\langle 1, 1, -3 \rangle$  in  $xyz$ -space.
- $(0, 5, 0)$  and  $\langle 0, 5, 0 \rangle$  in  $xyz$ -space.

**Definition 13.** Let  $P = (x_1, y_1, z_1)$  and  $Q = (x_2, y_2, z_2)$ . Then the vector with initial point  $P$  and terminal point  $Q$  is defined as

$$\overrightarrow{\mathbf{PQ}} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

**Problem 14.** Plot and sketch the points  $P$ ,  $Q$  and the vector  $\overrightarrow{\mathbf{PQ}}$  for each.

- $P = (1, 3)$ ,  $Q = (-3, 6)$  in the  $xy$ -plane
- $P = (3, 1)$ ,  $Q = (0, -2)$  in the  $xy$ -plane
- $P = (1, 1, 1)$ ,  $Q = (-3, -1, 3)$  in  $xyz$ -space
- $P = (-2, 0, 3)$ ,  $Q = (1, 3, -3)$  in  $xyz$ -space

**Definition 15.** The magnitude  $|\vec{\mathbf{v}}|$  of a vector  $\vec{\mathbf{v}}$  in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  is the distance between its initial and terminal points.

**Theorem 16.** The magnitude of  $\vec{\mathbf{v}} = \langle a, b \rangle$  is given by

$$|\vec{\mathbf{v}}| = \sqrt{a^2 + b^2}$$

The magnitude of  $\vec{\mathbf{v}} = \langle a, b, c \rangle$  is given by

$$|\vec{\mathbf{v}}| = \sqrt{a^2 + b^2 + c^2}$$

**Problem 17.** Evaluate the magnitude of the following vectors:

- $\langle 5, 5 \rangle$
- $\langle -4, 3 \rangle$
- $\langle 12, -5 \rangle$
- $\langle 3, 1, -2 \rangle$
- $\langle 4, -2, -4 \rangle$
- $\langle 8, 0, -6 \rangle$

### 12.2.1 Basic Vector Operations

**Definition 18.** **Vector addition** is defined component-wise as follows for  $\mathbb{R}^2$  and  $\mathbb{R}^3$

$$\vec{u} + \vec{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$$

$$\vec{u} + \vec{v} = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

**Definition 19.** A **scalar** is simply a real number by itself (as opposed to a vector of real numbers).

**Definition 20.** **Scalar multiplication of a vector** is defined component-wise as follows for  $\mathbb{R}^2$  and  $\mathbb{R}^3$ :

$$k\vec{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle$$

$$k\vec{u} = k\langle u_1, u_2, u_3 \rangle = \langle ku_1, ku_2, ku_3 \rangle$$

**Problem 21.** Sketch the following vectors.

- $\vec{\mathbf{u}} = \langle 1, -3 \rangle$ ,  $\vec{\mathbf{v}} = \langle 3, 1 \rangle$  and  $\vec{\mathbf{u}} + \vec{\mathbf{v}}$  in the  $xy$ -plane.
- $\vec{\mathbf{u}} = \langle 2, 0, 1 \rangle$ ,  $\vec{\mathbf{v}} = \langle -2, 4, 2 \rangle$  and  $\vec{\mathbf{u}} + \vec{\mathbf{v}}$  in  $xyz$ -space.
- $\vec{\mathbf{u}} = \langle 8, -2 \rangle$  and  $\frac{1}{2}\vec{\mathbf{u}}$  in the  $xy$ -plane.
- $\vec{\mathbf{u}} = \langle 5, 3, -1 \rangle$  and  $3\vec{\mathbf{u}}$  in  $xyz$ -space.

**Definition 22.** A vector  $\vec{\mathbf{v}}$  is a **unit vector** if  $|\vec{\mathbf{v}}| = 1$ .

**Theorem 23.** For any non-zero vector  $\vec{\mathbf{v}}$ , the vector

$$\frac{1}{|\vec{\mathbf{v}}|} \vec{\mathbf{v}} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

is a unit vector.

**Definition 24.** The **direction** of a vector  $\vec{\mathbf{v}}$  is the unit vector  $\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$ .

**Theorem 25.** Any vector  $\vec{\mathbf{v}}$  is the scalar product of its magnitude and direction:

$$\vec{\mathbf{v}} = |\vec{\mathbf{v}}| \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

**Problem 26.** Write the following vectors as the scalar product of their magnitude and direction:

- $\langle 5, 5 \rangle$
- $\langle -4, 3 \rangle$
- $\langle 12, -5 \rangle$
- $\langle 3, 1, -2 \rangle$
- $\langle 4, -2, -4 \rangle$
- $\langle 8, 0, -6 \rangle$

**Definition 27.** The **standard unit vectors** in  $\mathbb{R}^2$  are  $\hat{\mathbf{i}} = \langle 1, 0 \rangle$  and  $\hat{\mathbf{j}} = \langle 0, 1 \rangle$ , and any vector in  $\mathbb{R}^2$  can be expressed in **standard unit vector form**:

$$\langle a, b \rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$$

The **standard unit vectors** in  $\mathbb{R}^3$  are  $\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle$ ,  $\hat{\mathbf{j}} = \langle 0, 1, 0 \rangle$ , and  $\hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$ , and any vector in  $\mathbb{R}^3$  can be expressed in **standard unit vector form**:

$$\langle a, b, c \rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$$

**Note 28.** Since the  $xy$ -plane is the plane  $z = 0$  in  $xyz$ -space, we say the points  $(a, b) = (a, b, 0)$  and vectors  $\langle a, b \rangle = \langle a, b, 0 \rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$  are equal.

**Problem 29.** Write the following vectors in standard unit vector form.

- $\langle 5, 5 \rangle$
- $\langle -4, 3 \rangle$
- $\langle 12, -5 \rangle$
- $\langle 3, 1, -2 \rangle$
- $\langle 4, -2, -4 \rangle$
- $\langle 8, 0, -6 \rangle$

**Theorem 30.** The following properties hold for any two vectors  $\vec{u}$ ,  $\vec{v}$  and scalars  $a$ ,  $b$ .

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $0\vec{u} = \vec{0}$
- $1\vec{u} = \vec{u}$
- $a(b\vec{u}) = (ab)\vec{u}$
- $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
- $(a + b)\vec{u} = a\vec{u} + b\vec{u}$

**Definition 31. Vector subtraction** is defined as the addition of a negative:

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}) = \langle u_1 - v_1, u_2 - v_2 \rangle$$

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}) = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$$

Suggested Homework: Section 12.2 numbers 3, 5, 13, 14, 15, 19, 21, 24, 26



## 12.3 The Dot Product

**Definition 32.** Let  $\theta$  be the angle between two non-zero vectors  $\vec{u}$ ,  $\vec{v}$ . The **dot product**  $\vec{u} \cdot \vec{v}$  is the product of their lengths when projected into the same direction, obtained by this formula:

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$$

**Definition 33.** The dot product with a zero vector is always zero:

$$\vec{v} \cdot \vec{0} = \vec{0} \cdot \vec{v} = 0$$

**Theorem 34.** By the Law of Cosines:

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle = u_1v_1 + u_2v_2$$

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1v_1 + u_2v_2 + u_3v_3$$

**Definition 35.** Two vectors  $\vec{u}$ ,  $\vec{v}$  are **orthogonal** if  $\vec{u} \cdot \vec{v} = 0$ .

**Theorem 36.** Two non-zero vectors are orthogonal if the angle  $\theta$  between them is  $\frac{\pi}{2}$  radians.

**Theorem 37.** The following properties hold for any three vectors  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{w}$  and scalar  $c$ .

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $(c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) = c(\vec{u} \cdot \vec{v})$
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
- $\vec{u} \cdot \vec{u} = |\vec{u}|^2$

**Problem 38.** Solve for  $\cos \theta$  for the following pairs of vectors.

- $\vec{u} = \langle 4, -3 \rangle$   
 $\vec{v} = \langle 5, 12 \rangle$
- $\vec{u} = \langle 1, 4, 2 \rangle$   
 $\vec{v} = \langle 4, 1, -2 \rangle$
- $\vec{u} = \langle 0, 5, -11 \rangle$   
 $\vec{v} = \langle 2, 0, 0 \rangle$

**Definition 39.** The work  $W$  done by a force vector  $\vec{F}$  over a displacement vector  $\vec{D}$  is given by

$$W = \vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos \theta$$

Suggested Homework: Section 12.3 numbers 3, 5, 6, 7, 8, 9, 10, 11, 15, 17, 21, 27, 41, 42, 44

## 12.4 The Cross Product

**Definition 40.** For any two non-parallel vectors  $\vec{u}, \vec{v}$  in  $\mathbb{R}^3$ , the **Right-Hand Rule** gives a specific direction orthogonal to both: position  $\vec{u}$  with your right thumb and  $\vec{v}$  with your right index finger, and let your middle finger extend orthogonal to both to give this direction.

**Definition 41.** Let  $\theta$  be the angle between two non-zero vectors  $\vec{u}, \vec{v}$  in  $\mathbb{R}^3$ , and let  $\vec{n}$  be the direction given by the Right-Hand Rule. The **cross product**  $\vec{u} \times \vec{v}$  is the vector orthogonal to both which follows the Right-Hand Rule and has magnitude equal to the area of the parallelogram formed from both.

$$\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}|\sin\theta)\vec{n}$$

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta$$

**Definition 42.** The cross product with a zero vector is always the zero vector:

$$\vec{v} \times \vec{0} = \vec{0} \times \vec{v} = \vec{0}$$

**Theorem 43.** The following properties hold for any three vectors  $\vec{u}, \vec{v}, \vec{w}$  and scalars  $a, b$ .

- $(a\vec{u}) \times (b\vec{v}) = (ab)(\vec{u} \times \vec{v})$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$
- $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$

**Definition 44.** Two vectors  $\vec{u}, \vec{v}$  are **parallel** if  $\vec{u} \times \vec{v} = \vec{0}$ .

**Theorem 45.** Two non-zero vectors are parallel if the angle  $\theta$  between them is 0 or  $\pi$  radians.

**Definition 46.** The cross products of the standard unit vectors are given as follows:

- $\hat{i} \times \hat{j} = \hat{k}$
- $\hat{j} \times \hat{k} = \hat{i}$
- $\hat{k} \times \hat{i} = \hat{j}$

**Definition 47.** A **determinant** is a short hand for writing certain commonly occurring algebraic expressions:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

**Theorem 48.** By breaking up  $\vec{u}$ ,  $\vec{v}$  into standard unit vectors:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle$$

**Problem 49.** Use the cross product to find a vector normal to both  $\vec{u}$  and  $\vec{v}$ .

- $\vec{u} = \langle 4, -3, 0 \rangle$   
 $\vec{v} = \langle 2, 6, -3 \rangle$
- $\vec{u} = \langle 1, 4, 2 \rangle$   
 $\vec{v} = \langle 4, 1, -2 \rangle$
- $\vec{u} = \langle 0, 5, -11 \rangle$   
 $\vec{v} = \langle 2, 0, 0 \rangle$

**Definition 50.** The torque  $\tau$  done by a force vector  $\vec{F}$  on an arm given by  $\vec{D}$  is given by

$$\tau = |\vec{F} \times \vec{D}| = |\vec{F}||\vec{D}| \sin \theta$$

**Theorem 51.** The volume of a parallelepiped determined by the vectors  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{w}$ , is given by the **triple scalar product**

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Suggested Homework: Section 12.4 numbers 1 – 3, 17, 19, 28, 29, 33, 35