## Absolute Convergence, Ratio, & Root Test 11.6

**Definition 80** (Absolutely Convergent). A series  $\sum a_n$  is called absolutely convergent if the series  $\sum |a_n|$  is convergent.

Theorem 81. Absolutely convergent series are convergent.

Definition 82 (Conditionally Convergent). A series is called conditionally convergent if it is convergent but NOT absolutely convergent.

**Problem 83.** Determine whether whether or not  $\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n^2}$  is absolutely convergent, conditionally convergent, or divergent.

$$\left| \frac{(-1)^{n-1}}{n^2} \right| = \left| \frac{1}{n^2} \right| + \left$$

**Problem 84.** Determine whether whether or not  $\sum_{i=1}^{\infty} \frac{(-1)^{n-1}}{n}$  is absolutely convergent, conditionally convergent, or divergent.

**Problem 85.** Determine whether whether or not  $\sum_{n=0}^{\infty} \frac{\cos(n)}{n^2}$  is absolutely convergent, con-

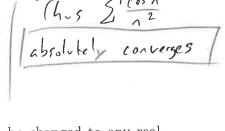
ditionally convergent, or divergent. (Pirectly) Compare Sleas with Sinz! 

Theorem 86. The value of a conditionally convergent series can be changed to any real number by changing the order of its terms. The value of an absolutely convergent series cannot.





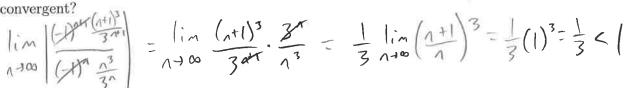




**Theorem 87** (Ratio Test). Let  $\sum_{n=1}^{\infty} a_n$  be a series. Then

- if  $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$  then the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.
- if  $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$  or  $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$  diverges to  $\infty$ , then the series  $\sum_{n=1}^{\infty} a_n$  is divergent.
- if  $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$  then no conclusion can be drawn from this test.

**Problem 88.** Determine whether  $\sum_{n=0}^{\infty} (-1)^n \frac{n^3}{3^n}$  is convergent or divergent. Is it absolutely



**Problem 89.** Determine whether  $\sum_{n=1}^{\infty} \frac{n!}{n^n}$  is convergent or divergent. Is it absolutely con-

vergent?

$$\begin{vmatrix} i m \\ n \neq 00 \end{vmatrix} = \begin{vmatrix} (n+1)! \\ (n+1)^{n+1} \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)^{n+1} \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)^{n+1} \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)^{n+1} \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)^{n+1} \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)^{n+1} \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)^{n+1} \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)^{n+1} \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)^{n+1} \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)^{n+1} \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)! \\ (n+1)! \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)! \\ (n+1)! \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)! \\ (n+1)! \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)! \\ (n+1)! \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)! \\ (n+1)! \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+1)! \\ (n+1)! \\ (n+1)! \\ (n+1)! \end{vmatrix} = \begin{vmatrix} i m \\ (n+1)! \\ (n+$$

Thus Sin!

**Theorem 90** (Root Test). Let  $\sum_{n=1}^{\infty} a_n$  be a series. Then

- If  $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$  then  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.
- if  $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$  or  $\lim_{n\to\infty} \sqrt[n]{|a_n|}$  diverges to  $\infty$ , then the series  $\sum_{n=1}^{\infty} a_n$  is divergent.
- $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ , then no conclusion can be drawn from this test.

Theorem 91.  $\lim_{n\to\infty} \sqrt[n]{n} = 1$ 

**Problem 92.** Determine whether  $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$  is convergent or divergent. Is it absolutely convergent?

lim (2+3) = lim 2+3 = 3</

Thus  $2\left(\frac{2n+3}{3n+2}\right)^n$  is absolutely convergent.

**Problem 93.** Determine whether  $\sum_{n=1}^{\infty} \frac{n+1}{n^{2n}}$  is convergent or divergent. Is it absolutely  $\alpha$ 

$$\frac{1}{1} \frac{1}{1} \frac{1}{1} \frac{1}{1} = \frac{1}{1} \frac{1}{1} \frac{1}{1} \frac{1}{1} \frac{1}{1} \frac{1}{1} = \frac{1}{1} \frac{$$

Suggested Problems: Section 11.6 numbers 3, 5, 7, 9, 10, 11 - 13, 16, 17, 19, 21, 23, 27, 28

## 11.7 Strategies for Testing Series

The only thing I really have to say here is that practice makes better. If you do enough problems, eventually you will get an intuition for what will work in what situation. Nevertheless, here is a list of tests that could come in handy.

/AKA	\
Divergence	)
Test	-

Test	When to Use	Conclusion
Geometric Series	$\sum_{k=1}^{\infty} ar^k$	Converges to $\frac{a}{1-r}$ if $ r  < 1$ ;
		diverges if $ r  \geq 1$ .
→ k <sup>th</sup> Term Test	All Series	If $\lim_{k\to\infty} a_k \neq 0$ , the series diverges.
Integral Test	$\sum_{k=1}^{\infty} a_k$ where $f(k) = a_k$ and	$\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x) dx$
	$f$ is continuous, decreasing, and $f(x) \ge 0$	both converge or both diverge.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^{\overline{\nu}}}$	Converges for $p > 1$ , diverges for $p \le 1$ .
Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ , where $0 \le a_k \le b_k$	If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
		If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ , where	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$
	$a_k, b_k > 0$ and $\lim_{k \to \infty} \frac{a_k}{b_k} = L > 0$	both converge or both diverge.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ where $a_k > 0$ for all $k$	If $\lim_{k\to\infty} a_k = 0$ and $a_{k+1} \le a_k$ for all $k$ ,
		then the series converges.
Absolute Convergence	Series with some positive and some	If $\sum_{k=1}^{\infty}  a_k $ converges, then
	negative terms (including alternating series)	$\sum_{k=1}^{\infty} a_k$ converges (absolutely).
Ratio Test		For $\lim_{k\to\infty} \left  \frac{a_{k+1}}{a_k} \right  = L$ ,
	Any Series (especially those involving exponentials and/or factorials)	if $L < 1$ , $\sum_{k=1}^{\infty} a_k$ converges absolutely,
		if $L > 1$ , $\sum_{k=1}^{\infty} a_k$ diverges,
		if $L=1$ , no conclusion.
Root Test	Any Series (especially those involving exponentials)	For $\lim_{k\to\infty} \sqrt[k]{ a_k } = L$ ,
		if $L < 1$ , $\sum_{k=1}^{\infty} a_k$ converges absolutely,
		if $L > 1$ , $\sum_{k=1}^{\infty} a_k$ diverges,
		if $L=1$ , no conclusion.

**Problem 94.** Determine whether the series  $\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$  converges or diverges.



**Problem 95.** Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$$
 converges or diverges.



$$\frac{\text{lim t (omp Test)}}{\sqrt{n^{3+1}}} = \frac{1}{1} \frac{n^{3/2} \sqrt{n^{3+1}}}{\sqrt{n^{3+1}}} = \frac{1}{1} \frac{n^{3/2} \sqrt{n^{3+1$$

**Problem 96.** Determine whether the series  $\sum_{n=1}^{\infty} ne^{-n^2}$  converges or diverges.



$$\lim_{n\to\infty} \left| \frac{n}{e^n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0 < 1$$

**Problem 97.** Determine whether the series  $\sum_{n=0}^{\infty} (-1)^n \frac{n^3}{n^4+1}$  converges or diverges.



$$\frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{1+1}} > \frac{1}{\sqrt{1+1}} = \frac{1}$$

Problem 98. Determine whether the series  $\sum_{k=1}^{\infty} \frac{2^k}{k!}$  converges or diverges.

**Problem 99.** Determine whether the series 
$$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$$
 converges or diverges.

$$\frac{1}{7+3^{1}} \leq \frac{1}{3^{1}} = \left(\frac{1}{3}\right)^{2}$$