Chapter 6

Applications of Integrals

6.1 Area Between Curves

6.1.1 Area with Respect to the x-axis

Recall 1. If $f(x)$ is a continuous function and $a \leq b$, then the definite integral $\displaystyle\int_a^b f(x)\,dx$ represents the "net area" between the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$. ("Net area" means the area above the x-axis minus the area below the x-axis.)

Theorem 2. If f, g are continuous functions of x such that $f(x) \leq g(x)$ for all $a \leq x \leq b$, then the area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ is

$$A = \int_a^b g(x) - f(x)\,dx$$

Problem 3. Find the area of the region bounded above by $y = e^x$, below by $y = x$, and on the sides by $x = 0$ and $x = 1$.

Problem 4. Find the area bounded by $y = x^2$ and $y = 2x - x^2$.

Problem 5. Find the area bounded by $y = \sin(x)$ and $y = \cos(x)$ from $x = 0$ to $x = \frac{\pi}{2}$.

6.1.2 Area with Respect to the y-axis

Theorem 6. If f, g are continuous functions of y such that $f(y) \leq g(y)$ for all $c \leq y \leq d$, then the area A of the region bounded by the curves $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$ is

$$A = \int_c^d g(y) - f(y)\, dy$$

Problem 7. Find the area enclosed by $y = x - 1$ and $y^2 = 2x + 6$.

Problem 8. Find the area enclosed by $x = 2y - y^2$ and $x = y^2 - 4y$.

Suggested Homework: Section 6.1 numbers 1, 4, 12, 24, 27, 44, 50

6.2 Volumes by Cross-Sections

Definition 9. The three-dimensional solid obtained by moving a planar shape along a line perpendicular to the plane is called a **cylinder**.

Definition 10. The volume V of a cylinder with base area B and height h is defined to be

$$V = Bh$$

Definition 11. The volume of a solid positioned between $x = a$ and $x = b$ with cross-sectional areas given by $A(x)$ for each x-value between a and b is defined to be

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_{i,n})\,\Delta x_n = \int_a^b A(x)\,dx$$

Problem 12. Show that the volume V of a pyramid with height h and a square base with side length s is $V = \frac{1}{3}s^2h$.

Definition 13. A **solid of revolution** is the result of rotating a shape around a line (called the **axis of revolution**).

Theorem 14 (Disc Method)**.** Suppose that a solid of revolution is formed from a shape positioned flush against a horizontal or vertical axis of revolution.

If the axis of revolution is horizontal, and $R(x)$ gives the distance from the axis to the outside of the shape being rotated for each value of x, then the volume of the solid of revolution from $x = a$ to $x = b$ is

$$V = \int_a^b \pi[R(x)]^2 \, dx$$

If the axis of revolution is vertical, and $R(y)$ gives the distance from the axis to the outside of the shape being rotated for each value of y, then the volume of the solid of revolution from $y = c$ to $y = d$ is

$$V = \int_c^d \pi[R(y)]^2 \, dy$$

Problem 15. Show that the volume of a sphere of radius r is $V = \dfrac{4}{3}\pi r^3$.

Problem 16. Find the volume of the solid obtained by rotating the shape bounded by $y = \sqrt{x}$, $y = 0$, $x = 0$, and $x = 1$ about the x-axis.

Problem 17. Find the volume of the solid obtained by rotating the shape bounded by $y = x^3$, $x = 0$, $y = 0$, and $y = 8$ about the y-axis.

Theorem 18 (Washer Method)**.** Suppose that a solid of revolution is formed from a shape not positioned flush against a horizontal or vertical axis of revolution.

If the axis of revolution is horizontal, $R(x)$ gives the distance from the axis to the outside of the shape being rotated for each value of x, and $r(x)$ gives the distance from the axis to the inside of the shape being rotated for each value of x, then the volume of the solid of revolution from $x = a$ to $x = b$ is

$$V = \int_a^b \pi[R(x)]^2 - \pi[r(x)]^2 \, dx$$

(The similar formula works for y values and a vertial axis of revolution.)

Problem 19. Find the volume of the solid obtained by rotating the triangle with vertices at $(1, 1)$, $(3, 1)$, and $(1, 2)$ around the x-axis.

Problem 20. Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ about the line $x = -1$.

Suggested Homework: Section 6.2 numbers 4, 5, 12, 15, 19, 25, 42, 54, 56, 59

6.3 Volume by Cylindrical Shell

Theorem 21 (Shell Method)**.** Suppose that a solid of revolution is formed from a shape positioned flush against a horizontal or vertical axis of revolution.

If the axis of revolution is vertical with equation $x = a$, $h(x)$ gives the height of the shape between the axis $x = a$ to the shape's edge $x = b$, and $C(x) = 2\pi(x - a)$ is the circumference of a cylindrical shell formed from the shape at each x-value, then the volume of the solid of revolution is

$$V = \int_a^b C(x)h(x)\,dx = \int_a^b 2\pi(x-a)h(x)\,dx$$

If the axis of revolution is horizontal with equation $y = c$, and $w(y)$ gives the width of the shape between the axis $y = c$ to the shape's edge $y = d$, and $C(y) = 2\pi(y - c)$ is the circumference of a cylindrical shell formed from the shape at each y-value, then the volume of the solid of revolution is

$$V = \int_c^d C(y)w(y)\,dy = \int_c^d 2\pi(y-c)w(y)\,dy$$

Problem 22. Find the volume of the solid obtained by rotating the shape with bounds $y = 2x^2 - x^3$ and $y = 0$ about the y-axis.

Problem 23. Find the volume of the solid obtained by rotating the shape with bounds $x = y$ and $x = y^2$ about the x-axis.

Problem 24. Find the volume of the solid obtained by rotating the shape with bounds $y = \sqrt{x}$, $y = 0$, and $x = 4$ about the y-axis.

Problem 25. Find the volume of the solid obtained by rotating the shape with bounds $y = x - x^2$ and $y = 0$ about the line $x = 2$.

Suggested Homework: Section 6.3 numbers 2, 7, 9, 17, 19, 37

6.4 Work

Definition 26. The **work** W done by a constant **force** F exerted on an object over a distance d is given by the equation

$$W = Fd$$

Theorem 27. The work W done by a variable force $F(x)$ exerted on an object over the distance from $x = a$ to $x = b$ is given by the equation

$$W = \int_a^b F(x)\,dx$$

Problem 28. A vehicle is moved from mile marker $x = 1$ to mile marker $x = 3$ with a force of $F(x) = x^2 + 2x$ tons at a given position x on the interstate. How much work is done in moving the vehicle in this way?

Problem 29. **Hooke's Law** tells us that a spring with spring constant k requires $F(x) = kx$ units of force to stretch the spring x units beyond its natural length. If a force of 40 newtons is required to stretch a spring from its natural length of 10 meters to 15 meters, what is the value of the spring's constant k, and how much work is required to stretch the spring further from 15 meters to 18 meters?

Problem 30. A 200 pound cable hangs 100 feet from the top of a building. How much work is required to retract the cable to the top of the building?

Problem 31. A tank in the shape of an upside-down cone has a height of 16 meters and radius of 4 meters. Assuming that the density of the water is 1000 kg/m^3, compute the amount of work required to completely fill the tank by pumping water into the tip of the tank.

(No Suggested Homework)