11.6 Absolute Convergence, Ratio, & Root Test

Definition 80 (Absolutely Convergent). A series $\sum a_n$ is called **absolutely convergent** if the series $\sum |a_n|$ is convergent.

Theorem 81. Absolutely convergent series are convergent.

Definition 82 (Conditionally Convergent). A series is called **conditionally convergent** if it is convergent but NOT absolutely convergent.

Problem 83. Determine whether whether or not $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent, conditionally convergent, or divergent.

$\sum_1 \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_1 \frac{1}{n^2} \leftarrow$ converges (p-Series)

Thus $\sum_1 \frac{(-1)^{n-1}}{n^2}$ $\boxed{\text{absolutely converges}}$.

Problem 84. Determine whether whether or not $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is absolutely convergent, conditionally convergent, or divergent.

$\sum_1 \left|\frac{(-1)^{n-1}}{n}\right| = \sum_1 \frac{1}{n} \leftarrow$ diverges (Harmonic)

$\sum_1 \frac{(-1)^{n-1}}{n}$ is an alternating series.

- $\frac{1}{n} > \frac{1}{n+1}$
- $\lim_{n\to\infty} \frac{1}{n} = 0$

Thus $\sum_1 \frac{(-1)^{n-1}}{n}$ converges by AST, and $\boxed{\text{conditionally converges}}$

Problem 85. Determine whether whether or not $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ is absolutely convergent, conditionally convergent, or divergent.

(Directly) Compare $\sum_1 \frac{|\cos n|}{n^2}$ with $\sum_1 \frac{1}{n^2}$!

$\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$

Since $\sum_1 \frac{1}{n^2}$ converges, the smaller $\sum_1 \frac{|\cos n|}{n^2}$ converges.

Thus $\sum_1 \frac{\cos n}{n^2}$ $\boxed{\text{absolutely converges}}$

Theorem 86. The value of a conditionally convergent series can be changed to any real number by changing the order of its terms. The value of an absolutely convergent series cannot.

Theorem 87 (Ratio Test). Let $\sum_{n=1}^{\infty} a_n$ be a series. Then

- if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ diverges to ∞, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ then no conclusion can be drawn from this test.

Problem 88. Determine whether $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ is convergent or divergent. Is it absolutely convergent?

$$\lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(n+1)^3}{3^{n+1}}}{\frac{\cancel{(-1)^n}\, n^3}{3^n}}\right| = \lim_{n\to\infty}\frac{(n+1)^3}{\cancel{3^{n+1}}}\cdot\frac{\cancel{3^n}}{n^3} = \frac{1}{3}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}(1)^3 = \frac{1}{3} < 1$$

Thus $\sum (-1)^n\frac{n^3}{3^n}$ $\boxed{\text{absolutely converges}}$

Problem 89. Determine whether $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ is convergent or divergent. Is it absolutely convergent?

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right| = \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} = \lim_{n\to\infty}\frac{\cancel{n!}\cancel{(n+1)}}{(n+1)^n\cancel{(n+1)}}\cdot\frac{n^n}{\cancel{n!}}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}\left(\approx\frac{1}{2.7}\right) < 1$$

(Cal I Fact! $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$)

Thus $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ $\boxed{\text{absolutely converges}}$

Theorem 90 (Root Test). Let $\sum_{n=1}^{\infty} a_n$ be a series. Then

- If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ diverges to ∞, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then no conclusion can be drawn from this test.

Theorem 91. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Problem 92. Determine whether $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$ is convergent or divergent. Is it absolutely convergent? 10

$$\lim_{n\to\infty}\left|\left(\frac{2n+3}{3n+2}\right)^n\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{2n+3}{3n+2} = \frac{2}{3} < 1$$

Thus $\sum \left(\frac{2n+3}{3n+2}\right)^n$ is $\boxed{\text{absolutely convergent}}$.

Problem 93. Determine whether $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n^{2n}}$ is convergent or divergent. Is it absolutely convergent? 10

$$\lim_{n\to\infty}\left|\frac{n+1}{n^{2n}}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{\sqrt[n]{n+1}}{n^2} = \lim_{n\to\infty}\frac{1}{n^2} = 0 < 1$$

Thus $\sum_{n=1}^{\infty}\frac{n+1}{n^{2n}}$ is $\boxed{\text{absolutely convergent}}$.

Suggested Problems: Section 11.6 numbers 3, 5, 7, 9, 10, 11 – 13, 16, 17, 19, 21, 23, 27, 28

11.7 Strategies for Testing Series

The only thing I really have to say here is that practice makes better. If you do enough problems, eventually you will get an intuition for what will work in what situation. Nevertheless, here is a list of tests that could come in handy.

(AKA Divergence Test)

Test	When to Use	Conclusion
Geometric Series	$\sum_{k=1}^{\infty} ar^k$	Converges to $\frac{a}{1-r}$ if $\lvert r\rvert < 1$; diverges if $\lvert r\rvert \geq 1$.
k^{th} Term Test	All Series	If $\lim_{k\to\infty} a_k \neq 0$, the series diverges.
Integral Test	$\sum_{k=1}^{\infty} a_k$ where $f(k) = a_k$ and f is continuous, decreasing, and $f(x) \geq 0$	$\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$ **both** converge or **both** diverge.
***p*-series**	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	Converges for $p > 1$, diverges for $p \leq 1$.
Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $0 \leq a_k \leq b_k$	If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $a_k, b_k > 0$ and $\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ **both** converge or **both** diverge.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ where $a_k > 0$ for all k	If $\lim_{k\to\infty} a_k = 0$ and $a_{k+1} \leq a_k$ for all k, then the series converges.
Absolute Convergence	Series with some positive and some negative terms (including alternating series)	If $\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges, then $\sum_{k=1}^{\infty} a_k$ converges (absolutely).
Ratio Test	Any Series (especially those involving exponentials and/or factorials)	For $\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely, if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.
Root Test	Any Series (especially those involving exponentials)	For $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely, if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.

Problem 94. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$ converges or diverges.

(Divergence Test)

$$\lim_{n\to\infty} \frac{n-1}{2n+1} = \frac{1}{2} \neq 0$$

Thus $\sum \frac{n-1}{2n+1}$ $\boxed{\text{diverges}}$.

Problem 95. Determine whether the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ converges or diverges.

(Limit Comp Test)

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n^3+1}}{3n^3+4n^2+2} = \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n^3}\sqrt{1+\frac{1}{n^3}}}{3n^3+4n^2+2}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^3}\sqrt{1+\frac{1}{n^3}}}{\cancel{n^3}(3+\frac{4}{n}+\frac{2}{n^3})} = \frac{\sqrt{1}}{3} = \frac{1}{3} > 0$$

Thus $\sum \frac{1}{n^3}$ and $\sum \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ both $\boxed{\text{Converge}}$.

Problem 96. Determine whether the series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges.

(Root Test)

$$\lim_{n\to\infty} \left|\frac{n}{e^{n^2}}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{1/n}}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0 < 1$$

Thus $\sum \frac{n}{e^{n^2}} = \sum ne^{-n^2}$ (absolutely) $\boxed{\text{Converges}}$.

Problem 97. Determine whether the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$ converges or diverges.

(Alt. Series Test)

- $\frac{n^3}{n^4+1} = \frac{1}{n+\frac{1}{n^3}} > \frac{1}{n+1+\frac{1}{(n+1)^3}} = \frac{1}{n+\frac{(n+1)^3+1}{(n+1)^3}}$ is non-increasing
- $\lim_{n\to\infty} \frac{n^3}{n^4+1} = 0$

Thus $\sum (-1)^n \frac{n^3}{n^4+1}$ $\boxed{\text{Converges}}$.

Problem 98. Determine whether the series $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ converges or diverges.

(Ratio Test)

$$\lim_{k\to\infty}\left|\frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}}\right| = \lim_{k\to\infty}\frac{2^{k+1}}{(k+1)!}\frac{k!}{2^k} = \lim_{k\to\infty}\frac{\cancel{2^k}(2)}{\cancel{k!}(k+1)}\frac{\cancel{k!}}{\cancel{2^k}} = \lim_{k\to\infty}\frac{2}{k+1} = 0 < 1$$

Thus $\sum \frac{2^k}{k!}$ $\boxed{\text{converges}}$.

Problem 99. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ converges or diverges.

(Direct Comp. Test)

$$\frac{1}{2+3^n} \leq \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$$

Since $\sum \left(\frac{1}{3}\right)^n$ converges, the smaller $\sum \frac{1}{2+3^n}$ also $\boxed{\text{converges}}$.

Suggested Problems: Section 11.7 numbers 1, 2, 5 – 9, 11, 13, 14, 15 – 18, 23, 25 – 34, 37