11.3 The Integral Test

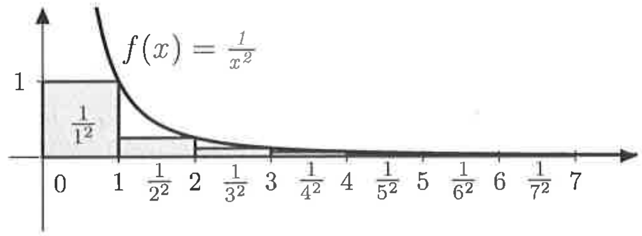

Figure 11.3: Integral Test

Theorem 65. Suppose that f is a continuous, positive, non-increasing function on $[1, \infty)$ such that $a_n = f(n)$. If $\lim_{t\to\infty} \int_1^t f(x)\,dx$ exists, then $\sum_{n=1}^{\infty} a_n$ converges. Otherwise, $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 66 (p-Series)**.** The p**-Series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

Problem 67. Determine whether or not the following is convergent or divergent:

- $\sum_{n=1}^{\infty} \frac{1}{n^3}$ $\boxed{\text{conv}}$
- $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ $\boxed{\text{div}}$
- $\sum_{n=1}^{\infty} n^{-4/3} = \sum_{1} \frac{1}{n^{4/3}}$ $\boxed{\text{conv}}$

Theorem 68. $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ diverges.

No Suggested Problems.

11.4 The Comparison Test

Theorem 69 (Direct Comparison Test). Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

- If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ is also convergent.
- If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

Theorem 70. If $0 < q < 1$ and $Q > 1$, then the following inequalities hold for sufficiently large n:

$$\frac{1}{n^n} < \frac{1}{n!} < \frac{1}{Q^n} < \frac{1}{n^Q} < \frac{1}{n} < \frac{1}{n^q} < 1 < n^q < n < n^Q < Q^n < n! < n^n$$

Problem 71. Use the Direct Comparison Test to determine whether $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges.

$$\frac{5}{2n^2+4n+3} \leq \frac{5}{2n^2} \left(= \frac{5/2}{n^2} \right)$$

(smaller)

Since the bigger $\sum_1^\infty \frac{5}{2n^2}$ converges (by p-Series Test), the smaller $\sum_1^\infty \frac{5}{2n^2+4n+3}$ also $\boxed{\text{converges}}$.

Problem 72. Use the Direct Comparison Test to determine whether $\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-3}$ converges or diverges.

(bigger)

$$\frac{1}{n} = \frac{n^2}{n^3} \leq \frac{n^2+1}{n^3} \leq \frac{n^2+1}{n^3-3}$$

(Smaller)

Since the smaller $\sum_1^\infty \frac{1}{n}$ diverges, the bigger $\sum_1^\infty \frac{n^2+1}{n^3-3}$ also $\boxed{\text{diverges}}$.

Theorem 73 (Limit Comparison Test). Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms and $c \in \mathbb{R}$. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$

then either both series converge or both series diverge.

Problem 74. Use the Limit Comparison Test to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges or diverges. 9 10

Compare with $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ (converges by geometric series)

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\left(\frac{1}{2}\right)^n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = 1 > 0$$

Thus $\sum \frac{1}{2^n-1}$ also $\boxed{\text{converges}}$.

Problem 75. Use the Limit Comparison Test to determine whether $\displaystyle\sum_{n=0}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ converges or diverges. 9 10

Compare with $\sum \frac{n^2}{\sqrt{n^5}} = \sum \frac{n^2}{n^{5/2}} = \sum \frac{1}{n^{1/2}}$ (diverges by p-Series)

$$\lim_{n\to\infty} \frac{\frac{2n^2+3n}{\sqrt{5+n^5}}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{\sqrt{5+n^5}} = \lim_{n\to\infty} \frac{\cancel{n^{5/2}}\left(2+\cancel{\frac{3}{n}}\right)}{\cancel{\sqrt{n^5}}\sqrt{\cancel{\frac{5}{n^5}}+5}} = \frac{2}{\sqrt{5}} > 0$$

Thus $\sum \frac{2n^2+3n}{\sqrt{5+n^5}}$ also $\boxed{\text{diverges}}$.

Suggested Problems: Section 11.4 numbers 3, 4, 5, 7, 14, 15, 17, 21, 23, 29, 30

11.5 Alternating Series

Theorem 76 (Alternating Series Test). If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ where each term in the sequence (b_n) is positive satisfies

- (b_n) is non-increasing
- $\lim_{n\to\infty} b_n = 0$

then the series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges.

Problem 77. Determine whether the **alternating harmonic series** $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges or diverges.

$= \sum (-1)^{n-1}\frac{1}{n}$ $\cancel{10}$

- $\frac{1}{n} \geq \frac{1}{n+1}$ (non-increasing)
- $\lim_{n\to\infty}\frac{1}{n} = 0$.

Thus $\sum \frac{(-1)^{n-1}}{n}$ $\boxed{\text{converges}}$.

Problem 78. Determine whether $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ converges or diverges.

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- $\frac{n^2}{n^3+1}$ is decreasing since $f(x) = \frac{x^2}{x^3+1} \Rightarrow f'(x) = \frac{2x(x^3+1)-x^2(3x^2)}{(x^3+1)^2} = \frac{2x-x^4}{(x^3+1)^2}$ $= \frac{x(2-x^3)}{(x^3+1)^2}$ is negative for $x > \sqrt[3]{2}$.
- $\lim_{n\to\infty}\frac{n^2}{n^3+1} = \lim_{n\to\infty}\frac{2n}{3n^2} = 0$

Thus $\sum(-1)^{n+1}\frac{n^2}{n^3+1}$ converges.

Problem 79. Determine whether $\sum_{n=1}^{\infty}\frac{(-1)^n 3n}{4n-1}$ converges or diverges.

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(AST fails if you try it.)

$\lim_{n\to\infty}\frac{(-1)^n 3n}{4n-1}$ diverges since even n limits to $3/4$ and odd n limits to $-3/4$.

Thus $\sum\frac{(-1)^n 3n}{4n-1}$ $\boxed{\text{diverges}}$ by the Divergence Test (Thm 59).

Suggested Problems: Section 11.5 numbers 2 – 6, 8, 9, 11, 13, 17, 19