Instructor: James Hammer
MATH 1620-(120)(146)

Final Exam
Date: April 29, 2013

Your Name:

Answer the following questions to the best of of your ability. Show all work. Answers without work will not receive credit. No calculators, cell phones, or other electronic devices are permitted for this examination. GOOD LUCK!

This test is out of 200 points.

MULTIPLE CHOICE: Circle the option that BEST fits the statement. (4pts each)

1. Suppose that **u**, **v**, and **w** are vectors. Each of the following is a vector, scalar, or is an undefined operation. Which are vectors? There may be more than one.

   A. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ B. $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ C. $(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{u}\cdot\mathbf{w})$ D. $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ E. $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$

   A: scalar, vector · (vector); B: vector, (scalar) vector; C: undefined, (vector) · (scalar); D: Vector, (vector) × vector; E: undefined, (scalar) · vector

2. Which of the following are unit vectors? There may be more than one.

   A. $\langle 1,1,1\rangle$ B. $\left\langle \frac{1}{\sqrt{13}},\frac{2}{\sqrt{13}},\frac{3}{\sqrt{13}}\right\rangle$ C. $\left\langle \frac{1}{3},\frac{1}{3},\frac{-1}{3}\right\rangle$ D. $\left\langle \frac{-2}{\sqrt{13}},\frac{-1}{\sqrt{13}},\frac{-3}{\sqrt{13}}\right\rangle$

   E. None of these

   A: mag $=\sqrt{3}$; B: mag $=\sqrt{\frac{14}{13}}$; C: mag $=\sqrt{\frac{1}{3}}$; D: mag $=\sqrt{\frac{14}{13}}$

3. For which values of b are the vectors $\langle 1,3,b\rangle$ and $\langle b,-b,2b\rangle$ perpendicular?

   A. $b=2$ B. $b=1,2$ C. $b=0,2$ D. $b=0,1$ E. $b=1$

   $\langle 1,3,b\rangle\cdot\langle b,-b,2b\rangle=0$
   $b-3b+2b^2=0$
   $2b^2-2b=0$
   $2b(b-1)=0$

4. After a u-substitution, the definite integral $\int_1^2 (3x^2+1)\sqrt{x^3+x}\,dx$ equals which of the following?

   x-values; converted to u-values

   A. $\int_{u=1}^{u=2}\sqrt{u}\,du$ B. $\int_{u=2}^{u=10}\sqrt{u}\,du$ C. $\frac{2}{3}\int_{u=1}^{u=2}u^{3/2}\,du$ D. $\frac{2}{3}\int_{u=2}^{u=10}u^{3/2}\,du$

   E. Not Here

5. After one application of integration by parts, the indefinite integral $\int x^3e^x\,dx$ could equal which of the following?

   $u=x^3$ $v=e^x$
   $du=3x^2dx$ $dv=e^x dx$

   A. $x^3e^x-\frac{1}{3}\int x^2e^x\,dx$ B. $x^3e^x-3\int x^2e^x\,dx$ C. $x^3e^x-\int x^2e^x\,dx$ D. $\frac{x^4e^x}{4}-\frac{1}{3}\int x^2e^x\,dx$

6. Which of the following are improper integrals?

   I.) $\int_0^1\frac{1}{\sqrt{x}}\,dx$ II.) $\int_{-1}^1\frac{1}{\sqrt{x}}\,dx$ III.) $\int_0^3\frac{1}{x-2}\,dx$ IV.) $\int_{-1}^1\frac{1}{1+x^2}\,dx$

   I: $\frac{1}{0}$; II: $\frac{1}{0}$; III: $\frac{1}{2-2}$; IV: Always defined

   A. I, III B. II, III C. I, II D. I, II, III E. I, II, III, IIII

7. If the region bounded by the curves $y=2-x$, $y=0$, $x=0$ is rotated about the y-axis and sliced horizontally, the slices will have which of the following shapes?

   A. Disks B. Washers C. Cylindrical Shells D. Rectangular Solids E. None of These

8. The series, $3/2+3/4+3/8+3/16+\cdots$ is of which type? $=\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^n=\sum_{n=0}^{\infty}ar^n$

   A. Geometric B. P-series C. Harmonic D. Alternating E. None of these

9. What is the radius of convergence for the series $\sum_{n=0}^{\infty}\frac{(x-1)^n}{3^n}$?

   A. 0 B. 1 C. 2 D. 3 E. 4 F. ∞

   Root Test

   $\lim_{n\to\infty}\left|\frac{(x-1)^n}{3^n}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\frac{|x-1|}{3}<1$

   $|x-1|<3$

   $-2<x<4$

10. Suppose that $a_n = \dfrac{2n}{4n+1}$. Then

A. the sequence $\{a_n\}$ converges and the series $\sum_{n=1}^{\infty} a_n$ converges.

$\lim_{n\to\infty} \frac{2n}{4n+1} = \frac{2}{4} = \frac{1}{2} \neq 0$

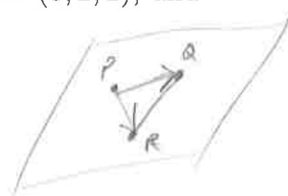
B. the sequence $\{a_n\}$ converges and the series $\sum_{n=1}^{\infty} a_n$ diverges. (by "k-th term" Divergence Test)

C. the sequence $\{a_n\}$ diverges and the series $\sum_{n=1}^{\infty} a_n$ converges.

D. the sequence $\{a_n\}$ diverges and the series $\sum_{n=1}^{\infty} a_n$ diverges.

SELF ANSWER: Use any properties and formulas to solve the following:

11. Consider the triangle whose vertices are at the points $P = (1,1,0)$, $Q = (3,2,2)$, and $R = (4,2,1)$.



(a) (4 pts.) Find the vector $\overrightarrow{PQ}$.

$\overrightarrow{PQ} = \langle 3-1, 2-1, 2-0 \rangle$
$= \boxed{\langle 2, 1, 2 \rangle}$

(b) (4 pts.) Find the distance between points Q and R.

$d(Q,R) = \sqrt{(4-3)^2+(2-2)^2+(1-2)^2}$
$= \sqrt{1+0+1}$
$= \boxed{\sqrt{2}}$

(c) (4 pts.) Find a *unit vector* in the direction from P to R.

$\overrightarrow{PR} = \langle 4-1, 2-1, 1-0 \rangle$
$= \langle 3, 1, 1 \rangle$

$|\overrightarrow{PR}| = \sqrt{9+1+1} = \sqrt{11}$

Unit vector $= \dfrac{\overrightarrow{PR}}{|\overrightarrow{PR}|} = \boxed{\left\langle \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}} \right\rangle}$

12. For the following integrals state the best method of integration and the corresponding appropriate information for the method you selected. You should only select numerical if none of the other methods is applicable. **Do not evaluate.** (4pts each)

Method	**Corresponding Information for Method**
Substitution	u
Integration by parts	u and dv
Trig substitution	appropriate trig substitution
Partial fractions	Form of decomposition (don't solve for constants)
Trigonometric Integrals	Appropriate Identity

Function	**Method**	**Corresponding information**
$\int x^6 \ln x \, dx$	Int by Parts	Let $u = \ln x \quad dv = x^6 dx$
$\int \frac{x^2}{1+x^3} \, dx$	Substitution	Let $u = 1+x^3$
$\int \ln x \, dx$	Int by Parts	Let $u = \ln x \quad dv = dx$
$\int \frac{1}{\sqrt{9-4x^2}} \, dx$	Trig Substitution	Let $9-4x^2 = 9-9\sin^2\theta = 9\cos^2\theta$ $(4x^2 = 9\sin^2\theta)$
$\int \frac{1}{\sqrt{3t+1}} \, dt$	Substitution	Let $u = 3t+1$
$\int \sin^5(x)\cos^2(x) \, dx$	Trig Integrals	$\sin^5(x)\cos^2(x) = (\sin^2 x)^2 \cos^2 x \sin x$ $= (1-\cos^2 x)^2 \cos^2 x \sin x$ Let $u = \cos\theta$ $du = -\sin\theta \, d\theta$
$\int \frac{x+1}{x^3+x} \, dx$	Partial Fractions	$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

13. Evaluate the following integrals. Show all work.

(a) (8 pts.) $\int 2x\sqrt{4x^2+7}\,dx$ $= \int \frac{1}{4}\sqrt{u}\,du$

Let $u = 4x^2+7$ $= \int \frac{1}{4} u^{1/2} du$

$du = 8x\,dx$ $= \frac{1}{4}\left(\frac{2}{3}u^{3/2}\right) + C$

$\frac{1}{4}du = 2x\,dx$ $= \boxed{\frac{1}{6}(4x^2+7)^{3/2} + C}$

(b) (8 pts.) $\int \frac{2x}{(x-2)(x+3)}\,dx$

$$\frac{2x}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$$

$$2x = A(x+3) + B(x-2)$$

$$2x + (0) = (A+B)x + (3A-2B)$$

$2 = A+B \Rightarrow A = 2-B$ $\rightarrow$ $A = \frac{4}{5}$

$3A - 2B = 0 \Rightarrow 3(2-B) - 2B = 0$

$6 - 3B - 2B = 0$

$6 = 5B$

$B = 6/5$

$= \int \frac{4/5}{x-2} + \frac{6/5}{x+3}\,dx$

$= \boxed{4/5 \ln|x-2| + 6/5 \ln|x+3| + C}$

(c) (8 pts.) $\int e^2\,dt$ $\left(= \int \text{constant } dt\right)$

$= \boxed{e^2 t + C}$

(d) (8 pts.) $\int r^4 \ln(r)\,dr$ $= \frac{1}{5}r^5 \ln r - \int \frac{1}{5} r^4 dr$

Let $u = \ln r$ $v = \frac{1}{5}r^5$ $= \boxed{\frac{1}{5}r^5 \ln r - \frac{1}{25}r^5 + C}$

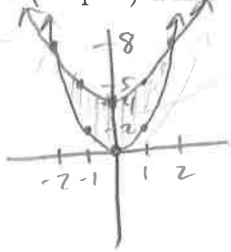
$du = \frac{1}{r}dr$ $dv = r^4 dr$

(e) (8 pts.) $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ $= \lim_{b\to\infty} \int_1^b x^{-1/2} dx = \lim_{b\to\infty} \left[2x^{1/2}\right]_1^b$

$= \lim_{b\to\infty} \left(2b^{1/2} - 2\right) \text{"} = \infty \text{"}$ $\boxed{\text{diverges}}$ OR $\boxed{\text{DNE}}$

OR $\boxed{\infty}$

14. (12 pts.) Find the area of the region bounded by the curves $y = 2x^2$ and $y = x^2 + 4$.



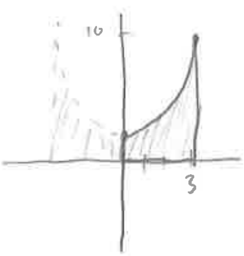
x	y
-2	8
-1	2
0	0
1	2
2	8

x	y
-2	8
-1	5
0	4
1	5
2	8

$$\text{Area} = \int_{-2}^{2} (\text{Above}) - (\text{Below})\, dx$$
$$= \int_{-2}^{2} (x^2+4) - (2x^2)\, dx$$
$$= \int_{-2}^{2} 4 - x^2\, dx$$
$$= \left[4x - \tfrac{1}{3}x^3\right]_{-2}^{2}$$
$$= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right)$$
$$= 16 - \tfrac{16}{3}$$
$$= \boxed{\tfrac{32}{3}}$$

15. (16 pts.) Find the volume formed by revolving the region bounded by the curves $y = x^2 + 1$, $y = 0$, and $x = 3$ about the y-axis.

and $x=0$



Disk/Washer

$$V = \int_0^1 \pi(3)^2\, dy + \int_1^{10} \pi(3)^2 - \pi\left(\sqrt{y-1}\right)^2 dy$$
$$= \int_0^1 9\pi\, dy + \int_1^{10} 9\pi - \pi y + \pi\, dy$$
$$= \left[9\pi y\right]_0^1 + \left[10\pi y - \tfrac{\pi}{2}y^2\right]_1^{10}$$
$$= 9\pi - 0 + (100\pi - 50\pi) - \left(10\pi - \tfrac{\pi}{2}\right)$$
$$= 9\pi + 50\pi - 10\pi + \tfrac{\pi}{2} = \boxed{\tfrac{99\pi}{2}}$$

Shell

$$V = \int_0^3 2\pi(x)(x^2+1)\, dx$$
$$= 2\pi \int_0^3 x^3 + x\, dx$$
$$= 2\pi \left[\tfrac{x^4}{4} + \tfrac{x^2}{2}\right]_0^3$$
$$= 2\pi \left[\left(\tfrac{81}{4} + \tfrac{9}{2}\right) - \cancel{(0+0)}\right]$$
$$= 2\pi\left(\tfrac{99}{4}\right)$$
$$= \boxed{\tfrac{99\pi}{2}}$$

16. (8 pts.) Find the Taylor series for $f(x) = \sin(x)$ centered about 0.

(aka Maclaurin Series)

$$f^{(0)}(x) = \sin x \rightarrow f^{(0)}(0) = 0$$
$$f^{(1)}(x) = \cos x \rightarrow f^{(1)}(0) = 1$$
$$f^{(2)}(x) = -\sin x \rightarrow \cancel{f^{(2)}(0) = 0}$$
$$f^{(3)}(x) = -\cos x \rightarrow f^{(3)}(0) = -1$$

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$
$$= \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1}$$
$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}$$

- Remove evens
- Use $(2n+1)$ for odds

Need Hooke's Law: $F(x)=kx$

17. (8 pts.) Suppose that 2 J of work is needed to stretch a spring from its natural length of 30 cm to a length of 42 cm. How much work is needed to stretch the spring from 35 cm to 40 cm?

Convert to x

0 12 5 10

$$W=\int_a^b kx\,dx$$

$$W=\left[\frac{k}{2}x^2\right]_a^b$$

$$W=\frac{k}{2}(b^2-a^2)$$

From 30 to 42

$$2=\frac{k}{2}(12^2-0^2)$$

$$2=\frac{k}{2}(144)$$

$$\frac{4}{144}=k$$

$$\frac{1}{36}=k$$

From 35 to 40

$$W=\frac{1/36}{2}(10^2-5^2)$$

$$=\frac{1}{72}(100-25)$$

$$=\frac{75}{72}$$

$$=\boxed{\frac{25}{24}\,(J)}$$

18. (8 pts.) Determine whether the sequence converges or diverges. If it converges, find the limit.

(a) $\{n^2e^{-n}\}$

$\left(\frac{\infty}{\infty}\right)$ $\left(\frac{\infty}{\infty}\right)$ $\left(\frac{2}{\infty}\right)$

$$\lim_{n\to\infty}\frac{n^2}{e^n}\overset{LH}{=}\lim_{n\to\infty}\frac{2n}{e^n}\overset{LH}{=}\lim_{n\to\infty}\frac{2}{e^n}=\boxed{0}$$ so $\boxed{\text{Converges}}$

(b) (8 pts.) $\left\{\sqrt[n]{3^{1+3n}}\right\}$

$$\lim_{n\to\infty}\sqrt[n]{3^{1+3n}}=\lim_{n\to\infty}\left(3^{1+3n}\right)^{\frac{1}{n}}=\lim_{n\to\infty}3^{\cancel{1/n}+3}=3^3=\boxed{27}$$

0

so $\boxed{\text{conv.}}$

Note, this problem should have had more space to solve.

19. (8 pts.) Find the exact length of the curve $y = \ln(1-x^2)$ from $x = 0$ to $x = 1/2$.

$\frac{dy}{dx} = \frac{-2x}{1-x^2}$

$L = \int_0^{1/2} \sqrt{1 + \frac{4x^2}{1-2x^2+x^4}}\, dx$

$= \int_0^{1/2} \sqrt{\frac{1+2x^2+x^4}{1-2x^2+x^4}}\, dx$

$= \int_0^{1/2} \frac{1+x^2}{1-x^2}\, dx$

$= \int_0^{1/2} -1 + \frac{2}{1-x^2}\, dx$ (Use Long division)

$= \int_0^{1/2} -1 + \frac{1}{1-x} + \frac{1}{1+x}\, dx$ (Use Partial Fractions)

$= \left[-x + \ln|1-x| + \ln|1+x| \right]_0^{1/2}$

$= \boxed{-\frac{1}{2} + \ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right)}$

$= \boxed{-\frac{1}{2} + \ln\left(\frac{3}{4}\right)}$

20. Use the vectors $\mathbf{a} = \langle 6, 1, -2 \rangle$, $\mathbf{b} = \langle 1, 8, 1 \rangle$, and $\mathbf{c} = \langle 2, 4, 0 \rangle$ to evaluate the following expressions.

(a) (8 pts.) $\mathbf{a} + \mathbf{b} - \mathbf{c}$

$= \langle 6+1-2, 1+8-4, -2+1-0 \rangle$

$= \boxed{\langle 5, 5, -1 \rangle}$

(b) (8 pts.) $\mathbf{a} \times \mathbf{c}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 1 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

$= \hat{i} \begin{vmatrix} 1 & -2 \\ 4 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 6 & -2 \\ 2 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 6 & 1 \\ 2 & 4 \end{vmatrix}$

$= \hat{i}(0-(-8)) - \hat{j}(0-(-4)) + \hat{k}(24-2)$

$= \boxed{8\hat{i} - 4\hat{j} + 22\hat{k}}$

$= \boxed{\langle 8, -4, 22 \rangle}$

(c) (8 pts.) $\mathbf{b} \cdot \mathbf{c}$

$= (1)(2) + (8)(4) + \overset{0}{\cancel{(1)(0)}}$

$= 2 + 32$

$= \boxed{34}$

MATH 1620 Final Exam

Name: **Date:** 29 April 2014

Instructions:

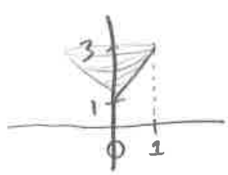
- You will have 2.5 hours to take this exam.
- There are 12 problems worth between 10 and 35 points each.
- You must show your work.
- Full credit will be awarded only when correct answers accompany proper justification.
- Your grade for the Final Exam will be the sum of the points you earn, out of 200 total points.

Problem 1 (10 pts)	Problem 2 (10 pts)	Problem 3 (10 pts)	Problem 4 (15 pts)
Problem 5 (15 pts)	Problem 6 (15 pts)	Problem 7 (20 pts)	Problem 8 (20 pts)
Problem 9 (20 pts)	Problem 10 (25 pts)	Problem 11 (30 pts)	Problem 12 (35 pts)

1 (10 pts) Find the exact area of the surface obtained by rotating the graph of $y = 2x + 1$ between $x = 0$ and $x = 1$ about the x-axis.

$f(x) =$

$f'(x) = 2$



$$L = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$
$$= \int_0^1 2\pi(2x+1)\sqrt{1+2^2}\,dx$$
$$= 2\pi\sqrt{5}\int_0^1 2x+1\,dx$$
$$= 2\pi\sqrt{5}\left[x^2+x\right]_0^1$$
$$= 2\pi\sqrt{5}\left[(1+1)-\cancel{(0+0)}\right]$$
$$= \boxed{4\pi\sqrt{5}}$$

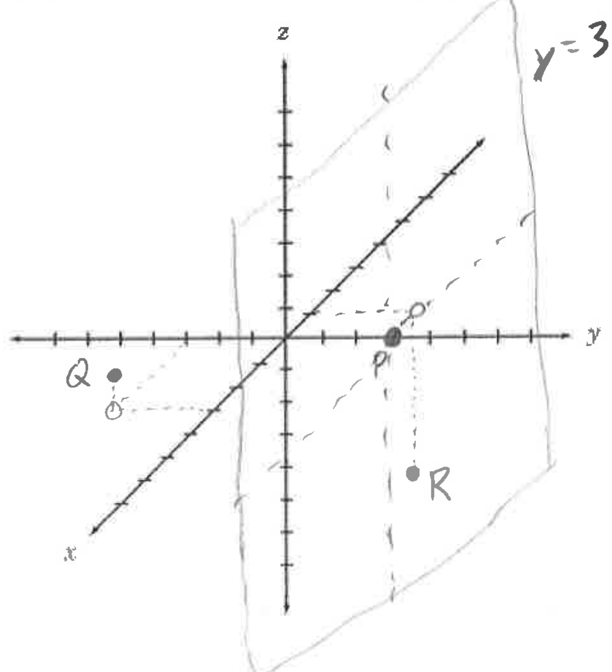
2 (10 pts) Consider the points $P(0, 3, 0)$, $Q(3, -3, 1)$, and $R(-1, 3, -5)$ and the equation $y = 3$ in $\mathbb{R}^3$.

(a) (3 pts) Plot the points P, Q, and R.

(b) (5 pts) Sketch the graph of $y = 3$.

(c) (2 pts) Which of the three points P, Q, and R lie on the graph of $y = 3$?

$$\boxed{\begin{array}{l} P=(0,3,0) \\ R=(-1,3,-5) \end{array}}$$



3 (10 pts) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is absolutely convergent, conditionally convergent, or divergent.

Typo: ~~n=0~~ n=1

Abs Con?

$\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$

diverges by Harmonic Series

Cond Con?

$\sum \frac{(-1)^n}{n}$

Alt. Series Test:

- $\frac{1}{n} \geq \frac{1}{n+1}$ (non-increasing)
- $\lim_{n\to\infty} \frac{1}{n} = 0$ (limits to zero)

So $\sum \frac{(-1)^n}{n}$ converges.

Since $\sum \frac{(-1)^n}{n}$ converges, but doesn't absolutely converge, it $\boxed{\text{Conditionally converges}}$.

4 (15 pts) A variable force of $f(x) = xe^x$ Newtons moves an object along a straight line when it is x meters from the origin. Calculate the work done moving the object from $x = 0$ meters to $x = \ln 2$ meters. (Note: the unit of measurement of work in this example is Joules.)

$$W = \int_a^b F(x)\,dx$$

$$= \int_0^{\ln 2} \underbrace{x}_{u}\underbrace{e^x\,dx}_{dv}$$

Let

$u = x \qquad v = e^x$

$du = dx \qquad dv = e^x dx$

$$= \left[xe^x - \int e^x dx\right]_0^{\ln 2}$$

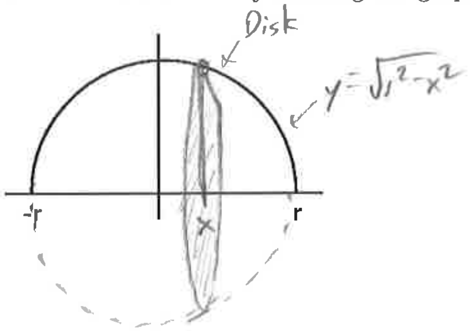
$$= \left[xe^x - e^x\right]_0^{\ln 2}$$

$$= \left((\ln 2)\cancel{e}^{\ln 2} - \cancel{e}^{\ln 2}\right) - \left(\cancel{0}e^0 - e^0\right)$$

$$= (2\ln 2 - 2) - (-1)$$

$$= \boxed{2\ln 2 - 1}$$

5 (15 pts) Let r be a positive constant. Derive the volume formula $V = \frac{4}{3}\pi r^3$ for a sphere by finding the volume of the solid obtained by rotating the graph of $y = \sqrt{r^2 - x^2}$ from $x = -r$ to $x = r$ about the x-axis.



$$V = \int_{-r}^{r} \pi (R(x))^2 \, dx$$

$$= \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \pi \int_{-r}^{r} r^2 - x^2 \, dx$$

$$= \pi \left[r^2 x - \frac{1}{3} x^3 \right]_{-r}^{r}$$

$$= \pi \left[\left(r^3 - \frac{1}{3} r^3\right) - \left(-r^3 + \frac{1}{3} r^3\right) \right]$$

$$= \pi \left[\frac{2}{3} r^3 - \left(-\frac{2}{3} r^3\right) \right]$$

$$= \pi \left(\frac{4}{3} r^3 \right)$$

$$\boxed{V = \frac{4}{3} \pi r^3}$$ ✓

6 (15 pts) Determine whether the series $\sum_{n=0}^{\infty} \frac{4n}{1+n+n^2+n^3}$ converges or diverges.

DCT

$$\frac{4n}{1+n+n^2+n^3} \le \frac{4n}{n^3} = \frac{4}{n^2}$$

Since $\sum \frac{4}{n^2}$ converges (by p-Series, $p>1$), the smaller $\sum \frac{4n}{1+n+n^2+n^3}$ also $\boxed{\text{converges}}$.

LCT (Compare with $\sum \frac{n}{n^3} = \sum \frac{1}{n^2}$, which converges by p-Series.)

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{4n}{1+n+n^2+n^3}} = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{1+n+n^2+n^3}{4n} = \lim_{n\to\infty} \frac{1+n+n^2+n^3}{4n^3}$$

$$= \frac{1}{4} > 0.$$

So $\sum \frac{1}{n^2}$ and $\sum \frac{4n}{1+n+n^2+n^3}$ match, and thus both series $\boxed{\text{converge}}$.

7 (20 pts) Evaluate $\int \frac{4x}{1+x+x^2+x^3}\, dx$.

(Factor denominator:)

$$(1+x)+(x^2+x^3) = \underline{1}(1+x)+x^2(1+x)$$

$$= (1+x^2)(1+x)$$

irreduceable quadratic   linear

(Partial Fractions)

$$\frac{4x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$4x = A(x^2+1)+(Bx+C)(x+1)$$

$$(0)x^2+(4)x+(0) = (A+B)x^2+(B+C)x+(A+C)$$

x^2: $0=A+B \to B=-A \to B=2$

x: $4=B+C \to 4=-A-A \to -2A=4 \to A=-2$

Const: $0=A+C \to C=-A \to C=2$

(Solve)

$$= \int \frac{-2}{x+1} + \frac{2x+2}{x^2+1}\, dx$$

$$= \int \frac{-2}{x+1} + \frac{2x}{x^2+1} + \frac{2}{x^2+1}\, dx$$

$$= \boxed{-2\ln|x+1| + \ln|x^2+1| + 2\,\mathrm{Arctan}(x) + C}$$

Can also use Ratio

$\downarrow$

8 (20 pts) Find the interval of convergence and radius of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$. (Typo)

Root Test

$$\lim_{n\to\infty}\left|\frac{(2x+1)^n}{n}\right|^{1/n} = \lim_{n\to\infty}\frac{|2x+1|^{\cancel{n/n}}}{n^{1/n}} = \frac{|2x+1|}{\cancel{1}} = |2x+1| < 1$$

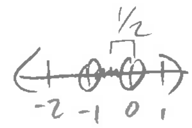
$$\downarrow$$

$$-1 < 2x+1 < 1$$

$$-2 < 2x < 0$$

$$\boxed{-1 \leq x < 0}$$

$$\boxed{R = \tfrac{1}{2}}$$



(Check endpoints)

x = -1

$$\sum \frac{(-2+1)^n}{n} = \sum \frac{(-1)^n}{n}$$

Converges (by AST)

x = 0

$$\sum \frac{(0+1)^n}{n} = \sum \frac{1}{n}$$

diverges by Harmonic/p-Series

9 (20 pts) In linear algebra, a set of vectors is called **orthonormal** if the vectors in that set are (i) orthogonal to each other and (ii) unit vectors.
Starting with $\vec{u} = \langle 1, 0, -1\rangle$ and $\vec{v} = \langle -5, -5, -5\rangle$, we will create an orthonormal set of 3 vectors using the method below.

(a) (5 pts) First, verify $\vec{u}$ and $\vec{v}$ are orthogonal. ← Dot prod. is 0.

$$\vec{u}\cdot\vec{v} = (-5)+(0)+(5)$$
$$= 0 \quad \checkmark$$

Use cross-prod.

(b) (10 pts) Next, find a third vector $\vec{w}$ orthogonal to $\vec{u}$ and $\vec{v}$.

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ -5 & -5 & -5 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 0 & -1 \\ -5 & -5\end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & -1 \\ -5 & -5\end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 0 \\ -5 & -5\end{vmatrix}$$
$$= \hat{\imath}(0-5) - \hat{\jmath}(-5-5) + \hat{k}(-5-0)$$
$$= \boxed{-5\hat{\imath} + 10\hat{\jmath} - 5\hat{k}}$$
$$= \boxed{\langle -5, 10, -5\rangle}$$

(c) (5 pts) Finally, find unit vectors $\vec{u}_1$, $\vec{v}_1$, and $\vec{w}_1$ having the same directions as $\vec{u}$, $\vec{v}$, and $\vec{w}$ respectively. (The set $\{\vec{u}_1, \vec{v}_1, \vec{w}_1\}$ is orthonormal.)

$$\vec{u} = \langle 1, 0, -1\rangle$$
$$|\vec{u}| = \sqrt{1+0+1} = \sqrt{2}$$
$$\vec{u}_1 = \frac{\vec{u}}{|\vec{u}|} = \frac{\langle 1, 0, -1\rangle}{\sqrt{2}} = \boxed{\left\langle \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right\rangle}$$
OR
$$= \boxed{\left\langle \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right\rangle}$$

$$\vec{v} = \langle -5, -5, -5\rangle$$
$$|\vec{v}| = \sqrt{25+25+25} = \sqrt{75} = \sqrt{3\cdot 25} = 5\sqrt{3}$$
$$\vec{v}_1 = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -5, -5, -5\rangle}{5\sqrt{3}} = \boxed{\left\langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\rangle}$$
OR
$$= \boxed{\left\langle -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right\rangle}$$

$$\vec{w} = \langle -5, 10, -5\rangle$$
$$|\vec{w}| = \sqrt{25+100+25} = \sqrt{150} = \sqrt{6\cdot 25} = 5\sqrt{6}$$
$$\vec{w}_1 = \frac{\vec{w}}{|\vec{w}|} = \frac{\langle -5, 10, -5\rangle}{5\sqrt{6}} = \boxed{\left\langle -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right\rangle}$$
OR
$$= \boxed{\left\langle -\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}\right\rangle}$$

10 (25 pts) Evaluate $\int \frac{\cos x - 1}{x}\,dx$ using series.

(*Note*: Please show and explain your work appropriately when computing the series. You will not receive full credit if you use a memorized series.)

(Explained elsewhere in packet.)

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\frac{\cos x - 1}{x} = -\frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \frac{x^7}{8!} - \cdots$$

$$\int \frac{\cos x - 1}{x}\,dx = \int -\frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \frac{x^7}{8!} - \cdots\,dx$$

$$= \boxed{-\frac{x^2}{2!(2)} + \frac{x^4}{4!(4)} - \frac{x^6}{6!(6)} + \frac{x^8}{8!(8)} - \cdots}$$

OR

$$= \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!(2n)} x^{2n}}$$

11 (30 pts) Evaluate $\int \underbrace{6y}_{u} \underbrace{\sec^2 y \tan^2 y \, dy}_{dv}$.

$u = 6y \qquad v = \frac{1}{3}\tan^3 y$

$du = 6\,dy \qquad dv = \sec^2 y \tan^2 y$

(To find v:)

$\int \sec^2 y \tan^2 y \, dy$

$= \int \tan^2 y \sec^2 y \, dy$

Let $u = \tan y$
$du = \sec^2 y \, dy$

$= \int u^2 \, du$

$= \frac{1}{3}u^3 + C$

$= \frac{1}{3}\tan^3 y + C$

$= (6y)(\frac{1}{3}\tan^3 y) - \int (\frac{1}{3}\tan^3 y)(6)\,dy$

$= 2y\tan^3 y - \int 2\tan^3 y \, dy$

$= 2y\tan^3 y - \int 2\tan y(\tan^2 y)\,dy$

$= 2y\tan^3 y - \int 2\tan y(\sec^2 y + 1)\,dy$

$= 2y\tan^3 y - \left(\int 2\tan y \sec^2 y \, dy + \int 2\tan y \, dy\right)$

(Let $u = \tan y$)

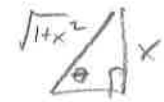
$= 2y\tan^3 y - \int 2u\,du - 2\int \tan y \, dy$

$= 2y\tan^3 y - u^2 - 2\ln|\sec y| + C$

$= \boxed{2y\tan^3 y - \tan^2 y - 2\ln|\sec y| + C}$

$\tan\theta = \frac{x}{1} = \frac{opp}{adj}$

12 (35 pts) Find the exact length of the graph of $y = \frac{1}{2}x^2$ between $x = 0$ and $x = 1$.



$\sec\theta = \frac{hyp}{adj} = \sqrt{1+x^2}$

$\frac{dy}{dx} = x$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + x^2}\, dx$$

Let $1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$

$x^2 = \tan^2\theta$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$= \int_{x=0}^{x=1} \sqrt{\sec^2\theta}\, \sec^2\theta\, d\theta$$

$$L = \int_{x=0}^{x=1} \sec^3\theta\, d\theta$$

$$= \int_{x=0}^{x=1} \sec\theta\,(1 + \tan^2\theta)\, d\theta$$

$$= \int_{x=0}^{x=1} \sec\theta\, d\theta + \int_{x=0}^{x=1} \sec\theta \tan^2\theta\, d\theta$$

(Int by parts)

$$= \Big[\ln|\sec\theta + \tan\theta|\Big]_{x=0}^{x=1} + \Big[\sec\theta\tan\theta - \int \sec^3\theta\, d\theta\Big]_{x=0}^{x=1}$$

$$= \Big[\ln|\sec\theta + \tan\theta| + \sec\theta\tan\theta\Big]_{x=0}^{x=1} - \int_{x=0}^{x=1} \sec^3\theta\, d\theta$$

$$L = \Big[\ln|\sec\theta + \tan\theta| + \sec\theta\tan\theta\Big]_{x=0}^{x=1} - L$$

$$\& \ L = \frac{1}{2}\Big[\ln|\sec\theta + \tan\theta| + \sec\theta\tan\theta\Big]_{x=0}^{x=1}$$

$$L = \frac{1}{2}\Big[\ln\left|\sqrt{1+x^2} + x\right| + \sqrt{1+x^2}\,(x)\Big]_0^1$$

$$= \frac{1}{2}\Big[\left(\ln|\sqrt{2}+1| + \sqrt{2}(1)\right) - \left(\cancel{\ln|\sqrt{1}+0|} + \cancel{\sqrt{1}(0)}\right)\Big]$$

$$= \boxed{\frac{1}{2}\ln|\sqrt{2}+1| + \frac{\sqrt{2}}{2}}$$

#1 $y = \sin x$ $y = \sin 2x$ $x = 0$ $x = \frac{\pi}{2}$

$\sin x = \sin 2x$ * Double angle formula

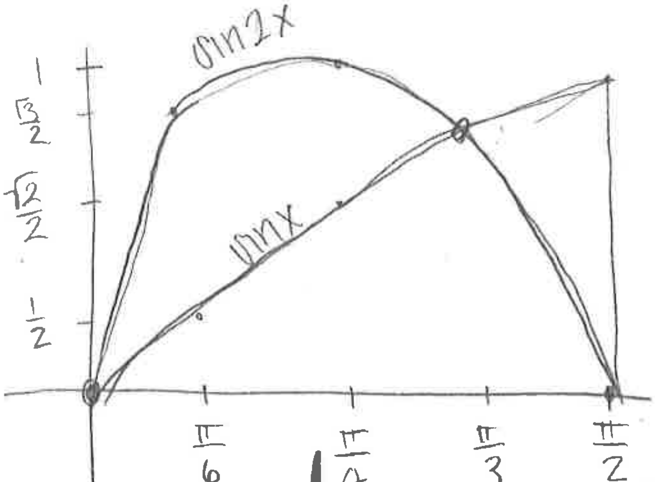
$\sin x = 2\cos x \sin x$

$0 = 2\cos x \sin x - \sin x$

$0 = (2\cos x - 1)\sin x$

$\sin x = 0$ $\cos x = \frac{1}{2}$

$x = 0$ $x = \frac{\pi}{3}$



$$\int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)\,dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sin x - \sin 2x)\,dx$$

$$-\frac{1}{2}\cos 2x + \cos x \Big/_0^{\frac{\pi}{3}} + \left(-\cos x + \frac{1}{2}\cos 2x \Big/_{\frac{\pi}{3}}^{\frac{\pi}{2}}\right)$$

$$-\frac{1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{3} - \left(-\frac{1}{2}\cos(0) + \cos(0)\right)$$

$$-\cos\frac{\pi}{2} + \frac{1}{2}\cos(\pi) - \left(-\cos\left(\frac{\pi}{3}\right) + \frac{1}{2}\cos\left(2\frac{\pi}{3}\right)\right)$$

$$\frac{1}{4} + \cancel{\frac{1}{2} + \frac{1}{2} - 1}$$

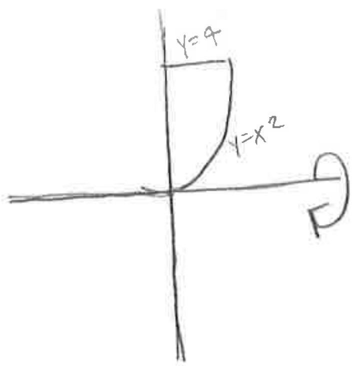
$$0 - \frac{1}{2} - \left(\frac{-1}{2} - \frac{1}{4}\right)$$

$$0 \cancel{- \frac{1}{2} + \frac{1}{2}} + \frac{1}{4}$$

$$\frac{1}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}$$

#2 $y=x^2$ $\quad 0 \le x \le 2$

$y=4$ $\quad x=0$ $\quad$ ↻ x-axis



Disk ; ↻ x-axis →

- vertical cuts
- not flush

r_{inner} $y=x^2$ $\qquad \int_0^2 (\pi 4^2 - \pi(x^2)^2)\,dx$

R_{outer} $y=4$

$$\pi \int_0^2 (16 - x^4)\,dx$$

$$\pi\left[16x - \frac{1}{5}x^5 \Big|_0^2\right]$$

$$\pi\left[16(2) - \frac{1}{5}(2)^5\right] - 0$$

$$\pi\left[32 - \frac{32}{5}\right] = \frac{148\pi}{5}$$

Shell ; ↻ x-axis

horizontal cuts

$y = x^2 \longrightarrow x = \sqrt{y}$

$r = y \quad h = \sqrt{y}$

$$\int_0^4 2\pi y\sqrt{y}\,dy = \int_0^4 2\pi y^{\frac{3}{2}}\,dy$$

$$2\pi \frac{2}{5} y^{\frac{5}{2}} \Big|_0^4 = \frac{4\pi}{5} 4^{\frac{5}{2}} = \frac{4\pi(32)}{5} = \frac{148\pi}{5}$$

F.K.

#3) $\int \tan^7 x \sec^3 x \, dx \rightarrow$

*remember
$\boxed{1 + \tan^2 x = \sec^2 x}$

$\int \tan^7 x \sec^2 x \sec x \, dx \rightarrow \int \tan^7 x [1 + \tan^2 x] \sec x \, dx$

$\int [\tan^7 x + \tan^9 x] \sec x \, dx \rightarrow \int \tan^7 x \sec x + \tan^9 x \sec x \, dx$

$$\boxed{\frac{1}{8} \tan^8 x + \frac{1}{10} \tan^{10} x + C}$$

#4) $\int \frac{x}{\sqrt{x^2 - 7}} \, dx$

$u = x^2 - 7$

$du = 2x\,dx \rightarrow dx = \frac{1}{2x} du$

$\int \frac{\cancel{x}}{\sqrt{u}} \cdot \frac{1}{2\cancel{x}} du \rightarrow \frac{1}{2} \int (u)^{-1/2}$

$u^{1/2} + C$

$$\boxed{\sqrt{x^2 - 7} + C}$$

5) $\int \frac{dx}{x^3 + x} \rightarrow \int \frac{dx}{x(x^2+1)}$

$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} \rightarrow 1 = A(x^2+1) + (Bx + C)x$

$1 = Ax^2 + A + Bx^2 + Cx$

$\int \frac{1}{x} dx + \int \frac{-x}{x^2+1} dx$

$(A + B)x^2 + Cx + A = 1$

$C = 0$

$A = 1$

$B = -1$

FK

5(II)

(#5 cont'd)

$\ln|x| - \int \frac{x}{x^2+1}\,dx$

$u = x^2+1$

$du = 2x\,dx \qquad dx = \frac{1}{2x}\,du$

$\ln|x| - \int \frac{\cancel{x}}{u} \cdot \frac{1}{2\cancel{x}}\,du$

$\boxed{\ln|x| - \frac{1}{2}\ln|x^2+1| + C}$

F.R.

#6) $\int x \sec x \tan x \, dx$ $\quad u = x \quad dv = \sec x \tan x \, dx$

$du = dx \quad v = \sec x$

$= x \sec x - \int \sec x \, dx \rightarrow x \sec x - \ln|\sec x + \tan x| + C$

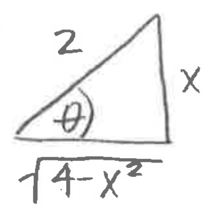
#7) $\int \frac{x^2}{(4-x^2)^{3/2}} \, dx$ $\quad x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$

$\int \frac{(2\sin\theta)^2}{(4-4\sin^2\theta)^{3/2}} \cdot 2\cos\theta \, d\theta \rightarrow \int \frac{8\sin^2\theta \cos\theta \, d\theta}{(4\cos^2\theta)^{3/2}}$

$\int \frac{8\sin^2\theta \cos\theta \, d\theta}{(2\cos\theta)^3} \rightarrow \int \frac{\cancel{8}\sin^2\theta \cancel{\cos\theta}}{\cancel{8}\cos^{\cancel{3}2}\theta} \, d\theta \rightarrow \int \frac{1-\cos^2\theta}{\cos^2\theta} \, d\theta$

$\int \sec^2\theta - 1 \, d\theta = \tan\theta - \theta + C$ $\quad \sin\theta = \frac{x}{2}$



$\frac{x}{\sqrt{4-x^2}} - \arcsin\left(\frac{x}{2}\right) + C$

#8) $\int_0^4 \frac{\ln x}{\sqrt{x}} \, dx$ $\quad u = \sqrt{x} \quad u^2 = x$

$du = \frac{1}{2\sqrt{x}} \, dx$

$s = \ln u \quad dt = du$

$ds = \frac{1}{u} du \quad t = u$

$\int_0^4 \frac{2\ln u}{\cancel{u}} \cdot 2\cancel{u} \, du \rightarrow 4\int_0^4 \ln u \, du$

$4\left[\sqrt{x} \cdot \ln\sqrt{x} - \sqrt{x} + C\right]$

$4\left[u\ln u - \int \frac{1}{u} u \, du\right] \rightarrow 4[u\ln u - u] \rightarrow 4[2 \cdot \ln 2 - 2$

Final Review

#9) $\lim_{n\to\infty} \frac{2+n^3}{1+2n^3} = \frac{1}{2}$

10) $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ $\quad \lim_{n\to\infty} \frac{\frac{n}{n^3+1}}{\frac{n}{n^3}} = \lim_{n\to\infty} \frac{n}{n^3+1} \cdot \frac{n^3}{n}$

$= \lim_{n\to\infty} \frac{n^4}{n^4+n} = 1$ $\quad \sum \frac{n}{n^3}$ converges (p-series)

$\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ converges

11) $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$ $\quad \lim_{n\to\infty} \frac{\frac{(n+1)^3}{5^{n+1}}}{\frac{n^3}{5^n}}$ $\quad \lim_{n\to\infty} \frac{(n+1)^3}{5^{\cancel{n+1}}} \cdot \frac{\cancel{5^n}}{n^3} = \frac{1}{5}$

$\frac{1}{5} < 1$ converges (ratio)

12) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{5^n\, n!}$

$\lim_{n\to\infty} \frac{\cancel{1\cdot 3\cdot 5\cdots(2n-1)}(2n+1)}{5^{\cancel{n+1}}\,(n+1)\cancel{!}} \cdot \frac{\cancel{5^n\, n!}}{\cancel{1\cdot 3\cdot 5\cdots(2n-1)}}$

$\lim_{n\to\infty} \frac{2n+1}{5(n+1)} = \frac{2}{5} < 1$ convergent ratio test!

final Review

13) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1}$

① $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$ ✓

② $\frac{\sqrt{n}}{n+1}$ $n = 1, 2, \ldots$ positive? ✓

③ is $\frac{\sqrt{n+1}}{n+2} < \frac{\sqrt{n}}{n+1}$ ✓

converges A.S.T

14) sum (geometric)

$\sum_{n=1}^{\infty} = \frac{a}{1-r}$ $\quad \sum ar^n$

$\sum_{n=1}^{\infty} \frac{(2)^{2n+1}}{5^n} \Rightarrow \sum_{n=1}^{\infty} \frac{2^{2n} \cdot 2}{5^n} \Rightarrow \sum_{n=1}^{\infty} \frac{(2^2)^n \cdot 2}{5^n}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{4^n}{5^n} \cdot 2 \Rightarrow \sum_{n=1}^{\infty} 2\left(\frac{4}{5}\right)^n$ $\quad \frac{4}{5} = r \quad 2 = a$

sum: $\frac{2}{\frac{5}{5} - \frac{4}{5}} = \frac{2}{\frac{1}{5}} = \boxed{10}$ ✓

15) interval of conv. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}$

$\lim_{n\to\infty} \left| \frac{(x+2)^{\cancel{n+1}}}{(n+1)4^{\cancel{n+1}}} \cdot \frac{n \cdot \cancel{4^n}}{\cancel{(x+2)^n}} \right| = \lim_{n\to\infty} \left| \frac{(x+2) \cdot n}{n+1 \cdot 4} \right| < 1$

$\frac{|x+2|}{4} < 1 \qquad |x+2| < 4 \qquad -4 < x+2 < 4$

$\boxed{-6 < x < 2}$

Final Review

16) $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \Longrightarrow \frac{x^2}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^{n+2}$

17) $f(x) = \ln(1-x)$ $\quad f(0) = \ln(1) = 0$ $\quad \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$

$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad -1 < x \leq 1$

$f'(x) = \frac{-1}{(1-x)}$ $\quad f'(0) = -1$ $\quad \frac{0x^0}{0!} - \frac{(+1)}{1!}x^1 - \frac{1}{2!}x^2 - \frac{2}{3!}x^3 - \frac{6x^4}{4!}$

$f''(x) = \frac{-1}{(1-x)^2}$ $\quad f''(0) = 1$

$f'''(x) = \frac{-2}{(1-x)^3}$ $\quad f'''(0) = -2$

$f^{IV}(x) = \frac{-6}{(1-x)^4}$ $\quad f^{IV}(0) = 6$ $\quad 0 - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4!}$

$\sum_{n=1}^{\infty} \frac{x^n}{n}$

-OR-

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-x)^n}{n} \Longrightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(-1)} \cdot \frac{(-1)^n x^n}{n}$

$= \sum_{n=1}^{\infty} \frac{[(-1)(-1)]^n x^n}{(-1)n} = \sum_{n=1}^{\infty} \frac{(-1)x^n}{n}$

9.2

18) $\int \frac{e^x}{x}\,dx \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\int \frac{e^x}{x} = \int \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

19) $\cancel{x^2} + \cancel{y^2} + z^2 + 4\cancel{x} + 6\cancel{y} - 10z = -2$

$$\underbrace{x^2+4x+\underline{4}} + \underbrace{y^2+6y+\underline{9}} + \underbrace{z^2-10z+\underline{25}} = -2+\underline{4}+\underline{9}+\underline{25}$$

$$(x+2)^2 + (y+3)^2 + (z-5)^2 = 36$$

$(-2,-3,5) \qquad R=6$

20) $\langle 3,2,x\rangle$ & $\langle 2x,4,x\rangle$ orthagonal if $u \cdot v = 0$

$$6x + 8 + x^2 = 0$$

$(x+4)(x+2)=0 \qquad x=-4$ or $x=-2$

21) $A(1,0,0) \quad B(2,0,-1) \quad C(1,4,3)$

a) $\overrightarrow{AB} = \langle 2-1, 0-0, -1-0\rangle = \langle 1,0,-1\rangle$

$\overrightarrow{AC} = \langle 1-1, 4-0, 3-0\rangle = \langle 0,4,3\rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 4 & 3 \end{vmatrix} = i\begin{vmatrix} 0 & -1 \\ 4 & 3 \end{vmatrix} - j\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} + k\begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix}$$

$$i(0+4) - j(3-0) + k(4-0) = 4i - 3j + 4k$$

b) $\frac{\sqrt{16+9+16}}{2} = \frac{\sqrt{41}}{2}$

FR

22 $(4,-1,2)$ & $(1,1,5)$

$v = \langle 1-4,\ 1+1,\ 5-2 \rangle = \langle -3, 2, 3 \rangle$

$r = \langle 1,1,5 \rangle + t\langle -3,2,3 \rangle$

$x = 1-3t \quad y = 1+2t \quad z = 5+3t$

23 $\overset{P}{(3,-1,1)} \quad \overset{Q}{(4,0,2)} \quad \overset{R}{(6,3,1)}$

$\overrightarrow{PQ} = \langle 1,1,1 \rangle \quad \overrightarrow{PR} = \langle 3,4,0 \rangle \rightarrow \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 3 & 4 & 0 \end{vmatrix} = i\begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix}$

$= i(0-4) - j(0-3) + k(4-3)$

$= -4i + 3j + k = \vec{N}$

$= -4(x-3) + 3(y+1) + (z-1) = 0$

OR $-4x + 3y + z = -14$

24 $x = t^2 + 4t \quad y = 2-t$

$t = 2-y$

$x = (2-y)^2 + 4(2-y)$

$x = y^2 - 8y + 12$

~~25 $x = \ln t$ $y = 1+t^2$ @ $t=1$ $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1/t} = 2t^2 \Big|_{t=1}$ $\boxed{2}$~~

~~26 $x = t\cos t$ $y = t\sin t$~~

~~$dy/dt = \sin t + t\cos t$~~

F. R.

27) length of curve $\quad L = \sqrt{(x')^2 + (y')^2}$

$$\int_0^2 \sqrt{6t^2 + 4t^2}\, dt = \int_0^2 \sqrt{10t^2}\, dt \Rightarrow \int_0^2 \sqrt{10}\, t$$

$$\frac{\sqrt{10}}{2} t^2 \Big|_0^2 \qquad \frac{\cancel{2}\sqrt{10}}{\cancel{2}} - 0 = 2\sqrt{10}$$

29) $y = \frac{1}{6}(x^2+4)^{3/2}$

$$y' = \frac{1}{\cancel{4}}(x^2+4)^{1/2} \cdot \cancel{2}x = \frac{1}{2}(x^2+4)^{1/2} x$$

$$\int_1^3 \sqrt{1 + \left(\frac{1}{2}(x^2+4)^{1/2} x\right)^2}\, dx \to \int_1^3 \sqrt{1 + \left[\tfrac{1}{4}(x^2+4)x^2\right]}\, dx$$

$$\int_1^3 \sqrt{1 + x^4/4 + x^2}\, dx \to \int_1^3 \sqrt{4 + x^4 + 4x^2}\, dx$$

$$\int_1^3 \sqrt{(x^2+2)^2}\, dx \to \int_1^3 x^2 + 2 \to \frac{x^3}{3} + 2x \Big|_1^3$$

$$\frac{3^{\cancel{3}2}}{\cancel{3}} + 2 \cdot 3 - \left[\tfrac{1}{3} + 2\right] = 9 + 6 - \tfrac{1}{3} + 2 = 17 - \tfrac{1}{3}$$

30) $r = \csc\theta = 1 = \frac{1}{\sin\theta} = r\sin\theta = 1 \quad y = 1$ ✓

31) $x^2 + y^2 = 2cx \to r^2 = 2r\cos\theta \to r = 2\cos\theta$

Calculus II Spring 2010 — Final

Your Name:

Instructor: Steven Clontz

Draw a box around your final answer. You must show all work to receive credit. Each question is worth 5 points.

You may choose one problem from questions #1-4 to skip by drawing an X through it.
You will receive full credit for the question you skip.

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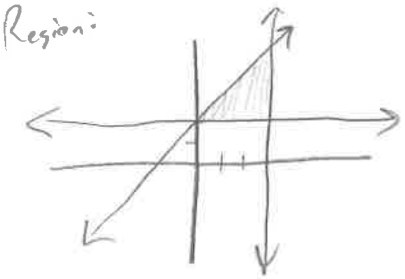
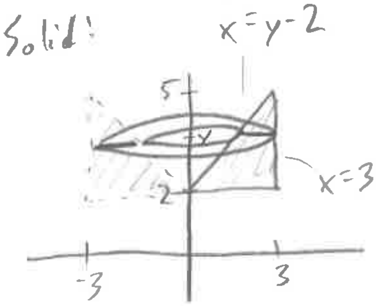
1. Use the washer method to find the volume of the solid obtained by rotating the region bounded by $y = 2$, $y = x + 2$, and $x = 3$ around the line $y = 1$.

Region:



Solid:



$$V = \int_2^5 \pi (R(y))^2 - \pi (r(y))^2 \, dy$$

$$= \int_2^5 \pi (3)^2 - \pi (y-2)^2 \, dy$$

$$= \int_2^5 9\pi - \pi (y^2 - 4y + 4) \, dy$$

$$= \int_2^5 -\pi y^2 + 4\pi y + 5\pi \, dy$$

$$= \left[-\frac{\pi}{3} y^3 + 2\pi y^2 + 5\pi y \right]_2^5$$

$$= \left(-\frac{\pi}{3}(125) + 2\pi(25) + 5\pi(5) \right) - \left(-\frac{\pi}{3}(8) + 2\pi(4) + 5\pi(2) \right)$$

$$= -\frac{\pi}{3}(117) + 2\pi(21) + 5\pi(3)$$

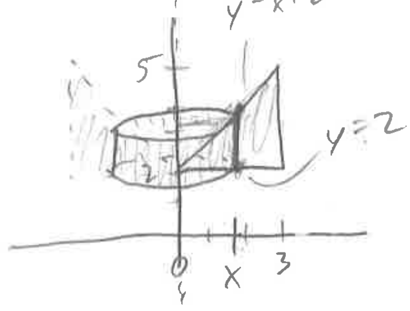
$$= -39\pi + 42\pi + 15\pi$$

$$= \boxed{18\pi}$$

---

2. Now use the cylindrical shell method to find the volume of the solid from question #1.

Solid: Axis



$$V=\int_0^3 2\pi(r(x))(h(x))\,dy$$

$$=\int_0^3 2\pi(x)((x+\cancel{2})-\cancel{2})\,dy$$

$$=\int_0^3 2\pi x^2\,dy$$

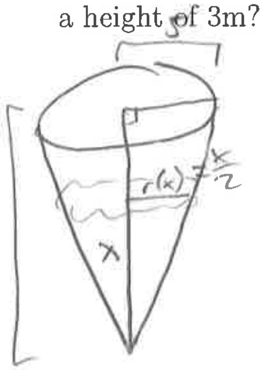
$$=\left[\frac{2}{3}\pi x^3\right]_0^3$$

$$=\frac{2}{3}\pi(27)-\cancel{\frac{2}{3}\pi(0)}$$

$$=\boxed{18\pi}$$

[redacted]

4. A conical tank of height 10m and radius 5m stands on its point. A liquid weighing 20,000 N/m^3 is pumped into the tank from its point. How much work is done in filling the tank to a height of 3m?



$\frac{10}{5} = \frac{x}{r(x)}$

$2r(x) = x$

$r(x) = \frac{x}{2}$

$F(x) = (\text{Volume @ } x)(\text{Density})$

$= \left(\frac{1}{3}\pi r^2 h\right)(20{,}000)$

$= \frac{1}{3}\pi \left(\frac{x}{2}\right)^2 (x)(20{,}000)$

$= \frac{5000}{3}\pi x^3$

$W = \int_0^3 F(x)\,dx$

$= \int_0^3 \frac{5000}{3}\pi x^3\,dx$

$= \left[\frac{1250}{3}\pi x^4\right]_0^3$

$= \frac{1250}{\cancel{3}}\pi(\cancel{81}\ 27) - 0$

$$\begin{array}{r} 1250 \\ \times\ 27 \\ \hline 8750 \\ 2500 \\ \hline 33750 \end{array}$$

$= \boxed{33750\pi}$

You may choose one problem from questions #5-8 to skip by drawing an X through it.

You will receive full credit for the question you skip.

5. Find $\int \cos(x)e^{2x}dx$

$u = e^{2x}$ $\quad v = \sin x$

$du = 2e^{2x}dx$ $\quad dv = \cos x\, dx$

$= e^{2x}\sin x - \int 2e^{2x}\sin x\, dx$

$u = 2e^{2x}$ $\quad v = -\cos x$

$du = 4e^{2x}dx$ $\quad dv = \sin x\, dx$

$= e^{2x}\sin x - \left[-2e^{2x}\cos x - \int -4e^{2x}\cos x\, dx\right]$

$\int e^{2x}\cos x\,dx = e^{2x}\sin x + 2e^{2x}\cos x - 4\int e^{2x}\cos x\, dx$

$5\int e^{2x}\cos x\,dx = e^{2x}\sin x + 2e^{2x}\cos x\ (+C)$

$\int e^{2x}\cos x\,dx = \boxed{\frac{1}{5}e^{2x}\sin x + \frac{2}{5}e^{2x}\cos x + C}$

6. Find $\int \frac{1}{1+9v^2}dv$

Let $1+9v^2 = 1+\tan^2\theta = \sec^2\theta$

$9v^2 = \tan^2\theta$

$3v = \tan\theta \longrightarrow \theta = \operatorname{Arctan}(3v)$

$v = \frac{1}{3}\tan\theta$

$dv = \frac{1}{3}\sec^2\theta\, d\theta$

$= \int \frac{1}{\cancel{\sec^2\theta}}\frac{1}{3}\cancel{\sec^2\theta}\, d\theta$

$= \int \frac{1}{3}d\theta$

$= \frac{1}{3}\theta + C$

$= \boxed{\frac{1}{3}\operatorname{Arctan}(3v) + C}$

7. Find $\int \frac{x^3-2x^2+5x-4}{x^2-2x}\,dx$.

$$x^2-2x+0\,\overline{)\,\cancel{x^3}-\cancel{2x^2}+5x-4}\quad x+0+\frac{5x-4}{x^2-2x+0}$$

$$-(\cancel{x^3}-\cancel{2x^2}+0x)$$

$$5x-4$$

$$=\int x+\frac{5x-4}{x^2-2x}\,dx$$

$$\frac{5x-4}{x(x-2)}=\frac{A}{x}+\frac{B}{x-2}$$

$$5x-4=A(x-2)+Bx$$

$$(5)x+(-4)=(A+B)x+(-2A)$$

$$5=A+B \rightarrow B=3$$

$$-4=-2A\rightarrow A=2$$

$$=\int x+\frac{2}{x}+\frac{3}{x-2}\,dx$$

$$=\boxed{\frac{1}{2}x^2+2\ln|x|+3\ln|x-2|+C}$$

8. Evaluate $\int_2^{\infty}\frac{2}{\theta^2}\,d\theta$.

$$=\lim_{b\to\infty}\int_2^b 2\theta^{-2}\,d\theta$$

$$=\lim_{b\to\infty}\left[-2\theta^{-1}\right]_2^b$$

$$=\lim_{b\to\infty}-\frac{2}{b}+\frac{2}{2}$$

$$=0+1=\boxed{1}$$

You may choose one problem from questions #9-13 to skip by drawing an X through it.

You will receive full credit for the question you skip.

9. Find $\lim_{n\to\infty} \frac{\ln(n^2+3)}{\ln(n)}$.

$$\overset{LH}{=} \lim_{n\to\infty} \frac{\frac{2n}{n^2+3}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2+3}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}} \frac{2}{1+\cancel{3/n^2}}$$

$$= \boxed{2}$$

10. Show that the series $\sum_{k=1}^{\infty} \frac{k^2-1}{k^4}$ converges or diverges.

DCT

$$\frac{k^2-1}{k^4} \leq \frac{k^2}{k^4} = \frac{1}{k^2}$$

Since $\sum \frac{1}{k^2}$ converges (by p-Series, $p>1$), the smaller $\sum \frac{k^2+1}{k^4}$ $\boxed{\text{converges}}$.

LCT Compare to $\sum \frac{1}{k^2}$, which converges (by p-Series, $p>1$).

$$\lim_{k\to\infty} \frac{\frac{1}{k^2}}{\frac{k^2-1}{k^4}} = \lim_{k\to\infty} \frac{1}{k^2}\frac{k^4}{k^2-1} = \lim_{k\to\infty} \frac{k^4}{k^4-k^2} = 1 > 0$$

So both $\sum \frac{1}{k^2}$ and $\sum \frac{k^2-1}{k^2}$ match, and thus $\boxed{\text{converge}}$.

Algebra

$$\sum \frac{k^2-1}{k^4} = \sum \frac{k^2}{k^4} - \sum \frac{1}{k^4} = \sum \frac{1}{k^2} - \sum \frac{1}{k^4} = \underset{\text{(by p-Series)}}{\text{finite}} - \underset{\text{(by p-Series)}}{\text{finite}}$$

Thus $\sum \frac{k^2-1}{k^4}$ $\boxed{\text{converges}}$.

11. Show that the series $\sum_{i=0}^{\infty} \frac{2^i}{3^{i+1}}$ converges or diverges.

$\boxed{\text{Geometric}}$

$$= \sum_{i=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^i$$

Since $\left|\frac{2}{3}\right| = \frac{2}{3} < 1$,

$$\left(= \frac{1/3}{1-2/3} = \frac{1/3}{1/3} = 1 \right)$$

Thus $\sum_{i=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^i$ $\boxed{\text{converges}}$ (to 1.)

$\boxed{\text{Root}}$

$$\lim_{i\to\infty} \left|\frac{2^i}{3^{i+1}}\right|^{1/i}$$

$$= \lim_{i\to\infty} \frac{2}{3^{1+\cancel{1/i}}}$$

$$= \frac{2}{3} < 1$$

Thus $\sum \frac{2^i}{3^{i+1}}$

$\boxed{\text{converges}}$

$\boxed{\text{Ratio}}$

$$\lim_{i\to\infty} \left| \frac{\frac{2^{i+1}}{3^{i+2}}}{\frac{2^i}{3^{i+1}}} \right|$$

$$= \lim_{i\to\infty} \frac{2^{\cancel{i+1}}}{3^{\cancel{i+2}}} \cdot \frac{\cancel{3^{i+1}}}{\cancel{2^i}}$$

$$= \frac{2}{3} < 1$$

Thus $\sum \frac{2^i}{3^{i+1}}$

$\boxed{\text{converges}}$

12. Show that the series $\sum_{n=1}^{\infty} \frac{1}{1+\ln(n)} - \frac{1}{1+\ln(n+1)}$ converges or diverges.

$\boxed{\text{By definition}}$

$$= \lim_{n\to\infty} \sum_{n=1}^{n} \frac{1}{1+\ln(n)} - \frac{1}{1+\ln(n+1)}$$

$$= \lim_{n\to\infty} \left[\left(\frac{1}{1+\ln 1} - \cancel{\frac{1}{1+\ln 2}} \right) + \left(\cancel{\frac{1}{1+\ln 2}} - \cancel{\frac{1}{1+\ln 3}} \right) + \cdots + \left(\cancel{\frac{1}{1+\ln n}} - \frac{1}{1+\ln(n+1)} \right) \right]$$

$$= \lim_{n\to\infty} \frac{1}{1+\underbrace{\cancel{\ln 1}}_{0}} - \underbrace{\cancel{\frac{1}{1+\ln(n+1)}}}_{0}$$

$$= \frac{1}{1} = \boxed{1}$$

Thus the series $\boxed{\text{converges}}$ (to 1).

13. Show that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges absolutely, converges conditionally, or diverges.

(Check Absolute convergence)

$\sum \left| \frac{(-1)^n}{n+1} \right| = \sum \frac{1}{n+1}$ diverges.

(Check conditional convergence)

$\sum \frac{(-1)^n}{n+1}$

Alt Series Test:

- $\frac{1}{n+1} \geq \frac{1}{(n+1)+1}$ (non-increasing) ✓
- $\lim_{n \to \infty} \frac{1}{n+1} = 0$ (limits to 0) ✓

Therefore $\sum \frac{(-1)^n}{n+1}$ converges.

Since it converges, but doesn't absolutely converge, it $\boxed{\text{conditionally converges}}$.

You may choose one problem from questions #14-20 to skip by drawing an X through it.
You will receive full credit for the question you skip.

14. Find the interval and radius of convergence for $\sum_{n=2}^{\infty} \frac{(2x-5)^n}{n-1}$.

$\boxed{\text{Root}}$ $\lim_{n\to\infty} \left|\frac{(2x-5)^n}{n-1}\right|^{1/n}$

$= \lim_{n\to\infty} \frac{|2x-5|}{\cancel{(n-1)^{1/n}}\,1}$

$= |2x-5| < 1$

OR $\boxed{\text{Ratio}}$ $\lim_{n\to\infty} \left|\frac{\frac{(2x-5)^{n+1}}{\cancel{n+1}}}{\frac{(2x-5)^n}{n-1}}\right|$

$= \lim_{n\to\infty} \left|\frac{(2x-5)^{\cancel{n+1}}}{\cancel{n}} \frac{\cancel{n-1}}{\cancel{(2x-5)^n}}\right|$

$= |2x-5| < 1$

$-1 < 2x-5 < 1$

$4 < 2x < 6$

$\boxed{2 \leq x < 3}$

$\boxed{R = 1/2}$

$\frac{1}{2}$

1 2 3 4

Check endpoints

Let $x=2$

$\sum \frac{(4-5)^n}{n-1}$

$= \sum \frac{(-1)^n}{n-1}$

converges by AST.

Let $x=3$

$\sum \frac{(6-5)^n}{n-1}$

$= \sum \frac{1}{n-1}$

diverges b/c harmonic series

15. Find the Maclaurin Series generated by $f(x) = \frac{1}{e^x}$. $= e^{-x}$

$f^{(0)}(x) = e^{-x} \rightarrow f^{(0)}(0) = e^0 = 1$

$f^{(1)}(x) = -e^{-x} \rightarrow f^{(1)}(0) = -e^0 = -1$

So

$$e^{-x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n} \left(= \boxed{1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots} \right)$$

---

16. Express $x^2 \cos(x)$ as a power series.

Let $f(x) = \cos x$

$f^{(0)}(x) = \cos x \rightarrow f^{(0)}(0) = 1$

$f^{(1)}(x) = -\sin x \rightarrow f^{(1)}(0) = 0$

$f^{(2)}(x) = -\cos x \rightarrow f^{(2)}(0) = -1$

$f^{(3)}(x) = \sin x \rightarrow f^{(3)}(0) = 0$

$(-1)^n$

Remove odds $(2n+1)$

So

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} \quad \left(\text{b/c } f^{(2n+1)}(0) = 0 \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$x^2 \cos x = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n+2}}$$