7.3 Trigonometric Substitution

Strategy 21. With square roots and other troublesome factors, it sometimes helps to substitute trigonometric functions in order to use their identities for cancellation.

Expression	Substitution	Differential	Fact to Use
$a^2 - x^2$	$x = a\sin(\theta) \Rightarrow x^2 = a^2\sin^2(\theta)$	$dx = a\cos(\theta)d\theta$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$a^2 + x^2$	$x = a\tan(\theta) \Rightarrow x^2 = a^2\tan^2(\theta)$	$dx = a\sec^2(\theta)d\theta$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$x^2 - a^2$	$x = a\sec(\theta) \Rightarrow x^2 = a^2\sec(\theta)$	$dx = a\sec(\theta)\tan(\theta)d\theta$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

Problem 22. ~~Prove~~ $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x) + C.$

Let $1+x^2 = 1+\tan^2\theta = \sec^2\theta$

$x^2 = \tan^2\theta$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$x = \tan\theta \Rightarrow \arctan(x) = \theta$

$= \int \frac{1}{\cancel{\sec^2\theta}} \cancel{\sec^2\theta}\, d\theta$

$= \int d\theta$

$= \theta + C$

$= \arctan(x) + C$ □.

Problem 23. ~~Evaluate~~ $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx.$

Let $9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$

$x^2 = 9\sin^2\theta$

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$\left(\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \quad 1 + \cot^2\theta = \csc^2\theta\right)$

$= \int \frac{\sqrt{\cancel{9}\cos^2\theta}}{\cancel{9}\sin^2\theta} \cancel{3}\cos\theta\, d\theta$

$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$

$= \int \cot^2\theta\, d\theta$

$= \int \csc^2\theta - 1\, d\theta$

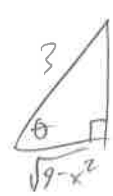
$= -\cot\theta - \theta + C$

$= \boxed{-\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C}$

$x = 3\sin\theta$

$\frac{x}{3} = \sin\theta$

$\arcsin\left(\frac{x}{3}\right) = \theta$



$\cot\theta = \frac{adj}{opp} = \frac{\sqrt{9-x^2}}{x}$

Problem 24. Evaluate $\int \frac{2x}{x^2+1}\,dx$.

(Smart way w/out trig:)

Let $u = x^2+1 \Rightarrow du = 2x\,dx$

$$= \int \frac{du}{u} = \ln|u| + C = \boxed{\ln|x^2+1| + C}$$

OR

Trig way

Let $x^2+1 = \tan^2\theta + 1 = \sec^2\theta$

$x^2 = \tan^2\theta$

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$$= \int \frac{2\tan\theta}{\cancel{\sec^2\theta}}\cancel{\sec^2\theta}\,d\theta$$

$$= 2\ln|\sec\theta| + C$$

$\tan\theta = \frac{x}{1} = \frac{\text{opp}}{\text{adj}}$

$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2+1}}{1}$

$$= \boxed{\cancel{2}\ln\left(\cancel{\sqrt{x^2+1}}\right) + C}$$

$$= \boxed{\ln(x^2+1) + C}$$

Problem 25. Evaluate $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

Let $x^2+4 = 4\tan^2\theta + 4 = 4\sec^2\theta$

$x^2 = 4\tan^2\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$$= \int \frac{1}{4\tan^2\theta\sqrt{\cancel{4}\sec^2\theta}}\cancel{2}\sec^{\cancel{2}}\theta\,d\theta$$

$$= \int \frac{\sec\theta}{4\tan^2\theta}\,d\theta$$

$$= \int \frac{\cos^{\cancel{2}}\theta}{4\sin^2\theta\cancel{\cos\theta}}\,d\theta$$

Let $u = \sin\theta$

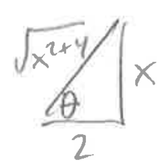
$du = \cos\theta\,d\theta$

$$= \int \frac{du}{4u^2}$$

$$= \int \frac{1}{4}u^{-2}\,d\theta$$

$$= -\frac{1}{4}u^{-1} + C$$

$$= -\frac{1}{4\sin\theta} + C$$



$$= \boxed{-\frac{\sqrt{x^2+4}}{4x} + C}$$

$\sin\theta = \frac{x}{\sqrt{x^2+4}}$

Problem 26. Evaluate $\int \frac{x}{\sqrt{x^2+4}}\,dx$.

Smart way:

Let $u=x^2+4$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$=\int \frac{1/2}{\sqrt{u}}\,dx$$

$$=\int \frac{1}{2}u^{-1/2}\,dx$$

$$=\frac{\cancel{1/2}\,u^{1/2}}{\cancel{1/2}}+C$$

$$=\boxed{\sqrt{x^2+4}+C}$$

Trig Way

Let $x^2+4=4\tan^2\theta+4=4\sec^2\theta$

$x^2=4\tan^2\theta$

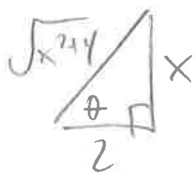
$x=2\tan\theta$

$dx=2\sec^2\theta\,d\theta$

$$=\int \frac{2\tan\theta}{\sqrt{\cancel{4}\sec^{\cancel{2}}\theta}}\cancel{2}\sec^{\cancel{2}}\theta\,d\theta$$

$$=\int 2\sec\theta\tan\theta\,d\theta$$

$$=2\sec\theta+C$$



$$=\cancel{2}\,\frac{\sqrt{x^2+4}}{\cancel{2}}+C$$

$$=\boxed{\sqrt{x^2+4}+C}$$

Problem ~~27.~~ Evaluate $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$.

Hard & Smart way:

Let $u=4x^2+9$

$x^2=\frac{u-9}{4}$

$du=8x\,dx$

$\frac{1}{8}du=x\,dx$

$$=\int_0^{3\sqrt{3}/2}\frac{x^2}{(4x^2+9)^{3/2}}\,x\,dx$$

$$=\int_{x=0}^{x=\frac{3\sqrt{3}}{2}}\frac{\frac{u-9}{4}}{u^{3/2}}\,\frac{1}{8}\,du$$

$$=\int_{x=0}^{x=\frac{3\sqrt{3}}{2}}\frac{u-9}{32u^{3/2}}\,du$$

$$=\int_{x=0}^{x=\frac{3\sqrt{3}}{2}}\frac{1}{32}u^{-1/2}-\frac{9}{32}u^{-3/2}\,du$$

$$=\left[\frac{1}{16}u^{1/2}+\frac{9}{16}u^{-1/2}\right]_{x=0}^{x=\frac{3\sqrt{3}}{2}}$$

$$=\left[\frac{\sqrt{4x^2+9}}{16}+\frac{9}{16\sqrt{4x^2+9}}\right]_0^{3\sqrt{3}/2}$$

$$=\left(\frac{\sqrt{4(\frac{27}{4})+9}}{16}+\frac{9}{16\sqrt{4(\frac{27}{4})+9}}\right)-\left(\frac{\sqrt{9}}{16}+\frac{9}{16\sqrt{9}}\right)$$

$$=\left(\frac{6}{16}+\frac{9}{16(6)}\right)-\left(\frac{3}{16}+\frac{9}{16(3)}\right)$$

$$=\frac{36+9-18-18}{16\cdot 6}=\frac{9}{16\cdot 6}=\frac{3}{16\cdot 2}$$

$$=\boxed{3/32}$$

Trig Way

Let $4x^2+9=9\tan^2\theta+9=9\sec^2\theta$

$4x^2=9\tan^2\theta$

$2x=3\tan\theta$

$x=\frac{3}{2}\tan\theta$

$dx=\frac{3}{2}\sec^2\theta\,d\theta$

$$=\int_{x=0}^{x=\frac{3\sqrt{3}}{2}}\frac{\frac{27}{8}\tan^3\theta}{(9\sec^2\theta)^{3/2}}\,\frac{3}{2}\sec^2\theta\,d\theta$$

$$=\int_{x=0}^{x=\frac{3\sqrt{3}}{2}}\frac{\cancel{27}/8\,\tan^3\theta}{\cancel{27}\sec^{\cancel{3}}\theta}\,3/2\sec^{\cancel{2}}\theta\,d\theta$$

$$=\frac{3}{16}\int_{x=0}^{x=3\sqrt{3}/2}\frac{\tan^3\theta}{\sec\theta}\,d\theta$$

$$=\frac{3}{16}\int_{x=0}^{x=3\sqrt{3}/2}\frac{\tan\theta(\sec^2\theta-1)}{\sec\theta}\,d\theta$$

$$=\frac{3}{16}\int_{x=0}^{x=\frac{3\sqrt{3}}{2}}\sec\theta\tan\theta-\sin\theta\,d\theta$$

$$=\left[\frac{3}{16}(\sec\theta+\cos\theta)\right]_{x=0}^{x=\frac{3\sqrt{3}}{2}}$$

$=$ etc.

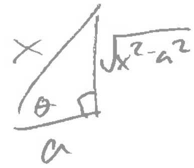
Problem 28. Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}}$ for $a > 0$.

Let $x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

$x^2 = a^2\sec^2\theta$

$x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$



$\sec\theta = \frac{x}{a} = \frac{\text{hyp}}{\text{adj}}$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{x^2-a^2}}{a}$

$$= \int \frac{\cancel{a}\sec\theta\,\cancel{\tan\theta}}{\cancel{\sqrt{a^2\tan^2\theta}}}\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C}$$

$$= \ln\left|x + \sqrt{x^2-a^2}\right| - \cancel{\ln a} + C$$

$$= \boxed{\ln\left|x + \sqrt{x^2+a^2}\right| + C}$$

Problem 29. Evaluate $\displaystyle\int \frac{1}{\sqrt{3 - 2x - x^2}}\, dx$. *Hint* Complete the square.

$$= \int \frac{1}{\sqrt{4-(x+1)^2}}\, dx$$

Let $4-(x+1)^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$

$(x+1)^2 = 4\sin^2\theta$

$x+1 = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$$= \int \frac{1}{\cancel{\sqrt{4\cos^2\theta}}}\cancel{2\cos\theta}\, d\theta$$

$$= \int 1\, d\theta$$

$$= \theta + C$$

$\left(\sin\theta = \frac{x+1}{2} \qquad \theta = \operatorname{Arcsin}\left(\frac{x+1}{2}\right)\right)$

$$= \boxed{\operatorname{Arcsin}\left(\frac{x+1}{2}\right)}$$

Suggested Homework: Section 7.3 numbers 2, 4, 5, 7, 9, 10, 11, 16, 22