Chapter 11

Sequences and Series

11.0 Prepositional Logic

All of the definitions in this section are adaptations of Irving M. Copi's book *Symbolic Logic*.

Definition 1 (Proposition)**.** A **proposition** is a statement which is either true or false.

Example 2. Britney is a goat. This statement has a definite truth value. It is either true or false, whether or not one can tell the truth value is a different story.

Definition 3 (Negation)**.** The **negation** of a statement P is a statement denoted $\neg P$ which has the opposite truth value of P.

Definition 4 (Argument)**.** An **argument** is a group of propositions, one of which is claimed to follow from another, providing grounds for truth.

Definition 5 (Structure of an Argument)**.** An argument is normally presented as a **conditional statement**. That is, it is of the form "If something is a car then it is a vehicle." The statement that goes with the "If" clause is called the **hypothesis** while the statement that goes with the "Then" clause is called the **conclusion**. For ease of notation, we typically call the hypothesis P and the conclusion Q and denote the argument "If P then Q" by $P \Rightarrow Q$. Statements of this form are false only when the premise is true and the conclusion is false. Another way of saying $P \Rightarrow Q$ is "P implies Q"

Definition 6 (Truth Table)**.** A **truth table** is an array which lists all of the possible truth values for a given argument.

Definition 7. The truth table for "P and Q" ($P \wedge Q$), "P or Q" ($P \vee Q$), and "P implies Q" ($P \Rightarrow Q$) is:

P	Q	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Definition 8 (Converse)**.** The **converse** of a conditional statement $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.

Definition 9 (Contrapositive)**.** The **contrapositive** of a conditional statement $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$.

Definition 10 (Logical Equivalence)**.** Two propositions are called (logically) **equivalent** if given the truth values of all subpropositions, both propositions share the same truth value.

Problem 11. Show that $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ are equivalent.

Problem 12. Show that $P \Rightarrow Q$ is not equivalent to $Q \Rightarrow P$.

Problem 13. Give a "real life example" of propositions P, Q such that $P \Rightarrow Q$, (and thus $\neg Q \Rightarrow \neg P$,) but $Q \not\Rightarrow P$.

Problem 14. Give a "mathematical example" of propositions P, Q such that $P \Rightarrow Q$, (and thus $\neg Q \Rightarrow \neg P$,) but $Q \not\Rightarrow P$.

11.1 Sequences

Definition 15 (Sequence)**.** [1][2] A **sequence** is a function whose domain is a final set of integers. If s is a sequence, we usually write its value at n as s_n instead of $s(n)$. We may denote a sequence as $(s_0, s_1, \dots)$ or $\{s_3, s_4, \dots\}$ or $\langle s_{-1}, s_0, \dots\rangle$ or $\{s_n\}_{n=1}^{\infty}$ or simply s_n (depending on its domain). If its domain is not given, we usually assume it to be $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{W} = \{0, 1, \dots\}$, or some other final set of integers which is always defined for the sequence definition.

Problem 16. Write the first five terms of the following sequences:

- $\left\{\dfrac{n}{n+1}\right\}$
- $\left(\dfrac{(-1)^n (n+1)}{3^n}\right)_{n=0}^{\infty}$
- $\left\langle \sqrt{n-3} \right\rangle$
- $\left\{\cos\left(\dfrac{n\pi}{6}\right)\right\}_{n=1}^{\infty}$

Problem 17. Find a general formula for the sequence $\left\{\dfrac{3}{5}, \dfrac{-4}{25}, \dfrac{5}{125}, \dfrac{-6}{625}, \dfrac{7}{3125}, \dots\right\}$.

[1] Definition based on Steven R. Lay's book *Analysis With an Introduction to Proof.*

[2] A final set of integers starts at some n, and contains every bigger integer. As examples: $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{W} = \{0, 1, 2, \dots\}$, $\{4, 5, 6, \dots\}$, $\{-2, -1, 0, 1, \dots\}$, etc.

Note 18. Some sequences do not have a simple defining equation.

Example 19. The n^{th} term of the decimals of e. The sequence of decimals of e look like $\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots\}$.

Note 20. On the other hand, there are sequences that do have a closed form definition that are simply not easy to find.

Example 21. The Fibbonacci Sequence is defined as $f_1 = f_2 = 1$ and for $n \geq 3, f_n = f_{n-1} + f_{n-2}$. This has a closed form definition of $\left(\frac{\varphi^n - \psi^n}{\sqrt{5}}\right)$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

Problem 22. Visualize the sequence $\left(\frac{n}{n+1}\right)$.

Problem 23. What, if anything, does it seem like the sequence $\left(\frac{n}{n+1}\right)$ is approaching?

Definition 24 (Limit of a Sequence)**.** A sequence (a_n) has the **limit** L if we can make the terms of (a_n) arbitrarily close to L as we like my taking n to be sufficiently large. If (a_n) has a limit L, then we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$.

Definition 25 (Convergence and Divergence)**.** If $\lim_{n\to\infty} a_n$ exists, then we say that the sequence **converges**. Otherwise, we say that the sequence **diverges** or is **divergent**.

Theorem 26 (Subset of a Continuous Function)**.** If $\lim_{x\to\infty} f(x) = L$, and $f(n) = a_n$ wherever n is in the domain of the sequence, then $\lim_{n\to\infty} a_n = L$.

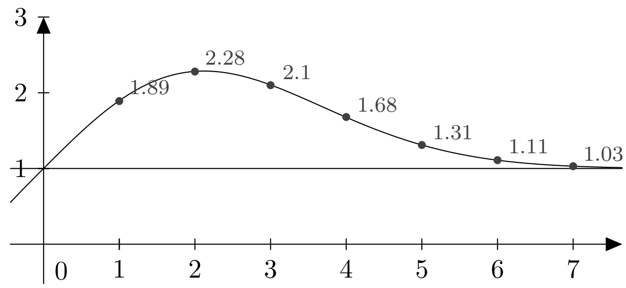

Figure 11.1: Sequence as a Subset of a Function

Corollary 27. If $\langle a_n\rangle_{n=N'}^{\infty}$ converges (diverges) for some choice of initial N', then $\langle a_n\rangle_{n=N}^{\infty}$ converges (diverges) for any choice of N where a_n is defined for all $n \geq N$.

Properties 28. If (a_n) and (b_n) are convergent sequences and $c \in \mathbb{R}$, then the following properties hold:

- $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$
- $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ as long as $\lim_{n\to\infty} b_n \neq 0$
- $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ for $p > 0$ and $a_n > 0$.

Theorem 29 (Squeeze Theorem)**.** If $a_n \leq b_n \leq c_n$ for all $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

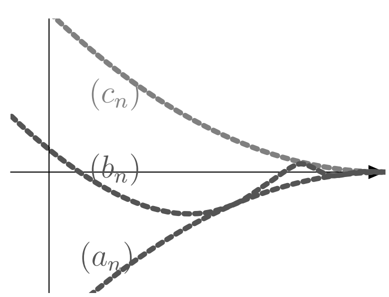

Figure 11.2: Squeeze Theorem

Corollary 30. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Problem 31. Determine whether the sequence $\left(\frac{n}{n+1}\right)$ is convergent or divergent. If it is convergent, what does it converge to?

Problem 32. Determine whether the sequence $\left(\frac{n}{\sqrt{10+n}}\right)$ is convergent or divergent. If it is convergent, what does it converge to?

Problem 33. Determine whether the sequence $((-1)^n)$ is convergent or divergent. If it is convergent, what does it converge to?

Problem 34. Determine whether the sequence $\left(\frac{\ln(n)}{n}\right)$ is convergent or divergent. If it is convergent, what does it converge to?

Problem 35. Determine whether the sequence $\left(\frac{(-1)^n}{n}\right)$ is convergent or divergent. If it is convergent, what does it converge to?

Theorem 36. If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Problem 37. Find $\lim_{n\to\infty} \sin(\pi n)$

Problem 38. Show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$. *Hint* show that $0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$

Theorem 39 (Preliminary for Geometric Series)**.** The sequence (r^n) is convergent if $-1 \leq r \leq 1$ and divergent otherwise.

Definition 40 (Monotone)**.** A sequence (a_n) is called **non-decreasing** if $a_n \leq a_{n+1}$ for all $n \geq 1$. Similarly, a sequence (a_n) is called **non-increasing** if $a_n \geq a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either non-decreasing or non-increasing.

Problem 41. Show that $\left(\frac{3}{n+5}\right)$ is decreasing.

Problem 42. Show that $\left(\dfrac{n}{n^2+1}\right)$ is decreasing.

Definition 43 (Bounded)**.** A sequence (a_n) is **bounded above** if there exists an $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \geq 1$. A sequence (a_n) is **bounded below** if there exists an $m \in \mathbb{R}$ such that $a_n \geq m$ for all $n \geq 1$. If a sequence is bounded above or bounded below then the sequence is said to be **bounded**.

Theorem 44. Every bounded monotonic sequence is convergent.

Suggested Problems: Section 11.1 numbers 5, 9, 13 – 15, 23 – 29, 33, 35, 37, 41, 42, 44, 49, 50, 53, 56

11.2 Series

Observation 45. What do we mean when we write $\pi = 3.1415926535\ldots$? It is a convenient way to write the following:

$$\pi = \frac{3}{10^0} + \frac{1}{10^1} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \frac{3}{10^9} + \frac{5}{10^{10}} + \cdots.$$

Definition 46 (Series)**.** Adding up the terms in an infinite sequence is a **series**. That is to say, given a sequence $(a_n)_{n=1}^{\infty}$, the series would be denoted as

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \ldots$$

Definition 47 (Partial Sum)**.** Let (a_n) be a sequence. The **partial sums** of the sequence are

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.
\end{aligned}$$

Definition 48 (Definition of Series)**.** If $s_n = \sum_{i=1}^{n} a_i$ is the n^{th} partial sum of the sequence $(a_i)_{i=1}^{\infty}$, then the **value** or **sum** of the series $\sum_{i=1}^{\infty} a_i$ is defined to be the limit of its sequence of partial sums:

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} s_n$$

whenever the limit exists.

Definition 49 (Series Convergence & Divergence)**.** The series $\sum_{i=1}^{\infty} a_i$ **converges** or **diverges** based on whether its sequence of partial sums

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} s_n$$

converges or diverges.

Problem 50. Determine whether or not the series $\sum_{i=1}^{\infty} a_i$ converges or diverges, given its n^{th} partial sum $s_n = a_1 + a_2 + \cdots + a_n = \dfrac{2n^2}{3n^2+5}$.

Note The above problem does not say anything about the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2}{3n^2+5}$.

Theorem 51 (Geometric Series)**.** The **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

The geometric series is divergent if $|r| \geq 1$.

Problem 52. Show that the geometric series converges when $|r| < 1$ and diverges otherwise.

Hint Show that its n^{th} partial sum is $s_n = a\frac{1-r^n}{1-r}$.

Problem 53. Find the sum of the series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

Problem 54. Is $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

Problem 55. Compute the sum of the series $\sum_{n=0}^{\infty} ar^n$ for $|r| < 1$.

Theorem 56 (Harmonic Series)**.** The **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

Problem 57. Show that the harmonic series diverges.

Problem 58. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)}$ converges and find its sum.

Hint Show that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

Theorem 59 (Divergence Test)**.** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Problem 60. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2}{3n^2+5}$ diverges.

Problem 61. Write the contrapositive of the Divergence Test.

Problem 62. Use the contrapositive of the Divergence Test to show that the sequence $\langle (0.6)^n \rangle$ converges.

Problem 63. Find the sum of $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n\,(n+1)} + \frac{1}{2^n} \right)$

Suggested Problems: Section 11.2 numbers 3, 15, 17, 18, 21, 27, 29, 30, 31, 33, 38, 45