7.4 Partial Fraction Decomposition

Strategy 30. If the degree in the numerator is greater than or equal to the degree in the denominator, use long division.

Problem 31. Evaluate $\int \frac{x^3+x^2-4}{x-1}\,dx$.

x^2+2x+2

$x-1\,\big)\,\cancel{x^3}+x^2+0x-4$

$-(\cancel{x^3}-x^2)$

$2\cancel{x^2}+0x$

$-(\cancel{2x^2}-2x)$

$2\cancel{x}-4$

$-(\cancel{2x}-2)$

-2

$$= \int x^2+2x+2-\frac{2}{x-1}\,dx$$

$$= \boxed{\frac{1}{3}x^3+x^2+2x-2\ln|x-1|+C}$$

Problem 32. Evaluate $\int \frac{x^4+1}{x^2+1}\,dx$.

$x^2 \quad -1$

$x^2+0x+1\,\big)\,\cancel{x^4}+0x^3+0x^2+0x+1$

$-(\cancel{x^4}+x^2)$

$-\cancel{x^2}+1$

$-(-\cancel{x^2}-1)$

2

$$= \int x^2-1+\frac{2}{x^2+1}\,dx$$

$$= \boxed{\frac{1}{3}x^3-x+2\,\mathrm{Arctan}(x)+C}$$

Theorem 33 (Fundamental Theorem of Algebra). Every polynomial of real numbers is factorable into linear terms $(ax+b)$ and irreducible quadratics (ax^2+bx+c).

Recall 34. To add fractions, we find a common denominator:

$$\frac{2}{x}+\frac{1}{x+1}=\frac{2(x+1)}{x(x+1)}+\frac{1(x)}{(x+1)(x)}=\frac{2x+2}{x(x+1)}+\frac{x}{x(x+1)}=\frac{2x+2+x}{x(x+1)}=\frac{3x+2}{x^2+x}.$$

More work is required to undo this process.

Strategy 35. If the denominator is the product of distinct linear factors, the fraction may be split into the sum of fractions with constant numerators and distinct linear factor denominators:

$$\frac{-x^2+14x+6}{2x^3+7x^2+3} = \frac{A}{x} + \frac{B}{2x+1} + \frac{C}{x+3}$$

$$-x^2+14x+6 = A(2x+1)(x+3) + Bx(x+3) + Cx(2x+1)$$

$$(-1)x^2 + (14)x + (6) = (2A+B+2C)x^2 + (7A+3B+C)x + (3A)$$

$$-1 = 2A+B+2C \qquad 14 = 7A+3B+C \qquad 6 = 3A$$

Problem 36. Evaluate $\displaystyle\int \frac{-x^2+14x+6}{2x^3+7x^2+3}\,dx$.

$A=2$

$-1 = 4+B+2C \to -1 = 4+B+2(-3B) \to -5 = -5B \to B=1$

$14 = 14+3B+C \to C=-3B \to C=-3$

$= \displaystyle\int \frac{2}{x} + \frac{1}{2x+1} - \frac{3}{x+3}\,dx$

$= \boxed{2\ln|x| + \frac{1}{2}\ln|2x+1| - 3\ln|x+3| + C}$

Problem 37. Evaluate $\int \frac{14x^2-8x+5}{(x+1)(1-2x)(x-2)}\,dx$. $= \int \frac{-3}{x+1}+\frac{-2}{1-2x}+\frac{-5}{x-2}\,dx$

$= \boxed{-3\ln|x+1|+\ln|1-2x|-5\ln|x-2|+C}$

$$\frac{14x^2-8x+5}{(x+1)(1-2x)(x-2)} = \frac{A}{x+1}+\frac{B}{1-2x}+\frac{C}{x-2}$$

$$14x^2-8x+5 = A(1-2x)(x-2)+B(x+1)(x-2)+C(x+1)(1-2x)$$

$$14x^2-8x+5 = -2Ax^2+5Ax-2A+Bx^2-Bx-2B-2Cx^2-Cx+C$$

x^2: $14=-2A+B-2C \rightarrow 6=3A-3C \rightarrow A=C+2 \rightarrow A=-7-2B$ $\quad A=-3 \quad C=-5$

x: $-8=5A-B-C \rightarrow$

Const: $5=-2A-2B+C \rightarrow 5=-2C-4-2B+C$

$9=-C-2B$

$C=-9-2B$

$14=-2(-7-2B)+B-2(-9-2B)$

$14=14+4B+B+18+4B$

$-18=9B$

$-2=B$

Problem 38. Find the value of the series $\sum_{n=2}^{\infty}\frac{1}{n^2+5n+6}$ by using partial fraction decomposition to compute the partial sums.

$$\frac{1}{(n+2)(n+3)} = \frac{A}{n+2}+\frac{B}{n+3}$$

$$1 = A(n+3)+B(n+2)$$

$$0n+1 = (A+B)n+(3A+2B)$$

$0=A+B \rightarrow A=-B \rightarrow A=1$

$1=3A+2B \rightarrow 1=-3B+2B$

$B=-1$

$$\sum_{n=2}^{\infty}\frac{1}{n+2}-\frac{1}{n+3} = \lim_{n\to\infty}\left(\left(\frac{1}{2+2}-\cancel{\frac{1}{2+3}}\right)+\left(\cancel{\frac{1}{3+2}}+\cancel{\frac{1}{3+3}}\right)+\cancel{\cdots}+\cancel{\left(\frac{1}{n+2}\right.}-\left.\frac{1}{n+3}\right)\right)$$

$$= \lim_{n\to\infty}\frac{1}{4}-\frac{1}{n+3}$$

$$= \boxed{\frac{1}{4}}$$

Strategy 39. If the denominator has a repeated linear factor, use an additional fraction with a constant numerator and a higher power for each repeated linear factor:

$$\frac{4x}{x^3-x^2-x+1}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

Problem 40. Evaluate $\displaystyle\int \frac{4x}{x^3-x^2-x+1}\,dx.$ $= \displaystyle\int \frac{-1}{x+1}+\frac{1}{x-1}+\frac{2}{(x-1)^2}\,dx$

$$x^3-x^2-x+1 = x^2(x-1)-1(x-1) = (x^2-1)(x-1) = (x+1)(x-1)^2$$

$$= \boxed{-\ln|x+1|+\ln|x-1|-\frac{2}{x-1}+C}$$

$$\frac{4x}{(x+1)(x-1)^2}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$4x = A(x-1)^2+B(x+1)(x-1)+C(x+1)$$

$$0x^2+4x+0 = Ax^2-2Ax+A+Bx^2-B+Cx+C$$

x^2: $0=A+B \to A=-B$ $\to 4=-2A+2 \to A=-1$

x: $4=-2A+C \to 4=2B+C \to 4=2C \to C=2$ $B=1$

Const: $0=A-B+C \to 0=-2B+C$

Problem 41. Evaluate $\displaystyle\int \frac{6x^2-7x-2}{x^3-2x^2}\,dx.$ $= \displaystyle\int \frac{4}{x}+\frac{1}{x^2}+\frac{2}{x-2}\,dx$

$$= \boxed{4\ln|x|-\frac{1}{x}+2\ln|x-2|+C}$$

$$\frac{6x^2-7x-2}{x^2(x-2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}$$

$$6x^2-7x-2 = A(x)(x-2)+B(x-2)+Cx^2$$

$$6x^2-7x-2 = Ax^2-2Ax+Bx-2B+Cx^2$$

x^2: $6=A+C \to 6=4+C \to C=2$

x: $-7=-2A+B \to -7=-2A+1 \to A=4$

Const: $-2=-2B \to B=1$

Strategy 42. If the denominator has an irreducible quadratic, use a linear factor for the numerator:

$$\frac{3x^2 - x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Problem 43. Evaluate $\int \frac{3x^2 - x + 4}{x^3 + 4x}\, dx$. $= \int \frac{1}{x} + \frac{2x-1}{x^2+4}\, dx$

$= \int \frac{1}{x} + \frac{2x}{x^2+4} - \frac{1}{x^2+4}\, dx$

$= \boxed{\ln|x| + \ln|x^2+4| - \frac{1}{2}\operatorname{Arctan}\left(\frac{x}{2}\right) + C}$

$\frac{3x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

$3x^2-x+4 = A(x^2+4) + (Bx+C)x$

$3x^2-x+4 = Ax^2 + 4A + Bx^2 + Cx$

$x^2: 3 = A + B \rightarrow B = 2$

$x: -1 = C$

Const: $4 = 4A \rightarrow A = 1$

$\int \frac{1}{x^2+4}\, dx$

Let $x^2+4 = 4\tan^2\theta + 4 = 4\sec^2\theta$

$x^2 = 4\tan^2\theta$

$x = 2\tan\theta \rightarrow \tan\theta = \frac{x}{2}$

$dx = 2\sec^2\theta\, d\theta$

$= \int \frac{1}{\cancel{4}_2 \cancel{\sec^2\theta}} \cancel{2}\cancel{\sec^2\theta}\, d\theta$

$= \int \frac{1}{2}\, d\theta$

$= \frac{1}{2}\theta + C$

$= \frac{1}{2}\operatorname{Arctan}\left(\frac{x}{2}\right) + C$

Strategy 44. If the denominator has a repeated irreducable quadractic factor, use an additional fraction with a linear numerator and a higher power for each repeated irreducable quadratic:

$$\frac{2 - 2x + 4x^2 - 2x^3}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

Problem 45. Evaluate $\int \frac{2 - 2x + 4x^2 - 2x^3}{x(x^2 + 1)^2}\, dx$ $= \int \frac{2}{x} + \frac{-2x-2}{x^2+1} + \frac{2x}{(x^2+1)^2}\, dx$

$-2x^3 + 4x^2 - 2x + 2 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)x$

$0x^4 - 2x^3 + 4x^2 - 2x + 2 = Ax^4 + 2Ax^2 + A + Bx^4 + Cx^3 + Bx^2 + Cx + Dx^2 + Ex$

$x^4: 0 = A + B \rightarrow B = -2$

$x^3: -2 = C$

$x^2: 4 = 2A + B + D \longrightarrow 4 = 4 + -2 + D$

$x: -2 = C + E \rightarrow E = 0$ $\quad D = 2$

Const: $2 = A$

$= \boxed{2\ln|x| - \ln|x^2+1| - 2\operatorname{Arctan}(x) - \frac{1}{x^2+1} + C}$

Suggested Homework: Section 7.4 numbers 2, 3, 12, 15, 16, 17, 18, 19, 26, 27