Chapter 7

Techniques of Integration

7.1 Integration by Parts

Problem 1. Prove that $\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$. *Hint* Use the product rule and work backwards.

$\frac{d}{dx}[f(x)g(x)] = g(x)f'(x) + f(x)g'(x)$

$\frac{d}{dx}[f(x)g(x)] - g(x)f'(x) = f(x)g'(x)$

$f(x)g(x) - \int g(x)f'(x)dx = \int f(x)g'(x)dx$

Theorem 2 (Integration by Parts). Given two continuous, differentiable functions $f(x)$ and $g(x)$,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

If $u = f(x)$ and $v = g(x)$, then we can write this as

$$\int u\,dv = uv - \int v\,du$$

Problem 3. Evaluate $\int \underbrace{x}_{u} \underbrace{\sin(x)\,dx}_{dv}$.

$u = x \qquad v = -\cos x$

$du = dx \qquad dv = \sin x\,dx$

$= -x\cos x + \int + \cos x\,dx$

$= \boxed{-x\cos x + \sin x + C}$

Problem 4. Evaluate $\int \underbrace{\ln(x)}_{u} \underbrace{dx}_{dv}$.

$u = \ln x \qquad v = x$
$du = \frac{1}{x}dx \qquad dv = dx$

$= x\ln x - \int \cancel{x}\frac{1}{\cancel{x}}dx$
$= \boxed{x\ln x - x + C}$

Problem 5. Evaluate $\int \underbrace{t^2}_{u} \underbrace{e^t\, dt}_{dv}$.

$u = t^2 \qquad v = e^t$
$du = 2t\,dt \qquad dv = e^t dt$

$= t^2e^t - \int \underbrace{2t}_{u} \underbrace{e^t dt}_{dv}$

$u = 2t \qquad v = e^t$
$du = 2\,dt \qquad dv = e^t dt$

$= t^2e^t + \left(-2te^t + \int 2e^t dt\right)$
$= \boxed{t^2e^t - 2te^t + 2e^t + C}$

Problem 6. Evaluate $\int_0^1 \underbrace{\arctan(x)}_{u}\,\underbrace{dx}_{dv}$.

$u = \arctan x \qquad v = x$

$du = \frac{1}{1+x^2}\,dx \qquad dv = dx$

$= \left[x\arctan x - \int \frac{x}{1+x^2}\,dx\right]_0^1$

$= \left[x\arctan x - \frac{1}{2}\ln|1+x^2|\right]_0^1$

$= \left(1\arctan 1 - \frac{1}{2}\ln(2)\right) - \cancel{\left(0 - \frac{1}{2}\ln 1\right)}^{\,0}$

$= \boxed{\frac{\pi}{4} - \frac{1}{2}\ln 2}$

Problem 7. Evaluate $\int \underbrace{e^x}_{u}\,\underbrace{\sin(x)\,dx}_{dv}$.

$u = e^x \qquad v = -\cos x$

$dv = e^x dx \qquad dv = \sin x\,dx$

$= -e^x\cos x + \int e^x \cos x\,dx$

$u = e^x \qquad v = \sin x$

$dv = e^x dx \qquad dv = \cos x\,dx$

$= -e^x\cos x + e^x\sin x - \int e^x \sin x\,dx$

$2\int e^x \sin x\,dx = -e^x\cos x + e^x \sin x + C$

$\int e^x \sin x\,dx = \boxed{\frac{-e^x\cos x + e^x\sin x}{2} + C}$

Suggested Homework: Section 7.1 numbers 1 – 4, 7, 10 – 12, 21, 24, 29, 30, 31

7.2 Trigonometric Integrals

7.2.1 Products of Powers of Sine and Cosine

Strategy 8. There are three types of integrals of the form $\int \sin^m(x)\cos^n(x)dx$:

I. **The power on $\sin(x)$ is odd.**
Apply $\sin^{2n+1}(x) = (\sin^2(x))^n \sin(x) = (1-\cos^2(x))^n \sin(x)$ and use the substitution $u = \cos(x)$.

II. **The power on $\cos(x)$ is odd.**
Apply $\cos^{2n+1}(x) = (\cos^2(x))^n \cos(x) = (1-\sin^2(x))^n \cos(x)$ and use the substitution $u = \sin(x)$.

III. **Both powers are even.**
Apply both $\cos^{2n}(x) = \left(\frac{1+\cos(2x)}{2}\right)^n$ and $\sin^{2n}(x) = \left(\frac{1-\cos(2x)}{2}\right)^n$ to reduce the exponents in the integral.

Problem 9. Evaluate $\displaystyle\int \cos^3(x)\,dx$.

$= \int (1-\sin^2 x)\cos x\,dx$ (Let $u = \sin x$)

$= \int 1-u^2\,du$

$= u - \frac{1}{3}u^3 + C$

$= \boxed{\sin x - \frac{1}{3}\sin^3 x + C}$

Problem 10. Evaluate $\displaystyle\int \sin^5(x)\cos^2(x)\,dx$.

$= \int (1-\cos^2 x)^2 \cos^2 x \sin x\,dx$ (Let $u = \cos x$)

$= \int (1-u^2)^2 u^2 (-du)$

$= \int -u^6 + 2u^4 - u^2\,du$

$= -\frac{u^7}{7} + \frac{2u^5}{5} - \frac{u^3}{3} + C = \boxed{\frac{-\cos^7 x}{7} + \frac{2\cos^5 x}{5} - \frac{\cos^3 x}{3} + C}$

Problem 11. Evaluate $\int_0^{\frac{\pi}{4}} \sin^2(x)\,dx$.

$$= \int_0^{\pi/4} \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$$

$$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi/4}$$

$$= \left(\pi/8 - \frac{1}{4}\sin\frac{\pi}{2}\right) - \cancel{\left(0 - \frac{1}{4}\sin 0\right)}^{0}$$

$$= \boxed{\frac{\pi}{8} - \frac{1}{4}}$$

Problem 12. Evaluate $\int \sin^4(x)\,dx$.

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^2 dx$$

$$= \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x \, dx$$

$$= \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \int \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x \, dx$$

$$= \boxed{\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C}$$

7.2.2 Products of Powers of Tangent and Secant

Strategy 13. To evaluate an integral of the form $\int \tan^m(x)\sec^n(x)\,dx$:

- If n is even,
  - Save a factor of $\sec^2(x)$ and use $\sec^2(x) = 1 + \tan^2(x)$ on the rest.
  - Use the u substitution $u = \tan(x)$.
- If m is odd,
  - Save a factor of $\sec(x)\tan(x)$ and use $\tan^2(x) = \sec^2(x) - 1$ on the rest.
  - Use the u substitution $u = \sec(x)$.

Problem 14. Evaluate $\int \tan^6(x)\sec^4(x)\,dx$.

$$= \int \tan^6 x\,(1 + \tan^2 x)\sec^2 x\,dx$$ (Let $u = \tan x$)

$$= \int u^6(1 + u^2)\,du$$

$$= \int u^6 + u^8\,du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C = \boxed{\frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C}$$

Problem 15. Evaluate $\int \tan^5(\theta)\sec^7(\theta)\,d\theta$.

$$= \int (\sec^2\theta - 1)^2 \sec^6\theta \sec\theta \tan\theta \, d\theta$$

(let $u = \sec\theta$)

$$= \int (u^2-1)^2 u^6 \, du$$

$$= \int u^{10} - 2u^8 + u^6 \, du$$

$$= \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C$$

$$= \boxed{\frac{\sec^{11}x}{11} - \frac{2\sec^9 x}{9} + \frac{\sec^7 x}{7} + C}$$

Recall 16. $\int \tan(x)\,dx = \ln|\sec(x)| + c$ and $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + c$

Problem 17. Evaluate $\int \tan^3(x)\,dx$.

$$= \int (\sec^2 x - 1)\tan x \, dx$$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$$= \boxed{\frac{1}{2}\tan^2 x - \ln|\sec x| + C}$$

Problem 18. Use Integration by Parts to evaluate $\int \sec^3(x)\,dx$.

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$\left(\begin{array}{ll} u = \sec x & v = \tan x \\ du = \sec x \tan x\,dx & dv = \sec^2 x\,dx \end{array}\right)$$

$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$$

$$= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx$$

$$= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x\,dx = \boxed{\frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C}$$

Recall 19.

$$\begin{aligned}
\sin(A)\cos(B) &= \tfrac{1}{2}\left[\sin(A-B) + \sin(A+B)\right] \\
\sin(A)\sin(B) &= \tfrac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \\
\cos(A)\cos(B) &= \tfrac{1}{2}\left[\cos(A-B) + \cos(A+B)\right].
\end{aligned}$$

Problem 20. Evaluate $\int \sin(4x)\cos(5x)\,dx$.

$$= \int \frac{1}{2}\left[\sin(-x) + \sin(9x)\right]dx$$

$$= \boxed{\frac{1}{2}\cos(-x) - \frac{1}{18}\cos(9x) + C}$$

Suggested Homework: Section 7.2 numbers 1, 3, 5 – 7, 10, 11, 15, 21, 23, 25, 27, 29, 38