Chapter 8

Further Applications of Integrals

8.1 Arc Length

Theorem 1 (Arc Length)**.** If $\frac{df}{dx}$ is continuous on $[a, b]$, then the length of the curve $y = f(x)$ where $a \le x \le b$ is

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (\Delta f)^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta f}{\Delta x}\right)^2}\, \Delta x = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$$

Problem 2. Prove that the circumference of a circle with radius r is $C = 2\pi r$.

Problem 3. Find the length of the arc on the curve $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.

Problem 4. Find the length of the arc of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

Theorem 5 (Arc Length Function)**.** If $\frac{df}{dx}$ is continuous, then the **arc length function** with initial point $(a, f(a))$ for the curve $y = f(x)$ is

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, dt$$

Problem 6. Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln(x)$ taking $(1, 1)$ as the initial point.

Suggested Problems Section 8.1 numbers 1, 2, 5, 7, 8, 10, 11, 13, 14, 19, 20, 35

8.2 Area of a Surface of Revolution

Theorem 7 (Surface Area)**.** Let f be a positive function with continuous derivative. Then the area of the surface obtained by rotating the curve $y = f(x)$ from $a \leq x \leq b$ about the x-axis is

$$SA = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

Problem 8. Prove that the surface area of a sphere with radius r is given by $SA = 4\pi r^2$

Problem 9. Find the area of the surface generated by rotating the arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ about the y-axis.

Problem 10. Find the area of the surface generated by rotating $y = e^x$ from $0 \leq x \leq 1$ about the x-axis.

Problem 11 (Gabriel's Horn)**.** Show that the solid obtained by rotating the region bounded by the curve $y = \frac{1}{x}$ and lines $y = 0$, $x = 1$ about the x-axis has finite volume but infinite surface area.

Suggested Homework: Section 8.2 numbers 5, 6, 7, 9, 13, 14, 16