11.10 Taylor and Maclaurin Series

Theorem 117 (Taylor Series). If f has a power series representation at a (that is $f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$ for $|x - a| < R$) then its coefficients are of the form

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

That is to say that if f has a power series representation then it can be written in the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

and has radius of convergence R.

Definition 118 (Maclaurin Series). A Taylor Series centered at $a = 0$ is called a **Maclaurin Series**. That is to say that a Maclaurin Series can be written as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Problem 119. Find the Maclaurin Series of $f(x) = e^x$ and find its radius of convergence.

$f^{(0)}(x) = e^x$
$f^{(1)}(x) = e^x$

$f^{(n)}(x) = e^x \to f^{(n)}(0) = e^0 = 1$

$$e^x = f(x) = \sum_{n=0}^{\infty} \frac{\cancel{f^{(n)}(0)}^{\,1}}{n!} x^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}} \quad \left(= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right)$$

Radius of Conv.

Ratio Test

$$\lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim_{n\to\infty} \frac{|x|^{\cancel{n+1}}}{(n+1)\cancel{n!}} \frac{\cancel{n!}}{\cancel{|x|^n}} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$ for all x. Conv. for "$|x| < \infty$".

$\boxed{\text{Radius is "infinite"}}$

Problem 120. Find the Maclaurin Series for $\sin(x)$.

$$f^{(0)}(x) = \sin x \to f^{(0)}(0) = 0$$
$$f^{(1)}(x) = \cos x \to f^{(1)}(0) = 1$$
$$f^{(2)}(x) = -\sin x \to f^{(2)}(0) = 0$$
$$f^{(3)}(x) = -\cos x \to f^{(3)}(0) = -1$$

(Since evens are all zero, need only odd terms in series)

($\langle 1, -1, 1, -1, \dots \rangle$ is alt. sequence $(-1)^n$)

$$\sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}} \quad \left(= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right)$$

Problem 121. Find the Maclaurin Series for $\cos(x)$.

$$f^{(0)}(x) = \cos x \to f^{(0)}(0) = 1$$
$$f^{(1)}(x) = -\sin x \to f^{(1)}(0) = 0$$
$$f^{(2)}(x) = -\cos x \to f^{(2)}(0) = -1$$
$$f^{(3)}(x) = \sin x \to f^{(3)}(0) = 0$$

$$\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}} \quad \left(= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right)$$

Problem 122. Find a power series representing $f(x) = xe^x$.

$$xe^x = x\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}}$$

Problem 123. Give a polynomial which approximates $\sin(x)$ near $x = \frac{\pi}{3}$ by finding the Taylor Series for $\sin(x)$ centered at $\frac{\pi}{3}$ and then writing out its first four non-zero terms. (This is sometimes called a Taylor Polynomial.)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{3})}{n!}\left(x-\frac{\pi}{3}\right)^n = f^{(0)}\left(\frac{\pi}{3}\right)\cancel{\left(x-\frac{\pi}{3}\right)^0} + f^{(1)}\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right)^1 + \frac{f^{(2)}(\frac{\pi}{3})}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{f^{(3)}(\frac{\pi}{3})}{3!}\left(x-\frac{\pi}{3}\right)^3$$

$$= \boxed{\frac{\sqrt{3}}{2} + \frac{1}{2}\left(x-\frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}\left(x-\frac{\pi}{3}\right)^2 - \frac{1}{12}\left(x-\frac{\pi}{3}\right)^3}$$

$f^{(0)}(x) = \sin x \rightarrow f^{(0)}\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$f^{(1)}(x) = \cos x \rightarrow f^{(1)}\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$f^{(2)}(x) = -\sin x \rightarrow f^{(2)}\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$f^{(3)}(x) = -\cos x \rightarrow f^{(3)}\left(\frac{\pi}{3}\right) = -\frac{1}{2}$

Suggested Problems: Section 11.10 numbers 3, 4, 6 – 8, 15, 16, 27, 28, 30, 32, 36