Chapter 6

Applications of Integrals

6.1 Area Between Curves

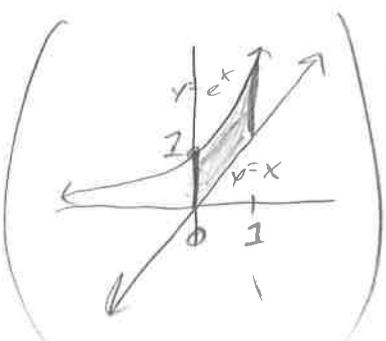
6.1.1 Area with Respect to the x-axis

Recall 1. If $f(x)$ is a continuous function and $a \leq b$, then the definite integral $\int_a^b f(x)\,dx$ represents the "net area" between the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$. ("Net area" means the area above the x-axis minus the area below the x-axis.)

Theorem 2. If f, g are continuous functions of x such that $f(x) \leq g(x)$ for all $a \leq x \leq b$, then the area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ is

$$A = \int_a^b \underbrace{g(x)}_{\text{Top}} - \underbrace{f(x)}_{\text{Bottom}}\,dx$$

Problem 3. Find the area of the region bounded above by $y = e^x$, below by $y = x$, and on the sides by $x = 0$ and $x = 1$.



$$A = \int_0^1 e^x - x\,dx$$

$$= \left[e^x - \frac{x^2}{2}\right]_0^1$$

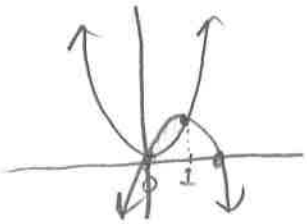
$$= \left(e - \frac{1}{2}\right) - (1 - 0)$$

$$= \boxed{e - \frac{3}{2}}$$

$(\approx 1.2\text{ish})$

Problem 4. Find the area bounded by $y = x^2$ and $y = 2x - x^2$.

$$\left(\begin{aligned} x^2 &= 2x - x^2 \\ 2x^2 - 2x &= 0 \\ 2x(x-1) &= 0 \end{aligned}\right)$$
$x=0$ $x=1$



$$A = \int_0^1 \overset{\text{Top}}{(2x - x^2)} - \overset{\text{Bottom}}{(x^2)}\,dx$$

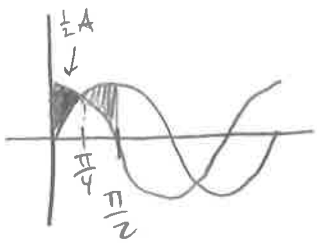
$$= \int_0^1 2x - 2x^2\,dx$$

$$= \left[x^2 - \frac{2}{3}x^3\right]_0^1$$

$$= \left(1 - \tfrac{2}{3}\right) - \cancel{(0-0)}$$

$$= \boxed{1/3}$$

Problem 5. Find the area bounded by $y = \sin(x)$ and $y = \cos(x)$ from $x = 0$ to $x = \frac{\pi}{2}$.



$$\frac{1}{2}A = \int_0^{\pi/4} \cos x - \sin x\,dx$$

$$= \Big[\sin x + \cos x\Big]_0^{\pi/4}$$

$$= \left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)\right]$$

$$\frac{1}{2}A = \sqrt{2} - 1$$

$$A = \boxed{2\sqrt{2} - 2}$$

$$\left(\text{OR}\quad A = \int_0^{\pi/4} \cos x - \sin x\,dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x\,dx = 2\sqrt{2} - 2\right)$$

6.1.2 Area with Respect to the y-axis

Theorem 6. If f, g are continuous functions of y such that $f(y) \leq g(y)$ for all $c \leq y \leq d$, then the area A of the region bounded by the curves $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$ is

$$A = \int_c^d g(y) - f(y)\, dy$$

$y = \sqrt{2x+6}$

Problem 7. Find the area enclosed by $y = x - 1$ and $y^2 = 2x + 6$.

$x = y+1 \qquad x = \frac{y^2-6}{2}$

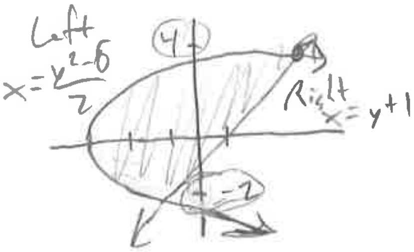
$y+1 = \frac{y^2-6}{2}$

$2y+2 = y^2-6$

$0 = y^2-2y-8$

$0 = (y-4)(y+2)$

$y=4 \quad y=-2$



$$A = \int_{-2}^{4} (y+1) - \left(\frac{y^2-6}{2}\right) dy$$

$$= \int_{-2}^{4} -\frac{y^2}{2} + y + 4\, dy$$

$$= \left[-\frac{y^3}{6} + \frac{y^2}{2} + 4y\right]_{-2}^{4}$$

$$= \left(-\frac{64}{6} + 8 + 8\right) - \left(\frac{8}{6} + 2 - 8\right)$$

$$= -\frac{72}{6} + 22$$

$$= -12 + 22$$

$$= \boxed{10}$$

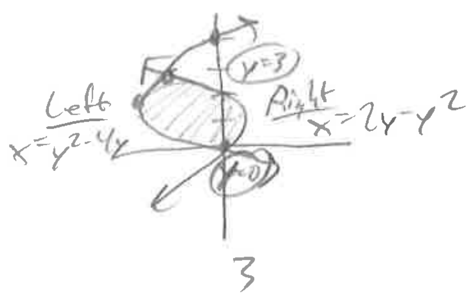
Problem 8. Find the area enclosed by $x = 2y - y^2$ and $x = y^2 - 4y$.

$2y - y^2 = y^2 - 4y$

$0 = 2y^2 - 6y$

$0 = 2y(y-3)$

$y=0 \quad y=3$



$$A = \int_0^3 (2y-y^2) - (y^2-4y)\, dy$$

$$= \int_0^3 -2y^2 + 6y\, dy$$

$$= \left[-\frac{2}{3}y^3 + 3y^2\right]_0^3$$

$$= \left(-\frac{2}{3}(27) + 27\right) \cancel{- (0+0)}$$

$$= \boxed{9}$$

Suggested Homework: Section 6.1 numbers 1, 4, 12, 24, 27, 44, 50

6.2 Volumes by Cross-Sections

Definition 9. The three-dimensional solid obtained by moving a planar shape along a line perpendicular to the plane is called a **cylinder**.

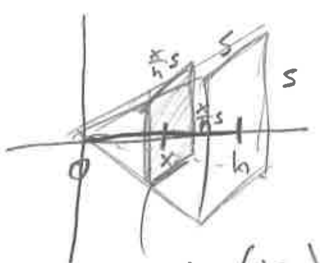
Definition 10. The volume V of a cylinder with base area B and height h is defined to be

$$V = Bh$$

Definition 11. The volume of a solid positioned between $x = a$ and $x = b$ with cross-sectional areas given by $A(x)$ for each x-value between a and b is defined to be

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_{i,n})\,\Delta x_n = \int_a^b A(x)\,dx$$

Problem 12. Show that the volume V of a pyramid with height h and a square base with side length s is $V = \frac{1}{3}s^2h$.



$$A(x) = \left(\frac{x}{h}s\right)^2$$

$$= \frac{s^2}{h^2}x^2$$

$$V = \int_0^h \left(\frac{s^2}{h^2}x^2\right)dx$$

$$= \frac{s^2}{h^2}\int_0^h x^2\,dx$$

$$= \frac{s^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h$$

$$= \frac{1}{3}\frac{s^2}{\cancel{h^2}}h^{\cancel{3}}$$

$$\boxed{V = \frac{1}{3}s^2h}$$

Definition 13. A **solid of revolution** is the result of rotating a shape around a line (called the **axis of revolution**).

Theorem 14 (Disc Method). Suppose that a solid of revolution is formed from a shape positioned flush against a horizontal or vertical axis of revolution.

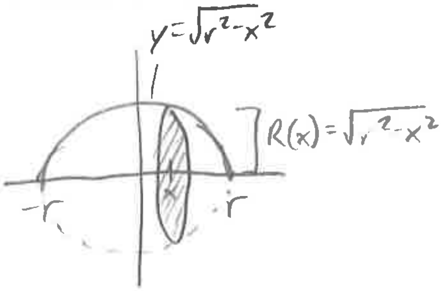
If the axis of revolution is horizontal, and $R(x)$ gives the distance from the axis to the outside of the shape being rotated for each value of x, then the volume of the solid of revolution from $x = a$ to $x = b$ is

$$V = \int_a^b \pi[R(x)]^2\,dx$$

If the axis of revolution is vertical, and $R(y)$ gives the distance from the axis to the outside of the shape being rotated for each value of y, then the volume of the solid of revolution from $y = c$ to $y = d$ is

$$V = \int_c^d \pi[R(y)]^2\,dy$$

Problem 15. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.



$$V=\int_{-r}^{r}\pi\left(\sqrt{r^2-x^2}\right)^2dx$$

$$=\int_{-r}^{r}\pi r^2-\pi x^2\,dx$$

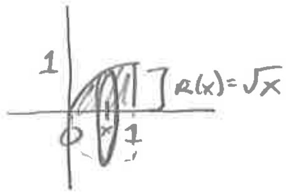
$$=\left[\pi r^2x-\pi\frac{x^3}{3}\right]_{-r}^{r}$$

$$=\left(\pi r^3-\pi\frac{r^3}{3}\right)-\left(-\pi r^3+\pi\frac{r^3}{3}\right)$$

$$=2\pi r^3-\frac{2}{3}\pi r^3$$

$$\boxed{V=\frac{4}{3}\pi r^3}$$

Problem 16. Find the volume of the solid obtained by rotating the shape bounded by $y = \sqrt{x}$, $y = 0$, $x = 0$, and $x = 1$ about the x-axis.



$$V = \int_0^1 \pi (\sqrt{x})^2 \, dx$$

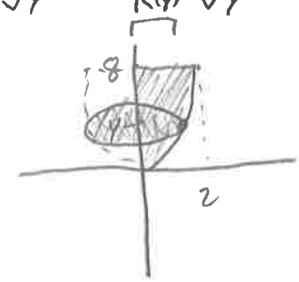
$$= \int_0^1 \pi x \, dx$$

$$= \left[\frac{\pi x^2}{2} \right]_0^1$$

$$= \boxed{\pi/2}$$

Problem 17. Find the volume of the solid obtained by rotating the shape bounded by $y = x^3$, $x = 0$, $y = 0$, and $y = 8$ about the y-axis.

$x = \sqrt[3]{y}$ $R(y) = \sqrt[3]{y}$



$$V = \int_0^8 \pi \left(y^{\frac{1}{3}}\right)^2 dy$$

$$= \int_0^8 \pi y^{2/3} \, dy$$

$$= \left[\pi \frac{3}{5} y^{5/3} \right]_0^8$$

$$= \pi \frac{3}{5} (8)^{5/3} \cancel{- 0}$$

$$= \pi \frac{3}{5} (32)$$

$$= \boxed{\frac{96\pi}{5}}$$

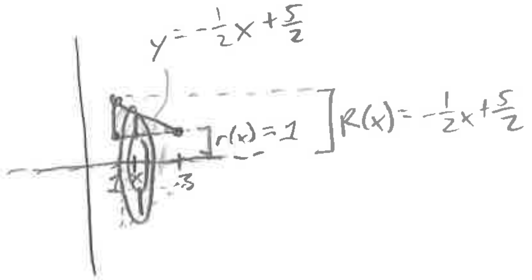
Theorem 18 (Washer Method)**.** Suppose that a solid of revolution is formed from a shape not positioned flush against a horizontal or vertical axis of revolution.

If the axis of revolution is horizontal, $R(x)$ gives the distance from the axis to the outside of the shape being rotated for each value of x, and $r(x)$ gives the distance from the axis to the inside of the shape being rotated for each value of x, then the volume of the solid of revolution from $x = a$ to $x = b$ is

$$V = \int_a^b \pi[R(x)]^2 - \pi[r(x)]^2 \, dx$$

(The similar formula works for y values and a vertial axis of revolution.)

Problem 19. Find the volume of the solid obtained by rotating the triangle with vertices at $(1, 1)$, $(3, 1)$, and $(1, 2)$ around the x-axis.



$$V = \int_1^3 \pi\left(-\tfrac{1}{2}x+\tfrac{5}{2}\right)^2 - \pi(1)^2 \, dx$$

$$= \int_1^3 \pi\left(\tfrac{1}{4}x^2 - \tfrac{5}{2}x + \tfrac{25}{4}\right) - \pi(1) \, dx$$

$$= \pi \int_1^3 \tfrac{1}{4}x^2 - \tfrac{5}{2}x + \tfrac{21}{4} \, dx$$

$$= \pi\left[\tfrac{1}{12}x^3 - \tfrac{5}{4}x^2 + \tfrac{21}{4}x\right]_1^3$$

$$= \pi\left[\left(\tfrac{9}{4} - \tfrac{45}{4} + \tfrac{63}{4}\right) - \left(\tfrac{1}{12} - \tfrac{5}{4} + \tfrac{21}{4}\right)\right]$$

$$= \pi\left[\tfrac{27}{4} - \tfrac{49}{12}\right]$$

$$= \boxed{\tfrac{8}{3}\pi}$$

Problem 20. Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ about the line $x = -1$.

$x = \sqrt{y}$

$$V = \int_0^1 \pi(\sqrt{y}+1)^2 - \pi(y+1)^2 \, dy$$

$$= \int_0^1 \pi(y + 2\sqrt{y} + 1) - \pi(y^2 + 2y + 1) \, dy$$

$$= \pi \int_0^1 -y^2 - y + 2y^{\frac{1}{2}} \, dy$$

$$= \pi\left[-\tfrac{1}{3}y^3 - \tfrac{1}{2}y^2 + \tfrac{4}{3}y^{3/2}\right]_0^1$$

$$= \pi\left(-\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{4}{3}\right)$$

$$= \boxed{\tfrac{\pi}{2}}$$

Suggested Homework: Section 6.2 numbers 4, 5, 12, 15, 19, 25, 42, 54, 56, 59

6.3 Volume by Cylindrical Shell

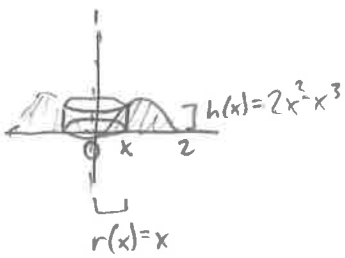
Theorem 21 (Shell Method). If the axis of revolution for a solid of revolution is vertical, and the cylindrical shell formed by rotating a vertical line segment within the shape at a fixed x value has height $h(x)$ and radius $r(x)$, then the volume of the solid of revolution is

$$V = \int_a^b 2\pi r(x)h(x)\,dx$$

If the axis of revolution for a solid of revolution is horizontal, and the cylindrical shell formed by rotating a horizontal line segment within the shape at a fixed y value has height $h(y)$ and radius $r(y)$, then the volume of the solid of revolution is

$$V = \int_c^d 2\pi r(y)h(y)\,dy$$

Problem 22. Find the volume of the solid obtained by rotating the shape with bounds $y = 2x^2 - x^3$ and $y = 0$ about the y-axis.



$$V = \int_0^2 2\pi(x)(2x^2-x^3)\,dx$$

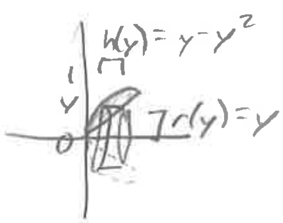
$$= \int_0^2 4\pi x^3 - 2\pi x^4\,dx$$

$$= \left[\pi x^4 - \frac{2}{5}\pi x^5\right]_0^2$$

$$= 16\pi - \frac{64}{5}\pi$$

$$= \boxed{\frac{16}{5}\pi}$$

Problem 23. Find the volume of the solid obtained by rotating the shape with bounds $x = y$ and $x = y^2$ about the x-axis.



$$V = \int_0^1 2\pi(y)(y-y^2)\,dy$$

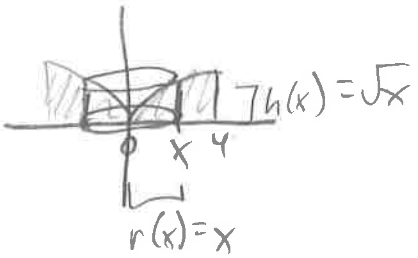
$$= \int_0^1 2\pi y^2 - 2\pi y^3\,dy$$

$$= \left[\frac{2}{3}\pi y^3 - \frac{1}{2}\pi y^4\right]_0^1$$

$$= \frac{2}{3}\pi - \frac{1}{2}\pi$$

$$= \boxed{\frac{\pi}{6}}$$

Problem 24. Find the volume of the solid obtained by rotating the shape with bounds $y = \sqrt{x}$, $y = 0$, and $x = 4$ about the y-axis.



$$V = \int_0^4 2\pi(x)(\sqrt{x})\,dx$$

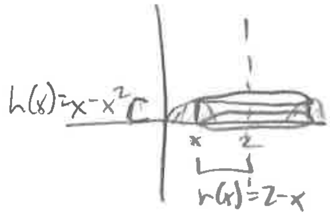
$$= \int_0^4 2\pi x^{3/2}\,dx$$

$$= \left[\frac{4}{5}\pi x^{5/2}\right]_0^4$$

$$= \frac{4}{5}\pi(2)^5$$

$$= \boxed{\frac{128}{5}\pi}$$

Problem 25. Find the volume of the solid obtained by rotating the shape with bounds $y = x - x^2$ and $y = 0$ about the line $x = 2$.



$$V = \int_0^1 2\pi(2-x)(x-x^2)\,dx$$

$$= \int_0^1 4\pi x - 6\pi x^2 + 2\pi x^3\,dx$$

$$= \left[2\pi x^2 - 2\pi x^3 + \frac{1}{2}\pi x^4\right]_0^1$$

$$= (\cancel{2\pi} - \cancel{2\pi} + \frac{1}{2}\pi) - \cancel{(0 - 0 + 0)}$$

$$= \boxed{\frac{\pi}{2}}$$

Suggested Homework: Section 6.3 numbers 2, 7, 9, 17, 19, 37