Chapter 7

Techniques of Integration

7.1 Integration by Parts

Problem 1. Prove that $\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$. *Hint* Use the product rule and work backwards.

Theorem 2 (Integration by Parts)**.** Given two continuous, differentiable functions $f(x)$ and $g(x)$,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

If $u = f(x)$ and $v = g(x)$, then we can write this as

$$\int u\,dv = uv - \int v\,du$$

Problem 3. Evaluate $\int x\sin(x)\,dx$.

Problem 4. Evaluate $\int \ln(x)\, dx$.

Problem 5. Evaluate $\int t^2 e^t\, dt$.

Problem 6. Evaluate $\int_0^1 \arctan(x)\,dx$.

Problem 7. Evaluate $\int e^x \sin(x)\,dx$.

Suggested Homework: Section 7.1 numbers $1-4$, 7, $10-12$, 21, 24, 29, 30, 31

7.2 Trigonometric Integrals

7.2.1 Products of Powers of Sine and Cosine

Strategy 8. There are three types of integrals of the form $\int \sin^m(x)\cos^n(x)dx$:

I. **The power on $\sin(x)$ is odd.**

Apply $\sin^{2n+1}(x) = (\sin^2(x))^n \sin(x) = (1-\cos^2(x))^n \sin(x)$ and use the substitution $u = \cos(x)$.

II. **The power on $\cos(x)$ is odd.**

Apply $\cos^{2n+1}(x) = (\cos^2(x))^n \cos(x) = (1-\sin^2(x))^n \cos(x)$ and use the substitution $u = \sin(x)$.

III. **Both powers are even.**

Apply both $\cos^{2n}(x) = \left(\frac{1+\cos(2x)}{2}\right)^n$ and $\sin^{2n}(x) = \left(\frac{1-\cos(2x)}{2}\right)^n$ to reduce the exponents in the integral.

Problem 9. Evaluate $\displaystyle\int \cos^3(x)\,dx$.

Problem 10. Evaluate $\displaystyle\int \sin^5(x)\cos^2(x)\,dx$.

Problem 11. Evaluate $\int_0^{\frac{\pi}{4}} \sin^2(x)\,dx$.

Problem 12. Evaluate $\int \sin^4(x)\,dx$.

7.2.2 Products of Powers of Tangent and Secant

Strategy 13. To evaluate an integral of the form $\int \tan^m(x)\sec^n(x)\,dx$:

- If n is even,
  - Save a factor of $\sec^2(x)$ and use $\sec^2(x) = 1 + \tan^2(x)$ on the rest.
  - Use the u substitution $u = \tan(x)$.
- If m is odd,
  - Save a factor of $\sec(x)\tan(x)$ and use $\tan^2(x) = \sec^2(x) - 1$ on the rest.
  - Use the u substitution $u = \sec(x)$.

Problem 14. Evaluate $\int \tan^6(x)\sec^4(x)\,dx$.

Problem 15. Evaluate $\int \tan^5(\theta) \sec^7(\theta)\, d\theta$.

Recall 16. $\int \tan(x)\, dx = \ln|\sec(x)| + c$ and $\int \sec(x)\, dx = \ln|\sec(x) + \tan(x)| + c$

Problem 17. Evaluate $\int \tan^3(x)\, dx$.

Problem 18. Use Integration by Parts to evaluate $\int \sec^3(x)\,dx$.

Recall 19.

$$\begin{aligned}
\sin(A)\cos(B) &= \tfrac{1}{2}\left[\sin(A-B)+\sin(A+B)\right]\\
\sin(A)\sin(B) &= \tfrac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]\\
\cos(A)\cos(B) &= \tfrac{1}{2}\left[\cos(A-B)+\cos(A+B)\right].
\end{aligned}$$

Problem 20. Evaluate $\int \sin(4x)\cos(5x)\,dx$.

Suggested Homework: Section 7.2 numbers 1, 3, 5 – 7, 10, 11, 15, 21, 23, 25, 27, 29, 38

7.3 Trigonometric Substitution

Strategy 21. With square roots and other troublesome factors, it sometimes helps to substitute trigonometric functions in order to use their identities for cancellation.

Expression	Substitution	Differential	Fact to Use
$a^2 - x^2$	$x = a\sin(\theta) \Rightarrow x^2 = a^2\sin^2(\theta)$	$dx = a\cos(\theta)d\theta$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$a^2 + x^2$	$x = a\tan(\theta) \Rightarrow x^2 = a^2\tan^2(\theta)$	$dx = a\sec^2(\theta)d\theta$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$x^2 - a^2$	$x = a\sec(\theta) \Rightarrow x^2 = a^2\sec(\theta)$	$dx = a\sec(\theta)\tan(\theta)d\theta$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

Problem 22. Prove $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$.

Problem 23. Evaluate $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

Problem 24. Evaluate $\displaystyle\int \frac{2x}{x^2+1}\,dx$.

Problem 25. Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

Problem 26. Evaluate $\int \frac{x}{\sqrt{x^2+4}}\,dx$.

Problem 27. Evaluate $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{\left(4x^2+9\right)^{3/2}}\,dx$.

Problem 28. Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$ for $a>0$.

Problem 29. Evaluate $\displaystyle\int \frac{1}{\sqrt{3-2x-x^2}}\,dx$. *Hint* Complete the square.

Suggested Homework: Section 7.3 numbers 2, 4, 5, 7, 9, 10, 11, 16, 22