7.8 Improper Integrals

Definition 46 (Improper Integral with Infinite Bounds). An integral with at least one infinite bound is computed as the limit of definite integrals:

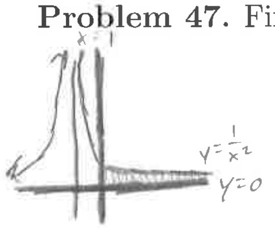
$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx \text{ for any value of } c$$

These equalities hold only when the given limits exist. In that case, the improper integral **converges**; otherwise, the improper integral **diverges**.

Problem 47. Find the area bounded by the curves $y = \frac{1}{x^2}$, $x = 1$, and $y = 0$.



$$\int_1^\infty \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \int_1^t x^{-2}\,dx$$

$$= \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t$$

$$= \cancel{\lim_{t\to\infty} -\frac{1}{t}} + \frac{1}{1}$$

$$= \boxed{1}$$

Problem 48. Determine whether $\int_1^\infty \frac{1}{2\sqrt{x}}\,dx$ is convergent or divergent. If it converges give its value.

$$= \lim_{t\to\infty} \int_1^t \frac{1}{2} x^{-1/2}\,dx$$

$$= \lim_{t\to\infty} \left[x^{1/2}\right]_1^t$$

$$= \lim_{t\to\infty} \sqrt{t} - \sqrt{1}$$

$\boxed{\text{diverges}}$ (to ∞)

Problem 49. Determine whether $\int_{-\infty}^{0} xe^x\,dx$ is convergent or divergent. If it converges give its value.

$$= \lim_{t\to-\infty} \int_t^0 \underbrace{x}_{u}\,\underbrace{e^x\,dx}_{dv}$$

$u = x \quad v = e^x$

$du = dx \quad dv = e^x dx$

$$= \lim_{t\to-\infty}\left[xe^x - \int e^x dx\right]_t^0$$

$$= \lim_{t\to-\infty}\left[xe^x - e^x\right]_t^0$$

$$= \lim_{t\to-\infty}\left[(0-1) - (te^t - e^t)\right]$$

$$= -1 - \lim_{t\to-\infty} te^t + \cancel{\lim_{t\to-\infty} e^t}^{\,0}$$

$$= -1 - \lim_{t\to-\infty}\frac{t}{e^{-t}}$$

$$\overset{LH}{=} -1 - \cancel{\lim_{t\to-\infty}\frac{1}{-e^{-t}}}$$

$$= \boxed{-1}$$

$\boxed{\text{Converges}}$

Problem 50. Compute $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx$.

$$= \int_{-\infty}^{0}\frac{1}{1+x^2}dx + \int_0^{\infty}\frac{1}{1+x^2}dx$$

$$= \lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}dx + \lim_{t\to\infty}\int_0^t\frac{1}{1+x^2}dx$$

$$= \lim_{t\to-\infty}\left[\text{Arctan}\,x\right]_t^0 + \lim_{t\to\infty}\left[\text{Arctan}\,x\right]_0^t$$

$$= \lim_{t\to-\infty}\left[\cancel{\text{Arctan}\,0} - \text{Arctan}\,t\right] + \lim_{t\to\infty}\left[\text{Arctan}\,t - \cancel{\text{Arctan}\,0}\right]$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$$

$$= \boxed{\pi}$$

$\boxed{\text{Converges}}$

Problem 51. Use the Integral Test to show that $\sum_{n=0}^{\infty}\frac{1}{1+n^2}$ converges.

$$\int_0^{\infty}\frac{1}{1+x^2}dx = \lim_{t\to\infty}\left[\text{Arctan}\,x\right]_0^t = \frac{\pi}{2} \quad \text{(See prev. problem)}$$

Since $\int_0^{\infty}\frac{1}{1+x^2} = \frac{\pi}{2}$, $\sum_{n=0}^{\infty}\frac{1}{1+n^2}$ $\boxed{\text{Converges}}$ (to something unknown).

(Converges)

Definition 52 (Improper Integral of Function Undefined within Interval). An integral of a function undefined for a point within the interval of integration is computed as the limit of definite integrals:

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \text{ for any value of } c$$

These equalities hold only when the given limits exist. In that case, the improper integral **converges**; otherwise, the improper integral **diverges**.

Problem 53. Determine whether $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ converges or diverges. If it converges, give its value.

$$= \lim_{t\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}}\,dx$$

$\left(\frac{1}{\sqrt{2-2}} \text{ DNE}\right)$

$$= \lim_{t\to 2^+}\left[2(x-2)^{\frac{1}{2}}\right]_t^5$$

$$= \lim_{t\to 2^+}\left(2\sqrt{5-2}\right)-\left(2\sqrt{t-2}\right)$$

$$= 2\sqrt{3} - \cancel{2\sqrt{2-2}}$$

$$= \boxed{2\sqrt{3}} \quad \boxed{\text{converges}}$$

Problem 54. Determine whether $\int_0^{\frac{\pi}{2}} \sec(x)\,dx$ converges or diverges. If it converges, give its value.

$$= \lim_{t\to \frac{\pi}{2}^-}\int_0^t \sec(x)\,dx$$

$\boxed{\text{diverges}}$

$\left(\frac{1}{\cos\frac{\pi}{2}} = \frac{1}{0} \text{ DNE}\right)$

$$= \lim_{t\to \frac{\pi}{2}^-}\Big[\ln\left|\sec(x)+\tan(x)\right|\Big]_0^t$$

$$= \lim_{t\to \frac{\pi}{2}^-} \ln\left|\sec(t)+\tan(t)\right| - \cancel{\ln\left|\overset{1}{\sec(0)}+\overset{0}{\tan(0)}\right|}^{\,0}$$

"$= \ln\left|\infty+\infty\right| = \infty$"

Problem 55. Determine whether $\int_0^1 \ln(x)\,dx$ converges or diverges. If it converges, give its value.

$\left(\begin{array}{c}\ln(0)\\ \text{ONE}\end{array}\right)$

$$= \lim_{t\to 0^+} \int_t^1 \ln(x)\,dx$$

$$= \lim_{t\to 0^+} \Big[x\ln|x| - x\Big]_t^1$$

$$= \lim_{t\to 0^+} \big(\cancel{1\ln|1|}^{\,0} - 1\big) - \big(t\ln|t| - t\big)$$

$$= -1 - \Big(\lim_{t\to 0^+} t\ln|t| - \overset{0}{\cancel{t}}\Big) \quad (0\cdot -\infty)$$

$$= -1 - \left(\lim_{t\to 0^+} \frac{\ln|t|}{1/t}\right) \quad \left(\frac{-\infty}{\infty}\right)$$

$$\overset{(L.H.)}{=} -1 - \left(\lim_{t\to 0^+} \frac{1/t}{-1/t^2}\right)$$

$$= -1 + \cancel{\lim_{t\to 0^+} t}^{\,0}$$

$$= \boxed{-1} \quad \boxed{\text{converges}}$$

Problem 56. Determine whether $\int_0^3 \frac{1}{x-1}\,dx$ converges or diverges. If it converges, give its value.

$\left(\begin{array}{c}\frac{1}{1-1}\\ \text{ONE}\end{array}\right)$

$$= \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx$$

$$= \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx + \lim_{t\to 1^-} \int_t^3 \frac{1}{x-1}\,dx$$

$$\lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \Big[\ln|x-1|\Big]_0^t$$

$$= \lim_{t\to 1^-} \Big[\ln|t-1| - \cancel{\ln|0-1|}^{\,0}\Big]$$

$$\text{“} = \ln|1-1| = \ln|0| = -\infty \text{”}$$

$\boxed{\text{diverges}}$

(Don't have to check 2nd integral if 1st diverges.)

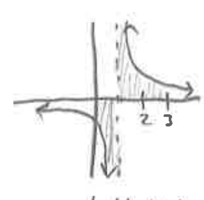
Tip! Avoid this!

$$\cancel{\int_0^3 \frac{1}{x-1}\,dx = \Big[\ln|x-1|\Big]_0^3}$$

$$\cancel{= \ln|3-1| - \ln|0-1|}$$

$$\cancel{= \ln 2}$$

Wrong because assumes infinite areas cancel:



$$-\infty + \infty + \ln 2 \neq \ln 2$$

Theorem 57 (Comparison Test for Integrals). Suppose that f and g are continuous functions with $0 \leq g(x) \leq f(x)$ for sufficiently large x.

- If the larger $\int_a^\infty f(x)\,dx$ is convergent, then the smaller $\int_a^\infty g(x)\,dx$ is also convergent.
- If the smaller $\int_a^\infty g(x)\,dx$ is divergent, then the larger $\int_a^\infty f(x)\,dx$ is also divergent.

Problem 58. Determine whether $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges or diverges.

$$\int_0^\infty e^{-x^2}dx = \underbrace{\int_0^1 e^{-x^2}dx}_{\text{finite}} + \underbrace{\int_1^\infty e^{-x^2}dx}_{?}$$

(Let $u = -x^2$
$du = -2x\,dx$
$-du = 2x\,dx$)

(Need to multiply by a $2x$.
Makes number bigger when $1 \leq x < \infty$.)

$$\int_1^\infty e^{-x^2}dx \leq \int_1^\infty 2xe^{-x^2}dx = \lim_{t\to\infty}\left[-e^{-x^2}\right]_1^t = \lim_{t\to\infty} \overset{0}{\cancel{-e^{-t^2}}} + e^{-1} = \frac{1}{e}$$

Since $\int_1^\infty 2xe^{-x^2}dx = \frac{1}{e}$ (converges), the smaller $\int_1^\infty e^{-x^2}dx$ $\boxed{\text{converges}}$ (between 0 and $\frac{1}{e}$).

Problem 59. Determine whether $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ converges or diverges.

$$\int_1^\infty \frac{1+e^{-x}}{x}dx \geq \int_1^\infty \frac{1\cancel{+e^{-x}}}{x}dx = \int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\left[\ln|x|\right]_1^t$$

(e^{-x} makes this complicated, but is a small #)

$$= \lim_{t\to\infty} \ln|t| \overset{0}{\cancel{-\ln|1|}}$$

$$\text{“} = \ln\infty = \infty \text{”}$$

diverges.

Since $\int_1^\infty \frac{1}{x}dx$ diverges, the bigger $\int_1^\infty \frac{1+e^{-x}}{x}dx$ also $\boxed{\text{diverges}}$.

Suggested Homework: Section 7.8 numbers 1, 7, 9, 13, 14 – 16, 18, 25, 27 – 33, 35, 49 – 52, 54, 55, 57