Chapter 6

Applications of Integrals

6.1 Area Between Curves

6.1.1 Area with Respect to the x-axis

Recall 1. If $f(x)$ is a continuous function and $a \leq b$, then the definite integral $\int_a^b f(x)\,dx$ represents the "net area" between the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$. ("Net area" means the area above the x-axis minus the area below the x-axis.)

Theorem 2. If f, g are continuous functions of x such that $f(x) \leq g(x)$ for all $a \leq x \leq b$, then the area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ is

$$A = \int_a^b g(x) - f(x)\,dx$$

Problem 3. Find the area of the region bounded above by $y = e^x$, below by $y = x$, and on the sides by $x = 0$ and $x = 1$.

Problem 4. Find the area bounded by $y = x^2$ and $y = 2x - x^2$.

Problem 5. Find the area bounded by $y = \sin(x)$ and $y = \cos(x)$ from $x = 0$ to $x = \frac{\pi}{2}$.

6.1.2 Area with Respect to the y-axis

Theorem 6. If f, g are continuous functions of y such that $f(y) \leq g(y)$ for all $c \leq y \leq d$, then the area A of the region bounded by the curves $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$ is

$$A = \int_c^d g(y) - f(y)\, dy$$

Problem 7. Find the area enclosed by $y = x - 1$ and $y^2 = 2x + 6$.

Problem 8. Find the area enclosed by $x = 2y - y^2$ and $x = y^2 - 4y$.

Suggested Homework: Section 6.1 numbers 1, 4, 12, 24, 27, 44, 50

6.2 Volumes by Cross-Sections

Definition 9. The three-dimensional solid obtained by moving a planar shape along a line perpendicular to the plane is called a **cylinder**.

Definition 10. The volume V of a cylinder with base area B and height h is defined to be

$$V = Bh$$

Definition 11. The volume of a solid positioned between $x = a$ and $x = b$ with cross-sectional areas given by $A(x)$ for each x-value between a and b is defined to be

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_{i,n})\,\Delta x_n = \int_a^b A(x)\,dx$$

Problem 12. Show that the volume V of a pyramid with height h and a square base with side length s is $V = \frac{1}{3}s^2h$.

Definition 13. A **solid of revolution** is the result of rotating a shape around a line (called the **axis of revolution**).

Theorem 14 (Disc Method)**.** Suppose that a solid of revolution is formed from a shape positioned flush against a horizontal or vertical axis of revolution.

If the axis of revolution is horizontal, and $R(x)$ gives the distance from the axis to the outside of the shape being rotated for each value of x, then the volume of the solid of revolution from $x = a$ to $x = b$ is

$$V = \int_a^b \pi[R(x)]^2\,dx$$

If the axis of revolution is vertical, and $R(y)$ gives the distance from the axis to the outside of the shape being rotated for each value of y, then the volume of the solid of revolution from $y = c$ to $y = d$ is

$$V = \int_c^d \pi[R(y)]^2\,dy$$

Problem 15. Show that the volume of a sphere of radius r is $V = \dfrac{4}{3}\pi r^3$.

Problem 16. Find the volume of the solid obtained by rotating the shape bounded by $y = \sqrt{x}$, $y = 0$, $x = 0$, and $x = 1$ about the x-axis.

Problem 17. Find the volume of the solid obtained by rotating the shape bounded by $y = x^3$, $x = 0$, $y = 0$, and $y = 8$ about the y-axis.

Theorem 18 (Washer Method)**.** Suppose that a solid of revolution is formed from a shape not positioned flush against a horizontal or vertical axis of revolution.

If the axis of revolution is horizontal, $R(x)$ gives the distance from the axis to the outside of the shape being rotated for each value of x, and $r(x)$ gives the distance from the axis to the inside of the shape being rotated for each value of x, then the volume of the solid of revolution from $x = a$ to $x = b$ is

$$V = \int_a^b \pi[R(x)]^2 - \pi[r(x)]^2 \, dx$$

(The similar formula works for y values and a vertial axis of revolution.)

Problem 19. Find the volume of the solid obtained by rotating the triangle with vertices at $(1, 1)$, $(3, 1)$, and $(1, 2)$ around the x-axis.

Problem 20. Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ about the line $x = -1$.

Suggested Homework: Section 6.2 numbers 4, 5, 12, 15, 19, 25, 42, 54, 56, 59