Chapter 8

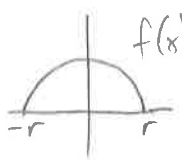
Further Applications of Integrals

8.1 Arc Length

Theorem 1 (Arc Length). If $\frac{df}{dx}$ is continuous on $[a, b]$, then the length of the curve $y = f(x)$ where $a \leq x \leq b$ is

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (\Delta f)^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta f}{\Delta x}\right)^2}\, \Delta x = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$$

Problem 2. Prove that the circumference of a semi circle with radius r is $C = \cancel{2}\pi r$.



$f(x) = \sqrt{r^2 - x^2}$

$f'(x) = \frac{-x}{\sqrt{r^2-x^2}}$

$$L = \int_{-r}^{r} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\, dx$$

(Trig sub)

$$= \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2-x^2}}\, dx = \int_{-r}^{r} \sqrt{\frac{r^2 \cancel{-x^2+x^2}}{r^2-x^2}}\, dx = \int_{-r}^{r} \frac{r}{\sqrt{r^2-x^2}}\, dx = \left[r \operatorname{Arcsin}\left(\frac{x}{r}\right)\right]_{-r}^{r}$$

$$= \left(r\frac{\pi}{2}\right) - \left(-r\frac{\pi}{2}\right) = \boxed{\pi r}$$

Problem 3. Find the length of the arc on the curve $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$. (Leave as integral.)

$f(x) = y = x^{3/2}$

$f'(x) = \frac{3}{2}x^{1/2}$

$$L = \int_1^4 \sqrt{1 + \left(\tfrac{3}{2}x^{1/2}\right)^2}\, dx = \boxed{\int_1^4 \sqrt{1 + \tfrac{9}{4}x}\, dx} = (\text{internet}) = \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$$

$$\approx 7.63$$

Problem 4. Find the length of the arc of the parabola $y^2 = x$ from $(0,0)$ to $(1,1)$.

$f(x) = y = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$$L = \int_0^1 \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx = \boxed{\int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx} = (\text{internet}) = \frac{1}{8}\left(4\sqrt{5} + \ln(9 + 4\sqrt{5})\right)$$

$$\approx 1.47$$

Theorem 5 (Arc Length Function). If $\frac{df}{dx}$ is continuous, then the **arc length function** with initial point $(a, f(a))$ for the curve $y = f(x)$ is

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, dt$$

Problem 6. Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln(x)$ taking $(1,1)$ as the initial point.

$f(x) = x^2 - \frac{1}{8}\ln(x)$

$f'(x) = 2x - \frac{1}{8x}$

$f'(t) = 2t - \frac{1}{8t}$

$$\boxed{s(x) = \int_1^x \sqrt{1 + \left(2t - \frac{1}{8t}\right)^2}\, dt}$$

(You'll learn better ways & applications for this in Cal III.)

Suggested Problems Section 8.1 numbers 1, 2, 5, 7, 8, 10, 11, 13, 14, 19, 20, 35

8.2 Area of a Surface of Revolution

Theorem 7 (Surface Area). Let f be a positive function with continuous derivative. Then the area of the surface obtained by rotating the curve $y = f(x)$ from $a \leq x \leq b$ about the x-axis is

$$SA = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

Problem 8. Prove that the surface area of a sphere with radius r is given by $SA = 4\pi r^2$

$y = \sqrt{r^2 - x^2} = f(x)$

$f'(x) = \frac{-x}{\sqrt{r^2-x^2}}$

$$A = \int_{-r}^{r} 2\pi\left(\sqrt{r^2-x^2}\right)\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx$$

$$= \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$

$$= \int_{-r}^{r} 2\pi\cancel{\sqrt{r^2-x^2}}\sqrt{\frac{r^2-x^2+x^2}{\cancel{r^2-x^2}}}\,dx$$

$$= \int_{-r}^{r} 2\pi\sqrt{r^2}\,dx$$

$$= \int_{-r}^{r} 2\pi r\,dx$$

$$= \left[2\pi r x\right]_{-r}^{r}$$

$$= (2\pi r^2) - (-2\pi r^2)$$

$$= \boxed{4\pi r^2}$$

Problem 9. Find the area of the surface generated by rotating the arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ about the y-axis.

$x = \sqrt{y}$

$f(y) = \sqrt{y}$

$f'(y) = \frac{1}{2\sqrt{y}}$

$$A = \int_c^d 2\pi f(y)\sqrt{1+(f'(y))^2}\,dy$$

$$= \int_1^4 2\pi\sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy$$

$$= \int_1^4 2\pi\sqrt{y}\sqrt{1+\frac{1}{4y}}\,dy$$

$$= \int_1^4 \cancel{2}\pi\cancel{\sqrt{y}}\,\frac{\sqrt{4y+1}}{\sqrt{\cancel{4}\cancel{y}}}\,dy$$

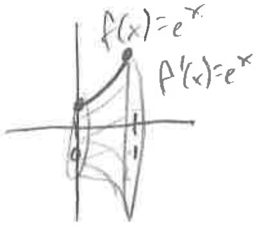
$$= \int_1^4 \pi\sqrt{4y+1}\,dy$$

$\left(\uparrow \text{ u-sub } u = 4y+1\right)$

$$= \left[\frac{\pi}{6}(4y+1)^{3/2}\right]_1^4$$

$$\boxed{\frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)}$$

Problem 10. Find the area of the surface generated by rotating $y = e^x$ from $0 \le x \le 1$ about the x-axis.



$$A = \boxed{\int_0^1 2\pi e^x \sqrt{1+e^{2x}}\,dx} \quad \text{Important part}$$

$u = e^x$

$$= \int_{x=0}^{x=1} 2\pi \sqrt{1+u^2}\,du$$

$\tan\theta = u$

$$= \int_{x=0}^{x=1} 2\pi \sec^3\theta\,d\theta$$

Int. by Parts w/ cycling

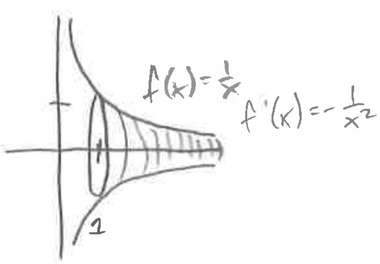
$$= \pi\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_{x=0}^{x=1}$$

Sub back for x

$$= \pi\left[\sqrt{1+e^{2x}}\,e^x + \ln\left|\sqrt{1+e^{2x}} + e^x\right|\right]_0^1$$

$$= \pi\left[\left(\sqrt{1+e^2}\,e + \ln\left|\sqrt{1+e^2}+e\right|\right) - \left(\sqrt{2} + \ln\left|\sqrt{2}+1\right|\right)\right] \approx 22.9$$

Problem 11 (Gabriel's Horn). Show that the solid obtained by rotating the region bounded by the curve $y = \frac{1}{x}$ and lines $y = 0$, $x = 1$ about the x-axis has finite volume but infinite surface area.



$$V = \int_1^\infty \pi\left[R(x)\right]^2dx$$

$$= \lim_{b\to\infty}\int_1^b \pi\left(\frac{1}{x}\right)^2dx$$

$$= \pi\lim_{b\to\infty}\int_1^b x^{-2}\,dx$$

$$= \pi\lim_{b\to\infty}\left[-x^{-1}\right]_1^b$$

$$= \pi\lim_{b\to\infty}\left(\cancel{-\frac{1}{b}}+\frac{1}{1}\right)$$

$$\boxed{V = \pi}$$

$$A = \int_1^\infty 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$= \int_1^\infty 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$$

$$\ge \int_1^\infty 2\pi\frac{1}{x}(1)\,dx$$

$$= 2\pi\lim_{b\to\infty}\int_1^b x^{-1}\,dx$$

$$= 2\pi\lim_{b\to\infty}\left[\ln x\right]_1^b$$

$$= 2\pi\lim_{b\to\infty}\left[\ln b - \cancel{\ln 1}\right)$$

$$\text{“}= 2\pi\,(\infty)\text{”}$$

$$\boxed{A \text{ is infinite}}$$

Suggested Homework: Section 8.2 numbers 5, 6, 7, 9, 13, 14, 16