Chapter 11

Sequences and Series

11.0 Prepositional Logic

All of the definitions in this section are adaptations of Irving M. Copi's book *Symbolic Logic*.

Definition 1 (Proposition)**.** A **proposition** is a statement which is either true or false.

Example 2. Britney is a goat. This statement has a definite truth value. It is either true or false, whether or not one can tell the truth value is a different story.

Definition 3 (Negation)**.** The **negation** of a statement P is a statement denoted $\neg P$ which has the opposite truth value of P.

Definition 4 (Argument)**.** An **argument** is a group of propositions, one of which is claimed to follow from another, providing grounds for truth.

Definition 5 (Structure of an Argument)**.** An argument is normally presented as a **conditional statement**. That is, it is of the form "If something is a car then it is a vehicle." The statement that goes with the "If" clause is called the **hypothesis** while the statement that goes with the "Then" clause is called the **conclusion**. For ease of notation, we typically call the hypothesis P and the conclusion Q and denote the argument "If P then Q" by $P \Rightarrow Q$. Statements of this form are false only when the premise is true and the conclusion is false. Another way of saying $P \Rightarrow Q$ is "P implies Q"

Definition 6 (Truth Table)**.** A **truth table** is an array which lists all of the possible truth values for a given argument.

Definition 7. The truth table for "P and Q" ($P \wedge Q$), "P or Q" ($P \vee Q$), and "P implies Q" ($P \Rightarrow Q$) is:

P	Q	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Definition 8 (Converse)**.** The **converse** of a conditional statement $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.

Definition 9 (Contrapositive)**.** The **contrapositive** of a conditional statement $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$.

Definition 10 (Logical Equivalence)**.** Two propositions are called (logically) **equivalent** if given the truth values of all subpropositions, both propositions share the same truth value.

Problem 11. Show that $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ are equivalent.

Problem 12. Show that $P \Rightarrow Q$ is not equivalent to $Q \Rightarrow P$.

Problem 13. Give a "real life example" of propositions P, Q such that $P \Rightarrow Q$, (and thus $\neg Q \Rightarrow \neg P$,) but $Q \not\Rightarrow P$.

Problem 14. Give a "mathematical example" of propositions P, Q such that $P \Rightarrow Q$, (and thus $\neg Q \Rightarrow \neg P$,) but $Q \not\Rightarrow P$.

11.1 Sequences

Definition 15 (Sequence)**.** [1,2] A **sequence** is a function whose domain is a final set of integers. If s is a sequence, we usually write its value at n as s_n instead of $s(n)$. We may denote a sequence as $(s_0, s_1, \dots)$ or $\{s_3, s_4, \dots\}$ or $\langle s_{-1}, s_0, \dots\rangle$ or $\{s_n\}_{n=1}^{\infty}$ or simply s_n (depending on its domain). If its domain is not given, we usually assume it to be $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{W} = \{0, 1, \dots\}$, or some other final set of integers which is always defined for the sequence definition.

Problem 16. Write the first five terms of the following sequences:

- $\left\{\dfrac{n}{n+1}\right\}$
- $\left(\dfrac{(-1)^n (n+1)}{3^n}\right)_{n=0}^{\infty}$
- $\left\langle\sqrt{n-3}\right\rangle$
- $\left\{\cos\left(\dfrac{n\pi}{6}\right)\right\}_{n=1}^{\infty}$

Problem 17. Find a general formula for the sequence $\left\{\dfrac{3}{5}, \dfrac{-4}{25}, \dfrac{5}{125}, \dfrac{-6}{625}, \dfrac{7}{3125}, \dots\right\}$.

[1] Definition based on Steven R. Lay's book *Analysis With an Introduction to Proof.*

[2] A final set of integers starts at some n, and contains every bigger integer. As examples: $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{W} = \{0, 1, 2, \dots\}$, $\{4, 5, 6, \dots\}$, $\{-2, -1, 0, 1, \dots\}$, etc.

Note 18. Some sequences do not have a simple defining equation.

Example 19. The n^{th} term of the decimals of e. The sequence of decimals of e look like $\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots\}$.

Note 20. On the other hand, there are sequences that do have a closed form definition that are simply not easy to find.

Example 21. The Fibbonacci Sequence is defined as $f_1 = f_2 = 1$ and for $n \geq 3, f_n = f_{n-1} + f_{n-2}$. This has a closed form definition of $\left(\frac{\varphi^n - \psi^n}{\sqrt{5}}\right)$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

Problem 22. Visualize the sequence $\left(\frac{n}{n+1}\right)$.

Problem 23. What, if anything, does it seem like the sequence $\left(\frac{n}{n+1}\right)$ is approaching?

Definition 24 (Limit of a Sequence)**.** A sequence (a_n) has the **limit** L if we can make the terms of (a_n) arbitrarily close to L as we like my taking n to be sufficiently large. If (a_n) has a limit L, then we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$.

Definition 25 (Convergence and Divergence)**.** If $\lim_{n\to\infty} a_n$ exists, then we say that the sequence **converges**. Otherwise, we say that the sequence **diverges** or is **divergent**.

Theorem 26 (Subset of a Continuous Function)**.** If $\lim_{x\to\infty} f(x) = L$, and $f(n) = a_n$ wherever n is in the domain of the sequence, then $\lim_{n\to\infty} a_n = L$.

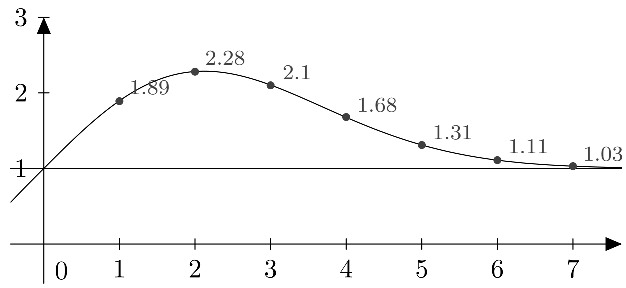

Figure 11.1: Sequence as a Subset of a Function

Corollary 27. If $\langle a_n\rangle_{n=N'}^{\infty}$ converges (diverges) for some choice of initial N', then $\langle a_n\rangle_{n=N}^{\infty}$ converges (diverges) for any choice of N where a_n is defined for all $n \geq N$.

Properties 28. If (a_n) and (b_n) are convergent sequences and $c \in \mathbb{R}$, then the following properties hold:

- $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$
- $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ as long as $\lim_{n\to\infty} b_n \neq 0$
- $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ for $p > 0$ and $a_n > 0$.

Theorem 29 (Squeeze Theorem)**.** If $a_n \leq b_n \leq c_n$ for all $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

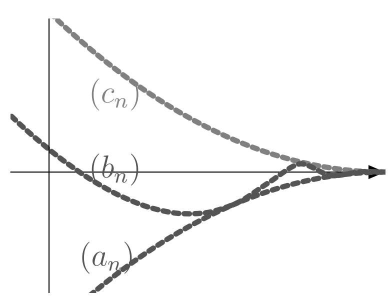

Figure 11.2: Squeeze Theorem

Corollary 30. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Problem 31. Determine whether the sequence $\left(\frac{n}{n+1}\right)$ is convergent or divergent. If it is convergent, what does it converge to?

Problem 32. Determine whether the sequence $\left(\frac{n}{\sqrt{10+n}}\right)$ is convergent or divergent. If it is convergent, what does it converge to?

Problem 33. Determine whether the sequence $((-1)^n)$ is convergent or divergent. If it is convergent, what does it converge to?

Problem 34. Determine whether the sequence $\left(\frac{\ln(n)}{n}\right)$ is convergent or divergent. If it is convergent, what does it converge to?

Problem 35. Determine whether the sequence $\left(\frac{(-1)^n}{n}\right)$ is convergent or divergent. If it is convergent, what does it converge to?

Theorem 36. If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Problem 37. Find $\lim_{n\to\infty} \sin(\pi n)$

Problem 38. Show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$. *Hint* show that $0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$

Theorem 39 (Preliminary for Geometric Series)**.** The sequence (r^n) is convergent if $-1 \leq r \leq 1$ and divergent otherwise.

Definition 40 (Monotone)**.** A sequence (a_n) is called **non-decreasing** if $a_n \leq a_{n+1}$ for all $n \geq 1$. Similarly, a sequence (a_n) is called **non-increasing** if $a_n \geq a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either non-decreasing or non-increasing.

Problem 41. Show that $\left(\frac{3}{n+5}\right)$ is decreasing.

Problem 42. Show that $\left(\dfrac{n}{n^2+1}\right)$ is decreasing.

Definition 43 (Bounded)**.** A sequence (a_n) is **bounded above** if there exists an $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \geq 1$. A sequence (a_n) is **bounded below** if there exists an $m \in \mathbb{R}$ such that $a_n \geq m$ for all $n \geq 1$. If a sequence is bounded above or bounded below then the sequence is said to be **bounded**.

Theorem 44. Every bounded monotonic sequence is convergent.

Suggested Problems: Section 11.1 numbers 5, 9, 13 – 15, 23 – 29, 33, 35, 37, 41, 42, 44, 49, 50, 53, 56

11.2 Series

Observation 45. What do we mean when we write $\pi = 3.1415926535\ldots$? It is a convenient way to write the following:

$$\pi = \frac{3}{10^0} + \frac{1}{10^1} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \frac{3}{10^9} + \frac{5}{10^{10}} + \cdots .$$

Definition 46 (Series)**.** Adding up the terms in an infinite sequence is a **series**. That is to say, given a sequence $(a_n)_{n=1}^{\infty}$, the series would be denoted as

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \ldots$$

Definition 47 (Partial Sum)**.** Let (a_n) be a sequence. The **partial sums** of the sequence are

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i. \end{aligned}$$

Definition 48 (Definition of Series)**.** If $s_n = \sum_{i=1}^{n} a_i$ is the n^{th} partial sum of the sequence $(a_i)_{i=1}^{\infty}$, then the **value** or **sum** of the series $\sum_{i=1}^{\infty} a_i$ is defined to be the limit of its sequence of partial sums:

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} s_n$$

whenever the limit exists.

Definition 49 (Series Convergence & Divergence)**.** The series $\sum_{i=1}^{\infty} a_i$ **converges** or **diverges** based on whether its sequence of partial sums

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} s_n$$

converges or diverges.

Problem 50. Determine whether or not the series $\sum_{i=1}^{\infty} a_i$ converges or diverges, given its n^{th} partial sum $s_n = a_1 + a_2 + \cdots + a_n = \dfrac{2n^2}{3n^2+5}$.

Note The above problem does not say anything about the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2}{3n^2+5}$.

Theorem 51 (Geometric Series)**.** The **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

The geometric series is divergent if $|r| \geq 1$.

Problem 52. Show that the geometric series converges when $|r| < 1$ and diverges otherwise.

Hint Show that its n^{th} partial sum is $s_n = a\frac{1-r^n}{1-r}$.

Problem 53. Find the sum of the series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

Problem 54. Is $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

Problem 55. Compute the sum of the series $\sum_{n=0}^{\infty} ar^n$ for $|r| < 1$.

Theorem 56 (Harmonic Series)**.** The **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

Problem 57. Show that the harmonic series diverges.

Problem 58. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)}$ converges and find its sum.

Hint Show that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

Theorem 59 (Divergence Test)**.** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Problem 60. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2}{3n^2+5}$ diverges.

Problem 61. Write the contrapositive of the Divergence Test.

Problem 62. Use the contrapositive of the Divergence Test to show that the sequence $\langle (0.6)^n \rangle$ converges.

Problem 63. Find the sum of $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n\,(n+1)} + \frac{1}{2^n} \right)$

Theorem 64. If $\displaystyle\sum_{i=1}^{\infty} a_i$ converges to S, then $\displaystyle\sum_{i=1}^{\infty} ca_i$ converges to cS for all real numbers c. If $\displaystyle\sum_{i=1}^{\infty} a_i$ diverges, then $\displaystyle\sum_{i=1}^{\infty} ca_i$ diverges for all real numbers $c \neq 0$.

Suggested Problems: Section 11.2 numbers 3, 15, 17, 18, 21, 27, 29, 30, 31, 33, 38, 45

11.3 The Integral Test

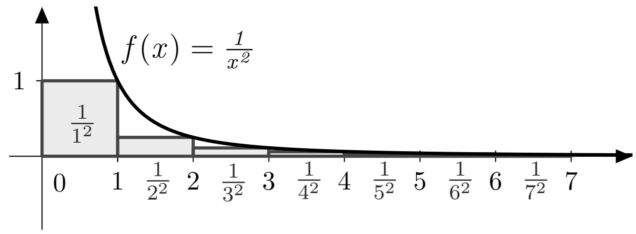

Figure 11.3: Integral Test

Theorem 65. Suppose that f is a continuous, positive, non-increasing function on $[1, \infty)$ such that $a_n = f(n)$. If $\lim_{t\to\infty} \int_1^t f(x)\,dx$ exists, then $\sum_{n=1}^{\infty} a_n$ converges. Otherwise, $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 66 (p-Series)**.** The p**-Series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Problem 67. Determine whether or not the following is convergent or divergent:

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$.
- $\displaystyle\sum_{n=1}^{\infty} n^{-4/3}$

Theorem 68. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges.

No Suggested Problems.

11.4 The Comparison Test

Theorem 69 (Direct Comparison Test)**.** Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

- If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ is also convergent.
- If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

Theorem 70. If $0 < q < 1$ and $Q > 1$, then the following inequalities hold for sufficiently large n:

$$\frac{1}{n^n} < \frac{1}{n!} < \frac{1}{Q^n} < \frac{1}{n^Q} < \frac{1}{n} < \frac{1}{n^q} < 1 < n^q < n < n^Q < Q^n < n! < n^n$$

Problem 71. Use the Direct Comparison Test to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

Problem 72. Use the Direct Comparison Test to determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{n^2 + 1}{n^3 - 3}$ converges or diverges.

Theorem 73 (Limit Comparison Test)**.** Suppose that $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ are series with positive terms and $c \in \mathbb{R}$. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$

then either both series converge or both series diverge.

Problem 74. Use the Limit Comparison Test to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges or diverges.

Problem 75. Use the Limit Comparison Test to determine whether $\displaystyle\sum_{n=0}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

Suggested Problems: Section 11.4 numbers 3, 4, 5, 7, 14, 15, 17, 21, 23, 29, 30

11.5 Alternating Series

Theorem 76 (Alternating Series Test)**.** If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ where each term in the sequence (b_n) is positive satisfies

- (b_n) is non-increasing
- $\lim\limits_{n\to\infty} b_n = 0$

then the series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges.

Problem 77. Determine whether the **alternating harmonic series** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges or diverges.

Problem 78. Determine whether $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ converges or diverges.

Problem 79. Determine whether $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 3n}{4n-1}$ converges or diverges.

Suggested Problems: Section 11.5 numbers 2 – 6, 8, 9, 11, 13, 17, 19

11.6 Absolute Convergence, Ratio, & Root Test

Definition 80 (Absolutely Convergent)**.** A series $\sum a_n$ is called **absolutely convergent** if the series $\sum |a_n|$ is convergent.

Theorem 81. Absolutely convergent series are convergent.

Definition 82 (Conditionally Convergent)**.** A series is called **conditionally convergent** if it is convergent but NOT absolutely convergent.

Problem 83. Determine whether whether or not $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent, conditionally convergent, or divergent.

Problem 84. Determine whether whether or not $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is absolutely convergent, conditionally convergent, or divergent.

Problem 85. Determine whether whether or not $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ is absolutely convergent, conditionally convergent, or divergent.

Theorem 86. The value of a conditionally convergent series can be changed to any real number by changing the order of its terms. The value of an absolutely convergent series cannot.

Theorem 87 (Ratio Test)**.** Let $\sum_{n=1}^{\infty} a_n$ be a series. Then

- if $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- if $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ diverges to ∞, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- if $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ then no conclusion can be drawn from this test.

Problem 88. Determine whether $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is convergent or divergent. Is it absolutely convergent?

Problem 89. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent or divergent. Is it absolutely convergent?

Theorem 90 (Root Test)**.** Let $\sum_{n=1}^{\infty} a_n$ be a series. Then

- If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ diverges to ∞, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then no conclusion can be drawn from this test.

Theorem 91. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Problem 92. Determine whether $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ is convergent or divergent. Is it absolutely convergent?

Problem 93. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^{2n}}$ is convergent or divergent. Is it absolutely convergent?

Suggested Problems: Section 11.6 numbers 3, 5, 7, 9, 10, 11 − 13, 16, 17, 19, 21, 23, 27, 28

11.7 Strategies for Testing Series

The only thing I really have to say here is that practice makes better. If you do enough problems, eventually you will get an intuition for what will work in what situation. Nevertheless, here is a list of tests that could come in handy.

Test	**When to Use**	**Conclusion**
Geometric Series	$\sum_{k=1}^{\infty} ar^k$	Converges to $\frac{a}{1-r}$ if $\lvert r\rvert < 1$; diverges if $\lvert r\rvert \geq 1$.
k^{th} **Term Test**	All Series	If $\lim_{k\to\infty} a_k \neq 0$, the series diverges.
Integral Test	$\sum_{k=1}^{\infty} a_k$ where $f(k) = a_k$ and f is continuous, decreasing, and $f(x) \geq 0$	$\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$ **both** converge or **both** diverge.
p**-series**	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	Converges for $p > 1$, diverges for $p \leq 1$.
Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $0 \leq a_k \leq b_k$	If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $a_k, b_k > 0$ and $\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ **both** converge or **both** diverge.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ where $a_k > 0$ for all k	If $\lim_{k\to\infty} a_k = 0$ and $a_{k+1} \leq a_k$ for all k, then the series converges.
Absolute Convergence	Series with some positive and some negative terms (including alternating series)	If $\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges, then $\sum_{k=1}^{\infty} a_k$ converges (absolutely).
Ratio Test	Any Series (especially those involving exponentials and/or factorials)	For $\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely, if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.
Root Test	Any Series (especially those involving exponentials)	For $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely, if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.

Problem 94. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$ converges or diverges.

Problem 95. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ converges or diverges.

Problem 96. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges.

Problem 97. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$ converges or diverges.

Problem 98. Determine whether the series $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ converges or diverges.

Problem 99. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ converges or diverges.

Suggested Problems: Section 11.7 numbers 1, 2, 5 − 9, 11, 13, 14, 15 − 18, 23, 25 − 34, 37

11.8 Power Series

Definition 100 (Power Series). A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 x^0 + c_1 x^1 + x_2 x^2 + \cdots$$

where c_i represent coefficients and x denotes our variable.

Definition 101 (Power Series Centered at a). A **power series centered at** a is a series of the form

$$\sum_{n=1}^{\infty} c_n (x-a)^n .$$

Note 102. When $x = a$, all of the terms are 0; so, naturally the power series centered at a always converges when $x = a$.

Problem 103. For what values of x is the series $\sum_{n=0}^{\infty} n! x^n$ convergent?

Problem 104. For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

Theorem 105 (Radius of Convergence)**.** For a given power series $\sum_{n=1}^{\infty} c_n (x-a)^n$, there are only three possibilities:

- The series converges only when $x = a$,
- The series converges for all $x \in \mathbb{R}$, and
- There is a positive real number R such that the series converges if $|x-a| < R$ and diverges when $|x-a| > R$.

This number R is called the **radius of convergence**.

Problem 106. Find the radius of convergence for the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+2}}$.

Problem 107. Find the radius of convergence for the series $\displaystyle\sum_{n=0}^{\infty} \frac{n (x+2)^n}{3^{n+1}}$

Suggested Problems: Section 11.8 numbers 3, 5, 7, 9, 10, 11, 13, 16, 18, 19

11.9 Representation of Functions as Power Series

Recall 108. The geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

converges when $|x| < 1$ and diverges otherwise.

Problem 109. Express $\dfrac{1}{1+x^2}$ as a power series and find its radius of convergence.

Problem 110. Find a power series representation for $\dfrac{1}{1-3x}$ and find its radius of convergence.

Problem 111. Find a power series representation of $\dfrac{1}{x+2}$ and find its radius of convergence.

Problem 112. Find a power series representation of $\dfrac{x^3}{1-x}$ and find its radius of convergence.

Theorem 113 (Integration and Differentiation)**.** If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

has the following derivative and integral defined within the same radius of convergence:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$$

$$\int f(x)\,dx = \left(c_0(x-a) + \frac{1}{2}c_1(x-a)^2 + \frac{1}{3}c_2(x-a)^3 + \dots\right) + C = \left(\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}\right) + C$$

Problem 114. Express $\dfrac{1}{(1-x)^2}$ as a power series.

Problem 115. Find a power series representation for $\ln(1+x)$.

Problem 116. Find a power series representation for $f(x) = \arctan(x)$

Suggested Problems: Section 11.9 numbers 6, 7, 9, 10, 12, 17, 27, 39

11.10 Taylor and Maclaurin Series

Theorem 117 (Taylor Series)**.** If f has a power series representation at a (that is $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for $|x-a| < R$) then its coefficients are of the form

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

That is to say that if f has a power series representation then it can be written in the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

and has radius of convergence R.

Definition 118 (Maclaurin Series)**.** A Taylor Series centered at $a = 0$ is called a **Maclaurin Series**. That is to say that a Maclaurin Series can be written as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Problem 119. Find the Maclaurin Series of $f(x) = e^x$ and find its radius of convergence.

Problem 120. Find the Maclaurin Series for $\sin(x)$.

Problem 121. Find a power series representing $f(x) = xe^x$.

Problem 122. Represent $\sin(x)$ as a Taylor Series centered at $\frac{\pi}{3}$.

Problem 123. Give a polynomial which approximates $\cos(x)$ near $x = 0$ by finding the Maclaurin Series for $\cos(x)$ and then writing out its first four non-zero terms.

Suggested Problems: Section 11.10 numbers 3, 4, $6-8$, 15, 16, 27, 28, 30, 32, 36