Calculus II - Fall 2014 - Mr. Clontz - Midterm Exam

Name: ______________________________ 9am / 10am

- If you completed the practice midterm, turn it in before beginning this exam.
- This exam is closed-note and closed-book.
- The withdrawal deadline is the evening of Tuesday, October 7. If you need me to post your grade to Canvas before the deadline, please mark this circle:
  ◯ POST GRADE BEFORE WITHDRAWAL DEADLINE

Good luck! Here are the series tests in case you need them:

Test	**When to Use**	**Conclusion**
Geometric Series	$\sum_{k=1}^{\infty} ar^k$	Converges to $\frac{a}{1-r}$ if $\lvert r\rvert < 1$; diverges if $\lvert r\rvert \geq 1$.
Divergence Test	All Series	If $\lim_{k\to\infty} a_k \neq 0$, the series diverges.
Integral Test	$\sum_{k=1}^{\infty} a_k$ where $f(k) = a_k$ and f is continuous, decreasing, and $f(x) \geq 0$	$\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$ **both** converge or **both** diverge.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	Converges for $p > 1$, diverges for $p \leq 1$.
Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $0 \leq a_k \leq b_k$	If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $a_k, b_k > 0$ and $\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ **both** converge or **both** diverge.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ where $a_k > 0$ for all k	If $\lim_{k\to\infty} a_k = 0$ and $a_{k+1} \leq a_k$ for all k, then the series converges.
Absolute Convergence	Series with some positive and some negative terms (including alternating series)	If $\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges, then $\sum_{k=1}^{\infty} a_k$ converges (absolutely).
Ratio Test	Any Series (especially those involving exponentials and/or factorials)	For $\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely, if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.
Root Test	Any Series (especially those involving exponentials)	For $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely, if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.

Multiple Choice (10 points total)

Please only mark the correct choice for each question.

1. (3 points) Nick Saban wrote the following[1]:

$$\text{"Since } \lim_{n\to\infty} \frac{n}{n^2+1} = 0, \text{ the series } \sum_{n=0}^{\infty} \frac{n}{n^2+1} \text{ converges."}$$

Why is this horribly wrong?

- ○ The limit $\lim_{n\to\infty} \frac{n}{n^2+1}$ is $\frac{1}{2}$, not 0.
- ○ Since $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$, the series $\sum_{n=0}^{\infty} \frac{n}{n^2+1}$ diverges.
- ○ The Divergence Test requires that the limit be different from 0, and cannot prove that a series converges.
- ○ The Divergence Test doesn't work on a series with only positive terms.

2. (3 points) Integration by parts is the reverse version of which rule?

- ○ Chain Rule ...... $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$
- ○ Power Rule ...... $\frac{d}{dx}[x^p] = px^{-1}$
- ○ Product Rule ...... $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$
- ○ Exponential Rule ...... $\frac{d}{dx}[b^x] = b^x \ln b$

3. (4 points) Since $\sin(x)$ has the MacLaurin Series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, which of these is the best approximating polynomial for the value of $\sin(x)$ when x is close to 0?

- ○ $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$
- ○ $1 + x^2 + x^3 + x^4$
- ○ $x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$
- ○ $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

[1] I can't back that up, but I feel like he would, y'know?

Full Solutions (90 points total)

Please show all work and draw a $\boxed{\text{box}}$ around your final answer, if appropriate. Solutions will be graded according to the rubrics given in the practice midterm.

1. (10 points) Find a general formula for the sequence $\left\{\frac{3}{2}, -\frac{4}{4}, \frac{5}{8}, -\frac{6}{16}, \frac{7}{32}, \dots\right\}$.

2. (10 points) Does the series $\sum_{n=1}^{\infty} \frac{(-2)^{n-1}}{3^n}$ converge or diverge? If it converges, give its sum.

3. (10 points) Determine whether or not $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is absolutely convergent, conditionally convergent, or divergent.

4. (10 points) Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{3n^2}{n^3+1}$ converges or diverges.

5. (10 points) Determine whether the series $\sum_{n=2}^{\infty} \frac{\sqrt{n-1}}{2+n^2}$ converges or diverges.

6. (10 points) For what values of x is the series $\displaystyle\sum_{n=0}^{\infty} \frac{(1-x)^n}{n+1}$ convergent? What is its radius of convergence?

7. (10 points) Give a power series representing the function $f(x) = \frac{1}{1+3x}$ and its radius of convergence.

8. (10 points) Find the Maclaurin series representing the “hyperbolic cosine” function

$$f(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

9. (10 points) Evaluate $\int 4xe^x \, dx$.

10. (5 points) (BONUS - no partial credit)

A common mistake I see Calculus I students do when taking derivatives is the following:

$$\frac{d}{dx}\left[x^2 \sin(x)\right] \neq \frac{d}{dx}\left[x^2\right]\frac{d}{dx}\left[\sin(x)\right] = 2x\cos(x)$$

instead of using the product rule to get the correct answer $x^2\cos(x) + 2x\sin(x)$.

Prove that this "freshman product rule"

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g'(x)$$

actually works if $g(x) = e^{\int \frac{f'(x)}{f'(x)-f(x)}\,dx}$. (An example is when $f(x) = g(x) = e^{2x}$.)