6.4 Work

Definition 26. The **work** W done by a constant **force** F exerted on an object over a distance d is given by the equation

$$W = Fd$$

Theorem 27. The work W done by a variable force $F(x)$ exerted on an object over the distance from $x = a$ to $x = b$ is given by the equation

$$W = \int_a^b F(x)\, dx$$

Problem 28. A vehicle is moved from mile marker $x = 1$ to mile marker $x = 3$ with a force of $F(x) = x^2 + 2x$ tons at a given position x on the interstate. How much work is done in moving the vehicle in this way?

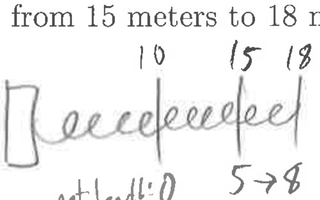
$$W = \int_1^3 x^2 + 2x \, dx$$

$$= \left[\frac{1}{3}x^3 + x^2\right]_1^3$$

$$= \left(\frac{1}{3}(27) + 9\right) - \left(\frac{1}{3}(1) + 1\right)$$

$$= 18 - \frac{4}{3} = \boxed{16\frac{2}{3}} = \boxed{\frac{50}{3}} \text{ (mile-tons)}$$

Problem 29. **Hooke's Law** tells us that a spring with spring constant k requires $F(x) = kx$ units of force to stretch the spring x units beyond its natural length. If a force of 40 newtons is required to stretch a spring from its natural length of 10 meters to 15 meters, what is the value of the spring's constant k, and how much work is required to stretch the spring further from 15 meters to 18 meters?



$$F(5) = k(5) = 40$$

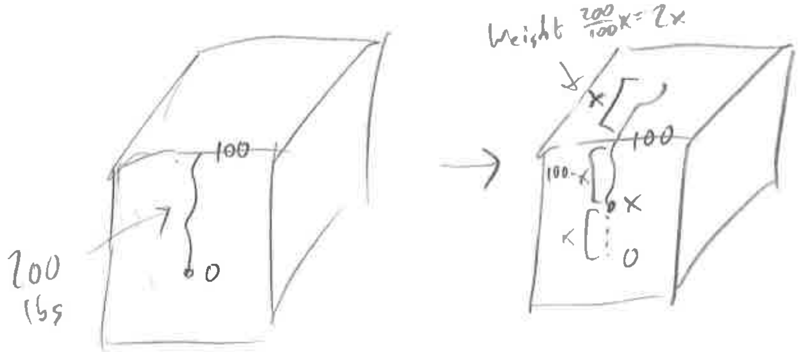
$$\boxed{k = 8}$$

$$W = \int_5^8 F(x)\, dx = \left[4x^2\right]_5^8$$

$$= \int_5^8 8x \, dx = 256 - 100$$

$$= \boxed{156} \text{ (N-m OR J)}$$

Problem 30. A 200 pound cable hangs 100 feet from the top of a building. How much work is required to retract the cable to the top of the building?



$F(x) =$ weight after lifting to x

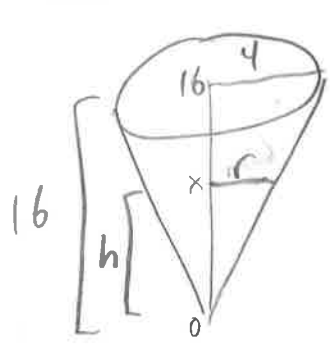
$= 200 - 2x$

$$W = \int_0^{100} 200 - 2x \, dx = \left[200x - x^2\right]_0^{100}$$

$$= 20000 - 10000$$

$$= \boxed{10000} \text{ (ft-lbs)}$$

Problem 31. A tank in the shape of an upside-down cone has a height of 16 meters and radius of 4 meters. Assuming that the density of the water is 1000 kg/m^3, compute the amount of work required to completely fill the tank by pumping water into the tip of the tank.



(From similar triangles)

$h = x$ $\quad \frac{x}{16} = \frac{r}{4} \rightarrow r = \frac{1}{4}x$

$$V(x) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{4}x\right)^2 x = \frac{1}{48}\pi x^3$$

$$F(x) = \text{weight of water} = \left(\text{volume of water}\right)\left(\text{density}\right) = V(x) \cdot 1000$$

$$= \frac{1000}{48}\pi x^3 = \frac{125}{6}\pi x^3$$

$$W = \int_0^{16} \frac{125}{6}\pi x^3 \, dx = \left[\frac{125}{18}\pi x^4\right]_0^{16} = \boxed{\frac{1024000}{3}\pi}$$

(Arithmetic harder than I meant to make it.)

(No Suggested Homework)