Chapter 7

Techniques of Integration

7.1 Integration by Parts

Problem 1. Prove that $\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$. *Hint* Use the product rule and work backwards.

Theorem 2 (Integration by Parts)**.** Given two continuous, differentiable functions $f(x)$ and $g(x)$,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

If $u = f(x)$ and $v = g(x)$, then we can write this as

$$\int u\,dv = uv - \int v\,du$$

Problem 3. Evaluate $\int x\sin(x)\,dx$.

Problem 4. Evaluate $\int \ln(x)\,dx$.

Problem 5. Evaluate $\int t^2 e^t\,dt$.

Problem 6. Evaluate $\int_0^1 \arctan(x)\, dx$.

Problem 7. Evaluate $\int e^x \sin(x)\, dx$.

Suggested Homework: Section 7.1 numbers $1 - 4$, 7, $10 - 12$, 21, 24, 29, 30, 31

7.2 Trigonometric Integrals

7.2.1 Products of Powers of Sine and Cosine

Strategy 8. There are three types of integrals of the form $\int \sin^m(x)\cos^n(x)dx$:

I. **The power on $\sin(x)$ is odd.**

Apply $\sin^{2n+1}(x) = (\sin^2(x))^n \sin(x) = (1-\cos^2(x))^n \sin(x)$ and use the substitution $u = \cos(x)$.

II. **The power on $\cos(x)$ is odd.**

Apply $\cos^{2n+1}(x) = (\cos^2(x))^n \cos(x) = (1-\sin^2(x))^n \cos(x)$ and use the substitution $u = \sin(x)$.

III. **Both powers are even.**

Apply both $\cos^{2n}(x) = \left(\frac{1+\cos(2x)}{2}\right)^n$ and $\sin^{2n}(x) = \left(\frac{1-\cos(2x)}{2}\right)^n$ to reduce the exponents in the integral.

Problem 9. Evaluate $\displaystyle\int \cos^3(x)\,dx$.

Problem 10. Evaluate $\displaystyle\int \sin^5(x)\cos^2(x)\,dx$.

Problem 11. Evaluate $\int_0^{\frac{\pi}{4}} \sin^2(x)\,dx$.

Problem 12. Evaluate $\int \sin^4(x)\,dx$.

7.2.2 Products of Powers of Tangent and Secant

Strategy 13. To evaluate an integral of the form $\int \tan^m(x)\sec^n(x)\,dx$:

- If n is even,
  - Save a factor of $\sec^2(x)$ and use $\sec^2(x) = 1 + \tan^2(x)$ on the rest.
  - Use the u substitution $u = \tan(x)$.
- If m is odd,
  - Save a factor of $\sec(x)\tan(x)$ and use $\tan^2(x) = \sec^2(x) - 1$ on the rest.
  - Use the u substitution $u = \sec(x)$.

Problem 14. Evaluate $\int \tan^6(x)\sec^4(x)\,dx$.

Problem 15. Evaluate $\int \tan^5(\theta) \sec^7(\theta)\, d\theta$.

Recall 16. $\int \tan(x)\, dx = \ln|\sec(x)| + c$ and $\int \sec(x)\, dx = \ln|\sec(x) + \tan(x)| + c$

Problem 17. Evaluate $\int \tan^3(x)\, dx$.

Problem 18. Use Integration by Parts to evaluate $\int \sec^3(x)\,dx$.

Recall 19.

$$\begin{aligned}
\sin(A)\cos(B) &= \tfrac{1}{2}\left[\sin(A-B)+\sin(A+B)\right]\\
\sin(A)\sin(B) &= \tfrac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]\\
\cos(A)\cos(B) &= \tfrac{1}{2}\left[\cos(A-B)+\cos(A+B)\right].
\end{aligned}$$

Problem 20. Evaluate $\int \sin(4x)\cos(5x)\,dx$.

Suggested Homework: Section 7.2 numbers 1, 3, 5 − 7, 10, 11, 15, 21, 23, 25, 27, 29, 38

7.3 Trigonometric Substitution

Strategy 21. With square roots and other troublesome factors, it sometimes helps to substitute trigonometric functions in order to use their identities for cancellation.

Expression	Substitution	Differential	Fact to Use
$a^2 - x^2$	$x = a\sin(\theta) \Rightarrow x^2 = a^2\sin^2(\theta)$	$dx = a\cos(\theta)d\theta$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$a^2 + x^2$	$x = a\tan(\theta) \Rightarrow x^2 = a^2\tan^2(\theta)$	$dx = a\sec^2(\theta)d\theta$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$x^2 - a^2$	$x = a\sec(\theta) \Rightarrow x^2 = a^2\sec(\theta)$	$dx = a\sec(\theta)\tan(\theta)d\theta$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

Problem 22. Prove $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$.

Problem 23. Evaluate $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

Problem 24. Evaluate $\int \frac{2x}{x^2+1}\,dx$.

Problem 25. Evaluate $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

Problem 26. Evaluate $\int \frac{x}{\sqrt{x^2+4}}\,dx$.

Problem 27. Evaluate $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{\left(4x^2+9\right)^{3/2}}\,dx$.

Problem 28. Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$ for $a>0$.

Problem 29. Evaluate $\displaystyle\int \frac{1}{\sqrt{3-2x-x^2}}\,dx$. *Hint* Complete the square.

Suggested Homework: Section 7.3 numbers 2, 4, 5, 7, 9, 10, 11, 16, 22

7.4 Partial Fraction Decomposition

Strategy 30. If the degree in the numerator is greater than or equal to the degree in the denominator, use long division.

Problem 31. Evaluate $\int \frac{x^3+x^2-4}{x-1}\,dx$.

Problem 32. Evaluate $\int \frac{x^4+1}{x^2+1}\,dx$.

Theorem 33 (Fundamental Theorem of Algebra)**.** Every polynomial of real numbers is factorable into linear terms $(ax+b)$ and irreducible quadratics (ax^2+bx+c).

Recall 34. To add fractions, we find a common denominator:

$$\frac{2}{x}+\frac{1}{x+1}=\frac{2(x+1)}{x(x+1)}+\frac{1(x)}{(x+1)(x)}=\frac{2x+2}{x(x+1)}+\frac{x}{x(x+1)}=\frac{2x+2+x}{x(x+1)}=\frac{3x+2}{x^2+x}.$$

More work is required to undo this process.

Strategy 35. If the denominator is the product of distinct linear factors, the fraction may be split into the sum of fractions with constant numerators and distinct linear factor denominators:

$$\frac{-x^2+14x+6}{2x^3+7x^2+3} = \frac{A}{x} + \frac{B}{2x+1} + \frac{C}{x+3}$$

$$-x^2+14x+6 = A(2x+1)(x+3) + Bx(x+3) + Cx(2x+1)$$

$$(-1)x^2+(14)x+(6) = (2A+B+2C)x^2 + (7A+3B+C)x + (3A)$$

$$-1 = 2A+B+2C \qquad 14 = 7A+3B+C \qquad 6 = 3A$$

Problem 36. Evaluate $\displaystyle\int \frac{-x^2+14x+6}{2x^3+7x^2+3}\,dx$.

Problem 37. Evaluate $\displaystyle\int \frac{14x^2 - 8x + 5}{(x+1)(1-2x)(x-2)}\,dx$.

Problem 38. Find the value of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 + 5n + 6}$ by using partial fraction decomposition to compute the partial sums.

Strategy 39. If the denominator has a repeated linear factor, use an additional fraction with a constant numerator and a higher power for each repeated linear factor:

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Problem 40. Evaluate $\displaystyle\int \frac{4x}{x^3 - x^2 - x + 1}\,dx$.

Problem 41. Evaluate $\displaystyle\int \frac{6x^2 - 7x - 2}{x^3 - 2x^2}\,dx$.

Strategy 42. If the denominator has an irreducible quadratic, use a linear factor for the numerator:

$$\frac{3x^2 - x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Problem 43. Evaluate $\int \frac{3x^2 - x + 4}{x^3 + 4x}\,dx$.

Strategy 44. If the denominator has a repeated irreducable quadractic factor, use an additional fraction with a linear numerator and a higher power for each repeated irreducable quadratic:

$$\frac{2 - 2x + 4x^2 - 2x^3}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

Problem 45. Evaluate $\int \frac{2 - 2x + 4x^2 - 2x^3}{x(x^2 + 1)^2}\,dx$

Suggested Homework: Section 7.4 numbers 2, 3, 12, 15, 16, 17, 18, 19, 26, 27

7.8 Improper Integrals

Definition 46 (Improper Integral with Infinite Bounds)**.** An integral with at least one infinite bound is computed as the limit of definite integrals:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx \text{ for any value of } c$$

These equalities hold only when the given limits exist. In that case, the improper integral **converges**; otherwise, the improper integral **diverges**.

Problem 47. Find the area bounded by the curves $y = \frac{1}{x^2}$, $x = 1$, and $y = 0$.

Problem 48. Determine whether $\int_1^\infty \frac{1}{2\sqrt{x}}\,dx$ is convergent or divergent. If it converges give its value.

Problem 49. Determine whether $\int_{-\infty}^{0} xe^x \, dx$ is convergent or divergent. If it converges give its value.

Problem 50. Compute $\int_{-\infty}^{\infty} \frac{1}{1+x^2}$.

Problem 51. Use the Integral Test to show that $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ converges.

Definition 52 (Improper Integral of Function Undefined within Interval)**.** An integral of a function undefined for a point within the interval of integration is computed as the limit of definite integrals:

$$\int_a^b f(x)\,dx = \lim_{t\to b}\int_a^t f(x)\,dx$$

$$\int_a^b f(x)\,dx = \lim_{t\to a}\int_t^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \text{ for any value of } c$$

These equalities hold only when the given limits exist. In that case, the improper integral **converges**; otherwise, the improper integral **diverges**.

Problem 53. Determine whether $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ converges or diverges. If it converges, give its value.

Problem 54. Determine whether $\int_0^{\frac{\pi}{2}} \sec(x)\,dx$ converges or diverges. If it converges, give its value.

Problem 55. Determine whether $\int_0^1 \ln(x)\,dx$ converges or diverges. If it converges, give its value.

Problem 56. Determine whether $\int_0^3 \frac{1}{x-1}\,dx$ converges or diverges. If it converges, give its value.

Theorem 57 (Comparison Test for Integrals)**.** Suppose that f and g are continuous functions with $0 \leq g(x) \leq f(x)$ for sufficiently large x.

- If the larger $\int_a^\infty f(x)\,dx$ is convergent, then the smaller $\int_a^\infty g(x)\,dx$ is also convergent.
- If the smaller $\int_a^\infty g(x)\,dx$ is divergent, then the larger $\int_a^\infty f(x)\,dx$ is also divergent.

Problem 58. Determine whether $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges or diverges.

Problem 59. Determine whether $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ converges or diverges.

Suggested Homework: Section 7.8 numbers 1, 7, 9, 13, 14 – 16, 18, 25, 27 – 33, 35, 49 – 52, 54, 55, 57