# Chapter 12

# Vectors and the Geometry of Space

## 12.1 Two and Three Dimensional Space

**Definition 1.** Let $\mathbb{R}$ be the collection of real numbers, let $\mathbb{R}^2$ be the collection of all **ordered pairs** of real numbers, and let $\mathbb{R}^3$ be the collection of all **ordered triples** of real numbers.

$\mathbb{R}$ is known as the **real line**, $\mathbb{R}^2$ is known as the **real plane** or the $xy$**-plane**, and $\mathbb{R}^3$ is known as **real (3D) space** or $xyz$**-space**.

**Definition 2.** The **distance** between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in $\mathbb{R}^2$ is given by the formula

$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The **distance** between two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in $\mathbb{R}^3$ is given by the formula

$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

**Problem 3.** Plot and find the distance between the following pairs of points:

- $(-2, 6)$ and $(3, -6)$
- $(0, 0, 0)$ and $(4, 2, 4)$
- $(3, 7, -2)$ and $(-1, 7, 1)$
- $(8, 2, 1)$ and $(4, -2, 7)$

**Definition 4.** **Simple lines** in $\mathbb{R}^2$ are given by the relations $x = a$, and $y = b$ for real numbers $a, b$.

**Simple planes** in $\mathbb{R}^3$ are given by the relations $x = a$, $y = b$, $z = c$ for real numbers $a, b, c$.

**Definition 5.** A **circle** in $\mathbb{R}^2$ is the set of all points a fixed distance (called its **radius**) from a fixed point (called its **center**). For a center $(a, b)$ and radius $r$, the equation for a circle is

$$(x - a)^2 + (y - b)^2 = r^2$$

A **sphere** in $\mathbb{R}^3$ is the set of all points a fixed distance (called its **radius**) from a fixed point (called its **center**). For a center $(a, b, c)$ and radius $r$, the equation for a sphere is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

**Question 6.** Sketch the following curves and surfaces.

- $x = 3$ in the $xy$-plane and $xyz$-space.
- $y = -1$ in the $xy$-plane and $xyz$-space.
- $z = 0$ in $xyz$-space.
- $(x - 2)^2 + (y + 1)^2 = 9$ in the $xy$-plane.
- $x^2 + y^2 + z^2 = 4$ in $xyz$-space.
- $x^2 + (y - 1)^2 + z^2 = 1$ in $xyz$-space.

Suggested Homework: Section 12.1 numbers 4, 6, 7, 8, 10, 11, 12, 14, 15, 16

## 12.2 Vectors

**Definition 7** (Vector)**.** A **vector** $\vec{\mathbf{v}}$ is a mathematical object that stores a **magnitude** (a nonnegative real number often thought of as length) and **direction**. Two vectors are **equal** if and only if they have the same magnitude and direction.

**Definition 8.** The **zero vector** $\vec{\mathbf{0}}$ has zero magnitude and no direction. (This is the only vector without a direction.)

**Definition 9.** For a given point $P = (a, b)$ in $\mathbb{R}^2$, its **position vector** is given by $\vec{\mathbf{P}} = \langle a, b\rangle$: the vector from the origin $(0, 0)$ to the point $P = (a, b)$.

For a given point $P = (a, b, c)$ in $\mathbb{R}^3$, its **position vector** is given by $\vec{\mathbf{P}} = \langle a, b, c\rangle$: the vector from the origin $(0, 0, 0)$ to the point $P = (a, b, c)$.

**Theorem 10.** Two vectors are equal if and only if they share the same magnitude and direction as a common position vector.

**Definition 11.** Since all vectors are equal to some position vector $\langle a, b\rangle$ or $\langle a, b, c\rangle$, we usually define vectors by a position vector written in this **component form**. Since the component form of a vector stores the same information as a point, we will use both interchangeably, that is, $\langle a, b\rangle = (a, b) \in \mathbb{R}^2$ and $\langle a, b, c\rangle = (a, b, c) \in \mathbb{R}^3$ (although we usually sketch them differently).

**Problem 12.** Plot the following points and position vectors.

- $(1, 3)$ and $\langle 1, 3\rangle$ in the $xy$-plane.
- $(-2, 5)$ and $\langle -2, 5\rangle$ in the $xy$-plane.
- $(1, 1, -3)$ and $\langle 1, 1, -3\rangle$ in $xyz$-space.
- $(0, 5, 0)$ and $\langle 0, 5, 0\rangle$ in $xyz$-space.

**Definition 13.** Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$. Then the vector with initial point $P$ and terminal point $Q$ is defined as

$$\overrightarrow{\mathbf{PQ}} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

**Problem 14.** Plot and sketch the points $P$, $Q$ and the vector $\overrightarrow{\mathbf{PQ}}$ for each.

- $P = (1, 3)$, $Q = (-3, 6)$ in the $xy$-plane
- $P = (3, 1)$, $Q = (0, -2)$ in the $xy$-plane
- $P = (1, 1, 1)$, $Q = (-3, -1, 3)$ in $xyz$-space
- $P = (-2, 0, 3)$, $Q = (1, 3, -3)$ in $xyz$-space

**Definition 15.** The magnitude $|\vec{\mathbf{v}}|$ of a vector $\vec{\mathbf{v}}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is the distance between its initial and terminal points.

**Theorem 16.** The magnitude of $\vec{\mathbf{v}} = \langle a, b \rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{a^2 + b^2}$$

The magnitude of $\vec{\mathbf{v}} = \langle a, b, c \rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{a^2 + b^2 + c^2}$$

**Problem 17.** Evaluate the magnitude of the following vectors:

- $\langle 5, 5\rangle$
- $\langle -4, 3\rangle$
- $\langle 12, -5\rangle$
- $\langle 3, 1, -2\rangle$
- $\langle 4, -2, -4\rangle$
- $\langle 8, 0, -6\rangle$

### 12.2.1 Basic Vector Operations

**Definition 18. Vector addition** is defined component-wise as follows for $\mathbb{R}^2$ and $\mathbb{R}^3$

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1, u_2\rangle + \langle v_1, v_2\rangle = \langle u_1 + v_1, u_2 + v_2\rangle$$

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1, u_2, u_3\rangle + \langle v_1, v_2, v_3\rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle$$

**Definition 19.** A **scalar** is simply a real number by itself (as opposed to a vector of real numbers).

**Definition 20. Scalar multiplication of a vector** is defined component-wise as follows for $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$k\vec{\mathbf{u}} = k\langle u_1, u_2\rangle = \langle ku_1, ku_2\rangle$$

$$k\vec{\mathbf{u}} = k\langle u_1, u_2, u_3\rangle = \langle ku_1, ku_2, ku_3\rangle$$

**Problem 21.** Sketch the following vectors.

- $\vec{\mathbf{u}} = \langle 1, -3 \rangle$, $\vec{\mathbf{v}} = \langle 3, 1 \rangle$ and $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ in the $xy$-plane.
- $\vec{\mathbf{u}} = \langle 2, 0, 1 \rangle$, $\vec{\mathbf{v}} = \langle -2, 4, 2 \rangle$ and $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ in $xyz$-space.
- $\vec{\mathbf{u}} = \langle 8, -2 \rangle$ and $\frac{1}{2}\vec{\mathbf{u}}$ in the $xy$-plane.
- $\vec{\mathbf{u}} = \langle 5, 3, -1 \rangle$ and $3\vec{\mathbf{u}}$ in $xyz$-space.

**Definition 22.** A vector $\vec{\mathbf{v}}$ is a **unit vector** if $|\vec{\mathbf{v}}| = 1$.

**Theorem 23.** For any non-zero vector $\vec{\mathbf{v}}$, the vector

$$\frac{1}{|\vec{\mathbf{v}}|}\vec{\mathbf{v}} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

is a unit vector.

**Definition 24.** The **direction** of a vector $\vec{\mathbf{v}}$ is the unit vector $\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$.

**Theorem 25.** Any vector $\vec{\mathbf{v}}$ is the scalar product of its magnitude and direction:

$$\vec{\mathbf{v}} = |\vec{\mathbf{v}}| \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

**Problem 26.** Write the following vectors as the scalar product of their magnitude and direction:

- $\langle 5, 5\rangle$
- $\langle -4, 3\rangle$
- $\langle 12, -5\rangle$
- $\langle 3, 1, -2\rangle$
- $\langle 4, -2, -4\rangle$
- $\langle 8, 0, -6\rangle$

**Definition 27.** The **standard unit vectors** in $\mathbb{R}^2$ are $\widehat{\mathbf{i}} = \langle 1, 0\rangle$ and $\widehat{\mathbf{j}} = \langle 0, 1\rangle$, and any vector in $\mathbb{R}^2$ can be expressed in **standard unit vector form**:

$$\langle a, b\rangle = a\widehat{\mathbf{i}} + b\widehat{\mathbf{j}}$$

The **standard unit vectors** in $\mathbb{R}^3$ are $\widehat{\mathbf{i}} = \langle 1, 0, 0\rangle$, $\widehat{\mathbf{j}} = \langle 0, 1, 0\rangle$, and $\widehat{\mathbf{k}} = \langle 0, 0, 1\rangle$, and any vector in $\mathbb{R}^3$ can be expressed in **standard unit vector form**:

$$\langle a, b, c\rangle = a\widehat{\mathbf{i}} + b\widehat{\mathbf{j}} + c\widehat{\mathbf{k}}$$

**Note 28.** Since the $xy$-plane is the the plane $z = 0$ in $xyz$-space, we say the points $(a, b) = (a, b, 0)$ and vectors $\langle a, b\rangle = \langle a, b, 0\rangle = a\widehat{\mathbf{i}} + b\widehat{\mathbf{j}} + 0\widehat{\mathbf{k}}$ are equal.

Suggested Homework: Section 12.2 numbers 3, 5, 13, 14, 15, 19, 21, 24, 26