Chapter 11

Sequences and Series

11.0 Prepositional Logic

All of the definitions in this section are adaptations of Irving M. Copi's book *Symbolic Logic*.

Definition 1 (Proposition). A **proposition** is a statement which is either true or false.

Example 2. Britney is a goat. This statement has a definite truth value. It is either true or false, whether or not one can tell the truth value is a different story.

Definition 3 (Negation). The **negation** of a statement P is a statement denoted $\neg P$ which has the opposite truth value of P.

Definition 4 (Argument). An **argument** is a group of propositions, one of which is claimed to follow from another, providing grounds for truth.

Definition 5 (Structure of an Argument). An argument is normally presented as a **conditional statement**. That is, it is of the form "If something is a car then it is a vehicle." The statement that goes with the "If" clause is called the **hypothesis** while the statement that goes with the "Then" clause is called the **conclusion**. For ease of notation, we typically call the hypothesis P and the conclusion Q and denote the argument "If P then Q" by $P \Rightarrow Q$. Statements of this form are false only when the premise is true and the conclusion is false. Another way of saying $P \Rightarrow Q$ is "P implies Q"

Definition 6 (Truth Table). A **truth table** is an array which lists all of the possible truth values for a given argument.

Definition 7. The truth table for "P and Q" ($P \wedge Q$), "P or Q" ($P \vee Q$), and "P implies Q" ($P \Rightarrow Q$) is:

P	Q	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Definition 8 (Converse). The **converse** of a conditional statement $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.

Definition 9 (Contrapositive). The **contrapositive** of a conditional statement $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$.

Definition 10 (Tautological Equivalence). Two arguments are called **tautologically equivalent** (or **tautologies** for short) if given the truth values for the constituent statements, the truth values for each argument is the same. We use the symbol $A_1 \equiv A_2$ to denote that A_1 is tautologically equivalent to A_2.

Problem 11. Show that $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ are tautologically equivalent. We sometimes write this as $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

$P \Rightarrow Q$ / $\neg Q \Rightarrow \neg P$ have the same truth values for any value of P, Q.

Problem 12. Show that $P \Rightarrow Q$ is not tautologically equivalent to $Q \Rightarrow P$.

If P is false and Q is true, then $P \Rightarrow Q$, but $Q \not\Rightarrow P$.

(Could also draw truth table.)

Problem 13. Give a "real life example" of propositions P, Q such that $P \Rightarrow Q$, (and thus $\neg Q \Rightarrow \neg P$,) but $Q \not\Rightarrow P$.

P: "the shirt is black" Q: "the shirt is not white"

$P \Rightarrow Q$: If the shirt is black, then the shirt is not white.

Always true.

$Q \Rightarrow P$: If the shirt is not white, then the shirt is black.

False for a pink shirt.

Problem 14. Give a "mathematical example" of propositions P, Q such that $P \Rightarrow Q$, (and thus $\neg Q \Rightarrow \neg P$,) but $Q \not\Rightarrow P$.

P: $x = 2$ Q: $x^2 = 4$

$P \Rightarrow Q$: If $x = 2$, then $x^2 = 4$.

Always true.

$Q \Rightarrow P$: If $x^2 = 4$, then $x = 2$.

False when $x = -2$.

11.1 Sequences

Definition 15 (Sequence). [1,2] A **sequence** is a function whose domain is a final set of integers. If s is a sequence, we usually write its value at n as s_n instead of $s(n)$. We may denote a sequence as $(s_0, s_1, \dots)$ or $\{s_3, s_4, \dots\}$ or $\langle s_{-1}, s_0, \dots \rangle$ or $\{s_n\}_{n=1}^{\infty}$ or simply s_n (depending on its domain). If its domain is not given, we usually assume it to be $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{W} = \{0, 1, \dots\}$, or some other final set of integers which is always defined for the sequence definition.

Problem 16. Write the first five terms of the following sequences:

- $\left\{\frac{n}{n+1}\right\} = \left\{\frac{0}{0+1}, \frac{1}{1+1}, \frac{2}{2+1}, \frac{3}{3+1}, \frac{4}{4+1}, \dots\right\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$
- $\left(\frac{(-1)^n (n+1)}{3^n}\right)_{n=0}^{\infty} = \left(\frac{(-1)^0(0+1)}{3^0}, \frac{(-1)^1(1+1)}{3^1}, \dots\right) = \left(1, -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots\right)$
- $\langle \sqrt{n-3} \rangle = \langle \sqrt{3-3}, \sqrt{4-3}, \dots \rangle = \langle \sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \dots \rangle$
- $\left\{\cos\left(\frac{n\pi}{6}\right)\right\}_{n=1}^{\infty} = \left\{\cos\left(\frac{\pi}{6}\right), \cos\left(\frac{2\pi}{6}\right), \dots\right\} = \left\{\frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, \dots\right\}$

Problem 17. Find a general formula for the sequence $\left\{\frac{3}{5}, \frac{-4}{25}, \frac{5}{125}, \frac{-6}{625}, \frac{7}{3125}, \dots\right\}$.

$$\left\{\frac{(-1)^{n+1}(n+2)}{5^n}\right\}_{n=1}^{\infty} \quad \text{OR} \quad \left\{\frac{(-1)^n(n+3)}{5^{n+1}}\right\}_{n=0}^{\infty}$$

[1] Definition based on Steven R. Lay's book *Analysis With an Introduction to Proof*.

[2] A final set of integers starts at some n, and contains every bigger integer. As examples: $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{W} = \{0, 1, 2, \dots\}$, $\{4, 5, 6, \dots\}$, $\{-2, -1, 0, 1, \dots\}$, etc.

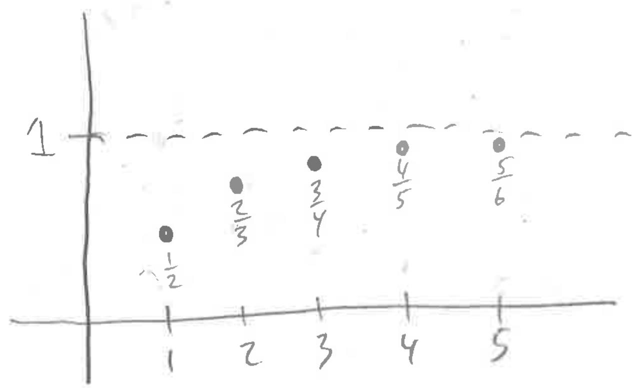
Note 18. Some sequences do not have a simple defining equation.

Example 19. The n^{th} term of the decimals of e. The sequence of decimals of e look like $\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \dots\}$.

Note 20. On the other hand, there are sequences that do have a closed form definition that are simply not easy to find.

Example 21. The Fibbonacci Sequence is defined as $f_1 = f_2 = 1$ and for $n \geq 3, f_n = f_{n-1} + f_{n-2}$. This has a closed form definition of $\left(\frac{\varphi^n - \psi^n}{\sqrt{5}}\right)$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

Problem 22. Visualize the sequence $\left(\frac{n}{n+1}\right)$.



Problem 23. What, if anything, does it seem like the sequence $\left(\frac{n}{n+1}\right)$ is approaching?

It looks like it's approaching 1, but I can't know for sure from just a picture.

Definition 24 (Limit of a Sequence). A sequence (a_n) has the **limit** L if we can make the terms of (a_n) arbitrarily close to L as we like my taking n to be sufficiently large. If (a_n) has a limit L, then we write $\lim\limits_{n\to\infty} a_n = L$ or $a_n \to L$.

Definition 25 (Convergence and Divergence). If $\lim\limits_{n\to\infty} a_n$ exists, then we say that the sequence **converges**. Otherwise, we say that the sequence **diverges** or is **divergent**.

Theorem 26 (Subset of a Continuous Function). If $\lim\limits_{x\to\infty} f(x) = L$, and $f(n) = a_n$ wherever n is in the domain of the sequence, then $\lim\limits_{n\to\infty} a_n = L$.

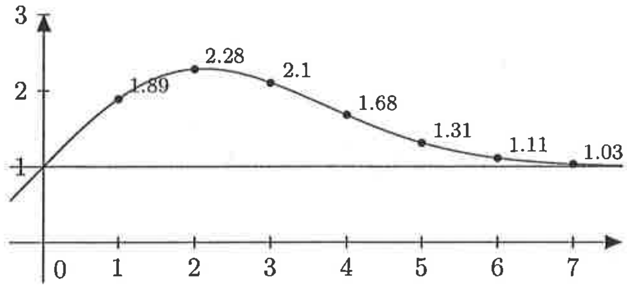


Figure 11.1: Sequence as a Subset of a Function

Corollary 27. If $\langle a_n\rangle_{n=N'}^{\infty}$ converges (diverges) for some choice of initial N', then $\langle a_n\rangle_{n=N}^{\infty}$ converges (diverges) for any choice of N where a_n is defined for all $n \geq N$.

Properties 28. If (a_n) and (b_n) are convergent sequences and $c \in \mathbb{R}$, then the following properties hold:

- $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$
- $\lim\limits_{n\to\infty} ca_n = c \lim\limits_{n\to\infty} a_n$
- $\lim\limits_{n\to\infty} a_n b_n = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$
- $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ as long as $\lim\limits_{n\to\infty} b_n \neq 0$
- $\lim\limits_{n\to\infty} a_n^p = \left(\lim\limits_{n\to\infty} a_n\right)^p$ for $p > 0$ and $a_n > 0$.

Theorem 29 (Squeeze Theorem). If $a_n \leq b_n \leq c_n$ for all $n \geq n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

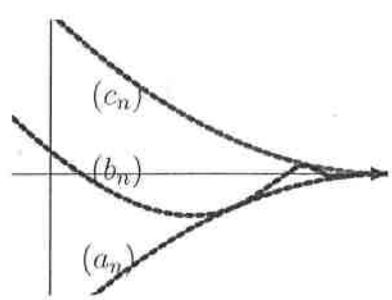


Figure 11.2: Squeeze Theorem

Corollary 30. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Problem 31. Determine whether the sequence $\left(\frac{n}{n+1}\right)$ is convergent or divergent. If it is convergent, what does it converge to?

Let $f(x) = \frac{x}{x+1}$.

$$\lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{1+\lim_{x\to\infty}\frac{1}{x}} = \frac{1}{1+0} = 1$$

Since $\lim_{x\to\infty} \frac{x}{x+1} = 1$, $\lim_{n\to\infty} \frac{n}{n+1} = \boxed{1}$ and $\boxed{\text{converges}}$.

Problem 32. Determine whether the sequence $\left(\frac{n}{\sqrt{10+n}}\right)$ is convergent or divergent. If it is convergent, what does it converge to?

Let $f(x) = \frac{x}{\sqrt{10+x}}$

(Looks like $\frac{\infty}{\infty}$)

(L'Hopital's Rule)

$$\lim_{x\to\infty} \frac{x}{\sqrt{10+x}} = \lim_{x\to\infty} \frac{\frac{d}{dx}[x]}{\frac{d}{dx}[\sqrt{10+x}]} = \lim_{x\to\infty} \frac{1}{\frac{1}{2}(10+x)^{-1/2}} = \lim_{x\to\infty} 2\sqrt{10+x}$$

$\left(\text{“}= \infty\text{”}\right)$

Since $\lim_{x\to\infty} \frac{x}{\sqrt{x+10}}$ DNE, $\lim_{n\to\infty} \frac{n}{\sqrt{n+10}}$ $\boxed{\text{diverges}}$.

Problem 33. Determine whether the sequence $((-1)^n)$ is convergent or divergent. If it is convergent, what does it converge to?

$$\left((-1)^n\right)_{n=0}^{\infty} = (1, -1, 1, -1, 1, -1, \ldots)$$

Since the sequence never trends toward a single number, it $\boxed{\text{diverges}}$.

Problem 34. Determine whether the sequence $\left(\frac{\ln(n)}{n}\right)$ is convergent or divergent. If it is convergent, what does it converge to?

Let $f(x) = \frac{\ln x}{x}$

(Looks like $\frac{\infty}{\infty}$)

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Since $\lim_{x\to\infty} \frac{\ln x}{x} = 0$, $\lim_{n\to\infty} \frac{\ln n}{n} = \boxed{0}$ and $\boxed{\text{converges}}$.

Problem 35. Determine whether the sequence $\left(\frac{(-1)^n}{n}\right)$ is convergent or divergent. If it is convergent, what does it converge to?

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty}\frac{1}{n} = 0$$

(See Cor 30)

Since $\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right| = 0$, $\lim_{n\to\infty}\frac{(-1)^n}{n} = \boxed{0}$ and $\boxed{\text{converges}}$.

OR

$$\frac{-1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$$

(Squeeze Theorem)

$$\lim_{n\to\infty}\frac{-1}{n} \leq \lim_{n\to\infty}\frac{(-1)^n}{n} \leq \lim_{n\to\infty}\frac{1}{n}$$

$$0 \leq \lim_{n\to\infty}\frac{(-1)^n}{n} \leq 0$$

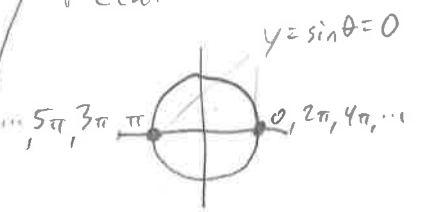
Thus $\lim_{n\to\infty}\frac{(-1)^n}{n} = \boxed{0}$ and $\boxed{\text{converges}}$.

Theorem 36. If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

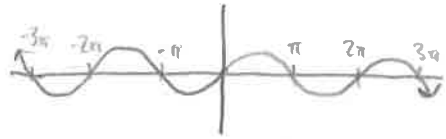
Problem 37. Find $\lim_{n\to\infty}\sin(\pi n)$

$$\lim_{n\to\infty}\sin(\pi n) = \lim_{n\to\infty} 0 = \boxed{0}$$ and $\boxed{\text{converges}}$.

Recall:



OR



So $\sin(\pi n) = 0$ when n is an integer

Problem 38. Show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$. *Hint* show that $0 \le \frac{n!}{n^n} \le \frac{1}{n}$

$0 \le \frac{n!}{n^n}$ since n is positive.

$\frac{n!}{n^n} = \frac{1 \times 2 \times 3 \times \cdots \times n}{n \times n \times n \times \cdots \times n} \le \frac{1}{n}$ since $\frac{2 \times 3 \times \cdots \times n}{n \times n \times \cdots \times n}$ is less than 1.

So:

$$0 \le \frac{n!}{n^n} \le \frac{1}{n}$$

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{n!}{n^n} \le \lim_{n\to\infty} \frac{1}{n}$$

$$0 \le \lim_{n\to\infty} \frac{n!}{n^n} \le 0$$

Thus

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

Theorem 39 (Preliminary for Geometric Series)**.** The sequence (r^n) is convergent if $-1 \le r \le 1$ and divergent otherwise.

Definition 40 (Monotone)**.** A sequence (a_n) is called **non-decreasing** if $a_n \le a_{n+1}$ for all $n \ge 1$. Similarly, a sequence (a_n) is called **non-increasing** if $a_n \ge a_{n+1}$ for all $n \ge 1$. A sequence is **monotonic** if it is either non-decreasing or non-increasing.

Problem 41. Show that $\left(\frac{3}{n+5}\right)$ is decreasing. (Let $a_n = \frac{3}{n+5}$.)

$$a_n = \frac{3}{n+5} > \frac{3}{n+6} = \frac{3}{(n+1)+5} = a_{n+1}.$$

Problem 42. Show that $\left(\frac{n}{n^2+1}\right)$ is decreasing. Let $a_n = \frac{n}{n^2+1} = \frac{1}{n+\frac{1}{n}}$.

$$a_n = \frac{1}{n+\frac{1}{n}} \geq \frac{1}{n+\frac{n+2}{n+1}} = \frac{1}{(n+1)+\frac{1}{n+1}} = a_{n+1}.$$

OR

Let $f(x) = \frac{x}{x^2+1}$.

Then $f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$

$$= \frac{1-x^2}{(x^2+1)^2} < 0$$

for $x > 1$.

Since f is a decreasing function, $\left(\frac{n}{n^2+1}\right)$ is a decreasing sequence.

Definition 43 (Bounded). A sequence (a_n) is **bounded above** if there exists an $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \geq 1$. A sequence (a_n) is **bounded below** if there exists an $m \in \mathbb{R}$ such that $a_n \geq m$ for all $n \geq 1$. If a sequence is bounded above or bounded below then the sequence is said to be **bounded**.

Theorem 44. Every bounded monotonic sequence is convergent.

Suggested Homework: Section 11.1 numbers 5, 9, 13 – 15, 23 – 29, 33, 35, 37, 41, 42, 44, 49, 50, 53, 56