Calculus II - Fall 2014 - Mr. Clontz - Midterm Exam

Name: Answer Key 9am / 10am

- If you completed the practice midterm, turn it in before beginning this exam.
- This exam is closed-note and closed-book.
- The withdrawal deadline is the evening of Tuesday, October 7. If you need me to post your grade to Canvas before the deadline, please mark this circle:
  ○ POST GRADE BEFORE WITHDRAWAL DEADLINE

Good luck! Here are the series tests in case you need them:

Test	When to Use	Conclusion
Geometric Series	$\sum_{k=1}^{\infty} ar^k$	Converges to $\frac{a}{1-r}$ if $\lvert r\rvert < 1$; diverges if $\lvert r\rvert \geq 1$.
Divergence Test	All Series	If $\lim_{k\to\infty} a_k \neq 0$, the series diverges.
Integral Test	$\sum_{k=1}^{\infty} a_k$ where $f(k) = a_k$ and f is continuous, decreasing, and $f(x) \geq 0$	$\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$ **both** converge or **both** diverge.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	Converges for $p > 1$, diverges for $p \leq 1$.
Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $0 \leq a_k \leq b_k$	If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $a_k, b_k > 0$ and $\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ **both** converge or **both** diverge.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ where $a_k > 0$ for all k	If $\lim_{k\to\infty} a_k = 0$ and $a_{k+1} \leq a_k$ for all k, then the series converges.
Absolute Convergence	Series with some positive and some negative terms (including alternating series)	If $\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges, then $\sum_{k=1}^{\infty} a_k$ converges (absolutely).
Ratio Test	Any Series (especially those involving exponentials and/or factorials)	For $\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely, if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.
Root Test	Any Series (especially those involving exponentials)	For $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely, if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.

Multiple Choice (10 points total)

Please only mark the correct choice for each question.

1. (3 points) Nick Saban wrote the following[1]:

$$\text{“Since } \lim_{n\to\infty} \frac{n}{n^2+1} = 0, \text{ the series } \sum_{n=0}^{\infty} \frac{n}{n^2+1} \text{ converges.”}$$

   Why is this horribly wrong?

   - ○ The limit $\lim_{n\to\infty} \frac{n}{n^2+1}$ is $\frac{1}{2}$, not 0.
   - ○ Since $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$, the series $\sum_{n=0}^{\infty} \frac{n}{n^2+1}$ diverges.
   - ● The Divergence Test requires that the limit be different from 0, and cannot prove that a series converges.
   - ○ The Divergence Test doesn't work on a series with only positive terms.

2. (3 points) Integration by parts is the reverse version of which rule?
   - ○ Chain Rule ...... $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$
   - ○ Power Rule ...... $\frac{d}{dx}[x^p] = px^{-1}$
   - ● Product Rule ...... $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$
   - ○ Exponential Rule ...... $\frac{d}{dx}[b^x] = b^x \ln b$

3. (4 points) Since $\sin(x)$ has the MacLaurin Series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, which of these is the best approximating polynomial for the value of $\sin(x)$ when x is close to 0?
   - ○ $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$
   - ○ $1 + x^2 + x^3 + x^4$
   - ● $x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$
   - ○ $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

[1] I can't back that up, but I feel like he would, y'know?

Full Solutions (90 points total)

Please show all work and draw a $\boxed{\text{box}}$ around your final answer, if appropriate. Solutions will be graded according to the rubrics given in the practice midterm.

1. (10 points) Find a general formula for the sequence $\left\{\frac{3}{2}, -\frac{4}{4}, \frac{5}{8}, -\frac{6}{16}, \frac{7}{32}, \dots\right\}$.

$$\boxed{\left\{\frac{(-1)^{n+1}(n+2)}{2^n}\right\}_{n=1}^{\infty}} \quad \text{OR} \quad \boxed{\left\{\frac{(-1)^{n}(n+3)}{2^{n+1}}\right\}_{n=0}^{\infty}}$$

or possibly others...

2. (10 points) Does the series $\sum_{n=1}^{\infty} \frac{(-2)^{n-1}}{3^n}$ converge or diverge? If it converges, give its sum.

$$= \sum_{n=1}^{\infty} \underbrace{\left(\tfrac{1}{3}\right)}_{a}\underbrace{\left(-\tfrac{2}{3}\right)}_{r}^{n-1} = \frac{1/3}{1-(-2/3)} = \frac{1/3}{5/3} = \boxed{1/5}$$

$\boxed{\text{converges}}$ to

$$\left(|r| = |-2/3| < 1 \right)$$

3. (10 points) Determine whether or not $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \leftarrow \underline{\text{divergent}} \ p\text{-Series}$$

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is an alternating series:

- $\frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}}$ (non-increasing)
- $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$

thus it <u>converges</u> by A.S.T.

Therefore, the series is $\boxed{\text{conditionally convergent.}}$

$\left(\begin{array}{l} \text{Compare to} \\ \sum \frac{n^2}{n^3} = \sum \frac{1}{n}, \text{ divergent} \end{array} \right)$

4. (10 points) Determine whether the series $\sum_{n=0}^{\infty} \frac{3n^2}{n^3+1}$ converges or diverges.

LCT

$$\lim_{n\to\infty} \frac{\frac{3n^2}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n^3}{n^3+1} = 3 > 0$$

Thus the series match, so $\sum_{n=0}^{\infty} \frac{3n^2}{n^3+1}$ also $\boxed{\text{diverges}}$.

DCT

$$\frac{2}{n} = \frac{3n^2}{3/2\, n^3} = \frac{3n^2}{n^3 + \frac{1}{2}n^3} \leq \frac{3n^2}{n^3+1}$$

Since $\sum \frac{2}{n}$ diverges, the larger $\sum \frac{3n^2}{n^3+1}$ also $\boxed{\text{diverges}}$.

Integral

$$\lim_{b\to\infty} \int_0^b \frac{3x^2}{x^3+1}\,dx = \lim_{b\to\infty} \Big[\ln|x^3+1| \Big]_0^b = \lim_{b\to\infty} \ln|x^3+1| \text{ DNE } (\infty)$$

Thus $\sum \frac{3n^2}{n^3+1}$ $\boxed{\text{diverges}}$.

$$\left(\text{Compare to } \sum \frac{\sqrt{n}}{n^2} = \sum \frac{1}{n^{3/2}}\right)$$

conv. by p-series

5. (10 points) Determine whether the series $\sum_{n=2}^{\infty} \frac{\sqrt{n-1}}{2+n^2}$ converges or diverges.

DCT

$$\frac{\sqrt{n-1}}{2+n^2} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

Since $\sum \frac{1}{n^{3/2}}$ converges, the smaller $\sum \frac{\sqrt{n-1}}{2+n^2}$ also $\boxed{\text{converges}}$.

LCT

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n-1}}{2+n^2}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n-1}}{2+n^2} = \lim_{n\to\infty} \frac{\cancel{n^{3/2}}\sqrt{n-1}}{\cancel{n^{3/2}}\left(\cancel{\frac{2}{n^{3/2}}}+\sqrt{n}\right)} = \sqrt{\lim_{n\to\infty} \frac{n-1}{n}} = \sqrt{1} = 1 > 0$$

Both series match, so $\sum \frac{\sqrt{n-1}}{2+n^2}$ also $\boxed{\text{converges}}$.

6. (10 points) For what values of x is the series $\sum_{n=0}^{\infty} \frac{(1-x)^n}{n+1}$ convergent? What is its radius of convergence?

Root Test

$$\lim_{n\to\infty} \left| \frac{(1-x)^n}{n+1} \right|^{1/n} = \lim_{n\to\infty} \frac{|1-x|}{(n+1)^{1/n}} = |1-x| < 1$$

radius of conv. 1

OR Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(1-x)^{n+1}}{n+2}}{\frac{(1-x)^n}{n+1}} \right| = \lim_{n\to\infty} \frac{|1-x|^{\cancel{n+1}}}{n+2} \frac{n+1}{\cancel{|1-x|^n}} = |1-x| \left(\lim_{n\to\infty} \frac{n+1}{n+2}\right) = |1-x| < 1$$

(Either way...)

$$-1 < 1-x < 1$$

$$-1 < x-1 < 1$$

$$0 < x \leq 2$$

Case $x=0$

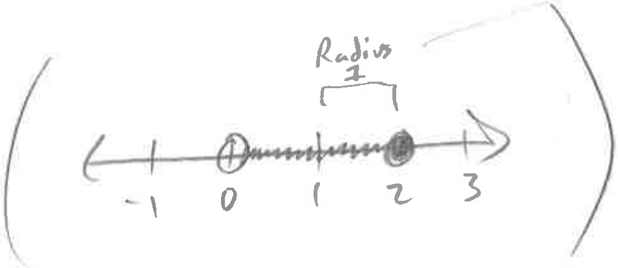
$$\sum \frac{(1)^n}{n+1} = \sum \frac{1}{n+1}$$

diverges

Case $x=2$

$$\sum \frac{(-1)^n}{n+1}$$

converges



7. (10 points) Give a power series representing the function $f(x) = \frac{1}{1+3x}$ and its radius of convergence.

$$f(x) = \frac{1}{1-(-3x)} = \sum_{n=0}^{\infty} (1)(-3x)^n = \boxed{\sum_{n=0}^{\infty} (-3)^n x^n}$$

$$|-3x| < 1$$

$$|x| < \frac{1}{3}$$

$\boxed{\text{radius} = \frac{1}{3}}$

8. (10 points) Find the Maclaurin series representing the "hyperbolic cosine" function

$$f(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$f^{(0)}(x) = \frac{e^x + e^{-x}}{2}$

$f^{(1)}(x) = \frac{e^x - e^{-x}}{2}$

$f^{(2)}(x) = \frac{e^x + e^{-x}}{2}$

$f^{(3)}(x) = \frac{e^x - e^{-x}}{2}$

$f^{(0)}(0) = \frac{1+1}{2} = 1$

$f^{(1)}(0) = \frac{1-1}{2} = 0$

$f^{(2)}(0) = \frac{1+1}{2} = 1$

$f^{(3)}(0) = \frac{1-1}{2} = 0$

Evens: 1
Odds: 0

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{\overset{1}{\cancel{f^{(2n)}(0)}}}{(2n)!} x^{2n} = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$$

(since odd derivatives are zero)

9. (10 points) Evaluate $\int \underbrace{4x}_{u}\underbrace{e^x\,dx}_{dv}$.

Let $u = 4x$ $\quad v = e^x$

$du = 4dx$ $\quad dv = e^x dx$

$= \underbrace{4x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\underbrace{4dx}_{du}$

$= 4xe^x - 4\int e^x dx$

$= \boxed{4xe^x - 4e^x + C}$

10. (5 points) (BONUS - no partial credit)

A common mistake I see Calculus I students do when taking derivatives is the following:

$$\frac{d}{dx}\left[x^2\sin(x)\right] \neq \frac{d}{dx}\left[x^2\right]\frac{d}{dx}\left[\sin(x)\right] = 2x\cos(x)$$

instead of using the product rule to get the correct answer $x^2\cos(x) + 2x\sin(x)$.

Prove that this "freshman product rule"

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g'(x)$$

actually works if $g(x) = e^{\int \frac{f'(x)}{f'(x)-f(x)}\,dx}$. (An example is when $f(x) = g(x) = e^{2x}$.)

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\,g'(x) + f'(x)\,g(x)$$

$$= f(x)\left(\frac{f'(x)}{f'(x)-f(x)}\,e^{\int\frac{f'(x)}{f'(x)-f(x)}dx}\right) + f'(x)\,e^{\int\frac{f'(x)}{f'(x)-f(x)}dx}$$

$$= \frac{\cancel{f(x)f'(x)}}{f'(x)-f(x)}\,e^{\int\frac{f'(x)}{f'(x)-f(x)}dx} + \frac{f'(x)f'(x)-\cancel{f'(x)f(x)}}{f'(x)-f(x)}\,e^{\int\frac{f'(x)}{f'(x)-f(x)}dx}$$

$$= f'(x)\left(\frac{f'(x)}{f'(x)-f(x)}\,e^{\int\frac{f'(x)}{f'(x)-f(x)}dx}\right)$$

$$= f'(x)\,g'(x) \quad \square$$