11.8 Power Series

Definition 100 (Power Series). A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 x^0 + c_1 x^1 + x_2 x^2 + \cdots$$

where c_i represent coefficients and x denotes our variable.

Definition 101 (Power Series Centered at a). A **power series centered at** a is a series of the form

$$\sum_{n=1}^{\infty} c_n (x-a)^n .$$

Note 102. When $x = a$, all of the terms are 0; so, naturally the power series centered at a always converges when $x = a$.

Problem 103. For what values of x is the series $\sum_{n=0}^{\infty} n! x^n$ convergent?

Ratio Test

$$\lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = \lim_{n\to\infty} \frac{\cancel{n!}(n+1)\cancel{|x|^n}|x|}{\cancel{n!}\cancel{|x|^n}} = \lim_{n\to\infty} (n+1)|x|$$

$\xrightarrow{x=0} 0 < 1$ converges

$\xrightarrow{x\neq 0} \infty > 1$ diverges

$$\boxed{x = 0}$$

Problem 104. For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

Root Test

$$\lim_{n\to\infty} \left| \frac{(x-3)^n}{n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|x-3|}{n^{1/n}} = |x-3| < 1$$

$$-1 < x-3 < 1$$

$$2 < x < 4$$

Case $x=2$

$$\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{converges}$$

Case $x=4$

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges}$$

$$\boxed{2 \le x < 4}$$

Theorem 105 (Radius of Convergence). For a given power series $\sum_{n=1}^{\infty} c_n (x-a)^n$, there are only three possibilities:

- The series converges only when $x = a$,
- The series converges for all $x \in \mathbb{R}$, and
- There is a positive real number R such that the series converges if $|x-a| < R$ and diverges when $|x-a| > R$.

This number R is called the **radius of convergence**.

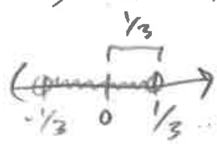
Problem 106. Find the radius of convergence for the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+2}}$.

Root Test

$$\lim_{n\to\infty}\left|\frac{(-3)^n x^n}{(n+2)^{1/2}}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{3|x|}{\cancel{(n+2)^{\frac{1}{2n}}}} = 3|x| < 1$$

$$|x| < \frac{1}{3} \rightarrow \boxed{\text{Radius} = 1/3}$$

$$-\frac{1}{3} < x < \frac{1}{3}$$



Problem 107. Find the radius of convergence for the series $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

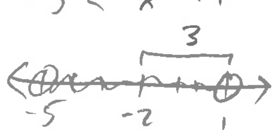
Root Test

$$\lim_{n\to\infty}\left|\frac{n(x+2)^n}{3^{n+1}}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{\cancel{n^{1/n}}|x+2|}{3^{1+\cancel{1/n}}} = \frac{|x+2|}{3} < 1$$

$$|x+2| < 3 \rightarrow \boxed{\text{Radius} = 3}$$

$$-3 < x+2 < 3$$

$$-5 < x < 1$$



Suggested Problems: Section 11.8 numbers 3, 5, 7, 9, 10, 11, 13, 16, 18, 19

11.9 Representation of Functions as Power Series

Recall 108. The geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

converges when $|x| < 1$ and diverges otherwise.

Problem 109. Express $\dfrac{1}{1+x^2}$ as a power series and find its radius of convergence.

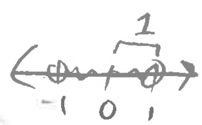
$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n x^{2n}}$$

$$|-x^2| < 1$$

$$|x|^2 < 1$$

$$|x| < \sqrt{1} = 1$$

$$-1 < x < 1$$



$$\boxed{\text{Radius} = 1}$$

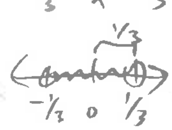
Problem 110. Find a power series representation for $\dfrac{1}{1-3x}$ and find its radius of convergence.

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty}(3x)^n = \boxed{\sum_{n=0}^{\infty} 3^n x^n}$$

$$|3x| < 1$$

$$|x| < \frac{1}{3}$$

$$-\frac{1}{3} < x < \frac{1}{3}$$



$$\boxed{\text{Radius} = \frac{1}{3}}$$

Problem 111. Find a power series representation of $\frac{1}{x+2}$ and find its radius of convergence.

$$= \frac{1}{1-(-x-1)} = \sum_{n=0}^{\infty} (-x-1)^n$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n (x+1)^n}$$

$$|-x-1| < 1$$

$$|x+1| < 1 \longrightarrow \boxed{\text{Radius} = 1}$$

$$\left(\begin{array}{c} -1 < x+1 < 1 \\ -2 < x < 0 \end{array} \right)$$

OR

$$= \frac{\frac{1}{2}}{1-(-\frac{x}{2})} = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)\left(-\tfrac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} (-1)^n \frac{1}{2^n} x^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n}$$

$$\left|-\frac{x}{2}\right| < 1$$

$$\frac{|x|}{2} < 1$$

$$|x| < 2 \longrightarrow \boxed{\text{Radius} = 2}$$

$$\left(-2 < x < 2 \right)$$

Problem 112. Find a power series representation of $\frac{x^3}{1-x}$ and find its radius of convergence.

$$= \sum_{n=0}^{\infty} (x^3)(x)^n = \boxed{\sum_{n=0}^{\infty} x^{n+3}} \overset{\text{OR}}{=} \boxed{\sum_{n=3}^{\infty} x^n}$$

$$|x| < 1 \longrightarrow \boxed{\text{Radius} = 1}$$

$$\left(-1 < x < 1 \right)$$

Theorem 113 (Integration and Differentiation). If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

has the following derivative and integral defined within the same radius of convergence:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$$

$$\int f(x)\,dx = \left(c_0(x-a) + \frac{1}{2}c_1(x-a)^2 + \frac{1}{3}c_2(x-a)^3 + \dots\right) + C = \left(\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}\right) + C$$

Problem 114. Express $\dfrac{1}{(1-x)^2}$ as a power series.

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right]$$

$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] \xrightarrow{\text{OR}} = \frac{d}{dx}\left[1 + x + x^2 + x^3 + \cdots\right]$$

OR

$$= \sum_{n=0}^{\infty} \frac{d}{dx}\left[x^n\right]$$

$$= \boxed{\sum_{n=0}^{\infty} n x^{n-1}}$$

$$= 0 + 1 + 2x + 3x^2 + \cdots$$

$$= \boxed{\sum_{n=1}^{\infty} n x^{n-1}}$$

$$\overset{\text{OR}}{=} \boxed{\sum_{n=0}^{\infty} (n+1) x^n}$$

Problem 115. Find a power series representation for $\ln(1+x)$.

$$\ln(1+x) = \int \frac{1}{1+x}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx \xrightarrow{OR} = \int 1 - x + x^2 - x^3 + \cdots \, dx$$

$\downarrow$ OR

$$= \sum_{n=0}^{\infty} (-1)^n \int x^n \, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

$$\ln(1+0) = \cancel{\sum_{n=0}^{\infty} (-1)^n \frac{0^{n+1}}{n+1}} + C$$

$$0 = C$$

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C$$

$$\ln(1+0) = \cancel{0 - 0 + 0 - 0 + \cdots} + C$$

$$0 = C$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}}$$

Problem 116. Find a power series representation for $f(x) = \arctan(x)$

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx$$

(Can also do infinite sum as above.)

$$= \sum_{n=0}^{\infty} (-1)^n \int x^{2n} \, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$$\arctan(0) = \cancel{\sum_{n=0}^{\infty} (-1)^n \frac{0^{2n+1}}{2n+1}} + C$$

$$0 = C$$

$$\boxed{\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

Suggested Problems: Section 11.9 numbers 6, 7, 9, 10, 12, 17, 27, 39