Chapter 12

Vectors and the Geometry of Space

12.1 Operations in Two and Three Dimensional Space

12.1.1 Points

Definition 1. Let $\mathbb{R}$ be the collection of real numbers, let $\mathbb{R}^2$ be the collection of all **ordered pairs** of real numbers, and let $\mathbb{R}^3$ be the collection of all **ordered triples** of real numbers.

$\mathbb{R}$ is known as the **real line**, $\mathbb{R}^2$ is known as the **real plane** or the xy**-plane**, and $\mathbb{R}^3$ is known as **real (3D) space** or xyz**-space**.

Definition 2. The **distance** between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in $\mathbb{R}^2$ is given by the formula

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The **distance** between two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in $\mathbb{R}^3$ is given by the formula

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Problem 3. Plot and find the distance between the following pairs of points:

- $(-2, 6)$ and $(3, -6)$
- $(0, 0, 0)$ and $(4, 2, 4)$
- $(3, 7, -2)$ and $(-1, 7, 1)$
- $(8, 2, 1)$ and $(4, -2, 7)$

Definition 4. **Simple lines** in $\mathbb{R}^2$ are given by the relations $x = a$, and $y = b$ for real numbers a, b.

Simple planes in $\mathbb{R}^3$ are given by the relations $x = a$, $y = b$, $z = c$ for real numbers a, b, c.

Definition 5. A **circle** in $\mathbb{R}^2$ is the set of all points a fixed distance (called its **radius**) from a fixed point (called its **center**). For a center (a, b) and radius r, the equation for a circle is

$$(x-a)^2 + (y-b)^2 = r^2$$

A **sphere** in $\mathbb{R}^3$ is the set of all points a fixed distance (called its **radius**) from a fixed point (called its **center**). For a center (a, b, c) and radius r, the equation for a sphere is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Question 6. Sketch the following curves and surfaces.

- $x = 3$ in the xy-plane and xyz-space.
- $y = -1$ in the xy-plane and xyz-space.
- $z = 0$ in xyz-space.
- $(x-2)^2 + (y+1)^2 = 9$ in the xy-plane.
- $x^2 + y^2 + z^2 = 4$ in xyz-space.
- $x^2 + (y-1)^2 + z^2 = 1$ in xyz-space.

12.1.2 Vectors

Definition 7 (Vector)**.** A **vector** is a mathematical object that stores a **magnitude** (often thought of as length) and **direction**. Two vectors are **equal** if and only if they have the same magnitude and direction.

Definition 8. For a given point $P = (a, b)$ in $\mathbb{R}^2$, its **position vector** is given by $\overrightarrow{\mathbf{P}} = \langle a, b\rangle$: the vector from the origin $(0, 0)$ to the point $P = (a, b)$.

For a given point $P = (a, b, c)$ in $\mathbb{R}^3$, its **position vector** is given by $\overrightarrow{\mathbf{P}} = \langle a, b, c\rangle$: the vector from the origin $(0, 0, 0)$ to the point $P = (a, b, c)$.

Theorem 9. Two vectors are equal if and only if they share the same magnitude and direction as a common position vector.

Definition 10. Since all vectors are equal to some position vector $\langle a, b\rangle$ or $\langle a, b, c\rangle$, we usually define vectors by a position vector written in this **component form**. Since the component form of a vector stores the same information as a point, we will use both interchangeably, that is, $\langle a, b\rangle = (a, b) \in \mathbb{R}^2$ and $\langle a, b, c\rangle = (a, b, c) \in \mathbb{R}^3$ (although we usually sketch them differently).

Problem 11. Plot the following points and position vectors.

- $(1, 3)$ and $\langle 1, 3\rangle$ in the xy-plane.
- $(-2, 5)$ and $\langle -2, 5\rangle$ in the xy-plane.
- $(1, 1, -3)$ and $\langle 1, 1, -3\rangle$ in xyz-space.
- $(0, 5, 0)$ and $\langle 0, 5, 0\rangle$ in xyz-space.

Definition 12. Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$. Then the vector with initial point P and terminal point Q is defined as

$$\overrightarrow{\mathbf{PQ}} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Problem 13. Plot and sketch the points P, Q and the vector $\overrightarrow{\mathbf{PQ}}$ for each.

- $P = (1, 3)$, $Q = (-3, 6)$ in the xy-plane
- $P = (-2, 0, 3)$, $Q = (1, 3, -3)$ in xyz-space

Definition 14. The magnitude of a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is the distance between its initial and terminal points.

Theorem 15. The magnitude of $\langle a, b \rangle$ is given by $\sqrt{a^2 + b^2}$, and the magnitude of $\langle a, b, c \rangle$ is given by $\sqrt{a^2 + b^2 + c^2}$.

Problem 16. Give the magitude of $\overrightarrow{\mathbf{PQ}}$ for each bullet in the previous problem.

12.1.3 Operations

Definition 17. Vector addition is defined component-wise as follows for $\mathbb{R}^2$ and $\mathbb{R}^3$

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$$

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

Definition 18. A **scalar** is simply a real number by itself (as opposed to a vector of real numbers).

Definition 19. Scalar multiplication of a vector is defined component-wise as follows for $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$k\vec{\mathbf{u}} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle$$

$$k\vec{\mathbf{u}} = k\langle u_1, u_2, u_3 \rangle = \langle ku_1, ku_2, ku_3 \rangle$$

Definition 20 (Unit Vector)**.** A **unit vector** is a vector whose magnitude is 1. Note that we can given a vector $\vec{\mathbf{v}}$, we can form a unit vector $\widehat{\mathbf{v}}$ by dividing by the magnitude of $\vec{\mathbf{v}}$. That is to say, Let $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then

$$\widehat{\mathbf{v}} = \frac{1}{|\vec{\mathbf{v}}|} \langle v_1, v_2, v_3 \rangle .$$

Definition 21 (Standard Vectors)**.** Any vector can be denoted as the linear combination of the **standard unit vectors** $\widehat{\mathbf{i}} = \langle 1, 0, 0 \rangle$, $\widehat{\mathbf{j}} = \langle 0, 1, 0 \rangle$, and $\widehat{\mathbf{k}} = \langle 0, 0, 1 \rangle$. So given a vector $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$, one can express it with respect to the standard vectors as

$$\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle = v_1\widehat{\mathbf{i}} + v_2\widehat{\mathbf{j}} + v_3\widehat{\mathbf{k}}.$$

This text, however, will more often than not use the angle brace notation.

Definition 22 (Dot Product)**.** Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then the dot product or Euclidean Inner Product as it is sometimes referred is

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1v_1 + u_2v_2 + u_3v_3 = |\vec{\mathbf{u}}| \, |\vec{\mathbf{v}}| \cos(\theta) .$$

Theorem 23. Two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are **orthogonal** if and only if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.

Problem 24. Show that if two non-zero vector are orthogonal then $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.

Definition 25 (Cross Product)**.** Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then the cross product is the determinant of the following matrix:

$$\begin{aligned}
\vec{\mathbf{u}} \times \vec{\mathbf{v}} &= \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \widehat{\mathbf{i}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \widehat{\mathbf{j}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \widehat{\mathbf{k}} \\
&= \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle .
\end{aligned}$$

Observation 26. The cross product of two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ gives us a vector that is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

Definition 27 (Equivalent to Cross Product)**.** Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then the cross product can also be defined as

$$|\vec{\mathbf{u}} \times \vec{\mathbf{v}}| = |\vec{\mathbf{u}}| \, |\vec{\mathbf{v}}| \sin(\theta) .$$

Problem 28. Show that is two non-zero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel if and only if $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$.

Suggested Homework: Section 12.1 numbers 4, 6, 7, 8, 10, 11, 12, 14, 15, 16
Section 12.2 numbers 3, 5, 13, 14, 15, 19, 21, 24, 26
Section 12.3 numbers 3, 5, 6, 7, 8, 9, 10, 11, 15, 17, 21, 27, 41, 42, 44
Section 12.4 numbers $1 - 3$, 17, 19, 28, 29, 33, 35

12.5 Equations in 3-Space

Equation 29 (Parametrization of a Line)**.** Let $O = (0,0,0)$ be the origin in $\mathbb{R}^3, P_0 = (x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$, and $\vec{\mathbf{v}} = \langle A, B, C\rangle$ be a vector in $\mathbb{R}^3$ parallel to the line being parametrized. Then the line through P_0 parallel to $\vec{\mathbf{v}}$ is

$$\vec{\mathbf{r}}(t) = \overrightarrow{\mathbf{OP_0}} + t\vec{\mathbf{v}} \qquad t \in \mathbb{R}.$$

This can also be written as

$$x = x_0 + At, \qquad y = y_0 + Bt, \qquad z = z_0 + Ct \qquad t \in \mathbb{R}.$$

or as the symmetric equation

$$\frac{x - x_0}{A} = \frac{y - y_o}{B} = \frac{z - z_0}{C}.$$

Equation 30 (Parametrization of a Line Segment)**.** Let O denote the origin, P be the initial point of a line segment, and Q be the terminal point of a line segment. Then the line segment $\overline{PQ}$ can be parametrized as

$$\vec{\mathbf{r}}(t) = (1-t)\overrightarrow{\mathbf{OP}} + t\overrightarrow{\mathbf{OQ}} \qquad 0 \le t \le 1.$$

Problem 31. Find a vector equation and parametric equation for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\langle 1, 4, -2\rangle$.

Problem 32. Find the parametric Equation of the line segment from $(2, 4, -3)$ to $(3, -1, 1)$.

Equation 33 (Planes)**.** Let $P_0 = (x_0, y_0, z_0)$ be a point in the plane and $\vec{\mathbf{n}} = \langle a, b, c\rangle$ be a vector normal to the plane. Then the equation of the plane is

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0)\,.$$

Suggested Homework: Section 12.5 numbers 3, 4, 6, 7, 17, 19, 24, 27, 31, 32