Chapter 12

Vectors and the Geometry of Space

12.1 Two and Three Dimensional Space

Definition 1. Let $\mathbb{R}$ be the collection of real numbers, let $\mathbb{R}^2$ be the collection of all **ordered pairs** of real numbers, and let $\mathbb{R}^3$ be the collection of all **ordered triples** of real numbers.

$\mathbb{R}$ is known as the **real line**, $\mathbb{R}^2$ is known as the **real plane** or the xy**-plane**, and $\mathbb{R}^3$ is known as **real (3D) space** or xyz**-space**.

Definition 2. The **distance** between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in $\mathbb{R}^2$ is given by the formula

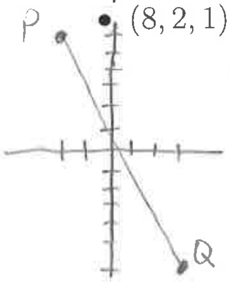
$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The **distance** between two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in $\mathbb{R}^3$ is given by the formula

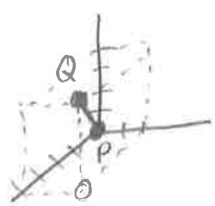
$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Problem 3. Plot and find the distance between the following pairs of points:

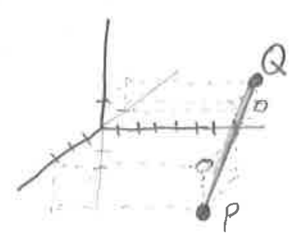
- $\overset{P}{(-2,6)}$ and $\overset{Q}{(3,-6)}$
- $\overset{P}{(0,0,0)}$ and $\overset{Q}{(4,2,4)}$
- $\overset{P}{(3,7,-2)}$ and $\overset{Q}{(-1,7,1)}$
- $\overset{P}{(8,2,1)}$ and $\overset{Q}{(4,-2,7)}$



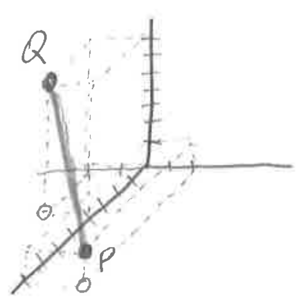
$$d(P,Q) = \sqrt{(3+2)^2 + (-6-6)^2} = \sqrt{5^2 + 12^2} = \sqrt{25+144} = \sqrt{169} = \boxed{13}$$



$$d(P,Q) = \sqrt{(4-0)^2 + (2-0)^2 + (4-0)^2} = \sqrt{4^2+2^2+4^2} = \sqrt{16+4+16} = \sqrt{36} = \boxed{6}$$



$$d(P,Q) = \sqrt{(-1-3)^2 + \cancel{(7-7)^2} + (1- -2)^2} = \sqrt{4^2+3^2} = \sqrt{25} = \boxed{5}$$



$$d(P,Q) = \sqrt{(8-4)^2 + (-2-2)^2 + (7-1)^2} = \sqrt{4^2+4^2+6^2} = \sqrt{16+16+36} = \sqrt{68} = \boxed{2\sqrt{17}}$$

Definition 4. **Simple lines** in $\mathbb{R}^2$ are given by the relations $x = a$, and $y = b$ for real numbers a, b.

Simple planes in $\mathbb{R}^3$ are given by the relations $x = a$, $y = b$, $z = c$ for real numbers a, b, c.

Definition 5. A **circle** in $\mathbb{R}^2$ is the set of all points a fixed distance (called its **radius**) from a fixed point (called its **center**). For a center (a, b) and radius r, the equation for a circle is

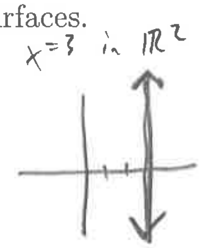
$$(x-a)^2 + (y-b)^2 = r^2$$

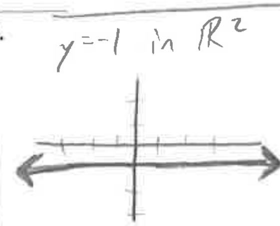
A **sphere** in $\mathbb{R}^3$ is the set of all points a fixed distance (called its **radius**) from a fixed point (called its **center**). For a center (a, b, c) and radius r, the equation for a sphere is

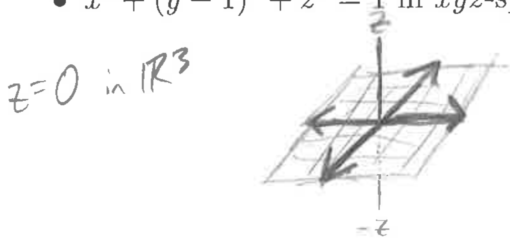
$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

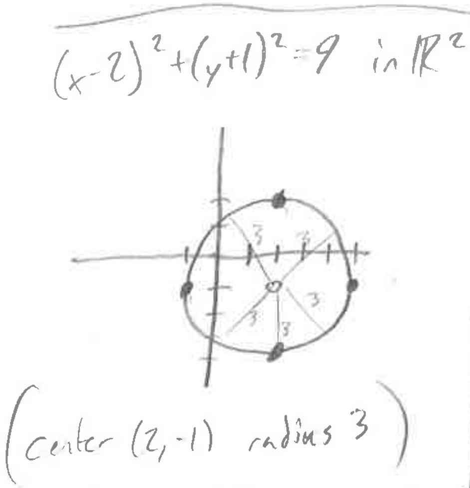
Question 6. Sketch the following curves and surfaces.

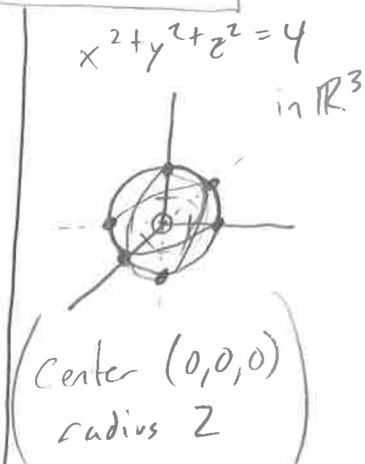
- $x = 3$ in the xy-plane and xyz-space.
- $y = -1$ in the xy-plane and xyz-space.
- $z = 0$ in xyz-space.
- $(x-2)^2 + (y+1)^2 = 9$ in the xy-plane.
- $x^2 + y^2 + z^2 = 4$ in xyz-space.
- $x^2 + (y-1)^2 + z^2 = 1$ in xyz-space.

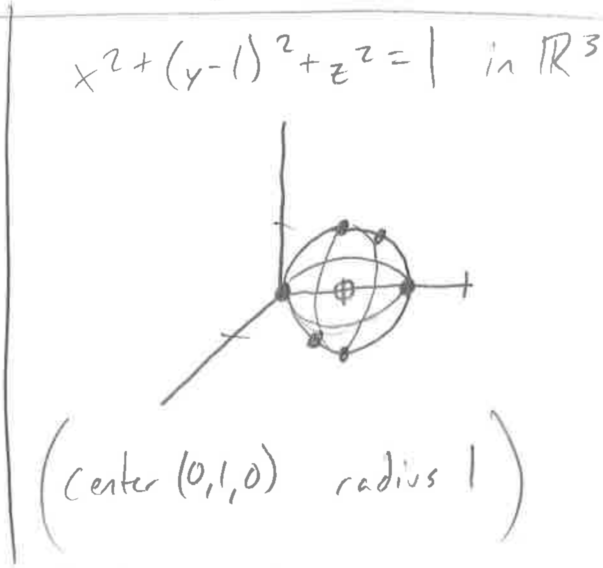












Suggested Homework: Section 12.1 numbers 4, 6, 7, 8, 10, 11, 12, 14, 15, 16

12.2 Vectors

Definition 7 (Vector). A **vector** $\vec{\mathbf{v}}$ is a mathematical object that stores a **magnitude** (a nonnegative real number often thought of as length) and **direction**. Two vectors are **equal** if and only if they have the same magnitude and direction.

Definition 8. The **zero vector** $\vec{\mathbf{0}}$ has zero magnitude and no direction. (This is the only vector without a direction.)

Definition 9. For a given point $P = (a, b)$ in $\mathbb{R}^2$, its **position vector** is given by $\vec{\mathbf{P}} = \langle a, b\rangle$: the vector from the origin $(0, 0)$ to the point $P = (a, b)$.

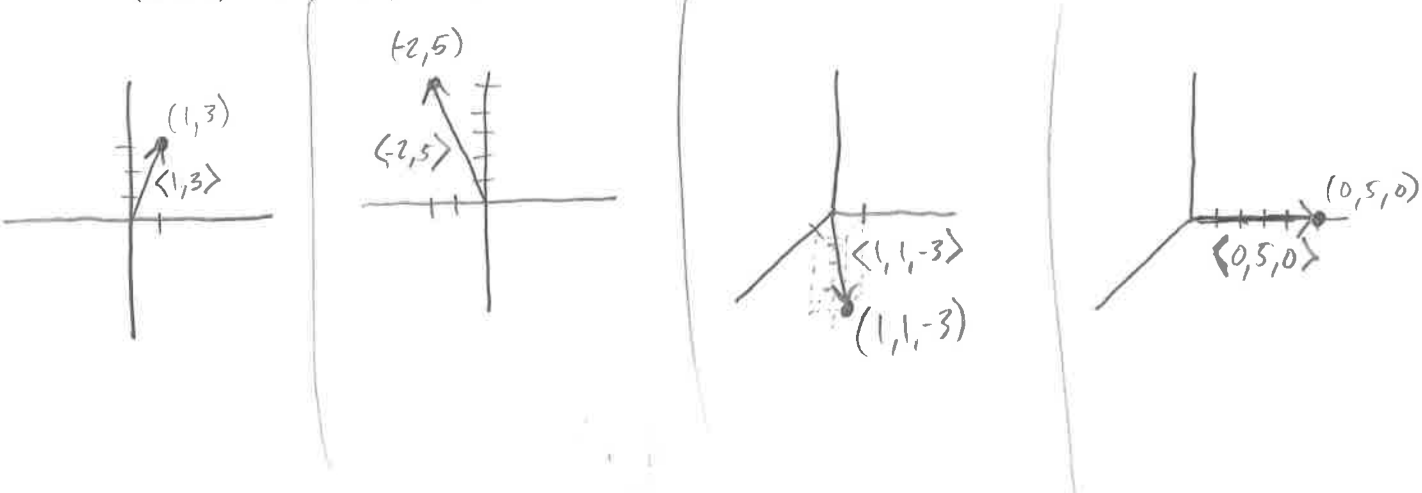
For a given point $P = (a, b, c)$ in $\mathbb{R}^3$, its **position vector** is given by $\vec{\mathbf{P}} = \langle a, b, c\rangle$: the vector from the origin $(0, 0, 0)$ to the point $P = (a, b, c)$.

Theorem 10. Two vectors are equal if and only if they share the same magnitude and direction as a common position vector.

Definition 11. Since all vectors are equal to some position vector $\langle a, b\rangle$ or $\langle a, b, c\rangle$, we usually define vectors by a position vector written in this **component form**. Since the component form of a vector stores the same information as a point, we will use both interchangeably, that is, $\langle a, b\rangle = (a, b) \in \mathbb{R}^2$ and $\langle a, b, c\rangle = (a, b, c) \in \mathbb{R}^3$ (although we usually sketch them differently).

Problem 12. Plot the following points and position vectors.

- $(1, 3)$ and $\langle 1, 3\rangle$ in the xy-plane.
- $(-2, 5)$ and $\langle -2, 5\rangle$ in the xy-plane.
- $(1, 1, -3)$ and $\langle 1, 1, -3\rangle$ in xyz-space.
- $(0, 5, 0)$ and $\langle 0, 5, 0\rangle$ in xyz-space.

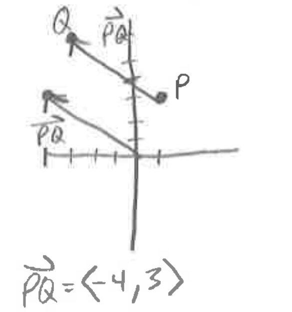


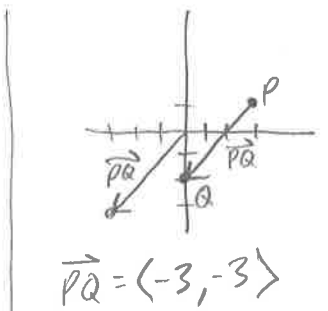
Definition 13. Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$. Then the vector with initial point P and terminal point Q is defined as

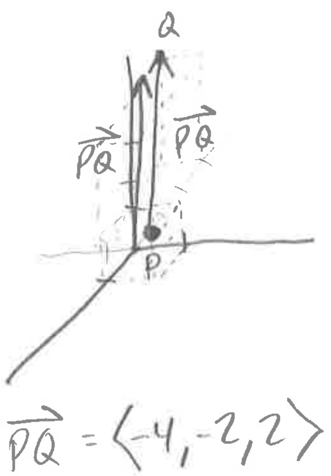
$$\overrightarrow{\mathbf{PQ}} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

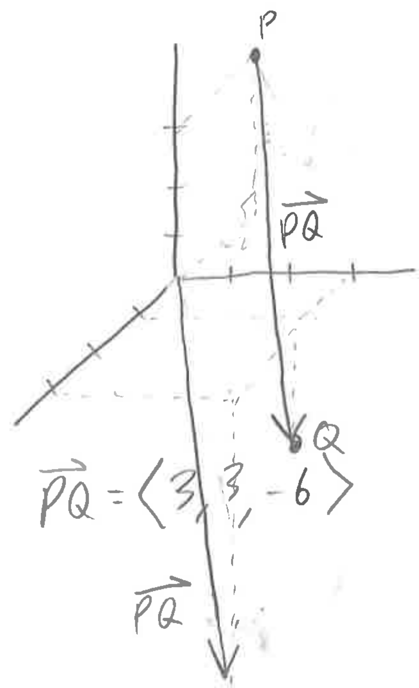
Problem 14. Plot and sketch the points P, Q and the vector $\overrightarrow{\mathbf{PQ}}$ for each.

- $P = (1, 3)$, $Q = (-3, 6)$ in the xy-plane
- $P = (3, 1)$, $Q = (0, -2)$ in the xy-plane
- $P = (1, 1, 1)$, $Q = (-3, -1, 3)$ in xyz-space
- $P = (-2, 0, 3)$, $Q = (1, 3, -3)$ in xyz-space









Definition 15. The magnitude $|\vec{\mathbf{v}}|$ of a vector $\vec{\mathbf{v}}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is the distance between its initial and terminal points.

Theorem 16. The magnitude of $\vec{\mathbf{v}} = \langle a, b \rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{a^2 + b^2}$$

The magnitude of $\vec{\mathbf{v}} = \langle a, b, c \rangle$ is given by

$$|\vec{\mathbf{v}}| = \sqrt{a^2 + b^2 + c^2}$$

Problem 17. Evaluate the magnitude of the following vectors:

- $|\langle 5,5\rangle| = \sqrt{25+25} = 5\sqrt{2}$
- $|\langle -4,3\rangle| = \sqrt{16+9} = 5$
- $|\langle 12,-5\rangle| = \sqrt{144+25} = \sqrt{169} = 13$
- $|\langle 3,1,-2\rangle| = \sqrt{9+1+4} = \sqrt{14}$
- $|\langle 4,-2,-4\rangle| = \sqrt{16+4+16} = \sqrt{36} = 6$
- $|\langle 8,0,-6\rangle| = \sqrt{64+0+36} = \sqrt{100} = 10$

12.2.1 Basic Vector Operations

Definition 18. Vector addition is defined component-wise as follows for $\mathbb{R}^2$ and $\mathbb{R}^3$

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1, u_2\rangle + \langle v_1, v_2\rangle = \langle u_1 + v_1, u_2 + v_2\rangle$$

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle u_1, u_2, u_3\rangle + \langle v_1, v_2, v_3\rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle$$

Definition 19. A **scalar** is simply a real number by itself (as opposed to a vector of real numbers).

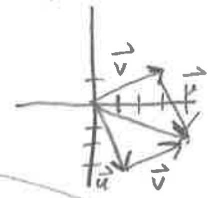
Definition 20. Scalar multiplication of a vector is defined component-wise as follows for $\mathbb{R}^2$ and $\mathbb{R}^3$:

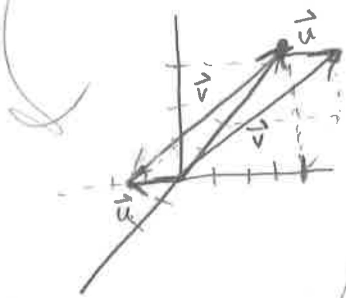
$$k\vec{\mathbf{u}} = k\langle u_1, u_2\rangle = \langle ku_1, ku_2\rangle$$

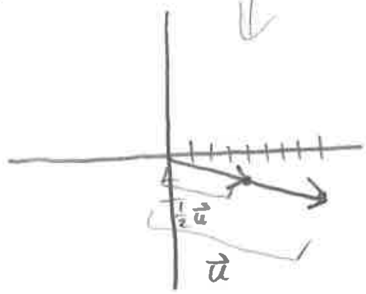
$$k\vec{\mathbf{u}} = k\langle u_1, u_2, u_3\rangle = \langle ku_1, ku_2, ku_3\rangle$$

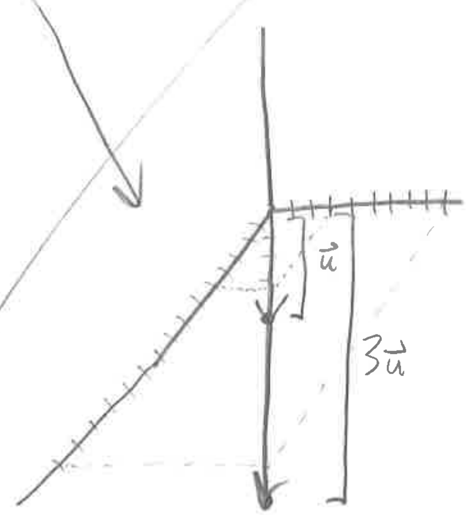
Problem 21. Sketch the following vectors.

- $\vec{\mathbf{u}} = \langle 1, -3\rangle$, $\vec{\mathbf{v}} = \langle 3, 1\rangle$ and $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ $= \langle 4, -2\rangle$ in the xy-plane.
- $\vec{\mathbf{u}} = \langle 2, 0, 1\rangle$, $\vec{\mathbf{v}} = \langle -2, 4, 2\rangle$ and $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ $= \langle 0, 4, 3\rangle$ in xyz-space.
- $\vec{\mathbf{u}} = \langle 8, -2\rangle$ and $\frac{1}{2}\vec{\mathbf{u}}$ $= \langle 4, -1\rangle$ in the xy-plane.
- $\vec{\mathbf{u}} = \langle 5, 3, -1\rangle$ and $3\vec{\mathbf{u}}$ $= \langle 15, 9, -3\rangle$ in xyz-space.









Definition 22. A vector $\vec{\mathbf{v}}$ is a **unit vector** if $|\vec{\mathbf{v}}| = 1$.

Theorem 23. For any non-zero vector $\vec{\mathbf{v}}$, the vector

$$\frac{1}{|\vec{\mathbf{v}}|}\vec{\mathbf{v}} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

is a unit vector.

Definition 24. The **direction** of a vector $\vec{\mathbf{v}}$ is the unit vector $\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$.

Theorem 25. Any vector $\vec{\mathbf{v}}$ is the scalar product of its magnitude and direction:

$$\vec{\mathbf{v}} = |\vec{\mathbf{v}}|\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

Problem 26. Write the following vectors as the scalar product of their magnitude and direction:

Magnitudes in Prob 17

- $\langle 5,5\rangle = 5\sqrt{2}\left\langle \frac{5}{5\sqrt{2}}, \frac{5}{5\sqrt{2}}\right\rangle = 5\sqrt{2}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle = 5\sqrt{2}\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$
- $\langle -4,3\rangle = 5\left\langle -\frac{4}{5}, \frac{3}{5}\right\rangle$
- $\langle 12,-5\rangle = 13\left\langle \frac{12}{13}, -\frac{5}{13}\right\rangle$
- $\langle 3,1,-2\rangle = \sqrt{14}\left\langle \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}\right\rangle = \sqrt{14}\left\langle \frac{3\sqrt{14}}{14}, \frac{\sqrt{14}}{14}, -\frac{\sqrt{14}}{7}\right\rangle$
- $\langle 4,-2,-4\rangle = 6\left\langle \frac{4}{6}, -\frac{2}{6}, \frac{4}{6}\right\rangle = 6\left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle$
- $\langle 8,0,-6\rangle = 10\left\langle \frac{8}{10}, 0, -\frac{6}{10}\right\rangle = 10\left\langle \frac{4}{5}, 0, -\frac{3}{5}\right\rangle$

Definition 27. The **standard unit vectors** in $\mathbb{R}^2$ are $\hat{\mathbf{i}} = \langle 1,0\rangle$ and $\hat{\mathbf{j}} = \langle 0,1\rangle$, and any vector in $\mathbb{R}^2$ can be expressed in **standard unit vector form**:

$$\langle a,b\rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$$

The **standard unit vectors** in $\mathbb{R}^3$ are $\hat{\mathbf{i}} = \langle 1,0,0\rangle$, $\hat{\mathbf{j}} = \langle 0,1,0\rangle$, and $\hat{\mathbf{k}} = \langle 0,0,1\rangle$, and any vector in $\mathbb{R}^3$ can be expressed in **standard unit vector form**:

$$\langle a,b,c\rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$$

Note 28. Since the xy-plane is the the plane $z=0$ in xyz-space, we say the points $(a,b) = (a,b,0)$ and vectors $\langle a,b\rangle = \langle a,b,0\rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$ are equal.

Suggested Homework: Section 12.2 numbers 3, 5, 13, 14, 15, 19, 21, 24, 26