

## MATH 2242 (Calculus IV) Course Outline — Vector Calculus (Marsden)

### 1.5 $n$ -Dimensional Euclidean Space

- $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$
- Addition
  - $\langle x_1, x_2, \dots, x_n \rangle + \langle y_1, y_2, \dots, y_n \rangle = \langle x_1 + y_1, x_2 + y_2, \dots, x_n + y_n \rangle$
- Scalar multiplication
  - $\alpha \langle x_1, x_2, \dots, x_n \rangle = \langle \alpha x_1, \alpha x_2, \dots, \alpha x_n \rangle$
- Inner/Dot Product
  - $\langle x_1, x_2, \dots, x_n \rangle \cdot \langle y_1, y_2, \dots, y_n \rangle = \sum_{i=1}^n x_i y_i$
- Norm/Length/Magnitude
  - $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$
- Standard basis vectors
  - $\mathbf{e}_1 = \langle 1, 0, \dots, 0 \rangle, \mathbf{e}_2 = \langle 0, 1, \dots, 0 \rangle, \dots, \mathbf{e}_n = \langle 0, 0, \dots, 1 \rangle$
- Theorems
  - $(\alpha \mathbf{x} + \beta \mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z})$
  - Prove the above theorem.
  - $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
  - $\mathbf{x} \cdot \mathbf{x} \geq 0$
  - $\mathbf{x} \cdot \mathbf{x} = 0$  if and only if  $\mathbf{x} = \mathbf{0}$
  - $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$  (the Cauchy-Schwarz inequality)
  - Prove the Cauchy-Schwarz inequality.
  - $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$  (the triangle inequality)
  - Prove the triangle inequality.
- Matrices
  - $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

- Addition  $A + B$
- Scalar Multiplication  $\alpha A$
- Transposition  $A^T$

• Vectors as Matrices

- $\mathbf{a} = \langle a_1, a_2, \dots, a_n \rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$
- $\mathbf{a}^T = [a_1 \ a_2 \ \cdots \ a_n]$

• Matrix Multiplication

- If  $A$  has  $m$  rows and  $B$  has  $n$  columns, then  $M = AB$  is an  $m \times n$  matrix.
- Coordinate  $ij$  of  $M = AB$  is given by  $m_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j$  where  $\mathbf{a}_i^T$  is the  $i$ th row of  $A$  and  $\mathbf{b}_j$  is the  $j$ th column of  $B$ .
- (Example 4) Compute  $AB$  and  $BA$  for

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

- (Example 5) Compute  $AB$  for

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

• Matrices as Linear Transformations

- An  $m \times n$  matrix  $A$  gives a function from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ :  $\mathbf{x} \mapsto A\mathbf{x}$
- This linear transformation satisfies  $A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y}$
- (Example 7) Express  $A\mathbf{x}$  where  $x = \langle x_1, x_2, x_3 \rangle$  and  $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$ .

- (Example) Compute where the points  $(-1, -1, 0)$ ,  $(0, 1, 0)$ ,  $(1, -1, 1)$ , and  $(2, 1, 1)$  in  $\mathbb{R}^3$  get mapped to in  $\mathbb{R}^4$  by  $A\mathbf{x}$  from the previous example. Then plot the projections of the original points in  $\mathbb{R}^3$  onto their first two coordinates in  $\mathbb{R}^2$ , and compare this with the projection plot of their images in  $\mathbb{R}^4$  onto their first two coordinates in  $\mathbb{R}^2$ .
- Identity and Inverse
  - $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$
  - If  $AA^{-1} = A^{-1}A = I_n$ , then  $A$  is invertable and  $A^{-1}$  is its inverse.
- Determinant
  - Let  $A_i$  be the submatrix of  $A$  with the first column and  $i$ th row removed. Then  $\det(A) = \sum_{i=1}^n (-1)^{i+1} a_{1i} \det(A_i)$
  - This is equivalent to  $\det(A) = \sum_{\sigma \in S^n} \text{sgn}(\sigma) \prod_{1 \leq i \leq n} a_{i, \sigma_i}$  where  $S^n$  is the collection of all permutations of elements 1 to  $n$  and  $\text{sgn}(\sigma)$  is 1 when  $\sigma$  is obtained by an even number of swaps, and  $-1$  when  $\sigma$  is obtained by an odd number of swaps.
  - An  $n \times n$  matrix is invertable if and only if its determinant is nonzero.
- HW: 1-18, 21-24

## 2.3 Differentiation

- Functions  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ 
  - $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$
  - $\mathbf{f}(\mathbf{x}) = \langle f_1(\mathbf{x}), \dots, f_m(\mathbf{x}) \rangle$  where  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$
- Partial Derivative Matrix
  - $\mathbf{Df}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{bmatrix}$
  - We say  $\mathbf{f}$  is differentiable at  $\mathbf{x}$  if  $\mathbf{f}(\mathbf{x} + \mathbf{h}) \approx \mathbf{f}(\mathbf{x}) + [\mathbf{Df}(\mathbf{x})]\mathbf{h}$  for all  $\mathbf{h}$  near  $\mathbf{0}$ .
  - (Example) Let  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by  $\mathbf{f}(x, y) = \langle x^2 + y^2, xy \rangle$ , and let  $\mathbf{T} = \mathbf{Df}(1, 0)$ . Compute  $\mathbf{f}(1.1, -0.1)$  and  $\mathbf{f}(1, 0) + \mathbf{T}\langle 0.1, -0.1 \rangle$ .

- If each  $\frac{\partial f_i}{\partial x_j} : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous function near  $\mathbf{x}$ , then we say  $\mathbf{f}$  is strongly differentiable or class  $C^1$  at  $\mathbf{x}$ .
- Gradient
  - If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , then the gradient vector function  $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is defined by  $\nabla f(\mathbf{x}) = (\mathbf{D}f(\mathbf{x}))^T = \langle \frac{\partial f}{\partial x_1}(\mathbf{x}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{x}) \rangle$
  - $[\mathbf{D}f(\mathbf{x})]\mathbf{h} = \nabla f(\mathbf{x}) \cdot \mathbf{h}$
- Linearizations and Tangent Hyperplanes
  - Letting  $\mathbf{y} = \mathbf{x} + \mathbf{h}$  and  $\mathbf{y}_0 = \mathbf{x}$ , we have  $\mathbf{f}(\mathbf{y}) \approx \mathbf{f}(\mathbf{y}_0) + [\mathbf{D}\mathbf{f}(\mathbf{y}_0)](\mathbf{y} - \mathbf{y}_0)$  for differentiable  $\mathbf{f}$ .
  - For  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and a point  $\mathbf{x}_0 \in \mathbb{R}^n$ , let the linearization of  $\mathbf{f}$  at  $\mathbf{x}_0$  be  $\mathbf{L}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + [\mathbf{D}\mathbf{f}(\mathbf{x}_0)](\mathbf{x} - \mathbf{x}_0)$ . Note  $\mathbf{L}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x})$  for  $\mathbf{x}$  near  $\mathbf{x}_0$ .
  - (Example 5) Find the linearization  $L(x, y)$  of  $f(x, y) = x^2 + y^4 + e^{xy}$  at the point  $(1, 0)$ , and observe that this gives the equation of a tangent plane to the surface  $z = f(x, y)$  at the point  $(1, 0, 2)$ .
- HW: 1-3, 5-21

## 2.5 Properties of the Derivative

- Sum/Product/Quotient Rules
  - $\mathbf{D}[\alpha \mathbf{f}] = \alpha \mathbf{D}\mathbf{f}$
  - $\mathbf{D}[\mathbf{f} + \mathbf{g}] = \mathbf{D}\mathbf{f} + \mathbf{D}\mathbf{g}$
  - $\mathbf{D}[fg] = g\mathbf{D}f + f\mathbf{D}g$
  - $\mathbf{D}[\frac{f}{g}] = \frac{g\mathbf{D}f - f\mathbf{D}g}{g^2}$
  - Sketch proofs for strongly differentiable  $f, g$ .
- Chain Rule
  - $\mathbf{D}[\mathbf{f} \circ \mathbf{g}] = [\mathbf{D}\mathbf{f}](\mathbf{g})\mathbf{D}\mathbf{g}$
  - (Example) Find the rate of change of  $f(x, y) = x^2 + y^2$  along the path  $\mathbf{c}(t) = \langle t^2, t \rangle$  when  $t = 1$ .
  - (Example 2) Verify the Chain Rule for  $f(u, v, w) = u^2 + v^2 - w$  and  $\mathbf{g}(x, y, z) = \langle x^2y, y^2, e^{-xz} \rangle$ .
  - (Example 3) Compute  $\mathbf{D}[\mathbf{f} \circ \mathbf{g}](1, 1)$  where  $\mathbf{f}(u, v) = \langle u + v, u, v^2 \rangle$  and  $\mathbf{g}(x, y) = \langle x^2 + 1, y^2 \rangle$ .
- HW: 6-13, 15-16

### 3.2 Taylor's Theorem

- First-Order Taylor Formula
  - $f(\mathbf{x} + \mathbf{h}) \approx f(\mathbf{x}) + [\mathbf{D}f(\mathbf{x})]\mathbf{h}$  or  $f(\mathbf{x}) \approx f(\mathbf{x}_0) + [\mathbf{D}f(\mathbf{x}_0)](\mathbf{x} - \mathbf{x}_0)$
  - Alternate form:  $f(\mathbf{x} + \mathbf{h}) \approx f(\mathbf{x}) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x})h_i$  or  $f(\mathbf{x}) \approx f(\mathbf{x}_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i - x_{0,i})$
- Second-Order Taylor Formula
  - $f(\mathbf{x} + \mathbf{h}) \approx f(\mathbf{x}) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x})h_i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x})h_i h_j$
  - $f(\mathbf{x}) \approx f(\mathbf{x}_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i - x_{0,i}) + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)(x_i - x_{0,i})(x_j - x_{0,j})$
  - (Example 3) Find linear and quadratic functions of  $x, y$  which approximate  $f(x, y) = \sin(xy)$  near the point  $(1, \pi/2)$ .
- HW: 1-12

### 4.3 Vector Fields

- Vector Fields
  - A vector field is a map  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  assigning an  $n$ -dimensional vector to each point in  $\mathbb{R}^n$
  - (Example 1) The velocity field of a fluid may be modeled as a vector field.
  - (Example 2) Sketch the rotary motion given by the vector field  $\mathbf{V}(x, y) = \langle -y, x \rangle$ .
- Gradient Vector Fields
  - $\nabla f = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle$
  - (Example) The derivative of a scalar function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  in the direction given by a unit vector  $\mathbf{v}$  is given by  $\nabla f \cdot \mathbf{v}$ . Show that the maximum value of a directional derivative for a fixed point is given by  $\|\nabla f\|$  and attained by the direction  $\frac{1}{\|\nabla f\|} \nabla f$ .
  - (Example 4) If temperature is given by  $T(x, y, z)$ , then the energy or heat flux field is given by  $\mathbf{J} = -k\nabla T$  where  $k$  is the conductivity of the body. Level sets are called isotherms.
  - (Example 5) The gravitational potential of bodies with mass  $m, M$  is given by  $V = -\frac{mMG}{r}$  where  $G$  is the gravitational constant and  $r$  is the distance between the bodies, and the gravitational force field is given by  $\mathbf{F} = -\nabla V$ . Show that  $\mathbf{F} = -\frac{mMG}{r^3}\mathbf{r}$ , where  $\mathbf{r}$  is the vector pointing from the center of mass  $M$  to the center of mass  $m$ .

- A vector field  $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is conservative iff there exists a potential function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\mathbf{F} = \nabla f$ .
- (Example) Show that  $\mathbf{W} = \langle 2y + 1, 2x \rangle$  is conservative.
- (Example 7) Show that  $\mathbf{V} = \langle y, -x \rangle$  is not conservative.
- Flow Lines
  - A flow line for a vector field  $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a path  $\mathbf{c} : \mathbb{R} \rightarrow \mathbb{R}^n$  satisfying  $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$ .
  - (Example 8) Show that  $\mathbf{c}(t) = \langle \cos t, \sin t \rangle$  is a flow line for  $\mathbf{F} = \langle -y, x \rangle$ , and find some other flow lines.
- HW: 1-22

## 4.4 Divergence and Curl

- Divergence
  - The divergence of a vector field  $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is denoted by  $\text{div } \mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}$  and defined by  $\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}$
  - (Examples 3-5) Compute the divergences of  $\mathbf{F} = \langle x, y \rangle$ ,  $\mathbf{G} = \langle -x, -y \rangle$  and  $\mathbf{H} = \langle -y, x \rangle$  at any point on  $\mathbb{R}^2$ . How does divergence correspond with the motion described by the vector field plots?
  - (Example) Compute the divergence of  $\mathbf{F} = \langle x^2, y \rangle$  various points and interpret those values against a plot of the vector field.
- Curl
  - The curl of a three-dimensional vector field  $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is denoted by  $\text{curl } \mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  and defined by  $\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \rangle$
  - The scalar curl of a two-dimensional vector field  $\mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is denoted by  $\text{scurl } \mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}$  and defined by  $\text{scurl } \mathbf{F} = \text{curl } \mathbf{F} \cdot \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$
  - (Example) Compute the scalar curl of  $\mathbf{F} = \langle x, y \rangle$ ,  $\mathbf{G} = \langle -x, -y \rangle$  and  $\mathbf{H} = \langle -y, x \rangle$  at every point in  $\mathbb{R}^2$ . How does this scalar curl correspond with the motion described by the vector field plots?
  - (Example) Compute the curl of  $\mathbf{F} = \langle y, -x, z \rangle$  at every point in  $\mathbb{R}^3$ . How does curl correspond with the motion described by the vector field plot?
- Facts about  $\nabla f$ ,  $\text{div } \mathbf{F}$ ,  $\text{curl } \mathbf{F}$ 
  - The curl of a conservative field is zero:  $\text{curl } \nabla f = \nabla \times (\nabla f) = \mathbf{0}$ .
  - (Example) Prove the above theorem.

- (Example) Prove that  $\mathbf{F} = \langle x^2 + z, y - z, z^3 + 3xy \rangle$  is not a conservative field.
- The divergence of a curl field is zero:  $\operatorname{div} \operatorname{curl} \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$
- Many identities on pg. 255 of Marsden text.
- (Example) Sketch proof of identity #8:  $\operatorname{div} (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$ .
- HW: 1-37

## 5.3 The Double Integral Over More General Regions

- Hypervolume
  - The hypervolume  $HV_1$  of an interval  $[a, b]$  in  $\mathbb{R}$  is just its length  $b - a$ .
  - The hypervolume of a well-behaved bounded subset  $D \subseteq \mathbb{R}^{n+1}$  is defined for each  $i \in \{1, \dots, n+1\}$  by

$$HV_{n+1} = \int_{x_i \in I} HV(Z_i) dx_i = \int_{x_i=a}^{x_i=b} HV_n(D_i) dx_i$$

where  $I = [a, b]$  is an interval containing all values  $x_i$  included in the  $i$ th coordinate of  $D$ , and  $D_i$  is the projection of all points in  $D$  onto  $\mathbb{R}^n$  by removing the  $i$ th coordinate.

- (Example) For  $n = 1$  and  $D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, f(x) \leq y \leq g(x)\}$ , we have that

$$HV_2 = A = \int_{x \in [a, b]} g(x) - f(x) dx = \int_a^b g(x) - f(x) dx$$

(assuming  $f(x) \leq g(x)$  whenever  $a \leq x \leq b$ ).

- (Example) For  $n = 2$  and  $D \subseteq \mathbb{R}^3$  including values of  $x$  between  $a$  and  $b$ , we have that

$$HV_3 = V = \int_{x=a}^{x=b} A(x) dx$$

$$HV_3 = V = \int_{y=c}^{y=d} A(y) dy$$

where  $A(x)$  is the area of the cross-section of  $D$  taken by fixing each value of  $x$  (or similar for  $y$ ).

- Double Integrals

- For a bounded region  $D \subseteq \mathbb{R}^2$  and continuous  $f : D \rightarrow \mathbb{R}$ , the double integral

$$\iint_D f \, dA$$

is defined to be the volume of  $D^\uparrow = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D, 0 \leq z \leq f(x, y)\}$  minus the volume of  $D_\downarrow = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D, f(x, y) \leq z \leq 0\}$  (sometimes called net volume or signed volume).

- Assuming  $f(x, y) \geq 0$ , we may apply the definition of volume above to get

$$\iint_D F \, dA = \int_{x=a}^{x=b} A(x) \, dx$$

And if each cross section  $A(x)$  is described by  $\phi_1(x) \leq y \leq \phi_2(x)$  and  $0 \leq z \leq f(x, y)$ , we have that

$$\iint_D F \, dA = \int_{x=a}^{x=b} A(x) \, dx = \int_{x=a}^{x=b} \left[ \int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x, y) \, dy \right] dx$$

- Similarly, if  $D$  is described by  $c \leq y \leq d$  and  $\psi_1(y) \leq x \leq \psi_2(y)$ , then

$$\iint_D F \, dA = \int_{y=c}^{y=d} \left[ \int_{x=\psi_1(y)}^{x=\psi_2(y)} f(x, y) \, dx \right] dy$$

- The above holds even when  $f(x, y) \geq 0$  doesn't hold.

- Iterated integrals

- An iterated integral is a shorthand for the expansion of two or more nested integrals, e.g.:

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) \, dy \, dx = \int_{x=a}^{x=b} \left[ \int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x, y) \, dy \right] dx$$

- (Example) Sketch the region of integration for  $\int_0^\pi \int_{-x}^x \cos(y) \, dy \, dx$ , evaluate it, and interpret it as the signed volume of a region in  $\mathbb{R}^3$ .
- (Example) Express  $\iint_R (12x^3y - 1) \, dA$  where  $R$  is the rectangle with vertices  $(0, 0), (3, 0), (3, 2), (0, 2)$  as an iterated integral, then evaluate it.
- (Example) Express  $\iint_T (12x^3y - 1) \, dA$  where  $T$  is the triangle with vertices  $(0, 0), (1, 0), (1, 1)$  as an iterated integral, then evaluate it.

- Applications

- $\iint_D 1 \, dA$  is the area of  $D$
- $\frac{1}{A(D)} \iint_D f(x, y) \, dA$  is the average value of the function  $f$  restricted to  $D$

- HW: 1-17



## 5.4 Changing the Order of Integration

- Rectangular regions of integration

- For constant bounds of integration:

$$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

- (Example) Verify that  $\int_0^1 \int_1^2 x^2 + 2xy dy dx = \int_1^2 \int_0^1 x^2 + 2xy dx dy$ .

- Nonrectangular regions of integration

- Bounds of integration cannot be directly swapped; however, by interpreting the region of integration new bounds may be found in the other order.

- (Example) Verify that  $\int_0^4 \int_0^{\frac{4-y}{2}} x + y dx dy$  and  $\int_0^2 \int_0^{4-2x} x + y dy dx$  share the same region of integration and are equal.

- (Example) Evaluate  $\int_1^e \int_0^{\log x} \frac{(2x-e)\sqrt{1+e^y}}{e-e^y} dy dx$ . (Note that this is technically improper, but that this does not effect the solution.)

- Estimating double integrals

- If  $g(x, y) \leq f(x, y) \leq h(x, y)$  for  $(x, y) \in D$ , then  $\iint_D g(x, y) dA \leq \iint_D f(x, y) dA \leq \iint_D h(x, y) dA$ .

- (Example 3) Prove that  $\frac{1}{\sqrt{3}} \leq \iint_D \frac{1}{\sqrt{1+x^6+y^8}} dA \leq 1$  where  $D$  is the unit square.

- HW: 1-15

## 5.5 The Triple Integral

- Triple Integrals

- For a bounded region  $D \subseteq \mathbb{R}^3$  and continuous  $f : D \rightarrow \mathbb{R}$ , the triple integral

$$\iiint_D f dV$$

is defined to be the hypervolume of  $D^\uparrow = \{(x, y, z, w) \in \mathbb{R}^3 : (x, y, z) \in D, 0 \leq w \leq f(x, y, z)\}$  minus the hypervolume of  $D_\downarrow = \{(x, y, z, w) \in \mathbb{R}^3 : (x, y, z) \in D, f(x, y, z) \leq w \leq 0\}$ .

- Applications

- $\iiint_D 1 dV$  is the volume of  $D$

- $\frac{1}{V(D)} \iiint_D f(x, y, z) dV$  is the average value of the function  $f$  restricted to  $D$
- If  $\rho(x, y, z)$  gives the density of a solid at the coordinate  $(x, y, z)$ , then  $\iiint_D \rho(x, y, z) dV$  calculates its overall mass.

- Rectangular Boxes

- If  $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ , then

$$\begin{aligned} \iiint_B f dV &= \int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x, y, z) dx dy dz \\ &= \int_{a_2}^{b_2} \int_{a_1}^{b_1} \int_{a_3}^{b_3} f(x, y, z) dz dx dy \\ &= \text{etc.} \end{aligned}$$

- (Example) Write  $\iiint_D e^{x+y+z} dV$  where  $D = [0, 4] \times [0, 2] \times [1, 3]$  as a few different iterated integrals, then evaluate one.

- General regions of integration

- If  $E \subseteq \mathbb{R}^2$  and  $D = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in E, \gamma_1(x, y) \leq z \leq \gamma_2(x, y)\}$ , then

$$\iiint_D f(x, y, z) dV = \iint_E \left[ \int_{\gamma_1(x, y)}^{\gamma_2(x, y)} f(x, y, z) dz \right] dA$$

(and similar for  $x, y$  instead of  $z$ ).

- (Example 5) Express  $\iiint_W x dV$  where  $W$  is the solid for which  $x, y, z$  are positive and  $x^2 + y^2 \leq z \leq 2$  as a few different iterated integrals.
- (Example 6) Express  $\iiint_W x dV$  where  $W$  is the solid in  $\mathbb{R}^3$  above the triangle with vertices  $(0, 0, 0)$ ,  $(1, 0, 0)$ ,  $(1, 1, 0)$  and between the surfaces  $z = x^2 + y^2$  and  $z = 2$  as an iterated integral, then evaluate it.

- HW: 1-22, 25-28

## 1.4 Cylindrical and Spherical Coordinates

- Transformation of variables

- A transformation of variables is a function  $\mathbf{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .
- (Example) Sketch the integer lattice on the  $uv$  plane and its image in the  $xy$  plane for the transformation of variables  $\mathbf{T}(u, v) = (x, y) = (u, u + v)$ .

- Polar Coordinates

- $\mathbf{p}(r, \theta) = (r \cos \theta, r \sin \theta)$
- $r^2 = x^2 + y^2, \tan \theta = \frac{y}{x}$

- (Example) Convert  $A = \mathbf{p}(4, 2\pi/3)$  from polar to Cartesian. Convert  $B = (3, -3)$  from Cartesian to polar. Plot both.
- (Example) Express the curves  $x = \sqrt{4 - y^2}$  and  $y = 3$  in terms of polar coordinates.
- Cylindrical Coordinates
  - $\mathbf{c}(r, \theta, z) = (r \cos \theta, r \sin \theta, z)$
  - Usually, assume  $r \geq 0$  and  $0 \leq \theta \leq 2\pi$
  - $r^2 = x^2 + y^2$ ,  $\tan \theta = \frac{y}{x}$
  - (Example 1) Convert  $A = \mathbf{c}(8, 2\pi/3, -3)$  from cylindrical to Cartesian. Convert  $B = (6, 6, 8)$  from Cartesian to cylindrical. Plot both.
  - (Example) Express the surfaces  $x^2 + y^2 = 9$  and  $z^2 = x^2 + y^2$  in terms of cylindrical coordinates.
- Spherical Coordinates
  - $\mathbf{s}(\rho, \theta, \phi) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)$
  - Usually, assume  $\rho \geq 0$ ,  $0 \leq \theta \leq 2\pi$ , and  $0 \leq \phi \leq \pi$
  - $\rho^2 = x^2 + y^2 + z^2$ ,  $\tan \theta = \frac{y}{x}$ ,  $\tan \phi = \frac{r}{z} = \frac{\sqrt{x^2 + y^2}}{z}$
  - (Example 2) Convert  $A = (1, -1, 1)$  from Cartesian to spherical. Convert  $B = \mathbf{s}(3, \pi/6, \pi/4)$  from spherical to Cartesian. Convert  $C = (2, -3, 6)$  from Cartesian to spherical. Convert  $D = \mathbf{s}(1, -\pi/2, \pi/4)$  from spherical to Cartesian. Plot all four.
  - (Example 3) Express the surfaces  $xz = 1$  and  $x^2 + y^2 - z^2 = 1$  in terms of spherical coordinates.
- HW: 1-12, 15-16

## 6.1 The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

- Images of regions by transformations
  - (Example 1) Find the image of the rectangle  $[0, 1] \times [0, 2\pi]$  in the  $r\theta$  plane under the polar coordinate transformation  $\mathbf{p}$ .
  - (Example 2) Find the image of the square  $[-1, 1]^2 = [-1, 1] \times [-1, 1]$  in the  $uv$  plane under the transformation  $\mathbf{T}(u, v) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \langle u, v \rangle$
- One-to-one and Onto

- A one-to-one transformation sends each point in the domain to a distinct point in the range.
- An onto transformation sends something in the domain onto each point of the range.
- (Example 3) Show that the polar coordinate transformation  $\mathbf{p}$  is onto but not one-to-one.
- (Example 4) Show that the transformation  $\mathbf{T}$  from example 2 is both one-to-one and onto.
- (Example 5) Show that  $\mathbf{T}(u, v) = (u, 0)$  is neither one-to-one nor onto.
- (Example 7) Find a rectangle in the  $r\theta$  plane which maps onto the region  $\{(x, y) : x, y \geq 0, a^2 \leq x^2 + y^2 \leq b^2\}$  in the Cartesian plane by the polar coordinate transformation.
- Linear transformations
  - Transformations  $\mathbf{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined by  $\mathbf{T}(\mathbf{u}) = A\mathbf{u}$  for an  $n$ -dimensional matrix  $A$  are called linear transformations.
  - (Example 6) Find a region in the  $uv$  plane which maps onto the square with vertices  $(1, 0), (0, 1), (-1, 0), (0, -1)$  in the  $xy$  plane by the linear transformation given in Example 2.
  - Transformations  $\mathbf{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined by  $\mathbf{T}(\mathbf{u}) = A\mathbf{u} + \mathbf{x}_0$  for an  $n$ -dimensional matrix  $A$  and  $n$ -dimensional vector  $\mathbf{x}_0$  are called affine transformations. (Every linear transformation is affine.)
  - (Example) Find an affine transformation which maps the unit square in the  $uv$  plane onto the square with vertices  $(1, 0), (0, 1), (-1, 0), (0, -1)$  in the  $xy$  plane.
  - An affine transformation is both one-to-one and onto exactly when  $\det A \neq 0$ .
  - (Example) Use this fact to reinvestigate examples 4 and 5.
  - (Example) Prove that affine transformations send parallelograms to parallelograms.
- HW: 1-13

## 6.2 The Change of Variables Theorem

- Affine transformations of areas
  - (Example) Prove that the area of the image of the unit square under a linear transformation with matrix  $M$  is given by  $|\det M|$ . (Hint: the area of a parallelogram determined by two vectors  $\mathbf{v}_1, \mathbf{v}_2$  is given by  $\|\mathbf{v}_1 \times \mathbf{v}_2\|$ .)

- An affine transformation with matrix  $M$  transforms hypervolumes by a factor of  $|\det M|$ .
- (Example) Verify this fact for the parallelogram with vertices  $(2, 0), (3, 1), (1, 3), (0, 2)$  in the  $uv$  plane and its image in the  $xy$  plane under the transformation  $\mathbf{T}(u, v) = (2u + v + 3, v - u - 2)$ .
- Put another way,  $\iint_D 1 \, dA = \iint_{D^*} |\det M| \, dA$ .
- Affine transformations of single/double/triple integrals
  - (Example) Let  $x = T(u) = mu + x_0$ . Use substitution to prove that if the image of  $[c_1, c_2]$  under  $T$  is  $[b_1, b_2]$ , then  $\int_{b_1}^{b_2} f(x) \, dx = \int_{c_1}^{c_2} f(T(u))|m| \, du$ .
  - (Example) Use the previous fact to show that  $\int_0^4 \sqrt{2x+1} \, dx = \int_1^9 \frac{1}{2}\sqrt{u} \, du$
  - For any 2D affine transformation  $\mathbf{T}$  with matrix  $M$  transforming  $D^*$  to  $D$ ,  $\iint_D f(x, y) \, dA = \iint_{D^*} f(\mathbf{T}(u, v))|\det M| \, dA$ .
  - (Example) Use a linear transformation to prove that  $\int_0^2 \int_{y/2}^{(y+4)/2} 2y \, dx \, dy = 4 \int_0^1 \int_0^1 4v \, dv \, du$  and compute both integrals directly to verify.
  - (Example) Compute  $\iint_D (x+y)(x-y-2) \, dA$  where  $T$  is the triangle with vertices  $(4, 2), (3, 1), (2, 2)$ .
  - For any 3D affine transformation  $\mathbf{T}$  with matrix  $M$  transforming  $D^*$  to  $D$ ,  $\iiint_D f(x, y, z) \, dV = \iiint_{D^*} f(\mathbf{T}(u, v, w))|\det M| \, dV$ .
- Jacobian
  - The Jacobian  $\frac{\partial \mathbf{T}}{\partial \mathbf{u}}$  of a transformation is defined to be the determinant of its partial derivative matrix:  $\det(\mathbf{DT})$ .
  - (Example) Prove that for an affine transformation  $\mathbf{T}$  with matrix  $M$  that  $\mathbf{DT} = M$  and therefore  $\frac{\partial \mathbf{T}}{\partial \mathbf{u}} = \det M$ .
  - For any 2D transformation  $\mathbf{T}$  transforming  $D^*$  to  $D$ ,  $\iint_D f(\mathbf{x}) \, dA = \iint_{D^*} f(\mathbf{T}(\mathbf{u}))\left|\frac{\partial \mathbf{T}}{\partial \mathbf{u}}\right| \, dA$ .
  - For any 3D transformation  $\mathbf{T}$  transforming  $D^*$  to  $D$ ,  $\iiint_D f(\mathbf{x}) \, dV = \iiint_{D^*} f(\mathbf{T}(\mathbf{u}))\left|\frac{\partial \mathbf{T}}{\partial \mathbf{u}}\right| \, dV$ .
  - (Example) Use a 2D transformation to compute  $\iint_D e^x \cos(\pi e^x) \, dA$  where  $D$  is the region bounded by  $y = 0, y = e^x - 2, y = \frac{e^x - 1}{2}$ .
- Polar, cylindrical, spherical change of variables
  - Polar coordinates:  $\iint_D f(x, y) \, dA = \iint_{D^*} f(r \cos \theta, r \sin \theta) r \, dA$
  - Cylindrical coordinates:  $\iiint_D f(x, y, z) \, dV = \iiint_{D^*} f(r \cos \theta, r \sin \theta, z) r \, dV$
  - Spherical coordinates:  $\iiint_D f(x, y, z) \, dV = \iiint_{D^*} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, dV$
  - (Example 4) Evaluate  $\iint_D \log(x^2 + y^2) \, dA$  where  $D$  is the region in the first quadrant between the circles  $x^2 + y^2 = a^2$  and  $x^2 + y^2 = b^2$  for  $0 < a < b$ .

- (Example 6) Evaluate  $\iiint_W \exp[(x^2+y^2+z^3)^{3/2}] dV$  where  $W$  is unit ball centered at the origin.
- (Example) Find a formula for the volume of a cone with radius  $R$  and height  $H$ .
- (Example 7) Find a formula for the volume of a sphere with radius  $R$ .
- HW: 1-8, 11, 13-16, 21, 23-28

## 7.1 The Path Integral

- Path Integral with respect to Arclength
  - Recall that for a curve  $C$  defined by  $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^n$ , the arclength function  $s : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $s(t) = \int_0^t \|\mathbf{r}(\tau)\| d\tau$  gives the length of the curve from 0 to  $t$ .
  - (Example) Prove that  $C = \pi D$  gives the circumference of a circle with diameter  $D$ .
  - If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a function defined along the curve  $C$  defined by  $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^n$  for  $t \in [a, b]$ , then

$$\int_C f ds = \int_a^b f(\mathbf{r}(t)) \frac{ds}{dt} dt$$

where  $\frac{ds}{dt} = \|\frac{d\mathbf{r}}{dt}\|$ .

- (Example 1) Find the average value of the function  $f(x, y, z) = x^2 + y^2 + z^2$  along the portion of the helix given by  $\mathbf{c}(t) = \langle \cos t, \sin t, t \rangle$  for  $t \in [0, 2\pi]$ .
- (Example 2) The base of a fence is given by the curve  $\mathbf{c}(t) = \langle 30 \cos^3 t, 30 \sin^3 t \rangle$ , and the height of the fence is given by  $f(x, y) = 1 + \frac{y}{3}$ . How much paint is required to cover both sides of this fence?
- HW: 1-15, 17-19, 21-23, 25-27

## 7.2 Line Integrals

- Line Integral with respect to a Curve
  - If  $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a vector field defined along the curve  $C$  defined by  $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^n$  for  $t \in [a, b]$ , then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \frac{d\mathbf{r}}{dt} dt$$

- The work done in moving an object along the curve  $C$  defined by  $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^n$  for  $t \in [a, b]$  using a force vector field  $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is given by

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt$$

- (Example) An object is pushed around the unit circle with a force  $\langle -y, x \rangle$  at each point  $(x, y)$ . Compute the precise work done in pushing the box around 3 full counter-clockwise rotations.
- (Example 1) Let  $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$  for  $t \in [0, 2\pi]$  define the curve  $C$ , and define the vector field  $\mathbf{F}$  by  $\mathbf{F}(\mathbf{x}) = \mathbf{x} \cdot \langle 1, 1, 1 \rangle$ . Compute  $\int_C \mathbf{F} \cdot d\mathbf{c}$ .
- (Example 5) Let  $C$  be a circle in the  $yz$  plane centered at the origin. Show that no work is done by a force  $\mathbf{F} = \langle x^3, y, z \rangle$  acting on an object moving around the circle.

- Line integrals with respect to variables

- If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a function defined along the curve  $C$  defined by  $\mathbf{c} : \mathbb{R} \rightarrow \mathbb{R}^n$  for  $t \in [a, b]$ , then for  $1 \leq i \leq n$

$$\int_C f dx_i = \int_a^b f(\mathbf{c}(t)) \frac{dx_i}{dt} dt$$

where  $\mathbf{r} = \langle x_1, x_2, \dots \rangle$ .

- (Example) Compute  $\int_C xy dy$  where  $C$  is the parabola defined by  $\mathbf{c}(t) = \langle t, t^2, 1 \rangle$  for  $t \in [0, 1]$ .
- Note that

$$\int_C \mathbf{F} \cdot d\mathbf{c} = \sum_{i=1}^n \int_C F_i \cdot dx_i$$

- (Example 2) Evaluate and interpret  $\int_C x^2 dx + xy dy + dz$  where  $C$  is the parabola defined by  $\mathbf{c}(t) = \langle t, t^2, 1 \rangle$  for  $t \in [0, 1]$ .

- Reparametrizations and partitions

- The value of  $\int_C f ds$  is independent of the choice of parametrization  $\mathbf{r}(t)$  regardless of orientation.
- The value of  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of the choice of parametrization  $\mathbf{r}(t)$  provided it respects the orientation of  $C$ .
- If  $C$  and  $-C$  represent the same curve with opposite orientations, then  $\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \mathbf{F} \cdot d\mathbf{r}$ .
- If  $C = C_1 + C_2$ , then  $\int_C f ds = \int_{C_1} f ds + \int_{C_2} f ds$  and  $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ .

- (Example 11) Compute  $\int_C x^2 dx + xy dy$  where  $C$  is the perimeter of the unit square oriented counter-clockwise.
- HW: 1-9, 11-14, 16-20

## 7.3 Parametrized Surfaces

- Parametrization of a Surface
  - Let  $S \subseteq \mathbb{R}^3$  be a surface and  $D \subseteq \mathbb{R}^2$  be a two-dimensional region. Then  $\Phi : D \rightarrow S$  is a parametrization of  $S$  by  $D$ .
  - (Example) Show that the surface given by  $z = f(x, y)$  has the parametrization  $\Phi(x, y) = \langle x, y, f(x, y) \rangle$ .
  - (Example 1) Show that the plane passing through the point  $P \in \mathbb{R}^3$  and normal to the vector  $\mathbf{a} \times \mathbf{b}$  has a parametrization  $\Phi(u, v) = \mathbf{P} + \mathbf{a}u + \mathbf{b}v$ .
  - (Example 2) Show that the cone  $z = \sqrt{x^2 + y^2}$  has a parametrization  $\Phi(r, \theta) = \langle r \cos \theta, r \sin \theta, r \rangle$  for  $r \geq 0, 0 \leq \theta \leq 2\pi$ .
  - Surfaces which are conveniently described using cylindrical or spherical coordinates may be easily parameterized by adapting the relevant transformation.
  - (Example) Use the cylindrical and spherical transformations to find parametrizations of the cone  $z = \sqrt{x^2 + y^2}$ .
- Tangent and Normal Vectors to a Surface
  - The tangent plane to a surface parameterized by  $\Phi$  at the point  $\Phi(\mathbf{u}_0)$  has parameterization  $\mathbf{L}(\mathbf{u}) = \Phi(\mathbf{u}_0) + [\mathbf{D}\Phi(\mathbf{u}_0)]\mathbf{u}$ .
  - (Example 3) Find a parameterization of the plane tangent to the surface defined by  $\Phi(u, v) = \langle u \cos v, u \sin v, u^2 + v^2 \rangle$  at the point  $(1, 0, 1)$ .
  - The columns  $\frac{\partial \Phi}{\partial u}(u_0, v_0), \frac{\partial \Phi}{\partial v}(u_0, v_0)$  give tangent vectors to the surface, and their cross-product is normal to the surface.
  - (Example) Find an equation in  $x, y, z$  for the tangent plane in Example 3.
  - (Example) Find a parameterization for the sphere centered at the origin with radius 3. Then describe the plane tangent to it at the point  $(1, -2, 2)$ .
- HW: 1-3, 7-11, 13-15

## 7.4 Area of a Surface

- Definition of Surface Area



- The area of a surface parametrized by  $\Phi$  with domain  $D$  is given by  $\iint_D \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| dA$
  - (Example) Verify that this definition matches the area of the rectangle given by the vectors  $\langle 3, 0, -4 \rangle$  and  $\langle 0, -2, 0 \rangle$ .
  - (Example 1) Show that the surface area of a cone with slant length  $L$  and radius  $R$  is given by the formula  $A = \pi R^2 + \pi RL$ .
  - (Example 2) Find that the area of a helicoid parameterized by  $\Phi(r, \theta) = \langle r \cos \theta, r \sin \theta, \theta \rangle$  is equal to  $2\pi \int_0^1 \sqrt{r^2 + 1} dr$ .
  - TODO
- HW: tba

## Remaining Topics

- 7.5 Integrals of Scalar Functions Over Surfaces
- 7.6 Surface Integrals of Vector Fields
- 8.1 Green's Theorem
- 8.2 Stokes' Theorem
- 8.3 Conservative Fields
- 8.4 Gauss' Theorem