**MATH 2242 (Calculus IV) Course Outline** — Vector Calculus (Marsden)

# 1.5 $n$-Dimensional Euclidean Space

- $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$
- Addition
  - $\langle x_1, x_2, \ldots, x_n\rangle + \langle y_1, y_2, \ldots, y_n\rangle = \langle x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\rangle$
- Scalar multiplication
  - $\alpha\langle x_1, x_2, \ldots, x_n\rangle = \langle \alpha x_1, \alpha x_2, \ldots, \alpha x_n\rangle$
- Inner/Dot Product
  - $\langle x_1, x_2, \ldots, x_n\rangle \cdot \langle y_1, y_2, \ldots, y_n\rangle = \sum_{i=1}^n x_i y_i$
- Norm/Length/Magnitude
  - $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$
- Standard basis vectors
  - $\mathbf{e}_1 = \langle 1, 0, \ldots, 0\rangle$, $\mathbf{e}_2 = \langle 0, 1, \ldots, 0\rangle$, $\ldots$, $\mathbf{e}_n = \langle 0, 0, \ldots, 1\rangle$
- Theorems
  - $(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z})$
  - Prove the above theorem.
  - $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
  - $\mathbf{x} \cdot \mathbf{x} \geq 0$
  - $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$
  - $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ (the Cauchy-Schwarz inequality)
  - Prove the Cauchy-Schwarz inequality.
  - $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (the triangle inequality)
  - Prove the triangle inequality.
- Matrices
  - $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$

- Addition $A + B$
- Scalar Mutiplication $\alpha A$
- Transposition $A^T$

• Vectors as Matrices

- $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$
- $\mathbf{a}^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$

• Matrix Multiplication

- If $A$ has $m$ rows and $B$ has $n$ columns, then $M = AB$ is an $m \times n$ matrix.
- Coordinate $ij$ of $M = AB$ is given by $m_{ij} = \mathbf{a_i} \cdot \mathbf{b_j}$ where $\mathbf{a_i}^T$ is the $i$th row of $A$ and $\mathbf{b_j}$ is the $j$th column of $B$.
- (Example 4) Compute $AB$ and $BA$ for

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

- (Example 5) Compute $AB$ for

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

• Matrices as Linear Transformations

- An $m \times n$ matrix $A$ gives a function from $\mathbb{R}^n$ to $\mathbb{R}^m$: $\mathbf{x} \mapsto A\mathbf{x}$
- This linear transformation satsifies $A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y}$
- (Example 7) Express $A\mathbf{x}$ where $x = \langle x_1, x_2, x_3 \rangle$ and $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

  - (Example) Compute where the points $(-1,-1,0)$, $(0,1,0)$, $(1,-1,1)$, and $(2,1,1)$ in $\mathbb{R}^3$ get mapped to in $\mathbb{R}^4$ by $A\mathbf{x}$ from the previous example. Then plot the projections of the original points in $\mathbb{R}^3$ onto their first two coordinates in $\mathbb{R}^2$, and compare this with the projection plot of their images in $\mathbb{R}^4$ onto their first two coordinates in $\mathbb{R}^2$.

- Identity and Inverse
  - $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$
  - If $AA^{-1} = A^{-1}A = I_n$, then $A$ is invertable and $A^{-1}$ is its inverse.

- Determinant
  - Let $A_i$ be the submatrix of $A$ with the first column and $i$th row removed. Then $\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \det(A_i)$
  - This is equivalent to $\det(A) = \sum_{\sigma \in S^n} \operatorname{sgn}(\sigma) \prod_{1 \leq i \leq n} a_{i,\sigma_i}$ where $S^n$ is the collection of all permutations of elements 1 to $n$ and $\operatorname{sgn}(\sigma)$ is 1 when $\sigma$ is obtained by an even number of swaps, and $-1$ when $\sigma$ is obtained by an odd number of swaps.
  - An $n \times n$ matrix is invertable if and only if its determinant is nonzero.

- *HW: 1-18, 21-24*

## 2.3 Differentiation

- Functions $\mathbb{R}^n \to \mathbb{R}^m$
  - $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$
  - $\mathbf{f}(\mathbf{x}) = \langle f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}) \rangle$ where $f_i : \mathbb{R}^n \to \mathbb{R}$

- Partial Derivative Matrix
  - $\mathbf{Df}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{bmatrix}$
  - We say $\mathbf{f}$ is differentiable at $\mathbf{x}$ if $\mathbf{f}(\mathbf{x}+\mathbf{h}) \approx \mathbf{f}(\mathbf{x}) + [\mathbf{Df}(\mathbf{x})]\mathbf{h}$ for all $\mathbf{h}$ near $\mathbf{0}$.
  - (Example) Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mathbf{f}(x,y) = \langle x^2+y^2, xy \rangle$, and let $\mathbf{T} = \mathbf{Df}(1,0)$. Compute $\mathbf{f}(1.1,-0.1)$ and $\mathbf{f}(1,0) + \mathbf{T}\langle 0.1, -0.1 \rangle$.

- If each $\frac{\partial f_i}{\partial x_j} : \mathbb{R}^n \to \mathbb{R}$ is a continuous function near $\mathbf{x}$, then we say $\mathbf{f}$ is strongly differentiable or class $C^1$ at $\mathbf{x}$.

- Gradient
  - If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient vector function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\nabla f(\mathbf{x}) = (\mathbf{D}f(\mathbf{x}))^T = \langle \frac{\partial f}{\partial x_1}(\mathbf{x}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{x})\rangle$
  - $[\mathbf{D}f(\mathbf{x})]\mathbf{h} = \nabla f(\mathbf{x}) \cdot \mathbf{h}$

- Linearizations and Tangent Hyperplanes
  - Letting $\mathbf{y} = \mathbf{x} + \mathbf{h}$ and $\mathbf{y}_0 = \mathbf{x}$, we have $\mathbf{f}(\mathbf{y}) \approx \mathbf{f}(\mathbf{y}_0) + [\mathbf{Df}(\mathbf{y_0})](\mathbf{y} - \mathbf{y}_0)$ for differentiable $\mathbf{f}$.
  - For $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and a point $\mathbf{x}_0 \in \mathbb{R}^n$, let the linearization of $\mathbf{f}$ at $x_0$ be $\mathbf{L}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + [\mathbf{Df}(\mathbf{x_0})](\mathbf{x} - \mathbf{x}_0)$. Note $\mathbf{L}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x})$ for $\mathbf{x}$ near $\mathbf{x}_0$.
  - (Example 5) Find the linearization $L(x, y)$ of $f(x, y) = x^2 + y^4 + e^{xy}$ at the point $(1, 0)$, and observe that this gives the equation of a tangent plane to the surface $z = f(x, y)$ at the point $(1, 0, 2)$.

- *HW: 1-3, 5-21*

## 2.5 Properties of the Derivative

- Sum/Product/Quotient Rules
  - $\mathbf{D}[\alpha\mathbf{f}] = \alpha\mathbf{Df}$
  - $\mathbf{D}[\mathbf{f} + \mathbf{g}] = \mathbf{Df} + \mathbf{Dg}$
  - $\mathbf{D}[fg] = g\mathbf{D}f + f\mathbf{D}g$
  - $\mathbf{D}[\frac{f}{g}] = \frac{g\mathbf{D}f - f\mathbf{D}g}{g^2}$
  - Sketch proofs for strongly differentiable $f, g$.

- Chain Rule
  - $\mathbf{D}[\mathbf{f} \circ \mathbf{g}] = [\mathbf{Df}](\mathbf{g})\mathbf{Dg}$
  - (Example) Find the rate of change of $f(x, y) = x^2+y^2$ along the path $\mathbf{c}(t) = \langle t^2, t\rangle$ when $t = 1$.
  - (Example 2) Verify the Chain Rule for $f(u, v, w) = u^2 + v^2 - w$ and $\mathbf{g}(x, y, z) = \langle x^2y, y^2, e^{-xz}\rangle$.
  - (Example 3) Compute $\mathbf{D}[\mathbf{f} \circ \mathbf{g}](1, 1)$ where $\mathbf{f}(u, v) = \langle u + v, u, v^2\rangle$ and $\mathbf{g}(x, y) = \langle x^2 + 1, y^2\rangle$.

- *HW: 6-13, 15-16*

## 3.2 Taylor's Theorem

- First-Order Taylor Formula
  - $f(\mathbf{x}+\mathbf{h}) \approx f(\mathbf{x}) + [\mathbf{D}f(\mathbf{x})]\mathbf{h}$ or $f(\mathbf{x}) \approx f(\mathbf{x}_0) + [\mathbf{D}f(\mathbf{x}_0)](\mathbf{x}-\mathbf{x}_0)$
  - Alternate form: $f(\mathbf{x}+\mathbf{h}) \approx f(\mathbf{x})+\sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x})h_i$ or $f(\mathbf{x}) \approx f(\mathbf{x}_0)+\sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i-x_{0,i})$
- Second-Order Taylor Formula
  - $f(\mathbf{x}+\mathbf{h}) \approx f(\mathbf{x}) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x})h_i + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x})h_i h_j$
  - $f(\mathbf{x}) \approx f(\mathbf{x}_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i - x_{0,i}) + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)(x_i - x_{0,i})(x_j - x_{0,j})$
  - (Example 3) Find linear and quadratic functions of $x, y$ which approximate $f(x, y) = \sin(xy)$ near the point $(1, \pi/2)$.
- *HW: 1-12*

## 4.3 Vector Fields

- Vector Fields
  - A vector field is a map $f : \mathbb{R}^n \to \mathbb{R}^n$ assinging an $n$-dimensional vector to each point in $\mathbb{R}^n$
  - (Example 1) The velocity field of a fluid may be modeled as a vector field.
  - (Example 2) Sketch the rotary motion given by the vector field $\mathbf{V}(x, y) = \langle -y, x\rangle$.
- Gradient Vector Fields
  - $\nabla f = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\rangle$
  - (Example) The derivative of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$ in the direction given by a unit vector $\mathbf{v}$ is given by $\nabla f \cdot \mathbf{v}$. Show that the maximum value of a directional derivative for a fixed point is given by $\|\nabla f\|$ and attained by the direction $\frac{1}{\|\nabla f\|}\nabla f$.
  - (Example 4) If temperature is given by $T(x, y, z)$, then the energy or heat flux field is given by $\mathbf{J} = -k\nabla T$ where $k$ is the conductivity of the body. Level sets are called isotherms.
  - (Example 5) The gravitational potential of bodies with mass $m, M$ is given by $V = -\frac{mMG}{r}$ where $G$ is the gravitational constant and $r$ is the distance between the bodies, and the gravitational force field is given by $\mathbf{F} = -\nabla V$. Show that $\mathbf{F} = -\frac{mMG}{r^3}\mathbf{r}$, where $\mathbf{r}$ is the vector pointing from the center of mass $M$ to the center of mass $m$.

- A vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is conservative iff there exists a potential function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.
- (Example) Show that $\mathbf{W} = \langle 2y + 1, 2x \rangle$ is conservative.
- (Example 7) Show that $\mathbf{V} = \langle y, -x \rangle$ is not conservative.

- Flow Lines
  - A flow line for a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a path $\mathbf{c} : \mathbb{R} \to \mathbb{R}^n$ satisfying $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$.
  - (Example 8) Show that $\mathbf{c}(t) = \langle \cos t, \sin t \rangle$ is a flow line for $\mathbf{F} = \langle -y, x \rangle$, and find some other flow lines.

- *HW: 1-22*

## 4.4 Divergence and Curl

- Divergence
  - The divergence of a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is denoted by $\operatorname{div} \mathbf{F} : \mathbb{R}^n \to \mathbb{R}$ and defined by $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$
  - (Examples 3-5) Compute the divergences of $\mathbf{F} = \langle x, y \rangle$, $\mathbf{G} = \langle -x, -y \rangle$ and $\mathbf{H} = \langle -y, x \rangle$ at any point on $\mathbb{R}^2$. How does divergence correspond with the motion described by the vector field plots?
  - (Example) Compute the divergence of $\mathbf{F} = \langle x^2, y \rangle$ various points and interpret those values against a plot of the vector field.

- Curl
  - The curl of a three-dimensional vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is denoted by $\operatorname{curl} \mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ and defined by $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \rangle$
  - The scalar curl of a two-dimensional vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is denoted by $\operatorname{scurl} \mathbf{F} : \mathbb{R}^2 \to \mathbb{R}$ and defined by $\operatorname{scurl} \mathbf{F} = \operatorname{curl} \mathbf{F} \cdot \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$
  - (Example) Compute the scalar curl of $\mathbf{F} = \langle x, y \rangle$, $\mathbf{G} = \langle -x, -y \rangle$ and $\mathbf{H} = \langle -y, x \rangle$ at every point in $\mathbb{R}^2$. How does this scalar curl correspond with the motion described by the vector field plots?
  - (Example) Compute the curl of $\mathbf{F} = \langle y, -x, z \rangle$ at every point in $\mathbb{R}^3$. How does curl correspond with the motion described by the vector field plot?

- Facts about $\nabla f$, $\operatorname{div} \mathbf{F}$, $\operatorname{curl} \mathbf{F}$
  - The curl of a conservative field is zero: $\operatorname{curl} \nabla f = \nabla \times (\nabla f) = \mathbf{0}$.
  - (Example) Prove the above theorem.

- (Example) Prove that $\mathbf{F} = \langle x^2 + z, y - z, z^3 + 3xy \rangle$ is not a conservative field.
- The divergence of a curl field is zero: $\operatorname{div}\operatorname{curl}\mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$
- Many identities on pg. 255 of Marsden text.
- (Example) Sketch proof of identity #8: $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl}\mathbf{F} - \mathbf{F} \cdot \operatorname{curl}\mathbf{G}$.

- *HW: 1-37*

## 5.3 The Double Integral Over More General Regions

- Hypervolume
  - The hypervolume $HV_1$ of an interval $[a, b]$ in $\mathbb{R}$ is just its length $b - a$.
  - The hypervolume of a well-behaved bounded subset $D \subseteq \mathbb{R}^{n+1}$ is defined for each $i \in \{1, \dots, n+1\}$ by

$$HV_{n+1} = \int_{x_i \in I} HV(Z_i)\, dx_i = \int_{x_i=a}^{x_i=b} HV_n(D_i)\, dx_i$$

    where $I = [a, b]$ is an interval containing all values $x_i$ included in the $i$th coordinate of $D$, and $D_i$ is the projection of of all points in $D$ onto $\mathbb{R}^n$ by removing the $i$th coordinate.
  - (Example) For $n = 1$ and $D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, f(x) \leq y \leq g(x)\}$, we have that

$$HV_2 = A = \int_{x \in [a,b]} g(x) - f(x)\, dx = \int_a^b g(x) - f(x)\, dx$$

    (assuming $f(x) \leq g(x)$ whenever $a \leq x \leq b$).
  - (Example) For $n = 2$ and $D \subseteq R^3$ including values of $x$ between $a$ and $b$, we have that

$$HV_3 = V = \int_{x=a}^{x=b} A(x)\, dx$$

$$HV_3 = V = \int_{y=c}^{y=d} A(y)\, dx$$

    where $A(x)$ is the area of the cross-section of $D$ taken by fixing each value of $x$ (or similar for $y$).

- Double Integrals

- For a bounded region $D \subseteq \mathbb{R}^2$ and continuous $f : D \to \mathbb{R}$, the double integral

$$\iint_D f\,dA$$

is defined to be the volume of $D^{\uparrow} = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D, 0 \leq z \leq f(x,y)\}$ minus the volume of $D_{\downarrow} = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D, f(x,y) \leq z \leq 0\}$ (sometimes called net volume or signed volume).

- Assuming $f(x,y) \geq 0$, we may apply the definition of volume above to get

$$\iint_D F\,dA = \int_{x=a}^{x=b} A(x)\,dx$$

And if each cross section $A(x)$ is described by $\phi_1(x) \leq y \leq \phi_2(x)$ and $0 \leq z \leq f(x,y)$, we have that

$$\iint_D F\,dA = \int_{x=a}^{x=b} A(x)\,dx = \int_{x=a}^{x=b} \left[\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy\right] dx$$

- Similarly, if $D$ is described by $c \leq y \leq d$ and $\psi_1(y) \leq x \leq \psi_2(y)$, then

$$\iint_D F\,dA = \int_{y=c}^{y=d} \left[\int_{x=\psi_1(y)}^{x=\psi_2(y)} f(x,y)\,dx\right] dy$$

- The above holds even when $f(x,y) \geq 0$ doesn't hold.

• Iterated integrals

- An iterated integral is a shorthand for the expansion of two or more nested integrals, e.g.:

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx = \int_{x=a}^{x=b} \left[\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy\right] dx$$

- (Example) Sketch the region of integration for $\int_0^{\pi} \int_{-x}^{x} \cos(y)\,dy\,dx$, evaluate it, and interpret it as the signed volume of a region in $\mathbb{R}^3$.
- (Example) Express $\iint_R (12x^3y - 1)\,dA$ where $R$ is the rectangle with vertices $(0,0), (3,0), (3,2), (0,2)$ as an interated integral, then evaluate it.
- (Example) Express $\iint_T (12x^3y - 1)\,dA$ where $T$ is the triangle with vertices $(0,0), (1,0), (1,1)$ as an interated integral, then evaluate it.

• Applications

- $\iint_D 1\,dA$ is the area of $D$
- $\frac{1}{A(D)} \iint_D f(x,y)\,dA$ is the average value of the function $f$ restricted to $D$

• *HW: 1-17*

## 5.4 Changing the Order of Integration

- Rectangular regions of integration
  - For constant bounds of integration:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

  - (Example) Verify that $\int_0^1 \int_1^2 x^2 + 2xy\,dy\,dx = \int_1^2 \int_0^1 x^2 + 2xy\,dx\,dy$.
- Nonrectangular regions of integration
  - Bounds of integration cannot be directly swapped; however, by interpreting the region of integration new bounds may be found in the other order.
  - (Example) Verify that $\int_0^4 \int_0^{\frac{4-y}{2}} x + y\,dx\,dy$ and $\int_0^2 \int_0^{4-2x} x + y\,dy\,dx$ share the same region of integration and are equal.
  - (Example) Evaluate $\int_1^e \int_0^{\log x} \frac{(2x-e)\sqrt{1+e^y}}{e-e^y}\,dy\,dx$. (Note that this is technically improper, but that this does not effect the solution.)
- Estimating double integrals
  - If $g(x,y) \leq f(x,y) \leq h(x,y)$ for $(x,y) \in D$, then $\iint_D g(x,y)\,dA \leq \iint_D f(x,y)\,dA \leq \iint_D h(x,y)\,dA$.
  - (Example 3) Prove that $\frac{1}{\sqrt{3}} \leq \iint_D \frac{1}{\sqrt{1+x^6+y^8}}\,dA \leq 1$ where $D$ is the unit square.
- *HW: 1-15*

## 5.5 The Triple Integral

- Triple Integrals
  - For a bounded region $D \subseteq \mathbb{R}^3$ and continuous $f : D \to \mathbb{R}$, the triple integral

$$\iiint_D f\,dV$$

    is defined to be the hypervolume of $D^{\uparrow} = \{(x,y,z,w) \in \mathbb{R}^3 : (x,y,z) \in D, 0 \leq w \leq f(x,y,z)\}$ minus the hypervolume of $D_{\downarrow} = \{(x,y,z,w) \in \mathbb{R}^3 : (x,y,z) \in D, f(x,y,z) \leq w \leq 0\}$.
- Applications
  - $\iiint_D 1\,dV$ is the volume of $D$

  - $\frac{1}{V(D)}\iiint_D f(x,y,z)\,dV$ is the average value of the function $f$ restricted to $D$
  - If $\rho(x,y,z)$ gives the density of a solid at the coordinate $(x,y,z)$, then $\iiint_D \rho(x,y,z)\,dV$ calculates its overall mass.

- Rectangular Boxes
  - If $B=[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$, then

$$\begin{aligned}\iiint_B f\,dV &= \int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y,z)\,dx\,dy\,dz\\ &= \int_{a_2}^{b_2}\int_{a_1}^{b_1}\int_{a_3}^{b_3} f(x,y,z)\,dz\,dx\,dy\\ &= \text{etc.}\end{aligned}$$

  - (Example) Write $\iiint_D e^{x+y+z}\,dV$ where $D=[0,4]\times[0,2]\times[1,3]$ as a few different iterated integrals, then evaluate one.

- General regions of integration
  - If $E\subseteq\mathbb{R}^2$ and $D=\{(x,y,z)\in\mathbb{R}^3:(x,y)\in E,\gamma_1(x,y)\le z\le\gamma_2(x,y)\}$, then

$$\iiint_D f(x,y,z)\,dV=\iint_E\left[\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dz\right]dA$$

    (and similar for $x,y$ instead of $z$).
  - (Example 5) Express $\iiint_W x\,dV$ where $W$ is the solid for which $x,y,z$ are positive and $x^2+y^2\le z\le 2$ as a few different iterated integrals.
  - (Example 6) Express $\iiint_W x\,dV$ where $W$ is the solid in $\mathbb{R}^3$ above the triangle with vertices $(0,0,0),(1,0,0),(1,1,0)$ and between the surfaces $z=x^2+y^2$ and $z=2$ as an iterated integral, then evaluate it.

- *HW: 1-22, 25-28*

## 1.4 Cylindrical and Spherical Coordinates

- Transformation of variables
  - A transformation of variables is a function $\mathbf{T}:\mathbb{R}^n\to\mathbb{R}^n$.
  - (Example) Sketch the integer lattice on the $uv$ plane and its image in the $xy$ plane for the transformation of variables $\mathbf{T}(u,v)=(x,y)=(u,u+v)$.

- Polar Coordinates
  - $\mathbf{p}(r,\theta)=(r\cos\theta,r\sin\theta)$
  - $r^2=x^2+y^2$, $\tan\theta=\frac{y}{x}$

  - (Example) Convert $A = \mathbf{p}(4, 2\pi/3)$ from polar to Cartesian. Convert $B = (3, -3)$ from Cartesian to polar. Plot both.
  - (Example) Express the curves $x = \sqrt{4 - y^2}$ and $y = 3$ in terms of polar coordinates.

- Cylindrical Coordinates
  - $\mathbf{c}(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$
  - Usually, assume $r \geq 0$ and $0 \leq \theta \leq 2\pi$
  - $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$
  - (Example 1) Convert $A = \mathbf{c}(8, 2\pi/3, -3)$ from cylindrical to Cartesian. Convert $B = (6, 6, 8)$ from Cartesian to cylindrical. Plot both.
  - (Example) Express the surfaces $x^2 + y^2 = 9$ and $z^2 = x^2 + y^2$ in terms of cylindrical coordinates.

- Spherical Coordinates
  - $\mathbf{s}(\rho, \theta, \phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$
  - Usually, assume $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$
  - $\rho^2 = x^2 + y^2 + z^2$, $\tan\theta = \frac{y}{x}$, $\tan\phi = \frac{r}{z} = \frac{\sqrt{x^2+y^2}}{z}$
  - (Example 2) Convert $A = (1, -1, 1)$ from Cartesian to spherical. Convert $B = \mathbf{s}(3, \pi/6, \pi/4)$ from spherical to Cartesian. Convert $C = (2, -3, 6)$ from Cartesian to spherical. Convert $D = \mathbf{s}(1, -\pi/2, \pi/4)$ from spherical to Cartesian. Plot all four.
  - (Example 3) Express the surfaces $xz = 1$ and $x^2 + y^2 - z^2 = 1$ in terms of spherical coordiantes.

- *HW: 1-12, 15-16*

## 6.1 The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

- Images of regions by transformations
  - (Example 1) Find the image of the rectangle $[0, 1] \times [0, 2\pi]$ in the $r\theta$ plane under the polar coordinate transformation $\mathbf{p}$.
  - (Example 2) Find the image of the square $[-1, 1]^2 = [-1, 1] \times [-1, 1]$ in the $uv$ plane under the transformation $\mathbf{T}(u, v) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \langle u, v \rangle$

- One-to-one and Onto

  - A one-to-one transformation sends each point in the domain to a distinct point in the range.
  - An onto transformation sends something in the domain onto each point of the range.
  - (Example 3) Show that the polar coordinate transformation $\mathbf{p}$ is onto but not one-to-one.
  - (Example 4) Show that the transformation $\mathbf{T}$ from example 2 is both one-to-one and onto.
  - (Example 5) Show that $\mathbf{T}(u, v) = (u, 0)$ is neither one-to-one nor onto.
  - (Example 7) Find a rectangle in the $r\theta$ plane which maps onto the region $\{(x, y) : x, y \geq 0, a^2 \leq x^2 + y^2 \leq b^2\}$ in the Cartesian plane by the polar coordinate transformation.

- Linear transformations
  - Transformations $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathbf{T}(\mathbf{u}) = A\mathbf{u}$ for an $n$-dimensional matrix $A$ are called linear transformations.
  - (Example 6) Find a region in the $uv$ plane which maps onto the square with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$ in the $xy$ plane by the linear transformation given in Example 2.
  - Transformations $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathbf{T}(\mathbf{u}) = A\mathbf{u} + \mathbf{x}_0$ for an $n$-dimensional matrix $A$ and $n$-dimensional vector $\mathbf{x}_0$ are called affine transformations. (Every linear transformation is affine.)
  - (Example) Find an affine transformation which maps the unit square in the $uv$ plane onto the square with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$ in the $xy$ plane.
  - An affine transformation is both one-to-one and onto exactly when $\det A \neq 0$.
  - (Example) Use this fact to reinvestigate examples 4 and 5.
  - (Example) Prove that affine transformations send parallelograms to parallelograms.

- *HW: 1-13*

## 6.2 The Change of Variables Theorem

- Affine transformations of areas
  - (Example) Prove that the area of the image of the unit square under a linear transformation with matrix $M$ is given by $|\det M|$. (Hint: the area of a parallelogram determined by two vectors $\mathbf{v}_1, \mathbf{v}_2$ is given by $\|\mathbf{v}_1 \times \mathbf{v}_2\|$.)

  - An affine transformation with matrix $M$ transforms hypervolumes by a factor of $|\det M|$.
  - (Example) Verify this fact for the parallelogram with vertices $(2,0),(3,1),(1,3),(0,2)$ in the $uv$ plane and its image in the $xy$ plane under the transformation $\mathbf{T}(u,v) = (2u+v+3, v-u-2)$.
  - Put another way, $\iint_D 1\,dA = \iint_{D^*} |\det M|\,dA$.

- Affine transformations of single/double/triple integrals
  - (Example) Let $x = T(u) = mu + x_0$. Use substitution to prove that if the image of $[c_1, c_2]$ under $T$ is $[b_1, b_2]$, then $\int_{b_1}^{b_2} f(x)\,dx = \int_{c_1}^{c_2} f(T(u))|m|\,du$.
  - (Example) Use the previous fact to show that $\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \frac{1}{2}\sqrt{u}\,du$
  - For any 2D affine transformation $\mathbf{T}$ with matrix $M$ transforming $D^*$ to $D$, $\iint_D f(x,y)\,dA = \iint_{D^*} f(\mathbf{T}(u,v))|\det M|\,dA$.
  - (Example) Use a linear transformation to prove that $\int_0^2 \int_{y/2}^{(y+4)/2} 2y\,dx\,dy = 4\int_0^1\int_0^1 4v\,dv\,du$ and compute both integrals directly to verify.
  - (Example) Compute $\iint_D (x+y)(x-y-2)\,dA$ where $T$ is the triangle with vertices $(4,2)$, $(3,1)$, $(2,2)$.
  - For any 3D affine transformation $\mathbf{T}$ with matrix $M$ transforming $D^*$ to $D$, $\iint_D f(x,y,z)\,dV = \iint_{D^*} f(\mathbf{T}(u,v,w))|\det M|\,dV$.

- Jacobian
  - The Jacobian $\frac{\partial \mathbf{T}}{\partial \mathbf{u}}$ of a transformation is defined to be the determinant of its partial derivative matrix: $\det(\mathbf{DT})$.
  - (Example) Prove that for an affine transformation $\mathbf{T}$ with matrix $M$ that $\mathbf{DT} = M$ and therefore $\frac{\partial \mathbf{T}}{\partial \mathbf{u}} = \det M$.
  - For any 2D transformation $\mathbf{T}$ transforming $D^*$ to $D$, $\iint_D f(\mathbf{x})\,dA = \iint_{D^*} f(\mathbf{T}(\mathbf{u}))|\frac{\partial \mathbf{T}}{\partial \mathbf{u}}|\,dA$.
  - For any 3D transformation $\mathbf{T}$ transforming $D^*$ to $D$, $\iiint_D f(\mathbf{x})\,dV = \iiint_{D^*} f(\mathbf{T}(\mathbf{u}))|\frac{\partial \mathbf{T}}{\partial \mathbf{u}}|\,dV$.
  - (Example) Use a 2D transformation to compute $\iint_D e^x \cos(\pi e^x)\,dA$ where $D$ is the region bounded by $y=0$, $y = e^x - 2$, $y = \frac{e^x - 1}{2}$.

- Polar, cylindrical, spherical change of variables
  - Polar coordinates: $\iint_D f(x,y)\,dA = \iint_{D^*} f(r\cos\theta, r\sin\theta) r\,dA$
  - Cylindrical coordinates: $\iint_D f(x,y,z)\,dV = \iint_{D^*} f(r\cos\theta, r\sin\theta, z) r\,dV$
  - Spherical coordinates: $\iint_D f(x,y,z)\,dV = \iint_{D^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,dV$
  - (Example 4) Evaluate $\iint_D \log(x^2+y^2)\,dA$ where $D$ is the region in the first quadrant between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ for $0<a<b$.

- (Example 6) Evaluate $\iiint_W \exp[(x^2+y^2+z^3)^{3/2}]\,dV$ where $W$ is unit ball centered at the origin.
- (Example) Find a formula for the volume of a cone with radius $R$ and height $H$.
- (Example 7) Find a formula for the volume of a sphere with radius $R$.

- *HW: 1-8, 11, 13-16, 21, 23-28*

## 7.1 The Path Integral

- Path Integral with respect to Arclength
  - Recall that for a curve $C$ defined by $\mathbf{r}:\mathbb{R}\to\mathbb{R}^n$, the arclength function $s:\mathbb{R}\to\mathbb{R}$ defined by $s(t)=\int_0^t \|\mathbf{r}'(\tau)\|\,d\tau$ gives the length of the curve from 0 to $t$.
  - (Example) Prove that $C=\pi D$ gives the circumference of a circle with diameter $D$.
  - If $f:\mathbb{R}^n\to\mathbb{R}$ is a function defined along the curve $C$ defined by $\mathbf{r}:\mathbb{R}\to\mathbb{R}^n$ for $t\in[a,b]$, then
  $$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))\frac{ds}{dt}\,dt$$
  where $\frac{ds}{dt}=\|\frac{d\mathbf{r}}{dt}\|$.
  - (Example 1) Find the average value of the function $f(x,y,z)=x^2+y^2+z^2$ along the portion of the helix given by $\mathbf{c}(t)=\langle\cos t,\sin t,t\rangle$ for $t\in[0,2\pi]$.
  - (Example 2) The base of a fence is given by the curve $\mathbf{c}(t)=\langle 30\cos^3 t, 30\sin^3 t\rangle$, and the height of the fence is given by $f(x,y)=1+\frac{y}{3}$. How much paint is required to cover both sides of this fence?

- *HW: 1-15, 17-19, 21-23, 25-27*

## 7.2 Line Integrals

- Line Integral with respect to a Curve
  - If $\mathbf{F}:\mathbb{R}^n\to\mathbb{R}^n$ is a vector field defined along the curve $C$ defined by $\mathbf{r}:\mathbb{R}\to\mathbb{R}^n$ for $t\in[a,b]$, then
  $$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{c}(t))\frac{d\mathbf{r}}{dt}\,dt$$

- The work done in moving an object along the curve $C$ defined by $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ for $t \in [a, b]$ using a force vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot\, d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt$$

- (Example) An object is pushed around the unit circle with a force $\langle -y, x\rangle$ at each point $(x, y)$. Compute the precise work done in pushing the box around 3 full counter-clockwise rotations.
- (Example 1) Let $\mathbf{r}(t) = \langle \sin t, \cos t, t\rangle$ for $t \in [0, 2\pi]$ define the curve $C$, and define the vector field $\mathbf{F}$ by $\mathbf{F}(\mathbf{x}) = \mathbf{x} \cdot \langle 1, 1, 1\rangle$. Compute $\int_C \mathbf{F} \cdot\, d\mathbf{c}$.
- (Example 5) Let $C$ be a circle in the $yz$ plane centered at the origin. Show that no work is done by a force $\mathbf{F} = \langle x^3, y, z\rangle$ acting on an object moving around the circle.

- Line integrals with respect to variables

  - If $f : \mathbb{R}^n \to \mathbb{R}$ is a function defined along the curve $C$ defined by $\mathbf{c} : \mathbb{R} \to \mathbb{R}^n$ for $t \in [a, b]$, then for $1 \leq i \leq n$

$$\int_C f\, dx_i = \int_a^b f(\mathbf{c}(t)) \frac{dx_i}{dt}\, dt$$

  where $\mathbf{r} = \langle x_1, x_2, \ldots\rangle$.
  - (Example) Compute $\int_C xy\, dy$ where $C$ is the parabola defined by $\mathbf{c}(t) = \langle t, t^2, 1\rangle$ for $t \in [0, 1]$.
  - Note that

$$\int_C \mathbf{F} \cdot\, d\mathbf{r} = \sum_{i=1}^n \int_C F_i \cdot\, dx_i$$

  - (Example 2) Evaluate and interpret $\int_C x^2\, dx + xy\, dy + dz$ where $C$ is the parabola defined by $\mathbf{c}(t) = \langle t, t^2, 1\rangle$ for $t \in [0, 1]$.

- Reparametrizations and partitions

  - The value of $\int_C f\, ds$ is independent of the choice of parametrization $\mathbf{r}(t)$ regardless of orientation.
  - The value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the choice of parametrization $\mathbf{r}(t)$ provided it respects the orientation of $C$.
  - If $C$ and $-C$ represent the same curve with opposite orientations, then $\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \mathbf{F} \cdot\, d\mathbf{r}$.
  - If $C = C_1 + C_2$, then $\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds$ and $\int_C \mathbf{F} \cdot\, d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot\, d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot\, d\mathbf{r}$.

  - (Example 11) Compute $\int_C x^2\,dx + xy\,dy$ where $C$ is the perimeter of the unit square oriented counter-clockwise.

- *HW: 1-9, 11-14, 16-20*

## 8.1 Green's Theorem

- Green's Theorem
  - Let $\partial D$ be the c.c.w. oriented boundary of a simple region $D \subseteq \mathbb{R}^2$. Then $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \operatorname{scurl} \mathbf{F}\,dA = \iint_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\,dA = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA$.
  - Note that the book lets $\mathbf{F} = \langle F_1, F_2 \rangle = \langle P, Q \rangle$.
  - (Example 1) Verify Green's Theorem for $\mathbf{F} = \langle x, xy \rangle$ and $D = \{(x, y) : x^2 + y^2 \leq 1\}$.
  - (Example) Use Green's Theorem to prove that the area of $D$ is $\frac{1}{2} \int_{\partial D} x\,dy - y\,dx$.
  - (Example 3) Compute the work done using a force $\mathbf{F} = \langle xy^2, y + x \rangle$ in moving an object from the origin to $(1, 1)$ along the curve $y = x^2$ and then back to the origin along the line $y = x$.
- *HW: 1-6, 9-10, 15*

## 8.3 Conservative Fields

- Characterizations of Conservative Fields
  - These are all equivalent to $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ being conservative:
    1. There exists a potential function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.
    2. $\operatorname{curl} \mathbf{F} = 0$.
    3. $\mathbf{F}$ is path-independent: for any two curves $C_1, C_2$ which share starting and ending points, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.
    4. For any simple closed curve $C$, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.
    5. For any curve starting at $A$ and ending at $B$, and any potential function $f$ for $\mathbf{F}$: $\int_C \mathbf{F} \cdot d\mathbf{r} = [f]_A^B = f(B) - f(A)$.
  - (Example) Prove that (4) implies (3) above.
  - (7.2 Example 9) Evaluate $\int_C y\,dx + x\,dy$ where $C$ is the curve given by $\mathbf{r}(t) = (t^4/4, \sin^3(t\pi/2))$ for $t \in [0, 1]$.
  - (Example 4) Find $\int_C 2x \cos y\,dx - x^2 \sin y\,dy$ where $C$ is given by $\mathbf{r} : [1, 2] \to \mathbb{R}^2$ defined by $x = e^{t-1}, y = \sin(\pi/t)$.

- (Example 1) Show that $\int_C \langle y, z\cos yz + x, y\cos yz\rangle \cdot d\mathbf{r} = 0$ for any simple closed curve $C$.

- *HW: 1-2, 5-8, 10-11*

## Surface Integrals

- Definition
  - If $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar function defined on the surface $S$ defined by $\mathbf{\Phi} : \mathbb{R}^2 \to \mathbb{R}^n$ for $(u, v) \in D$, then

$$\iint_S f(\mathbf{x})\, dS = \iint_D f(\mathbf{\Phi}(u, v)) \left\| \frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v} \right\| dA$$

  - (Example) Use $A = \iint_S 1\, dS$ to derive a formula for the surface area of a cone in terms of its height $H$ and angle $\theta$.
  - If $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}$ is a scalar function defined on the surface $S$ defined by $\mathbf{\Phi} : \mathbb{R}^2 \to \mathbb{R}^n$ for $(u, v) \in D$ preserving orientation, then

$$\iint_S \mathbf{F}(\mathbf{x}) \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{\Phi}(u, v)) \cdot \left( \frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v} \right) dA$$

- Stokes' Theorem
  - If $\partial S$ is the positively oriented boundary of a surface $S$, then $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$.
  - (8.2 Example 2) Evaluate $\iint_S (3x^2 + 3y^2)\mathbf{k} \cdot d\mathbf{S}$ where $S$ is the portion of the plane $z = 1 - x - y$ above the unit disk $x^2 + y^2 \leq 1$ oriented toward positive values of $x, y, z$.

- Gauss'/Divergence Theorem
  - If $\partial W$ is the outward oriented boundary of a solid $W$, then $\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint_W \operatorname{div} \mathbf{F}\, dV$.
  - (8.3 Example 3) Evaluate $\iint_S \langle 2x, y^2, z^2\rangle \cdot d\mathbf{S}$ where $S$ is the outward oriented boundary of the unit sphere $x^2 + y^2 + z^2 = 1$.

- *HW: Review above examples*

## Overview of integration theorems

- $\int_a^b f'(x)\,dx = f(b) - f(a) = [f]_{\partial[a,b]}$
- $\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = [f]_{\partial C}$
- $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$
- $\iiint_W \operatorname{div} \mathbf{F}\, dV = \iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}$