MATH 2242 (Calculus IV) Course Outline

1.5 n-Dimensional Euclidean Space

- $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$
- Addition
    - $\langle x_1, x_2, \ldots, x_n\rangle + \langle y_1, y_2, \ldots, y_n\rangle = \langle x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\rangle$
- Scalar multiplication
    - $\alpha\langle x_1, x_2, \ldots, x_n\rangle = \langle \alpha x_1, \alpha x_2, \ldots, \alpha x_n\rangle$
- Standard basis vectors
    - $\mathbf{e}_1 = \langle 1, 0, \ldots, 0\rangle$, $\mathbf{e}_2 = \langle 0, 1, \ldots, 0\rangle$, $\ldots$, $\mathbf{e}_n = \langle 0, 0, \ldots, 1\rangle$
- Theorems
    - $(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z})$
    - Prove the above theorem.
    - $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
    - $\mathbf{x} \cdot \mathbf{x} \geq 0$
    - $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$
    - $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ (the Cauchy-Schwarz inequality)
    - Prove the Cauchy-Schwarz inequality.
    - $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (the triangle inequality)
    - Prove the triangle inequality.
- Matrices
    - $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$
    - Addition $A + B$
    - Scalar Mutiplication αA
    - Transposition A^T
- Vectors as Matrices

  - $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$
  - $\mathbf{a}^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$

- Matrix Multiplication
  - If A has m rows and B has n columns, then $M = AB$ is an $m \times n$ matrix.
  - Coordinate ij of $M = AB$ is given by $m_{ij} = \mathbf{a_i} \cdot \mathbf{b_j}$ where $\mathbf{a_i}^T$ is the ith row of A and $\mathbf{b_j}$ is the jth column of B.
  - (Example 4) Compute AB and BA for

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

  - (Example 5) Compute AB for

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Matrices as Linear Transformations
  - An $m \times n$ matrix A gives a function from $\mathbb{R}^n$ to $\mathbb{R}^m$: $\mathbf{x} \mapsto A\mathbf{x}$
  - This linear transformation satsifies $A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y}$
  - (Example 7) Express $A\mathbf{x}$ where $x = \langle x_1, x_2, x_3 \rangle$ and $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$. Then compute where the points $(3, -2, 1)$, $(1, 0, 1)$, $(-1, 1, 0)$, and $(-3, 3, 0)$ in $\mathbb{R}^3$ get mapped to in $\mathbb{R}^4$

- Identity and Inverse

  - $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$
  - If $AA^{-1} = A^{-1}A = I_n$, then A is invertable and A^{-1} is its inverse.

- Determinant
  - Let A_i be the submatrix of A with the first column and ith row removed. Then $\det(A) = \sum_{i=1}^{n}(-1)^{i+1}a_{1i}\det(A_i)$

- *Suggested HW: 1-18, 21-24*

2.3 Differentiation

- Functions $\mathbb{R}^n \to \mathbb{R}^m$
  - $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$
  - $\mathbf{f}(\mathbf{x}) = \langle f_1(\mathbf{x}), \dots, f_m(\mathbf{x})\rangle$ where $f_i : \mathbb{R}^n \to \mathbb{R}$

- Partial Derivative Matrix
  - $\mathbf{Df}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{bmatrix}$
  - Let $\mathbf{T} = \mathbf{Df}(\mathbf{x_0})$. We say $\mathbf{f}$ is differentiable at $\mathbf{x_0}$ if $\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x}_0) + \mathbf{Tx}$ for all $\mathbf{x}$ near $\mathbf{x}_0$.
  - $\mathbf{f}(\mathbf{x}_0) + \mathbf{Tx}$ is the equation of the tangent plane for $\mathbf{f}$ at $\mathbf{x}$.
  - (Example) Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mathbf{f}(x, y) = \langle x^2 + y^2, xy\rangle$, and let $\mathbf{T} = \mathbf{Df}(1, 0)$. Compute $\mathbf{f}(1.1, -0.1)$ and $\mathbf{f}(1, 0) + \mathbf{T}\langle 1.1, -0.1\rangle$.
  - If each $\frac{\partial f_i}{\partial x_j} : \mathbb{R}^n \to \mathbb{R}$ is a continuous function near $\mathbf{x}$, then $\mathbf{f}$ is strongly differentiable at $\mathbf{x}$.

- Gradient
  - If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient vector function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\nabla f(\mathbf{x}) = (\mathbf{D}f(\mathbf{x}))^T = \langle \frac{\partial f}{\partial x_1}(\mathbf{x}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{x})\rangle$
  - Let $\mathbf{T} = \mathbf{D}f(\mathbf{x}_0)$. Then $\mathbf{Tx} = \nabla f(\mathbf{x}_0) \cdot \mathbf{x}$

- *Suggested HW: 1-21*

Remaining Topics

- 2.4 Introduction to Paths and Curves
- 2.5 Properties of the Derivative
- 2.6 Gradients and Directional Derivatives
- 3.2 Taylor's Theorem
- 4.1 Acceleration and Newton's Second Law
- 4.2 Arc Length
- 4.3 Vector Fields
- 4.4 Divergence and Curl
- 5.3 The Double Integral Over More General Regions
- 5.4 Changing the Order of Integration
- 5.5 The Triple Integral
- 6.1 The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$
- 6.2 The Change of Variables Theorem
- 7.1 The Path Integral
- 7.2 Line Integrals
- 7.3 Parametrized Surfaces
- 7.4 Area of a Surface
- 7.5 Integrals of Scalar Functions Over Surfaces
- 7.6 Surface Integrals of Vector Fields
- 8.1 Green's Theorem
- 8.2 Stokes' Thoerem
- 8.3 Conservative Fields
- 8.4 Gauss' Theorem