MATH 1121 (Calculus for Engineering Technology) Course Outline

1.3 Rectangular Coordinates

- Illustrate the following concepts:
  - rectangular coordinate system,
  - x-axis,
  - y-axis,
  - origin,
  - quadrants,
  - coordinates
- Examples:
  - (Example 1) Plot $A = (2, 1)$ and $B = (-4, -3)$.
  - (Example 3) Three verticies of a rectangle are $A = (-3, -2)$, $B = (4, -2)$, $C = (4, 1)$. What is the fourth vertex?
- *HW: 1-9, 15-16, 21-24*

2.1 Some Basic Definitions

- Distance Formula
  - $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  - (Example 2) Find the distance between $(3, -1)$ and $(-2, -5)$.
- Slope Formula
  - $m = \frac{y_2 - y_1}{x_2 - x_1}$
  - (Example 3) Find the slope of the line joining $(3, -5)$, $(-2, -6)$.
  - (Example 4) Find the slope of the line joining $(3, 4)$, $(4, -6)$.
  - $m = \tan\theta$
  - (Example) Find the slope of the line with inclination $120°$.
- Identify parallel/perpendicular lines by slopes.
  - Parallel: $m_1 = m_2$
  - Perpendicular: $m_1 = -\frac{1}{m_2}$

- (Example 7) Prove that the triangle with vertices $A = (-5, 3)$, $B = (6, 0)$, and $C = (5, 5)$ is a right triangle.

- *HW: 1-20, 29-36*

2.2 The Straight Line

- Point-slope form
  - $y - y_1 = m(x - x_1)$
  - (Example 2) Find the equation of the line passing through $(2, -1)$ and $(6, 2)$.
- Slope-intercept form
  - $y = mx + b$
  - (Example 4) Find the slope and y-intercept of the straight line with equation $2y + 4x - 5 = 0$.
- *HW: 1-21, 33-40*

2.3 The Circle

- Definition
  - A circle is a collection of points equidistant from its center.
- Standard form
  - $(x - h)^2 + (y - k)^2 = r^2$
  - (Example 1) Sketch $(x - 1)^2 + (y + 2)^2 = 16$.
  - (Example 2) Find an equation for the circle with center $(2, 1)$ which passes through $(4, 8)$.
- General form
  - $x^2 + y^2 + Dx + Ey + F = 0$
  - (Example 4) Find the center and radius of the circle $x^2 + y^2 - 6x + 8y - 24 = 0$.
  - (Example) Find two functions whose graphs represent the circle with the previous equation.
- *HW: 1-32, 37-38*

2.4 The Parabola

- Definition
  - A parabola is a colleciton of points equidistant from a focus point and a directrix line.
  - The vertex of a parabola is the point closest to the focus and directix.
  - (Example 6) Find an equation for the parabola with focus $(2, 3)$ and directrix $(y = -1)$
- Standard forms with vertex at origin and horizontal/vertical directrix
  - $y^2 = 4px$ with directrix at $x = -p$ and focus at $(p, 0)$
  - $y^2 = -4px$ with directrix at $x = p$ and focus at $(-p, 0)$
  - $x^2 = 4py$ with directrix at $y = -p$ and focus at $(0, p)$
  - $x^2 = -4py$ with directrix at $y = p$ and focus at $(0, -p)$
  - (Example 2) Find an equation for the parabola with focus $(-2, 0)$ and directrix $(x = 2)$.
  - (Example 4) Find the focus and directrix of the parabola with equation $2x^2 = -9y$.
- *HW: 1-22, 25-28*

2.5 The Ellipse

- Definition
  - An ellipse is a collection of points where the sum of distances from two fixed points (called foci) is kept constant.
  - The two points furthest/closest apart from each other on an ellipse are the end-points of the major/minor axis.
  - The sum of distances between each point and the foci is the same as the length of the major axis. The major axis passes through both foci.
- Standard form with center at the origin
  - $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with foci given by $(c, 0), (-c, 0)$, where $a^2 - b^2 = c^2$
  - $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$, with foci given by $(0, c), (0, -c)$, where $a^2 - b^2 = c^2$
  - (Example 3) Sketch the ellipse with equation $4x^2 + 16y^2 = 64$, and compute the locations of its foci.

  - (Example 5) Find the equation of the ellipse centered at the origin with an end of its minor axis at $(2,0)$ and containing the point $(-1,\sqrt{6})$.

- *HW: 1-26*

2.6 The Hyperbola

- Definition
  - A hyperbola is a collection of points where the difference of distances from two fixed points (called foci) is kept constant.
  - Hyperbolas are split into two curves. The two closest points on opposite curves are called vertices and give the transverse axis.
- Standard form with center at the origin
  - $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, with foci given by $(c,0),(-c,0)$ and asymptotes $y=\pm\frac{bx}{a}$, where $a^2+b^2=c^2$
  - $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, with foci given by $(0,c),(0,-c)$ and asymptotes $x=\pm\frac{by}{a}$, where $a^2+b^2=c^2$
  - (Example 2) Sketch $\frac{y^2}{4}-\frac{x^2}{16}=1$, labeling its vertices, asymptotes, and foci.
  - (Example 3) Sketch $4x^2-9y^2=36$, labeling its vertices, asymptotes, and foci.
- Hypberbola with coordinate axis asymptotes
  - $xy=c^2$, with vertices given by (c,c) and $(-c,-c)$
  - $xy=-c^2$, with vertices given by $(c,-c)$ and $(-c,c)$
  - (Example 5) Sketch $xy=4$.
- *HW: 1-14, 17-24*

2.7 Translation of Axes

- Vertical/horizontal translation:
  - Shift right h: replace x with $x-h$.
  - Shift up k: replace y with $y-k$.
  - (Example 1) Give an equation of the parabola with vertex $(2,4)$ and focus $(4,4)$.
  - (Example 2) Sketch the curve with equation $\frac{(x-3)^2}{25}+\frac{(y+2)^2}{9}=1$.
- *HW: 1-36*

1.2 Algebraic Functions

- Definition of a function $y = f(x)$.
- Types of functions
  - Polynomials $P(x) = a_0 + a_1 x + \cdots + a_n x^n$
  - Rational functions $R(x) = \frac{P(x)}{Q(x)}$ for polynomials P, Q
  - (Example 1) Voltage equals current multiplied by resistance. If the voltage at time t is given by $E(t) = 2t^2 = y + 5$ and the resistance at time t is given by $R(t) = 3t + 20$, then find a function $I(t)$ which measures the current at time t. Identify it as a polynomial and/or rational function.
- Combinations of functions
  - Addition/Subtraction/Multiplication/Division
  - Compositions $f \circ g$ and $g \circ f$
  - (Example 2) Express $f+g$, $f \circ g$, and $g \circ f$ for the functions given by $f(x) = 2x^2 - 3$ and $g(x) = \sqrt{x+2}$.
- Domain/Range
  - The domain of a function is all real numbers which may be plugged into it without causing division by zero, even roots of negatives, or any other undefined operations.
  - The range of a function is all real numbers which may possibly be attained by the function.
  - (Example 5) Find the domain and range of $f(x) = x^2 + 2$ and $g(t) = \frac{1}{t+2}$.
  - (Example 7) Find the domain of $f(x) = 16\sqrt{x} + \frac{1}{x}$.
- Piecewise functions
  - Piecewise functions are defined differently for different parts of their domains.
  - (Example 9) Find the domain for

    $$f(t) = \begin{cases} 8 - 2t & 0 \leq t \leq 4 \\ 0 & t > 4 \end{cases}$$

    and compute $f(3), f(6), f(-1)$ if possible.
- Exponent laws
  - $a^m a^n = a^{m+n}$

  - $\frac{a^m}{a^n} = a^{m-n}$
  - $(a^m)^n = a^{mn}$
  - $(ab)^m = a^m b^m$
  - $(\frac{a}{b})^m = \frac{a^m}{b^m}$
  - $a^0 = 1$
  - $a^{-n} = \frac{1}{a^n}$
  - $a^{1/n} = \sqrt[n]{a}$
  - Note $\sqrt{a^2} = |a|$ but $\sqrt[3]{a^3} = a$
  - (Example 4) Simplify

$$f(x) = \frac{(3x^2-1)^{1/3}(2x) - (2x^3)(3x^2-1)^{-2/3}}{(3x^2-1)^{2/3}}$$

- *HW: 1-18, 21-34*

1.4 The Graph of a Function

- Definition
  - The graph of a function is the collection of all ordered pairs (x, y) such that $y = f(x)$
  - Graphing Method 1: using Chapter 2
  - Graphing Method 2: using xy chart
  - Vertical line test: the graph of any function hits every vertical line at most once
- Examples
  - (Example 1) Graph $f(x) = 3x - 5$.
  - (Example 3) Graph $f(x) = 1 + \frac{1}{x}$.
  - (Example 4) Graph $f(x) = \sqrt{x+1}$.
  - (Example 6) Graph

$$f(x) = \begin{cases} 2x+1 & x \leq 1 \\ 6 - x^2 & x > 1 \end{cases}$$

- *HW: 1-12, 37-40*

3.1 Limits

- Limits
  - $\lim_{x\to a} f(x) = L$ means that the value of $f(x)$ approaches L as the value of x approaches a in the domain of f.
  - (Example) Given $f(x) = x^2$, we may write the following chart of values

x	$f(x)$
1.9	3.61
1.99	3.9601
1.999	3.996001
2.001	4.004001
2.01	4.0401
2.1	4.41

    to infer that $\lim_{x\to 2} f(x) = 4$.
  - (Example) Given
    $$g(x) = \begin{cases} x^2 & x \neq 2 \\ -5 & x = 2 \end{cases}$$
    we have the same chart of values as before, so we assume $\lim_{x\to 2} g(x) = 4$.
  - (Example) Since $h(x) = \frac{x^3-2x^2}{x-2}$ equals x^2 for all values of x except 2, we have the same chart of values as before, and we assume $\lim_{x\to 2} h(x) = 4$.
  - (Example) By graphing $y = f(x)$, $y = g(x)$, and $y = h(x)$, we can see that the points on the graph approach the point $(2, 4)$ in all three cases.
- Continuity
  - A continuous function satsifies the equality $f(a) = \lim_{x\to a} f(x)$ for all numbers a in its domain. (The “just plug it in” rule.)
  - Intuitively: the graph of the function can be drawn without lifting your pencil on the intervals where it is defined
  - FACT: $f(x) = x$ is continuous, and any combination of continuous functions using $+$, $-$, $\times$, $/$, $\circ$, or powers is continuous (where it is defined).
  - (Example 3) $f(x) = \frac{1}{x-2}$ is continuous for its entire domain, but undefined at its asymptote $x = 2$.
  - (Example 5) By graphing
    $$f(x) = \begin{cases} x+2 & x < 1 \\ -\frac{x}{2}+5 & x \geq 1 \end{cases} \quad g(x) = \begin{cases} 2x-1 & x \leq 2 \\ -x+5 & x > 2 \end{cases}$$
    we see that f is continuous except for when $x = 1$, and g is continuous everywhere.

- Limits to $\pm\infty$
  - $\lim_{x\to\infty} f(x) = L$ means that the value of $f(x)$ approaches L as the value of x attains arbitrarily large postive values.
  - $\lim_{x\to-\infty} f(x) = L$ means that the value of $f(x)$ approaches L as the value of x attains arbitrarily large negative values.
  - (Example) Use a chart of values to infer that $\lim_{x\to\pm\infty} \frac{1}{x} = 0$.
  - (Example 14) Use a chart of values and algebraic manipulation to show that $\lim_{x\to\pm\infty} \frac{x^2+1}{2x^2+3} = \frac{1}{2}$.
- Evaluating limits analytically
  - For continuous functions, use the "just plug it in" rule.
  - (Example 10) Evaluate $\lim_{x\to 4} x^2 - 7$
  - For limits of the form $\frac{\text{nonzero}}{0}$, the limit is undefined.
  - (Example 9) Show $\lim_{x\to 2} \frac{1}{x-2}$ does not exist.
  - For limits of the form $\frac{0}{0}$, the limit is indeterminate: use canceling to determine its value.
  - (Example 11) Evaluate $\lim_{x\to 2} \frac{x^2-4}{x-2}$.
- *HW: 25-44*

3.3 The Derivative

- Secant and Tangent Lines
  - The slope of a secant line is given by $\frac{\Delta y}{\Delta x}$.
  - The slope of a tangent line is given by $\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$.
  - (Example) Find the slope of a few secant lines for $y = x^2$ about the point $(2, 4)$, use this to guess the slope of the tangent line at $(2, 4)$, then calculate the tangent slope directly from the limit.
- Derivative
  - The derivative $f'(x)$ or $\frac{d}{dx}[f(x)]$ of a function gives the slope of the tangent lines for each point on the graph $(x, f(x))$.
  - $f'(x) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$
  - (Example) Show that the derivative of $f(x) = x^2$ is $f'(x) = 2x$, then use this to find the slope of the tangent line at $(2, 4)$.
  - (Example 2) Prove that for $y = 6x - 2x^3$, $y' = \frac{dy}{dx} = 6 - 6x^2$.

  - (Example 4) Prove that for $g(x) = x^2 + \frac{1}{x+1}$, $g'(x) = 2x - \frac{1}{(x+1)^2}$.

- *HW: 1-24*

3.5 Derivatives of Polynomials

- Derivatives of Constants and Identity
  - $\frac{d}{dx}[c] = 0$
  - (Example 1) Calculate the $\frac{dy}{dx}$ for $y = -5$.
  - $\frac{d}{dx}[x] = 1$
  - (Example 3) Prove that if $y = x$ then $y' = 1$.
- Derivatives of x^n
  - (Example) Prove that if $f(x) = x^5$ then $f'(x) = 5x^4$.
  - $\frac{d}{dx}[x^p] = px^{p-1}$
  - (Example 2) Find the derivative of $y = x^3$.
  - (Example 4) Find $\frac{dv}{dr}$ where $v = r^{10}$.
- Constant Multiple Rule
  - $\frac{d}{dx}[cf(x)] = cf'(x)$
  - (Example 5) Find the derivative of $y = 3x^2$.
- Sum/Difference Rule
  - $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
  - (Example 7) Find the slope of a line tangent to the curve $y = 4x^7 - x^4$ at the point $(1, 3)$.
- *HW: 1-18*

3.4 The Derivative as an Instantaneous Rate of Change

- Interpretation of $\frac{du}{dv}$
  - The fraction $\frac{\Delta u}{\Delta v}$ represents the change in a variable u as compared to the change in another variable v.
  - Therefore the expression $\frac{du}{dv} = \lim_{\Delta v \to 0} \frac{\Delta u}{\Delta v}$ measures the instantaneous rate of change in u with respect to the rate of change in v.

- In particular, if s is the position of an object and t is the time, then $\frac{ds}{dt}$ is the instantaneous rate of change in position with respect to time, known as its velocity.
- (Example 3) Objects at sea level fall roughly $16t^2$ feet after t seconds from release. Note that after 4 seconds, the object has fallen 256 feet. Use the following chart to approximate the instantaneous downward velocity of the object 4 seconds after release, then compute it exactly using a derivative.

t	3	3.9	3.99	3.999	4
Δt from 4	1	0.1	0.01	0.001	0
s	144	243.36	254.7216	255.872016	256
Δs from 256	112	12.64	1.2784	0.127984	0
$\frac{\Delta s}{\Delta t}$	112	126.4	127.84	127.984	(DNE)

- (Example 5) A spherical balloon is being inflated. Find a formula for the instantaneous rate of change of volume with respect to its radius, then compute it when the radius is 2 meters. (Hint: $V = \frac{4}{3}\pi r^3$.)
- (3.5 Example 8) Suppose the displacement of a piston is $t^3 - 6t^2 + 8t$ centimeters after t seconds have elapsed. Find the position and velocity of the piston in one second intervals from $t = 0$ to $t = 4$.

- *HW **in section 3.5**: 25-32, 38-42*

3.6 Derivatives of Products and Quotients of Functions

- Product Rule
  - (Example 2) Find the derivative of the function $p(x) = (x^2 + 2)(3 - 2x)$.
  - Product Rule: $\frac{d}{dx}[f(x)g(x)] = g(x)f'(x) + f(x)g'(x)$
  - (Example 2 again) Verify the product rule.
  - (Example 3) Find $\frac{dy}{dx}$ where $y = (3 - x - 2x^2)(x^4 - x)$.
- Quotient Rule
  - (Example) Find the derivative of the function $q(x) = \frac{x^2+1}{x}$.
  - Quotient Rule: $\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)f'(x)+f(x)g'(x)}{[g(x)]^2}$
  - (Same Example) Verify the quotient rule.
  - (Example 4) Find the derivative of $h(x) = \frac{3-2x}{x^2+2}$.

- (Example 5) The stress S on a hollow tube with tension T, outer diameter D, and inner diameter d is given by the equation $S = \frac{16DT}{\pi(D^4-d^4)}$. Assume this tube has constant tension $T = 10$ and constant inner diameter $d = 1$. Find the rate of change stress increases with respect to an increasing outer diameter when $D = 2$.

• *HW: 1-28, 39-42*

3.7 The Derivative of a Power of a Function

• Chain Rule for Power Functions

- (Example 1) Find the derivative of the function $y = (3 - 2x)^3$.
- Chain Rule for Powers: $\frac{d}{dx}[(f(x))^p] = p(f(x))^{p-1}f'(x)$
- (Example 1 again) Verify the chain rule.
- (Example 5) Find the derivative of $y = 6\sqrt[3]{x^2}$.
- (Example 4) Find the derivative of $y = \sqrt{x^2+1}$.
- (Example 8) Evaluate the derivative of $y = \frac{x}{\sqrt{1-4x}}$ when $x = -2$.

• *HW: 1-24, 29-30, 35-38*

3.8 Differentiation of Implicit Functions

• Implicit Functions

- The expression $y = f(x)$ defines an explicit function.
- An equation with variables x, y may define y as an implicit function of x.
- (Example 1) Manipulate the equation $3x + 4y = 5$ which defines y as an implicit function of x so that y is defined as an explicit function of x.
- (Example) Give an explicit function which describes the part of the hyperbola centered at the origin with focus $(0, 5)$ and vertex $(0, 3)$ passing through the point $(-16/3, 5)$.

• Implicit Differentiation

- (Example) Find the slope of the line tangent to the hyperbola from the previous example at the point $(-16/3, 5)$.
- Implicit functions may be differentated directly by using the chain rule: differentiate y terms as you would x, but tack on a $\frac{dy}{dx}$ term each time you do.
- (Example) Find the slope of the line described in Example 1 using both implicit and explicit differentation.

  - (Example) Solve the hyperbola problem using implicit differentiation.
  - (Example 3) Find $\frac{dy}{dx}$ in terms of x, y where $3y^4 + xy^2 = 6 - 2x^3$.
  - (Example 5) Find the slope of a line tangent to the graph of $2y^3 + xy + 1 = 0$ at the point $(-3, 1)$.

- *HW: 1-25, 28-30*

3.8 Differentiation of Implicit Functions

- Implicit Functions
  - The expression $y = f(x)$ defines an explicit function.
  - An equation with variables x, y may define y as an implicit function of x.
  - (Example 1) Manipulate the equation $3x + 4y = 5$ which defines y as an implicit function of x so that y is defined as an explicit function of x.
  - (Example) Give an explicit function which describes the part of the hyperbola centered at the origin with focus $(0, 5)$ and vertex $(0, 3)$ passing through the point $(-16/3, 5)$.
- Implicit Differentiation
  - (Example) Find the slope of the line tangent to the hyperbola from the previous example at the point $(-16/3, 5)$.
  - Implicit functions may be differentated directly by using the chain rule: differentiate y terms as you would x, but tack on a $\frac{dy}{dx}$ term each time you do.
  - (Example) Find the slope of the line described in Example 1 using both implicit and explicit differentation.
  - (Example) Solve the hyperbola problem using implicit differentiation.
  - (Example 3) Find $\frac{dy}{dx}$ in terms of x, y where $3y^4 + xy^2 = 6 - 2x^3$.
  - (Example 5) Find the slope of a line tangent to the graph of $2y^3 + xy + 1 = 0$ at the point $(-3, 1)$.
- *HW: 1-25, 28-30*

3.9 Higher Derivatives

- Higher Derivatives
  - The derivative of a derivative is its second derivative. The derivative of a second derivative is its third derivative, etc.

- (Example 1) Find all higher derivatives of $y = 5x^3 - 2x$.
- (Example 3) Find the second derivative of $y = \frac{2}{1-x}$ when $x = -2$.
- (Example 4) Find y'' for the implicit function defined by $2x^2 + 3y^2 = 6$ in terms of x, y.

- Acceleration
  - If s is position defined in terms of t (time), then $s' = \frac{ds}{dt}$ is velocity and $s'' = \frac{d^2s}{dt^2}$ is acceleration.
  - (Example) The height of an object launched upward from the ground with an initial velocity of v_0 m/s is roughly $s = -4.9t^2 + v_0 t$ meters after t seconds. Find the velocity and acceleration of this object when $t = 1$ and $v_0 = 10$.
- *HW: 1-34, 37-38*

4.4 Related Rates

- Related Rates as Implicit Differentation
  - If the variables in an equation are functions of time, then we may use implicit differentiation to compare their rates of change with respect to time.
  - (Example 1) The voltage E of a certain thermocouple may be measured as $E = 2.8T + 0.006T^2$ where T is its temperature in Celcius. If the temperature of the thermocouple is increasing at a rate of 1° C/min, then how fast is the voltage increasing when the temperature is 100° C?
  - (Example 3) A spherical balloon is being filled at a rate of 2 cubic feet per minute. How fast is its radius growing when the radius is 3 feet long?
  - (Example 5) Two ships leave port at noon. Ship A travels west at 12 km/h, and ship B travels south at 16 km/h. Communication systems between the two ships only function when their relative speed is less than 20 km/h. How long after leaving port can the two ships communicate before their relative speed reaches 20 km/h?
- *HW: 1-24*

Remaining Topics

- 4.5 Using Derivatives in Curve Sketching
- 4.7 Applied Maximum and Minimum Problems

- 4.8 Differentials and Linear Approximations
- 5.1 Antiderivatives
- 5.2 The Indefinite Integral
- 5.3 The Area Under a Curve
- 5.4 The Definite Integral
- 7.1 The Trigonometric Functions
- 7.2 Basic Trigonometric Relations
- 7.3 Derivatives of the Sine and Cosine Functions
- 7.4 Derivatives of the Other Trigonometric Functions
- 8.1 Exponential and Logarithmic Functions
- 8.2 Derivative of the Logarithmic Functions
- 8.3 Derivative of the Exponentials Function
- 9.1 The General Power Formula
- 9.2 Basic Logarithmic Form
- 9.3 Exponential Form
- 9.4 Basic Trigonometric Forms