**MATH 2242 (Calculus IV) Course Outline** — Vector Calculus (Marsden)

## 1.5 $n$-Dimensional Euclidean Space

- $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$
- Addition
  - $\langle x_1, x_2, \ldots, x_n\rangle + \langle y_1, y_2, \ldots, y_n\rangle = \langle x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\rangle$
- Scalar multiplication
  - $\alpha\langle x_1, x_2, \ldots, x_n\rangle = \langle \alpha x_1, \alpha x_2, \ldots, \alpha x_n\rangle$
- Inner/Dot Product
  - $\langle x_1, x_2, \ldots, x_n\rangle \cdot \langle y_1, y_2, \ldots, y_n\rangle = \sum_{i=1}^n x_i y_i$
- Norm/Length/Magnitude
  - $\|\mathbf{x}\| = (\mathbf{x}\cdot\mathbf{x})^{1/2}$
- Standard basis vectors
  - $\mathbf{e}_1 = \langle 1, 0, \ldots, 0\rangle$, $\mathbf{e}_2 = \langle 0, 1, \ldots, 0\rangle$, $\ldots$, $\mathbf{e}_n = \langle 0, 0, \ldots, 1\rangle$
- Theorems
  - $(\alpha\mathbf{x} + \beta\mathbf{y})\cdot\mathbf{z} = \alpha(\mathbf{x}\cdot\mathbf{z}) + \beta(\mathbf{y}\cdot\mathbf{z})$
  - Prove the above theorem.
  - $\mathbf{x}\cdot\mathbf{y} = \mathbf{y}\cdot\mathbf{x}$
  - $\mathbf{x}\cdot\mathbf{x} \geq 0$
  - $\mathbf{x}\cdot\mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$
  - $|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ (the Cauchy-Schwarz inequality)
  - Prove the Cauchy-Schwarz inequality.
  - $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (the triangle inequality)
  - Prove the triangle inequality.
- Matrices
  - $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$

- Addition $A + B$
- Scalar Mutiplication $\alpha A$
- Transposition $A^T$

- Vectors as Matrices
  - $\mathbf{a} = \langle a_1, a_2, \dots, a_n \rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$
  - $\mathbf{a}^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$

- Matrix Multiplication
  - If $A$ has $m$ rows and $B$ has $n$ columns, then $M = AB$ is an $m \times n$ matrix.
  - Coordinate $ij$ of $M = AB$ is given by $m_{ij} = \mathbf{a_i} \cdot \mathbf{b_j}$ where $\mathbf{a_i}^T$ is the $i$th row of $A$ and $\mathbf{b_j}$ is the $j$th column of $B$.
  - (Example 4) Compute $AB$ and $BA$ for

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

  - (Example 5) Compute $AB$ for

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Matrices as Linear Transformations
  - An $m \times n$ matrix $A$ gives a function from $\mathbb{R}^n$ to $\mathbb{R}^m$: $\mathbf{x} \mapsto A\mathbf{x}$
  - This linear transformation satsifies $A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y}$
  - (Example 7) Express $A\mathbf{x}$ where $x = \langle x_1, x_2, x_3 \rangle$ and $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

  - (Example) Compute where the points $(-1,-1,0)$, $(0,1,0)$, $(1,-1,1)$, and $(2,1,1)$ in $\mathbb{R}^3$ get mapped to in $\mathbb{R}^4$ by $A\mathbf{x}$ from the previous example. Then plot the projections of the original points in $\mathbb{R}^3$ onto their first two coordinates in $\mathbb{R}^2$, and compare this with the projection plot of their images in $\mathbb{R}^4$ onto their first two coordinates in $\mathbb{R}^2$.

- Identity and Inverse
  - $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$
  - If $AA^{-1} = A^{-1}A = I_n$, then $A$ is invertable and $A^{-1}$ is its inverse.

- Determinant
  - Let $A_i$ be the submatrix of $A$ with the first column and $i$th row removed. Then $\det(A) = \sum_{i=1}^n (-1)^{i+1} a_{1i} \det(A_i)$
  - This is equivalent to $\det(A) = \sum_{\sigma \in S^n} \operatorname{sgn}(\sigma) \prod_{1 \leq i \leq n} a_{i,\sigma_i}$ where $S^n$ is the collection of all permutations of elements 1 to $n$ and $\operatorname{sgn}(\sigma)$ is 1 when $\sigma$ is obtained by an even number of swaps, and $-1$ when $\sigma$ is obtained by an odd number of swaps.
  - An $n \times n$ matrix is invertable if and only if its determinant is nonzero.

- *HW: 1-18, 21-24*

## 2.3 Differentiation

- Functions $\mathbb{R}^n \to \mathbb{R}^m$
  - $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$
  - $\mathbf{f}(\mathbf{x}) = \langle f_1(\mathbf{x}), \dots, f_m(\mathbf{x}) \rangle$ where $f_i : \mathbb{R}^n \to \mathbb{R}$

- Partial Derivative Matrix
  - $\mathbf{Df}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{bmatrix}$
  - We say $\mathbf{f}$ is differentiable at $\mathbf{x}$ if $\mathbf{f}(\mathbf{x}+\mathbf{h}) \approx \mathbf{f}(\mathbf{x}) + [\mathbf{Df}(\mathbf{x})]\mathbf{h}$ for all $\mathbf{h}$ near $\mathbf{0}$.
  - (Example) Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mathbf{f}(x,y) = \langle x^2+y^2, xy \rangle$, and let $\mathbf{T} = \mathbf{Df}(1,0)$. Compute $\mathbf{f}(1.1,-0.1)$ and $\mathbf{f}(1,0) + \mathbf{T}\langle 0.1, -0.1 \rangle$.

  - If each $\frac{\partial f_i}{\partial x_j} : \mathbb{R}^n \to \mathbb{R}$ is a continuous function near $\mathbf{x}$, then we say $\mathbf{f}$ is strongly differentiable or class $C^1$ at $\mathbf{x}$.

- Gradient
  - If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient vector function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\nabla f(\mathbf{x}) = (\mathbf{D}f(\mathbf{x}))^T = \langle \frac{\partial f}{\partial x_1}(\mathbf{x}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{x}) \rangle$
  - $[\mathbf{D}f(\mathbf{x})]\mathbf{h} = \nabla f(\mathbf{x}) \cdot \mathbf{h}$

- Linearizations and Tangent Hyperplanes
  - Letting $\mathbf{y} = \mathbf{x} + \mathbf{h}$ and $\mathbf{y}_0 = \mathbf{x}$, we have $\mathbf{f}(\mathbf{y}) \approx \mathbf{f}(\mathbf{y}_0) + [\mathbf{Df}(\mathbf{y_0})](\mathbf{y} - \mathbf{y}_0)$ for differentiable $\mathbf{f}$.
  - For $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and a point $\mathbf{x}_0 \in \mathbb{R}^n$, let the linearization of $\mathbf{f}$ at $x_0$ be $\mathbf{L}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + [\mathbf{Df}(\mathbf{x_0})](\mathbf{x} - \mathbf{x}_0)$. Note $\mathbf{L}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x})$ for $\mathbf{x}$ near $\mathbf{x}_0$.
  - (Example 5) Find the linearization $L(x, y)$ of $f(x, y) = x^2 + y^4 + e^{xy}$ at the point $(1, 0)$, and observe that this gives the equation of a tangent plane to the surface $z = f(x, y)$ at the point $(1, 0, 2)$.

- *HW: 1-3, 5-21*

## 2.5 Properties of the Derivative

- Sum/Product/Quotient Rules
  - $\mathbf{D}[\alpha\mathbf{f}] = \alpha\mathbf{Df}$
  - $\mathbf{D}[\mathbf{f} + \mathbf{g}] = \mathbf{Df} + \mathbf{Dg}$
  - $\mathbf{D}[fg] = g\mathbf{D}f + f\mathbf{D}g$
  - $\mathbf{D}[\frac{f}{g}] = \frac{g\mathbf{D}f - f\mathbf{D}g}{g^2}$
  - Sketch proofs for strongly differentiable $f, g$.

- Chain Rule
  - $\mathbf{D}[\mathbf{f} \circ \mathbf{g}] = [\mathbf{Df}](\mathbf{g})\mathbf{Dg}$
  - (Example) Find the rate of change of $f(x, y) = x^2 + y^2$ along the path $\mathbf{c}(t) = \langle t^2, t \rangle$ when $t = 1$.
  - (Example 2) Verify the Chain Rule for $f(u, v, w) = u^2 + v^2 - w$ and $\mathbf{g}(x, y, z) = \langle x^2 y, y^2, e^{-xz} \rangle$.
  - (Example 3) Compute $\mathbf{D}[\mathbf{f} \circ \mathbf{g}](1, 1)$ where $\mathbf{f}(u, v) = \langle u + v, u, v^2 \rangle$ and $\mathbf{g}(x, y) = \langle x^2 + 1, y^2 \rangle$.

- *HW: 6-13, 15-16*

## 3.2 Taylor's Theorem

- First-Order Taylor Formula
  - $f(\mathbf{x}+\mathbf{h}) \approx f(\mathbf{x}) + [\mathbf{D}f(\mathbf{x})]\mathbf{h}$ or $f(\mathbf{x}) \approx f(\mathbf{x}_0) + [\mathbf{D}f(\mathbf{x}_0)](\mathbf{x}-\mathbf{x}_0)$
  - Alternate form: $f(\mathbf{x}+\mathbf{h}) \approx f(\mathbf{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x})h_i$ or $f(\mathbf{x}) \approx f(\mathbf{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i - x_{0,i})$
- Second-Order Taylor Formula
  - $f(\mathbf{x}+\mathbf{h}) \approx f(\mathbf{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x})h_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x})h_i h_j$
  - $f(\mathbf{x}) \approx f(\mathbf{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i - x_{0,i}) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)(x_i - x_{0,i})(x_j - x_{0,j})$
  - (Example 3) Find linear and quadratic functions of $x, y$ which approximate $f(x,y) = \sin(xy)$ near the point $(1, \pi/2)$.
- *HW: 1-12*

## 4.3 Vector Fields

- Vector Fields
  - A vector field is a map $f : \mathbb{R}^n \to \mathbb{R}^n$ assinging an $n$-dimensional vector to each point in $\mathbb{R}^n$
  - (Example 1) The velocity field of a fluid may be modeled as a vector field.
  - (Example 2) Sketch the rotary motion given by the vector field $\mathbf{V}(x,y) = \langle -y, x\rangle$.
- Gradient Vector Fields
  - $\nabla f = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\rangle$
  - (Example) The derivative of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$ in the direction given by a unit vector $\mathbf{v}$ is given by $\nabla f \cdot \mathbf{v}$. Show that the maximum value of a directional derivative for a fixed point is given by $\|\nabla f\|$ and attained by the direction $\frac{1}{\|\nabla f\|}\nabla f$.
  - (Example 4) If temperature is given by $T(x,y,z)$, then the energy or heat flux field is given by $\mathbf{J} = -k\nabla T$ where $k$ is the conductivity of the body. Level sets are called isotherms.
  - (Example 5) The gravitational potential of bodies with mass $m, M$ is given by $V = -\frac{mMG}{r}$ where $G$ is the gravitational constant and $r$ is the distance between the bodies, and the gravitational force field is given by $\mathbf{F} = -\nabla V$. Show that $\mathbf{F} = -\frac{mMG}{r^3}\mathbf{r}$, where $\mathbf{r}$ is the vector pointing from the center of mass $M$ to the center of mass $m$.

  - A vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is conservative iff there exists a potential function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.
  - (Example) Show that $\mathbf{W} = \langle 2y + 1, 2x \rangle$ is conservative.
  - (Example 7) Show that $\mathbf{V} = \langle y, -x \rangle$ is not conservative.

- Flow Lines
  - A flow line for a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a path $\mathbf{c} : \mathbb{R} \to \mathbb{R}^n$ satisfying $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$.
  - (Example 8) Show that $\mathbf{c}(t) = \langle \cos t, \sin t \rangle$ is a flow line for $\mathbf{F} = \langle -y, x \rangle$, and find some other flow lines.

- *HW: 1-22*

## 4.4 Divergence and Curl

- Divergence
  - The divergence of a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is denoted by $\operatorname{div} \mathbf{F} : \mathbb{R}^n \to \mathbb{R}$ and defined by $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}$
  - (Examples 3-5) Compute the divergences of $\mathbf{F} = \langle x, y \rangle$, $\mathbf{G} = \langle -x, -y \rangle$ and $\mathbf{H} = \langle -y, x \rangle$ at any point on $\mathbb{R}^2$. How does divergence correspond with the motion described by the vector field plots?
  - (Example) Compute the divergence of $\mathbf{F} = \langle x^2, y \rangle$ various points and interpret those values against a plot of the vector field.

- Curl
  - The curl of a three-dimensional vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is denoted by $\operatorname{curl} \mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ and defined by $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \rangle$
  - The scalar curl of a two-dimensional vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is denoted by $\operatorname{scurl} \mathbf{F} : \mathbb{R}^2 \to \mathbb{R}$ and defined by $\operatorname{scurl} \mathbf{F} = \operatorname{curl} \mathbf{F} \cdot \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$
  - (Example) Compute the scalar curl of $\mathbf{F} = \langle x, y \rangle$, $\mathbf{G} = \langle -x, -y \rangle$ and $\mathbf{H} = \langle -y, x \rangle$ at every point in $\mathbb{R}^2$. How does this scalar curl correspond with the motion described by the vector field plots?
  - (Example) Compute the curl of $\mathbf{F} = \langle y, -x, z \rangle$ at every point in $\mathbb{R}^3$. How does curl correspond with the motion described by the vector field plot?

- Facts about $\nabla f$, $\operatorname{div} \mathbf{F}$, $\operatorname{curl} \mathbf{F}$
  - The curl of a conservative field is zero: $\operatorname{curl} \nabla f = \nabla \times (\nabla f) = \mathbf{0}$.
  - (Example) Prove the above theorem.

  - (Example) Prove that $\mathbf{F} = \langle x^2 + z, y - z, z^3 + 3xy \rangle$ is not a conservative field.
  - The divergence of a curl field is zero: $\operatorname{div} \operatorname{curl} \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$
  - Many identities on pg. 255 of Marsden text.
  - (Example) Sketch proof of identity #8: $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$.

- *HW: 1-37*

## 5.3 The Double Integral Over More General Regions

- Hypervolume
  - The hypervolume $HV_1$ of an interval $[a, b]$ in $\mathbb{R}$ is just its length $b - a$.
  - The hypervolume of a well-behaved bounded subset $D \subseteq \mathbb{R}^{n+1}$ is defined for each $i \in \{1, \ldots, n+1\}$ by

$$HV_{n+1} = \int_{x_i \in I} HV(Z_i)\, dx_i = \int_{x_i=a}^{x_i=b} HV_n(D_i)\, dx_i$$

    where $I = [a, b]$ is an interval containing all values $x_i$ included in the $i$th coordinate of $D$, and $D_i$ is the projection of of all points in $D$ onto $\mathbb{R}^n$ by removing the $i$th coordinate.
  - (Example) For $n = 1$ and $D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, f(x) \leq y \leq g(x)\}$, we have that

$$HV_2 = A = \int_{x \in [a,b]} g(x) - f(x)\, dx = \int_a^b g(x) - f(x)\, dx$$

    (assuming $f(x) \leq g(x)$ whenever $a \leq x \leq b$).
  - (Example) For $n = 2$ and $D \subseteq R^3$ including values of $x$ between $a$ and $b$, we have that

$$HV_3 = V = \int_{x=a}^{x=b} A(x)\, dx$$

$$HV_3 = V = \int_{y=c}^{y=d} A(y)\, dx$$

    where $A(x)$ is the area of the cross-section of $D$ taken by fixing each value of $x$ (or similar for $y$).

- Double Integrals

- For a bounded region $D \subseteq \mathbb{R}^2$ and continuous $F : D \to \mathbb{R}$, the double integral

$$\iint_D f\,dA$$

is defined to be the volume of $D^\uparrow = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D, 0 \leq z \leq F(x, y)\}$ minus the volume of $D_\downarrow = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D, F(x, y) \leq z \leq 0\}$ (sometimes called net volume or signed volume).

- Assuming $F(x, y) \geq 0$, we may apply the definition of volume above to get

$$\iint_D F\,dA = \int_{x=a}^{x=b} A(x)\,dx$$

And if each cross section $A(x)$ is described by $\phi_1(x) \leq y \leq \phi_2(x)$ and $0 \leq z \leq F(x, y)$, we have that

$$\iint_D F\,dA = \int_{x=a}^{x=b} A(x)\,dx = \int_{x=a}^{x=b} \left[\int_{y=\phi_1(x)}^{y=\phi_2(x)} F(x, y)\,dy\right] dx$$

- Similarly, if $D$ is described by $c \leq y \leq d$ and $\psi_1(y) \leq x \leq \psi_2(y)$, then

$$\iint_D F\,dA = \int_{y=c}^{y=d} \left[\int_{x=\psi_1(y)}^{x=\psi_2(y)} F(x, y)\,dx\right] dy$$

- The above holds even when $F(x, y) \geq 0$ doesn't hold.

- Iterated integrals
  - An iterated integral is a shorthand for the expansion of two or more nested integrals, e.g.:

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} F(x, y)\,dy\,dx = \int_{x=a}^{x=b} \left[\int_{y=\phi_1(x)}^{y=\phi_2(x)} F(x, y)\,dy\right] dx$$

  - (Example) Sketch the region of integration for $\int_0^\pi \int_{-x}^x \cos(y)\,dy\,dx$, evaluate it, and interpret it as the signed volume of a region in $\mathbb{R}^3$.
  - (Example) Express $\iint_R (12x^3y - 1)\,dA$ where $R$ is the rectangle with vertices $(0, 0), (3, 0), (3, 2), (0, 2)$ as an interated integral, then evaluate it.
  - (Example) Express $\iint_T (12x^3y - 1)\,dA$ where $T$ is the triangle with vertices $(0, 0), (1, 0), (1, 1)$ as an interated integral, then evaluate it.

- *HW: 1-17*

## Remaining Topics

- 5.4 Changing the Order of Integration
- 5.5 The Triple Integral
- 6.1 The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$
- 6.2 The Change of Variables Theorem
- 7.1 The Path Integral
- 7.2 Line Integrals
- 7.3 Parametrized Surfaces
- 7.4 Area of a Surface
- 7.5 Integrals of Scalar Functions Over Surfaces
- 7.6 Surface Integrals of Vector Fields
- 8.1 Green's Theorem
- 8.2 Stokes' Thoerem
- 8.3 Conservative Fields
- 8.4 Gauss' Theorem