MATH 2242 (Calculus IV) Course Outline — Vector Calculus (Marsden)

1.5 n-Dimensional Euclidean Space

- $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$

- Addition

    - $\langle x_1, x_2, \ldots, x_n \rangle + \langle y_1, y_2, \ldots, y_n \rangle = \langle x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n \rangle$

- Scalar multiplication

    - $\alpha \langle x_1, x_2, \ldots, x_n \rangle = \langle \alpha x_1, \alpha x_2, \ldots, \alpha x_n \rangle$

- Inner/Dot Product

    - $\langle x_1, x_2, \ldots, x_n \rangle \cdot \langle y_1, y_2, \ldots, y_n \rangle = \sum_{i=1}^{n} x_i y_i$

- Norm/Length/Magnitude

    - $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$

- Standard basis vectors

    - $\mathbf{e}_1 = \langle 1, 0, \ldots, 0 \rangle$, $\mathbf{e}_2 = \langle 0, 1, \ldots, 0 \rangle$, $\ldots$, $\mathbf{e}_n = \langle 0, 0, \ldots, 1 \rangle$

- Theorems

    - $(\alpha \mathbf{x} + \beta \mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z})$
    - Prove the above theorem.
    - $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
    - $\mathbf{x} \cdot \mathbf{x} \geq 0$
    - $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$
    - $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ (the Cauchy-Schwarz inequality)
    - Prove the Cauchy-Schwarz inequality.
    - $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (the triangle inequality)
    - Prove the triangle inequality.

- Matrices

    - $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$

- ■ Addition $A + B$
- ■ Scalar Mutiplication αA
- ■ Transposition A^T

- Vectors as Matrices

  - ■ $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

  - ■ $\mathbf{a}^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$

- Matrix Multiplication

  - ■ If A has m rows and B has n columns, then $M = AB$ is an $m \times n$ matrix.

  - ■ Coordinate ij of $M = AB$ is given by $m_{ij} = \mathbf{a_i} \cdot \mathbf{b_j}$ where $\mathbf{a_i}^T$ is the ith row of A and $\mathbf{b_j}$ is the jth column of B.

  - ■ (Example 4) Compute AB and BA for

  $$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

  $$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

  - ■ (Example 5) Compute AB for

  $$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

  $$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Matrices as Linear Transformations

  - ■ An $m \times n$ matrix A gives a function from $\mathbb{R}^n$ to $\mathbb{R}^m$: $\mathbf{x} \mapsto A\mathbf{x}$

  - ■ This linear transformation satsifies $A(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y}$

  - ■ (Example 7) Express $A\mathbf{x}$ where $x = \langle x_1, x_2, x_3 \rangle$ and $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

- ■ (Example) Compute where the points $(-1, -1, 0)$, $(0, 1, 0)$, $(1, -1, 1)$, and $(2, 1, 1)$ in $\mathbb{R}^3$ get mapped to in $\mathbb{R}^4$ by $A\mathbf{x}$ from the previous example. Then plot the projections of the original points in $\mathbb{R}^3$ onto their first two coordinates in $\mathbb{R}^2$, and compare this with the projection plot of their images in $\mathbb{R}^4$ onto their first two coordinates in $\mathbb{R}^2$.

- • Identity and Inverse

  - ■ $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

  - ■ If $AA^{-1} = A^{-1}A = I_n$, then A is invertable and A^{-1} is its inverse.

- • Determinant

  - ■ Let A_i be the submatrix of A with the first column and ith row removed. Then $\det(A) = \sum_{i=1}^n (-1)^{i+1} a_{1i} \det(A_i)$

  - ■ This is equivalent to $\det(A) = \sum_{\sigma \in S^n} \text{sgn}(\sigma) \prod_{1 \le i \le n} a_{i,\sigma_i}$ where S^n is the collection of all permutations of elements 1 to n and $\text{sgn}(\sigma)$ is 1 when σ is obtained by an even number of swaps, and -1 when σ is obtained by an odd number of swaps.

  - ■ An $n \times n$ matrix is invertable if and only if its determinant is nonzero.

- • *HW: 1-18, 21-24*

2.3 Differentiation

- • Functions $\mathbb{R}^n \to \mathbb{R}^m$

  - ■ $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$
  - ■ $\mathbf{f}(\mathbf{x}) = \langle f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}) \rangle$ where $f_i : \mathbb{R}^n \to \mathbb{R}$

- • Partial Derivative Matrix

  - ■ $\mathbf{Df}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{bmatrix}$

  - ■ We say $\mathbf{f}$ is differentiable at $\mathbf{x}$ if $\mathbf{f}(\mathbf{x} + \mathbf{h}) \approx \mathbf{f}(\mathbf{x}) + [\mathbf{Df}(\mathbf{x})]\mathbf{h}$ for all $\mathbf{h}$ near $\mathbf{0}$.

  - ■ (Example) Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mathbf{f}(x, y) = \langle x^2 + y^2, xy \rangle$, and let $\mathbf{T} = \mathbf{Df}(1, 0)$. Compute $\mathbf{f}(1.1, -0.1)$ and $\mathbf{f}(1, 0) + \mathbf{T}\langle 0.1, -0.1 \rangle$.

- If each $\frac{\partial f_i}{\partial x_j} : \mathbb{R}^n \to \mathbb{R}$ is a continuous function near $\mathbf{x}$, then we say $\mathbf{f}$ is strongly differentiable at $\mathbf{x}$.

- Gradient

  - If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient vector function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\nabla f(\mathbf{x}) = (\mathbf{D}f(\mathbf{x}))^T = \langle \frac{\partial f}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x}) \rangle$
  - $[\mathbf{D}f(\mathbf{x})]\mathbf{h} = \nabla f(\mathbf{x}) \cdot \mathbf{h}$

- *HW: 1-3, 5-21*

2.5 Properties of the Derivative

- Sum/Product/Quotient Rules

  - $\mathbf{D}[\alpha \mathbf{f}] = \alpha \mathbf{D}\mathbf{f}$
  - $\mathbf{D}[\mathbf{f} + \mathbf{g}] = \mathbf{D}\mathbf{f} + \mathbf{D}\mathbf{g}$
  - $\mathbf{D}[fg] = g\mathbf{D}f + f\mathbf{D}g$
  - $\mathbf{D}[\frac{f}{g}] = \frac{g\mathbf{D}f - f\mathbf{D}g}{g^2}$
  - Sketch proofs for strongly differentiable f, g.

- Chain Rule

  - $\mathbf{D}[\mathbf{f} \circ \mathbf{g}] = [\mathbf{D}\mathbf{f}](\mathbf{g})\mathbf{D}\mathbf{g}$
  - (Example) Find the rate of change of $f(x, y) = x^2 + y^2$ along the path $\mathbf{c}(t) = \langle t^2, t \rangle$ when $t = 1$.
  - (Example 2) Verify the Chain Rule for $f(u, v, w) = u^2 + v^2 - w$ and $\mathbf{g}(x, y, z) = \langle x^2 y, y^2, e^{-xz} \rangle$.
  - (Example 3) Compute $\mathbf{D}[\mathbf{f} \circ \mathbf{g}](1, 1)$ where $\mathbf{f}(u, v) = \langle u + v, u, v^2 \rangle$ and $\mathbf{g}(x, y) = \langle x^2 + 1, y^2 \rangle$.

- *HW: 6-13, 15-16*

3.2 Taylor's Theorem

- First-Order Taylor Formula

  - $f(\mathbf{x} + \mathbf{h}) \approx f(\mathbf{x}) + [\mathbf{D}f(\mathbf{x})]\mathbf{h}$
  - Alternate form: $f(\mathbf{x} + \mathbf{h}) \approx f(\mathbf{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x})h_i$

- Second-Order Taylor Formula

- $f(\mathbf{x} + \mathbf{h}) \approx f(\mathbf{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x})h_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x})h_i h_j$
- (Example 3) Find linear and quadratic functions of x, y which approximate $f(x, y) = \sin(xy)$ near the point $(1, \pi/2)$.

- *HW: 1-12*

4.3 Vector Fields

- Vector Fields

  - A vector field is a map $f : \mathbb{R}^n \to \mathbb{R}^n$ assinging an n-dimensional vector to each point in $\mathbb{R}^n$
  - (Example 1) The velocity field of a fluid may be modeled as a vector field.
  - (Example 2) Sketch the rotary motion given by the vector field $\mathbf{V}(x, y) = \langle -y, x \rangle$.

- Gradient Vector Fields

  - $\nabla f = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$
  - (Example) The directional derivative of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$ is given by $\nabla f \cdot \mathbf{v}$. Show that the maximum value of the directional derivative at a point is given by $\|\nabla f\|$ and attained by the direction $\frac{1}{\|\nabla f\|}\nabla f$.
  - (Example 4) If temperature is given by $T(x, y, z)$, then the energy or heat flux field is given by $\mathbf{J} = -k\nabla T$ where k is the conductivity of the body. Level sets are called isotherms.
  - (Example 5) The gravitational potential of bodies with mass m, M is given by $V = -\frac{mMG}{r}$ where G is the gravitational constant and r is the distance between the bodies, and the gravitational force field is given by $\mathbf{F} = -\nabla V$. Show that $\mathbf{F} = -\frac{mMG}{r^3}\mathbf{r}$, where $\mathbf{r}$ is the vector pointing from the center of mass M to the center of mass m.
  - A vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is conservative iff there exists a potential function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.
  - (Example) Show that $\mathbf{W} = \langle 2y + 1, 2x \rangle$ is conservative.
  - (Example 7) Show that $\mathbf{V} = \langle y, -x \rangle$ is not conservative.

- Flow Lines

  - A flow line for a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a path $\mathbf{c} : \mathbb{R} \to \mathbb{R}^n$ satisfying $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$.
  - (Example 8) Show that $\mathbf{c}(t) = \langle \cos t, \sin t \rangle$ is a flow line for $\mathbf{F} = \langle -y, x \rangle$, and find some other flow lines.

- *HW: 1-22*

Remaining Topics

- 4.4 Divergence and Curl

- 5.3 The Double Integral Over More General Regions

- 5.4 Changing the Order of Integration

- 5.5 The Triple Integral

- 6.1 The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

- 6.2 The Change of Variables Theorem

- 7.1 The Path Integral

- 7.2 Line Integrals

- 7.3 Parametrized Surfaces

- 7.4 Area of a Surface

- 7.5 Integrals of Scalar Functions Over Surfaces

- 7.6 Surface Integrals of Vector Fields

- 8.1 Green's Theorem

- 8.2 Stokes' Thoerem

- 8.3 Conservative Fields

- 8.4 Gauss' Theorem