MATH 2242 (Calculus IV) Course Outline

1.2 The Inner Product, Length, and Distance

- Inner/Dot Product
  - $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
- Norm/Magnitude/Length
  - $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$
  - Alternate dot product: $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$
- Normalization/Direction
  - $\frac{\mathbf{a}}{\|\mathbf{a}\|}$
- Distance
  - $\|\mathbf{b} - \mathbf{a}\|$
- Inequalities
  - $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|$
  - $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$

1.3 Matricies, Determinants, and the Cross Product

- Matrices
  - $\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$
  - $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$
- Determinants
  - $\det\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = x_{11}x_{22} - x_{12}x_{21}$
  - $\det\left(\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}\right)$
    $= x_{11}\det\left(\begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix}\right) - x_{12}\det\left(\begin{bmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{bmatrix}\right) + x_{13}\det\left(\begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}\right)$

  - $\det(A) = \sum_{i=1}^{n} (-1)^{i+1} x_{1i} \det(A_i)$

- Cross-Product
  - $\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle = \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}\right)$
  - $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$
  - $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ are mutually orthogonal and follow the right-hand-rule

- Triple Scalar Product
  - $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \det\left(\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix}\right)$

- Plane Equation
  - $\mathbf{n} \cdot (\mathbf{x} - \mathbf{P}) = 0$
  - $n_1(x - P_1) + n_2(y - P_2) + n_3(z - P_3) = 0$

1.5 n-Dimensional Euclidean Space

- $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$
- Addition
  - $\langle x_1, x_2, \ldots, x_n\rangle + \langle y_1, y_2, \ldots, y_n\rangle = \langle x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\rangle$
- Scalar multiplication
  - $\alpha\langle x_1, x_2, \ldots, x_n\rangle = \langle \alpha x_1, \alpha x_2, \ldots, \alpha x_n\rangle$
- Standard basis vectors
  - $\mathbf{e}_1 = \langle 1, 0, \ldots, 0\rangle$, $\mathbf{e}_2 = \langle 0, 1, \ldots, 0\rangle$, $\ldots$, $\mathbf{e}_n = \langle 0, 0, \ldots, 1\rangle$
- Theorems
  - $(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z})$
  - Prove the above theorem.
  - $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
  - $\mathbf{x} \cdot \mathbf{x} \geq 0$

- $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$
- $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ (the Cauchy-Schwarz inequality)
- Prove the Cauchy-Schwarz inequality.
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (the triangle inequality)
- Prove the triangle inequality.

- Matrices
  - $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$
  - Addition $A + B$
  - Scalar Mutiplication αA
  - Transposition A^T

- Vectors as Matrices
  - $\mathbf{a} = \langle a_1, a_2, \dots, a_n \rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$
  - $\mathbf{a}^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$

- Matrix Multiplication
  - If A has m rows and B has n columns, then $M = AB$ is an $m \times n$ matrix.
  - Coordinate ij of $M = AB$ is given by $m_{ij} = \mathbf{a_i} \cdot \mathbf{b_j}$ where $\mathbf{a_i}^T$ is the ith row of A and $\mathbf{b_j}$ is the jth column of B.
  - (Example 4) Compute AB and BA for

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

  - (Example 5) Compute AB for

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Matrices as Linear Transformations
  - An $m \times n$ matrix A gives a function from $\mathbb{R}^n$ to $\mathbb{R}^m$: $\mathbf{x} \mapsto A\mathbf{x}$.
  - (Example 7) Express $A\mathbf{x}$ where $x = \langle x_1, x_2, x_3 \rangle$ and $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$. Then compute where the point $(3, -2, 1)$ in $\mathbb{R}^3$ gets mapped to in $\mathbb{R}^4$
- Identity and Inverse
  - $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$
  - If $AA^{-1} = A^{-1}A = I_n$, then A is invertable and A^{-1} is its inverse.
- Determinant
  - Let A_i be the submatrix of A with the first column and ith row removed. Then $\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \det(A_i)$
- *Suggested HW: 1-18, 21-24*

Remaining Topics

- 2.1 The Geometry of Real-Valued Functions
- 2.3 Differentiation
- 2.4 Introduction to Paths and Curves
- 2.5 Properties of the Derivative
- 2.6 Gradients and Directional Derivatives
- 3.2 Taylor's Theorem
- 4.1 Acceleration and Newton's Second Law
- 4.2 Arc Length

- 4.3 Vector Fields
- 4.4 Divergence and Curl
- 5.3 The Double Integral Over More General Regions
- 5.4 Changing the Order of Integration
- 5.5 The Triple Integral
- 6.1 The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$
- 6.2 The Change of Variables Theorem
- 7.1 The Path Integral
- 7.2 Line Integrals
- 7.3 Parametrized Surfaces
- 7.4 Area of a Surface
- 7.5 Integrals of Scalar Functions Over Surfaces
- 7.6 Surface Integrals of Vector Fields
- 8.1 Green's Theorem
- 8.2 Stokes' Thoerem
- 8.3 Conservative Fields
- 8.4 Gauss' Theorem