① Does $\int_2^\infty \frac{32}{x^3}\,dx$ converge or diverge? If it converges, what is its value?

$$= \lim_{b\to\infty} \int_2^b 32x^{-3}\,dx$$

$$= \lim_{b\to\infty} \left[-16x^{-2}\right]_2^b$$

$$= \left(\lim_{b\to\infty} -\frac{16}{b^2}\right) + \frac{16}{2^2}$$

$$= 0 + \frac{16}{4}$$

$$= \boxed{4}$$ $\boxed{\text{converges}}$

② Does $\int_0^\infty \frac{2y}{y^2+3}\,dy$ converge or diverge? If it converges, what is its value?

$$= \lim_{b\to\infty} \int_0^b \frac{2y}{y^2+3}\,dy$$

Let $u = y^2+3$ $\quad y=b \Rightarrow u=b^2+3$

$du = 2y\,dy$ $\quad y=0 \Rightarrow u=0+3$

$$= \lim_{b\to\infty} \int_3^{b^2+3} \frac{1}{u}\,du$$

$$= \lim_{b\to\infty} \ln(b^2+3) - \ln(3)$$

$\boxed{\text{diverges}}$ (to infinity)

③ Does $\int_e^\infty \frac{1}{\ln(x^x)}\,dx$ converge or diverge? If it converges, what is its value?

$$= \lim_{b\to\infty} \int_e^b \frac{1}{x\ln x}\,dx$$

Let $u = \ln x$ $\quad x=b \Rightarrow u = \ln b$

$du = \frac{1}{x}dx$ $\quad x = e \Rightarrow u = \ln e = 1$

$$= \lim_{b\to\infty} \int_1^{\ln b} \frac{1}{u}\,du$$

$$= \lim_{b\to\infty} \ln(\ln b) - \ln(1)$$

$\boxed{\text{diverges}}$ (to infinity)

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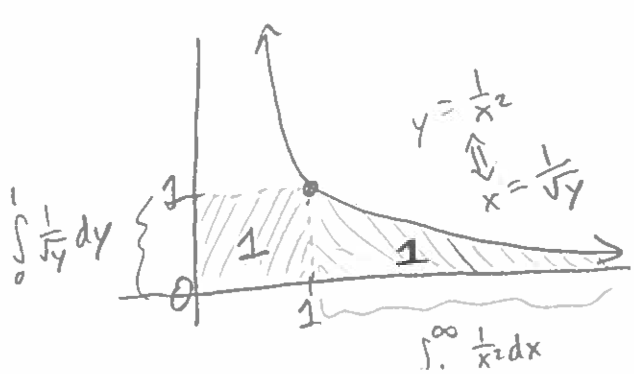
④ Show that $\int_1^\infty \frac{1}{x^2}dx + 1 = \int_0^1 \frac{1}{\sqrt{y}}dy$. Then illustrate why.

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\left[-x^{-1}\right]_1^b$$
$$= \left(\lim_{b\to\infty} -\frac{1}{b}\right) + \frac{1}{1}$$
$$= 0 + 1 = 1$$

$$\int_0^1 \frac{1}{\sqrt{y}}dy = \lim_{a\to 0^+}\left[2y^{1/2}\right]_a^1$$
$$= 2\sqrt{1} - 2\lim_{a\to 0^+}\sqrt{a}$$
$$= 2 - 0 = 2$$

Thus $\int_1^\infty \frac{1}{x^2}dx + 1 = 1 + 1 = 2 = \int_0^1 \frac{1}{\sqrt{y}}dy$. □



(5) Does $\sum_{n=0}^{\infty} \frac{2n}{n^2+3}$ converge or diverge?

From (2), $\int_0^{\infty} \frac{2x}{x^2+3}\,dx$ diverges, so by the Integral Test, $\sum_{n=0}^{\infty} \frac{2n}{n^2+3}$ also $\boxed{\text{diverges}}$.

(6) Does $\sum_{n=3}^{\infty} \frac{4}{n(\ln n)^3}$ converge or diverge?

$$\int_3^{\infty} \frac{4}{x(\ln x)^3}\,dx = \lim_{b\to\infty} \int_3^{b} \frac{4}{x(\ln x)^3}\,dx$$

Let $u = \ln x$ $\quad x=b \Rightarrow u=\ln b$

$du = \frac{1}{x}dx$ $\quad x=3 \Rightarrow u=\ln 3$

$$= \lim_{b\to\infty} \int_{\ln 3}^{\ln b} \frac{4}{u^3}\,du$$

$$= \lim_{b\to\infty} \left[-\frac{2}{u^2}\right]_{\ln 3}^{\ln b}$$

$$= \left(\lim_{b\to\infty} -\frac{2}{(\ln b)^2}\right) + \frac{2}{(\ln 3)^2}$$

$$= 0 + \frac{2}{(\ln 3)^2}$$

Since the integral converges, the series also $\boxed{\text{converges}}$.

⑦ Does $\sum_{n=-2}^{\infty} \frac{1}{e^n}$ converge or diverge?

$$\int_0^{\infty} \frac{1}{e^x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx$$

Let $u=-x$ $\quad x=b \Rightarrow u=-b$

$du=-dx$ $\quad x=0 \Rightarrow u=0$

$$= \lim_{b\to\infty} \int_0^{-b} -e^u\,du$$

$$= \lim_{b\to\infty} \left[-e^u\right]_0^{-b}$$

$$= \left(\lim_{b\to\infty} -\frac{1}{e^b}\right) + e^0$$

$$= 0 + 1 = 1$$

Since the integral converges, the series also $\boxed{\text{converges}}$.

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⑧ Show that $\int_1^{\infty} \frac{1}{x^2}\,dx \neq \sum_{n=1}^{\infty} \frac{1}{n^2}$, even though they both converge.

From ④, $\int_1^{\infty} \frac{1}{x^2}\,dx = 1$. Thus $\sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges.

$$\text{Thus } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \left(\frac{1}{4} + \frac{1}{9} + \ldots\right)$$

$$> \frac{1}{1} = 1 = \int_1^{\infty} \frac{1}{x^2}\,dx. \quad \square$$

(Fun Fact

In advanced calculus, it can be shown that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64493.$$
)

(9) Does $\displaystyle\sum_{k=100}^{\infty} \frac{5}{\sqrt[7]{k^6}}$ converge or diverge?

$$= 5 \sum_{k=100}^{\infty} \frac{1}{k^{6/7}} \leftarrow p$$

Since $p = 6/7 \leq 1$, the series $\boxed{\text{diverges}}$ by the p-Series Test.

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(10) Does $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n^2 - 8n + 16}$ converge or diverge?

$$= \sum_{n=5}^{\infty} \frac{1}{(n-4)^2}$$

$$= \sum_{n=5-4}^{\infty} \frac{1}{((n+\cancel{4})\cancel{-4})^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \leftarrow p$$

Since $p = 2 > 1$, the series $\boxed{\text{converges}}$ by the p-Series Test.

(11) Does $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$ converge or diverge?

$$\int_0^\infty \frac{e^x}{1+(e^x)^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{e^x}{1+(e^x)^2}\,dx$$

Let $u = e^x$ $\quad x=b \Rightarrow u = e^b$

$du = e^x dx$ $\quad x = 0 \Rightarrow u = 1$

$$= \lim_{b\to\infty} \int_1^{e^b} \frac{1}{1+u^2}\,du$$

$$= \lim_{b\to\infty} \Big[\tan^{\leftarrow}(u)\Big]_1^{e^b}$$

$$= \left(\lim_{b\to\infty} \tan^{\leftarrow}(e^b)\right) - \tan^{\leftarrow}(1)$$

$$= \pi/2 - \pi/4$$

$$= \pi/4$$

Since the integral converges, the series also

$\boxed{\text{converges}}$.