⑥ Find $\int \frac{\sqrt{x^2-16}}{x}\,dx$ where $x \geq 4$.

Let $x^2 - 16 = 16\sec^2\theta - 16 = 16\tan^2\theta$ $\rightarrow$ $\tan^2\theta = \frac{1}{16}x^2 - 1$

$x^2 = 16\sec^2\theta$

$x = 4\sec\theta \longrightarrow \frac{x}{4} = \sec\theta$

$dx = 4\sec\theta\tan\theta\,d\theta$ $\quad \theta = \sec^{-1}\left(\frac{x}{4}\right)$

$$= \int \frac{\sqrt{16\tan^2\theta}}{4\sec\theta}\, 4\sec\theta\tan\theta\,d\theta$$

$$= \int 4\tan^2\theta\,d\theta$$

$$= \int 4\sec^2\theta - 4\,d\theta$$

$$= 4\tan\theta - 4\theta + C$$

$$= \boxed{4\sqrt{\frac{1}{16}x^2 - 1} - 4\sec^{-1}\left(\frac{x}{4}\right) + C}$$

OR

$$= \boxed{\sqrt{x^2 - 16} - 4\sec^{-1}\left(\frac{x}{4}\right) + C}$$

(7) Find $\int \frac{1}{\sqrt{4t^2-1}}\,dt$ where $t > \frac{1}{2}$.

Let $4t^2-1 = \sec^2\theta - 1 = \tan^2\theta$, $\tan\theta = \sqrt{4t^2-1}$

$4t^2 = \sec^2\theta$

$2t = \sec\theta$

$t = \frac{1}{2}\sec\theta$

$dt = \frac{1}{2}\sec\theta\tan\theta\,d\theta$

$$= \int \frac{1}{\sqrt{\tan^2\theta}} \frac{1}{2}\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{2}\int \sec\theta\,d\theta$$

$$= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\frac{1}{2}\ln\left(2t + \sqrt{4t^2-1}\right) + C}$$

(8) Find $\int \frac{2}{\sqrt{1-4x^2}}\,dx$ without a trig sub.

(Looks like $\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C$.)

Let $4x^2 = u^2$

$2x = u$

$2dx = du$

$= \int \frac{1}{\sqrt{1-u^2}}\,du$

$= \sin^{-1}(u) + C$

$= \boxed{\sin^{-1}(2x) + C}$

(9) Find $\int \frac{2}{4+9x^2}\,dx$ without a trig sub.

(Looks like $\int \frac{1}{1+u^2}\,du = \tan^{-1}(u) + C$.)

$$= \int \frac{2}{4_2}\,\frac{1}{1+\frac{9}{4}x^2}\,dx$$

Let $\frac{9}{4}x^2 = u^2$

$\frac{3}{2}x = u$

$x = \frac{2}{3}u$

$dx = \frac{2}{3}\,du$

$$= \frac{1}{2}\,\frac{2}{3}\int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{3}\tan^{-1}(u) + C$$

$$= \boxed{\frac{1}{3}\tan^{-1}\left(\frac{3}{2}x\right) + C}$$