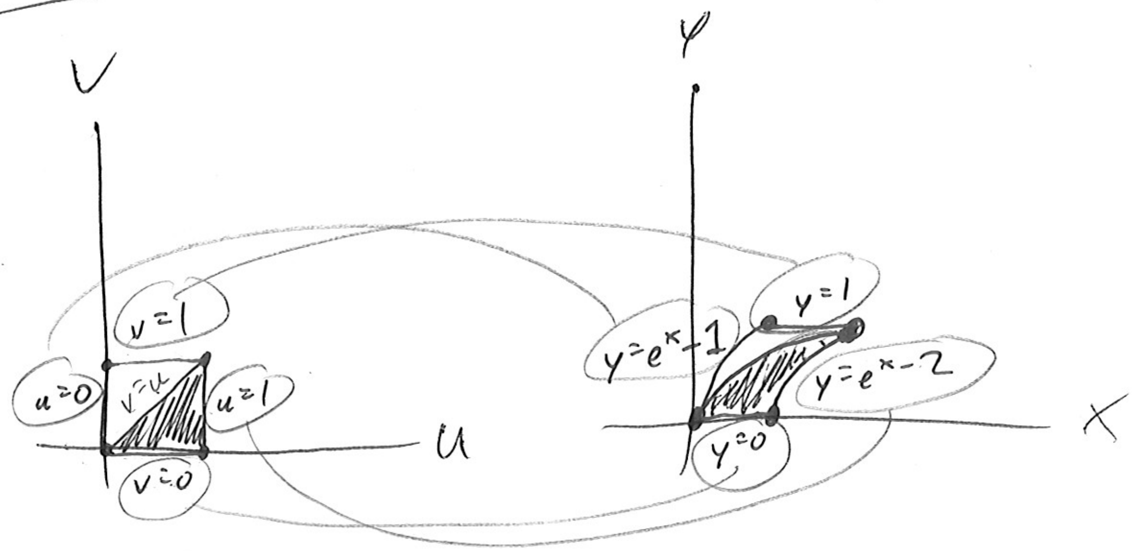
(Example) Use a 2D transformation to compute

$\iint_D e^x \cos(\pi e^x)\, dA$ where D is the region bounded by $y=0$, $y=e^x-2$, $y=\frac{e^x-1}{2}$. (Hint: find transformation from unit square to region bounded by $y=0$, $y=1$, $y=e^x-1$, $y=e^x-2$.)



$u=0 \Rightarrow y=e^x-1$ $\big) -1$

$u=1 \Rightarrow y=e^x-2$

$u\in[0,1] \Rightarrow y=\underbrace{e^x-1-1u}$

$v=0 \Rightarrow y=0$

$v=1 \Rightarrow y=1$

$v\in[0,1] \Rightarrow y=v$

$e^x-1-u=v$

$e^x=u+v+1 \longrightarrow x=\log_e(u+v+1)$

$$\underline{T}(u,v) = (x,y) = \left(\log_e(u+v+1),\ v\right)$$

Check:

If $v=0$, then $y=v=0$ $\checkmark$

If $u=1$, $x = \log_e(v+2)$

$$e^x = v+2$$

$$v = e^x - 2$$

$$y = e^x - 2 \quad \checkmark$$

If $v=u$, then $x = \log_e(2v+1)$

$$e^x = 2v+1$$

$$v = \frac{e^x - 1}{2}$$

$$y = \frac{e^x - 1}{2} \quad \checkmark$$

(FACT:

$$\iint_D f(x,y)\, dA = \iint_{D^*} f(\underline{T}(u,v)) \left| \frac{\partial \underline{T}}{\partial \underline{u}} \right| dA$$
)

$$\frac{\partial \underline{T}}{\partial \underline{u}} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \det \begin{pmatrix} \frac{1}{u+v+1}\cancel{(1+0+0)} & \frac{1}{u+v+1}\cancel{(0+1+0)} \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{u+v+1}$$

For points in D^*

$$\left| \frac{\partial \underline{T}}{\partial \underline{u}} \right| = \left| \frac{1}{u+v+1} \right|$$

$$= \frac{1}{u+v+1}$$

$$\iint_D e^x \cos(\pi e^x)\, dA = \iint_{D^*} \cancel{e}^{\underbrace{\cancel{\log_e}(u+v+1)}} \cos\left(\pi\, \cancel{e}^{\underbrace{\cancel{\log_e}(u+v+1)}}\right) \underbrace{\frac{1}{u+v+1}}_{\left|\frac{\partial I}{\partial u}\right|} dA$$

$$\times$$

$$= \int_{u=0}^{u=1}\int_{v=0}^{v=u} \cancel{(u+v+1)} \cos(\pi(u+v+1)) \frac{1}{\cancel{u+v+1}}\, dv\, du$$

$$= \int_0^1 \left[\int_0^u \cos(\pi u + \pi v + \pi)\, dv\right] du$$

$$= \int_0^1 \left[\frac{1}{\pi}\sin(\pi u + \pi v + \pi)\right]_0^u du$$

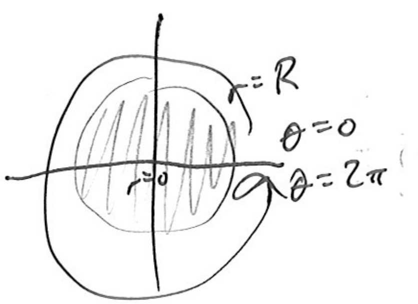
$$= \int_0^1 \frac{1}{\pi}\sin(2\pi u + \pi) - \frac{1}{\pi}\sin(\pi u + \pi)\, du$$

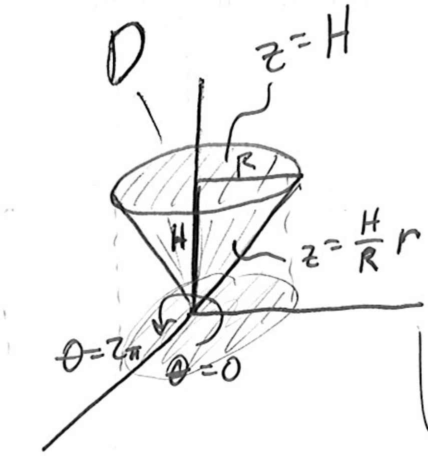
$$= \left[-\frac{1}{2\pi^2}\cos(2\pi u + \pi) + \frac{1}{\pi^2}\cos(\pi u + \pi)\right]_0^1$$

$$= \left(-\frac{1}{2\pi^2}\overset{(-1)}{\cancel{\cos(3\pi)}} + \frac{1}{\pi^2}\overset{1}{\cancel{\cos(2\pi)}}\right) - \left(-\frac{1}{2\pi^2}\overset{(-1)}{\cancel{\cos(\pi)}} + \frac{1}{\pi^2}\overset{(-1)}{\cancel{\cos(\pi)}}\right)$$

$$= +\cancel{\frac{1}{2\pi^2}} + \frac{1}{\pi^2} - \cancel{\frac{1}{2\pi^2}} + \frac{1}{\pi^2} = \boxed{2/\pi^2}$$

(Example) Prove that the formula for the volume of a cone with radius R and height H is $V = \frac{1}{3}\pi R^2 H$.





$$V = \iiint_D 1 \, dV$$

$$\left(\text{in } xyz \text{ coordinates} \right) = \underbrace{\int_{-}^{\sqrt{R^2+H^2}} \int_{y=-\sqrt{\ }}^{} \int_{\cancel{z=\sqrt{x^2+y^2}}}^{z=H} 1 \, dz\, dy\, dx}_{\text{awful}}$$

easier to describible with cylindrical...

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=R} \left[\int_{z=\frac{H}{R}r}^{z=H} \underbrace{1}_{f(\Phi(r,\theta,z))} \; \underbrace{\overset{\text{Always for cylindrical}}{r}}_{\left|\frac{\partial \Phi}{\partial(r,\theta,z)}\right|} \, dz \right] dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^R \Big[rz \Big]_{z=\frac{H}{R}r}^{z=H} dr\, d\theta$$

$$= \int_0^{2\pi} \left[\int_0^R Hr - \frac{H}{R} r^2 \, dr \right] d\theta$$

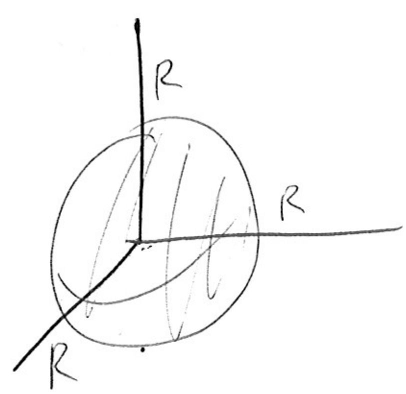
$$= \int_0^{2\pi} \left[\frac{H}{2} r^2 - \frac{H}{3R} r^3 \right]_0^R d\theta$$

$$= \int_0^{2\pi} \frac{HR^2}{2} - \frac{HR^{\cancel{3}2}}{3\cancel{R}} \, d\theta$$

$$= \int_0^{2\pi} \frac{HR^2}{6} \, d\theta$$

$$= \frac{2\pi}{6} R^2 H = \boxed{\frac{1}{3}\pi R^2 H} \checkmark$$

(Example 7) Prove that the formula for the volume of a sphere with radius R is $V = \frac{4}{3}\pi R^3$



$$V = \iiint_D 1 \, dV$$

$$\cancel{= \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{-\sqrt{R^2-y^2-x^2}}^{\sqrt{R^2-y^2-x^2}} 1 \, dz\,dy\,dx}$$

awful

Use spherical coordinates instead!

$$\int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \left[\int_{\rho=0}^{\rho=R} \underbrace{\cancel{1}}_{f(\underline{s}(\rho,\theta,\phi))} \underbrace{\rho^2 \sin\phi}_{\left|\frac{\partial \underline{s}}{\partial(\rho,\theta,\phi)}\right|} \, d\rho \right] d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \left[\frac{1}{3}\rho^3 \sin\phi \right]_0^R d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\pi} \frac{1}{3} R^3 \sin\phi \, d\phi \right] d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3} R^3 \cos\phi \right]_0^{\pi} d\theta$$

$$= \int_0^{2\pi} + \frac{1}{3} R^3 \overset{(+1)}{\cancel{\cos(\pi)}} + \frac{1}{3} R^3 \overset{(+1)}{\cancel{\cos 0}} \, d\theta$$

$$= \int_0^{2\pi} \frac{2}{3} R^3 \, d\theta = \boxed{\frac{4}{3}\pi R^3}$$

6.2 HW 1-6, 11, 13-14, 21, 26

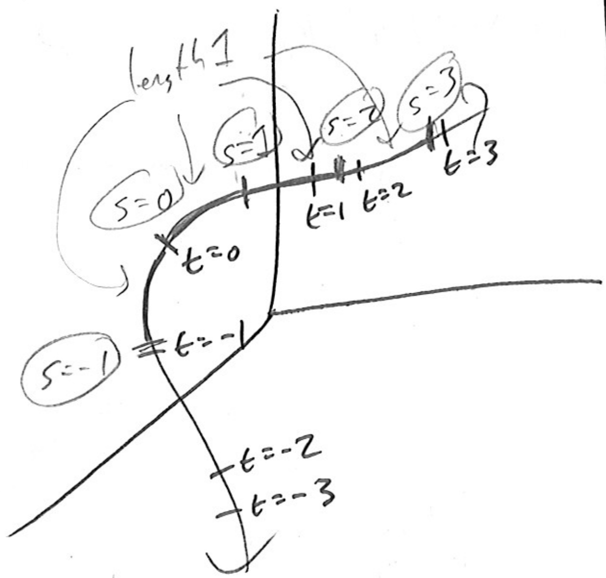
7.1 The Path Integral

A curve C defined by $\underline{r}: \mathbb{R} \to \mathbb{R}^n$ has an arclength function $s: \mathbb{R} \to \mathbb{R}$ defined by

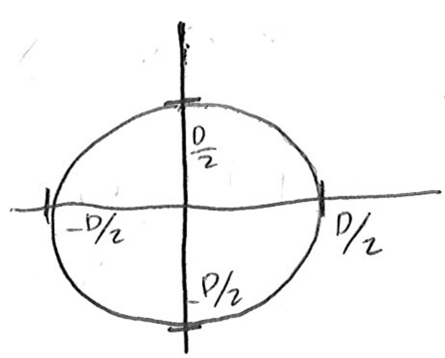
$$s(t) = \int_0^t \| \underline{r}'(\tau) \| \, d\tau$$

in 3D:

$$\int_0^t \sqrt{x'(\tau) + y'(\tau) + z'(\tau)} \, d\tau$$



(Example) Prove that $C = \pi D$.

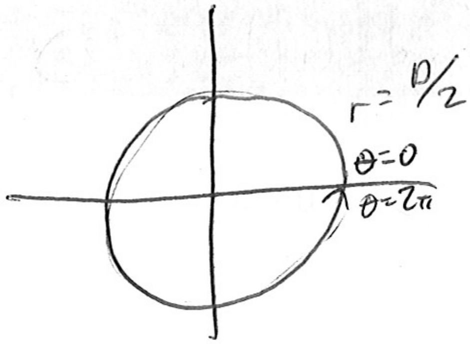


$$\underline{r}(t) = \left(\frac{D}{2}\cos t, \frac{D}{2}\sin t\right)$$

$$0 \le t \le 2\pi$$

OR think with polar: $\rho(r,\theta) = (r\cos\theta, r\sin\theta)$

$$\rho\left(\frac{D}{2},\theta\right) = \left(\frac{D}{2}\cos\theta, \frac{D}{2}\sin\theta\right)$$



Let $\underline{r}(\theta) = \left(\frac{D}{2}\cos\theta, \frac{D}{2}\sin\theta\right),\quad 0 \le \theta \le 2\pi$

$$\underline{r}'(\theta) = \left(-\frac{D}{2}\sin\theta, \frac{D}{2}\cos\theta\right)$$

$$\underline{r}'(\tau) = \left(-\frac{D}{2}\sin\tau, \frac{D}{2}\cos\tau\right)$$

$$\|\underline{r}'(\tau)\| = \sqrt{\frac{D^2}{4}\sin^2\tau + \frac{D^2}{4}\cos^2\tau} = \sqrt{\frac{D^2}{4}(1)} = \frac{D}{2}$$

Then

$$s(\theta) = \int_0^\theta \|\underline{r}'(\tau)\|\,d\tau$$

✓

$$s(\theta) = \int_0^\theta D/2\,d\tau = \left[\frac{D}{2}\tau\right]_0^\theta$$

$$s(\theta) = D/2\,\theta$$

So the circumference of the circle is

$$C = s(2\pi) = \frac{\cancel{D}}{\cancel{2}}(\cancel{2}\pi) = \pi D \checkmark$$

If $f:\mathbb{R}^n \to \mathbb{R}$ is a function defined along a curve C defined by $\underline{r}:[a,b] \to \mathbb{R}^n$, then

$$\int_C f(x,y,z)\, ds = \int_{t=a}^{t=b} f(\underline{r}(t)) \underbrace{\frac{ds}{dt}}_{\|\underline{r}'(t)\|} dt$$

The path integral over C

of the function f

with respect to arclength

This represents the net area of a "ribbon" formed by a base C and heights given by f.