MATH 2242 (Calculus IV) Course Outline — Vector Calculus (Marsden)

1.5 n-Dimensional Euclidean Space

- $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$
- Addition
  - $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$
- Scalar multiplication
  - $\alpha\,(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$
- Inner/Dot Product
  - $(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = \sum_{i=1}^{n} x_i y_i$
- Norm/Length/Magnitude
  - $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$
- Standard basis vectors
  - $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0)$, $\ldots$, $\mathbf{e}_n = (0, 0, \ldots, 1)$
- Theorems
  - $(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z})$
  - Prove the above theorem.
  - $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
  - $\mathbf{x} \cdot \mathbf{x} \geq 0$
  - $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$
  - $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ (the Cauchy-Schwarz inequality)
  - (Example) Prove the Cauchy-Schwarz inequality.
  - $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (the triangle inequality)
  - (Example) Prove the triangle inequality.
- Matrices
  - $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$

- Addition $A + B$
- Scalar Mutiplication αA
- Transposition A^T

- Vectors as Matrices
  - $\mathbf{a} = (a_1, a_2, \ldots, a_n) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$
  - $\mathbf{a}^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$

- Matrix Multiplication
  - If A has m rows and B has n columns, then $M = AB$ is an $m \times n$ matrix.
  - Coordinate ij of $M = AB$ is given by $m_{ij} = \mathbf{a_i} \cdot \mathbf{b_j}$ where $\mathbf{a_i}^T$ is the ith row of A and $\mathbf{b_j}$ is the jth column of B.
  - (Example 4) Compute AB and BA for

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

  - (Example 5) Compute AB for

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Matrices as Linear Transformations
  - An $m \times n$ matrix A gives a function from $\mathbb{R}^n$ to $\mathbb{R}^m$: $\mathbf{x} \mapsto A\mathbf{x}$
  - This linear transformation satsifies $A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y}$
  - (Example 7) Express $A\mathbf{x}$ where $x = (x_1, x_2, x_3)$ and $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

  - (Example) Compute where the points $(-1,-1,0)$, $(0,1,0)$, $(1,-1,1)$, and $(2,1,1)$ in $\mathbb{R}^3$ get mapped to in $\mathbb{R}^4$ by $A\mathbf{x}$ from the previous example. Then plot the projections of the original points in $\mathbb{R}^3$ onto their first two coordinates in $\mathbb{R}^2$, and compare this with the projection plot of their images in $\mathbb{R}^4$ onto their first two coordinates in $\mathbb{R}^2$.

- Identity and Inverse

  - The $n \times n$ identity matrix I satisfies $i_{jj} = 1$ and $i_{jk} = 0$ when $j \neq k$. That is: $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$
  - If $AA^{-1} = A^{-1}A = I$, then A is invertable and A^{-1} is its inverse.

- Determinant

  - Let A_i be the submatrix of A with the first column and ith row removed. Then $\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \det(A_i)$
  - (Example) Prove that
$$\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1$$
and
$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$
$$= (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_1 b_3 c_2 + a_2 b_1 c_3 + a_3 b_2 c_1)$$
  - (Example) Prove that the inverse of the matrix $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is $\frac{1}{\det A} \begin{bmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{bmatrix}$.
  - An $n \times n$ matrix is invertable if and only if its determinant is nonzero.

- *HW: 1-18, 21-24*

2.3 Differentiation

- Functions $\mathbb{R}^n \to \mathbb{R}^m$

  - $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$
  - $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ where $f_i : \mathbb{R}^n \to \mathbb{R}$

- Partial Derivative Matrix
  - $\mathbf{Df}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{bmatrix}$
  - We say $\mathbf{f}$ is differentiable at $\mathbf{x}_0$ if $\mathbf{f}(\mathbf{x}_0+\mathbf{h}) \approx \mathbf{f}(\mathbf{x}_0)+[\mathbf{Df}(\mathbf{x}_0)]\mathbf{h}$ whenever $\mathbf{h} \approx \mathbf{0}$.
  - (Example) Prove that this is equivalent to saying $\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x}_0)+[\mathbf{Df}(\mathbf{x}_0)](\mathbf{x}-\mathbf{x}_0)$ whenever $\mathbf{x} \approx \mathbf{x}_0$.
  - (Example) Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mathbf{f}(x, y) = \left(x^2 + y^2, xy\right)$, and let $\mathbf{T} = \mathbf{Df}(1, 0)$. Compute $\mathbf{f}(1.1, -0.1)$ and $\mathbf{f}(1, 0) + \mathbf{T}\,(0.1, -0.1)$.
  - If each $\frac{\partial f_i}{\partial x_j} : \mathbb{R}^n \to \mathbb{R}$ is a continuous function near $\mathbf{x}_0$, then we say $\mathbf{f}$ is strongly differentiable or class C^1 at $\mathbf{x}_0$. All C^1 functions are differentiable.
- Gradient
  - If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient vector function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\nabla f(\mathbf{x}) = (\mathbf{D}f(\mathbf{x}))^T = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{x})\right)$
  - $[\mathbf{D}f(\mathbf{x})]\mathbf{h} = \nabla f(\mathbf{x}) \cdot \mathbf{h}$
- Linearizations and Tangent Hyperplanes
  - For $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and a point $\mathbf{x}_0 \in \mathbb{R}^n$, let the linearization of $\mathbf{f}$ at $\mathbf{x}_0$ be $\mathbf{L}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + [\mathbf{Df}(\mathbf{x_0})](\mathbf{x} - \mathbf{x}_0)$. Note $\mathbf{f}(\mathbf{x}) \approx \mathbf{L}(\mathbf{x})$ whenever $\mathbf{x} \approx \mathbf{x}_0$.
  - (Example 5) Recall that the tangent plane to a surface $z = f(x, y)$ given by $f : \mathbb{R}^2 \to \mathbb{R}$ passing through $\mathbf{x}_0 \in \mathbb{R}^3$ is given by the normal vector ∇f. Show that $z = L(x, y)$ gives an equation for the tangent plane to the surface $z = x^2+y^4+e^{xy}$ at the point $(1, 0, 2)$.
- *HW: 1-3, 5-21*

2.5 Properties of the Derivative

- Sum/Product/Quotient Rules
  - $\mathbf{D}[\alpha\mathbf{f}] = \alpha\mathbf{Df}$
  - $\mathbf{D}[\mathbf{f} + \mathbf{g}] = \mathbf{Df} + \mathbf{Dg}$
  - $\mathbf{D}[fg] = g\mathbf{D}f + f\mathbf{D}g$
  - $\mathbf{D}[\frac{f}{g}] = \frac{g\mathbf{D}f-f\mathbf{D}g}{g^2}$
  - (Example) Prove the sum rule above.

- Chain Rule
  - $\mathbf{D}[\mathbf{f} \circ \mathbf{g}] = \mathbf{Df}(\mathbf{g})\mathbf{Dg}$
  - (Example) Find the rate of change of $f(x, y) = x^2 + y^2$ along the path $\mathbf{c}(t) = \left(t^2, t\right)$ when $t = 1$.
  - (Example 2) Verify the Chain Rule for $f(u, v, w) = u^2 + v^2 - w$ and $\mathbf{g}(x, y, z) = \left(x^2y, y^2, e^{-xz}\right)$.
  - (Example 3) Compute $\mathbf{D}[\mathbf{f} \circ \mathbf{g}](1, 1)$ where $\mathbf{f}(u, v) = \left(u + v, u, v^2\right)$ and $\mathbf{g}(x, y) = \left(x^2 + 1, y^2\right)$.
- *HW: 6-13, 15-16*

3.2 Taylor's Theorem

- Single-variable Taylor Series
  - $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$
    $= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f'(x_0)(x - x_0)^2 + \frac{1}{6}f'(x_0)(x - x_0)^3 + \ldots$
  - $f(x) \approx \sum_{n=0}^{m} \frac{f^{(n)}(x_0)}{n!}(x - x_0)$
- First-Order Taylor Formula
  - $f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{x}_0) + [\mathbf{D}f(\mathbf{x}_0)](\mathbf{x} - \mathbf{x}_0) = f(\mathbf{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i - x_{0i})$
- Second-Order Taylor Formula
  - $f(\mathbf{x}) \approx f(\mathbf{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i - x_{0,i}) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)(x_i - x_{0,i})(x_j - x_{0,j})$
  - (Example) Use the second-order Taylor formula for $f(x, y) = \sqrt{x + 2y}$ near the point $(2, 1)$ to approximate $\sqrt{4.05}$.
  - (Example 3) Find linear and quadratic functions of x, y which approximate $f(x, y) = \sin(xy)$ near the point $(1, \pi/2)$.
- *HW: 3-7, 12*

4.3 Vector Fields

- Vector Fields

  - A vector field is a map $f : \mathbb{R}^n \to \mathbb{R}^n$ assinging an n-dimensional vector to each point in $\mathbb{R}^n$
  - (Example 1) The velocity field of a fluid may be modeled as a vector field.
  - (Example 2) Sketch the rotary motion given by the vector field $\mathbf{V}(x, y) = (-y, x)$.

- Gradient Vector Fields
  - $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$
  - (Example) The derivative of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$ in the direction given by a unit vector $\mathbf{v}$ is given by $\nabla f \cdot \mathbf{v}$. Show that the maximum value of a directional derivative for a fixed point is given by $\|\nabla f\|$ and attained by the direction $\frac{1}{\|\nabla f\|}\nabla f$.
  - (Example 4) If temperature is given by $T(x, y, z)$, then the energy or heat flux field is given by $\mathbf{J} = -k\nabla T$ where k is the conductivity of the body. Level sets are called isotherms.
  - (Example 5) The gravitational potential of bodies with mass m, M is given by $V = -\frac{mMG}{r}$ where G is the gravitational constant and r is the distance between the bodies, and the gravitational force field is given by $\mathbf{F} = -\nabla V$. Show that $\mathbf{F} = -\frac{mMG}{r^3}\mathbf{r}$, where $\mathbf{r}$ is the vector pointing from the center of mass M to the center of mass m.
  - A vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is conservative iff there exists a potential function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.
  - (Example) Show that $\mathbf{W} = (2y + 1, 2x)$ is conservative.
  - (Example 7) Show that $\mathbf{V} = (y, -x)$ is not conservative.

- Flow Lines
  - A flow line for a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a path $\mathbf{c} : \mathbb{R} \to \mathbb{R}^n$ satisfying $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$.
  - (Example 8) Show that $\mathbf{c}(t) = (\cos t, \sin t)$ is a flow line for $\mathbf{F} = (-y, x)$, and find some other flow lines.

- *HW: 1-12, 17-21*

4.4 Divergence and Curl

- Divergence
  - The divergence of a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is denoted by $\operatorname{div} \mathbf{F} : \mathbb{R}^n \to \mathbb{R}$ and defined by $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$

  - (Examples 3-5) Compute the divergences of $\mathbf{F} = (x, y)$, $\mathbf{G} = (-x, -y)$ and $\mathbf{H} = (-y, x)$ at any point on $\mathbb{R}^2$. How does divergence correspond with the motion described by the vector field plots?
  - (Example) Compute the divergence of $\mathbf{F} = \left(x^2, y\right)$ various points and interpret those values against a plot of the vector field.

- Curl
  - The curl of a three-dimensional vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is denoted by $\operatorname{curl} \mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ and defined by $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$
  - The scalar curl of a two-dimensional vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is denoted by $\operatorname{scurl} \mathbf{F} : \mathbb{R}^2 \to \mathbb{R}$ and defined by $\operatorname{scurl} \mathbf{F} = \operatorname{curl} \mathbf{F} \cdot \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$
  - (Example) Compute the scalar curl of $\mathbf{F} = (x, y)$, $\mathbf{G} = (-x, -y)$ and $\mathbf{H} = (-y, x)$ at every point in $\mathbb{R}^2$. How does this scalar curl correspond with the motion described by the vector field plots?
  - (Example) Compute the curl of $\mathbf{F} = (y, -x, z)$ at every point in $\mathbb{R}^3$. How does curl correspond with the motion described by the vector field plot?

- Facts about ∇f, $\operatorname{div} \mathbf{F}$, $\operatorname{curl} \mathbf{F}$
  - The curl of a conservative field is zero: $\operatorname{curl} \nabla f = \nabla \times (\nabla f) = \mathbf{0}$.
  - (Example) Prove the above theorem.
  - (Example) Prove that $\mathbf{F} = \left(x^2 + z, y - z, z^3 + 3xy\right)$ is not a conservative field.
  - The divergence of a curl field is zero: $\operatorname{div} \operatorname{curl} \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$
  - Many identities on pg. 255 of Marsden text.
  - (Example) Sketch proof of identity #8: $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$.

- *HW: 1-4, 9-17, 22-25, 29-30*

5.3 The Double Integral Over More General Regions

- Hypervolume
  - The hypervolume $HV_1(D)$ of an interval $D = [a, b]$ in $\mathbb{R}$ is just its length $b - a$.
  - The hypervolume of a well-behaved bounded subset $D \subseteq \mathbb{R}^{n+1}$ is defined for each $n \in \{1, 2, \dots\}$ by

$$HV_{n+1}(D) = \int_{x_i \in I} HV(D_i)\, dx_i = \int_{x_i = a}^{x_i = b} HV_n(D_i)\, dx_i$$

    where $I = [a, b]$ is an interval containing all values x_i included in the ith coordinate of D, and D_i is the projection of of all points in D onto $\mathbb{R}^n$ by removing the ith coordinate.

- (Example) For $n = 1$ and $D = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b, f(x) \leq y \leq g(x)\}$, we have that

$$HV_2 = A = \int_{x \in [a,b]} g(x) - f(x)\,dx = \int_a^b g(x) - f(x)\,dx.$$

- (Example) For $n = 2$ and $D \subseteq R^3$ including values of x between a and b, we have that

$$HV_3 = V = \int_{x=a}^{x=b} A(x)\,dx$$

where $A(x)$ is the area of the cross-section of D taken by fixing each value of x (or similar for y).

- Double Integrals
  - For a bounded region $D \subseteq \mathbb{R}^2$ and continuous nonnegative $f : D \to \mathbb{R}$, the double integral

$$\iint_D f\,dA$$

  is defined to be the volume of $\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D, 0 \leq z \leq f(x,y)\}$.
  - We may apply the definition of volume above to get

$$\iint_D F\,dA = \int_{x=a}^{x=b} A(x)\,dx$$

  where D lies between the lines $x = a$ and $x = b$.
  - If D is described by $a \leq x \leq b$ and $\phi_1(x) \leq y \leq \phi_2(x)$, then

$$\iint_D F\,dA = \int_{x=a}^{x=b} A(x)\,dx = \int_{x=a}^{x=b} \left[\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy \right] dx$$

  - Similarly, if D is described by $c \leq y \leq d$ and $\psi_1(y) \leq x \leq \psi_2(y)$, then

$$\iint_D F\,dA = \int_{y=c}^{y=d} \left[\int_{x=\psi_1(y)}^{x=\psi_2(y)} f(x,y)\,dx \right] dy$$

  - If f is sometimes negative on the domain D, then $\iint_D f\,dA$ is the net volume between $z = f(x,y)$ and D (volume above the xy plane minus volume below) and the above formulas still hold.
- Iterated integrals

- An iterated integral is a shorthand for the expansion of two or more nested integrals, that is:

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx = \int_{x=a}^{x=b} \left[\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy \right] dx$$

- (Example) Sketch the region of integration for $\int_0^\pi \int_{-x}^{x} \cos(y)\,dy\,dx$, evaluate it, and interpret it as the signed volume of a region in $\mathbb{R}^3$.
- (Example) Express $\iint_R (12x^3y - 1)\,dA$ where R is the rectangle with vertices $(0,0),(3,0),(3,2),(0,2)$ as an interated integral, then evaluate it.
- (Example) Express $\iint_T (12x^3y - 1)\,dA$ where T is the triangle with vertices $(0,0),(1,0),(1,1)$ as an interated integral, then evaluate it.

- Applications
  - $\iint_D 1\,dA$ is the area of D
  - $\frac{1}{A(D)} \iint_D f(x,y)\,dA$ is the average value of the function f restricted to D
- *HW: 1-9*

5.4 Changing the Order of Integration

- Rectangular regions of integration
  - For constant bounds of integration:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

  - (Example) Verify that $\int_0^1 \int_1^2 x^2 + 2xy\,dy\,dx = \int_1^2 \int_0^1 x^2 + 2xy\,dx\,dy$.
- Nonrectangular regions of integration
  - Bounds of integration cannot be directly swapped; however, by interpreting the region of integration new bounds may be found in the other order.
  - (Example) Verify that $\int_0^4 \int_0^{\frac{4-y}{2}} x + y\,dx\,dy$ and $\int_0^2 \int_0^{4-2x} x + y\,dy\,dx$ share the same region of integration and are equal.
  - (Example) Evaluate $\int_1^e \int_0^{\log x} \frac{(2x-e)\sqrt{1+e^y}}{e-e^y}\,dy\,dx$. (Note that this is technically improper, but that this does not effect the solution.)
- Estimating double integrals

- If $g(x,y) \leq f(x,y) \leq h(x,y)$ for $(x,y) \in D$, then $\iint_D g(x,y)\,dA \leq \iint_D f(x,y)\,dA \leq \iint_D h(x,y)\,dA$.
- (Example 3) Prove that $\frac{1}{\sqrt{3}} \leq \iint_D \frac{1}{\sqrt{1+x^6+y^8}}\,dA \leq 1$ where D is the unit square.
- (Example) Prove that $e \leq \iint_D e^{x^2y+y}\,dA \leq \frac{e^2}{2}$ where D is the unit square.

• *HW: 1-5, 7-10*