MATH 2242 (Calculus IV) Course Outline — Vector Calculus (Marsden)

1.5 n-Dimensional Euclidean Space

- $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$
- Addition
  - $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$
- Scalar multiplication
  - $\alpha\,(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$
- Inner/Dot Product
  - $(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = \sum_{i=1}^{n} x_i y_i$
- Norm/Length/Magnitude
  - $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$
- Standard basis vectors
  - $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0)$, $\ldots$, $\mathbf{e}_n = (0, 0, \ldots, 1)$
- Theorems
  - $(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z})$
  - Prove the above theorem.
  - $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
  - $\mathbf{x} \cdot \mathbf{x} \geq 0$
  - $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$
  - $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ (the Cauchy-Schwarz inequality)
  - (Example) Prove the Cauchy-Schwarz inequality.
  - $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (the triangle inequality)
  - (Example) Prove the triangle inequality.
- Matrices
  - $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$

- Addition $A + B$
- Scalar Mutiplication αA
- Transposition A^T

- Vectors as Matrices
  - $\mathbf{a} = (a_1, a_2, \dots, a_n) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$
  - $\mathbf{a}^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$

- Matrix Multiplication
  - If A has m rows and B has n columns, then $M = AB$ is an $m \times n$ matrix.
  - Coordinate ij of $M = AB$ is given by $m_{ij} = \mathbf{a_i} \cdot \mathbf{b_j}$ where $\mathbf{a_i}^T$ is the ith row of A and $\mathbf{b_j}$ is the jth column of B.
  - (Example 4) Compute AB and BA for

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

  - (Example 5) Compute AB for

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Matrices as Linear Transformations
  - An $m \times n$ matrix A gives a function from $\mathbb{R}^n$ to $\mathbb{R}^m$: $\mathbf{x} \mapsto A\mathbf{x}$
  - This linear transformation satsifies $A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y}$
  - (Example 7) Express $A\mathbf{x}$ where $x = (x_1, x_2, x_3)$ and $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

  - (Example) Compute where the points $(-1,-1,0)$, $(0,1,0)$, $(1,-1,1)$, and $(2,1,1)$ in $\mathbb{R}^3$ get mapped to in $\mathbb{R}^4$ by $A\mathbf{x}$ from the previous example. Then plot the projections of the original points in $\mathbb{R}^3$ onto their first two coordinates in $\mathbb{R}^2$, and compare this with the projection plot of their images in $\mathbb{R}^4$ onto their first two coordinates in $\mathbb{R}^2$.

- Identity and Inverse
  - The $n \times n$ identity matrix I satisfies $i_{jj} = 1$ and $i_{jk} = 0$ when $j \neq k$. That is: $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$
  - If $AA^{-1} = A^{-1}A = I$, then A is invertable and A^{-1} is its inverse.

- Determinant
  - Let A_i be the submatrix of A with the first column and ith row removed. Then $\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \det(A_i)$
  - (Example) Prove that

$$\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1$$

    and

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

$$= (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_1 b_3 c_2 + a_2 b_1 c_3 + a_3 b_2 c_1)$$

  - (Example) Prove that the inverse of the matrix $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is $\frac{1}{\det A} \begin{bmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{bmatrix}$.
  - An $n \times n$ matrix is invertable if and only if its determinant is nonzero.

- *HW: 1-18, 21-24*

2.3 Differentiation

- Functions $\mathbb{R}^n \to \mathbb{R}^m$
  - $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$
  - $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ where $f_i : \mathbb{R}^n \to \mathbb{R}$

- Partial Derivative Matrix
  - $\mathbf{Df}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{bmatrix}$
  - We say $\mathbf{f}$ is differentiable at $\mathbf{x}_0$ if $\mathbf{f}(\mathbf{x}_0+\mathbf{h}) \approx \mathbf{f}(\mathbf{x}_0)+[\mathbf{Df}(\mathbf{x}_0)]\mathbf{h}$ whenever $\mathbf{h} \approx \mathbf{0}$.
  - (Example) Prove that this is equivalent to saying $\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x}_0)+[\mathbf{Df}(\mathbf{x}_0)](\mathbf{x}-\mathbf{x}_0)$ whenever $\mathbf{x} \approx \mathbf{x}_0$.
  - (Example) Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mathbf{f}(x, y) = \big(x^2 + y^2, xy\big)$, and let $\mathbf{T} = \mathbf{Df}(1, 0)$. Compute $\mathbf{f}(1.1, -0.1)$ and $\mathbf{f}(1, 0) + \mathbf{T}\,(0.1, -0.1)$.
  - If each $\frac{\partial f_i}{\partial x_j} : \mathbb{R}^n \to \mathbb{R}$ is a continuous function near $\mathbf{x}_0$, then we say $\mathbf{f}$ is strongly differentiable or class C^1 at $\mathbf{x}_0$. All C^1 functions are differentiable.
- Gradient
  - If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient vector function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\nabla f(\mathbf{x}) = (\mathbf{D}f(\mathbf{x}))^T = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x})\right)$
  - $[\mathbf{D}f(\mathbf{x})]\mathbf{h} = \nabla f(\mathbf{x}) \cdot \mathbf{h}$
- Linearizations and Tangent Hyperplanes
  - For $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and a point $\mathbf{x}_0 \in \mathbb{R}^n$, let the linearization of $\mathbf{f}$ at $\mathbf{x}_0$ be $\mathbf{L}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + [\mathbf{Df}(\mathbf{x_0})](\mathbf{x} - \mathbf{x}_0)$. Note $\mathbf{f}(\mathbf{x}) \approx \mathbf{L}(\mathbf{x})$ whenever $\mathbf{x} \approx \mathbf{x}_0$.
  - (Example 5) Recall that the tangent plane to a surface $z = f(x, y)$ given by $f : \mathbb{R}^2 \to \mathbb{R}$ passing through $\mathbf{x}_0 \in \mathbb{R}^3$ is given by the normal vector ∇f. Show that $z = L(x, y)$ gives an equation for the tangent plane to the surface $z = x^2+y^4+e^{xy}$ at the point $(1, 0, 2)$.
- *HW: 1-3, 5-21*

2.5 Properties of the Derivative

- Sum/Product/Quotient Rules
  - $\mathbf{D}[\alpha\mathbf{f}] = \alpha\mathbf{Df}$
  - $\mathbf{D}[\mathbf{f} + \mathbf{g}] = \mathbf{Df} + \mathbf{Dg}$
  - $\mathbf{D}[fg] = g\mathbf{D}f + f\mathbf{D}g$
  - $\mathbf{D}[\frac{f}{g}] = \frac{g\mathbf{D}f-f\mathbf{D}g}{g^2}$
  - (Example) Prove the sum rule above.

- Chain Rule
  - $\mathbf{D}[\mathbf{f} \circ \mathbf{g}] = \mathbf{Df}(\mathbf{g})\mathbf{Dg}$
  - (Example) Find the rate of change of $f(x, y) = x^2 + y^2$ along the path $\mathbf{c}(t) = \left(t^2, t\right)$ when $t = 1$.
  - (Example 2) Verify the Chain Rule for $f(u, v, w) = u^2 + v^2 - w$ and $\mathbf{g}(x, y, z) = \left(x^2 y, y^2, e^{-xz}\right)$.
  - (Example 3) Compute $\mathbf{D}[\mathbf{f} \circ \mathbf{g}](1, 1)$ where $\mathbf{f}(u, v) = \left(u + v, u, v^2\right)$ and $\mathbf{g}(x, y) = \left(x^2 + 1, y^2\right)$.
- *HW: 6-13, 15-16*

3.2 Taylor's Theorem

- Single-variable Taylor Series
  - $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$
    $= f(x_0) + f'(x_0)(x - x_0) + \dfrac{1}{2} f'(x_0)(x - x_0)^2 + \dfrac{1}{6} f'(x_0)(x - x_0)^3 + \dots$
  - $f(x) \approx \displaystyle\sum_{n=0}^{m} \frac{f^{(n)}(x_0)}{n!}(x - x_0)$
- First-Order Taylor Formula
  - $f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{x}_0) + [\mathbf{D}f(\mathbf{x}_0)](\mathbf{x} - \mathbf{x}_0) = f(\mathbf{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i - x_{0i})$
- Second-Order Taylor Formula
  - $f(\mathbf{x}) \approx f(\mathbf{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i - x_{0,i}) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)(x_i - x_{0,i})(x_j - x_{0,j})$
  - (Example) Use the second-order Taylor formula for $f(x, y) = \sqrt{x + 2y}$ near the point $(2, 1)$ to approximate $\sqrt{4.05}$.
  - (Example 3) Find linear and quadratic functions of x, y which approximate $f(x, y) = \sin(xy)$ near the point $(1, \pi/2)$.
- *HW: 3-7, 12*

4.3 Vector Fields

- Vector Fields

- A vector field is a map $f : \mathbb{R}^n \to \mathbb{R}^n$ assinging an n-dimensional vector to each point in $\mathbb{R}^n$
- (Example 1) The velocity field of a fluid may be modeled as a vector field.
- (Example 2) Sketch the rotary motion given by the vector field $\mathbf{V}(x, y) = (-y, x)$.

- Gradient Vector Fields
  - $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$
  - (Example) The derivative of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$ in the direction given by a unit vector $\mathbf{v}$ is given by $\nabla f \cdot \mathbf{v}$. Show that the maximum value of a directional derivative for a fixed point is given by $\|\nabla f\|$ and attained by the direction $\frac{1}{\|\nabla f\|}\nabla f$.
  - (Example 4) If temperature is given by $T(x, y, z)$, then the energy or heat flux field is given by $\mathbf{J} = -k\nabla T$ where k is the conductivity of the body. Level sets are called isotherms.
  - (Example 5) The gravitational potential of bodies with mass m, M is given by $V = -\frac{mMG}{r}$ where G is the gravitational constant and r is the distance between the bodies, and the gravitational force field is given by $\mathbf{F} = -\nabla V$. Show that $\mathbf{F} = -\frac{mMG}{r^3}\mathbf{r}$, where $\mathbf{r}$ is the vector pointing from the center of mass M to the center of mass m.
  - A vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is conservative iff there exists a potential function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.
  - (Example) Show that $\mathbf{W} = (2y + 1, 2x)$ is conservative.
  - (Example 7) Show that $\mathbf{V} = (y, -x)$ is not conservative.

- Flow Lines
  - A flow line for a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a path $\mathbf{c} : \mathbb{R} \to \mathbb{R}^n$ satisfying $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$.
  - (Example 8) Show that $\mathbf{c}(t) = (\cos t, \sin t)$ is a flow line for $\mathbf{F} = (-y, x)$, and find some other flow lines.

- *HW: 1-12, 17-21*

4.4 Divergence and Curl

- Divergence
  - The divergence of a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is denoted by $\operatorname{div} \mathbf{F} : \mathbb{R}^n \to \mathbb{R}$ and defined by $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$

- (Examples 3-5) Compute the divergences of $\mathbf{F} = (x, y)$, $\mathbf{G} = (-x, -y)$ and $\mathbf{H} = (-y, x)$ at any point on $\mathbb{R}^2$. How does divergence correspond with the motion described by the vector field plots?
- (Example) Compute the divergence of $\mathbf{F} = \left(x^2, y\right)$ various points and interpret those values against a plot of the vector field.

- Curl
  - The curl of a three-dimensional vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is denoted by $\operatorname{curl} \mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ and defined by $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$
  - The scalar curl of a two-dimensional vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is denoted by $\operatorname{scurl} \mathbf{F} : \mathbb{R}^2 \to \mathbb{R}$ and defined by $\operatorname{scurl} \mathbf{F} = \operatorname{curl} \mathbf{F} \cdot \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$
  - (Example) Compute the scalar curl of $\mathbf{F} = (x, y)$, $\mathbf{G} = (-x, -y)$ and $\mathbf{H} = (-y, x)$ at every point in $\mathbb{R}^2$. How does this scalar curl correspond with the motion described by the vector field plots?
  - (Example) Compute the curl of $\mathbf{F} = (y, -x, z)$ at every point in $\mathbb{R}^3$. How does curl correspond with the motion described by the vector field plot?

- Facts about ∇f, $\operatorname{div} \mathbf{F}$, $\operatorname{curl} \mathbf{F}$
  - The curl of a conservative field is zero: $\operatorname{curl} \nabla f = \nabla \times (\nabla f) = \mathbf{0}$.
  - (Example) Prove the above theorem.
  - (Example) Prove that $\mathbf{F} = \left(x^2 + z, y - z, z^3 + 3xy\right)$ is not a conservative field.
  - The divergence of a curl field is zero: $\operatorname{div} \operatorname{curl} \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$
  - Many identities on pg. 255 of Marsden text.
  - (Example) Sketch proof of identity #8: $\operatorname{div} (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$.

- *HW: 1-4, 9-17, 22-25, 29-30*

5.3 The Double Integral Over More General Regions

- Hypervolume
  - The hypervolume $HV_1(D)$ of an interval $D = [a, b]$ in $\mathbb{R}$ is just its length $b - a$.
  - The hypervolume of a well-behaved bounded subset $D \subseteq \mathbb{R}^{n+1}$ is defined for each $n \in \{1, 2, \dots\}$ by

$$HV_{n+1}(D) = \int_{x_i \in I} HV(D_i)\, dx_i = \int_{x_i=a}^{x_i=b} HV_n(D_i)\, dx_i$$

    where $I = [a, b]$ is an interval containing all values x_i included in the ith coordinate of D, and D_i is the projection of of all points in D onto $\mathbb{R}^n$ by removing the ith coordinate.

- (Example) For $n = 1$ and $D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, f(x) \leq y \leq g(x)\}$, we have that

$$HV_2 = A = \int_{x\in[a,b]} g(x) - f(x)\,dx = \int_a^b g(x) - f(x)\,dx.$$

- (Example) For $n = 2$ and $D \subseteq R^3$ including values of x between a and b, we have that

$$HV_3 = V = \int_{x=a}^{x=b} A(x)\,dx$$

where $A(x)$ is the area of the cross-section of D taken by fixing each value of x (or similar for y).

- Double Integrals
  - For a bounded region $D \subseteq \mathbb{R}^2$ and continuous nonnegative $f : D \to \mathbb{R}$, the double integral

$$\iint_D f\,dA$$

is defined to be the volume of $\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D, 0 \leq z \leq f(x, y)\}$.

  - We may apply the definition of volume above to get

$$\iint_D F\,dA = \int_{x=a}^{x=b} A(x)\,dx$$

where D lies between the lines $x = a$ and $x = b$.

  - If D is described by $a \leq x \leq b$ and $\phi_1(x) \leq y \leq \phi_2(x)$, then

$$\iint_D F\,dA = \int_{x=a}^{x=b} A(x)\,dx = \int_{x=a}^{x=b} \left[\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x, y)\,dy\right] dx$$

  - Similarly, if D is described by $c \leq y \leq d$ and $\psi_1(y) \leq x \leq \psi_2(y)$, then

$$\iint_D F\,dA = \int_{y=c}^{y=d} \left[\int_{x=\psi_1(y)}^{x=\psi_2(y)} f(x, y)\,dx\right] dy$$

  - If f is sometimes negative on the domain D, then $\iint_D f\,dA$ is the net volume between $z = f(x, y)$ and D (volume above the xy plane minus volume below) and the above formulas still hold.

- Iterated integrals

  - An iterated integral is a shorthand for the expansion of two or more nested integrals, that is:

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx = \int_{x=a}^{x=b} \left[\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy\right] dx$$

  - (Example) Sketch the region of integration for $\int_0^\pi \int_{-x}^{x} \cos(y)\,dy\,dx$, evaluate it, and interpret it as the signed volume of a region in $\mathbb{R}^3$.
  - (Example) Express $\iint_R (12x^3y - 1)\,dA$ where R is the rectangle with vertices $(0,0),(3,0),(3,2),(0,2)$ as an interated integral, then evaluate it.
  - (Example) Express $\iint_T (12x^3y - 1)\,dA$ where T is the triangle with vertices $(0,0),(1,0),(1,1)$ as an interated integral, then evaluate it.

- Applications
  - $\iint_D 1\,dA$ is the area of D
  - $\frac{1}{A(D)} \iint_D f(x,y)\,dA$ is the average value of the function f restricted to D
- Additivity
  - If $D \subseteq \mathbb{R}^2$ is the union of two subregions D_1, D_2 overlapping only on their boundary, then $\iint_D f\,dV = \iint_{D_1} f\,dV + \iint_{D_2} f\,dV$.
  - (Example) Prove that the area of the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$ is two by setting it up as a double integral, then using additivity to split it up into two or more subregions.
- *HW: 1-9*

5.4 Changing the Order of Integration

- Rectangular regions of integration
  - For constant bounds of integration:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

  - (Example) Verify that $\int_0^1 \int_1^2 x^2 + 2xy\,dy\,dx = \int_1^2 \int_0^1 x^2 + 2xy\,dx\,dy$.
- Nonrectangular regions of integration
  - Bounds of integration cannot be directly swapped; however, by interpreting the region of integration new bounds may be found in the other order.

  - (Example) Verify that $\int_0^4 \int_0^{\frac{4-y}{2}} x + y\,dx\,dy$ and $\int_0^2 \int_0^{4-2x} x + y\,dy\,dx$ share the same region of integration and are equal.
  - (Example) Evaluate $\int_1^e \int_0^{\log x} \frac{(2x-e)\sqrt{1+e^y}}{e-e^y}\,dy\,dx$.

- Estimating double integrals
  - If $g(x,y) \leq f(x,y) \leq h(x,y)$ for $(x,y) \in D$, then $\iint_D g(x,y)\,dA \leq \iint_D f(x,y)\,dA \leq \iint_D h(x,y)\,dA$.
  - (Example 3) Prove that $\frac{1}{\sqrt{3}} \leq \iint_D \frac{1}{\sqrt{1+x^6+y^8}}\,dA \leq 1$ where D is the unit square.
  - (Example) Prove that $e \leq \iint_D e^{x^2y+y}\,dA \leq \frac{e^2}{2}$ where D is the unit square.

- *HW: 1-5, 7-10*

5.5 The Triple Integral

- Triple Integrals
  - For a bounded region $D \subseteq \mathbb{R}^3$ and nonnegative $f : D \to \mathbb{R}$, the triple integral

$$\iiint_D f\,dV$$

    is defined to be the hypervolume of $\{(x,y,z,w) \in \mathbb{R}^4 : (x,y,z) \in D, 0 \leq w \leq f(x,y,z)\}$.

- Applications
  - $\iiint_D 1\,dV$ is the volume of D
  - $\frac{1}{V(D)} \iiint_D f(x,y,z)\,dV$ is the average value of the function f restricted to D
  - If $\rho(x,y,z)$ gives the density of a solid at the coordinate (x,y,z), then $\iiint_D \rho(x,y,z)\,dV$ calculates its overall mass.

- Rectangular Boxes
  - If $B = [a_1,b_1] \times [a_2,b_2] \times [a_3,b_3]$, then

$$\begin{aligned} \iiint_B f\,dV &= \int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x,y,z)\,dx\,dy\,dz \\ &= \int_{a_2}^{b_2} \int_{a_1}^{b_1} \int_{a_3}^{b_3} f(x,y,z)\,dz\,dx\,dy \\ &= \text{etc.} \end{aligned}$$

  - (Example) Write $\iiint_D e^{x+y+z}\,dV$ where $D = [0,4] \times [0,2] \times [1,3]$ as a few different iterated integrals, then evaluate one.

- General regions of integration
  - If $E \subseteq \mathbb{R}^2$ and $D = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in E, \gamma_1(x, y) \leq z \leq \gamma_2(x, y)\}$, then

$$\iiint_D f(x, y, z)\, dV = \iint_E \left[\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)\, dz \right] dA$$

    (and similar for x, y instead of z).
  - (Example 5) Express $\iiint_W x\, dV$ where W is the solid for which x, y, z are positive and $x^2 + y^2 \leq z \leq 2$ as a few different iterated integrals.
  - (Example 6) Express $\iiint_W x\, dV$ where W is the solid in $\mathbb{R}^3$ above the triangle with vertices $(0, 0, 0), (1, 0, 0), (1, 1, 0)$ in the xy plane, and also between the surfaces $z = x^2 + y^2$ and $z = 2$, as an iterated integral. Then evaluate it.
- Additivity
  - If $D \subseteq \mathbb{R}^3$ is the union of two subregions D_1, D_2 overlapping only on their boundary, then $\iiint_D f\, dV = \iiint_{D_1} f\, dV + \iiint_{D_2} f\, dV$.
- *HW: 1-6, 11-17, 25-28*

1.4 Cylindrical and Spherical Coordinates

- Transformation of variables
  - A transformation of variables is a function $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$.
  - (Example) Sketch the integer lattice on the uv plane and its image in the xy plane for the transformation of variables $\mathbf{T}(u, v) = (x, y) = (u, u + v)$.
- Polar Coordinates
  - $\mathbf{p}(r, \theta) = (r \cos \theta, r \sin \theta)$
  - $r^2 = x^2 + y^2$, $\tan \theta = \frac{y}{x}$
  - (Example) Convert $A = \mathbf{p}(4, 2\pi/3)$ from polar to Cartesian. Convert $B = (3, -3)$ from Cartesian to polar. Plot both in the $r\theta$ and xy planes.
  - (Example) Express the curves $x = \sqrt{4 - y^2}$ and $y = 3$ in terms of polar coordinates. Plot both in the $r\theta$ and xy planes.
- Cylindrical Coordinates
  - $\mathbf{c}(r, \theta, z) = (r \cos \theta, r \sin \theta, z)$
  - Usually, assume $r \geq 0$ and $0 \leq \theta \leq 2\pi$

- $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$
- (Example 1) Convert $A = \mathbf{c}(8, 2\pi/3, -3)$ from cylindrical to Cartesian. Convert $B = (6, 6, 8)$ from Cartesian to cylindrical. Plot both in xyz space.
- (Example) Express the surfaces $x^2 + y^2 = 9$ and $z^2 = x^2 + y^2$ in terms of cylindrical coordinates. Plot both in xyz space.

• Spherical Coordinates

- $\mathbf{s}(\rho, \theta, \phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$
- Usually, assume $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$
- $\rho^2 = x^2 + y^2 + z^2$, $\tan\theta = \frac{y}{x}$, $\tan\phi = \frac{r}{z} = \frac{\sqrt{x^2+y^2}}{z}$
- (Example 2) Convert $A = (1, -1, 1)$ from Cartesian to spherical. Convert $B = \mathbf{s}(3, \pi/6, \pi/4)$ from spherical to Cartesian. Convert $C = (2, -3, 6)$ from Cartesian to spherical. Convert $D = \mathbf{s}(1, -\pi/2, \pi/4)$ from spherical to Cartesian. Plot all four in xyz space.
- (Example 3) Express the surfaces $xz = 1$ and $x^2 + y^2 - z^2 = 1$ in terms of spherical coordiantes.

• *HW: 1-11, 15*

6.1 The Geometry of Maps from $\mathbb{R}^n$ to $\mathbb{R}^n$

• Images of regions by transformations

- (Example 1) Find the image of the rectangle $[0, 1] \times [0, 2\pi]$ in the $r\theta$ plane under the polar coordinate transformation $\mathbf{p}$.
- (Example 2) Find the image of the square $[-1, 1]^2 = [-1, 1] \times [-1, 1]$ in the uv plane under the transformation $\mathbf{T}(u, v) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} (u, v)$

• One-to-one and Onto

- A one-to-one transformation sends each point in the domain to a distinct point in the range.
- An onto transformation sends something in the domain onto each point of the range.
- (Example 3) Show that the polar coordinate transformation $\mathbf{p}$ is onto but not one-to-one.
- (Example 4) Show that the transformation $\mathbf{T}$ from example 2 is both one-to-one and onto.

- (Example 5) Show that $\mathbf{T}(u, v) = (u, 0)$ is neither one-to-one nor onto.
- (Example 7) Find a rectangle in the $r\theta$ plane which maps onto the region $\{(x, y) : x, y \geq 0, a^2 \leq x^2 + y^2 \leq b^2\}$ in the Cartesian plane by the polar coordinate transformation.

- Linear transformations
  - Transformations $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathbf{T}(\mathbf{u}) = A\mathbf{u}$ for an n-dimensional matrix A are called linear transformations.
  - (Example 6) Find a region in the uv plane which maps onto the square with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$ in the xy plane by the linear transformation given in Example 2.
  - Transformations $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathbf{T}(\mathbf{u}) = A\mathbf{u} + \mathbf{x}_0$ for an n-dimensional matrix A and n-dimensional vector $\mathbf{x}_0$ are called affine transformations. (Every linear transformation is affine.)
  - (Example) Find an affine transformation which maps the unit square in the uv plane onto the square with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$ in the xy plane.
  - An affine transformation is both one-to-one and onto exactly when $\det A \neq 0$.
  - (Example) Use this fact to reinvestigate examples 4 and 5.
- *HW: 1-4, 8, 10*

6.2 The Change of Variables Theorem

- Affine transformations of areas
  - An affine transformation with matrix M transforms hypervolumes by a factor of $|\det M|$.
  - (Example) Verify this fact for the parallelogram with vertices $(2, 0), (3, 1), (1, 3), (0, 2)$ in the uv plane and its image in the xy plane under the transformation $\mathbf{T}(u, v) = (2u + v + 3, v - u - 2)$.
  - Put another way, $\iint_D 1\, dA = \iint_{D^*} |\det M|\, dA$.
- Affine transformations of single/double/triple integrals
  - (Example) Let $x = T(u) = mu + x_0$. Use substitution to prove that if the image of $[c_1, c_2]$ under T is $[b_1, b_2]$, then $\int_{b_1}^{b_2} f(x)\, dx = \int_{c_1}^{c_2} f(T(u))|m|\, du$.
  - (Example) Use the previous fact to show that $\int_0^4 \sqrt{2x+1}\, dx = \int_1^9 \frac{1}{2}\sqrt{u}\, du$
  - For any 2D affine transformation $\mathbf{T}$ with matrix M transforming D^* to D, $\iint_D f(x, y)\, dA = \iint_{D^*} f(\mathbf{T}(u, v))|\det M|\, dA$.

  - (Example) Use an affine transformation to prove that $\int_0^2 \int_{y/2}^{(y+4)/2} 2y\,dx\,dy = \int_0^1 \int_0^1 16v\,dv\,du$ and compute both integrals directly to verify.
  - (Example) Compute $\iint_D (x+y)(x-y-2)\,dA$ where T is the triangle with vertices $(4,2)$, $(3,1)$, $(2,2)$.
  - For any 3D affine transformation $\mathbf{T}$ with matrix M transforming D^* to D, $\iint_D f(x,y,z)\,dV = \iint_{D^*} f(\mathbf{T}(u,v,w))|\det M|\,dV$.

- Jacobian
  - The Jacobian $\frac{\partial \mathbf{T}}{\partial \mathbf{u}}$ of a transformation is defined to be the determinant of its partial derivative matrix: $\det(\mathbf{DT})$.
  - (Example) Prove that for an affine transformation $\mathbf{T}$ with matrix M that $\mathbf{DT} = M$ and therefore $\frac{\partial \mathbf{T}}{\partial \mathbf{u}} = \det M$.
  - For any 2D transformation $\mathbf{T}$ transforming D^* to D, $\iint_D f(\mathbf{x})\,dA = \iint_{D^*} f(\mathbf{T}(\mathbf{u}))|\frac{\partial \mathbf{T}}{\partial \mathbf{u}}|\,dA$.
  - For any 3D transformation $\mathbf{T}$ transforming D^* to D, $\iiint_D f(\mathbf{x})\,dV = \iiint_{D^*} f(\mathbf{T}(\mathbf{u}))|\frac{\partial \mathbf{T}}{\partial \mathbf{u}}|\,dV$.
  - (Example) Use a 2D transformation to compute $\iint_D e^x \cos(\pi e^x)\,dA$ where D is the region bounded by $y=0$, $y=e^x-2$, $y=\frac{e^x-1}{2}$. (Hint: find a transformation from the unit square to the region bounded by $y=0$, $y=1$, $y=e^x-1$, $y=e^x-2$.)

- Polar, cylindrical, spherical change of variables
  - Polar coordinates: $\iint_D f(x,y)\,dA = \iint_{D^*} f(r\cos\theta, r\sin\theta)r\,dA$
  - Cylindrical coordinates: $\iint_D f(x,y,z)\,dV = \iint_{D^*} f(r\cos\theta, r\sin\theta, z)r\,dV$
  - Spherical coordinates: $\iint_D f(x,y,z)\,dV = \iint_{D^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,dV$
  - (Example) Compute the area of the triangle with vertices $(0,0), (\sqrt{3},1), (0,1)$ using polar coordinates.
  - (Example 4*) Evaluate $\iint_D \log_e(x^2+y^2)\,dA$ where D is the region in the first quadrant between the circles $x^2+y^2=1$ and $x^2+y^2=e^2$. (Hint: $\int \log_e x\;dx = x\log_e x + x + C$.)
  - (Example 6) Evaluate $\iiint_W \exp[(x^2+y^2+z^3)^{3/2}]\,dV$ where W is unit ball centered at the origin.
  - (Example) Prove that the formula for the volume of a cone with radius R and height H is $V = \frac{1}{3}\pi R^2 H$.
  - (Example 7) Prove that the formula for the volume of a sphere with radius R is $V = \frac{4}{3}\pi R^3$.

- *HW: 1-3, 5-6, 11, 13-14, 21, 26*

7.1 The Path Integral

- Path Integral with respect to Arclength
  - Recall that for a curve C defined by $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$, the arclength function $s : \mathbb{R} \to \mathbb{R}$ defined by $s(t) = \int_0^t \|\mathbf{r}'(\tau)\| \, d\tau$ gives the length of the curve from 0 to t.
  - (Example) Prove that $C = \pi D$ gives the circumference of a circle with diameter D.
  - If $f : C \to \mathbb{R}$ is a function defined along the curve C defined by $\mathbf{r} : [a, b] \to \mathbb{R}^n$, then
  $$\int_C f \, ds = \int_a^b f(\mathbf{r}(t)) \frac{ds}{dt} \, dt$$
  where $\frac{ds}{dt} = \|\frac{d\mathbf{r}}{dt}\|$. This represents the area of a ribbon with base C and height f at each point of C.
  - (Example 1) Find the average value of the function $f(x, y, z) = x^2 + y^2 + z^2$ along the portion of the helix given by $\mathbf{c}(t) = (\cos t, \sin t, t)$ for $t \in [0, 2\pi]$.
  - (Example 2) The base of a fence is given by the curve $\mathbf{c}(t) = \left(30 \cos^3 t, 30 \sin^3 t\right)$, and the height of the fence is given by $f(x, y) = 1 + \frac{y}{3}$. How much paint is required to cover both sides of this fence?
- *HW: 1-8, 10-13*

7.2 Line Integrals

- Line Integral with respect to a Curve
  - If $\mathbf{F} : C \to \mathbb{R}^n$ is a vector field defined along the curve C defined by $\mathbf{r} : [a, b] \to \mathbb{R}^n$, then
  $$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} \, dt$$
  represents the work done by a force $\mathbf{F}$ over the curve C.
  - (Example) An object is pushed around the unit circle with a force $(-y, x)$ at each point (x, y). Compute the work done in pushing the box around 3 full counter-clockwise rotations.
  - (Example 1) Let $\mathbf{r}(t) = (\sin t, \cos t, t)$ for $t \in [0, 2\pi]$ define the curve C, and define the vector field $\mathbf{F} = (x, y, z)$. Compute $\int_C \mathbf{F} \cdot d\mathbf{s}$.
  - (Example 5) Let C be a circle in the yz plane centered at the origin. Show that no work is done by a force $\mathbf{F} = \left(x^3, y, z\right)$ acting on an object moving around the circle.

- Line integrals with respect to variables
  - If $f : \mathbb{R}^n \to \mathbb{R}$ is a function defined along the curve C defined by $\mathbf{r} : [a, b] \to \mathbb{R}^n$, and $\mathbf{s} = (x_1, \dots, x_n)$,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C (F_1, \dots, F_n) \cdot (dx_1, \dots, dx_n) = \sum_{i=1}^{n} \int_C F_i \cdot dx_i$$

    where

$$\int_C f\, dx_i = \int_a^b f(\mathbf{c}(t)) \frac{dx_i}{dt}\, dt.$$

  - (Example 2) Evaluate and interpret $\int_C x^2\, dx + xy\, dy + dz$ where C is the parabola defined by $\mathbf{c}(t) = \left(t, t^2, 1\right)$ for $t \in [0, 1]$.
- Reparametrizations and partitions
  - The value of $\int_C f\, ds$ is independent of the choice of parametrization $\mathbf{r}(t)$ regardless of orientation.
  - The value of $\int_C \mathbf{F} \cdot d\mathbf{s}$ is independent of the choice of parametrization $\mathbf{r}(t)$ provided it respects the orientation of C.
  - If C and $-C$ represent the same curve with opposite orientations, then $\int_C \mathbf{F} \cdot d\mathbf{s} = -\int_{-C} \mathbf{F} \cdot d\mathbf{s}$.
  - If $C = C_1 + C_2$, then $\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds$ and $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$.
  - (Example 11) Compute $\int_C x^2\, dx + xy\, dy$ where C is the perimeter of the unit square oriented counter-clockwise.
- *HW: 1-5, 13, 17-18*

8.1 Green's Theorem

- Green's Theorem
  - Let ∂D be the c.c.w. oriented boundary of a simple region $D \subseteq \mathbb{R}^2$. Then $\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \operatorname{scurl} \mathbf{F}\, dA = \iint_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$.
  - (The book assumes $\mathbf{F} = (F_1, F_2) = (P, Q)$.)
  - (Example 1) Verify Green's Theorem for $\mathbf{F} = (x, xy)$ and $D = \{(x, y) : x^2 + y^2 \leq 1\}$.
  - (Example) Use Green's Theorem to prove that the area of D is $\frac{1}{2} \int_{\partial D} x\, dy - y\, dx$.

  - (Example 3) Compute the work done using a force $\mathbf{F} = \big(xy^2, y + x\big)$ in moving an object from the origin to $(1, 1)$ along the curve $y = x^2$ and then back to the origin along the line $y = x$.

- *HW: 1-6, 9-10, 15*

8.3 Conservative Fields

- Characterizations of Conservative Fields
  - These are all equivalent to $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ being conservative:
    (1) There exists a potential function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$, and for any curve starting at A and ending at B, $\int_C \mathbf{F} \cdot d\mathbf{s} = [f]_A^B = f(B) - f(A)$.
    (2) $\operatorname{curl} \mathbf{F} = \mathbf{0}$.
    (3) $\int \mathbf{F} \cdot d\mathbf{s}$ is path-independent: for any two curves C_1, C_2 which share starting and ending points, $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$.
    (4) For any simple closed curve C, $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$.
  - (Example) Prove that (1) implies (2) above.
  - (Example) Prove that (3) implies (4) above.
  - (7.2 Example 9) Evaluate $\int_C y\,dx + x\,dy$ where C is the curve given by $\mathbf{r}(t) = (t^4/4, \sin^3(t\pi/2))$ for $t \in [0, 1]$.
  - (Example 4) Find $\int_C 2x \cos y\,dx - x^2 \sin y\,dy$ where C is given by $\mathbf{r} : [1, 2] \to \mathbb{R}^2$ defined by $x = e^{t-1}, y = \sin(\pi/t)$.
  - (Example 1) Show that $\int_C (y, z \cos yz + x, y \cos yz) \cdot d\mathbf{s} = 0$ for any simple closed curve C.

- *HW: 1-2, 5-8, 10-11*

7.3 Parametrized Surfaces

- Parametrization of a Surface
  - Let $S \subseteq \mathbb{R}^3$ be a surface and $D \subseteq \mathbb{R}^2$ be a two-dimensional region. Then $\mathbf{\Phi} : D \to S$ is a parametrization of S by D.
  - (Example) Show that the surface given by $z = f(x, y)$ has the parametrization $\mathbf{\Phi}(x, y) = (x, y, f(x, y))$.
  - (Example 1) Show that the plane passing through the point $P \in \mathbb{R}^3$ and normal to the vector $\mathbf{a} \times \mathbf{b}$ has a parametrization $\mathbf{\Phi}(u, v) = \mathbf{P} + \mathbf{a}u + \mathbf{b}v$.

- (Example 2) Show that the cone $z = \sqrt{x^2 + y^2}$ has a parametrization $\mathbf{\Phi}(r, \theta) = (r\cos\theta, r\sin\theta, r)$ for $r \geq 0, 0 \leq \theta \leq 2\pi$.
- Surfaces which are conveniently described using cylindrical or spherical coordinates may be easily parameterized by adapting the relevant transformation.
- (Example) Use the cylindrical and spherical transformations to find parametrizations of the cone $z = \sqrt{x^2 + y^2}$.

- Tangent and Normal Vectors to a Surface
  - The tangent plane to a surface parameterized by $\mathbf{\Phi}$ at the point $\mathbf{\Phi}(\mathbf{u}_0)$ has parameterization

$$\mathbf{L}(\mathbf{u}) = \mathbf{\Phi}(\mathbf{u}_0) + [\mathbf{D\Phi}(\mathbf{u}_0)]\mathbf{u} = \mathbf{\Phi}(\mathbf{u}_0) + \frac{\partial \mathbf{\Phi}}{\partial u}(u_0, v_0)u + \frac{\partial \mathbf{\Phi}}{\partial v}(u_0, v_0)v.$$

  - (Example 3) Find a parameterization of the plane tangent to the surface defined by $\mathbf{\Phi}(u, v) = \left(u\cos v, u\sin v, u^2 + v^2\right)$ at the point $(1, 0, 1)$.
  - $\frac{\partial \mathbf{\Phi}}{\partial u}(u_0, v_0) \times \frac{\partial \mathbf{\Phi}}{\partial v}(u_0, v_0)$ is a normal vector to the surface.
  - (Example) Find an equation in x, y, z for the tangent plane in Example 3.
  - (Example) Find a parameterization for the sphere cenetered at the origin with radius 3. Show that the vector (x, y, z) is normal to the sphere for each point (x, y, z) on it. Then describe the plane tangent to it at the point $(1, -2, 2)$ as an equation of x, y, z.

- *HW: 1-3, 7-11*

7.4 Area of a Surface

- Definition of Surface Area
  - The area of a surface parametrized by $\mathbf{\Phi}$ with domain D is given by $\iint_D \|\frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v}\| \, dA$
  - (Example) Verify that this definition matches the area of the rectangle given by the vectors $(3, 0, -4)$ and $(0, -2, 0)$.
  - (Example 1) Show that the surface area of a cone with slant length L and radius R is given by the formula $A = \pi R^2 + \pi RL$.
  - (Example 2) Find that the area of a helicoid parameterized by $\mathbf{\Phi}(r, \theta) = (r\cos\theta, r\sin\theta, \theta)$ from $0 \leq \theta \leq 2\pi, 0 \leq r \leq 1$ is equal to $2\pi \int_0^1 \sqrt{r^2 + 1} \, dr$.
  - (Example) Prove that the surface area of a sphere of radius R is given by the formula $A = 4\pi R^2$.

- *HW: 3, 6-10*

7.5 Integrals of Scalar Functions over Surfaces

- Definition
  - If $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar function defined on the surface S defined by $\mathbf{\Phi} : D \to \mathbb{R}^n$ ($D \subseteq \mathbb{R}^2$), then

    $$\iint_S f(\mathbf{x})\, dS = \iint_D f(\mathbf{\Phi}(u,v)) \left\| \frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v} \right\| dA.$$

    This represents the area of a solid with base S and thickness f at each point of S.
  - (Example 1) Compute $\iint_S f\, dS$ where S is the helicoid parameterized by $\mathbf{\Phi}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$ from $0 \leq \theta \leq 2\pi, 0 \leq r \leq 1$ and $f(x,y,z) = \sqrt{x^2+y^2+1}$.
  - (Example 4) Compute $\iint_S x\, dS$ where S is the triangle with vertices $(1,0,0), (0,1,0), (0,0,1)$.
- *HW: 1-4, 6-7*

7.6 Surface Integrals of Vector Fields

- Definition
  - The orientation of a surface is given by a unit vector field $\mathbf{N}$ normal to each point on the surface.
  - We say a paramaterization $\mathbf{\Phi} : D \to \mathbb{R}^n$ preserves orientation if the orientation of the surface is given by unit vectors in the direction of $\frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v}$ at each point.
  - If $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}$ is a vector field defined on the surface S defined by $\mathbf{\Phi} : D \to \mathbb{R}^n$ ($D \subseteq \mathbb{R}^2$) preserving orientation, then

    $$\iint_S \mathbf{F}(\mathbf{x}) \cdot d\mathbf{S} = \iint_S (\mathbf{F}(\mathbf{x}) \cdot \mathbf{N})\, d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{\Phi}(u,v)) \cdot \left(\frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v} \right) dA.$$

    This represents the flux of the vector field passing through the surface S with regards to its orientation.
  - (Example 4) Suppose the temperature $T(x,y,z)$ of a point $(x,y,z) \in \mathbb{R}^3$ is given by $x^2+y^2+z^2$. Compute the heat flux $\iint_S -k\nabla T \cdot d\mathbf{S}$ across the unit circle oriented outward if $k = 1$.
  - (Example) Suppose fluid is moving according to the velocity field $\mathbf{F}(x,y,z) = (x,y,z)$ through the triangle with vertices $(1,0,0), (0,1,0), (0,0,1)$. Compute the flux of the velocity field through the triangle.
- *HW: 1-5, 13*

8.2 Stokes' Theorem

- Stokes' Theorem
  - For an oriented surface S, let ∂S be its boundary oriented counter-clockwise with respect to the orientation $\mathbf{N}$ of S.
  - $\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{N} \, dS = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.
  - (Example 1) Let $\mathbf{F} = (ye^z, xe^z, xye^z)$. Prove that $\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = 0$ for any surface S.
  - (Example 3) Rewrite $\iint_S (0, -ze^{xz}, -2) \cdot d\mathbf{S}$ as a definite integral of a single variable, where S is the surface $x^2 + y^2 + (z-1)^2 = 2$ above the xy plane and oriented upwards.

8.4 Gauss' Theorem

- Gauss'/Divergence Theorem
  - For a bounded region D in $\mathbb{R}^3$, let ∂D be its outward-oriented boundary surface.
  - $\iint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \operatorname{div} \mathbf{F} \, dV$.
  - (Example 3*) Evaluate $\iint_S \left(2y, zy, z^2\right) \cdot d\mathbf{S}$ where S is the outward oriented boundary of the unit sphere $x^2 + y^2 + z^2 = 1$.

Overview of integration theorems

- $\int_C \nabla f \cdot d\mathbf{s} = [f]_{\partial C}$
  $\int_{[a,b]} \frac{df}{dx} \, dx = [f]_{\partial [a,b]}$
- $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$
  $\iint_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k} \, dA = \int_{\partial D} \mathbf{F} \cdot d\mathbf{s}$
- $\iiint_D \operatorname{div} \mathbf{F} \, dV = \iint_{\partial D} \mathbf{F} \cdot d\mathbf{S}$