Recall the equation of the plane: ~~tangent to z = f(x,y) at~~

(Continued from last time)

$$\underbrace{\underline{n}}_{\substack{\text{normal}\\ \text{vector}\\ (A,B,C)}} \cdot \Big(\underbrace{\underline{x}}_{\substack{\text{arbitrary}\\ \text{point}\\ (x,y,z)}} - \underbrace{\underline{x}_0}_{\substack{\text{point on}\\ \text{plane}\\ (x_0,y_0,z_0)}}\Big) = 0$$

$$\Rightarrow A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

For $z=f(x,y)$, the normal vector is $\Big(\overset{A}{\frac{\partial f}{\partial x}(\underline{x}_0)}, \overset{B}{\frac{\partial f}{\partial y}(\underline{x}_0)}, \overset{C}{-1}\Big)$:

$$\frac{\partial f}{\partial x}(1,0)(x-1)+\frac{\partial f}{\partial y}(1,0)(y-0)-1(z-2)=0$$

Since $f(x,y)=x^2+y^4+e^{xy}$, $\quad \frac{\partial f}{\partial x}=2x+0+ye^{xy}$ & $\frac{\partial f}{\partial y}=0+4y^3+xe^{xy}$

$\frac{\partial f}{\partial x}(1,0)=2+0+0=2$ $\quad \frac{\partial f}{\partial y}(1,0)=0+0+1e^0=1$

$$2(x-1)+1(y-0)-1(z-2)=0$$

$$2x\cancel{-2}+y-z\cancel{+2}=0$$

$$\boxed{z=2x+y} \leftarrow \text{(Same as before)} \checkmark$$

End of 2.3 | 2.3 HW: 1-3, 5-21

2.5 Properties of the Derivative

Our favorite derivative rules hold for $\underline{D}\underline{f}$:

Constant Multiple ~~Scalar~~ Rule $\quad \underline{D}[\alpha \underline{f}] = \alpha(\underline{D}\underline{f})$

constant multiple

Sum Rule or Difference $\quad \underline{D}[\underline{f} \pm \underline{g}] = \underline{D}\underline{f} \pm \underline{D}\underline{g}$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad \underline{f}: \mathbb{R}^n \to \mathbb{R}^m$$

$\underline{g}$ (\# vars) (\# functions)

Product Rule $\quad \underline{D}[fg] = g\underline{D}f + f\underline{D}g$

Quotient Rule $\quad \underline{D}\left[\frac{f}{g}\right] = \frac{g\underline{D}f - f\underline{D}g}{g^2}$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad f: \mathbb{R}^n \to \mathbb{R}$$

g

(Example) Prove the Sum Rule $\underline{D}[\underline{f}+\underline{g}] = \underline{D}\underline{f} + \underline{D}\underline{g}$.

$$\underline{D}[\underline{f}+\underline{g}] = \underline{D}\left[(f_1,\dots,f_m)+(g_1,\dots,g_m)\right]$$

$$= \underline{D}\left[(f_1+g_1,\dots,f_m+g_m)\right]$$

$$= \begin{bmatrix} \frac{\partial(f_1+g_1)}{\partial x_1} & \cdots & \frac{\partial(f_1+g_1)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial(f_m+g_m)}{\partial x_1} & \cdots & \frac{\partial(f_m+g_m)}{\partial x_n} \end{bmatrix}$$

$\downarrow$ sum rule for scalar functions (from Cal 3)

$$= \begin{bmatrix} \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial g_1}{\partial x_1}\right) & \cdots & \left(\frac{\partial f_1}{\partial x_n} + \frac{\partial g_1}{\partial x_n}\right) \\ \vdots & & \vdots \\ \left(\frac{\partial f_m}{\partial x_1} + \frac{\partial g_m}{\partial x_1}\right) & \cdots & \left(\frac{\partial f_m}{\partial x_n} + \frac{\partial g_m}{\partial x_n}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} + \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial g_n} \end{bmatrix}$$

$$= \underline{D}\,\underline{f} + \underline{D}\,\underline{g} \qquad \square$$

The other big rule also works: Chain Rule:

Let $\underline{f}: \mathbb{R}^n \to \mathbb{R}^m$ and $\underline{g}: \mathbb{R}^p \to \mathbb{R}^n$.

Then $\underline{f} \circ \underline{g}: \mathbb{R}^p \to \mathbb{R}^m$ is defined by $\underline{f}(\underline{g}(\underline{x}))$, and

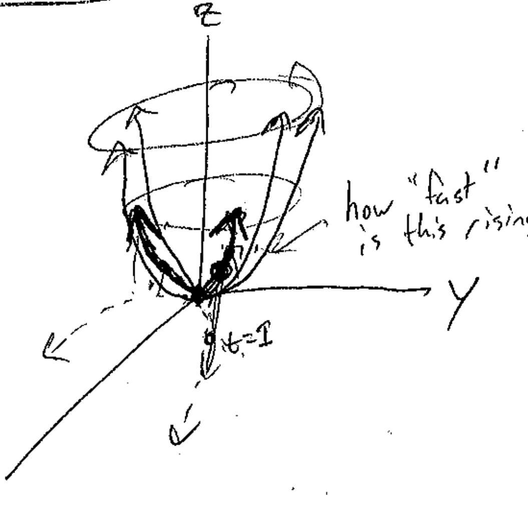
$$\underbrace{\underline{D}\left[\underline{f} \circ \underline{g}\right]}_{\substack{m \times p \\ \text{matrix}}} = \underbrace{\underline{D}\,\underline{f}\,(\underline{g})}_{\substack{m \times n \\ \text{matrix}}} \; \underbrace{\underline{D}\,\underline{g}}_{\substack{n \times p \\ \text{matrix}}} \quad \leftarrow \text{matrix multiplication}$$

(Compare with $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$...

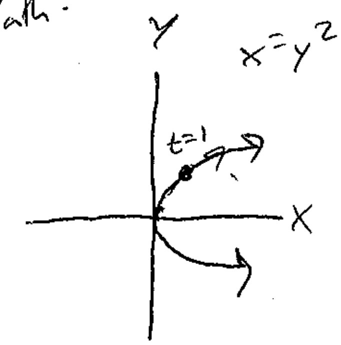
$$\frac{d}{dx}\left[f \circ g\right] = \frac{d}{dx}\left[f(g(x))\right] = \underbrace{f'(g(x))}_{\text{Deriv of Out}} \; \underbrace{g'(x)}_{\text{Deriv of Ins}} = \frac{df}{dg}\frac{dg}{dx}.$$
)

(Example) Find the rate of change of $f(x,y) = x^2 + y^2$ along the path $\underline{c}(t) = (\underset{c_1 = x}{t^2}, \underset{y = c_2}{t})$ when $t = 1$. $\Big\}$ find $\underline{D}[f \circ \underline{c}]$ @ $t=1$

Surface:



Path:



Two ways:

1st way

Chain Rule

$$\underline{D}[f \circ \underline{c}] = \underline{D}f(\underline{c})\, \underline{D}\,\underline{c}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}(\underline{c}) \begin{bmatrix} \frac{dc_1}{dt} \\ \frac{dc_2}{dt} \end{bmatrix}$$

$$= \begin{bmatrix} 2x & 2y \end{bmatrix}(\underset{\substack{\text{plug in} \\ x = t^2 \\ y = t}}{\underline{c}}) \begin{bmatrix} 2t \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2t^2 & 2t \end{bmatrix} \begin{bmatrix} 2t \\ 1 \end{bmatrix}$$

$$= 4t^3 + 2t$$

$$\underline{D}[\underline{f} \circ \underline{c}](1) = 4 + 2 = \boxed{6}$$

2nd way

Directly

$$\underline{D}[f\circ\underline{c}] = \cancel{\underline{D}[\ldots]}$$

$$= \underline{D}[x^2+y^2]$$

$$= \underline{D}[(t^2)^2+(t)^2]$$

$$= \underline{D}[t^4+t^2]$$

1 var, 1 function

$$= \frac{d}{dt}[t^4+t^2]$$

$$= 4t^3+2t$$

$$\underline{D}[f\circ\underline{c}](1) = 4+2 = \boxed{6}$$

(Example 2) Verify the Chain Rule for $f(u,v,w) = u^2+v^2-w$

and $\underbrace{\underline{g}(x,y,z) = (x^2y,\, y^2,\, e^{-xz})}_{\text{a.k.a.}}$.

$g_1 = u = x^2y$

$g_2 = v = y^2$

$g_3 = w = e^{-xz}$

Direct way:

$$f\circ\underline{g} = (\underset{\uparrow\atop u}{x^2y})^2 + (\underset{\uparrow\atop v}{y^2})^2 - (\underset{\uparrow\atop w}{e^{-xz}}) = x^4y^2+y^4-e^{-xz}$$

$$\underline{D}[f\circ g] = \begin{bmatrix} \frac{\partial(f\circ g)}{\partial x} & \frac{\partial(f\circ g)}{\partial y} & \frac{\partial(f\circ g)}{\partial z} \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 4x^3y^2 + ze^{-xz} & 2x^4y + 4y^3 & xe^{-xz} \end{bmatrix}}$$

Chain Rule: $\underline{D}[f\circ g] = \underline{D}f(g)\ \underline{D}g$ ← 3 vars, 3 functions

$$= \cancel{\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{bmatrix}}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{bmatrix}(g) \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} & \frac{\partial g_3}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 2u & 2v & -1 \end{bmatrix}(g) \begin{bmatrix} \equiv \end{bmatrix}$$

$$= \begin{bmatrix} 2(x^2y) & 2(y^2) & -1 \end{bmatrix} \begin{bmatrix} \equiv \end{bmatrix}$$

$$= \begin{bmatrix} 2x^2y & 2y^2 & -1 \end{bmatrix} \begin{bmatrix} 2xy & x^2 & 0 \\ 0 & 2y & 0 \\ -ze^{-xz} & 0 & -xe^{-xz} \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 4x^3y^2 + ze^{-xz} & 2x^4y + 4y^3 & xe^{-xz} \end{bmatrix}}$$

Same as before. ✓

(Example 3)

Compute $\underline{D}[\underline{f} \circ \underline{g}](1,1)$ where $\underline{f}(u,v) = (u+v, u, v^2)$ and $\underline{g}(x,y) = (x^2+1, y)$.

Direct way

$$\underline{h} = \underline{f} \circ \underline{g} = \underline{f}(\underset{u}{x^2+1}, \underset{v}{y})$$

$$= ((\underset{u}{(x^2+1)} + \underset{v}{(y)}), \underset{u}{(x^2+1)}, \underset{v}{(y)}^2)$$

$$= (\underbrace{x^2+y+1}_{h_1}, \underbrace{x^2+1}_{h_2}, \underbrace{y^2}_{h_3})$$

two vars
three functions

$$\underline{D}[\underline{f} \circ \underline{g}] = \underline{D}\underline{h} = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \\ \frac{\partial h_3}{\partial x} & \frac{\partial h_3}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 2x & 1 \\ 2x & 0 \\ 0 & 2y \end{bmatrix}$$

$$\underline{D}\underline{h}(1,1) = \boxed{\begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}}$$

Chain Rule

$$\underline{D}[\underline{f} \circ \underline{g}](1,1) = \underline{D}\underline{f}(\underline{g}(1,1))(\underline{D}\underline{g})(1,1)$$

$$\underline{g}(1,1) = (1+1, 1)$$
$$= (2,1)$$

$$= (\underline{D}\underline{f})(2,1)(\underline{D}\underline{g})(1,1)$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{bmatrix}(2,1) \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix}(1,1)$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{bmatrix}(\underset{u}{2}, \underset{v}{1}) \begin{bmatrix} 2x & 0 \\ 0 & 1 \end{bmatrix}(\underset{x}{1}, \underset{y}{1})$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 0+1 \\ 2+0 & 0+0 \\ 0+0 & 0+2 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}}$$

2.5 HW 6–13, 15–16