MATH 2242 - Fall 2015 - Dr. Clontz - Test 3

Name: Answers

- This test is worth 250 points toward your overall grade. Each problem is labeled with its value toward this total. Points earned beyond 250 will be counted as bonus.
- On multiple choice problems, you do not need to show your work. No partial credit will be given.
- On full response problems, show all of your work and give a complete solution. When in doubt, don't skip any steps. Partial credit will be given at the discretion of the professor.
- This exam is open notes, provided that these notes are completely in your own handwriting. The professor may take up notes you use with your test and return them after the test is graded.
- Calculators are not necessary to solve any questions on the test and are not allowed. Notes on electronic devices must be approved by the professor prior to the test day (e.g. for accomodations) and should be in airplane mode.
- Tests submitted after the end of 70 minutes will be deducted 25 points, with 25 more points deducted every following minute.

Multiple Choice (110 points total)

1. (20 points) Find the value of the Jacobian $\frac{\partial \mathbf{T}}{\partial \mathbf{u}}$ for the affine transformation $\mathbf{T}(u, v) = (4u - 5v + 7, 1 - 2v)$. $= \begin{bmatrix} 4 & -5 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

   $\det \begin{pmatrix} 4 & -5 \\ 0 & -2 \end{pmatrix} = -8 - 0 = -8$

   - ○ -10
   - ○ 0
   - ○ 4
   - ○ 9
   - ● None of these

2. (20 points) Recall that the length of a curve is given by the path integral $\int_C 1\, ds$. Which of these definite integrals gives the circumference of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$? Hint: use the fact that $\cos^2\theta + \sin^2\theta = 1$.

   $\cos^2\theta \quad \sin^2\theta$

   $x = a\cos\theta \quad y = b\sin\theta$

   $\frac{d\mathbf{r}}{dt} = (-a\sin\theta, b\cos\theta)$

   $\left\| \frac{d\mathbf{r}}{dt} \right\| = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

   $L = \int_0^{2\pi} \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_0^{2\pi} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$

   - ● $\int_0^{2\pi} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$
   - ○ $\int_0^{2\pi} b^2 - a^2\, d\theta$
   - ○ $\int_0^{\pi} a^2\cos^2\theta + b^2\sin^2\theta\, d\theta$
   - ○ $\int_0^{\pi} \frac{1}{\sqrt{a^2 - b^2}}\, d\theta$

3. (20 points) Which of these gives the value of $\int_C \nabla f \cdot d\mathbf{r}$ where C the curve given by $\mathbf{r}(t) = (t, t^2, t^3)$ from $t = 1$ to $t = 2$.

   $= [f]_A^B = f(\mathbf{r}(2)) - f(\mathbf{r}(1))$

   $= f(2,4,8) - f(1,1,1)$

   - ○ $\operatorname{curl} \nabla f \times (\mathbf{r}(2) - \mathbf{r}(1))$
   - ● $f(2, 4, 8) - f(1, 1, 1)$
   - ○ $\operatorname{div} \mathbf{r}(f(1, 1, 1)) + \operatorname{div} \mathbf{r}(f(2, 4, 8))$
   - ○ $\mathbf{r}(2) \cdot \nabla f(1, 1, 1) - \mathbf{r}(1) \cdot \nabla f(2, 4, 8)$

4. (20 points) Express $\int_{\partial D} \overset{F_1}{2xy}\,dx + \overset{F_2}{xy^2}\,dy$ where ∂D is the **clockwise** oriented boundary of the unit square $[0,1] \times [0,1]$ as a double iterated integral.

- ○ $\int_0^1 \int_0^{2x} y^2 + 2xy + 1\, dy\, dx$
- ● $\int_0^1 \int_0^1 -y^2 + 2x\, dx\, dy$
- ○ $\int_{-1}^0 \int_{-1}^0 2xy + x + xy^2\, dy\, dx$
- ○ $\int_{-1}^1 \int_{-1}^1 4y\, dx\, dy$

$$-\int_{\partial D} \underline{F} \cdot d\underline{r} = -\iint_D \text{curl}\, \underline{F}\, dA$$
$$= \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$$
$$= -\iint_D y^2 - 2x\, dA$$
$$= \iint_D -y^2 + 2x\, dA$$

5. (20 points) Let $\mathbf{F}$ be a vector field. Which of these is not equivalent to the rest?

- ○ $\mathbf{F}$ is conservative.
- ○ There exists a function f where $\mathbf{F} = \nabla f$.
- ● curl $\mathbf{F} = \mathbf{F}$. (curl $\underline{F} = \underline{0}$)
- ○ $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of C.

6. (10 points) (BONUS) Which of these is equal to the surface integral $\iint_S \langle x, y, z \rangle \cdot d\mathbf{S}$ where $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$?

- ○ π
- ○ 3π
- ● 4π
- ○ $\frac{9}{2}\pi$
- ○ 9π
- ○ 0
- ○ None of these.

$$\iint_S \underline{F} \cdot d\underline{S} = \iiint_B \text{div}\, \underline{F}\, dV$$
$$= \iiint_B \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\, dV$$
$$= 3 \iiint_B dV$$
$$= 3\left(\tfrac{4}{3}\pi (1)^3\right)$$
$$= 4\pi$$

Full Response (150 points total)

7. (30 points) Compute $\iiint_W (x^2+y^2)+z^2\,dV$ where $W = \{(x,y,z) : x^2+y^2 \le 4, 0 \le z \le 3\}$.

r^2

$$= \int_0^{2\pi}\int_0^2\int_0^3 (r^2+z^2)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2\int_0^3 r^3+z^2r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left[r^3z+\tfrac{1}{3}z^3r\right]_0^3 dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 3r^3+9r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{3}{4}r^4+\tfrac{9}{2}r^2\right]_0^2 d\theta$$

$$= \int_0^{2\pi} [12+9]\,d\theta$$

$$= \boxed{42\pi}$$

$\underline{r}(t) = (t, e^t)$ for $t \in [1,2]$

8. (30 points) Let C be the curve $y = e^x$ from $(1, e)$ to $(2, e^2)$. Prove that $\int_C xy\, ds = \int_1^2 xe^x\sqrt{1+e^{2x}}\, dx$.

$$\frac{d\underline{r}}{dt} = (1, e^t)$$

$$\frac{ds}{dt} = \left\|\frac{d\underline{r}}{dt}\right\| = \sqrt{1+(e^t)^2}$$

$$\int_C xy\, ds = \int_a^b xy \frac{ds}{dt}\, dt$$

$$= \int_1^2 te^t\sqrt{1+e^{2t}}\, dt$$

$$= \boxed{\int_1^2 xe^x\sqrt{1+e^{2x}}\, dx}$$

9. (30 points) Evaluate $\int_C \langle x, y\rangle \cdot \mathbf{dr}$, where C is the curve given by $\mathbf{r}(t) = (t^2, 3t)$ for $t \in [0, 1]$.

$\frac{dr}{dt}\,dt$

$\frac{dc}{dt}(t) = (2t, 3)$

$$= \int_0^1 \langle t^2, 3t\rangle \cdot \langle 2t, 3\rangle \, dt$$

$$= \int_0^1 2t^3 + 9t \, dt$$

$$= \left[\frac{1}{2}t^4 + \frac{9}{2}t^2\right]_0^1$$

$$= \frac{1}{2} + \frac{9}{2} = \boxed{5}$$

10. (30 points) Evaluate $\int_C \underbrace{\langle yz, xz, xy\rangle}_{\nabla f\ ?} \cdot d\mathbf{r}$ where C is the line segment from $(1,1,1)$ to $(1,2,3)$.

$$\frac{\partial f}{\partial x} = yz \qquad \frac{\partial f}{\partial y} = xz \qquad \frac{\partial f}{\partial z} = xy$$

$$f = xyz + \Phi \qquad f = xyz + \Phi \qquad f = xyz + \Phi$$

$$f = xyz$$

$$\int_C \nabla f \cdot d\underline{r} = [f]_A^B$$

$$= [xyz]_{(1,1,1)}^{(1,2,3)}$$

$$= 6 - 1 = \boxed{5}$$

11. (30 points) Evaluate $\int_C 2x\,dy - 2y\,dx$ where C is the counter-clockwise oriented boundary of a region with area 4.

R

$$= \int_C \langle \overset{F_1}{-2y}, \overset{F_2}{2x} \rangle \cdot \langle dx, dy \rangle$$

$$= \int_C \underline{F} \cdot d\underline{r}$$

$$\cancel{= \iint_R \ldots \, dA}$$

$$= \iint_R \operatorname{scurl} \underline{F} \, dA$$

$$= \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA$$

$$= \iint_R 2 - (-2) \, dA$$

$$= 4 \iint_R 1 \, dA$$

$$= 4(4) = \boxed{16}$$