(6.1 cont)

(Example 3 cont.) Show that p is not one-to-one.

(I need to show an example of two different polar coordinates which map to the same coordinate in xy plane.)

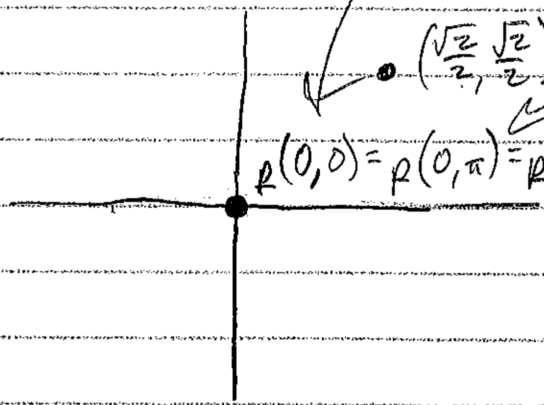
Lots of examples:

same Cartesian coordinate

different polar coordinates

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = p(1, 45^\circ) = p(1, 45^\circ - 360^\circ)$$

$$p(0,0) = p(0,\pi) = p(0,e) = (0,0)$$



Consider the polar coordinates $p(0,0)$ and $p(0,\pi)$.

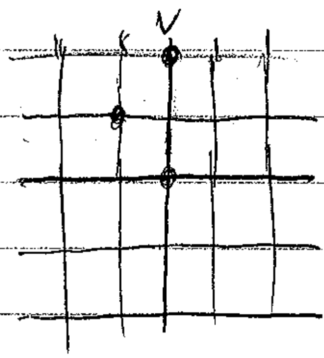
$$p(0,0) = (0\cos 0,\ 0 \sin 0) = (0,0)$$
$$p(0,\pi) = (0\cos\pi,\ 0\sin\pi) = (0,0) = p(0,0).$$

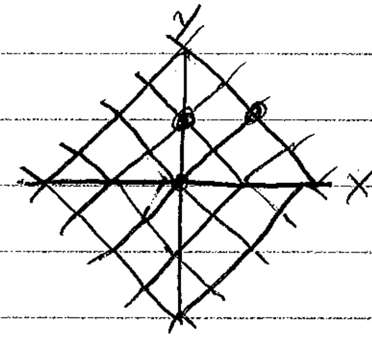
Therefore p is not one-to-one. $\square$

(Example 4) Show that $\underline{T}(u,v) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$

$= \left(\frac{1}{2}u + \frac{1}{2}v, \frac{1}{2}u - \frac{1}{2}v\right)$

is both onto and one-to-one.





Let $(x,y) \in \mathbb{R}^2$. Since $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$ has a nonzero determinant, it has an inverse $A^{-1} = \frac{1}{\det A}\begin{bmatrix} -1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$

$= \frac{1}{-1/2}\begin{bmatrix} -1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$

Let $(u,v) = A^{-1}(x,y)$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} x+y \\ x-y \end{bmatrix}$

$= (x+y, x-y).$

Then $\underline{T}(u,v) = A(u,v)$

$= \cancel{A A^{-1}}(x,y)$

$= (x,y).$ ✓

So $\underline{T}$ is onto.

To show $\underline{T}$ is one-to-one, suppose $\underline{T}(u_1,v_1) = \underline{T}(u_2,v_2)$.

Then

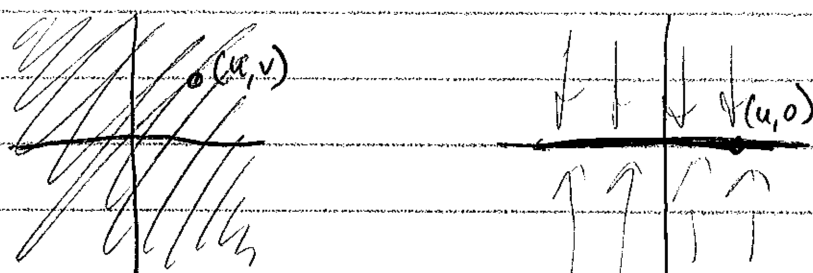
$$A\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = A\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$

$$\cancel{A^{-1}A}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \cancel{A^{-1}A}\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}.$$

So no two different points map by $\underline{T}$ to the same point.
So $\underline{T}$ is one-to-one. ✓

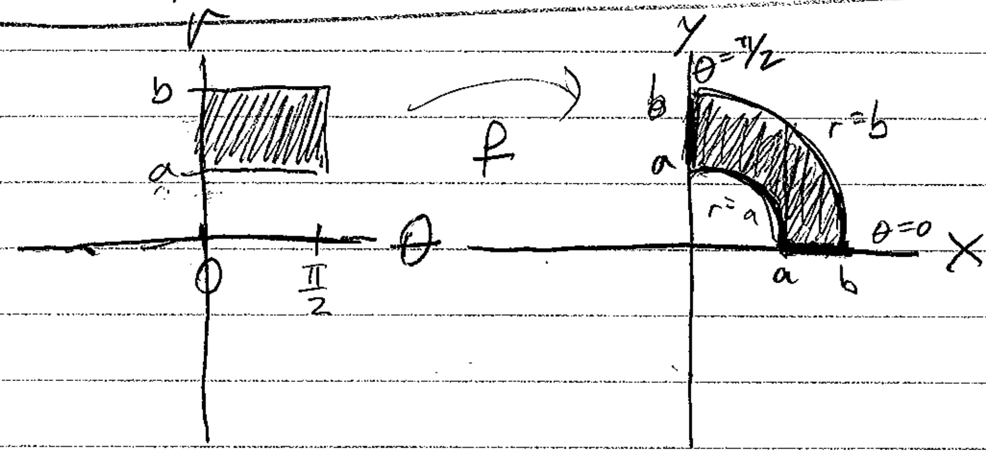
(Example 5) Show that $\underline{T}(u,v) = (u,0)$ is neither one-to-one nor onto.



$\underline{T}$ is not onto because $(1,1) \neq (u,0) = T(u,v)$ for all $(u,v) \in \mathbb{R}^2$. ✓

T is not one-to-one because $\underline{T}(1,1) = (1,0)$
and $\underline{T}(1,2) = (1,0)$. ✓

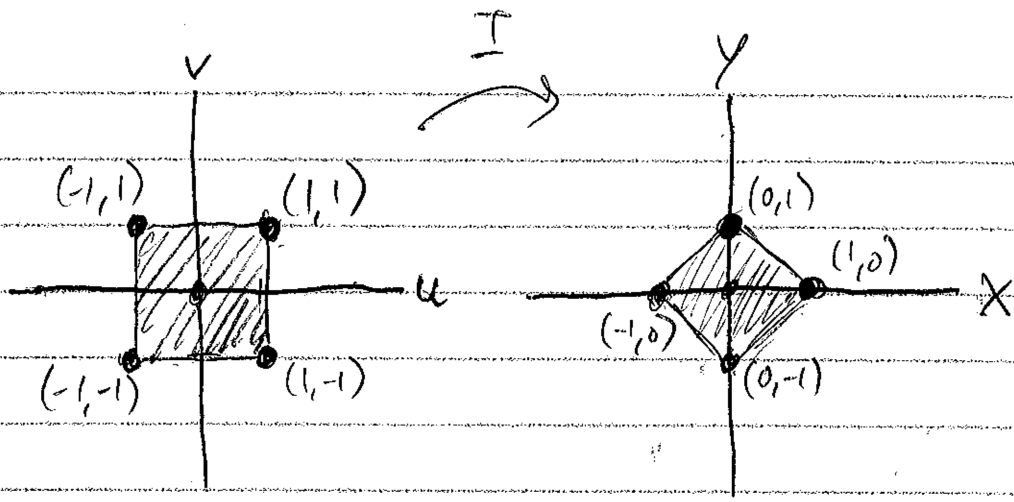
(Example 7) Find a rectangle in the $r\theta$ plane which maps onto the region $\{(x,y) : x,y \geq 0, a^2 \leq x^2+y^2 \leq b^2\}$ by the polar coordinate transformation $\underline{P}$.



$$\boxed{\{(r,\theta) : a \leq r \leq b, 0 \leq \theta \leq \frac{\pi}{2}\}}$$

A transformation $\underline{T} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\underline{T}(\underline{u}) = A\underline{u}$ for an $n \times n$ matrix A is called a linear transformation.

(Example 6) Find a region in the uv plane which maps onto the square with vertices $(1,0), (0,1), (-1,0), (0,-1)$ in the xy plane by the linear transformation $\underline{T} = A\,\underline{u}$ for $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$. $\left(\underline{T}(u,v) = \left(\frac{1}{2}u + \frac{1}{2}v, \frac{1}{2}u - \frac{1}{2}v\right)\right)$



$$\underline{T}(1,1) = \left(\tfrac{1}{2}+\tfrac{1}{2}, \tfrac{1}{2}-\tfrac{1}{2}\right) = (1,0) \qquad \underline{T}(1,-1) = \left(\tfrac{1}{2}-\tfrac{1}{2}, \tfrac{1}{2}+\tfrac{1}{2}\right) \quad (0,1)$$

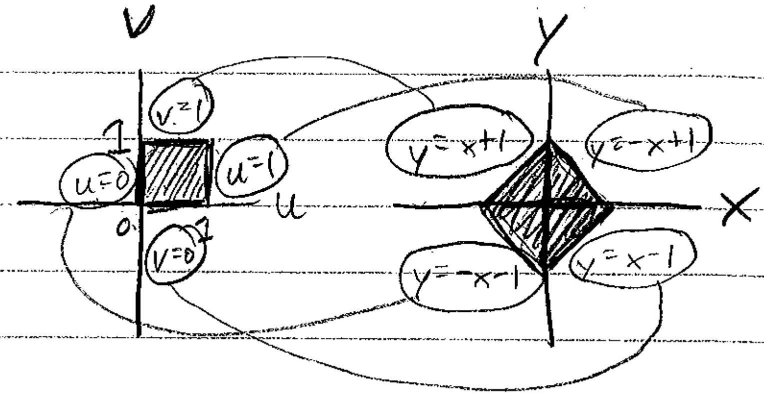
(etc.)

$$\boxed{\{(u,v) \in \mathbb{R}^2 : -1 \le u,v \le 1\}}$$

A transformation $\underline{T} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\underline{T}(\underline{u}) = A\underline{u} + \underline{x}_0$ for an $n \times n$ matrix A and $\underline{x}_0 \in \mathbb{R}^n$ is called an <u>affine transformation</u>.

(Every linear is affine by letting $\underline{x}_0 = \underline{0}$.)

(Example) Find an affine transformation which maps the unit square in uv plane onto the square with vertices $(1,0)$ $(0,1)$ $(-1,0)$ $(0,-1)$ in the xy plane.



$u=0 \Rightarrow y=-x-1$

$u=1 \Rightarrow y=-x+1$ $\quad +2$

For $u\in[0,1] \Rightarrow y=-x\underbrace{\underbrace{-1}_{u=0}+2u}_{u=1}$

solve for x, y in terms of u, v

$v=0 \Rightarrow y=x-1$

$v=1 \Rightarrow y=x+1$

For $v\in[0,1] \Rightarrow y=x-1+2v$

Set equal: $-x\cancel{-1}+2u=x\cancel{-1}+2v$

$2u-2v=2x$

$\boxed{x=u-v}$

$$\boxed{T(u,v)=(u-v+0,\ u+v-1)}$$

OR

Plug in x to find y: $y=-(u-v)-1+2u$

$y=-u+v-1+2u$

$\boxed{y=u+v-1}$

$$\boxed{T(u,v)=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}+\begin{bmatrix}0\\ -1\end{bmatrix}}$$

FACT An affine transformation $\underline{T}(\underline{u}) = A\underline{u} + \underline{x}_0$ is one-to-one and onto

if and only if

A has an inverse A^{-1}

if and only if

$\det A \neq 0$.

(6.1 HW) 1-4, 8, 10.

6.2 The Change of Variables Thm

FACT: An affine transformation with matrix M transforms hypervolumes by a factor of $|\det M|$.

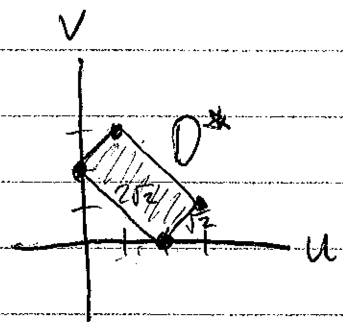
(Example) Verify this fact for the parallelogram with vertices $(2,0)$, $(3,1)$, $(1,3)$, $(0,2)$ in the uv plane & its image under the transformation $\underline{T}(u,v) = (2u+v+3, v-u-2)$.

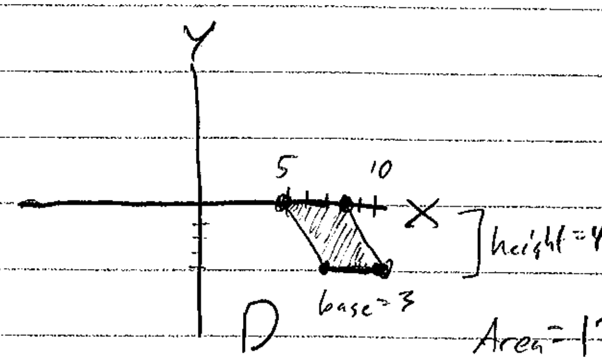
Note

$$\underline{T}(u,v) = \begin{array}{cc} & \begin{array}{cc} u & v \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

coefficients constants

So $M = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ and $\det M = 2-(-1) = 3$.





Area = 12

$$\text{Area} = 2\sqrt{2}(\sqrt{2}) = 4$$

$$\underline{T}(2,0) = (7,-4)$$
$$\underline{T}(3,1) = (10,-4)$$
$$\underline{T}(1,3) = (8,0)$$
$$\underline{T}(0,2) = (5,0)$$

So by geometry: $12 = 3(4)$.

Put another way:

$$\text{Area of } D = \underbrace{\iint_D 1 \, dA} = |\det M| \text{ Area of } D^* = \underbrace{\iint_{D^*} |\det M| \, dA}$$