8.3 (cont.)

These all characterize conservative fields $\underline{F}: \mathbb{R}^n \to \mathbb{R}^n$:

① There exists a potential function $f: \mathbb{R}^n \to \mathbb{R}$ such that $\underline{F} = \nabla f$, and for any curve C starting at A & ending at B,

$$\int_C \underline{F} \cdot d\underline{s} = [f]_A^B = f(B) - f(A).$$

(Fund Thm of Line Ints:

$$\int_A^B \nabla f \cdot d\underline{s} = [f]_A^B = f(B) - f(A).)$$

② $\text{curl } \underline{F} = \underline{0}$.

③ $\int \underline{F} \cdot d\underline{s}$ is path-independent: If C_1 & C_2 start at A & end at B, then

$$\int_{C_1} \underline{F} \cdot d\underline{s} = \int_{C_2} \underline{F} \cdot d\underline{s}.$$

④ For ~~all~~ any curve starting & ending at the same point, (a loop AKA simple closed curve)

$$\oint \underline{F} \cdot d\underline{s} = 0.$$

(Example)

Prove that ① implies ②.

Assume ①, so there exists a potential function $f: \mathbb{R}^n \to \mathbb{R}$ such that $\nabla f = \underline{F}$.

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (F_1, F_2, F_3)$$

Then

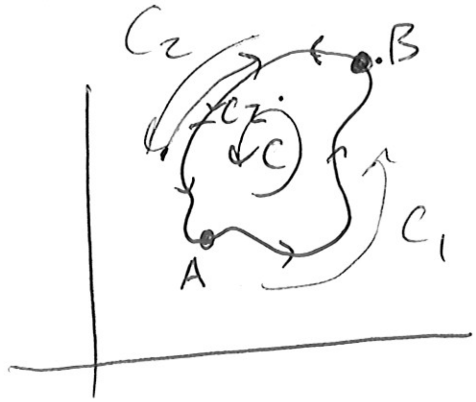
$$\begin{aligned}
\text{curl } \underline{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\
&= (f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy}) \\
&= (f_{yz} - f_{yz}, f_{xz} - f_{xz}, f_{xy} - f_{xy}) \\
&= (0, 0, 0) \\
&= \underline{0}. \quad \square
\end{aligned}$$

(Example)

Prove that 3 implies 4.

Assume that $\int \underline{F} \cdot d\underline{s}$ is path independent.

Let C be a loop:



Let A, B be points on C.

Let C_1 be portion of C from A to B.

Let C_2 be portion of C from B to A.

(So $-C_2$ is curve from A to B.)

$$\int_C \underline{F} \cdot d\underline{s} = \int_{C_1} \underline{F} \cdot d\underline{s} + \int_{C_2} \underline{F} \cdot d\underline{s}$$

$$= \int_{C_1} \underline{F} \cdot d\underline{s} - \int_{-C_2} \underline{F} \cdot d\underline{s}$$

Since C_1 & $-C_2$ start at A and end at B,

$$\int_{C_1} \underline{F} \cdot d\underline{s} = \int_{C_2} \underline{F} \cdot d\underline{s}$$

Therefore

$$\int_{C} \underline{F} \cdot d\underline{s} = \int_{C_1} \underline{F} \cdot d\underline{s} - \int_{C_1} \underline{F} \cdot d\underline{s}$$

$$= 0. \quad \square$$

(7.2 Example 9)

Evaluate $\int_C y\,dx + x\,dy$ where C is the curve given by $\underline{r}(t) = \Big(\underbrace{t^4/4}_{x},\ \underbrace{\sin^3\left(t\frac{\pi}{2}\right)}_{y}\Big)$ for $t \in [0,1]$.

Directly:

$$\int_C (y, x) \cdot d\underline{s} = \int_{t=0}^{t=1} \left(\sin^3\left(\tfrac{\pi}{2}t\right), t^4/4\right) \cdot \frac{d\underline{r}}{dt}\, dt$$

$$= \int_0^1 \left(\sin^3\left(\tfrac{\pi}{2}t\right), t^4/4\right) \cdot \left(t^3, 3\sin^2\left(t\tfrac{\pi}{2}\right)\left(\tfrac{\pi}{2}\cos\left(t\tfrac{\pi}{2}\right)\right)\right) dt$$

$$= \int_0^1 t^3 \sin^3\left(\tfrac{\pi}{2}t\right) + \tfrac{3\pi}{8} t^4 \sin^2\left(\tfrac{\pi}{2}t\right)\cos\left(\tfrac{\pi}{2}t\right) dt$$

awful

(integration by parts?
u-substitution?)

Using Conservative Fields

(First show that (y, x) is conservative.)

Let $\underline{F} = (y, x) = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$\frac{\partial f}{\partial x} = y$

$f = xy + \underset{\substack{\uparrow \\ \text{function of } y}}{\Phi}$

$\frac{\partial f}{\partial y} = x$

$f = xy + \underset{\substack{\uparrow \\ \text{function of } x}}{\Phi}$

$$f(x,y) = xy \cancel{+a}$$

Since $\nabla f = \underline{F}$, $\underline{F}$ is conservative.

So $\int_C \underline{F} \cdot d\underline{s} = \left[xy\right]_{A=\underline{r}(0)}^{B=\underline{r}(1)} = \left[xy\right]_{(0,0)}^{(1/4,1)} = \left(\frac{1}{4}\right)(1) - (0)(0)$

$= \boxed{1/4}$

8.3
(Example 4)

Find $\int_C \underbrace{2x\cos y}_{F_1}\,dx - \underbrace{x^2\sin y}_{F_2}\,dy$ where C is given by $x = e^{t-1}$, $y = \sin(\pi/t)$ for $t \in [1,2]$.

$$\underline{r}(t) = \left(e^{t-1}, \sin(\pi/t)\right)$$

Let $\underline{F} = \nabla f$.

$$\frac{\partial f}{\partial x} = 2x\cos y \qquad \frac{\partial f}{\partial y} = -x^2\sin y$$

$$f = x^2\cos y + \Phi \qquad f = +x^2(+\cos y) + \Phi$$

$$f = x^2\cos y \cancel{+0}$$

So $$\int_C \underline{F}\cdot d\underline{s} = \left[x^2\cos y\right]_{\underline{r}(1)}^{\underline{r}(2)}$$

$$= \left[x^2\cos y\right]_{(1,0)}^{(e,1)}$$

$$= e^2\cos 1 - 1\cancel{\cos 0}$$

$$= \boxed{e^2\cos(1) - 1}$$

(Example 1)

Prove that $\int_C (\underset{F_1}{y}, \underset{F_2}{z\cos(yz)+x}, \underset{F_3}{y\cos(yz)}) \cdot d\underline{s} = 0$

for any simple closed curve (AKA loop) C.

True for all conservative fields.

Method 1: Prove conservative by ~~[illegible]~~ finding a potential function

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = z\cos(yz)+x \qquad \frac{\partial f}{\partial z} = y\cos(yz)$$

$$f = xy + \underset{\uparrow \atop y\,\&\,z}{\Phi_1} \qquad f = \sin(yz)+xy+\underset{\uparrow \atop x\,\&\,z}{\Phi_2} \qquad f = \sin(yz)+\underset{\uparrow \atop x\,\&\,y}{\Phi_3}$$

$$f = xy + \sin(yz)$$

So f is a potential function for $\underline{F}$, so

$\underline{F}$ is conservative.

Therefore $\int_C \underline{F} \cdot d\underline{s} = 0$ for the loop C.

(Method 2) Prove Conservative by showing curl $\underline{E} = \underline{0}$.

$$\text{curl } \underline{E} = \left(\left(\frac{\partial F_3}{\partial y}\right) - \left(\frac{\partial F_2}{\partial z}\right), \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \left(\frac{\partial F_2}{\partial x}\right) - \left(\frac{\partial F_1}{\partial y}\right)\right)$$

$$= \left(\left(\cancel{\cos(yz)} + \cancel{z(-\sin(yz))y}\right) - \left(\cancel{\cos(yz)} + \cancel{z(-\sin(yz))y}\right) + 0,\right.$$

$$\left. 0 - 0, (0 + \cancel{1}) - (\cancel{1})\right)$$

$$= (0, 0, 0) = \underline{0}$$

Therefore $\underline{E}$ is conservative and

$\int_C \underline{E} \cdot d\underline{s} = 0$ for the loop C.

HW 8.3 1-2, 5-8, 10-11