1,5 cont.

2016-01-20

Determinant for an $n \times n$ matrix: $\left(\det([a]) = a\right)$

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \det(A_i)$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$$

For 2×2:

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = +a_{11}\det(a_{22}) - a_{12}\det(a_{21})$$

$$= \underbrace{a_{11}a_{22}}_{\text{downward diagonal}} - \underbrace{a_{12}a_{21}}_{\text{upward diagonal}}$$

Same thing

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \underbrace{(+1)\,a_{11}a_{22}}_{\sigma = \langle 1,2 \rangle} + \underbrace{(-1)\,a_{12}a_{21}}_{\sigma = \langle 2,1 \rangle}$$

For 3×3:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = +a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - \sim + \sim$$

$$= \Big(\underbrace{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}}_{\text{downward diagonals}}\Big)$$

$$- \Big(\underbrace{a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31}}_{\text{upward diagonals}}\Big)$$

(Example) Prove that the inverse of $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is

$$\frac{1}{\det A}\begin{bmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{bmatrix}.$$

Multiply them together:

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\begin{bmatrix} b_2/\det A & -a_2/\det A \\ -b_1/\det A & a_1/\det A \end{bmatrix} = \begin{bmatrix} \dfrac{\cancel{a_1b_2 - a_2b_1}}{\cancel{\det A}} & \dfrac{\cancel{-a_1a_2} + \cancel{a_1a_2}}{\det A} \\ \dfrac{\cancel{b_1b_2} - \cancel{b_1b_2}}{\det A} & \dfrac{\cancel{-b_1a_2 + a_1b_2}}{\cancel{\det A}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$$\boxed{\begin{array}{l}\text{HW 1.5} \\ \text{1-18, 21-24}\end{array}}$$

2.3 Differentiation

Functions from $\mathbb{R}^n \to \mathbb{R}^m$

Denoted $\underline{f}: \mathbb{R}^n \to \mathbb{R}^m$ and

$$\underline{f}(\underline{x}) = \underline{f}(x_1, \cdots, x_n)$$
$$= \left(f_1(\underline{x}), \cdots, f_m(\underline{x})\right)$$
$$= \left(f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n)\right)$$

For example: $\underline{f}: \mathbb{R}^2 \to \mathbb{R}^2$ may be defined by

$$\underline{f}(x_1, x_2) = \underline{f}(x, y) = (\underbrace{x+y}_{f_1(x,y)}, \underbrace{xy}_{f_2(x,y)})$$

So $$\underline{f}(0, -3) = (0 + (-3), (0)(-3))$$
$$= (-3, 0).$$

Partial Derivative Matrix:

$$\underline{D}\,\underline{f}(\underline{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Using the prev. example... $\underline{f} = (x+y, xy)$

$$\underline{D}\underline{f}(\underline{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 1+0 & 0+1 \\ y(1) & x(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ y & x \end{bmatrix}$$

We say $\underline{f}$ is differentiable at $\underline{x}_0$ if

$$\underline{f}(\underline{x}_0+\underline{h}) \approx \underline{f}(\underline{x}_0) + [\underline{D}\underline{f}(\underline{x}_0)]\,\underline{h} \quad \text{whenever } \underline{h} \approx \underline{0}.$$

Compare with...

$$f(x_0+h) \approx f(x_0) + f'(x_0)\,h \iff \cancel{f(x_0) \approx f(x_0+h)}$$

$$\iff f'(x_0)\,h \approx f(x_0+h) - f(x_0)$$

$$\iff f'(x_0) \approx \frac{f(x_0+h)-f(x_0)}{h}$$

because $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

(Example) Prove that the definition of differentiability is equivalent to saying

$$\underline{f}(\underline{x}) \approx \underline{f}(\underline{x}_0) + [\underline{D}\underline{f}(\underline{x}_0)](\underline{x} - \underline{x}_0) \text{ whenever } \underline{x} \approx \underline{x}_0$$

Let $\underline{x} = \underline{x}_0 + \underline{h}$. Then $\underline{h} = \underline{x} - \underline{x}_0$, and if $\underline{h} \approx \underline{0}$, then $\underline{x} - \underline{x}_0 \approx \underline{0}$ and $\underline{x} \approx \underline{x}_0$. $\square$

(Example) Let $\underline{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\underline{f}(x,y) = (x^2 + y^2, xy)$, and let $\underline{T} = \underline{D}\underline{f}(1,0)$. Compute $\underline{f}(1.1, -0.1)$ and $\underline{f}(1,0) + \underline{T}(0.1, -0.1)$.

$$\underline{f}(1.1, -0.1) = \boxed{(1.21 + 0.01, -0.11)}$$

$$= \boxed{(1.22, -0.11)}$$

$$\underline{T} = \underline{D}\underline{f}(1,0) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(1,0) & \frac{\partial f_1}{\partial y}(1,0) \\ \frac{\partial f_2}{\partial x}(1,0) & \frac{\partial f_2}{\partial y}(1,0) \end{bmatrix}$$

$$= \begin{bmatrix} 2x|_{(1,0)} & 2y|_{(1,0)} \\ y|_{(1,0)} & x|_{(1,0)} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{f}(1,0)+\underline{I}(0.1,-0.1)=$$

$$(1+0,0)+\begin{bmatrix}2&0\\0&1\end{bmatrix}(0.1,-0.1)$$

$$=\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}\cancel{2}&\cancel{0}\\\cancel{0}&\cancel{1}\end{bmatrix}\begin{bmatrix}0.1\\-0.1\end{bmatrix}$$

$$=\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0.2+0\\0+(-0.1)\end{bmatrix}$$

$$=\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0.2\\-0.1\end{bmatrix}=\boxed{\begin{bmatrix}1.2\\-0.1\end{bmatrix}}$$

$$=\boxed{(1.2,-0.1)}$$

In that example, $\underline{x}_0=(1,0)$, $\underline{h}=(0.1,-0.1)$, $\underline{x}=(1.1,-0.1)$.

Note $\underline{h}\approx\underline{0}$. So it's reasonable that

$$\underline{f}(\underline{x}_0+\underline{h}_0)=(1.22,-0.11)\approx(1.2,-0.1)=\underline{f}(\underline{x}_0)+[\underset{\text{OR } \underline{I}}{\underline{Df}(\underline{x}_0)}]\underline{h}$$

FACT: If $\frac{\partial f_i}{\partial x_j}$ is continuous near $\underline{x}_0$ for all values of i, j, then $\underline{f}$ is strongly differentiable a.k.a. class C^1.

Most of what we'll see is C^1.

Gradient

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function of multiple variables. Then,

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Ex: $f(x,y) = x^2 + 2xy + 3y^2 \implies \nabla f = \left(\overset{\partial f/\partial x}{2x+2y}, \overset{\partial f/\partial y}{2x+6y} \right).$

Compare this with:

$$\underline{D}f = \left[\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n} \right] = (\nabla f)^T$$

Ex: $\underline{D}f = \left[\underset{\partial f/\partial x}{2x+2y} \quad \underset{\partial f/\partial y}{2x+6y} \right]$

FACT: Let $\underline{h}$ be an n-dimensional vector. Then

$$[\underline{D}f]\,\underline{h} = \left[\partial f/\partial x_1 \; \cdots \; \partial f/\partial x_n \right] \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} = (\partial f/\partial x_1)(h_1) + \dots + \left(\frac{\partial f}{\partial x_n}\right)(h_n)$$

$$= \nabla f \cdot \underline{h}$$

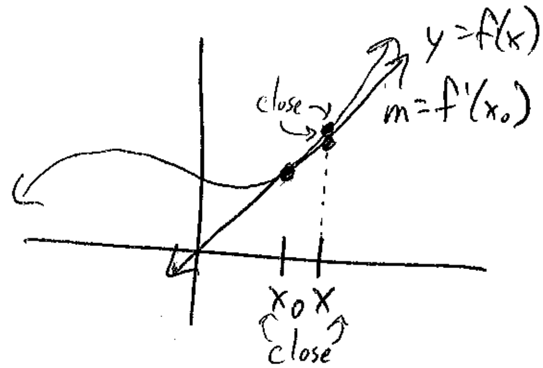
Linearizations & Tangent Hyperplanes

For $\underline{f}:\mathbb{R}^n \to \mathbb{R}^m$ and a point $\underline{x}_0 \in \mathbb{R}^n$, let the <u>linearization</u> of $\underline{f}$ at $\underline{x}_0$ be

$$\underline{L}(\underline{x}) = \underline{f}(\underline{x}_0) + [D\underline{f}(\underline{x}_0)](\underline{x}-\underline{x}_0).$$

So by definition, $\underline{f}(\underline{x}) \approx \underline{L}(\underline{x})$ whenever $\underline{x} \approx \underline{x}_0$ if $\underline{f}$ is differentiable.

(Recall from Cal I:



So the tangent line is the linearization function:

$$y=L(x)=f(x_0)+f'(x_0)(x-x_0)$$
)

(Example 5) Recall that the tangent plane to the surface $z=f(x,y)$ given by $f:\mathbb{R}^2\to\mathbb{R}$ passing thru $\underline{x}_0 \in \mathbb{R}^3$ is given by $\nabla f(\underline{x}_0)$. Show that $z=L(x,y)$ is the equation of the tangent plane to the surface $z=\underbrace{x^2+y^4+e^{xy}}_{f(x,y)}$ at the point $\underbrace{(1,0,2)}_{\underline{x}_0}$.

$$z = L(x,y)$$

$$= f(1,0) + \left(\underline{D}f(1,0)\right)\left(\underline{x} - (1,0)\right)$$

$$(x,y)-(1,0)$$

$$(x-1,\, y-0)$$

$$= \left(1 + 0 + e^{0}\right) + \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}_{@(1,0)} \begin{bmatrix} x-1 \\ y-0 \end{bmatrix}$$

$$= 2 + \begin{bmatrix} 2x + ye^{xy} & 4y^{3} + xe^{xy} \end{bmatrix}_{@(1,0)} \begin{bmatrix} x-1 \\ y-0 \end{bmatrix}$$

$$= 2 + \begin{bmatrix} 2+0 & 0+1e^{0} \end{bmatrix} \begin{bmatrix} x-1 \\ y-0 \end{bmatrix}$$

$$= 2 + \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x-1 \\ y-0 \end{bmatrix}$$

$$\boxed{z = 2 + 2(x-1) + 1(y-0)}$$

(to be concluded tomorrow...)