(4.4 cont.)

The curl of a 3D vector field $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is denoted by curl $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$ and defined by

$$\text{curl } \underline{F} = \nabla \times \underline{F}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_1, F_2, F_3)$$

$$= \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

$$= + \det \begin{pmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{pmatrix} \hat{i} - \det \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{pmatrix} \hat{j} + \det \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{pmatrix} \hat{k}$$

$$\text{curl } \underline{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

The scalar curl of a 2D vector field $\underline{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is denoted by scurl $\underline{F} : \mathbb{R}^2 \to \mathbb{R}$ and defined by

$$\underbrace{\text{scurl } \underline{F}}_{\hat{k} \text{ component of curl}} = (\text{curl } \underline{F}) \cdot \hat{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

A picture of a point where $\frac{\partial F_2}{\partial x}$ and $-\frac{\partial F_1}{\partial y}$ is positive:

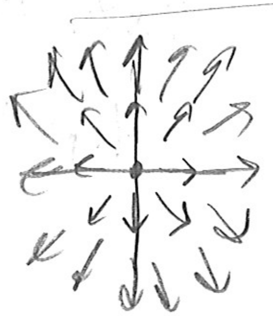
So we think of scurl $\underline{F}$ as measuring c.c.w. spinning.

(Example) Compute (scurl $\underline{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$) for
scalar curl

$\underline{F} = (x, y)$, $\underline{G} = (-x, -y)$, and $\underline{H} = (-y, x)$

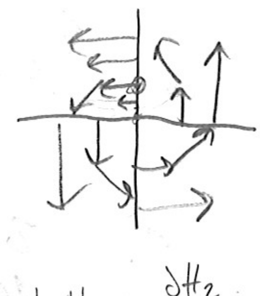
at an arbitary point of $\mathbb{R}^2$. How does this corespond with the plots of these vector fields?

$\underline{F}$



$\underline{G}$

$\underline{H}$



$$\text{scurl } \underline{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 - 0 = 0$$

$$\text{scurl } \underline{G} = \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = 0 - 0 = 0$$

$$\text{scurl } \underline{H} = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = 1 - (-1) = 2$$

(Example) Compute the curl of $\underline{F} = (\overset{F_1}{y}, \overset{F_2}{-x}, \overset{F_3}{z})$ at every point of $\mathbb{R}^3$. Compare the curl with the plot of $\underline{F}$.

$$\text{curl } \underline{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= (0 - 0, 0 - 0, -1 - 1)$$

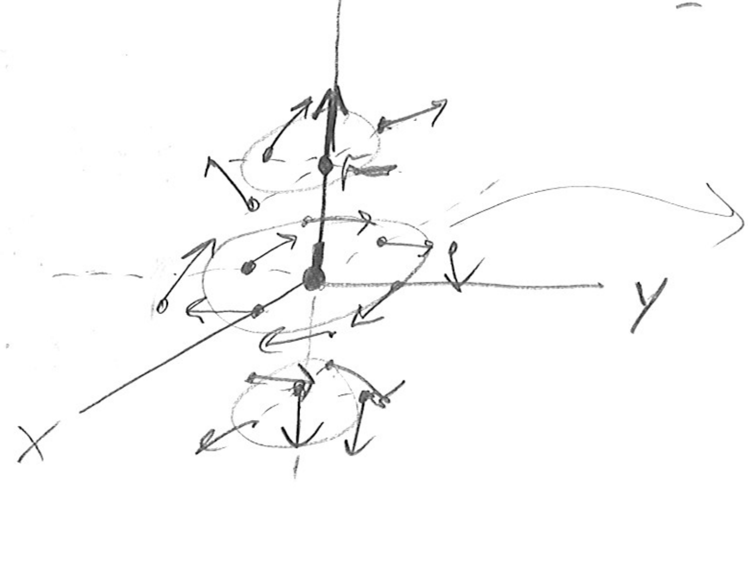
$$= (0, 0, -2)$$

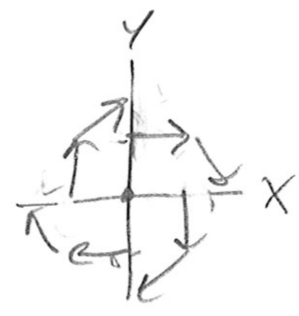
$\hat{\imath}$ is up

$\hat{\jmath}$ is up

c.c.w. spin if $\hat{k}$ is up

$\underline{F} = (y, -x, z)$

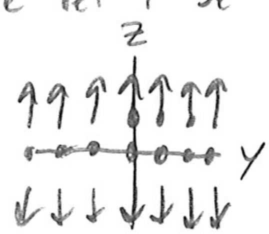




$\hat{k}$ is "up"

-2 makes sense, since spin is c.w.

But if we let $\hat{\imath}$ be up instead...



no spin, so 0 makes sense.

Fun Facts about ∇f, $\text{div}\,\underline{F} = \nabla \cdot \underline{F}$, and $\text{curl}\,\underline{F} = \nabla \times \underline{F}$

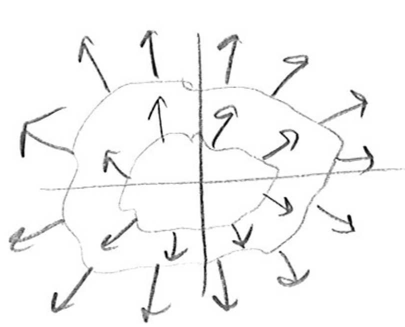
- The curl of a conservative field $\underline{F} = \nabla f$ is always zero: $\text{curl}(\nabla f) = \nabla \times (\nabla f) = \underline{0}$.

(Example) Prove it.

$$
\begin{aligned}
\text{curl}(\nabla f) &= \nabla \times \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
&= \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{pmatrix} \\
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \\
&= (0, 0, 0) \\
&= \underline{0} \quad \square
\end{aligned}
$$

Picture:

Since ∇f gives normal vectors to level curves...



... We don't expect any rotation, so $\text{curl}\,\underline{F} = 0$.

- The divergence of curl is always zero:

$$\operatorname{div}(\operatorname{curl} \underline{F}) = \nabla\cdot(\nabla\times\underline{F}) = 0$$

Rough idea:

If we're rotating, we aren't expanding or contracting

A bunch of other facts on ∇f, $\operatorname{div}\underline{F}$, $\operatorname{curl}\underline{F}$ may be found on pg. 255 of the textbook.

(Example) Sketch proof of idutity #8 from pg. 255:

$$\operatorname{div}(\underline{F}\times\underline{G}) = \underline{G}\cdot\operatorname{curl}\underline{F} - \underline{F}\cdot\operatorname{curl}\underline{G}$$

$$\underline{F}\times\underline{G} = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{pmatrix}$$

$$= \left(F_2G_3 - F_3G_2,\ F_3G_1 - F_1G_3,\ F_1G_2 - F_2G_1\right)$$

$$\operatorname{div}(\underline{F}\times\underline{G}) = \frac{\partial}{\partial x}(F_2G_3 - F_3G_2) + \frac{\partial}{\partial y}(\sim) + \frac{\partial}{\partial z}(\sim)$$

$$= G_3\frac{\partial F_2}{\partial x} + F_2\frac{\partial G_3}{\partial x} - G_2\frac{\partial F_3}{\partial x} - F_3\frac{\partial G_2}{\partial x} + (\sim) + (\sim)$$

$$= \cdots = (G_1, G_2, G_3)\cdot\underbrace{\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \sim, \sim\right)}_{\operatorname{curl} F} + (-F_1, -F_2, -F_3)\cdot\underbrace{(\sim, \sim, \sim)}_{\operatorname{curl} G}$$

(Example) Prove that $\underline{F} = (x^2+z, y-z, z^3+3xy)$ is not a conservative field.

$$\text{curl } \underline{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$
$$= (3x - (-1), 1 - 3y, 0 - 0)$$
$$= (3x+1, 1-3y, 0) \neq \underline{0}$$

Since curl of every conservative field is $\underline{0}$,

$\underline{F}$ can't be conservative. $\square$

4.4 HW: 1-4, 9-17, 22-25, 29-30