MATH 2242 (Calculus IV) Course Outline — Vector Calculus (Marsden)

1.5 n-Dimensional Euclidean Space

- $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$
- Addition
  - $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$
- Scalar multiplication
  - $\alpha\,(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$
- Inner/Dot Product
  - $(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = \sum_{i=1}^n x_i y_i$
- Norm/Length/Magnitude
  - $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$
- Standard basis vectors
  - $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0)$, $\ldots$, $\mathbf{e}_n = (0, 0, \ldots, 1)$
- Theorems
  - $(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z})$
  - Prove the above theorem.
  - $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
  - $\mathbf{x} \cdot \mathbf{x} \geq 0$
  - $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$
  - $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ (the Cauchy-Schwarz inequality)
  - (Example) Prove the Cauchy-Schwarz inequality.
  - $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (the triangle inequality)
  - (Example) Prove the triangle inequality.
- Matrices
  - $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$

- Addition $A + B$
- Scalar Mutiplication αA
- Transposition A^T

- Vectors as Matrices
  - $\mathbf{a} = (a_1, a_2, \ldots, a_n) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$
  - $\mathbf{a}^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$

- Matrix Multiplication
  - If A has m rows and B has n columns, then $M = AB$ is an $m \times n$ matrix.
  - Coordinate ij of $M = AB$ is given by $m_{ij} = \mathbf{a_i} \cdot \mathbf{b_j}$ where $\mathbf{a_i}^T$ is the ith row of A and $\mathbf{b_j}$ is the jth column of B.
  - (Example 4) Compute AB and BA for

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

  - (Example 5) Compute AB for

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Matrices as Linear Transformations
  - An $m \times n$ matrix A gives a function from $\mathbb{R}^n$ to $\mathbb{R}^m$: $\mathbf{x} \mapsto A\mathbf{x}$
  - This linear transformation satsifies $A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y}$
  - (Example 7) Express $A\mathbf{x}$ where $x = (x_1, x_2, x_3)$ and $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

  - (Example) Compute where the points $(-1,-1,0)$, $(0,1,0)$, $(1,-1,1)$, and $(2,1,1)$ in $\mathbb{R}^3$ get mapped to in $\mathbb{R}^4$ by $A\mathbf{x}$ from the previous example. Then plot the projections of the original points in $\mathbb{R}^3$ onto their first two coordinates in $\mathbb{R}^2$, and compare this with the projection plot of their images in $\mathbb{R}^4$ onto their first two coordinates in $\mathbb{R}^2$.

- Identity and Inverse
  - The $n \times n$ identity matrix I satisfies $i_{jj} = 1$ and $i_{jk} = 0$ when $j \neq k$. That is: $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$
  - If $AA^{-1} = A^{-1}A = I$, then A is invertable and A^{-1} is its inverse.

- Determinant
  - Let A_i be the submatrix of A with the first column and ith row removed. Then $\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \det(A_i)$
  - (Example) Prove that

$$\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1$$

    and

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

$$= (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_1 b_3 c_2 + a_2 b_1 c_3 + a_3 b_2 c_1)$$

  - (Example) Prove that the inverse of the matrix $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is $\frac{1}{\det A} \begin{bmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{bmatrix}$.
  - An $n \times n$ matrix is invertable if and only if its determinant is nonzero.

- *HW: 1-18, 21-24*

2.3 Differentiation

- Functions $\mathbb{R}^n \to \mathbb{R}^m$
  - $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$
  - $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ where $f_i : \mathbb{R}^n \to \mathbb{R}$

- Partial Derivative Matrix
  - $\mathbf{Df}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{bmatrix}$
  - We say $\mathbf{f}$ is differentiable at $\mathbf{x}_0$ if $\mathbf{f}(\mathbf{x}_0+\mathbf{h}) \approx \mathbf{f}(\mathbf{x}_0)+[\mathbf{Df}(\mathbf{x}_0)]\mathbf{h}$ whenever $\mathbf{h} \approx \mathbf{0}$.
  - (Example) Prove that this is equivalent to saying $\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x}_0)+[\mathbf{Df}(\mathbf{x}_0)](\mathbf{x}-\mathbf{x}_0)$ whenever $\mathbf{x} \approx \mathbf{x}_0$.
  - (Example) Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mathbf{f}(x,y) = \big(x^2+y^2, xy\big)$, and let $\mathbf{T} = \mathbf{Df}(1,0)$. Compute $\mathbf{f}(1.1,-0.1)$ and $\mathbf{f}(1,0) + \mathbf{T}\,(0.1,-0.1)$.
  - If each $\frac{\partial f_i}{\partial x_j} : \mathbb{R}^n \to \mathbb{R}$ is a continuous function near $\mathbf{x}_0$, then we say $\mathbf{f}$ is strongly differentiable or class C^1 at $\mathbf{x}_0$. All C^1 functions are differentiable.

- Gradient
  - If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient vector function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\nabla f(\mathbf{x}) = (\mathbf{D}f(\mathbf{x}))^T = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{x})\right)$
  - $[\mathbf{D}f(\mathbf{x})]\mathbf{h} = \nabla f(\mathbf{x}) \cdot \mathbf{h}$

- Linearizations and Tangent Hyperplanes
  - For $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and a point $\mathbf{x}_0 \in \mathbb{R}^n$, let the linearization of $\mathbf{f}$ at $\mathbf{x}_0$ be $\mathbf{L}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + [\mathbf{Df}(\mathbf{x_0})](\mathbf{x}-\mathbf{x}_0)$. Note $\mathbf{f}(\mathbf{x}) \approx \mathbf{L}(\mathbf{x})$ whenever $\mathbf{x} \approx \mathbf{x}_0$.
  - (Example 5) Recall that the tangent plane to a surface $z = f(x,y)$ given by $f : \mathbb{R}^2 \to \mathbb{R}$ passing through $\mathbf{x}_0 \in \mathbb{R}^3$ is given by the normal vector ∇f. Show that $z = L(x,y)$ gives an equation for the tangent plane to the surface $z = x^2+y^4+e^{xy}$ at the point $(1,0,2)$.

- *HW: 1-3, 5-21*

2.5 Properties of the Derivative

- Sum/Product/Quotient Rules
  - $\mathbf{D}[\alpha\mathbf{f}] = \alpha\mathbf{Df}$
  - $\mathbf{D}[\mathbf{f}+\mathbf{g}] = \mathbf{Df}+\mathbf{Dg}$
  - $\mathbf{D}[fg] = g\mathbf{D}f + f\mathbf{D}g$
  - $\mathbf{D}[\frac{f}{g}] = \frac{g\mathbf{D}f - f\mathbf{D}g}{g^2}$
  - (Example) Prove the sum rule above.

- Chain Rule
  - $\mathbf{D}[\mathbf{f} \circ \mathbf{g}] = \mathbf{Df}(\mathbf{g})\mathbf{Dg}$
  - (Example) Find the rate of change of $f(x, y) = x^2 + y^2$ along the path $\mathbf{c}(t) = (t^2, t)$ when $t = 1$.
  - (Example 2) Verify the Chain Rule for $f(u, v, w) = u^2 + v^2 - w$ and $\mathbf{g}(x, y, z) = (x^2 y, y^2, e^{-xz})$.
  - (Example 3) Compute $\mathbf{D}[\mathbf{f} \circ \mathbf{g}](1, 1)$ where $\mathbf{f}(u, v) = (u + v, u, v^2)$ and $\mathbf{g}(x, y) = (x^2 + 1, y^2)$.
- *HW: 6-13, 15-16*

3.2 Taylor's Theorem

- Single-variable Taylor Series
  - $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$
    $= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f'(x_0)(x - x_0)^2 + \frac{1}{6}f'(x_0)(x - x_0)^3 + \dots$
  - $f(x) \approx \sum_{n=0}^{m} \frac{f^{(n)}(x_0)}{n!}(x - x_0)$
- First-Order Taylor Formula
  - $f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{x}_0) + [\mathbf{D}f(\mathbf{x}_0)](\mathbf{x} - \mathbf{x}_0) = f(\mathbf{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i - x_{0i})$
- Second-Order Taylor Formula
  - $f(\mathbf{x}) \approx f(\mathbf{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i - x_{0,i}) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)(x_i - x_{0,i})(x_j - x_{0,j})$
  - (Example) Use the second-order Taylor formula for $f(x, y) = \sqrt{x + 2y}$ near the point $(2, 1)$ to approximate $\sqrt{4.05}$.
  - (Example 3) Find linear and quadratic functions of x, y which approximate $f(x, y) = \sin(xy)$ near the point $(1, \pi/2)$.
- *HW: 3-7, 12*

4.3 Vector Fields

- Vector Fields

  - A vector field is a map $f : \mathbb{R}^n \to \mathbb{R}^n$ assinging an n-dimensional vector to each point in $\mathbb{R}^n$
  - (Example 1) The velocity field of a fluid may be modeled as a vector field.
  - (Example 2) Sketch the rotary motion given by the vector field $\mathbf{V}(x, y) = (-y, x)$.

- Gradient Vector Fields
  - $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$
  - (Example) The derivative of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$ in the direction given by a unit vector $\mathbf{v}$ is given by $\nabla f \cdot \mathbf{v}$. Show that the maximum value of a directional derivative for a fixed point is given by $\|\nabla f\|$ and attained by the direction $\frac{1}{\|\nabla f\|}\nabla f$.
  - (Example 4) If temperature is given by $T(x, y, z)$, then the energy or heat flux field is given by $\mathbf{J} = -k\nabla T$ where k is the conductivity of the body. Level sets are called isotherms.
  - (Example 5) The gravitational potential of bodies with mass m, M is given by $V = -\frac{mMG}{r}$ where G is the gravitational constant and r is the distance between the bodies, and the gravitational force field is given by $\mathbf{F} = -\nabla V$. Show that $\mathbf{F} = -\frac{mMG}{r^3}\mathbf{r}$, where $\mathbf{r}$ is the vector pointing from the center of mass M to the center of mass m.
  - A vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is conservative iff there exists a potential function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.
  - (Example) Show that $\mathbf{W} = (2y + 1, 2x)$ is conservative.
  - (Example 7) Show that $\mathbf{V} = (y, -x)$ is not conservative.

- Flow Lines
  - A flow line for a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a path $\mathbf{c} : \mathbb{R} \to \mathbb{R}^n$ satisfying $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$.
  - (Example 8) Show that $\mathbf{c}(t) = (\cos t, \sin t)$ is a flow line for $\mathbf{F} = (-y, x)$, and find some other flow lines.

- *HW: 1-12, 17-21*

4.4 Divergence and Curl

- Divergence
  - The divergence of a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is denoted by $\operatorname{div}\mathbf{F} : \mathbb{R}^n \to \mathbb{R}$ and defined by $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}$

  - (Examples 3-5) Compute the divergences of $\mathbf{F} = (x, y)$, $\mathbf{G} = (-x, -y)$ and $\mathbf{H} = (-y, x)$ at any point on $\mathbb{R}^2$. How does divergence correspond with the motion described by the vector field plots?
  - (Example) Compute the divergence of $\mathbf{F} = \left(x^2, y\right)$ various points and interpret those values against a plot of the vector field.

- Curl
  - The curl of a three-dimensional vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is denoted by $\operatorname{curl} \mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ and defined by $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$
  - The scalar curl of a two-dimensional vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is denoted by $\operatorname{scurl} \mathbf{F} : \mathbb{R}^2 \to \mathbb{R}$ and defined by $\operatorname{scurl} \mathbf{F} = \operatorname{curl} \mathbf{F} \cdot \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$
  - (Example) Compute the scalar curl of $\mathbf{F} = (x, y)$, $\mathbf{G} = (-x, -y)$ and $\mathbf{H} = (-y, x)$ at every point in $\mathbb{R}^2$. How does this scalar curl correspond with the motion described by the vector field plots?
  - (Example) Compute the curl of $\mathbf{F} = (y, -x, z)$ at every point in $\mathbb{R}^3$. How does curl correspond with the motion described by the vector field plot?

- Facts about ∇f, $\operatorname{div} \mathbf{F}$, $\operatorname{curl} \mathbf{F}$
  - The curl of a conservative field is zero: $\operatorname{curl} \nabla f = \nabla \times (\nabla f) = \mathbf{0}$.
  - (Example) Prove the above theorem.
  - (Example) Prove that $\mathbf{F} = \left(x^2 + z, y - z, z^3 + 3xy\right)$ is not a conservative field.
  - The divergence of a curl field is zero: $\operatorname{div} \operatorname{curl} \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$
  - Many identities on pg. 255 of Marsden text.
  - (Example) Sketch proof of identity #8: $\operatorname{div} (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$.

- *HW: 1-4, 9-17, 22-25, 29-30*

5.3 The Double Integral Over More General Regions

- Hypervolume
  - The hypervolume $HV_1(D)$ of an interval $D = [a, b]$ in $\mathbb{R}$ is just its length $b - a$.
  - The hypervolume of a well-behaved bounded subset $D \subseteq \mathbb{R}^{n+1}$ is defined for each $n \in \{1, 2, \ldots\}$ by

$$HV_{n+1}(D) = \int_{x_i \in I} HV(D_i)\, dx_i = \int_{x_i = a}^{x_i = b} HV_n(D_i)\, dx_i$$

where $I = [a, b]$ is an interval containing all values x_i included in the ith coordinate of D, and D_i is the projection of of all points in D onto $\mathbb{R}^n$ by removing the ith coordinate.

- (Example) For $n = 1$ and $D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, f(x) \leq y \leq g(x)\}$, we have that

$$HV_2 = A = \int_{x \in [a,b]} g(x) - f(x)\, dx = \int_a^b g(x) - f(x)\, dx.$$

- (Example) For $n = 2$ and $D \subseteq R^3$ including values of x between a and b, we have that

$$HV_3 = V = \int_{x=a}^{x=b} A(x)\, dx$$

  where $A(x)$ is the area of the cross-section of D taken by fixing each value of x (or similar for y).

- Double Integrals
  - For a bounded region $D \subseteq \mathbb{R}^2$ and continuous nonnegative $f : D \to \mathbb{R}$, the double integral

$$\iint_D f\, dA$$

    is defined to be the volume of $\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D, 0 \leq z \leq f(x, y)\}$.
  - We may apply the definition of volume above to get

$$\iint_D F\, dA = \int_{x=a}^{x=b} A(x)\, dx$$

    where D lies between the lines $x = a$ and $x = b$.
  - If D is described by $a \leq x \leq b$ and $\phi_1(x) \leq y \leq \phi_2(x)$, then

$$\iint_D F\, dA = \int_{x=a}^{x=b} A(x)\, dx = \int_{x=a}^{x=b} \left[\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x, y)\, dy \right] dx$$

  - Similarly, if D is described by $c \leq y \leq d$ and $\psi_1(y) \leq x \leq \psi_2(y)$, then

$$\iint_D F\, dA = \int_{y=c}^{y=d} \left[\int_{x=\psi_1(y)}^{x=\psi_2(y)} f(x, y)\, dx \right] dy$$

  - If f is sometimes negative on the domain D, then $\iint_D f\, dA$ is the net volume between $z = f(x, y)$ and D (volume above the xy plane minus volume below) and the above formulas still hold.

- Iterated integrals

  - An iterated integral is a shorthand for the expansion of two or more nested integrals, that is:

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx = \int_{x=a}^{x=b} \left[\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy\right] dx$$

  - (Example) Sketch the region of integration for $\int_0^\pi \int_{-x}^{x} \cos(y)\,dy\,dx$, evaluate it, and interpret it as the signed volume of a region in $\mathbb{R}^3$.
  - (Example) Express $\iint_R (12x^3y-1)\,dA$ where R is the rectangle with vertices $(0,0),(3,0),(3,2),(0,2)$ as an interated integral, then evaluate it.
  - (Example) Express $\iint_T (12x^3y-1)\,dA$ where T is the triangle with vertices $(0,0),(1,0),(1,1)$ as an interated integral, then evaluate it.

- Applications
  - $\iint_D 1\,dA$ is the area of D
  - $\frac{1}{A(D)} \iint_D f(x,y)\,dA$ is the average value of the function f restricted to D

- Additivity
  - If $D \subseteq \mathbb{R}^2$ is the union of two subregions D_1, D_2 overlapping only on their boundary, then $\iint_D f\,dV = \iint_{D_1} f\,dV + \iint_{D_2} f\,dV$.
  - (Example) Prove that the area of the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$ is two by setting it up as a double integral, then using additivity to split it up into two or more subregions.

- *HW: 1-9*

5.4 Changing the Order of Integration

- Rectangular regions of integration
  - For constant bounds of integration:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

  - (Example) Verify that $\int_0^1 \int_1^2 x^2+2xy\,dy\,dx = \int_1^2 \int_0^1 x^2+2xy\,dx\,dy$.

- Nonrectangular regions of integration
  - Bounds of integration cannot be directly swapped; however, by interpreting the region of integration new bounds may be found in the other order.

- (Example) Verify that $\int_0^4 \int_0^{\frac{4-y}{2}} x + y\,dx\,dy$ and $\int_0^2 \int_0^{4-2x} x + y\,dy\,dx$ share the same region of integration and are equal.
- (Example) Evaluate $\int_1^e \int_0^{\log x} \frac{(2x-e)\sqrt{1+e^y}}{e-e^y}\,dy\,dx$.

- Estimating double integrals
  - If $g(x,y) \leq f(x,y) \leq h(x,y)$ for $(x,y) \in D$, then $\iint_D g(x,y)\,dA \leq \iint_D f(x,y)\,dA \leq \iint_D h(x,y)\,dA$.
  - (Example 3) Prove that $\frac{1}{\sqrt{3}} \leq \iint_D \frac{1}{\sqrt{1+x^6+y^8}}\,dA \leq 1$ where D is the unit square.
  - (Example) Prove that $e \leq \iint_D e^{x^2y+y}\,dA \leq \frac{e^2}{2}$ where D is the unit square.
- *HW: 1-5, 7-10*

5.5 The Triple Integral

- Triple Integrals
  - For a bounded region $D \subseteq \mathbb{R}^3$ and nonnegative $f : D \to \mathbb{R}$, the triple integral

$$\iiint_D f\,dV$$

    is defined to be the hypervolume of $\{(x,y,z,w) \in \mathbb{R}^4 : (x,y,z) \in D, 0 \leq w \leq f(x,y,z)\}$.
- Applications
  - $\iiint_D 1\,dV$ is the volume of D
  - $\frac{1}{V(D)} \iiint_D f(x,y,z)\,dV$ is the average value of the function f restricted to D
  - If $\rho(x,y,z)$ gives the density of a solid at the coordinate (x,y,z), then $\iiint_D \rho(x,y,z)\,dV$ calculates its overall mass.
- Rectangular Boxes
  - If $B = [a_1,b_1] \times [a_2,b_2] \times [a_3,b_3]$, then

$$\begin{aligned} \iiint_B f\,dV &= \int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x,y,z)\,dx\,dy\,dz \\ &= \int_{a_2}^{b_2} \int_{a_1}^{b_1} \int_{a_3}^{b_3} f(x,y,z)\,dz\,dx\,dy \\ &= \text{etc.} \end{aligned}$$

  - (Example) Write $\iiint_D e^{x+y+z}\,dV$ where $D = [0,4] \times [0,2] \times [1,3]$ as a few different iterated integrals, then evaluate one.

- General regions of integration
  - If $E \subseteq \mathbb{R}^2$ and $D = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in E, \gamma_1(x, y) \leq z \leq \gamma_2(x, y)\}$, then

$$\iiint_D f(x,y,z)\,dV = \iint_E \left[\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dz\right] dA$$

    (and similar for x, y instead of z).
  - (Example 5) Express $\iiint_W x\,dV$ where W is the solid for which x, y, z are positive and $x^2 + y^2 \leq z \leq 2$ as a few different iterated integrals.
  - (Example 6) Express $\iiint_W x\,dV$ where W is the solid in $\mathbb{R}^3$ above the triangle with vertices $(0,0,0), (1,0,0), (1,1,0)$ in the xy plane, and also between the surfaces $z = x^2 + y^2$ and $z = 2$, as an iterated integral. Then evaluate it.
- Additivity
  - If $D \subseteq \mathbb{R}^3$ is the union of two subregions D_1, D_2 overlapping only on their boundary, then $\iiint_D f\,dV = \iiint_{D_1} f\,dV + \iiint_{D_2} f\,dV$.
- *HW: 1-6, 11-17, 25-28*

1.4 Cylindrical and Spherical Coordinates

- Transformation of variables
  - A transformation of variables is a function $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$.
  - (Example) Sketch the integer lattice on the uv plane and its image in the xy plane for the transformation of variables $\mathbf{T}(u, v) = (x, y) = (u, u + v)$.
- Polar Coordinates
  - $\mathbf{p}(r, \theta) = (r\cos\theta, r\sin\theta)$
  - $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$
  - (Example) Convert $A = \mathbf{p}(4, 2\pi/3)$ from polar to Cartesian. Convert $B = (3, -3)$ from Cartesian to polar. Plot both in the $r\theta$ and xy planes.
  - (Example) Express the curves $x = \sqrt{4 - y^2}$ and $y = 3$ in terms of polar coordinates. Plot both in the $r\theta$ and xy planes.
- Cylindrical Coordinates
  - $\mathbf{c}(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$
  - Usually, assume $r \geq 0$ and $0 \leq \theta \leq 2\pi$

- $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$
- (Example 1) Convert $A = \mathbf{c}(8, 2\pi/3, -3)$ from cylindrical to Cartesian. Convert $B = (6, 6, 8)$ from Cartesian to cylindrical. Plot both in xyz space.
- (Example) Express the surfaces $x^2 + y^2 = 9$ and $z^2 = x^2 + y^2$ in terms of cylindrical coordinates. Plot both in xyz space.

- Spherical Coordinates
  - $\mathbf{s}(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$
  - Usually, assume $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$
  - $\rho^2 = x^2 + y^2 + z^2$, $\tan\theta = \frac{y}{x}$, $\tan\phi = \frac{r}{z} = \frac{\sqrt{x^2+y^2}}{z}$
  - (Example 2) Convert $A = (1, -1, 1)$ from Cartesian to spherical. Convert $B = \mathbf{s}(3, \pi/6, \pi/4)$ from spherical to Cartesian. Convert $C = (2, -3, 6)$ from Cartesian to spherical. Convert $D = \mathbf{s}(1, -\pi/2, \pi/4)$ from spherical to Cartesian. Plot all four in xyz space.
  - (Example 3) Express the surfaces $xz = 1$ and $x^2 + y^2 - z^2 = 1$ in terms of spherical coordiantes.

- *HW: 1-12, 15-16*

6.1 The Geometry of Maps from $\mathbb{R}^n$ to $\mathbb{R}^n$

- Images of regions by transformations
  - (Example 1) Find the image of the rectangle $[0, 1] \times [0, 2\pi]$ in the $r\theta$ plane under the polar coordinate transformation $\mathbf{p}$.
  - (Example 2) Find the image of the square $[-1, 1]^2 = [-1, 1] \times [-1, 1]$ in the uv plane under the transformation $\mathbf{T}(u, v) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} (u, v)$

- One-to-one and Onto
  - A one-to-one transformation sends each point in the domain to a distinct point in the range.
  - An onto transformation sends something in the domain onto each point of the range.
  - (Example 3) Show that the polar coordinate transformation $\mathbf{p}$ is onto but not one-to-one.
  - (Example 4) Show that the transformation $\mathbf{T}$ from example 2 is both one-to-one and onto.

  - (Example 5) Show that $\mathbf{T}(u, v) = (u, 0)$ is neither one-to-one nor onto.
  - (Example 7) Find a rectangle in the $r\theta$ plane which maps onto the region $\{(x, y) : x, y \geq 0, a^2 \leq x^2 + y^2 \leq b^2\}$ in the Cartesian plane by the polar coordinate transformation.

- Linear transformations
  - Transformations $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathbf{T}(\mathbf{u}) = A\mathbf{u}$ for an n-dimensional matrix A are called linear transformations.
  - (Example 6) Find a region in the uv plane which maps onto the square with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$ in the xy plane by the linear transformation given in Example 2.
  - Transformations $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathbf{T}(\mathbf{u}) = A\mathbf{u} + \mathbf{x}_0$ for an n-dimensional matrix A and n-dimensional vector $\mathbf{x}_0$ are called affine transformations. (Every linear transformation is affine.)
  - (Example) Find an affine transformation which maps the unit square in the uv plane onto the square with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$ in the xy plane.
  - An affine transformation is both one-to-one and onto exactly when $\det A \neq 0$.
  - (Example) Use this fact to reinvestigate examples 4 and 5.
- *HW: 1-4, 8, 10*

6.2 The Change of Variables Theorem

- Affine transformations of areas
  - An affine transformation with matrix M transforms hypervolumes by a factor of $|\det M|$.
  - (Example) Verify this fact for the parallelogram with vertices $(2, 0), (3, 1), (1, 3), (0, 2)$ in the uv plane and its image in the xy plane under the transformation $\mathbf{T}(u, v) = (2u + v + 3, v - u - 2)$.
  - Put another way, $\iint_D 1\, dA = \iint_{D^*} |\det M|\, dA$.
- Affine transformations of single/double/triple integrals
  - (Example) Let $x = T(u) = mu + x_0$. Use substitution to prove that if the image of $[c_1, c_2]$ under T is $[b_1, b_2]$, then $\int_{b_1}^{b_2} f(x)\, dx = \int_{c_1}^{c_2} f(T(u))|m|\, du$.
  - (Example) Use the previous fact to show that $\int_0^4 \sqrt{2x + 1}\, dx = \int_1^9 \frac{1}{2}\sqrt{u}\, du$
  - For any 2D affine transformation $\mathbf{T}$ with matrix M transforming D^* to D, $\iint_D f(x, y)\, dA = \iint_{D^*} f(\mathbf{T}(u, v))|\det M|\, dA$.

  - (Example) Use an affine transformation to prove that $\int_0^2 \int_{y/2}^{(y+4)/2} 2y\,dx\,dy = \int_0^1 \int_0^1 16v\,dv\,du$ and compute both integrals directly to verify.
  - (Example) Compute $\iint_D (x+y)(x-y-2)\,dA$ where T is the triangle with vertices $(4,2)$, $(3,1)$, $(2,2)$.
  - For any 3D affine transformation $\mathbf{T}$ with matrix M transforming D^* to D, $\iint_D f(x,y,z)\,dV = \iint_{D^*} f(\mathbf{T}(u,v,w))|\det M|\,dV$.

- Jacobian

  - The Jacobian $\frac{\partial \mathbf{T}}{\partial \mathbf{u}}$ of a transformation is defined to be the determinant of its partial derivative matrix: $\det(\mathbf{DT})$.
  - (Example) Prove that for an affine transformation $\mathbf{T}$ with matrix M that $\mathbf{DT} = M$ and therefore $\frac{\partial \mathbf{T}}{\partial \mathbf{u}} = \det M$.
  - For any 2D transformation $\mathbf{T}$ transforming D^* to D, $\iint_D f(\mathbf{x})\,dA = \iint_{D^*} f(\mathbf{T}(\mathbf{u}))|\frac{\partial \mathbf{T}}{\partial \mathbf{u}}|\,dA$.
  - For any 3D transformation $\mathbf{T}$ transforming D^* to D, $\iiint_D f(\mathbf{x})\,dV = \iiint_{D^*} f(\mathbf{T}(\mathbf{u}))|\frac{\partial \mathbf{T}}{\partial \mathbf{u}}|\,dV$.
  - (Example) Use a 2D transformation to compute $\iint_D e^x \cos(\pi e^x)\,dA$ where D is the region bounded by $y = 0$, $y = e^x - 2$, $y = \frac{e^x - 1}{2}$.

- Polar, cylindrical, spherical change of variables

  - Polar coordinates: $\iint_D f(x,y)\,dA = \iint_{D^*} f(r\cos\theta, r\sin\theta) r\,dA$
  - Cylindrical coordinates: $\iint_D f(x,y,z)\,dV = \iint_{D^*} f(r\cos\theta, r\sin\theta, z) r\,dV$
  - Spherical coordinates: $\iint_D f(x,y,z)\,dV = \iint_{D^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,dV$
  - (Example 4) Evaluate $\iint_D \log(x^2+y^2)\,dA$ where D is the region in the first quadrant between the circles $x^2+y^2 = a^2$ and $x^2+y^2 = b^2$ for $0 < a < b$.
  - (Example 6) Evaluate $\iiint_W \exp[(x^2+y^2+z^3)^{3/2}]\,dV$ where W is unit ball centered at the origin.
  - (Example) Prove that the formula for the volume of a cone with radius R and height H is $V = \frac{1}{3}\pi R^2 H$.
  - (Example 7) Prove that the formula for the volume of a sphere with radius R is $V = \frac{4}{3}\pi R^3$.

- *HW: 1-6, 11, 13-14, 21, 26*

7.1 The Path Integral

- Path Integral with respect to Arclength

  - Recall that for a curve C defined by $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$, the arclength function $s : \mathbb{R} \to \mathbb{R}$ defined by $s(t) = \int_0^t \|\mathbf{r}'(\tau)\|\,d\tau$ gives the length of the curve from 0 to t.

- (Example) Prove that $C = \pi D$ gives the circumference of a circle with diameter D.
- If $f : \mathbb{R}^n \to \mathbb{R}$ is a function defined along the curve C defined by $\mathbf{r} : [a, b] \to \mathbb{R}^n$, then
$$\int_C f\, ds = \int_a^b f(\mathbf{r}(t)) \frac{ds}{dt}\, dt$$
where $\frac{ds}{dt} = \|\frac{d\mathbf{r}}{dt}\|$. This represents the area of a ribbon with base C and height f at each point of C.
- (Example 1) Find the average value of the function $f(x, y, z) = x^2 + y^2 + z^2$ along the portion of the helix given by $\mathbf{c}(t) = (\cos t, \sin t, t)$ for $t \in [0, 2\pi]$.
- (Example 2) The base of a fence is given by the curve $\mathbf{c}(t) = \left(30\cos^3 t, 30\sin^3 t\right)$, and the height of the fence is given by $f(x, y) = 1 + \frac{y}{3}$. How much paint is required to cover both sides of this fence?

- *HW: 1-8, 10-13*

7.2 Line Integrals

- Line Integral with respect to a Curve
  - If $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field defined along the curve C defined by $\mathbf{r} : [a, b] \to \mathbb{R}^n$, then
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{c}(t)) \frac{d\mathbf{r}}{dt}\, dt$$
represents the work done by a force $\mathbf{F}$ over the curve C.
  - (Example) An object is pushed around the unit circle with a force $(-y, x)$ at each point (x, y). Compute the work done in pushing the box around 3 full counter-clockwise rotations.
  - (Example 1) Let $\mathbf{r}(t) = (\sin t, \cos t, t)$ for $t \in [0, 2\pi]$ define the curve C, and define the vector field $\mathbf{F} = (x, y, z)$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.
  - (Example 5) Let C be a circle in the yz plane centered at the origin. Show that no work is done by a force $\mathbf{F} = \left(x^3, y, z\right)$ acting on an object moving around the circle.
- Line integrals with respect to variables
  - Note that for $\mathbf{r} = (x_1, \ldots, x_n)$,
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (F_1, \ldots, F_n) \cdot (dx_1, \ldots, dx_n) = \sum_{i=1}^n \int_C F_i \cdot dx_i$$

- If $f : \mathbb{R}^n \to \mathbb{R}$ is a function defined along the curve C defined by $\mathbf{r} : [a,b] \to \mathbb{R}^n$, then for $1 \leq i \leq n$
$$\int_C f\,dx_i = \int_a^b f(\mathbf{c}(t))\frac{dx_i}{dt}\,dt$$
- (Example) Compute $\int_C xy\,dy$ where C is the parabola defined by $\mathbf{c}(t) = \big(t, t^2, 1\big)$ for $t \in [0,1]$.
- (Example 2) Evaluate and interpret $\int_C x^2\,dx + xy\,dy + dz$ where C is the parabola defined by $\mathbf{c}(t) = \big(t, t^2, 1\big)$ for $t \in [0,1]$.

- Reparametrizations and partitions
  - The value of $\int_C f\,ds$ is independent of the choice of parametrization $\mathbf{r}(t)$ regardless of orientation.
  - The value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the choice of parametrization $\mathbf{r}(t)$ provided it respects the orientation of C.
  - If C and $-C$ represent the same curve with opposite orientations, then $\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \mathbf{F} \cdot d\mathbf{r}$.
  - If $C = C_1 + C_2$, then $\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$ and $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.
  - (Example 11) Compute $\int_C x^2\,dx + xy\,dy$ where C is the perimeter of the unit square oriented counter-clockwise.

- *HW: 1-5, 13, 17-18*