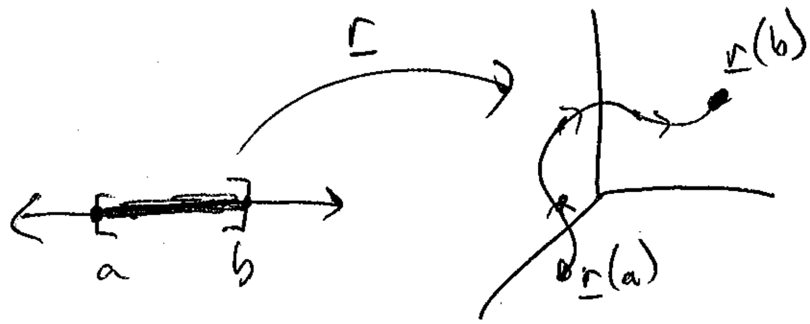
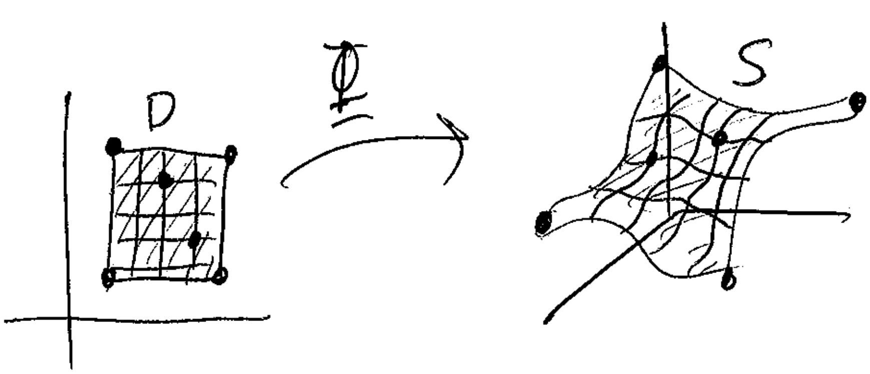
7.3 Parametrized Surfaces

Recall: A parameterization of a curve $C \subseteq \mathbb{R}^n$ is a continuous & 1-1 vector function $\underline{r} : [a,b] \to \mathbb{R}^n$ such that $\underline{r}(t)$ gives points on the curve C for $a \le t \le b$, $\underline{r}(a)$ is the starting point, $\underline{r}(b)$ is the ending point.

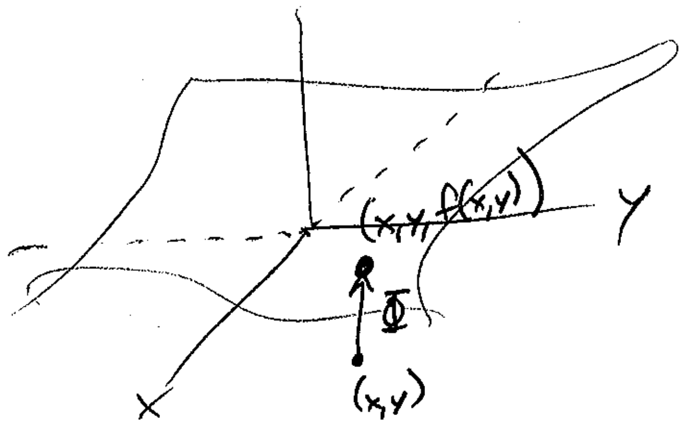


Definition

A parameterization of a surface $S \subseteq \mathbb{R}^n$ is a vector function $\underline{\Phi} : D \to \mathbb{R}^n$ such that $\underline{\Phi}(u,v)$ gives points on the surface S for each $(u,v) \in D$, where D is a region in $\mathbb{R}^2$.



(Example) Show that the surface given by $z = f(x,y)$ has the parameterization $\Phi(x,y) = (x, y, f(x,y))$.
OR $\Phi(u,v) = (u, v, f(u,v))$

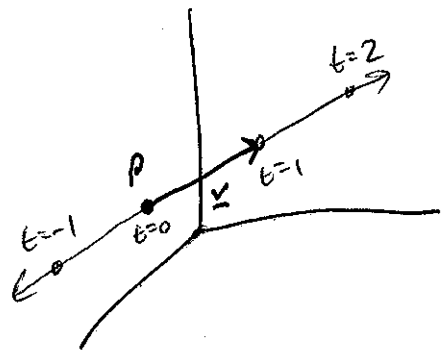


For each point (x,y,z) in the surface S, $z = f(x,y)$, so $(x,y,z) = (x,y,f(x,y)) = \Phi(x,y)$. $\square$

If we want the part of $z = f(x,y)$ where $x^2+y^2 \le 1$, then use $\Phi(x,y) = (x,y,f(x,y))$ with the same bounds:

Recall:

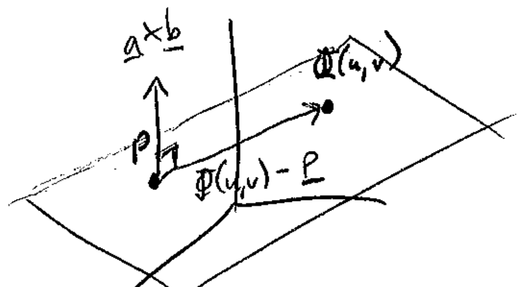
The line passing through $P \in \mathbb{R}^n$ and parallel to $\underline{v} \in \mathbb{R}^n$ has parameterization $\underline{r}(t) = \underline{P} + t\underline{v}$.



(Example 1) Show that the plane passing through $P \in \mathbb{R}^3$ and normal to the vector $\underline{a} \times \underline{b}$ has a parameterization

$$\Phi(u,v) = \underline{P} + u\underline{a} + v\underline{b}.$$

Note that $\underline{P} = \Phi(0,0)$.
Consider the point $\Phi(u,v)$ for a fixed $(u,v) \in \mathbb{R}^2$.



Then $\Phi(u,v) - \underline{P} = (\cancel{\underline{P}} + u\underline{a} + v\underline{b}) - (\cancel{\underline{P}}) = u\underline{a} + v\underline{b}$

So $(u\underline{a} + v\underline{b}) \cdot (\underline{a} \times \underline{b}) = u\underline{a} \cdot (\underline{a} \times \underline{b}) + v\underline{b} \cdot (\underline{a} \times \underline{b})$

$$= 0 + 0$$

$$= 0.$$

Thus $\underline{\Phi}(u,v)$ is on the plane orthogonal to $\underline{a} \times \underline{b}$.

(Example 2) Show that the cone $z = \sqrt{x^2+y^2}$ has a parameterization $\underline{\Phi}(r,\theta) = (r\cos\theta, r\sin\theta, r)$ for $r \geq 0$, $0 \leq \theta \leq 2\pi$.

Note ~~that $\underline{\Phi}(r,\theta) = \underline{c}(r,\theta,r)$~~

For any (r,θ), note that

$$z = r \quad \text{and} \quad \sqrt{x^2+y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$
$$= \sqrt{r^2}\,\cancel{\sqrt{\cos^2\theta + \sin^2\theta}}$$
$$= r.$$

Thus $z = \sqrt{x^2+y^2}$.

Surfaces with a simple cylindrical or spherical coordinate equation can usually be easily parameterizing using the coresponding transformation.

(Example) Parameterize the cone $z = \sqrt{x^2+y^2}$ using both ① cylindrical & ② spherical coordinates.

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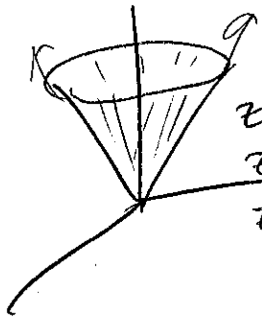
①



$z = \sqrt{x^2+y^2}$

$z = \sqrt{r^2}$

$z = r$ ← cylindrical equation

$\underline{\Phi}(r,\theta) = \underline{c}(r,\theta,r)$

$$\boxed{\underline{\Phi}(r,\theta) = (r\cos\theta, r\sin\theta, r) \qquad \begin{cases} r \geq 0 \\ 0 \leq \theta \leq 2\pi \end{cases}}$$

From transformation

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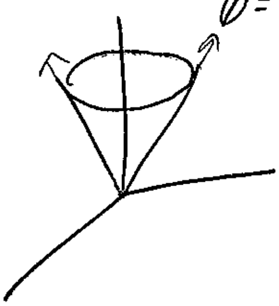
②



ϕ = constant

$z = \sqrt{x^2+y^2}$

$1 = \dfrac{\sqrt{x^2+y^2}}{z} = \tan\phi$

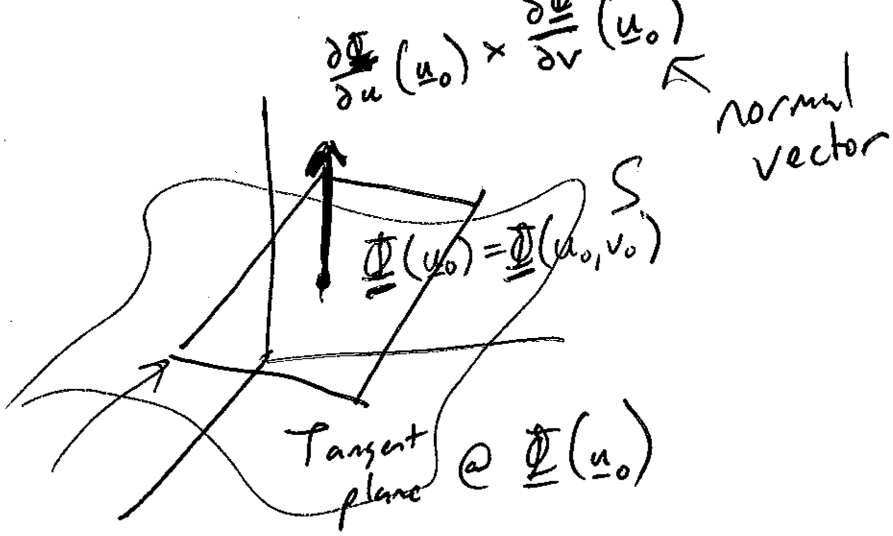
$\phi = \pi/4$

$\underline{\Phi}(\rho,\theta) = \underline{s}(\rho,\theta,\pi/4)$

$= (\rho \sin\pi/4 \cos\theta, \rho \sin\pi/4 \sin\theta, \rho\cos\pi/4)$

$$\boxed{\underline{\Phi}(\rho,\theta) = \left(\tfrac{\sqrt{2}}{2}\rho\cos\theta, \tfrac{\sqrt{2}}{2}\rho\sin\theta, \tfrac{\sqrt{2}}{2}\rho\right)} \quad \boxed{\begin{matrix}\rho \geq 0 \\ 0 \leq \theta \leq 2\pi\end{matrix}}$$

Suppose $\underline{\Phi}$ parameterizes the surface S:



This tangent plane has parameterization:

$$\underline{L}(\underline{u}) = \underline{L}(u,v) = \underline{\Phi}(u_0,v_0) + u\,\frac{\partial \underline{\Phi}}{\partial u}(\underline{u}_0) + v\,\frac{\partial \underline{\Phi}}{\partial v}(\underline{u}_0)$$

Recall the partial derivative matrix:

$$\underline{D}\,\underline{\Phi} = \begin{bmatrix} \frac{\partial \Phi_1}{\partial u} & \frac{\partial \Phi_1}{\partial v} \\ \frac{\partial \Phi_2}{\partial u} & \frac{\partial \Phi_2}{\partial v} \\ \frac{\partial \Phi_3}{\partial u} & \frac{\partial \Phi_3}{\partial v} \end{bmatrix}$$

$\frac{\partial \underline{\Phi}}{\partial u}$ $\qquad$ $\frac{\partial \underline{\Phi}}{\partial v}$

$$\underline{D}\,\underline{\Phi}(\underline{u}_0)\,\underline{u} = \begin{bmatrix} \frac{\partial \Phi_1}{\partial u} & \frac{\partial \Phi_1}{\partial v} \\ \frac{\partial \Phi_2}{\partial u} & \frac{\partial \Phi_2}{\partial v} \\ \frac{\partial \Phi_3}{\partial u} & \frac{\partial \Phi_3}{\partial v} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \Phi_1}{\partial u} u + \frac{\partial \Phi_1}{\partial v} v \\ \vdots \\ \vdots \end{bmatrix}$$

$$= u\,\frac{\partial \underline{\Phi}}{\partial u}(\underline{u}_0) + v\,\frac{\partial \underline{\Phi}}{\partial v}(\underline{u}_0)$$

So

$$\underbrace{\underline{L}(\underline{u}) = \underline{\Phi}(\underline{u}_0) + \underline{D}\,\underline{\Phi}(\underline{u}_0)\,\underline{u}}_{\text{Linearization of } \underline{\Phi}:\ \mathbb{R}^n \to \mathbb{R}^m}$$

(Example 3) Find a parameterization of the plane tangent to surface defined by

$$\underline{\Phi}(u,v) = (u\cos v,\ u\sin v,\ u^2+v^2)$$

at the point $(1,0,1)$.

$\cancel{\Phi(1,0,1)}$

Note

$$\underline{\Phi}(u,0) = (u,\ 0,\ u^2)$$

$$\underline{\Phi}\underbrace{(1,0)}_{\substack{\underline{u}_0 \\ (u_0,v_0)}} = (1,\ 0,\ 1)$$

Then

$$\frac{\partial \underline{\Phi}}{\partial u} = (\cos v,\ \sin v,\ 2u)$$

$$\frac{\partial \underline{\Phi}}{\partial u}(\underline{u}_0) = (\cos 0,\ \sin 0,\ 2(1))$$

$$= (1,\ 0,\ 2)$$

$$\frac{\partial \underline{\Phi}}{\partial v} = (-u\sin v,\ u\cos v,\ 2v)$$

$$\frac{\partial \underline{\Phi}}{\partial v}(\underline{u}_0) = (0,\ 1,\ 0)$$

Then

$$\underline{L}(u,v) = \underline{\Phi}(\underline{u}_0) + u \frac{\partial \underline{\Phi}}{\partial u}(\underline{u}_0) + v \frac{\partial \underline{\Phi}}{\partial v}(\underline{u}_0)$$

$$= (1,0,1) + u(1,0,2) + v(0,1,0)$$

$$\boxed{\underline{L}(u,v) = (1+u,\ v,\ 1+2u)}$$

(Example) Give the above tangent plane in terms of x, y, z.

The normal vector to the plane is

$$\frac{\partial \underline{\Phi}}{\partial u}(\underline{u}_0) \times \frac{\partial \underline{\Phi}}{\partial v}(\underline{u}_0) = (1,0,2) \times (0,1,0)$$

$$= \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= (-2, 0, 1)$$

The equation of a plane is

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

normal vector $(-2,0,1)$

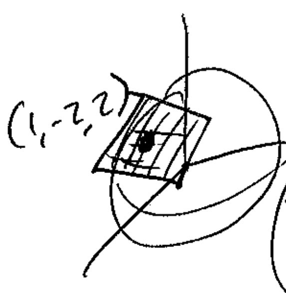
point on plane $\underline{\Phi}(\underline{u}_0) = (1,0,1)$

$$-2(x-1)+0(y-0)+1(z-1)=0$$

$$-2x+2+z-1=0$$

$$\boxed{-2x+z=-1}$$

(Example) Find a parameterization for the sphere centered at the origin with radius 3. ~~Then describe the plane tangent to it at the point (1,-2,2).~~



$\rho=3$

$\left(\begin{array}{c} x^2+y^2+z^2=9 \\ \rho^2=9 \end{array}\right)$

$$\Phi(\phi,\theta)=\underline{s}(3,\theta,\phi)$$
$$=(3\sin\phi\cos\theta,\ 3\sin\phi\sin\theta,\ 3\cos\phi)$$
$$0\le\phi\le\pi$$
$$0\le\theta\le 2\pi$$