Quiz ~~1~~ 3

Use

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 = \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \sum_{1\le i<j\le n}\left(x_i y_j - x_j y_i\right)^2$$

to prove $|\underline{x}\cdot\underline{y}| \le \|\underline{x}\|\|\underline{y}\|$.

Same as $(\underline{x}\cdot\underline{y})^2 \le \|\underline{x}\|^2\|\underline{y}\|^2$

To prove this...

$$(\underline{x}\cdot\underline{y})^2 = \left(\sum_{i=1}^{n} x_i y_i\right)^2$$

non-negative

$$= \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \sum_{\sim}\left(\sim\right)^2$$

$$\le \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

$$= (\underline{x}\cdot\underline{x})(\underline{y}\cdot\underline{y})$$

$$= \|\underline{x}\|^2\|\underline{y}\|^2. \quad \square$$

$$= \|\underline{x}\|^2 \|\underline{y}\|^2 - \sum_{\sim} (\sim)^2$$

$$\leq \|\underline{x}\|^2 \|\underline{y}\|^2 - 0 = \|\underline{x}\|^2 \|\underline{y}\|^2 \quad \square$$

5.4 (4c) Evaluate

$$\int_0^4 \left[\int_{y/2}^{2} e^{x^2}\,dx \right] dy .$$

Right curve: 2

Left curve: $y/2$

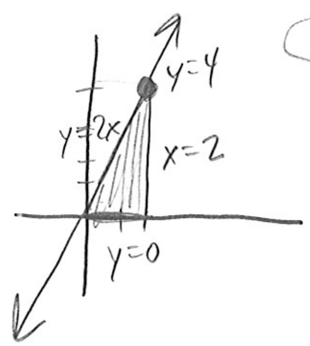
easier w/ resp. to y

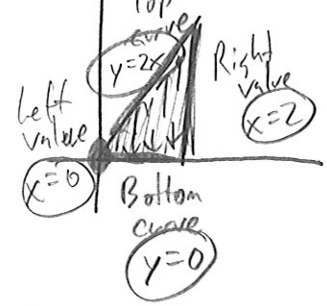
$$\cancel{= \int_{y/2}^{2} \int_0^4 e^{x^2}\,dy\,dx}$$

Can't just swap !!

$x = \frac{y}{2}$

$y = 2x$





$$= \int_0^2 \left[\int_0^{2x} e^{x^2}\,dy \right] dx = \int_0^2 \Big[e^{x^2} y \Big]_{y=0}^{y=2x} dx$$

$$= \int_0^2 \left(2x e^{x^2} - \cancel{0} \right) dx = \int_0^2 2x\, e^{x^2}\, dx$$

Let $u = x^2$

$du = 2x\,dx$

$$= \int_{x=0}^{x=2} e^{u} \, du = \left[e^{u} \right]_{x=0}^{x=2} = \left[e^{x^2} \right]_0^2$$

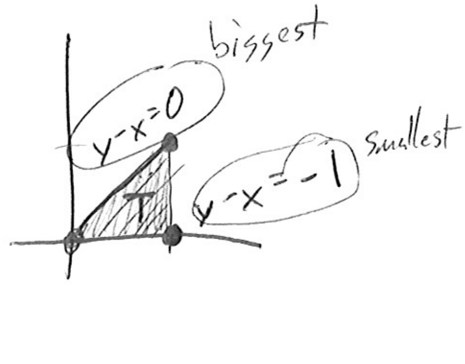
$$= e^4 - e^0 = \boxed{e^4 - 1}$$

5.4 8 Hint $\dfrac{\sin x}{1+(1)^4} \leq \dfrac{\sin x}{1+(xy)^4} \leq \dfrac{1}{1+(0)^4}$

for $(x,y) \in [0,1] \times [0,1]$

10 Hint $\dfrac{1}{3} = \dfrac{1}{(0)+3} \leq \dfrac{1}{(y-x)+3} \leq \dfrac{1}{(-1)+3} = \dfrac{1}{2}$

for $(x,y) \in T$



4.4 (23) Find div $\underline{F}$ and curl $\underline{F}$ for

$$\underline{F} = (\underbrace{e^{xz}}_{F_1}, \underbrace{\sin(xy)}_{F_2}, \underbrace{x^5y^3z^2}_{F_3}).$$

$$\begin{aligned} \text{div } \underline{F} &= \nabla \cdot \underline{F} \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= (ze^{xz}) + (x\cos(xy)) + x^5y^3(2z) \\ &= \boxed{ze^{xz} + x\cos(xy) + 2x^5y^3z} \end{aligned}$$

$$\begin{aligned} \text{curl } \underline{F} &= \nabla \times \underline{F} \\ &= \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{pmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \end{aligned}$$

$$= \boxed{\left((3x^5y^2z^2) - \cancel{(0)}, xe^{xz} - 5x^4y^3z^2, y\cos(xy) - \cancel{0} \right)}$$

4.4

22a Which of the vector fields in Exer. 13-16 could be gradient fields?

a.k.a. conservative

a.k.a. $\underline{F} = \nabla f$

FACT: $\underline{F}$ is conservative if and only if curl $\underline{F} = \underline{0}$.

(Why? curl$(\nabla f) = \nabla \times \nabla f = \underline{0}$)

So if our solutions were...

13 $\underline{F} = (x, y, z) \Rightarrow$ curl $\underline{F} = \underline{0}$

So $\underline{F}$ is conservative.

15 $\underline{F} = (x^2+y^2+z^2)(3\hat{i}+4\hat{j}+5\hat{k})$

$= (x^2+y^2+z^2)(3, 4, 5)$

$= (3x^2+3y^2+3z^2, 4x^2+4y^2+4z^2, 5x^2+5y^2+5z^2)$

$\Rightarrow$ curl $\underline{F} = (10y - 8z, 6z - 10x, 8x - 6y) \neq \underline{0}$

So $\underline{F}$ is not conservative.

4.3 21a Let $\underline{F} = (yz, xz, xy)$. Find $f: \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla f = \underline{F}$.

f is a potential function for $\underline{F}$.

$\frac{\partial f}{\partial x} = yz + 0$ $\quad$ $\frac{\partial f}{\partial y} = xz$ $\quad$ $\frac{\partial f}{\partial z} = xy$

$f = xyz + \Phi$ (any y or z stuff) $\quad$ $f = xyz + \Phi$ (x or z) $\quad$ $f = xyz + \Phi$ (x or y)

$$\boxed{f = xyz}$$

(Similar Example)

Find a potential function for

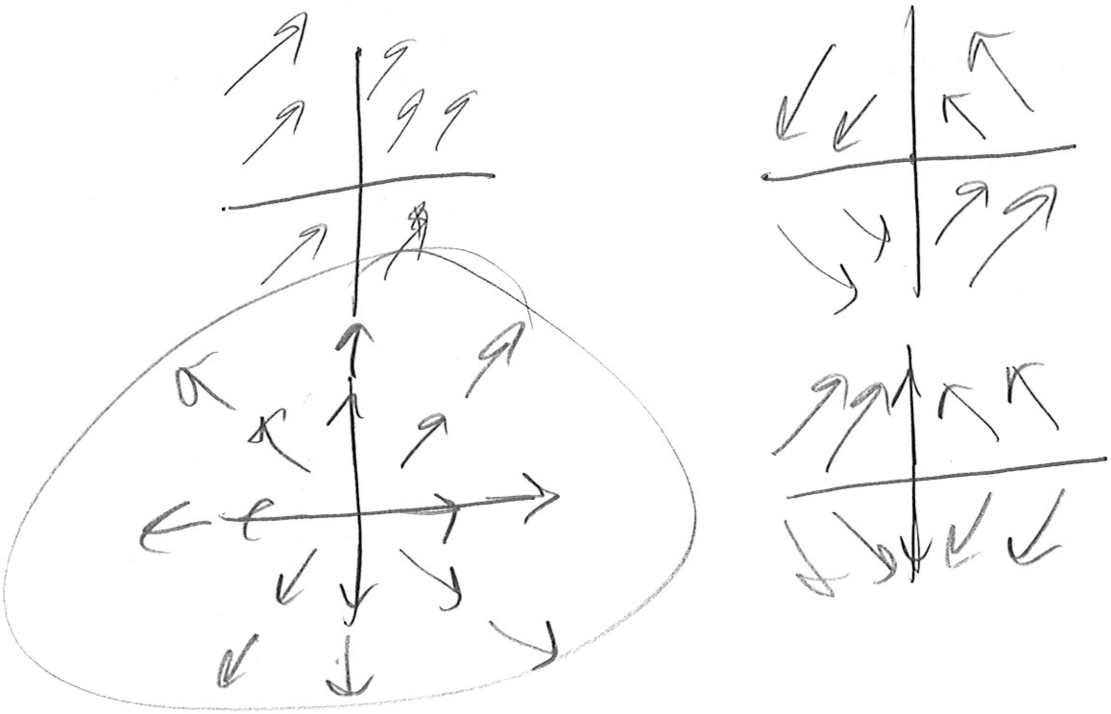
$$\underline{F} = \left(2xy + 3yz,\ x^2 + 3xz + 2e^y,\ 3xy\right)$$

$$\frac{\partial f}{\partial x} = 2xy + 3yz \qquad \frac{\partial f}{\partial y} = x^2 + 3xz + 2e^y \qquad \frac{\partial f}{\partial z} = 3xy$$

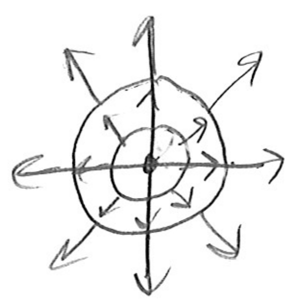
$$f = x^2y + 3xyz + \underset{2e^y}{\cancel{C}} \qquad f = x^2y + 3xyz + 2e^y + \underset{0}{\cancel{C}} \qquad f = 3xyz + \underset{x^2y + 2e^y}{\cancel{C}}$$

$$\boxed{f = x^2y + 3xyz + 2e^y}$$

Ex: Which of these is a plot of ∇f where $f(x,y) = x^2 + y^2$?



Option 1:



c	$f = c$
0	$x^2+y^2=0$
1	$x^2+y^2=1$
4	$x^2+y^2=4$

Option 2:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$
$$= (2x, 2y)$$