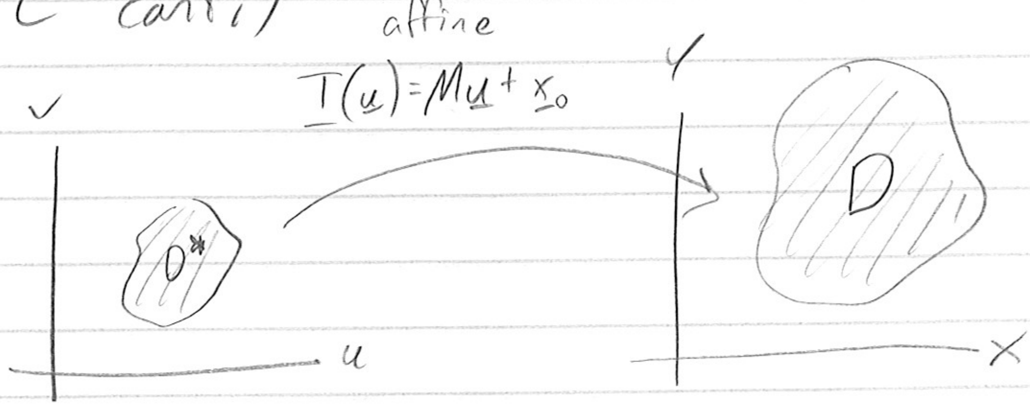
(6.2 cont.)



ratio of areas: $|\det M|$

Put another way:

$$(\text{Area of } D) = \iint_D 1\, dA = \iint_{D^*} |\det M|\, dA = |\det M|\,(\text{Area of } D^*)$$

(Example) Let $x = T(u) = mu + x_0$, $m > 0$.
Use substitution to prove that if the image of $[c_1, c_2]$ under T is $[b_1, b_2]$, then

$$\int_{b_1}^{b_2} f(x)\, dx = \int_{c_1}^{c_2} f(T(u))\,|m|\, du.$$

Since $x = mu + x_0$

$$\frac{dx}{du} = m$$

$$dx = m\, du$$

we have

$$\int_{b_1}^{b_2} f(x)\, dx = \int_{x=b_1}^{x=b_2} f(mu + x_0)\, m\, du$$

$$= \int_{c_1}^{c_2} f(T(u))\,|m|\, du \checkmark$$

(Example) Use the previous example to prove that

$$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \tfrac{1}{2}\sqrt{u}\,du$$

(Cal 1 way:

$$= \int_{x=0}^{x=4} \sqrt{u}\ \tfrac{1}{2}\,du$$

$u = 2x+1$

$du = 2dx$

$\tfrac{1}{2}du = dx$

$$= \int_{u=1}^{u=9} \tfrac{1}{2}\sqrt{u}\,du \checkmark \;)$$

We want $u = 2x+1$, so solve for x:

$$2x = u - 1$$

$$x = \underbrace{\tfrac{1}{2}}_{m}u - \underbrace{\tfrac{1}{2}}_{x_0} = T(u)$$

Since $T(u)$ transforms $\underbrace{[1,9]}_{u\text{-vals}}$ to $\underbrace{[0,4]}_{x\text{-vals}}$,

the theorem gives us:

$$\int_0^4 \underbrace{\sqrt{2x+1}}_{f(x)}\,dx = \int_1^9 \underbrace{\sqrt{2(\tfrac{1}{2}u-\tfrac{1}{2})+1}}_{f(T(u))}\ \Big|\underset{\substack{\uparrow\\ m}}{\tfrac{1}{2}}\Big|\,du$$

$$= \int_1^9 \tfrac{1}{2}\sqrt{u\cancel{-1+1}}\,du = \int_1^9 \tfrac{1}{2}\sqrt{u}\,du \checkmark$$

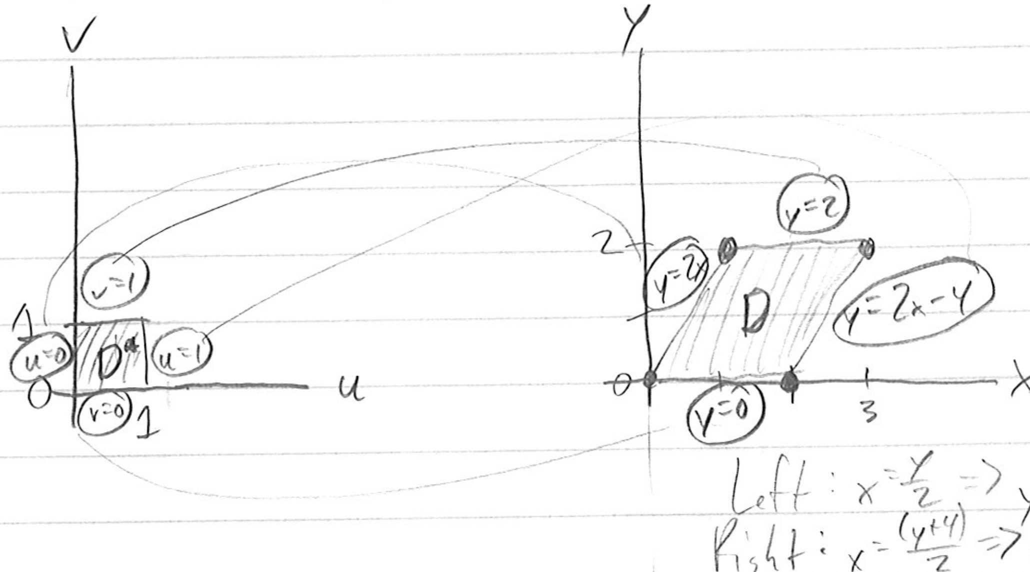
Theorem Let $\underline{T}(\underline{u}) = M\underline{u} + \underline{x}_0$ be an affine transformation mapping D^* in the uv plane to D in the xy plane, then

$$\iint_D f(\underline{x})\, dA = \iint_{D^*} f(\underline{T}(\underline{u}))\, |\det M|\, dA$$

(Example) Prove that $\int_0^2 \int_{y/2}^{(y+4)/2} 2y\, dx\, dy = \int_0^1 \int_0^1 16v\, dv\, du$

by the theorem & by computation.

Let D be the region of integration for $\int_0^2 \int_{y/2}^{(y+4)/2} 2y\,dx\,dy$:



Left: $x = \frac{y}{2} \Rightarrow y = 2x$
Right: $x = \frac{(y+4)}{2} \Rightarrow y = 2x - 4$

For $u = 0 \Rightarrow y = 2x + 0$
$u = 1 \Rightarrow y = 2x - 4$ $\quad -4$
For $u \in [0,1] \Rightarrow y = 2x - \underline{4}u$

For $v = 0 \Rightarrow y = 0$
$v = 1 \Rightarrow y = 2$ $\quad +2$
$v \in [0,1] \Rightarrow y = \underline{2}v$

So by setting y's equal...

$$2x - 4u = 2v$$
$$2x = 4u + 2v$$
$$x = 2u + v$$

And plugging in for x...

$$y = 2(2u+v) - 4u$$
$$y = \cancel{4u} + 2v - \cancel{4u}$$
$$y = 2v$$

← Or could have seen that from earlier

So let $\underline{T}(u,v) = (2u+v, 2v)$

$$= \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}}_{M} \begin{bmatrix} u \\ v \end{bmatrix} \cancel{+ \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

Then $\det M = \det \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = 4 - 0 = 4$

Therefore

$$\iint_D 2y \, dA = \iint_{D^*} \big(2(\underset{\uparrow\, y=2v}{2v})\big) \, \underset{\uparrow\, \det M}{|4|} \, dA$$

$$= \iint_D 16v \, dA$$

$$= \int_0^1 \int_0^1 16v \, dv\,du$$

To verify by computation...

$$\int_0^1 \left[\int_0^1 16v \, dv \right] du = \int_0^1 \left[8v^2 \right]_0^1 du$$

$$= \int_0^1 8 \, du$$

$$= 8$$

$$\int_0^2 \left[\int_{\frac{y}{2}}^{\frac{(y+4)}{2}} 2y \, dx \right] dy = \int_0^2 \left[2xy \right]_{y/2}^{(y+4)/2} dy$$

$$= \int_0^2 (\cancel{y}+4)y - \cancel{y^2} \, dy$$

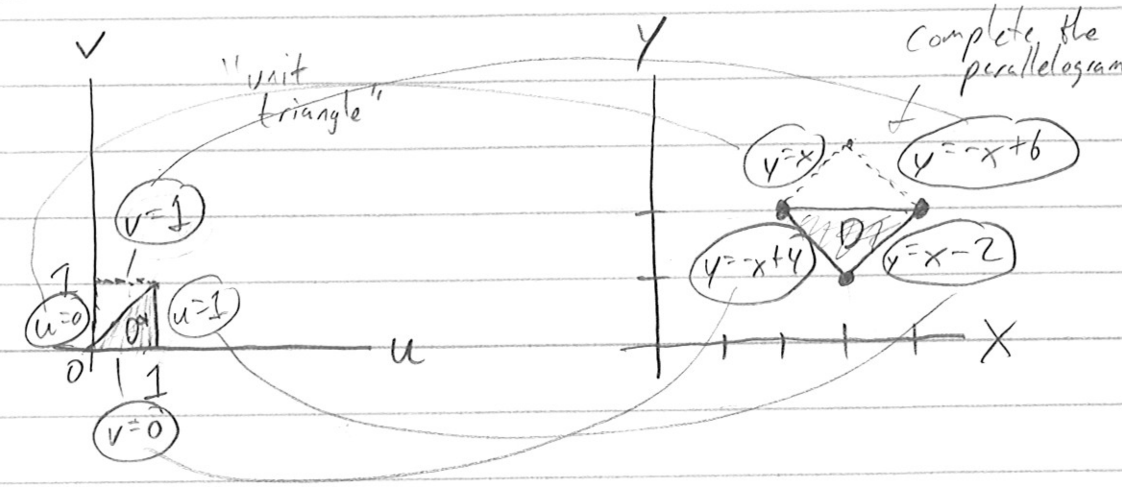
$$= \int_0^2 4y \, dy$$

$$= \left[2y^2 \right]_0^2$$

$$= 8 - 0 = 8$$

(Example) Compute $\iint_D (x+y)(x-y-2)\,dA$ where

D is the triangle with verts $(4,2)$ $(3,1)$ $(2,2)$.



$u=0 \Rightarrow y=x$

$u=1 \Rightarrow y=x-2$

$v=0 \Rightarrow y=-x+4$

$v=1 \Rightarrow y=-x+6$

$+2$

$y=x-2u$

$y=-x+4+2v$

$$x-2u=-x+4+2v$$
$$2x=2u+2v+4$$
$$x=u+v+2$$

$$y=(u+v+2)-2u$$
$$y=-u+v+2$$

$$T(u,v)=(u+v+2,\,-u+v+2)=\underbrace{\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}}_{M}\begin{bmatrix}u\\ v\end{bmatrix}+\begin{bmatrix}2\\ 2\end{bmatrix}$$

$$\det M = \det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 1-(-1) = 2$$

So

$$\iint_D (x+y)(x-y-2)\, dA$$

$$= \iint_{D^*} \Big((u+v+2)+(-u+v+2)\Big)\Big((u+v+2)-(-u+v+2)-2\Big)\,|2|\, dA$$

$$= \iint_{D^*} (2v+4)(2u-2)(2)\, dA$$

$$= \iint_{D^*} 8uv - 8v + 16u - 16 \; dA$$

$$= \underbrace{\int_0^1 \int_0^u}_{\text{Always for unit triangle}} \Big[8uv - 8v + 16u - 16 \; dv \Big] du$$

$$= \int_0^1 \Big[4uv^2 - 4v^2 + 16uv - 16v \Big]_0^u du$$

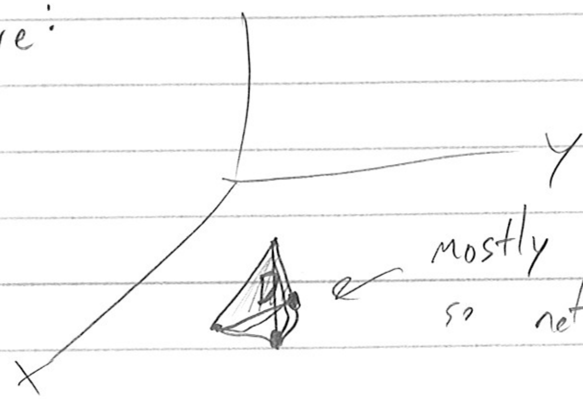
$$= \int_0^1 4u^3 - 4u^2 + 16u^2 - 16u \; du$$

$$= \int_0^1 4u^3 + 12u^2 - 16u \; du$$

$$= \Big[u^4 + 4u^3 - 8u^2 \Big]_0^1 = 1 + 4 - 8 = \boxed{-3}$$

(Note that, this time, I could have just found
$$\int_1^2 \int_{4-y}^{y+2} (x+y)(x-y-2)\,dx\,dy.$$)

Picture:



mostly below xy plane, so net volume is negative

It's also true in 3D:

$$\iiint_D f(x,y,z)\,dV = \iiint_{D^*} f(\underline{T}(u,v,w))\,|\det M|\,dV$$

where $\underline{T}(\underline{u}) = M\underline{u} + \underline{x}_0$ sends D^* to D.

The Jacobian $\frac{\partial T}{\partial \underline{u}} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$ of a transformation $\underline{T}: \mathbb{R}^n \to \mathbb{R}^n$ is defined to be the determinant of its partial derivative matrix:

$$\frac{\partial \underline{T}}{\partial \underline{u}} = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

(In $\mathbb{R}^3$)

(assuming $\underline{T}(\underline{u}) = (x(\underline{u}), y(\underline{u}), z(\underline{u}))$)

(Example) Prove that for $\underline{T}(\underline{u}) = M\underline{u} + \underline{x}_0$, then $\partial T / \partial \underline{u} = \det M$.

We need to show that

$$M = \underline{D}\,\underline{T} = \begin{bmatrix} \frac{\partial T_1}{\partial u_1} & \cdots & \frac{\partial T_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial T_n}{\partial u_1} & \cdots & \frac{\partial T_n}{\partial u_n} \end{bmatrix}$$

Consider the (i,j) coordinate of M: m_{ij}.

Then $T_i(\underline{u}) = M_i \cdot \underline{u} + (\underline{x}_0)_i$

$$= \cancel{m_{i1}u_1} + \cdots + \cancel{m_{in}u_n} + \cancel{(\underline{x}_0)_i}$$

$$\partial / \partial u_j \, T_i(\underline{u}) = m_{ij}$$

$$\partial T_i / \partial u_j = m_{ij}$$

Since $M = \underline{D}\,\underline{T}$, we have that

$$\det M = \det \underline{D}\,\underline{T} = \partial \underline{T} / \partial \underline{u}. \quad \square$$

- (Example) Use an affine transformation to prove that $\int_0^2 \int_{y/2}^{(y+4)/2} 2y\, dx\, dy = \int_0^1 \int_0^1 16v\, dv\, du$ and compute both integrals directly to verify.
- (Example) Compute $\iint_D (x+y)(x-y-2)\, dA$ where T is the triangle with vertices $(4,2)$, $(3,1)$, $(2,2)$.
- For any 3D affine transformation $\mathbf{T}$ with matrix M transforming D^* to D, $\iint_D f(x,y,z)\, dV = \iint_{D^*} f(\mathbf{T}(u,v,w))|\det M|\, dV$.

- Jacobian
  - The Jacobian $\frac{\partial \mathbf{T}}{\partial \mathbf{u}}$ of a transformation is defined to be the determinant of its partial derivative matrix: $\det(\mathbf{DT})$.
  - (Example) Prove that for an affine transformation $\mathbf{T}$ with matrix M that $\mathbf{DT} = M$ and therefore $\frac{\partial \mathbf{T}}{\partial \mathbf{u}} = \det M$.
  - For any 2D transformation $\mathbf{T}$ transforming D^* to D, $\iint_D f(\mathbf{x})\, dA = \iint_{D^*} f(\mathbf{T}(\mathbf{u}))|\frac{\partial \mathbf{T}}{\partial \mathbf{u}}|\, dA$.
  - For any 3D transformation $\mathbf{T}$ transforming D^* to D, $\iiint_D f(\mathbf{x})\, dV = \iiint_{D^*} f(\mathbf{T}(\mathbf{u}))|\frac{\partial \mathbf{T}}{\partial \mathbf{u}}|\, dV$.
  - (Example) Use a 2D transformation to compute $\iint_D e^x \cos(\pi e^x)\, dA$ where D is the region bounded by $y = 0$, $y = e^x - 2$, $y = \frac{e^x - 1}{2}$.

- Polar, cylindrical, spherical change of variables
  - Polar coordinates: $\iint_D f(x,y)\, dA = \iint_{D^*} f(r\cos\theta, r\sin\theta) r\, dA$
  - Cylindrical coordinates: $\iint_D f(x,y,z)\, dV = \iint_{D^*} f(r\cos\theta, r\sin\theta, z) r\, dV$
  - Spherical coordinates: $\iint_D f(x,y,z)\, dV = \iint_{D^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\, dV$
  - (Example 4) Evaluate $\iint_D \log(x^2 + y^2)\, dA$ where D is the region in the first quadrant between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ for $0 < a < b$.
  - (Example 6) Evaluate $\iiint_W \exp[(x^2+y^2+z^3)^{3/2}]\, dV$ where W is unit ball centered at the origin.
  - (Example) Prove that the formula for the volume of a cone with radius R and height H is $V = \frac{1}{3}\pi R^2 H$.
  - (Example 7) Prove that the formula for the volume of a sphere with radius R is $V = \frac{4}{3}\pi R^3$.

- *HW: 1-6, 11, 13-14, 21, 26*

7.1 The Path Integral

- Path Integral with respect to Arclength
  - Recall that for a curve C defined by $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$, the arclength function $s : \mathbb{R} \to \mathbb{R}$ defined by $s(t) = \int_0^t \|\mathbf{r}'(\tau)\|\, d\tau$ gives the length of the curve from 0 to t.

- (Example 5) Show that $\mathbf{T}(u, v) = (u, 0)$ is neither one-to-one nor onto.
- (Example 7) Find a rectangle in the $r\theta$ plane which maps onto the region $\{(x, y) : x, y \geq 0, a^2 \leq x^2 + y^2 \leq b^2\}$ in the Cartesian plane by the polar coordinate transformation.

- Linear transformations
  - Transformations $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathbf{T}(\mathbf{u}) = A\mathbf{u}$ for an n-dimensional matrix A are called linear transformations.
  - (Example 6) Find a region in the uv plane which maps onto the square with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$ in the xy plane by the linear transformation given in Example 2.
  - Transformations $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathbf{T}(\mathbf{u}) = A\mathbf{u} + \mathbf{x}_0$ for an n-dimensional matrix A and n-dimensional vector $\mathbf{x}_0$ are called affine transformations. (Every linear transformation is affine.)
  - (Example) Find an affine transformation which maps the unit square in the uv plane onto the square with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$ in the xy plane.
  - An affine transformation is both one-to-one and onto exactly when $\det A \neq 0$.
  - (Example) Use this fact to reinvestigate examples 4 and 5.

- *HW: 1-4, 8, 10*

6.2 The Change of Variables Theorem

- Affine transformations of areas
  - An affine transformation with matrix M transforms hypervolumes by a factor of $|\det M|$.
  - (Example) Verify this fact for the parallelogram with vertices $(2, 0), (3, 1), (1, 3), (0, 2)$ in the uv plane and its image in the xy plane under the transformation $\mathbf{T}(u, v) = (2u + v + 3, v - u - 2)$.
  - Put another way, $\iint_D 1\, dA = \iint_{D^*} |\det M|\, dA$.

- Affine transformations of single/double/triple integrals
  - (Example) Let $x = T(u) = mu + x_0$. Use substitution to prove that if the image of $[c_1, c_2]$ under T is $[b_1, b_2]$, then $\int_{b_1}^{b_2} f(x)\, dx = \int_{c_1}^{c_2} f(T(u))|m|\, du$.
  - (Example) Use the previous fact to show that $\int_0^4 \sqrt{2x+1}\, dx = \int_1^9 \frac{1}{2}\sqrt{u}\, du$
  - For any 2D affine transformation $\mathbf{T}$ with matrix M transforming D^* to D, $\iint_D f(x, y)\, dA = \iint_{D^*} f(\mathbf{T}(u, v))|\det M|\, dA$.