MATH 2242 - Fall 2015 - Dr. Clontz - Test 1

Name: Answers Section: ____________

- This test is worth 250 points toward your overall grade. Each problem is labeled with its value toward this total.
- On multiple choice problems, you do not need to show your work. No partial credit will be given.
- On full response problems, show all of your work and give a complete solution. When in doubt, don't skip any steps. Partial credit will be given at the discretion of the instructor.
- This exam is open notes, provided that these notes are completely in your own handwriting. The professor may take up notes you use with your test and return them after the test is graded.
- Tests submitted after the end of 70 minutes will be deducted 25 points, with 25 more points deducted every following minute.

Multiple Choice (100 points total)

1. (20 points) Evaluate AB given

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 0 & 1 \\ -1 & 4 \end{bmatrix}$$

- ○ $\begin{bmatrix} 0 & -11 \\ 4 & 2 \end{bmatrix}$
- ● $\begin{bmatrix} 3 & 1 \\ -13 & 19 \end{bmatrix}$
- ○ $\begin{bmatrix} 3 & 0 & 0 \\ 3 & 0 & 8 \end{bmatrix}$
- ○ $\begin{bmatrix} -1 & 2 & 0 \\ -9 & 0 & -4 \end{bmatrix}$
- ○ None of these

2. (20 points) Compute $\frac{\partial f}{\partial y}$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is defined by $f(x, y, z) = e^{xy} - 2yz^3$.

- ● $\frac{\partial f}{\partial y} = xe^{xy} - 2z^3$
- ○ $\frac{\partial f}{\partial y} = xye^{xy} - 6yz^2$
- ○ $\frac{\partial f}{\partial y} = \log(xy) - 2yz^2$
- ○ $\frac{\partial f}{\partial y} = e^x + ye^y - 6z^2$
- ○ None of these

3. (20 points) Let $\mathbf{f}(x,y) = \langle x^2, y-x\rangle$ and $\mathbf{g}(r,s,t) = \langle rs, e^t\rangle$. Which of these is equal to $\mathbf{D}[\mathbf{f}\circ\mathbf{g}]$?

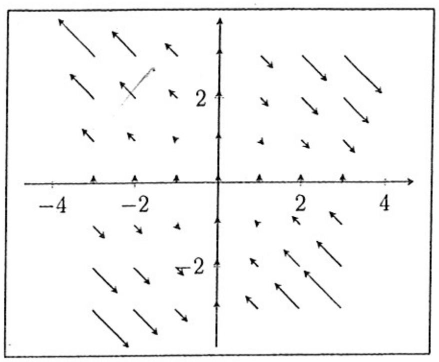
- ○ $\begin{bmatrix} 1 & 2rs \\ -1 & 0 \end{bmatrix}\begin{bmatrix} s & 0 & e^t \\ r & 0 & 0 \end{bmatrix}$
- ○ $\begin{bmatrix} 0 & 2rs \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & e^t \\ s & r & 0 \end{bmatrix}$
- ● $\begin{bmatrix} 2rs & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} s & r & 0 \\ 0 & 0 & e^t \end{bmatrix}$
- ○ $\begin{bmatrix} -1 & 1 \\ 2rs & 0 \end{bmatrix}\begin{bmatrix} r & s & 0 \\ e^t & 0 & 0 \end{bmatrix}$
- ○ None of these.

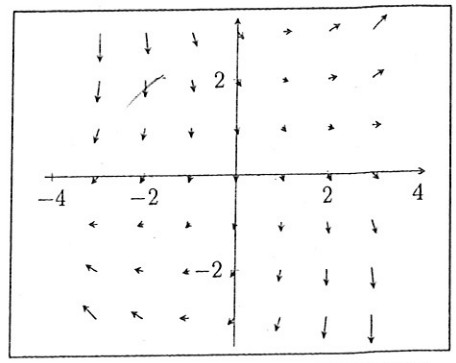
$$\underbrace{\begin{bmatrix} 2x & 0 \\ -1 & 1 \end{bmatrix}(g)}_{\underline{Df}} \underbrace{\begin{bmatrix} s & r & 0 \\ 0 & 0 & e^t \end{bmatrix}}_{\underline{Dg}}$$

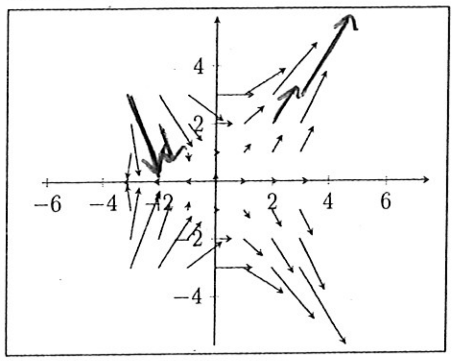
4. (20 points) Which of these functions is the linear approximation $L(x,y,z)$ of $f(x,y,z)$ near $(1,0,-2)$ given by Taylor's Theorem?

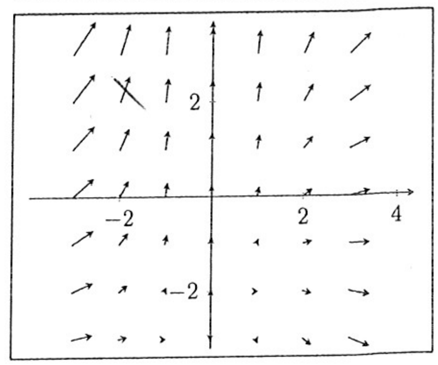
- ● $L(x,y,z) = f(1,0,-2) + \frac{\partial f}{\partial x}(1,0,-2)(x-1) + \frac{\partial f}{\partial y}(1,0,-2)(y) + \frac{\partial f}{\partial z}(1,0,-2)(z+2)$
- ○ $L(x,y,z) = \sum_{i=1}^{4} \frac{\partial f}{\partial x_i}(1,0,-2)(x_i - 1)$
- ○ $L(x,y,z) = \mathbf{D}[f](1,0,-2)\mathbf{D}[L]$
- ○ $L(x,y) = f(1,0,-2)\frac{\partial f}{\partial x}(1,0,-2)\frac{\partial f}{\partial y}(1,0,-2)\frac{\partial f}{\partial z}(1,0,-2)(x+y+z-1)$
- ○ None of these

5. (20 points) Which of the following vector field plots was generated from the gradient of $f(x, y) = x^2 + 3xy^2$? (Note that the plots are scaled down by a factor of 20, and zero vectors are shown as arrowheads.)









$$\nabla f = \langle 2x + 3y^2, 6xy \rangle$$

$$\nabla f(2,2) = \langle 4 + 12, 24 \rangle = \langle 16, 24 \rangle$$

$$\nabla f(3,3) = \langle 6 + 27, 54 \rangle = \langle 33, 54 \rangle$$

$$\nabla f(-2,2) = \langle -4 + 12, -24 \rangle = \langle 8, -24 \rangle$$

$$\nabla f(-3,3) = \langle -6 + 27, -54 \rangle = \langle 21, -54 \rangle$$

Full Response (150 points total)

6. (30 points) Verify the triangle inequality $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ where $\mathbf{x} = \langle 1, 2, 2, -4 \rangle$ and $\mathbf{y} = \langle 0, 3, 0, 4 \rangle$.

$$\|\underline{x}+\underline{y}\| = \|\langle 1,5,2,0\rangle\|$$
$$= \sqrt{1+25+4}$$
$$= \sqrt{30}$$

$$\|\underline{x}\|+\|\underline{y}\| = \sqrt{1+4+4+16} + \sqrt{9+16}$$
$$= \sqrt{25}+\sqrt{25}$$
$$= 10$$
$$= \sqrt{100}$$

Since $\sqrt{30} \leq \sqrt{100}$, $\|\underline{x}+\underline{y}\| \leq \|\underline{x}\|+\|\underline{y}\|$ in this case.

7. (30 points) Use the fact that $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{f}(x, y) = \langle x + y^2, 2xy \rangle$ is a differentiable function to explain why

$$\mathbf{f}(2.1, -1.1) \approx \langle 3, -4 \rangle + \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \langle 0.1, -0.1 \rangle = \langle 3.3, -4.6 \rangle$$

$$\underline{f}(2,-1) = \langle 2+1, 2(2)(-1) \rangle = \langle 3,-4 \rangle$$

$$\underline{Df} = \begin{bmatrix} 1 & 2y \\ 2y & 2x \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

By definition of differentiability,

$$\underline{f}(\underline{x}+\underline{h}) \approx f(\underline{x}) + \underline{Df}(\underline{x})\,\underline{h}$$

Let $\underline{x} = \langle 2,-1 \rangle$ and $\underline{h} = \langle 0.1, -0.1 \rangle$, then

$$\begin{aligned} f(2.1,-1.1) &= f(\underline{x}+\underline{h}) \\ &\approx f(\underline{x}) + \underline{Df}(\underline{x})\,\underline{h} \\ &= \langle 3,-4 \rangle + \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \langle 0.1,-0.1 \rangle \\ &= \langle 3,-4 \rangle + \langle 0.1+0.2, -0.2-0.4 \rangle \\ &= \langle 3.3, -4.6 \rangle. \end{aligned}$$

8. (30 points) Suppose the temperature at position (x, y, z) is given by $T(x, y, z) = 4x^2 + 4y^2 + z^2$. If a particle moves around the curve $\mathbf{c}(t) = \langle \sin t, -\cos t, 2t \rangle$, describe

- the temperature of the particle, $(T \circ \mathbf{c})(t)$

$$= T(\underline{c}(t)) = 4(\sin t)^2 + 4(-\cos t)^2 + (2t)^2$$

$$= 4\sin^2 t + 4\cos^2 t + 4t^2$$

$$= 4 + 4t^2$$

- the rate at which temperature is changing with respect to t, $\frac{d}{dt}[T \circ \mathbf{c}](t)$

$$= 8t$$

OR

$$= \underline{D}T(\underline{c}(t))\,\underline{D}\underline{c}$$

$$= [8x \quad 8y \quad 2z](\underline{c}(t)) \begin{bmatrix} \cos t \\ \sin t \\ 2 \end{bmatrix}$$

$$= \cancel{8\sin t\cos t - 8\sin t\cos t} + 8t$$

9. (30 points) What condition(s) must a flow line fitting the vector field $\mathbf{F} = \langle 1, 2x \rangle$ satisfy? Find one such flow line.

Let $\underline{c}(t)$ be a flow line. Then

$$\underline{c}'(t) = \underline{F}(\underline{c}(t))$$

$$\left\langle \frac{dc_1}{dt}, \frac{dc_2}{dt} \right\rangle = \langle 1, 2c_1 \rangle$$

So $\dfrac{dc_1}{dt} = 1$ is satisfied by $c_1 = t$

and $\dfrac{dc_2}{dt} = 2c_1 = 2t$ is satisfied by $c_2 = t^2$,

giving $\underline{c}(t) = \langle t, t^2 \rangle$ as one solution.

10. (30 points) Prove $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}(\mathbf{F}) + \mathbf{F}\cdot\nabla f$ for $f:\mathbb{R}^n \to \mathbb{R}$ and $\mathbf{F}:\mathbb{R}^n\to\mathbb{R}^n$.
(Recall that $\operatorname{div}(\mathbf{G}) = \sum_{i=1}^n \frac{\partial G_i}{\partial x_i}$ and $\nabla g = \langle \frac{\partial g_1}{\partial x_1}, \ldots, \frac{\partial g_n}{\partial x_n}\rangle$.)
Or, for half credit, verify that theorem for $f(x,y) = x^2+3y$ and $\mathbf{F} = \langle 2x^2y, x+y\rangle$.

$$\operatorname{div}(f\underline{F}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(fF_i)$$

$$= \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}F_i + f\frac{\partial F_i}{\partial x_i}\right)$$

$$= \sum_{i=1}^{n} F_i \frac{\partial f}{\partial x_i} + f\sum_{i=1}^{n}\frac{\partial F_i}{\partial x_i}$$

$$= \underline{F}\cdot\nabla f + f(\operatorname{div}\underline{F}).$$

OR

$f\underline{F} = \langle 2x^4y + 6x^2y^2, x^3 + x^2y + 3xy + 3y^2\rangle$

$\operatorname{div}(f\underline{F}) = 8x^3y + 12xy^2 + x^2 + 3x + 6y$

$\operatorname{div}(\underline{F}) = 4xy + 1$

$\nabla f = \langle 2x, 3\rangle$

$f\operatorname{div}\underline{F} + \underline{F}\cdot\nabla f = (x^2+3y)(4xy+1) + \langle 2x, 3\rangle\cdot\langle 2x^2y, x+y\rangle$

$= 4x^3y + x^2 + 12xy^2 + 3y + 4x^3y + 3x + 3y$

$= 8x^3y + 12xy^2 + x^2 + 3x + 6y$