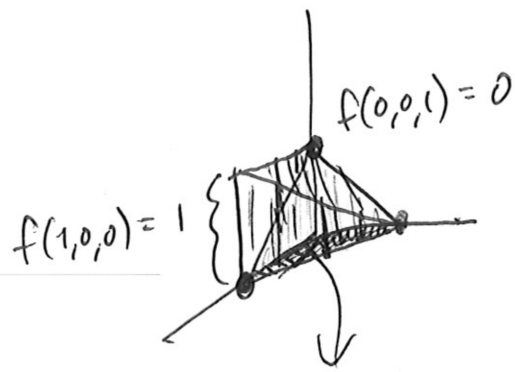
(7.5 cont)

(Example 4) Compute $\iint_S \underbrace{x}_{f(x,y,z)} \, dS$ where S is the triangle with vertices $(1,0,0)$, $(0,1,0)$, & $(0,0,1)$.



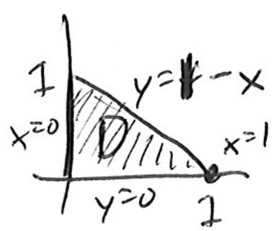
$$= \iint_D f(\Phi(u,v)) \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| dA$$

parameterization of S

$$x+y+z=1$$
$$z = 1-x-y$$

$$\left(\begin{array}{l} Ax+By+Cz=D \\ \begin{cases} A+0+0=D \\ 0+B+0=D \\ 0+0+C=D \end{cases} \end{array} \right)$$

$$\Phi(x,y) = \begin{array}{ccc} x & y & z \\ (x, & y, & 1-x-y) \end{array}$$



$$= \int_0^1 \int_0^{1-x} x \left\| \frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} \right\| dy \, dx$$

Replace x with x
y with y
z with 1-x-y

$$\frac{\partial \Phi}{\partial x} = (1, 0, -1)$$
$$\frac{\partial \Phi}{\partial y} = (0, 1, -1)$$

$$\frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} = (1, 1, 1)$$

$$\left\| \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} \right\| = \sqrt{1+1+1} = \sqrt{3}$$

$$= \int_0^1 \int_0^{1-x} \sqrt{3}\, x \, dy \, dx$$

$$= \int_0^1 \Big[\sqrt{3} x y \Big]_0^{1-x} dx$$

$$= \int_0^1 \left(\sqrt{3} x - \sqrt{3} x^2 \right) \cancel{+(0)} \, dx$$

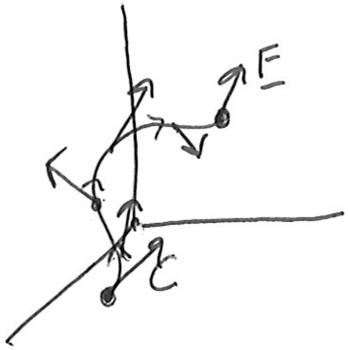
$$= \left[\frac{\sqrt{3}}{2} x^2 - \frac{\sqrt{3}}{3} x^3 \right]_0^1$$

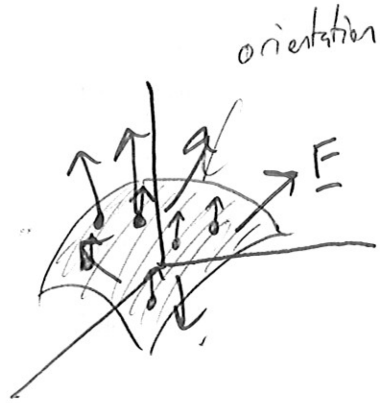
$$= \boxed{\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{3} \right)} \cancel{-(0+0)}$$

$$= \boxed{\sqrt{3}/6}$$

HW 7,5 1-4, 6-7

7.6 Surface Integrals of Vector Fields





Work or Flow $= \int_C \underline{F} \cdot d\underline{s}$

$= \int_{t=a}^{t=b} \underline{F}(\underline{r}(t)) \cdot \frac{dr}{dt}\, dt$

parameterization from $[a,b]$ onto C respecting orientation

Flux $= \iint_S \underline{F} \cdot d\underline{S}$

$= \iint_D \underline{F}(\underline{\Phi}(u,v)) \cdot \underbrace{\left(\frac{d\Phi}{du} \times \frac{d\Phi}{dv}\right)}_{}\, dA$

normal to surface

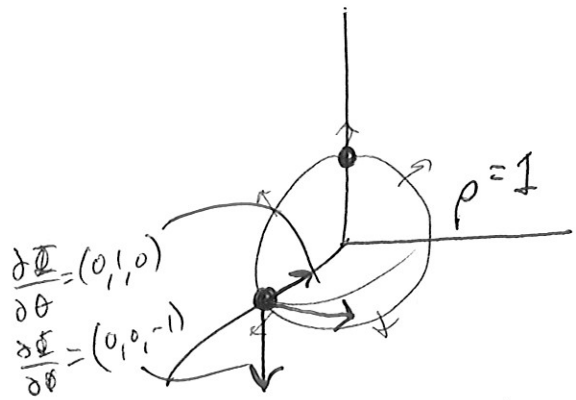
parametrization from D onto S respecting orientation

(Example 4) Suppose the temperature $T(x,y,z)$ of a point $(x,y,z) \in \mathbb{R}^3$ is given by $x^2+y^2+z^2$. Compute the heat flux $\iint_S -k\nabla T \cdot d\underline{S}$ across the unit sphere oriented ~~outward~~ inward or outward if $k=1$.

$$\nabla T = (2x+0+0,\ 0+2y+0,\ 0+0+2z)$$

$$-\nabla T = (-2x,\ -2y,\ -2z)$$

$$\iint_S (-2x, -2y, -2z) \cdot d\underline{S}$$



$$\underline{\Phi}(\theta, \phi) = \underline{s}(1, \theta, \phi)$$

$$= (\underbrace{\sin\phi \cos\theta}_{x},\ \underbrace{\sin\phi \sin\theta}_{y},\ \underbrace{\cos\phi}_{z})$$

$$\frac{\partial \Phi}{\partial \theta} = (-\sin\phi \sin\theta,\ \sin\phi \cos\theta,\ 0)$$

$$\frac{\partial \Phi}{\partial \phi} = (\cos\phi \cos\theta,\ \cos\phi \sin\theta,\ -\sin\phi)$$

$$\frac{\partial \Phi}{\partial \theta} \times \frac{\partial \Phi}{\partial \phi} = (-\sin^2\phi \cos\theta,\ -\sin^2\phi \sin\theta,\ -\sin\phi\cos\phi\sin^2\theta - \sin\phi\cos\phi\cos^2\theta)$$

$$= (-\sin^2\phi \cos\theta,\ -\sin^2\phi \sin\theta,\ -\sin\phi\cos\phi)$$

$$= \int_0^{2\pi}\int_0^{\pi} (-2\sin\phi\cos\theta,\ -2\sin\phi\sin\theta,\ -2\cos\phi) \cdot (-\sin^2\phi\cos\theta,\ -\sin^2\phi\sin\theta,\ -\sin\phi\cos\phi)\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} + 2\sin^3\phi\cos^2\phi + 2\sin^3\phi\sin^2\theta + 2\sin\phi\cos^2\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 2\sin^3\phi + 2\sin\phi\cos^2\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\left[\int_0^{\pi} 2\sin\phi \, d\phi\right] d\theta$$

$$= \int_0^{2\pi} \left[-2\cos\phi\right]_0^{\pi} d\theta$$

$$= \int_0^{2\pi} 2+2 \, d\theta = \boxed{8\pi}$$

(Example) Suppose fluid is moving according to the velocity field $F(x,y,z) = (x,y,z)$ through triangle w/ verts $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ oriented upwards. Compute the flux of this velocity field through the triangle.

$$\Phi(x,y) = (x, y, 1-x-y)$$

$$0 \le y \le 1-x$$

$$0 \le x \le 1$$

from 7.5 Example 4

$$\frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} = (1,1,1)$$

$$\iint_S (x, y, z) \cdot d\underline{S}$$

$$= \int_0^1 \int_0^{1-x} (x, y, 1-x-y) \cdot \underbrace{(1, 1, 1)}_{\frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y}} \, dy \, dx$$

replace x with x   y with y   z with 1-x-y

$$= \int_0^1 \int_0^{1-x} x \cancel{+y} + 1 - \cancel{x} \cancel{-y} \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} 1 \, dy \, dx = \cdots = \boxed{1/2}$$

HW 7.6 1-5, 13