(6.2 cont)

Jacobian:

$$\frac{\partial \underline{T}}{\partial \underline{u}} = \underbrace{\frac{\partial(x,y,z)}{\partial(u,v,w)}}_{\text{in } \mathbb{R}^3} = \det\left(\underline{D}\,\underline{T}\right) = \underbrace{\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}}_{\text{in } \mathbb{R}^3}.$$

For the affine transformation $\underline{T}(\underline{u}) = M\underline{u} + \underline{x}_0$,

$$\frac{\partial \underline{T}}{\partial \underline{u}} = \det(M) \quad \text{because} \quad \underline{D}\,\underline{T} = M.$$

Theorem: For a transformation $\underline{T}$ turning D^* into D,

$$\iint_D f(x,y)\, dA = \iint_{D^*} f(\underline{T}(u,v)) \left|\frac{\partial \underline{T}}{\partial \underline{u}}\right| dA$$

OR

$$\underbrace{\iiint_D f(x,y,z)\, dA}_{\substack{\text{Hard to set up}\\ \text{bounds of integration}\\ \text{for } D}} = \underbrace{\iiint_{D^*} f(\underline{T}(u,v,w)) \left|\frac{\partial \underline{T}}{\partial \underline{u}}\right| dA}_{\substack{\text{Easier to set up bounded of}\\ \text{integration for } D^*\\ \left(\text{often } \int\int_0 \int_0\right)}}$$

Jacobians of Polar, Cylindrical, & Spherical Transformations

$$\underline{p}(r,\theta) = (r\cos\theta, r\sin\theta)$$

$$\frac{\partial \underline{p}}{\partial(r,\theta)} = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}$$

$$= \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta)$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r\,\cancel{(\cos^2\theta + \sin^2\theta)}$$

$$\frac{\partial \underline{p}}{\partial(r,\theta)} = r$$

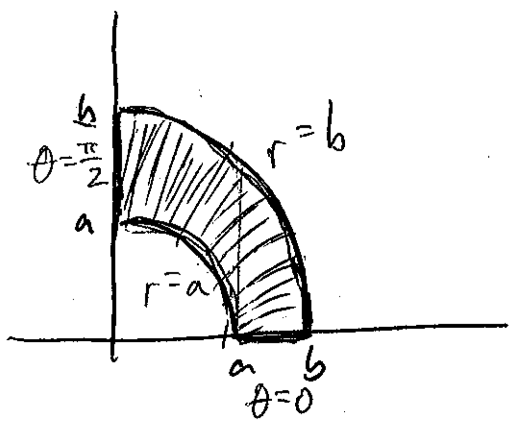
Similarly,

$$\frac{\partial \underline{c}}{\partial(r,\theta,z)} = r$$

& $$\frac{\partial \underline{s}}{\partial(\rho,\theta,\phi)} = \underbrace{\cdots}_{\text{lotta trig}} = \rho^2 \sin\phi .$$

(Example 4)

Evaluate $\iint_D \log_e(x^2+y^2)\,dA$ where D is the region in the 1st quadrant between the circles $x^2+y^2=a^2$ & $x^2+y^2=b^2$ for $0<a<b$.



If $r>0$

$$\iint_D \log_e(x^2+y^2)\,dA = \int_{\substack{\text{C.W.}\\ \theta}}^{\substack{\text{C.C.W.}\\ \theta}} \int_{\substack{\text{Inside}\\ \text{curve}}}^{\substack{\text{Outside}\\ \text{curve}}} \left(\begin{matrix}\text{replace } x,y\\ \text{with } r,\theta\end{matrix}\right) \, r \, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} \int_{r=a}^{r=b} \log_e(r^2)\, \underbrace{r\, dr}\, d\theta$$

Let $u = r^2$

$du = 2r\,dr$

$\frac{1}{2}du = r\,dr$

$$= \int_{\theta=0}^{\theta=\pi/2} \int_{u=a^2}^{u=b^2} \log_e(u) \left(\tfrac{1}{2}\, du\right) d\theta$$

$$= \frac{1}{2} \int_{0}^{\pi/2} \left[\int_{a^2}^{b^2} \log_e(u)\, du \right] d\theta$$

$$\cancel{= \frac{1}{2} \int_{0}^{\pi/2} \left[\frac{1}{u} \right]_{a^2}^{b^2} d\theta}$$

$$\cancel{= \frac{1}{2} \int_{0}^{\pi/2} 1/b^2 - 1/a^2 \, d\theta}$$

$$\cancel{= \left(1/2b^2 - 1/2a^2 \right) \int_{0}^{\pi/2} 1 \, d\theta}$$

$$\cancel{= \boxed{\left(1/2b^2 - 1/2a^2 \right)\left(\pi/2 \right)}}$$

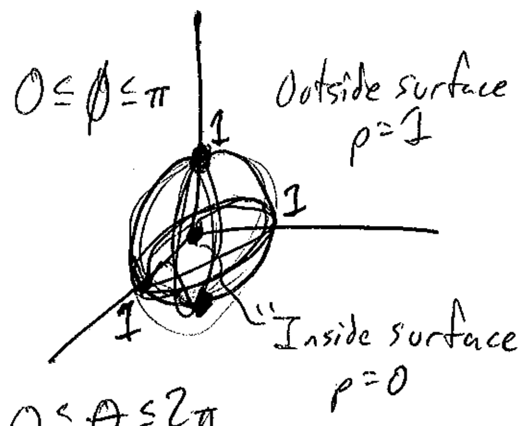
$$= \frac{1}{2} \int_{0}^{\pi/2} \left[u \log_e u - u \right]_{a^2}^{b^2} d\theta$$

$$= \frac{1}{2} \int_{0}^{\pi/2} \left(b^2 \log_e b^2 - b^2 \right) - \left(a^2 \log_e a^2 - a^2 \right) d\theta$$

$$= \boxed{\frac{\pi}{4} \left(b^2 \log_e b^2 - b^2 - a^2 \log_e a^2 + a^2 \right)}$$

(Example 6) Evaluate $\iiint\limits_W \exp\left((x^2+y^2+z^2)^{3/2}\right) dV$

where W is the unit ball centered at the origin.



when $0 \le \phi \le \pi$.

$$= \int_{\substack{\text{C.W.} \\ \theta}}^{\substack{\text{C.C.W.} \\ \theta}} \int_{\substack{\text{Top} \\ \phi}}^{\substack{\text{Bottom} \\ \phi}} \int_{\substack{\text{Inside surface} \\ \rho}}^{\substack{\text{Outside surface} \\ \rho}} \begin{pmatrix} \text{replace } x,y,z \text{ with} \\ \rho, \theta, \phi \end{pmatrix} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \exp\left((\rho^2)^{3/2}\right) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \left[\int_0^1 \exp(\rho^3) \, \rho^2 \sin\phi \, d\rho \right] d\phi \, d\theta$$

(let $u = \rho^3$)

$$= \int_0^{2\pi} \int_0^{\pi} \left[\frac{1}{3} \exp(\rho^3) \sin\phi \right]_0^1 d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{1}{3} e \sin\phi - \frac{1}{3} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3} e \cos\phi + \frac{1}{3} \cos\phi \right]_0^{\pi} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{3}e - \frac{1}{3}\right) - \left(-\frac{1}{3}e + \frac{1}{3}\right) d\theta$$

$$= \int_0^{2\pi} \frac{2}{3}e - \frac{2}{3} \, d\theta$$

$$= \left[\left[\frac{2}{3}e - \frac{2}{3}\right] \theta \right]_0^{2\pi}$$

$$= \boxed{\left(\frac{2}{3}e - \frac{2}{3}\right)(2\pi)}$$