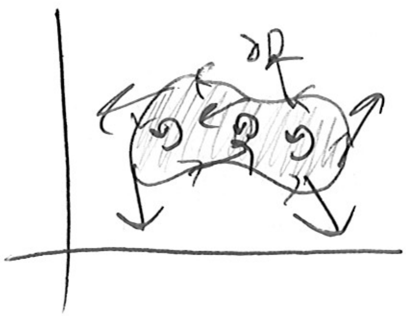
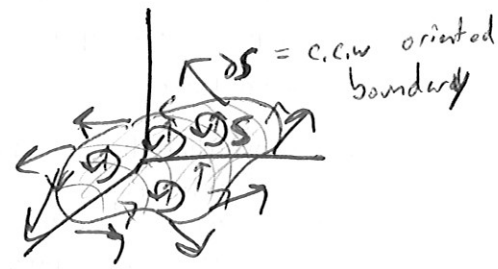
8.2 Stokes' Theorem

Green's Thm



$$\int_{\partial D} \underline{F}\cdot d\underline{s} = \iint_D \text{scurl}\,\underline{F}\; dA$$

Stokes' Thm



normal vectors

$$\int_{\partial S} \underline{F}\cdot d\underline{s} = \iint_S (\text{curl}\,\underline{F}\cdot \underline{N})\, dS$$

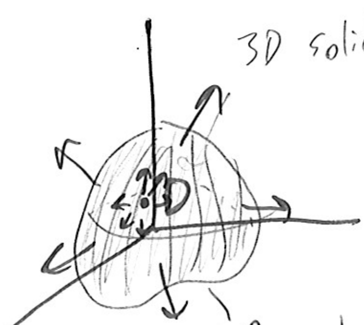
same

$$= \iint_S (\text{curl}\,\underline{F})\cdot d\underline{S}$$

(In general,
$$\iint_S \underline{F}\cdot d\underline{S} = \iint_S \underline{F}\cdot \underline{N}\; dS$$
)

AKA Divergence

8.4 Gauss' Theorem



∂D = outward-oriented $\underbrace{\text{boundary}}_{\text{surface}}$ of D

$$\cancel{\iiint} \iint_{\partial D} \underline{F} \cdot d\underline{S} = \iiint_{D} \operatorname{div} \underline{F} \, dV$$

(will add 8.2 & 8.4 examples to lecture notes on Moodle; Optional.)

Overview of Integration Theorems:

Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = \big[f(x)\big]_a^b = f(b) - f(a)$$

$$\int_{\underbrace{[a,b]}_{\substack{\text{entire}\\ \text{interval}}}} f'(x)\,dx = \big[f(x)\big]_{\underbrace{\partial[a,b]}_{\substack{\text{endpoints}\\ \text{aka}\\ \text{boundary}}}}$$

Fundamental Theorem of Line Integrals (8.3):

$$\int_{\underbrace{C}_{\substack{\text{entire}\\ \text{curve}}}} \nabla f \cdot d\underline{s} = \big[f\big]_{\underbrace{\partial C}_{\substack{\text{endpoints}\\ \text{aka}\\ \text{boundary}}}} = \big[f\big]_A^B$$

Green's & Stokes' Thm (8.1, 8.2):

$$\iint_{\underbrace{D}_{\substack{\text{entire}\\ \text{region}}}} \operatorname{scurl} \underline{F} \, dA = \int_{\underbrace{\partial D}_{\text{boundary}}} \underline{F} \cdot d\underline{s}$$

$$\iint_{\underbrace{S}_{\substack{\text{entire}\\ \text{surface}}}} \operatorname{curl} \underline{F} \cdot d\underline{S} = \int_{\underbrace{\partial S}_{\text{boundary}}} \underline{F} \cdot d\underline{s}$$

Gauss' aka
Divergence Thm (8.4)

$$\underbrace{\iiint_{D}}_{\text{entire solid}} \operatorname{div} \underline{F} \, dV = \underbrace{\iint_{\partial D}}_{\text{boundary}} \underline{F} \cdot d\underline{S}$$

Example Problems for 8.2 & 8.4

(8.2 Example 1) Let $\underline{F} = (ye^z, xe^z, xye^z)$.
Prove that $\int_{\partial S} \underline{F} \cdot d\underline{s} = 0$ for any surface S.

By Stokes' Thm,

$$\int_{\partial S} \underline{F} \cdot d\underline{s} = \iint_S \text{curl}\, \underline{F} \cdot d\underline{S}$$

$$= \iint_S \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \cdot d\underline{S}$$

$$= \iint_S \left(xe^z - xe^z, ye^z - ye^z, e^z - e^z \right) \cdot d\underline{S}$$

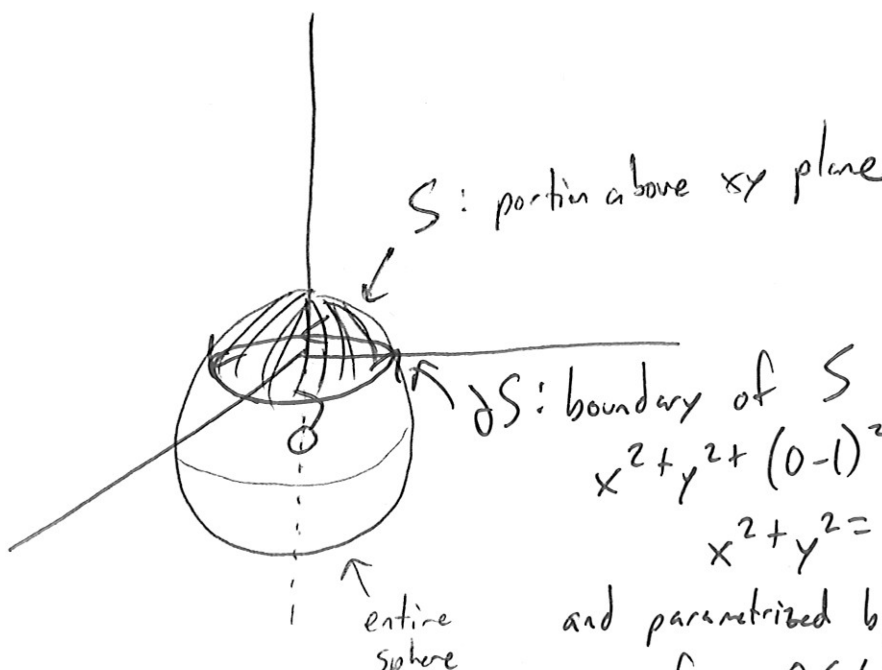
$$= \iint_S \underline{0} \cdot d\underline{S}$$

$$= 0. \quad \square$$

(Note that since curl $\underline{F} = \underline{0}$, and since the boundary of a surface is always a closed loop, the result also follows as $\underline{F}$ is conservative.)

(8.2 Example 3)

Express $\iint_S (0, -ze^{xz}, -2) \cdot d\underline{S}$ as an integral of a single variable, where S is the surface $x^2+y^2+(z-1)^2=2$ above the xy plane and oriented outwards.



∂S: boundary of S with equation

$$x^2+y^2+(0-1)^2=2$$
$$x^2+y^2=1$$

and parametrized by $\underline{r}(t)=(\cos t, \sin t, 0)$ for $0 \le t \le 2\pi$.

By Stokes' Theorem:

$$\iint_S \text{curl } \underline{F} \cdot d\underline{S} = \int_{\partial S} \underline{F} \cdot d\underline{s} \, .$$

So we must find $\underline{F}$ such that

$$\text{curl } \underline{F} = (0, -ze^{xz}, -2) \, .$$

(There are many possible examples, so we may assume things like $F_1 = 0$ to simplify our search. But, any candidate must satisfy the following equations...)

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = 0 \qquad \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = -ze^{xz} \qquad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -2$$

Let us assume that $F_1 = 0$. Then,

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = 0 \qquad \frac{\partial F_3}{\partial x} = ze^{xz} \qquad \frac{\partial F_2}{\partial x} = -2$$

Assuming $F_3 = e^{xz}$ and $F_2 = -2x$ satisfies all three equations. Therefore by Stokes' Theorem...

$$\iint_S (0, -ze^{xz}, -2) \cdot d\underline{S} = \int_{\partial S} (0, -2x, e^{xz}) \cdot d\underline{s}$$

$$= \int_0^{2\pi} (0, -2\cos t, e^0) \cdot \frac{d\underline{r}}{dt}\, dt$$

$$= \int_0^{2\pi} (0, -2\cos t, 1) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \boxed{\int_0^{2\pi} -2\cos^2 t \, dt}$$

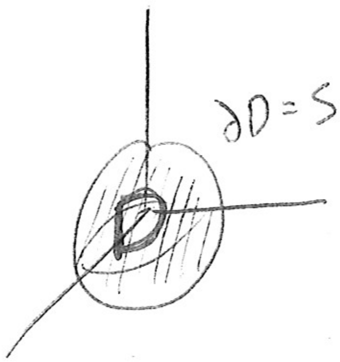
$$\left(= \int_0^{2\pi} -1 - \cos(2t)\, dt \right.$$

$$\left. = -2\pi. \right)$$

(8.4 Example 3#)

Evaluate $\iint\limits_S (2y, yz, z^2) \cdot d\underline{S}$ where S is the outward oriented boundary of the unit sphere $x^2+y^2+z^2=1$.

Let D be the unit ball $x^2+y^2+z^2 \leq 1$.



By Gauss'/Divergence Thm,

$$\iint\limits_S (2y, yz, z^2)\cdot d\underline{S} = \iiint\limits_D \operatorname{div}(2y, yz, z^2)\, dV$$

$$= \iiint\limits_D 0 + z + 2z \, dV$$

$$= \iiint\limits_D 3z \, dV$$

Spherical coordinate transformation

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} (3\rho\cos\phi)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

...

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} 3\rho^3 \sin\phi\cos\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \tfrac{3}{4}\sin\phi\cos\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{3}{8}\sin^2\phi\right]_0^{\pi} d\theta$$

$$= \boxed{0}$$