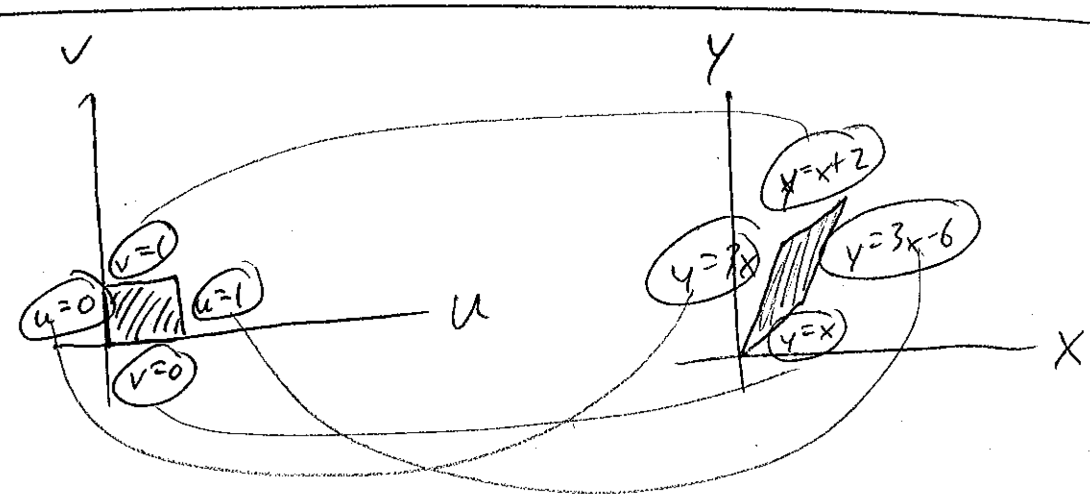
Quiz 8 #3 Find an affine transformation $\underline{T}(u,v)$ which maps the unit square in the uv plane to the parallelogram with sides given by $y=x$, $y=x+2$, $y=3x$, $y=3x-6$ in the xy plane.



$u=0 \Rightarrow y=3x$

$u=1 \Rightarrow y=3x-6$

$u\in[0,1] \Rightarrow y=3x-6u$

$v=0 \Rightarrow y=x$

$v=1 \Rightarrow y=x+2$

$v\in[0,1] \Rightarrow y=x+2v$

$3x-6u=x+2v$

$2x=6u+2v$

$x=3u+v$

$y=(3u+v)+2v$

$y=3u+3v$

$$\left(\underline{T}(u,v)=\begin{bmatrix}3 & 1\\ 3 & 3\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}\right)$$

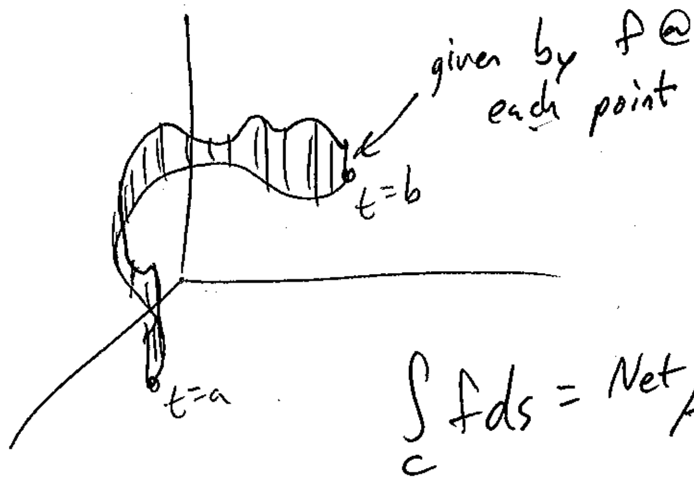
$$\boxed{\underline{T}(u,v)=(3u+v,\ 3u+3v)}$$

7.1 cont.

If $f: C \to \mathbb{R}$ is a function defined along the curve C, which defined by $\underline{r}: [a,b] \to \mathbb{R}^n$, then

$$\int_C f\,ds = \int_{t=a}^{t=b} f(\underline{r}(t)) \frac{ds}{dt}\, dt$$

$$\frac{ds}{dt} = \|\underline{r}'(t)\|$$



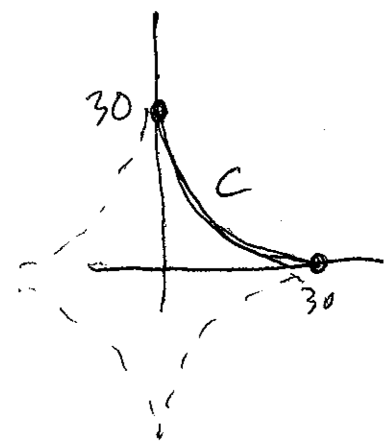
$$\int_C f\,ds = \text{Net Area}$$

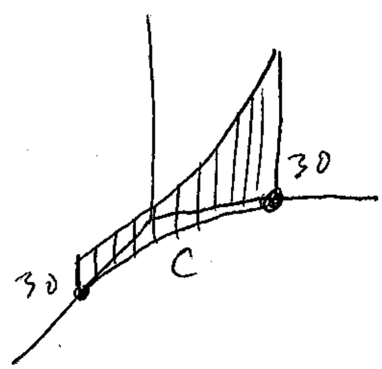
OR

(Example 2)

The base of a fence is along the curve $\underline{c}(t) = (30\cos^3 t,\ 30\sin^3 t)$ in the first quadrant;
$0 \le t \le \pi/2$

and the height of the fence is given by $f(x,y) = 1 + y/3$. How much paint is required to cover both sides of the fence?





$$\text{Area} = \int_C f(x,y)\, ds$$

$$= \int_C \left(1 + y/3\right) ds$$

$\|\underline{c}'(t)\|$

$$= \int_{t=0}^{t=\pi/2} \left(1 + \frac{30\sin^3 t}{3}\right) \frac{ds}{dt}\, dt$$

$$\underline{c}'(t) = \left(90\cos^2 t(-\sin t),\ 90\sin^2 t(\cos t)\right)$$

$$= \left(-90\cos^2 t\sin t,\ 90\sin^2 t\cos t\right)$$

$$= 90\cos t\sin t\left(-\cos t,\ \sin t\right)$$

$$\|\underline{c}'(t)\| = 90|\cos t||\sin t|\cancel{\sqrt{\cos^2 t+\sin^2 t}}$$

$$= 90\cos t\sin t$$

$$= \int_{t=0}^{t=\pi/2}\left(1+10\sin^3 t\right)\left(90\cos t\sin t\right)dt$$

$$= \int_0^{\pi/2} 90\cos t\sin t + 900\sin^4 t\cos t\, dt$$

$$= \left[45\sin^2 t + 180\sin^5 t\right]_0^{\pi/2}$$

$$= \left(45\cancel{\sin^2\pi/2} + 180\cancel{\sin^5\pi/2}\right) - \left(\cancel{45\sin^2 0} + \cancel{180\sin^5 0}\right)$$

Area $= 225$

Paint required: $225 \times 2 = \boxed{450}$ square units

(Example 1)

The average value of a function f over a curve C is given by $\cancel{\;\;}$ $\dfrac{1}{\text{length of } C} \displaystyle\int_C f \, ds$.

Find the average value of $f(x,y,z) = x^2+y^2+z^2$ along the portion of the helix given by

$$\underline{c}(t) = (\overset{x}{\cos t}, \overset{y}{\sin t}, \overset{z}{t}) \quad \text{for } t \in [0, 2\pi].$$

$$\underline{c}'(t) = (\overset{x'}{-\sin t}, \overset{y'}{\cos t}, \overset{z'}{1})$$

$$\|\underline{c}'(t)\| = \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1} + 1} = \sqrt{2}$$

$$\text{length of } C = \int_0^{2\pi} \|\underline{c}'(t)\| \, dt \quad \left(= \int_C 1 \, ds \right)$$

$$= \int_0^{2\pi} \sqrt{2} \, dt = \left[\sqrt{2} t \right]_0^{2\pi} = 2\sqrt{2}\pi$$

$$\int_C f\,ds = \int_C \left(x^2+y^2+z^2\right)\cdot\frac{ds}{dt}\,dt$$

$$= \int_0^{2\pi} \left(\underbrace{\cos^2 t + \sin^2 t}_{1} + t^2\right)\sqrt{2}\,dt$$

$$= \int_0^{2\pi} \sqrt{2} + \sqrt{2}\,t^2\,dt$$

$$= \left[\sqrt{2}\,t + \frac{\sqrt{2}}{3}t^3\right]_0^{2\pi}$$

$$= 2\sqrt{2}\pi + \frac{\sqrt{2}}{3}8\pi^3$$

$$\text{Avg val} = \frac{1}{\text{len of } C}\int_C f\,ds$$

$$= \frac{1}{2\sqrt{2}\pi}\left(\cancel{2\sqrt{2}\pi}^{\,1} + \frac{\cancel{\sqrt{2}}}{3}\cancel{8}^{\,4}\pi^{\cancel{3}\,2}\right)$$

$$= \boxed{1 + \frac{4}{3}\pi^2}$$

HW

7.1 1-8, 10-13

7.2 Line Integrals

If $\underline{F} : C \to \mathbb{R}^n$ is a vector field,

and C is a curve defined by $\underline{r} : [a,b] \to \mathbb{R}^n$,

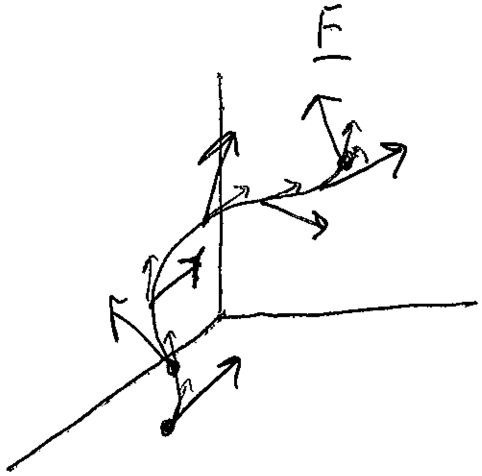
then

unit tangent vectors to curve C

$$\int_C \underline{F} \cdot d\underline{r} = \int_C \underbrace{(\underline{F} \cdot \underline{T})}_{} \, ds$$

"the line integral over C" "of the vector field $\underline{F}$" "with respect to the curve"

how much of $\underline{F}$ is in the direction of $\underline{T}$, aka the direction of the curve



= work done by force $\underline{F}$ over the curve C

Usually to compute the line integral, we use

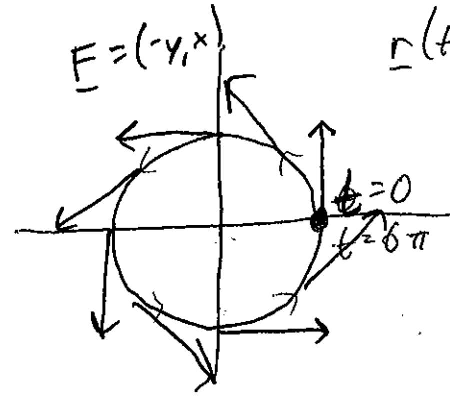
$$\int_C \underline{F} \cdot d\underline{r} = \int_{t=a}^{t=b} \Big(\underbrace{\underline{F}(\underline{r}(t))}_{\substack{\text{replace} \\ \text{all } x,y,z \\ \text{with } t}} \cdot \frac{d\underline{r}}{dt} \Big) dt$$

Note that the book uses the notation $\int_C \underline{F} \cdot d\underline{s}$

(Example) An object is pushed around the unit circle with a force of $\underline{F} = (-y, x)$ at each point (x,y). Compute the work done in pushing the box around the circle counter-clockwise 3 times.

$$\text{Work} = \int_C \underline{F} \cdot d\underline{r}$$

C ← three full rotations



$\underline{r}(t) = (\overset{x}{\cos t}, \overset{y}{\sin t})$

$$\frac{d\underline{r}}{dt} = (-\sin t, \cos t)$$

$$= \int_0^{6\pi} (-y, x) \cdot \frac{d\underline{r}}{dt}\, dt$$

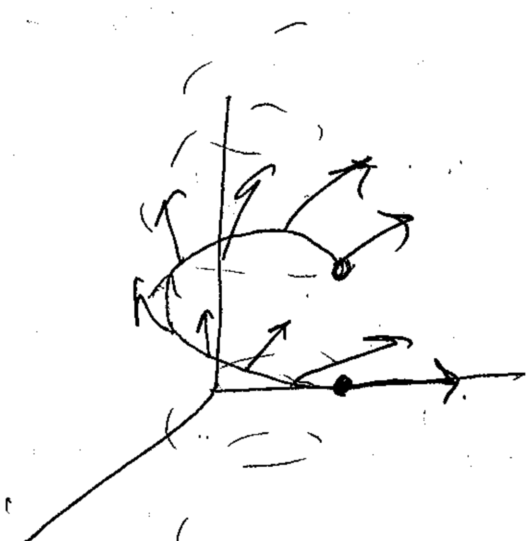
$$= \int_0^{6\pi} \left(-(\sin t), (\cos t)\right) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{6\pi} \sin^2 t + \cos^2 t \, dt$$

$$= \int_0^{6\pi} 1 \, dt = \left[t\right]_0^{6\pi} = \boxed{6\pi}$$

(Example 1) Let $\underline{r}(t) = (\overset{x}{\sin t}, \overset{y}{\cos t}, \overset{z}{t})$ for $t \in [0, 2\pi]$ define the curve C; let $\underline{F}(x,y,z) = (x,y,z)$ define a vector field on the curve. Compute $\int_C \underline{F} \cdot d\underline{r}$

$\underline{r}' = (\cos t, -\sin t, 1)$



$$= \int_0^{2\pi} (x,y,z) \cdot (\cos t, -\sin t, 1)\, dt$$

$$= \int_0^{2\pi} (\sin t, \cos t, t) \cdot (\cos t, -\sin t, 1)\, dt$$

$$= \int_0^{2\pi} \cancel{\sin t \cos t} - \cancel{\cos t \sin t} + t \, dt$$

$$= \int_0^{2\pi} t \, dt = \left[\frac{1}{2} t^2 \right]_0^{2\pi} = \boxed{2\pi^2}$$