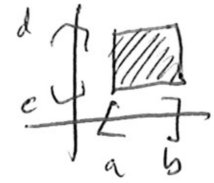
5.4 cont.

Rectangular regions: $\int_c^d\int_a^b f\,dx\,dy = \int_a^b\int_c^d f\,dy\,dx$



Nonrectangular ~~bounds~~ regions of integration:

$$\int_0^1\int_0^x \sin(e^{xy})\,dy\,dx \cancel{= \int_0^x\int_0^1 \sin(e^{xy})\,dx\,dy}$$

$= \underbrace{\text{something with } x}_{\text{not a volume}}$?

(Example) Swap the bounds of integration for

$$\int_0^4\int_0^{\frac{1}{2}(4-y)} x+y \; dx\,dy$$

and verify that both iterated integrals have the same value.

Evaluated directly...

$$= \int_0^4 \left[\frac{1}{2}x^2 + xy\right]_{x=0}^{x=\frac{1}{2}(4-y)} dy$$

$$= \int_0^4 \frac{1}{8}(4-y)^2 + \frac{1}{2}y(4-y)\,dy$$

$$= \int_0^4 \frac{1}{8}(16 - 8y + y^2) + \frac{1}{2}(4y - y^2)\,dy$$

$$= \int_0^4 2 - y + \frac{1}{8}y^2 + 2y - \frac{1}{2}y^2\,dy$$

$$= \int_0^4 2 + y - \frac{3}{8}y^2\,dy = \left[2y + \frac{1}{2}y^2 - \frac{1}{8}y^3\right]_0^4$$

$$= 8 + \frac{1}{2}(16) - \frac{1}{8}(64)$$

$$= 8 + 8 - 8 = \boxed{8}$$

To swap the bounds, first draw the region of integration!

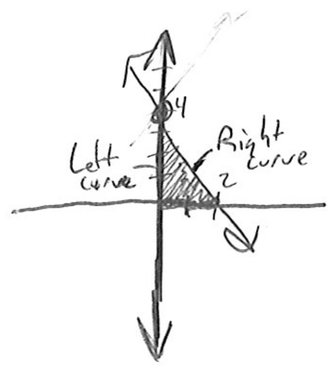
Top value, Right curve

$$\int_{y=0}^{y=4} \int_{x=0}^{x=\frac{1}{2}(4-y)} \sim\, dx\,dy$$

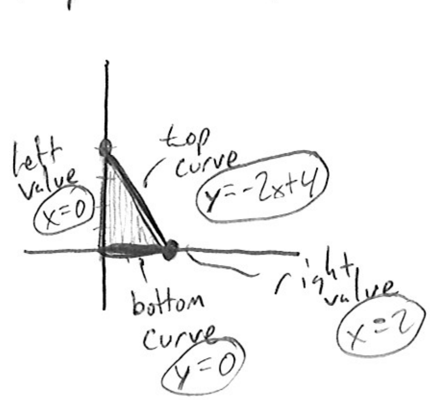
Bottom value, Left curve

$2x = 4 - y$

$y = -2x + 4$



Reinterpret the picture, swapping vertical/horizontal bounds



right value, top curve

$$\int_{x=0}^{x=2} \int_{y=0}^{y=-2x+4} \sim\, dy\,dx$$

left value, bottom curve

Should have the same value...

$$\int_0^2 \left[\int_0^{-2x+4} \underbrace{x+y}_{\substack{\text{stays}\\\text{the}\\\text{same}}} \, dy \right] dx$$

$$= \int_0^2 \left[xy + \frac{1}{2}y^2 \right]_{y=0}^{y=-2x+4} dx$$

$$= \int_0^2 x(-2x+4) + \frac{1}{2}(-2x+4)^2 \, dx$$

$$= \int_0^2 -2x^2 + 4x + \frac{1}{2}(4x^2 - 16x + 16) \, dx$$

$$= \int_0^2 \cancel{-2x^2} + 4x \cancel{+2x^2} - 8x + 8 \, dx$$

$$= \int_0^2 8 - 4x \, dx = \left[8x - 2x^2 \right]_0^2$$

$$= 16 - 8 = \boxed{8}$$

(Example) Evaluate $\displaystyle\int_{1}^{e}\int_{0}^{\log x} \frac{(2x-e)\sqrt{1+e^{y}}}{e-e^{y}}\,dy\,dx$.

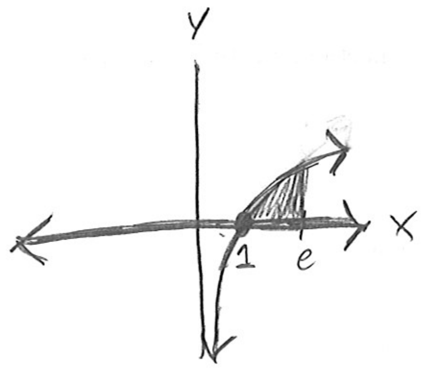
Idea! So much easier to integrate with respect to x...
let's swap the order so we can.

So,

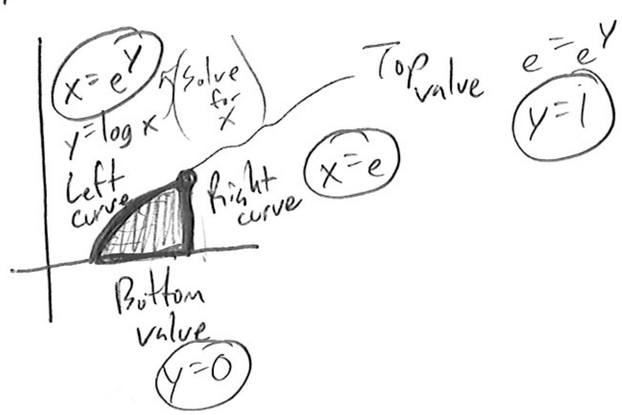
Right value $x=e$, Top curve $y=\log x$

$$\int_{x=1}^{x=e}\int_{y=0}^{y=\log x} \sim\, dy\,dx$$

Left value $x=1$, Bottom curve $y=0$



Reinterpret...



$$\int_{y=0}^{y=1}\int_{x=e^{y}}^{x=e} \frac{(2x-e)\sqrt{1+e^{y}}}{e-e^{y}}\,dx\,dy$$

$$= \int_{y=0}^{y=1} \frac{\sqrt{1+e^y}}{e-e^y} \left[\int_{x=e^y}^{x=e} 2x - e \, dx \right] dy$$

$$= \int_0^1 \frac{\sqrt{1+e^y}}{e-e^y} \left[x^2 - ex \right]_{x=e^y}^{x=e} dy$$

$$= \int_0^1 \frac{\sqrt{1+e^y}}{e-e^y} \left[\cancel{\left(e^2 - e^2\right)} - \left((e^y)^2 - ee^y \right) \right] dy$$

$$= \int_0^1 \frac{\sqrt{1+e^y}}{e-e^y} \left(ee^y - (e^y)^2 \right) dy$$

$$= \int_0^1 e^y \frac{\sqrt{1+e^y}}{\cancel{e-e^y}} \cancel{\left(e - e^y \right)} dy$$

$$= \int_0^1 \underbrace{e^y}\sqrt{1+e^y} \, \underbrace{dy}$$

Let $u = 1+e^y$

$du = e^y dy$

$$= \int_{y=0}^{y=1} \sqrt{u} \, \underbrace{du}$$

$$= \int_{y=0}^{y=1} u^{1/2} du$$

$$= \left[\frac{2}{3} u^{3/2} \right]_{y=0}^{y=1}$$

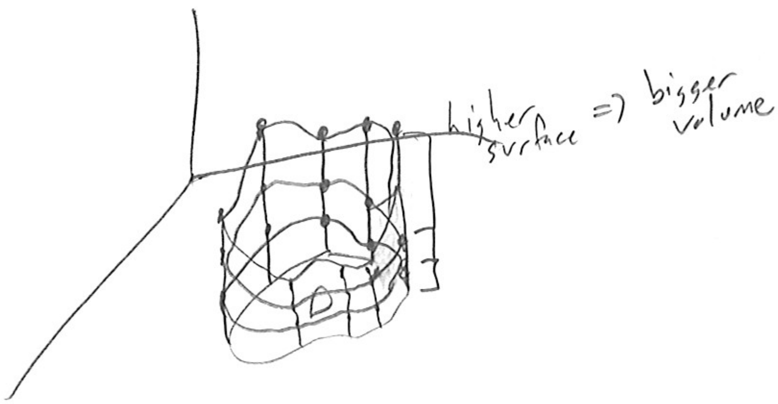
$$= \left[\frac{2}{3} (1+e^y)^{3/2} \right]_0^1$$

$$= \boxed{\frac{2}{3}(1+e)^{3/2} - \frac{2}{3}(2)^{3/2}}$$

Estimating double integrals:

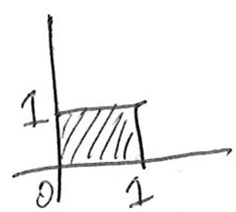
FACT. If $g(x,y) \le f(x,y) \le h(x,y)$ for $(x,y) \in D$,

then $\iint_D g(x,y)\,dA \le \iint_D f(x,y)\,dA \le \iint_D h(x,y)\,dA$



(Example 3) Prove that $\frac{1}{\sqrt{3}} \le \iint_D \frac{1}{\sqrt{1+x^6+y^8}}\,dA \le 1$

where D is the unit square $[0,1]\times[0,1]$



For $(x,y) \in D = [0,1]\times[0,1]$ ...

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{1+1^6+1^8}} \le \frac{1}{\sqrt{1+x^6+y^8}} \le \frac{1}{\sqrt{1+0^6+0^8}} = \frac{1}{\sqrt{1}} = 1$$

So

$$\iint_D \frac{1}{\sqrt{3}}\, dA \le \iint_D \frac{1}{\sqrt{1+x^6+x^8}}\, dA \le \iint_D 1\, dA$$

$$\frac{1}{\sqrt{3}}(1) = \frac{1}{\sqrt{3}} \text{Area}(D) \le \iint_D \frac{1}{\sqrt{1+x^6+x^8}}\, dA \le \text{Area}(D) = 1$$

Therefore,

$$\frac{1}{\sqrt{3}} \le \iint_D \frac{1}{\sqrt{1+x^6+y^8}}\, dA \le 1. \checkmark$$

(Example) Prove $e-1 \le \iint_D e^{(x^2y+y)}\, dA \le \frac{e^2}{2}-\frac{1}{2}$ where D is the unit square.

$$e^y = e^{0y+y} \le e^{x^2y+y} \le e^{1y+y} = e^{2y}$$

So

$$\int_0^1\int_0^1 e^y\, dy\, dx \le \iint_D e^{x^2y+y}\, dA \le \int_0^1\int_0^1 e^{2y}\, dy\, dx$$

$$\int_0^1 e-1\, dx \le \iint_D \sim dA \le \int_0^1 \frac{e^2}{2}-\frac{1}{2}\, dx$$

$$e-1 \le \iint_D \sim dA \le \frac{e^2}{2}-\frac{1}{2}. \checkmark$$

HW 5.4 1-5, 7-10