# 3.2 Taylor's Theorem

From Cal 2...

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$ ← Taylor Series

For example, $f(x)=e^x$ and $x_0=0$....

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$$

$$\approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$ ← Taylor Polynomial

↑ for $x \approx 0 = x_0$

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1st-order Taylor Polynomial:

$$f(x) \approx \frac{f^{(0)}(x_0)}{\cancel{0!}}\cancel{(x-x_0)^0} + \frac{f^{(1)}(x_0)}{\cancel{1!}}(x-x_0)^{\cancel{1}}$$

$$= f(x_0) + f'(x_0)(x-x_0) = L(x)$$

↑ linear approximation

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For $f:\mathbb{R}^n \to \mathbb{R}$, the 1st order Taylor Polynomial/Formula is given by

$$f(\underline{x}) \approx L(\underline{x}) = f(\underline{x}_0) + Df(\underline{x}_0)(\underline{x}-\underline{x}_0) \quad \text{for } \underline{x} \approx \underline{x}_0$$

$$= f(\underline{x}_0) + \left[\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right](\underline{x}_0)\begin{bmatrix} x_1 - x_{01} \\ \vdots \\ x_n - x_{0n} \end{bmatrix}$$

$$f(\underline{x}) \approx L(\underline{x}) = f(\underline{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{x}_0)\left(x_i - x_{0i}\right)$$

For $\underline{x} = (x, y) \ldots \underline{x}_0 = (x_0, y_0)$ (so $n = 2$)

$$f(x,y) \approx L(x,y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0)$$
$$+ \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

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To get better approximations, increase the order to two...

For $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) \approx \frac{f(x_0)}{\cancel{0!}} + \frac{f'(x_0)}{\cancel{1!}}(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \quad \text{when } x \approx x_0$$

For $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\underline{x}) \approx f(\underline{x}_0) + \cancel{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{x}_0)(\underline{x} - \underline{x}_0) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x}_0)(\underline{x} - \underline{x}_0)^2}$$
$$+ \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{x}_0)(x_i - x_{0i}) \quad \text{when } \underline{x}_0 \approx \underline{x}$$
$$+ \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x}_0)(x_i - x_{0i})(x_j - x_{0j})$$

So, when $n = 2$, we have $\underline{x} = (x, y)$ and $\underline{x}_0 = (x_0, y_0)$...

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$+\frac{1}{2}\frac{\partial^2 f}{\partial x\partial x}(x_0)(x-x_0)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y\partial y}(x_0,y_0)(y-y_0)^2$$

$$+\frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)}(x-x_0)(y-y_0) + \frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)}(y-y_0)(x-x_0)$$

same due to Mixed Derivative Thm

$$= f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x_0,y_0)(y-y_0)^2$$

$$+\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)(x-x_0)(y-y_0)$$

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(Example) Use the $2^{nd}$-order Taylor formula for $f(x,y)=\sqrt{x+2y}$ $=(x+2y)^{1/2}$ near the point $\underset{\mathbf{x}_0}{\mathbf{x}_0}=(\underset{x_0}{2},\underset{y_0}{1})$ to approximate $\sqrt{4.05}$.

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$$\frac{\partial f}{\partial x} = \tfrac{1}{2}(x+2y)^{-1/2}(1+0) = \frac{1}{2\sqrt{x+2y}} = \tfrac{1}{2}(x+2y)^{-1/2}$$

$$\frac{\partial f}{\partial y} = \cancel{\tfrac{1}{2}}(x+2y)^{-1/2}\cancel{(0+2)} = \frac{1}{\sqrt{x+2y}} = (x+2y)^{-1/2}$$

$$\frac{\partial f}{\partial x}(2,1) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$\frac{\partial f}{\partial y}(2,1) = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x^2} = \tfrac{1}{2}(-\tfrac{1}{2})(x+2y)^{-3/2}(1+0) = -\tfrac{1}{4}(x+2y)^{-3/2}$$

$$\frac{\partial^2 f}{\partial x^2}(2,1) = -\tfrac{1}{4}(4)^{-3/2} = -\frac{1}{4}\cdot\frac{1}{8} = -\frac{1}{32}$$

$$\frac{\partial^2 f}{\partial y^2} = -\cancel{\tfrac{1}{2}}(x+2y)^{-3/2}\cancel{(0+2)} = -(x+2y)^{-3/2}$$

$$\frac{\partial^2 f}{\partial y^2}(2,1) = -(4)^{-3/2} = -\frac{1}{8}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \cancel{2}\left(-\tfrac{1}{2}\right)\left(x+2y\right)^{-3/2}\cancel{(0+2)}$$

$$= -\tfrac{1}{2}\left(x+2y\right)^{-3/2}$$

$$\frac{\partial^2 f}{\partial x \partial y}(2,1) = -\tfrac{1}{2}\left(4\right)^{-3/2} = -\frac{1}{16}$$

Using the $2^{nd}$ order Taylor formula...

$$f(x,y) \approx f(2,1) + \underbrace{\tfrac{1}{4}}_{\frac{\partial f}{\partial x}(2,1)}(x-2) + \underbrace{\tfrac{1}{2}}_{\frac{\partial f}{\partial y}(2,1)}(y-1)$$

True when $(x,y) \approx (2,1)$

$$+\frac{1}{2}\underbrace{\left(-\frac{1}{32}\right)}_{\frac{\partial^2 f}{\partial x^2}(2,1)}(x-2)^2 + \frac{1}{2}\underbrace{\left(-\frac{1}{8}\right)}_{\frac{\partial^2 f}{\partial y^2}(2,1)}(y-1)^2 + \underbrace{\left(-\frac{1}{16}\right)}_{\frac{\partial^2 f}{\partial x \partial y}(2,1)}(x-2)(y-1).$$

As a result... when $x=2$ and $y=1.025$ ($\leftarrow$ other possibilities as well),

$$\sqrt{4.05} = \sqrt{x+2y} = f(x,y)$$

$$\approx \sqrt{4} + \cancel{\tfrac{1}{4}(2-2)} + \tfrac{1}{2}(1.025-1)$$

$$+\cancel{\frac{1}{2}\left(-\frac{1}{32}\right)(2-2)^2} + \frac{1}{2}\left(-\frac{1}{8}\right)(1.025-1)^2$$

$$+\cancel{\left(-\frac{1}{16}\right)(2-2)(1.025-1)}$$

$$= 2 + \frac{0.025}{2} - \frac{0.025^2}{16}$$

$$= 2.01246$$

From a calculator, $\sqrt{4.05} \approx 2.0124611 8$

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