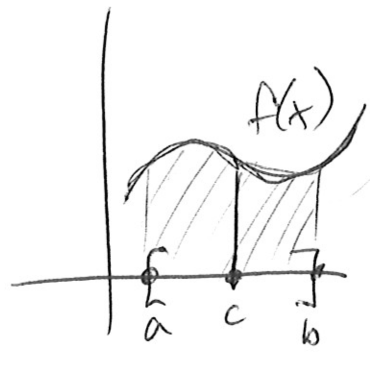
Note on 5.3 (Double Integrals)

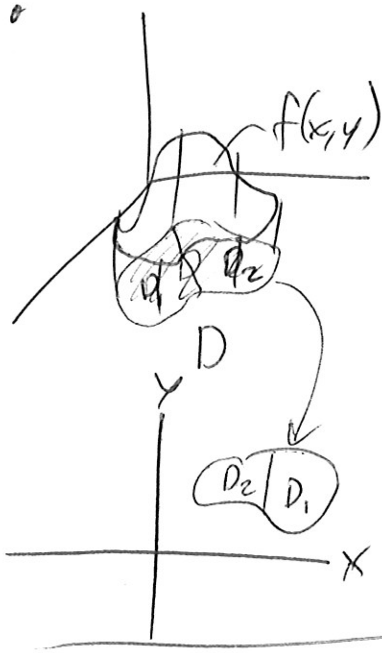
Additivity:

$\mathbb{R}^1$:



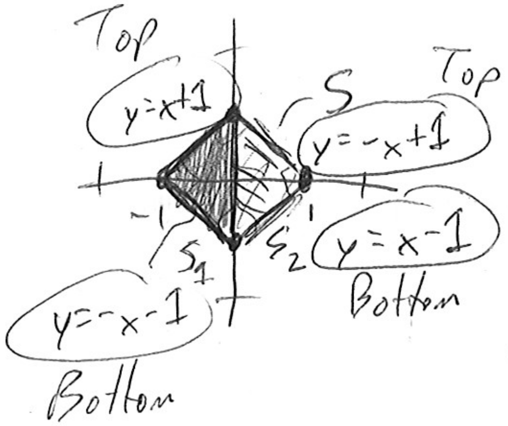
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$\mathbb{R}^2$:



$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

(Example) Prove that the area of the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$, & $(0,-1)$ is 2 by using the additivity of double integrals.



$$\text{Area} = \iint_S 1\, dA$$

(Hard way:)

$$= \int_{-1}^{1} \int_{|x|-1}^{1-|x|} 1\, dy\, dx$$

Better way:

$$= \iint_{S_1} 1\, dA + \iint_{S_2} 1\, dA$$

$$= \int_{-1}^{0} \left[\int_{-x-1}^{x+1} 1\, dy \right] dx + \int_{0}^{1} \left[\int_{x-1}^{-x+1} 1\, dy \right] dx$$

$$= \int_{-1}^{0} \Big[y \Big]_{-x-1}^{x+1} dx + \int_{0}^{1} \Big[y \Big]_{x-1}^{-x+1} dx$$

$$= \int_{-1}^{0} 2x+2\, dx + \int_{0}^{1} -2x+2\, dx$$

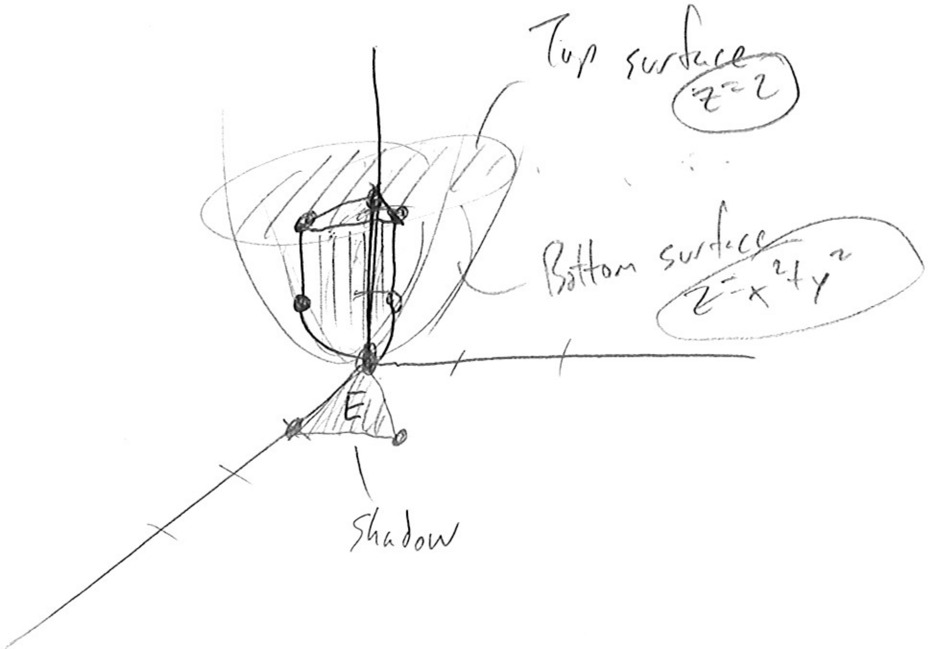
$$= \Big[x^2+2x \Big]_{-1}^{0} + \Big[-x^2+2x \Big]_{0}^{1}$$

$$= (0)-(1-2) + (-1+2)-(0)$$

$$= +1 \quad +1 \quad = \boxed{2}$$

5.5 cont.

(Example 6) Express $\iiint_W x\,dV$ where W is the solid above the triangle with vertices $(0,0,0)$, $(1,0,0)$, & $(1,1,0)$ in the xy plane, and also between the surfaces $z=x^2+y^2$ and $z=2$, as an iterated integral. Then evaluate it.



$$\iiint_W x\,dV = \iint_E \left[\int_{x^2+y^2}^{2} x\,dz \right] dA$$

$$= \int_{y=0}^{y=1} \int_{x=y}^{x=1} \left[\int_{z=x^2+y^2}^{z=2} x \, dz \right] dx \, dy$$

$$= \int_0^1 \int_y^1 \left(\int_{x^2+y^2}^{2} x \, dz \right) dx \, dy$$

$$= \int_0^1 \int_y^1 \Big[xz \Big]_{z=x^2+y^2}^{z=2} dx \, dy$$

$$= \int_0^1 \int_y^1 2x - x(x^2+y^2) \, dx \, dy$$

$$= \int_0^1 \left[\int_y^1 2x - xy^2 - x^3 \, dx \right] dy$$

$$= \int_0^1 \left[x^2 - \frac{1}{2}x^2 y^2 - \frac{1}{4}x^4 \right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(1 - \frac{1}{2}y^2 - \frac{1}{4}\right) + \left(-y^2 + \frac{1}{2}y^4 + \frac{1}{4}y^4\right) dy$$

$$= \int_0^1 \frac{3}{4} - \frac{3}{2}y^2 + \frac{3}{4}y^4 \, dy$$

$$= \left[\tfrac{3}{4}y - \tfrac{1}{2}y^3 + \tfrac{3}{20}y^5 \right]_0^1$$

$$= \left(\frac{3}{4} - \frac{1}{2} + \frac{3}{20} \right) \cancel{- (0)}$$

$$= \frac{15 - 10 + 3}{20} = \frac{8}{20} = \boxed{\frac{2}{5}}$$

Fun fact: ~~Additivity~~ Additivity holds for triple integrals also.

$$\iiint_D f(x,y,z)\,dV$$

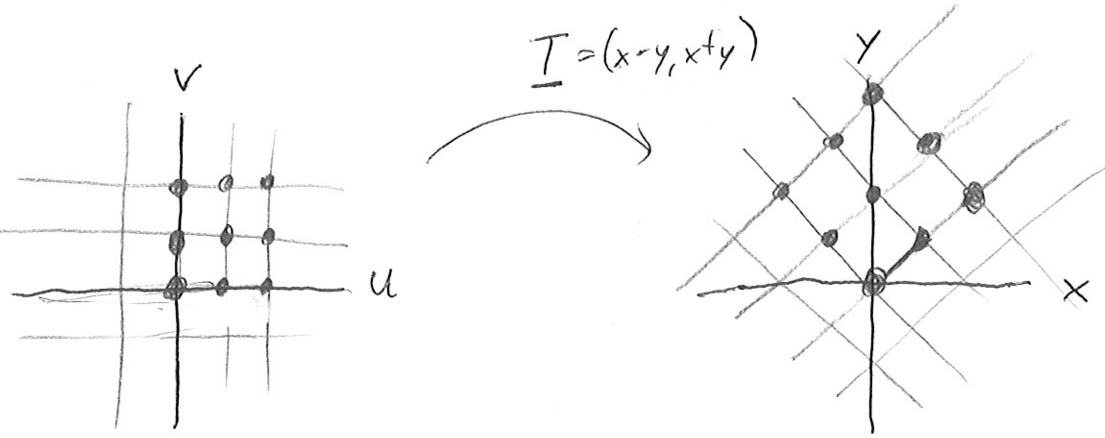
$$= \iiint_{D_1} f(x,y,z)\,dV + \iiint_{D_2} f(x,y,z)\,dV$$

5.5 HW
1-6, 11-17, 25-28

1.4 Polar, Cylindrical, & Spherical Coordinates

A transformation of variables is a function
$\underline{T}: \mathbb{R}^n \to \mathbb{R}^n$

(For example)

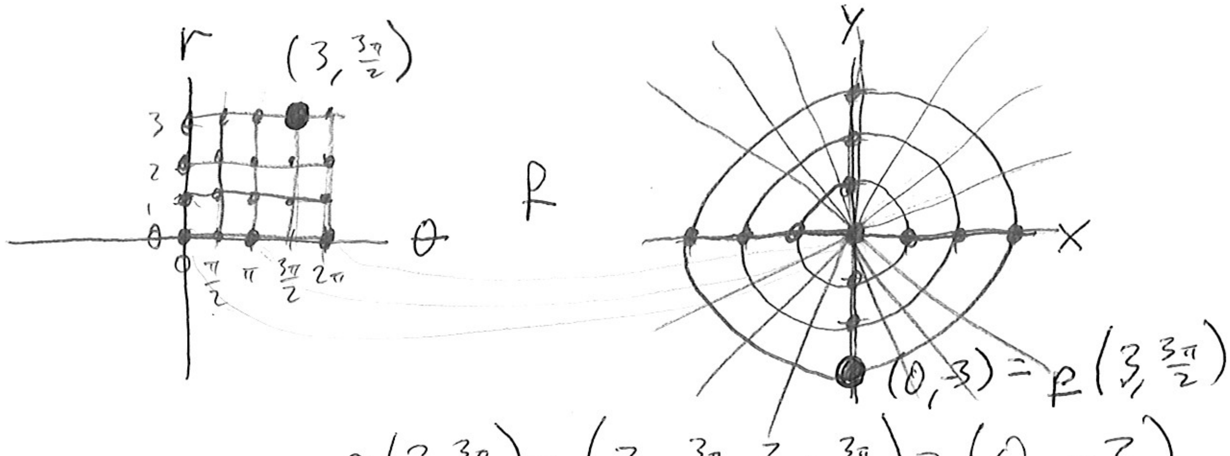


$\mathbb{R}^2 = \{(u,v) : u,v \in \mathbb{R}\}$ $\qquad$ $\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$

Polar Coordinates:

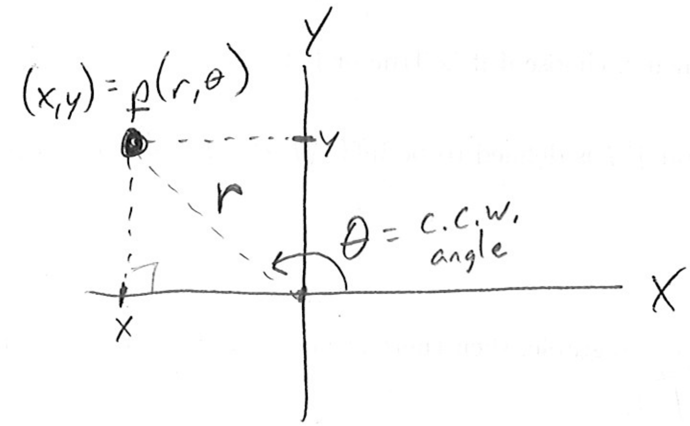
$\underline{P}: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\underline{P}(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$

(so $x = r\cos\theta$ $\quad y = r\sin\theta$.)



$\underline{P}(3, \frac{3\pi}{2}) = (3\cos\frac{3\pi}{2}, 3\sin\frac{3\pi}{2}) = (0, -3)$

In general...



Usually assume

$r \geq 0$

and

either $0 \leq \theta \leq 2\pi$

OR $-\pi \leq \theta \leq \pi$

FACTS!

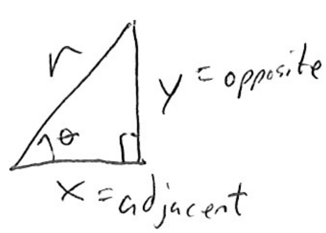
- $x^2 + y^2 = r^2$

(Because Pathagoram Theorem, OR

$$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2$$
$$= r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2\cancel{(\cos^2\theta + \sin^2\theta)}$$
$$= r^2$$
)

- $\tan\theta = \dfrac{y}{x}$

$r \neq 0$



(Because Trig, OR

$$\frac{y}{x} = \frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \tan\theta$$
)

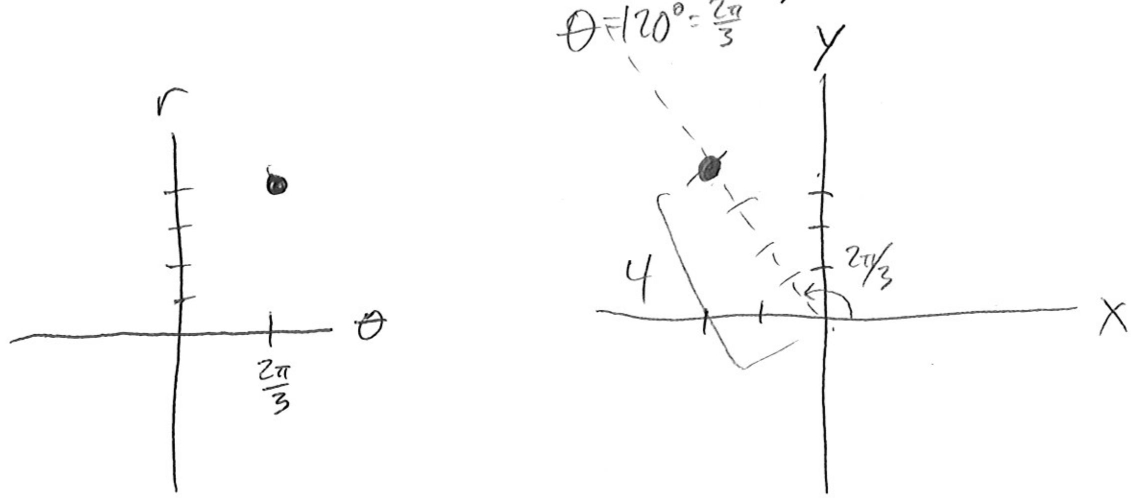
(Example) Convert $A = p\left(4, \frac{2\pi}{3}\right)$ from polar to Cartesian.

Plot in $r\theta$ and xy planes.

$$p\left(\underset{\substack{\uparrow \\ r}}{4}, \underset{\substack{\uparrow \\ \theta}}{\tfrac{2\pi}{3}}\right) = \left(4\cos\left(\tfrac{2\pi}{3}\right), 4\sin\left(\tfrac{2\pi}{3}\right)\right)$$

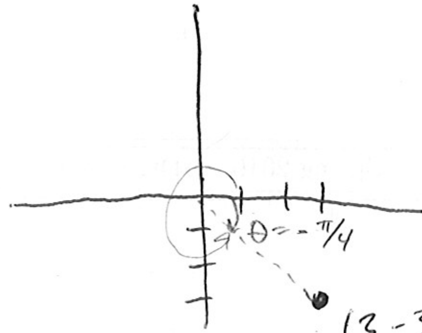
$$= \left(4\left(-\tfrac{1}{2}\right), 4\left(\tfrac{\sqrt{3}}{2}\right)\right)$$

$$= \left(-2, 2\sqrt{3}\right)$$



(Example) Convert $B = (3, -3)$ from Cartesian to Polar.

Plot in xy plane.



$(3, -3) = P\left(3\sqrt{2}, -\frac{\pi}{4}\right)$

Use the facts...

$$x^2 + y^2 = r^2$$
$$(3)^2 + (-3)^2 = r^2$$
$$18 = r^2$$
$$\sqrt{18} = r$$
$$3\sqrt{2} = r$$

$$\tan\theta = \frac{y}{x}$$
$$= \frac{-3}{3}$$
$$\tan\theta = -1$$
$$\frac{\sin\theta}{\cos\theta} = \frac{-1}{1}$$
$$\theta = -\pi/4$$