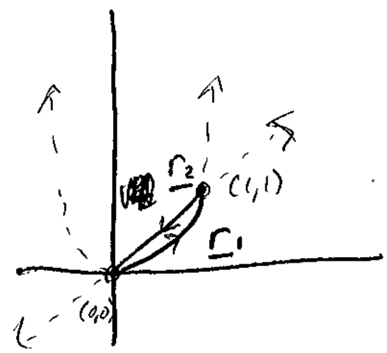
Parametrizing curves (Cal 3 review)

(Example) Parametrize the c.c.w. boundary of the region between $y = x$ and $y = x^2$.

Find a vector ~~function~~

$\underline{r} : \mathbb{R} \to \mathbb{R}^2$

for the curve.



For line segment

$$\underline{r}_2(t) = \underbrace{(1,1)}_{\text{start pt}} + t\,\underbrace{(-1,-1)}_{\text{vector from start}\to\text{end}}$$

$$\underline{r}_2(t) = (1-t,\ 1-t) \qquad 0 \le t \le 1$$

For piece of graph $y=f(x)$ Left to right

$$\underline{r}_1(t) = (\underset{\substack{\uparrow \\ x}}{t},\ \underset{\substack{\uparrow \\ y=f(x) \\ =x^2}}{t^2}) \qquad \underset{\substack{\uparrow \\ \text{least } x}}{0} \le t \le \underset{\substack{\uparrow \\ \text{greatest } x}}{1}$$

To combine:

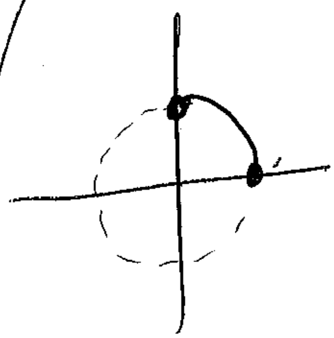
$$\boxed{\underline{r}(t) = \begin{cases} \underline{r}_1(t) & \text{for } 0 \le t \le 1 \\ \underline{r}_2(t-1) & \text{for } 1 \le t \le 2 \end{cases} = \begin{cases} (t, t^2) & \text{for } 0 \le t \le 1 \\ (2-t, 2-t) & \text{for } 1 \le t \le 2 \end{cases}}$$

What if you want $y=f(x)$ for $x\in[a,b]$ from Right to Left instead?

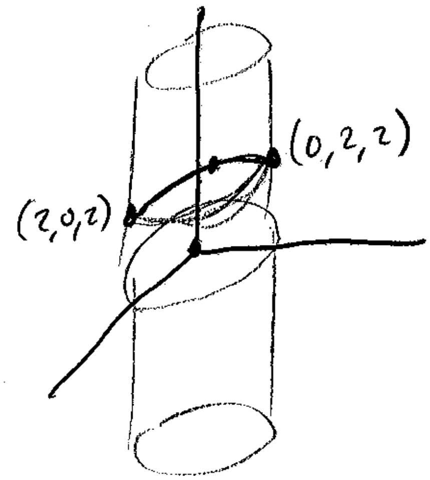
$$\underline{r}(t)=(-t,\ f(-t)) \quad -b\le t\le -a$$

curve from $(b,f(b))$ to $(a,f(a))$

(Example)
Parametrize the portion of the curve obtained by intersecting the cylinder $x^2+y^2=4$ and the plane $z=x+y$, between the points $(2,0,2)$ and $(0,2,2)$.



projection in xy plane



In $\mathbb{R}^2$, could use

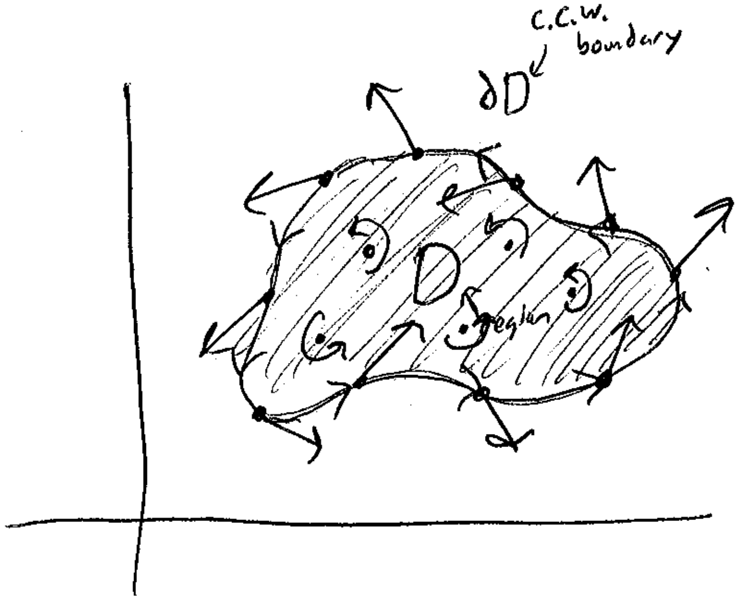
$$\underline{r}(t) = p(\underset{r}{\underset{\uparrow}{2}}, \underset{\theta}{\underset{\uparrow}{t}}) \quad \text{for } 0 \le \underset{\theta}{\underset{\uparrow}{t}} \le \pi/2$$

$$\underline{r}(t) = (\underset{x}{\underset{\uparrow}{2\cos t}}, \underset{y}{\underset{\uparrow}{2\sin t}}) \quad \text{for } 0 \le t \le \pi/2$$

So for 3D, let

$$\underline{r}(t) = (\underset{x}{\underset{\uparrow}{2\cos t}}, \underset{y}{\underset{\uparrow}{2\sin t}}, \underset{z=x+y}{2\cos t + 2\sin t}) \quad \text{for } 0 \le t \le \pi/2$$

8.1 Green's Theorem



$$\underbrace{\int_{\partial D} \underline{F} \cdot d\underline{s}}_{\substack{\text{Work done} \\ \text{by force } \underline{F} \\ \text{across } \partial D}} = \iint_D \underbrace{\text{scurl } \underline{F}}_{\substack{\text{tendency of} \\ \underline{F} \text{ to rotate} \\ \text{c.c.w. around} \\ \text{a point}}} \, dA$$

$$= \iint_D \left(\text{curl } \underline{F} \cdot \hat{k} \right) dA$$

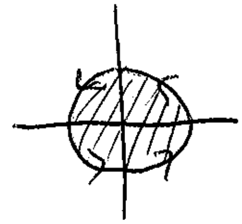
$$\boxed{\int_{\partial D} \underline{F} \cdot d\underline{s} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA}$$

In the book, $\underline{F} = (F_1, F_2) = (P, Q)$

(Example 1) Verify Green's Theorem for $\underline{F} = (x, xy)$ and $D = \{(x,y) : x^2 + y^2 \leq 1\}$.

Direct method:

$$\int_{\partial D} \underline{F} \cdot d\underline{s}$$



$$= \int_{t=0}^{t=2\pi} \underline{F}(\cos t, \sin t) \cdot \frac{d\underline{r}}{dt}\, dt$$

$\underline{r}(t) = \underline{p}(1, t)$
$= (\cos t, \sin t)$
$0 \leq t \leq 2\pi$

$\frac{d\underline{r}}{dt} = (-\sin t, \cos t)$

$$= \int_0^{2\pi} (\cos t, \cos t \sin t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} -\sin t \cos t + \cos^2 t \sin t\, dt$$

$$= \left[\frac{1}{2}\cos^2 t - \frac{1}{3}\cos^3 t\right]_0^{2\pi}$$

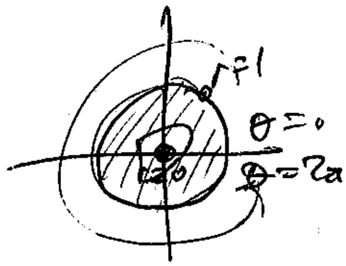
$$= \left(\frac{1}{2}\cancel{\cos^2(2\pi)} - \frac{1}{3}\cancel{\cos^3(2\pi)}\right) - \left(\frac{1}{2}\cancel{\cos^2(0)} - \frac{1}{3}\cancel{\cos^3(0)}\right)$$

$$= \boxed{0}$$

Green's Thm

$$\int_{\partial D} \underline{F} \cdot d\underline{s} = \iint_D \text{scurl } \underline{F} \, dA$$

$$= \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA$$



$$= \iint_D y - 0 \, dA$$

$$= \iint_D y \, dA$$

Use polar coordinate transformation:

$$= \underbrace{\int_0^{2\pi}}_{\substack{\theta \text{ bounds} \\ \left(\substack{\text{c.w.} \\ \text{to} \\ \text{c.c.w.}}\right)}} \underbrace{\int_0^1}_{\substack{r \text{ bounds} \\ \left(\substack{\text{inside} \\ \text{to} \\ \text{outside}}\right)}} \underbrace{(r\sin\theta)}_{y} \underbrace{(r)}_{\substack{\text{Jacobian for} \\ \text{polar}}} \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\int_0^1 \sin\theta \, r^2 \, dr \right] d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{3} \sin\theta \, r^3 \right]_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{3} \sin\theta \, d\theta$$

$$= \left[-\frac{1}{3} \cos\theta \right]_0^{2\pi} = \boxed{0}$$

$$= -\frac{1}{3} \cos 2\pi + \frac{1}{3} \cos 0 = -\frac{1}{3} + \frac{1}{3}$$

(Example) Use Green's Theorem to prove that the area of a region D is

$$\frac{1}{2}\int_{\partial D} x\,dy - y\,dx.$$

$$\left(\text{Area of } D = \iint_D 1\,dA\right)$$

$$\frac{1}{2}\int_{\partial D} x\,dy - y\,dx = \int_{\partial D} \underbrace{\left(-\tfrac{1}{2}y, \tfrac{1}{2}x\right)}_{\underline{F}} \cdot d\underline{s}^{(dx,dy)}$$

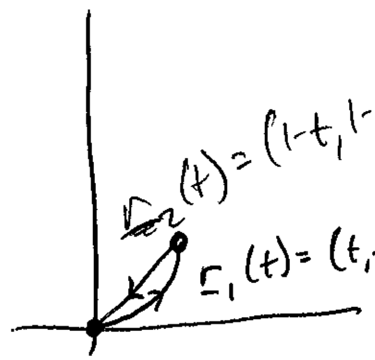
$$= \iint_D \operatorname{scurl} \underline{F}\, dA$$

$$= \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$$

$$= \iint_D \tfrac{1}{2} - \left(-\tfrac{1}{2}\right) dA$$

$$= \iint_D 1\, dA = \text{Area of } D. \quad \square$$

(Example 3) Compute the work done by using $\underline{F}=(xy^2,\ y+x)$ in moving an object from the origin to $(1,1)$ along $y=x^2$, and then back to the origin along $y=x$.



$\underline{r}_2(t)=(1-t,1-t)$ for $0\le t\le 1$, $\frac{d\underline{r}_2}{dt}=(-1,-1)$

$\underline{r}_1(t)=(t,t^2)$ for $0\le t\le 1$, $\frac{d\underline{r}_1}{dt}=(1,2t)$

Direct method:

$$\int_C \underline{F}\cdot d\underline{s} = \int_0^1 \underline{F}(\underline{r}_1(t))\cdot\frac{d\underline{r}_1}{dt}\,dt + \int_0^1 \underline{F}(\underline{r}_2(t))\cdot\frac{d\underline{r}_2}{dt}\,dt$$

$$=\int_0^1 \Big(\underbrace{(t)(t^2)^2}_{xy^2},\ \underbrace{(t^2)+(t)}_{y+x}\Big)\cdot(1,2t)\,dt$$

$$+\int_0^1 \Big(\underbrace{(1-t)(1-t)^2}_{xy^2},\ \underbrace{(1-t)+(1-t)}_{y+x}\Big)\cdot(-1,-1)\,dt$$

$$=\int_0^1 (t^5)+(2t^3+2t^2)\,dt + \int_0^1 (-1+3t-3t^2+t^3) + (2-2t)\,dt$$

$$=\int_0^1 t^5+2t^3+2t^2+1+t-3t^2+t^3\,dt$$

$$= \int_0^1 t^5 + 3t^3 - t^2 + t + 1 \, dt$$

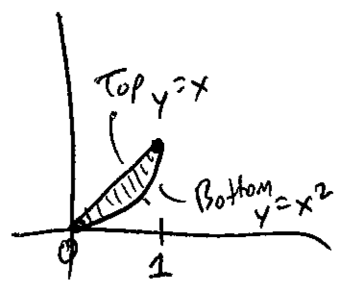
$$= \left[\frac{1}{6}t^6 + \frac{3}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + t \right]_0^1$$

$$= 1/6 + 3/4 - 1/3 + 1/2 + 1$$

$$= \frac{2 + 9 - 4 + 6 + 12}{12}$$

$$= \boxed{25/12}$$

Green's Theorem



$$\int_C \underline{F} \cdot d\underline{s} = \iint_D \text{scurl } \underline{F} \, dA$$

$$= \int_0^1 \int_{x^2}^{x} \text{scurl } \underline{F} \, dy \, dx$$

$$= \int_0^1 \int_{x^2}^{x} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dy \, dx$$

$$= \int_0^1 \left[\int_{x^2}^{x} 1 - 2xy \, dy \right] dx$$

$$= \int_0^1 \left[y - xy^2 \right]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^3) - (x^2 - x^5)\, dx$$

$$= \int_0^1 x - x^3 - x^2 + x^5 \, dx$$

$$= \left[\frac{1}{2}x^2 - \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{6}x^6 \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{4} - \frac{1}{3} + \frac{1}{6}$$

$$= \frac{6 - 3 - 4 + 2}{12} = \boxed{\frac{1}{12}}$$

8.1 HW 1-6, 9-10, 15

Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = \Big[\underset{\substack{\uparrow \\ \text{antiderivative}}}{F(x)}\Big]_a^b = F(b) - F(a)$$

Want:

$$\int_C \underline{F} \cdot d\underline{s} = \Big[\begin{matrix}\text{anti-derivative} \\ \text{thing?}\end{matrix}\Big]_{\text{Start pt}}^{\text{End pt}}$$

8.3 Conservative Fields

Four ~~definitions~~ characterizations for conservative fields $\underline{F}: \mathbb{R}^n \to \mathbb{R}^n$:

(1) There exists a potential function (antiderivative thing) $f: \mathbb{R}^n \to \mathbb{R}$ such that $\underline{F} = \nabla f$, and for any curve C starting at pt A and ending at pt B,

$$\int_C \underline{F} \cdot d\underline{s} = [f(\underline{x})]_A^B = f(B) - f(A).$$

(Fundamental Thm of Line Integrals:

$$\int_{\substack{C \\ \uparrow \\ \text{from } A \\ \text{to } B}} \nabla f \cdot d\underline{s} = f(B) - f(A).$$
)

(2) curl $\underline{F} = \underline{0}$.

(3) $\int \underline{F} \cdot d\underline{s}$ is path-independent: if C_1 & C_2 start and end at the same points as each other, then $\int_{C_1} \underline{F} \cdot d\underline{s} = \int_{C_2} \underline{F} \cdot d\underline{s}$

(4) For any simple closed curve C (starts & ends at the same point), $\int_C \underline{F} \cdot d\underline{s} = 0$.