~~Hello world.~~

1.5 (cont)

(Example) Prove C-S inequality: $|\underline{x}\cdot\underline{y}| \leq \|\underline{x}\|\|\underline{y}\|$.

Proof: Same as proving

$$0 \leq \|\underline{x}\|^2\|\underline{y}\|^2 - |\underline{x}\cdot\underline{y}|^2.$$

Applying definitions to the right-hand side we see that,

$$\|\underline{x}\|^2\|\underline{y}\|^2 - |\underline{x}\cdot\underline{y}|^2 = (\underline{x}\cdot\underline{x})(\underline{y}\cdot\underline{y}) - \left(\sum_{i=1}^{n} x_i y_i\right)^2$$

$$= \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)^2$$

$$= (x_1^2+\cdots+x_n^2)(y_1^2+\cdots y_n^2) - (x_1y_1+\cdots+x_ny_n)(x_1y_1+\cdots+x_ny_n)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i^2 y_j^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_i x_j y_j$$

$$= \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(x_i^2 y_j^2 - x_i y_i x_j y_j\right)$$

$$= \sum_{1\leq i<j\leq n} \left(x_i^2 y_j^2 - x_i y_i x_j y_j + x_j^2 y_i^2 - x_j y_j x_i y_i\right)$$

$$= \sum_{1 \le i < j \le n} \left(x_i^2 y_j^2 - 2 x_i y_i x_j y_j + x_j^2 y_i^2 \right)$$

$$= \sum_{1 \le i < j \le n} \left(x_i y_j - x_j y_i \right)^2 \ge 0. \quad \square$$

Triangle Inequality: $\|\underline{x} + \underline{y}\| \le \|\underline{x}\| + \|\underline{y}\|$

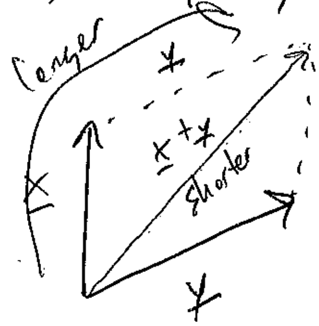
(Example) Prove it.

Proof: Consider the left side squared:

$$\begin{aligned}
\|\underline{x} + \underline{y}\|^2 &= (\underline{x} + \underline{y}) \cdot (\underline{x} + \underline{y}) \\
&= \underline{x} \cdot \underline{x} + \underline{x} \cdot \underline{y} + \underline{y} \cdot \underline{x} + \underline{y} \cdot \underline{y} \\
&= \|\underline{x}\|^2 + 2\, \underline{x} \cdot \underline{y} + \|\underline{y}\|^2 \\
&\le \|\underline{x}\|^2 + 2 |\underline{x} \cdot \underline{y}| + \|\underline{y}\|^2 \\
&\le \|\underline{x}\|^2 + 2 \|\underline{x}\| \|\underline{y}\| + \|\underline{y}\|^2 \\
&= \left(\|x\| + \|y\| \right)^2
\end{aligned}$$

which is the right side squared. $\square$

Sketch of the triangle inequality:



$$\|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$$

<u>Matrices</u> An $m \times n$ matrix is expressed as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Ex: $B = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 3 & 4 \end{bmatrix}$

then $b_{12} = 1$
$b_{31} = 3$, etc.

same dimensions

• Adding: $A+B$ is given by $(a+b)_{ij} = a_{ij} + b_{ij}$

Ex: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 0 & 1 \\ -2 & -2 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 5 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 3 \end{bmatrix}$

• Scalar Mult: αA is given by $(\alpha a)_i = \alpha(a_i)$

Ex: $\frac{1}{2}\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

• Transpose: A^T is given by $(a^T)_{ij} = a_{ji}$.

Ex: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

3 by 2   2 by 3

Vectors as Matrices

$$\underline{x} = (x_1, \cdots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

n-dim vector

$n \times 1$ dim matrix

$$\underline{x}^T = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$$

Matrix Multiplication

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} \text{Take dot product} \\ \text{of rows} \\ \text{with columns} \end{bmatrix}$$

$m \times n$ matrix — SAME — $n \times p$ matrix

$m \times p$ matrix

(Example 4)

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(3x3) (3x3)

Find AB.

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0+0+0 & 1+0+3 & 0+0+3 \\ 0+1+0 & 2+0+0 & 0+0+0 \\ 0+0+0 & 1+0+0 & 0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \checkmark.$$
(3×3)

(Example 5)

$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ (2×3) $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (3×3) Compute AB.

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 5 \\ 3 & 4 & 5 \end{bmatrix}$$
(2×3)

Note that in Example 4:

$$BA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & \sim \\ \sim & \\ \sim & \end{bmatrix} \neq \begin{bmatrix} 0 & \sim \\ \sim & \\ \sim & \end{bmatrix} = AB$$

And in Example 5:

two things

three things

$$BA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \cancel{\begin{bmatrix} & ? & \end{bmatrix}} \quad \text{DNE}$$

(3×3) ← NOT SAME → (2×3)

Matrices as Linear Transformations

Each matrix A (n×m dim) defines a special function $\mathbb{R}^m \to \mathbb{R}^n$:

$$\underline{x} \mapsto \underset{(n\times m)}{A}\ \underset{(m\times 1)}{\underline{x}}$$

same

m-dim vector
OR
m×1 dim matrix

It satisfies the property $(\alpha \underline{x} + \beta \underline{y}) \mapsto \alpha A \underline{x} + \beta A \underline{y}$

(Example 7) Express $A\underline{x}$ where $\underline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ and

$$A = \underset{(4\times 3)}{\begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}}$$

4-dim vector
OR
4×1 matrix

$$A\underline{x} = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 0x_2 + 3x_3 \\ -x_1 + 0x_2 + x_3 \\ 2x_1 + x_2 + 2x_3 \\ -x_1 + 2x_2 + 2x_3 \end{bmatrix}$$

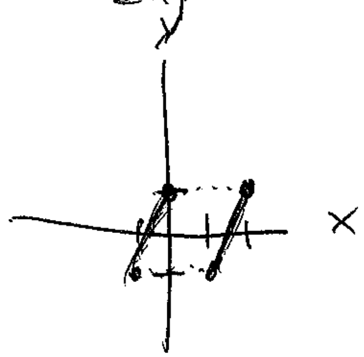
$$= (x_1 + 3x_3,\ -x_1 + x_3,\ 2x_1 + x_2 + 2x_3,\ -x_1 + 2x_2 + 2x_3)$$

(Example) Plug ~~and their images~~ $(-1,-1,0)$, $(0,1,0)$, $(1,-1,1)$, $(2,1,1)$ into $A\underline{x}$, then compare two-dimensional projections.

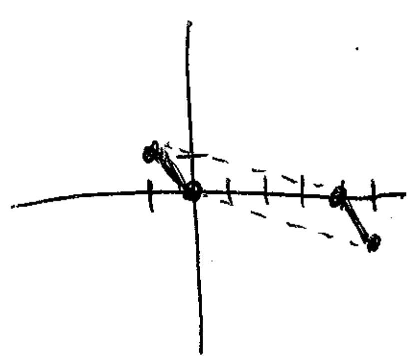
$$(-1,-1,0) \mapsto (-1+3(0), -(-1)+0, 2(-1)+(-1)+2(0), -(-1)+2(-1)+0)$$
$$= (-1, 1, -3, -1)$$

$$(0,1,0) \mapsto (0, 0, 1, 2)$$
$$(1,-1,1) \mapsto (4, 0, 3, -1)$$
$$(2,1,1) \mapsto (5, -1, 7, 2)$$

Input



Output



Identity & Inverse:

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$ is the $n \times n$ identity matrix.

FACT: $AI = IA = A$ for any $n \times n$ matrix A.

Let A^{-1} be the inverse of A if:

$$AA^{-1} = A^{-1}A = I.$$

FACT: Not all matrices have an inverse. Actually:

A has an inverse if and only if $\det(A) \neq 0$

Determinant:

$$\det A = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \det(A_i)$$ matrix where the 1st row and ith column is removed

$$= \sum_{\sigma \in S^n} \operatorname{sgn}(\sigma) \prod_{1 \le i \le n} a_{i\sigma_i}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} = + a_{11} \det \begin{bmatrix} a_{22} & \cdots & a_{n2} \\ \vdots & \ddots & \vdots \\ a_{2n} & \cdots & a_{nn} \end{bmatrix} - a_{12} \begin{bmatrix} \sim \end{bmatrix} + \cdots$$