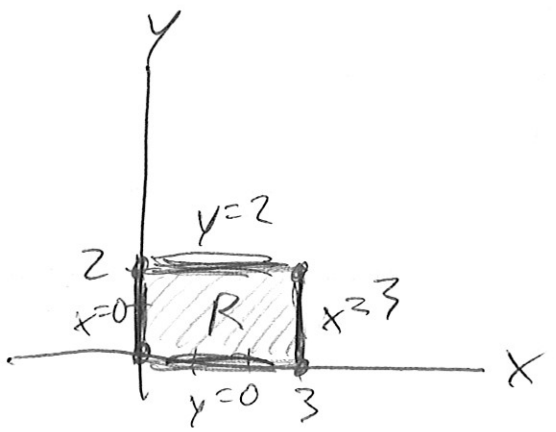
5.3 cont.

Double integral

(Example) Express $\iint_R (12x^3y - 1)\,dA$, where R is the rectangle with vertices $(0,0)$ $(3,0)$ $(3,2)$ and $(0,2)$ as an iterated integral, then evaluate.



Iterated integrals

Two options:

$$\int_{\text{Left value}}^{\text{Right value}} \int_{\text{Bottom curve}}^{\text{Top curve}} 12x^3y - 1\,dy\,dx = \int_{x=0}^{x=3}\left[\int_{y=0}^{y=2} 12x^3y - 1\,dy\right]dx$$

$$\int_{\text{Bottom value}}^{\text{Top value}} \int_{\text{Left curve}}^{\text{Right curve}} 12x^3y - 1\,dx\,dy = \int_{y=0}^{y=2}\left[\int_{x=0}^{x=3} 12x^3y - 1\,dx\right]dy$$

$$= \int_0^3 \left[\int_0^2 12x^3 y - 1 \, dy \right] dx$$

variable
(x fixed)

$$= \int_0^3 \left[\cancel{12}^{\,6} x^3 \left(\frac{y^2}{\cancel{2}} \right) - y \right]_0^2 dx$$

$$= \int_0^3 \left[6x^3 y^2 - y \right]_{y=0}^{y=2} dx$$

Plug in for y

$$= \int_0^3 \left(6x^3 (2)^2 - (2) \right) - \cancel{\left(6x^3 (0)^2 - 0 \right)} \, dx$$

$$= \int_0^3 24x^3 - 2 \, dx$$

$$= \left[6x^4 - 2x \right]_0^3 = \left(6(81) - 2(3) \right) - \cancel{(0 - 0)}$$

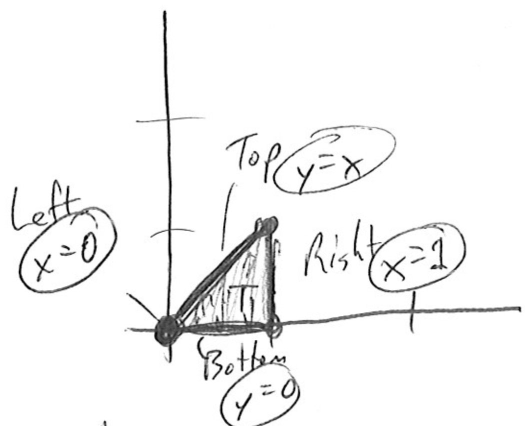
$$= 486 - 6$$

$$= \boxed{480}$$

You can check that $\int_{y=0}^{y=2} \int_{x=0}^{x=3} 12x^3 y - 1 \, dx \, dy = 480$ also.

(Example) Same problem for $\iint_T (12x^3y - 1)\, dA$

where T is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.



Outer bound Constants ONLY

Inner bound Variables OK

One option...

$$\int_{\text{Left value}}^{\text{Right val}} \int_{\text{Bottom curve}}^{\text{Top curve}} 12x^3y - 1\, dy\,dx = \int_{x=0}^{x=1}\left[\int_{y=0}^{y=x} 12x^3y - 1\, dy\right] dx$$

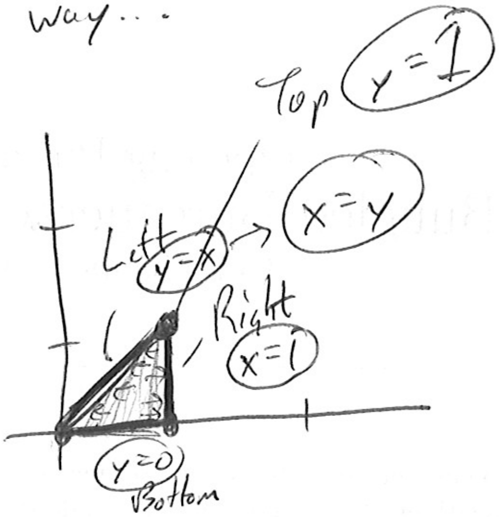
$$= \int_{x=0}^{x=1} \Big[6x^3y^2 - y\Big]_{y=0}^{y=x}\, dx$$

$$= \int_0^1 \left(6x^3(x)^2 - (x)\right) - \cancel{\left(6x^3(0)^2 - (0)\right)}\, dx$$

$$= \int_0^1 6x^5 - x\, dx = \left[x^6 - \frac{1}{2}x^2\right]_0^1$$

$$= \boxed{1/2}$$

Other way...

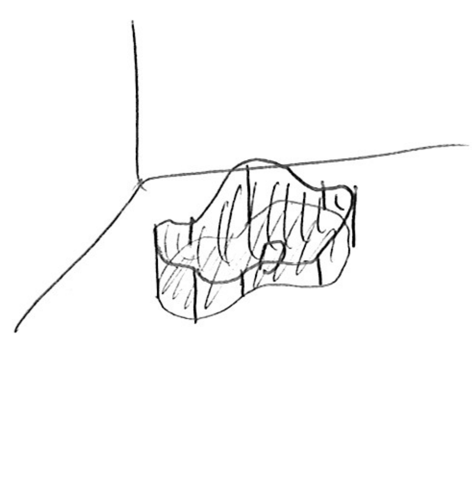


$$\int_{\text{Bottom value}}^{\text{Top value}} \int_{\text{Left curve}}^{\text{Right curve}} 12x^3y-1\,dx\,dy = \int_{y=0}^{y=1}\left[\int_{x=y}^{x=1} 12x^3y-1\,dx\right]dy$$

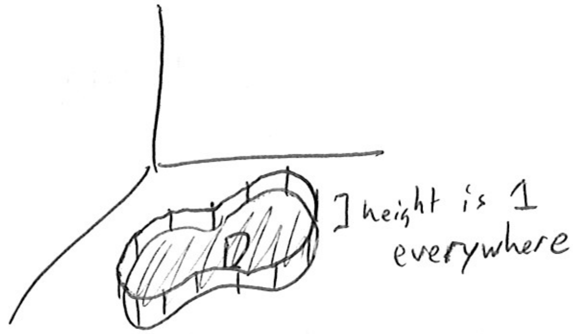
$$= \dots = 1/2$$

Applications:

- $\iint_D f(x,y)\,dA$ is the volume between $0 \le z \le f(x,y)$ (when $f(x,y)$ is nonnegative)



• Area: If $f(x,y) = 1 \ldots$

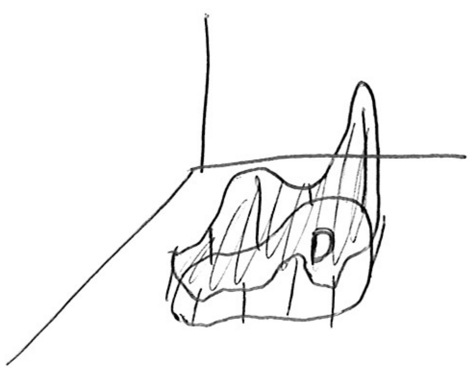


$$\text{Volume} = \iint_D f(x,y)\,dA = (\text{Area of } D)\cancel{(\text{height})}$$

$$\boxed{\iint_D 1\,dA = \text{Area of } D}$$

• Average Value:

(Compare to Avg val of $f(x)$ on $[a,b]$ $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$)



$$\boxed{\text{The Avg. Val. of } f(x,y) \text{ over } D = \text{The average height of the solid} = \frac{1}{\text{Area of } D}\iint_D f(x,y)\,dV}$$

5.3 HW: 1-9

5.4 Changing the Order of Integration

FACT: For constant bounds of integration (rectangular region of integration)...

$$\int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\,dy\,dx = \int_{y=c}^{y=d}\int_{x=a}^{x=b} f(x,y)\,dx\,dy$$

Otherwise, the bounds <u>cannot</u> be directly swapped.

No variables allowed!

$$\cancel{\int_0^1\int_0^x 12x^3y-1\,dy\,dx = \int_0^{x}\int_0^1 12x^3y-1\,dx\,dy}$$

From 5.3 we saw

$$\int_0^1\int_0^x 12x^3y-1\,dy\,dx = \int_0^1\int_y^1 12x^3y-1\,dx\,dy$$

Next time, we'll learn how to swap variable bounds by drawing the picture of the region of integration.

To be continued...