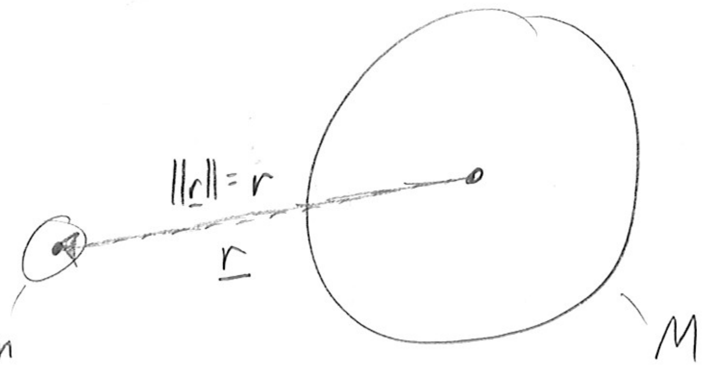
(4.3 cont.)

(Example 5) The gravitational potential of bodies with mass m, M is given by $V = -\frac{mMG}{r}$, where G is the gravitational constant and r is the distance between the bodies:



The gravitational force field is given by $\underline{F} = -\nabla V$. ($x, y, z$)
Show that $\underline{F} = -\frac{mMG}{r^3}\underline{r}$, where $\underline{r}$ is the vector pointing from M to m.

$$V = -\frac{mMG}{r}$$

$$\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

$$V(x,y,z) = -\frac{mMG}{\sqrt{x^2+y^2+z^2}} = -mMG(x^2+y^2+z^2)^{-1/2}$$

$$\underline{r} = (x,y,z)$$

$$= \left(+mMG\left(+\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-\frac{3}{2}}(2x), \sim, \sim\right)$$

$$= mMG(x^2+y^2+z^2)^{-3/2}(x,y,z)$$

$$\underline{F} = -\nabla V$$

$$= -mMG\left(x^2+y^2+z^2\right)^{-3/2}(x,y,z)$$

$$= -\frac{mMG}{\left(\sqrt{x^2+y^2+z^2}\right)^3}(x,y,z)$$

$$= -\frac{mMG}{\|\underline{r}\|^3}\underline{r} = -\frac{mMG}{r^3}\underline{r} \quad \checkmark$$

A vector field $\underline{F}: \mathbb{R}^n \to \mathbb{R}^n$ which may be defined as $\underline{F} = \nabla f$ for some "potential function" $f: \mathbb{R}^n \to \mathbb{R}$ is called <u>conservative</u>.

(Example) Show that $\underline{W} = (2y+1, 2x)$ is conservative.

If it is, there's $f: \mathbb{R}^2 \to \mathbb{R}$ where $\underline{W} = \nabla f$

$$(2y+1, 2x) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

So

$$\frac{\partial f}{\partial x} = 2y+1 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2x$$

$$f = 2xy + x + \phi(y) \qquad f = 2xy + \phi(x)$$

"phi"
function of integration

I need f to fit both formulas...

$$f(x,y) = \underline{2xy} + \underline{x}$$

(works because

$$f(x,y) = 2xy + x + \underbrace{0}_{\phi(y)} = 2xy + \underbrace{x}_{\phi(x)}.$$
)

Since $\underline{W} = \nabla f$, $\underline{W}$ is conservative.

(Example 7) Show that $\underline{V} = (y, -x)$ is not conservative.

If it was, then there exists $f: \mathbb{R}^2 \to \mathbb{R}$ where

$$\frac{\partial f}{\partial x} = y \quad \text{and} \quad \frac{\partial f}{\partial y} = -x$$

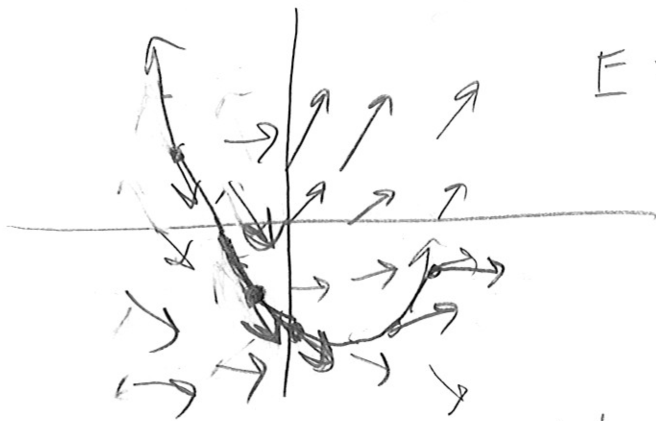
So then,

$$f = xy + \underbrace{\phi(y)}_{\text{just } y \text{ stuff}} \quad \text{and} \quad f = -xy + \underbrace{\phi(x)}_{\text{just } x \text{ stuff}}$$

Since the xy term cannot be positive & negative at the same time, f cannot exist. So $\underline{V}$ is not conservative.

Flow Lines

A *flow line* for a vector field $\underline{F}: \mathbb{R}^n \to \mathbb{R}^n$ is a path $\underline{c}: \mathbb{R} \to \mathbb{R}^n$ satisfying $\underline{c}'(t) = \underline{F}(\underline{c}(t))$.



$\underline{F}$ = tangent vectors to $\underline{c}$

$\underline{c}$ = path caused by moving along the "current" $\underline{F}$

(Example 8) Show that $\underline{c}(t) = (\cos t, \sin t)$ is flow line for $\underline{F} = (-y, x)$, and find some other flow lines.

Picture

$\left(\frac{1}{3}\underline{F}\right)$

flow lines seen to be c.c.w. circles centered @ origin

First show $\underline{c} = (\overset{x}{\cos t}, \overset{y}{\sin t})$ is a flow line.

$$\underline{c}'(t) = (-\sin t, \cos t).$$

$$\underline{F}(\underline{c}(t)) = (-y, x)$$

$$= (-(\sin t), (\cos t))$$

$$= (-\sin t, \cos t)$$

equal, so $\underline{c}$ is a flow line. ✓

Since any c.c.w. circle is of the form

$\alpha\underline{c} = (\overset{x}{\alpha\cos t}, \overset{y}{\alpha\sin t})$ for $\alpha > 0$, we see that

$$(\alpha\underline{c})'(t) = (\alpha(-\sin t), \alpha(\cos t))$$

$$= (-\alpha\sin t, \alpha\cos t)$$

and

$$\underline{F}(\underline{c}(t)) = (-(\alpha\sin t), (\alpha\cos t))$$

$$= (-\alpha\sin t, \alpha\cos t)$$

equal

So $\alpha\underline{c}$ is a flow line for $\underline{F}$ for all $\alpha > 0$.

By the way, we could find flow lines for $\underline{E}$ by using Diff EQ:

$$\underline{E}(\underline{c}(t)) = (-c_2, c_1) = \underline{c}'(t) = \left(\frac{dc_1}{dt}, \frac{dc_2}{dt}\right)$$

$$-c_2 = \frac{dc_1}{dt} \qquad c_1 = \frac{dc_2}{dt}$$

So the solution

$$c_1 = \alpha \cos t \qquad c_2 = \alpha \sin t$$

follows from the diff EQs.

4.3 HW: 1-12, 17-21

4.4 Divergence and Curl

The divergence of a vector field $\underline{F}:\mathbb{R}^n \to \mathbb{R}^n$ is denoted by $\text{div}\,\underline{F}:\mathbb{R}^n \to \mathbb{R}$ and defined by

$$\text{div}\,\underline{F} = \nabla \cdot \underline{F} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\cdot(F_1, \dots, F_n)$$

"nabla"
"upside down triangle thing"
"grad"

$$= \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$$

$$= \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$$

(Examples 3-5) Compute the divergences of $\underline{F}=(x,y)$, $\underline{G}=(-x,-y)$, and $\underline{H}=(-y,x)$ for an arbitrary point (x,y) of $\mathbb{R}^2$. How does divergence correspond with the motion shown by the vector field plots?

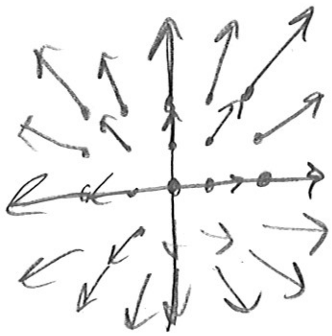
$$\text{div}\,\underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 1 + 1 = 2$$

$$\text{div}\,\underline{G} = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} = -1 - 1 = -2$$

$$\text{div}\,\underline{H} = \frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0 + 0 = 0$$

(constant across the fields)

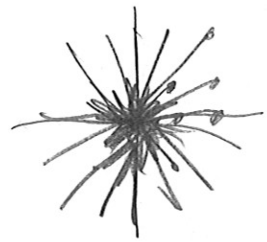
$\underline{F} = (x, y)$



$\operatorname{div} \underline{F} = 2$

positive divergence corresponds to outward expansion

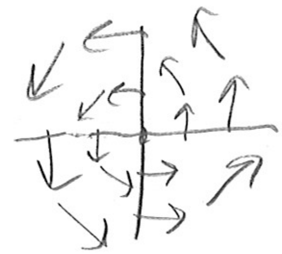
$\underline{G} = (-x, -y)$



$\operatorname{div} \underline{G} = -2$

negative divergence corresponds to inward contraction

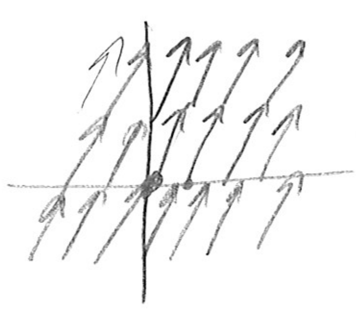
$\underline{H} = (-y, x)$



$\operatorname{div} \underline{H} = 0$

zero divergence means no expansion / contraction

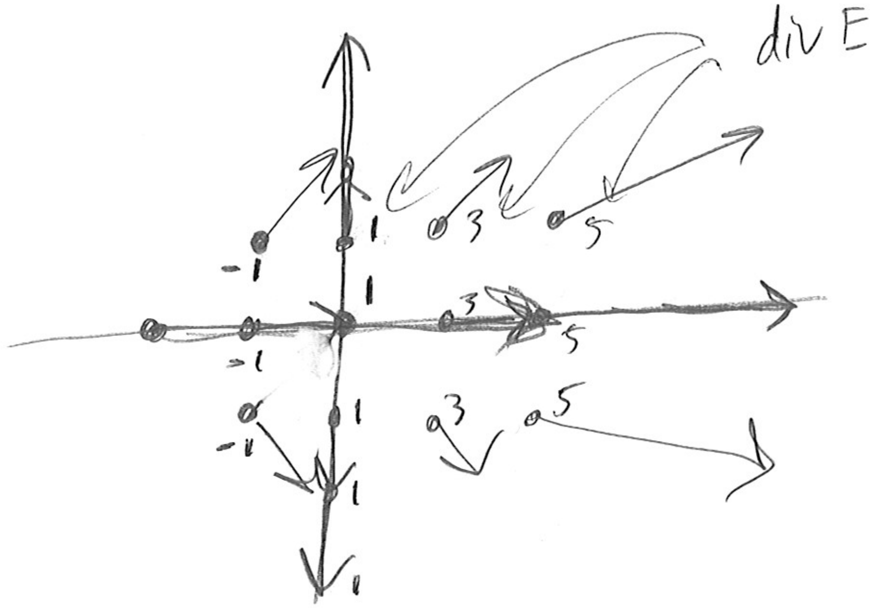
$\underline{C} = (1, 2)$



$$\operatorname{div} \underline{C} = 0 + 0$$
$$= 0$$

(Example) Plot div $\underline{E}$ for various points in the plot of

$$\underline{E} = (x^2, y)$$



$$\operatorname{div} \underline{E} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

$$= 2x + 1$$