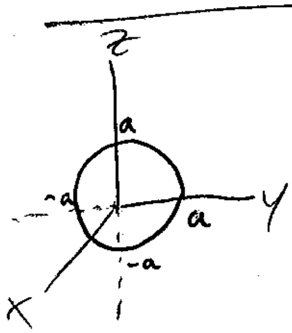
7.2 cont.

Change of notation: (Path integral of scalar function) $\int_C f\,ds$

(Line Integral of Vector Field) $\int_C \underline{F}\cdot d\underline{r} \rightarrow \int_C \underline{F}\cdot d\underline{s}$

$\underbrace{\int_C \underline{F}\cdot d\underline{r}}_{\text{Stewart's calculus}}$ $\quad$ $\underbrace{\int_C \underline{F}\cdot d\underline{s}}_{\text{Your textbook}}$

(Example 5) Let C be a circle in the yz plane centered at the origin. Show that no work is done by a force $\underline{F} = (x^3, y, z)$ acting on an object moving around the circle.



$$\underline{r}(t) = (\overset{x}{0}, \overset{y}{a\cos t}, \overset{z}{a\sin t}) \quad ? \le t \le ??$$

$$d\underline{r}/dt = (0, -a\sin t, a\cos t)$$

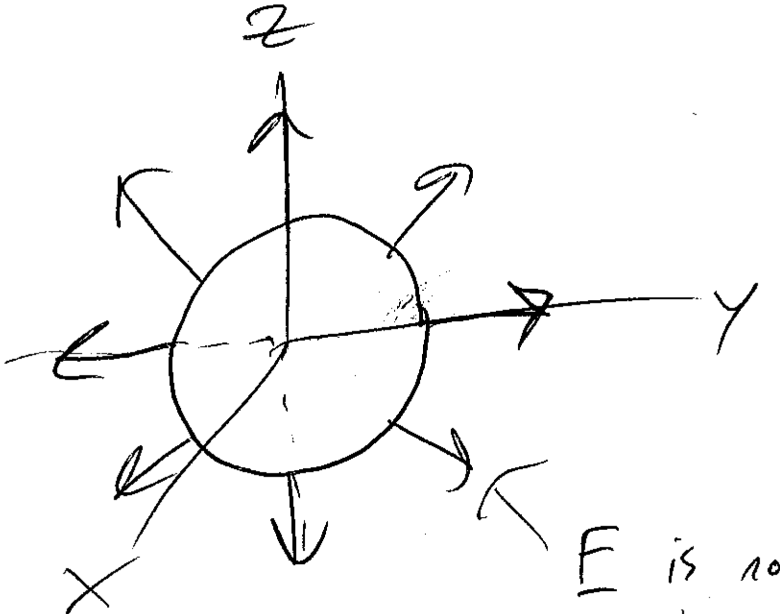
$$W = \int_C \underline{F}\cdot d\underline{s} = \int_{t=?}^{t=??} \underline{F}(\underline{r}(t))\cdot \frac{d\underline{r}}{dt}\,dt$$

$$= \int_{t=?}^{t=??} (\underset{x^3}{0^3}, \underset{y}{a\cos t}, \underset{z}{a\sin t})\cdot \frac{d\underline{r}}{dt}\,dt$$

$$= \int_{t=?}^{t=??} (0, a\cos t, a\sin t) \cdot (0, -a\sin t, a\cos t)\, dt$$

$$= \int_{t=?}^{t=??} 0 \cancel{- a^2 \sin t \cos t + a^2 \sin t \cos t}\, dt$$

$$W = 0$$



$\underline{E}$ is not pushing around or against circular motion.

So no work was done.

Let $d\underline{s} = (dx_1, dx_2, \cdots, dx_n)$ in $\mathbb{R}^n$

$= (dx, dy, dz)$ in $\mathbb{R}^3$

Then:

$$\int_C \underbrace{\underline{F}}_{\substack{\text{Vector}\\\text{field}}} \cdot d\underline{s} = \int_C (F_1, \cdots, F_n) \cdot (dx_1, \cdots, dx_n)$$

$$= \sum_{i=1}^{n} \int_C \underbrace{F_i}_{\substack{\text{scalar}\\\text{function}}} dx_i$$

$$= \int_C \Big(\underbrace{F_1}_{P} dx + \underbrace{F_2}_{Q} dy + \underbrace{F_3}_{R} dz \Big)$$

$$= \int_C F_1\, dx + \int_C F_2\, dy + \int_C F_3\, dz$$

Where $\displaystyle \int_C f\, dx_i = \int_{t=a}^{t=b} f(\underline{r}(t)) \underbrace{\frac{dx_i}{dt}}\, dt$

ith component of $d\underline{r}/dt$

(Example 2) Evaluate & interpret $\int_C x^2dx + xydy + dz$

where C is the parabola defined by $\underline{r}(t) = (\underbrace{t}_{x}, \underbrace{t^2}_{y}, \underbrace{1}_{z})$

for $t \in [0,1]$.

$$\frac{d\underline{r}}{dt} = (\underbrace{1}_{\frac{dx}{dt}}, \underbrace{2t}_{\frac{dy}{dt}}, \underbrace{0}_{\frac{dz}{dt}})$$

$$\int_C x^2 dx + xy\,dy + 1dz = \int_C (x^2, xy, 1)\cdot d\underline{s}$$

= Work done by $\underline{F} = (x^2, xy, 1)$ over the curve C.

Two ways to compute:

$$\int_C \underline{F}\cdot d\underline{s} = \int_{t=0}^{t=1} \underline{F}(\underline{r}(t))\cdot \frac{d\underline{r}}{dt}\,dt$$

$$= \int_{t=0}^{t=1} \left((t)^2, (t)(t^2), 1\right)\cdot\left(1, 2t, 0\right)dt$$

$$= \int_{t=0}^{t=1} t^2 + 2t^4 \cancel{+0}\,dt$$

$$= \left[\tfrac{1}{3}t^3 + \tfrac{2}{5}t^5\right]_0^1$$

$$= \tfrac{1}{3} + \tfrac{2}{5} = \boxed{\tfrac{11}{15}}$$

$$= \int_C x^2\,dx + \int_C xy\,dy + \int_C 1\,dz$$

$$= \int_{t=0}^{t=1} (t)^2 \underbrace{\frac{dx}{dt}}\,dt + \int_{t=0}^{t=1} (t)(t^2)\underbrace{\frac{dy}{dt}}\,dt + \int_{t=0}^{t=1} (1)\underbrace{\frac{dz}{dt}}\,dt$$

$$= \int_{t=0}^{t=1} t^2(1)\,dt + \int_{t=0}^{t=1} t^3(2t)\,dt + \cancel{\int_{t=0}^{t=1} 1(0)\,dt}$$

$$= \int_0^1 t^2\,dt + \int_0^1 2t^4\,dt$$

$$= \left[\frac{1}{3}t^3\right]_0^1 + \left[\frac{2}{5}t^5\right]_0^1$$

$$= 1/3 + 2/5 = \boxed{11/15}$$

FACTS about $\int_C f\,ds$ AND $\int_C \underline{F}\cdot d\underline{s}$

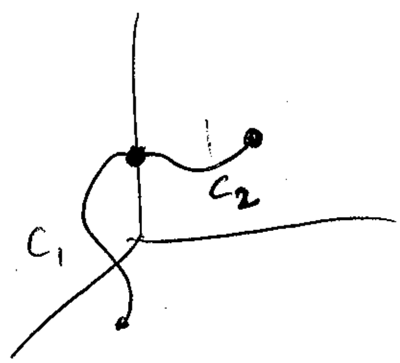
- The value of $\int_C f\,ds$ is independent of the chosen parameterization $(\underline{r}(t))$ for C.
- The value of $\int_C \underline{F}\cdot d\underline{s}$ is independent of the chosen parameterization $(\underline{r}(t))$ for C, provided that it preserves direction.

- If C, $-C$ represent the same curve, but different directions, then

$$\int_C \underline{F} \cdot d\underline{s} = -\int_{-C} \underline{F} \cdot d\underline{s}$$

$$\left(\text{but } \int_C f\, ds = +\int_{-C} f\, ds \right)$$

- If $C = \underline{C_1} + \underline{C_2}$



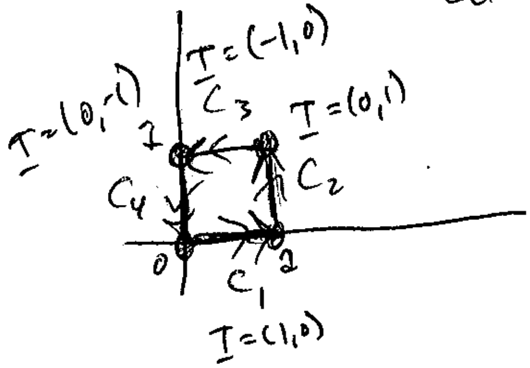
then,

$$\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds$$

$$\int_C \underline{F} \cdot d\underline{s} = \int_{C_1} \underline{F} \cdot d\underline{s} + \int_{C_2} \underline{F} \cdot d\underline{s}$$

(Example 11) Compute $\int_C x^2\,dx + xy\,dy$ where C is the perimeter of the unit square oriented c.c.w.

Let $\underline{F} = (x^2, xy)$



$$\int_C \underline{F}\cdot d\underline{s} = \int_{C_1} \underline{F}\cdot d\underline{s} + \int_{C_2} \underline{F}\cdot d\underline{s} + \int_{C_3} \underline{F}\cdot d\underline{s} + \int_{C_4} \underline{F}\cdot d\underline{s}$$

$$= \int_{C_1} \underline{F}\cdot\underline{T}\,ds + \int_{C_2} \underline{F}\cdot\underline{T}\,ds + \int_{C_3} \underline{F}\cdot\underline{T}\,ds + \int_{C_4} \underline{F}\cdot\underline{T}\,ds$$

$$= \int_{C_1} x^2 + 0\,ds + \int_{C_2} 0 + xy\,ds + \int_{C_3} -x^2 + 0\,ds + \int_{C_4} 0 - xy\,ds$$

$$= \cancel{\int_{x=0}^{x=1} x^2\,dx} + \int_{y=0}^{y=1} (1)y\,dy - \cancel{\int_{x=0}^{x=1} x^2\,dx} - \cancel{\int_{y=0}^{1} (0)y\,dy}$$

$$= \int_0^1 y\,dy = \left[\frac{1}{2}y^2\right]_0^1 = \boxed{1/2}$$

HW 7.2 1-5, 13, 17-18