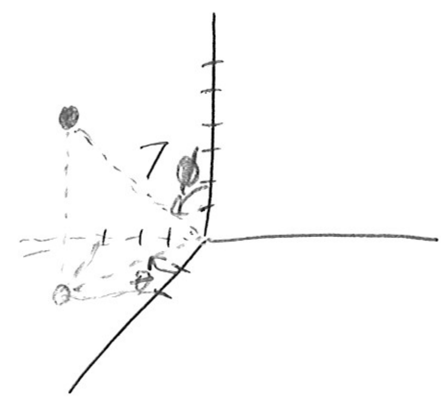
(1.4 cont.)

(Example 2c)

Convert $(2,-3,6)$ from Cartesian (x,y,z) to spherical $s(\rho,\theta,\phi)$.

Plot it.



$$x^2+y^2+z^2=\rho^2$$
$$4+9+36=\rho^2$$
$$\rho^2=49$$
$$\rho=7$$

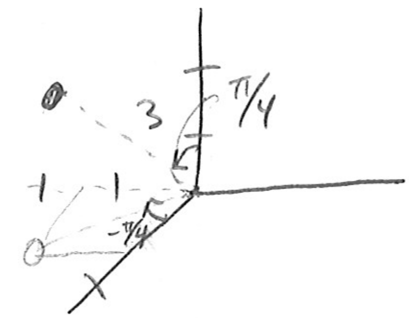
$$\tan\theta=\frac{y}{x}$$
$$\tan\theta=-\frac{3}{2}$$
$$\theta=\text{Arctan}\left(-\frac{3}{2}\right)$$
$$=-.98\text{ rad}$$

$$\tan\phi=\frac{r}{z}$$
$$=\frac{\sqrt{x^2+y^2}}{z}$$
$$\tan\phi=\sqrt{13}/6$$
$$\phi=\text{Arctan}\left(\sqrt{13}/6\right)$$
$$=$$
$$=.54\text{ rad}$$

$$\boxed{s(7,-0.98,0.54)}$$

(Example 2c*)

Convert $\left(-\frac{3}{2}, -\frac{3}{2}, \frac{3\sqrt{2}}{2}\right)$ from Cartesian to spherical & plot it.



$$x^2 + y^2 + z^2 = \rho^2$$

$$\left(\frac{9}{4}\right) + \left(\frac{9}{4}\right) + \left(\frac{9(2)}{4}\right) = \rho^2$$

$$\frac{\cancel{4}(9)}{\cancel{4}} = \rho^2$$

$$9 = \rho^2$$

$$\rho = 3$$

$$\tan\theta = \frac{y}{x}$$

$$= \frac{-3/2}{3/2}$$

$$\tan\theta = -1$$

$$\theta = -45^\circ$$

$$\theta = -\pi/4$$

$$\tan\phi = \frac{r}{z}$$

$$= \frac{\sqrt{x^2+y^2}}{z}$$

$$= \frac{\sqrt{9/4 + 9/4}}{3\sqrt{2}/2}$$

$$= \frac{\sqrt{18/4}}{\sqrt{18/4}}$$

$$= 1$$

$$\phi = \pi/4 = 45^\circ$$

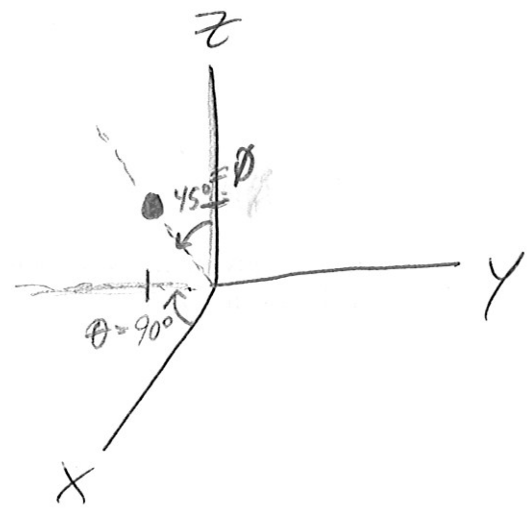
$$\boxed{S\left(3, -\pi/4, \pi/4\right)}$$

(Example 2d)

Convert $s\left(1, -\frac{\pi}{2}, \frac{\pi}{4}\right)$ from spherical to Cartesian.

$\rho \quad \theta \quad \phi$

Plot it.

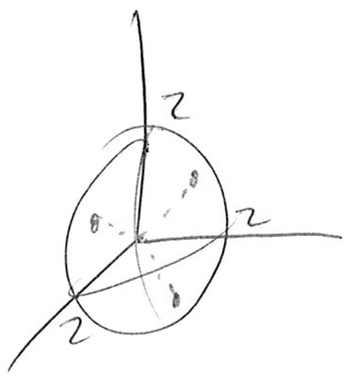


$$s\left(1, -\frac{\pi}{2}, \frac{\pi}{4}\right) = \left(\cancel{1 \sin\left(\frac{\pi}{4}\right)\cos\left(-\frac{\pi}{2}\right)},\ 1\left(\sin\frac{\pi}{4}\right)\sin\left(-\frac{\pi}{2}\right),\ 1\cos\left(\frac{\pi}{4}\right)\right)$$

$\rho \sin\phi \cos\theta \quad \rho \sin\phi \sin\theta \quad \rho\cos\phi$

$$= \boxed{\left(0,\ -\sqrt{2}/2,\ \sqrt{2}/2\right)}$$

(Example 3) Express the surfaces $x^2+y^2+z^2=4$, $xz=1$ and $x^2+y^2-z^2=1$ in terms of spherical coordinates.



$$\underbrace{x^2+y^2}+z^2=4$$

$$\rho^2=4$$

$$\boxed{\rho=2}$$

$$\boxed{0\le\theta\le 2\pi}$$

$$\boxed{0\le\phi\le\pi}$$

$$xz=1$$

$$(\rho\sin\phi\cos\theta)(\rho\cos\phi)=1$$

$$\boxed{\rho^2\sin\phi\cos\phi\cos\theta=1}$$

$$x^2+y^2-z^2=1$$

$$(\rho^2\sin^2\phi)\cos^2\theta+(\rho^2\sin^2\phi)\sin^2\theta$$

$$-\rho^2\cos^2\phi=1$$

$$(\rho^2\sin^2\phi)\cancel{(\cos^2\theta+\sin^2\theta)}$$

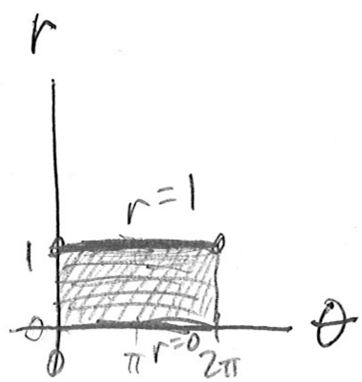
$$-\rho^2\cos^2\phi=1$$

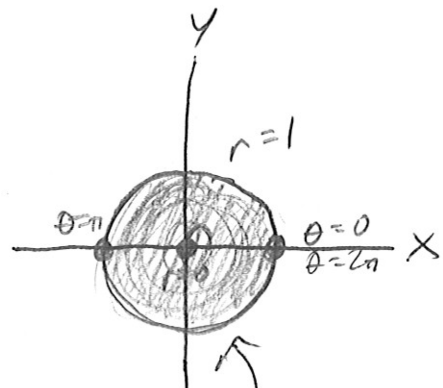
$$\boxed{\rho^2(\sin^2\phi-\cos^2\phi)=1}$$

1.4 HW 1-12, 15-16

6.1 The Geometry of Maps from $\mathbb{R}^n$ to $\mathbb{R}^n$

(Example 1) Find the image of the rectangle $[0,1] \times [0,2\pi]$ in the $r\theta$ plane under the polar transformation p.





$\boxed{\{(x,y) : 0 \le x^2+y^2 \le 1\}}$

$0 \le r \le 1$

$0 \le r^2 \le 1$

$0 \le x^2+y^2 \le 1$

(Example 2)

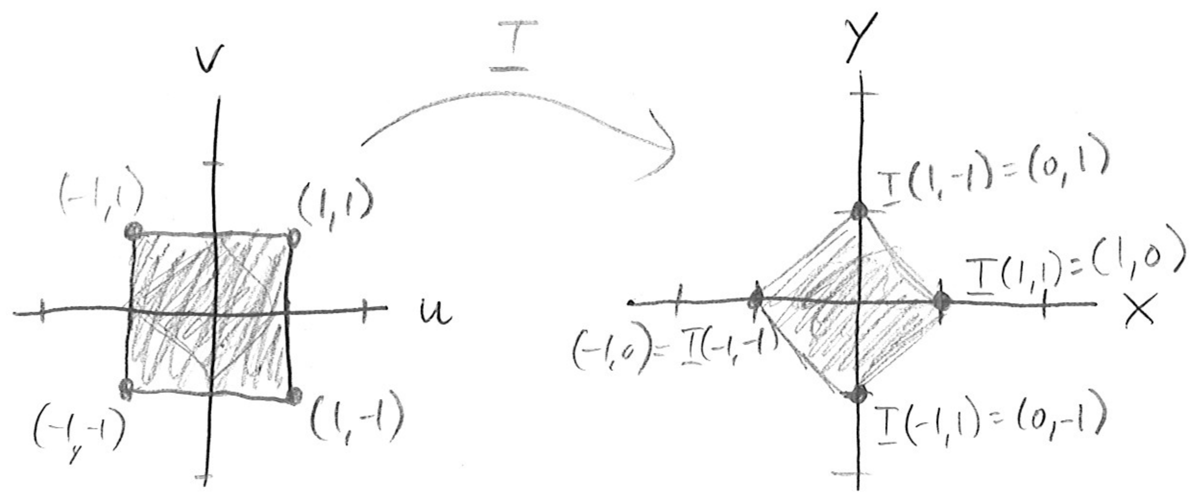
Find the image of the square $[-1,1]^2 = [-1,1]\times[-1,1]$ in the uv plane under the transformation

$$\underline{T}(u,v) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}(u,v).$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$= \begin{bmatrix} 1/2\,u + 1/2\,v \\ 1/2\,u - 1/2\,v \end{bmatrix}$$

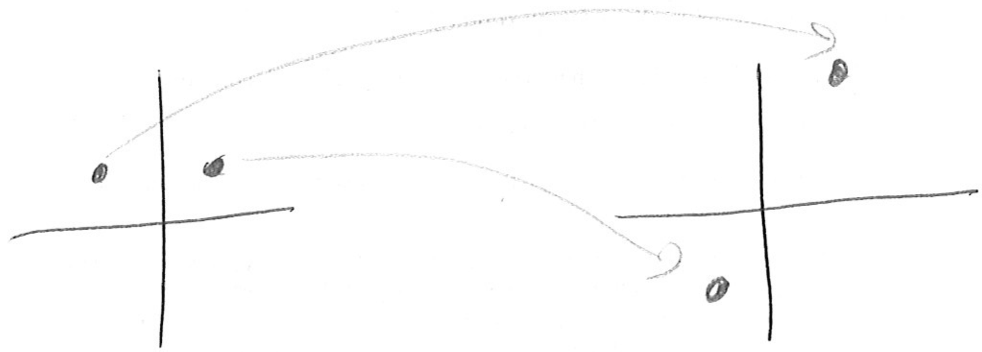
$$\underline{T}(u,v) = (\underbrace{1/2\,u + 1/2\,v}_{x},\ \underbrace{1/2\,u - 1/2\,v}_{y})$$



$$\underline{T}(1,1) = (1/2 + 1/2,\ 1/2 - 1/2)$$
$$= (1,\ 0)$$

Transformation Properties:

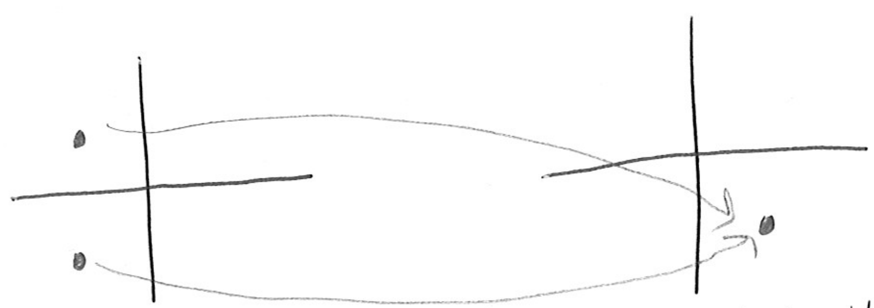
One-to-one:



Any two different points

Stay different under T.

Not one-to-one:



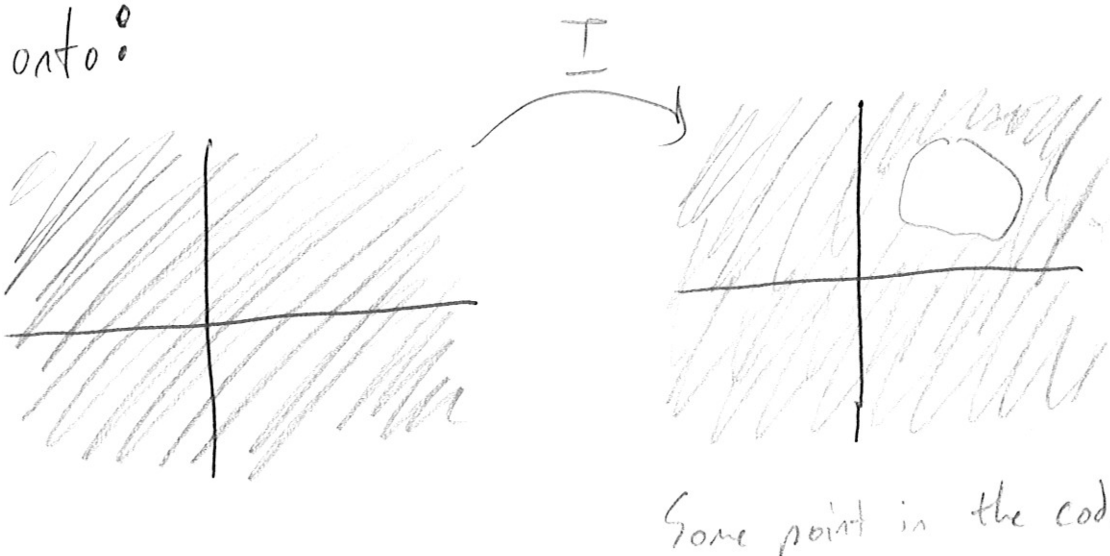
Can find two different points

get sent to the same point by T.

Onto:

Every point in the codomain is mapped to by $\underline{I}$ from some point.

Not onto:



Some point in the codomain is not mapped to by $\underline{I}$.

(Example 3)

Show that $f: \mathbb{R}^2 \to \mathbb{R}^2$ is onto, but not one-to-one.

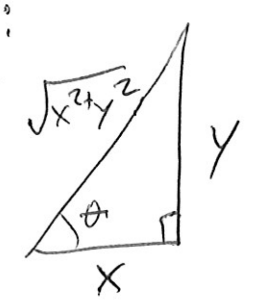
Let $(x,y) \in \mathbb{R}^2$. We want (r,θ) such that $f(r,\theta) = (x,y)$. So let (for $x \neq 0$),

$$r = \sqrt{x^2+y^2} \text{ and}$$

$$\theta = \operatorname{Arctan}\left(\frac{y}{x}\right).$$

Then $f\left(\sqrt{x^2+y^2}, \operatorname{Arctan}\left(\frac{y}{x}\right)\right) = \left(\sqrt{x^2+y^2}\cos\left(\operatorname{Arctan}\left(\frac{y}{x}\right)\right), \sqrt{x^2+y^2}\sin\left(\operatorname{Arctan}\left(\frac{y}{x}\right)\right)\right).$

If $\theta = \operatorname{Arctan}\frac{y}{x}$:



so $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$

and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

So $f\left(\sqrt{x^2+y^2}, \operatorname{Arctan}\left(\frac{y}{x}\right)\right) = \left(\cancel{\sqrt{x^2+y^2}}\frac{x}{\cancel{\sqrt{x^2+y^2}}}, \cancel{\sqrt{x^2+y^2}}\frac{y}{\cancel{\sqrt{x^2+y^2}}}\right)$

$= (x, y).$