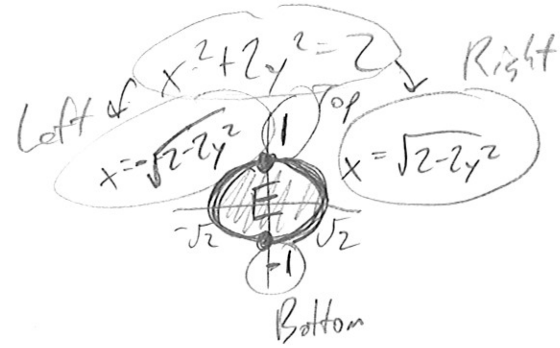
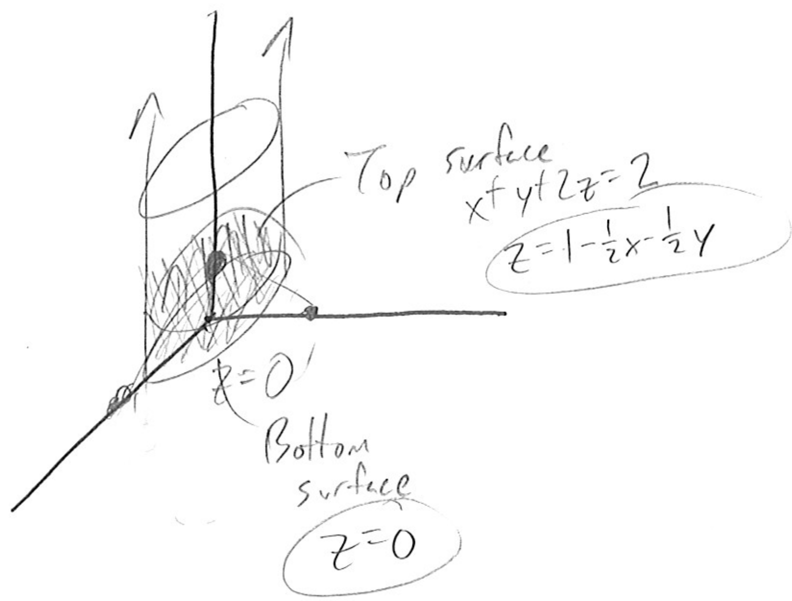
5.5 12 Find the volume of the region bounded by $x^2+2y^2=2$, $z=0$, and $x+y+2z=2$.

plane: $Ax+By+Cz=D$



$$V_D^{\text{of}} = \iiint_D 1\, dV$$



$$= \iint_{E} \left[\int_0^{1-\frac{1}{2}x-\frac{1}{2}y} 1\, dz \right] dA$$

shadow

$$= \int_{-1}^{1} \int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}} \int_0^{1-\frac{1}{2}x-\frac{1}{2}y} 1\, dz\, dx\, dy$$

$$= \int_{-1}^{1} \left[\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}} 1 - \frac{1}{2}x - \frac{1}{2}y \, dx \right] dy$$

$$= \int_{-1}^{1} \left[x - \frac{1}{4}x^2 - \frac{1}{2}xy \right]_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}} dy$$

$$= \int_{-1}^{1} \left(\sqrt{2-2y^2} - \cancel{\frac{1}{4}(2-2y^2)} - \frac{1}{2}y\sqrt{2-2y^2} \right) - \left(-\sqrt{2-2y^2} - \cancel{\frac{1}{4}(2-2y^2)} + \frac{1}{2}y\sqrt{2-2y^2} \right) dy$$

$$= \int_{-1}^{1} \underbrace{2\sqrt{2-2y^2}}_{\text{Trig sub}} - \underbrace{y\sqrt{2-2y^2}}_{\substack{\text{substitute} \\ u = 2-2y^2}} \, dy$$

$2 - 2y^2 = 2 - 2\sin^2\theta$

⋮

etc.

1.4 cont.

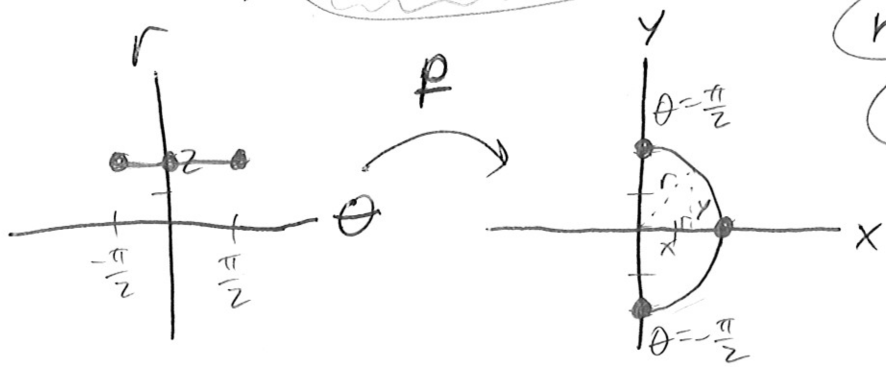
(Example) Express the curve $x = +\sqrt{4-y^2}$ in terms of polar coordinates. Plot the curve in both the $r\theta$ and xy planes.

$x^2 = 4 - y^2$ where $x \geq 0$

$x^2 + y^2 = 4 \longrightarrow r^2 = 4$

$r = 2$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$



Alternatively...

$p(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$

$x^2 + y^2 = 4$

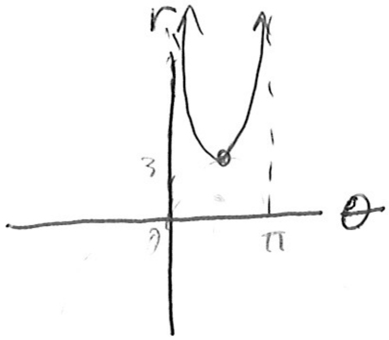
$(r\cos\theta)^2 + (r\sin\theta)^2 = 4$

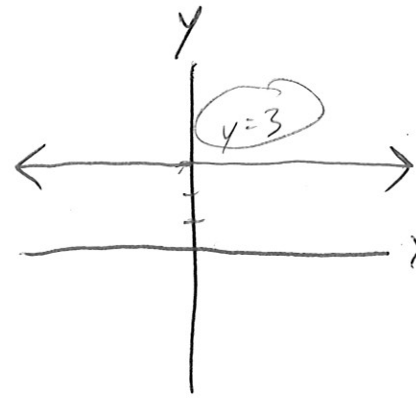
$r^2\cos^2\theta + r^2\sin^2\theta = 4$

$r^2(\cancel{\cos^2\theta + \sin^2\theta}) = 4$

$r^2 = 4$

(Example) Express the curve $y=3$ in terms of polar coordinates. Plot it in $r\theta$ and xy planes.

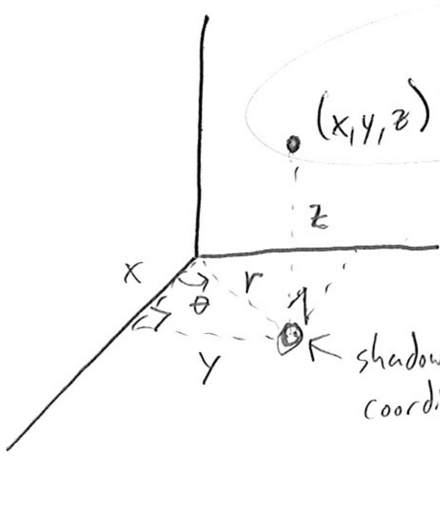




$$y=3$$
$$r\sin\theta=3$$
$$r=3\csc\theta \quad \text{for} \quad 0<\theta<\pi$$

Cylindrical Coordinates $\underline{c}:\mathbb{R}^3\to\mathbb{R}^3$

$$(x,y,z)=\underline{c}(r,\theta,z)=(r\cos\theta,\ r\sin\theta,\ z)$$



← shadow has polar coordinates $\underline{p}(r,\theta)$

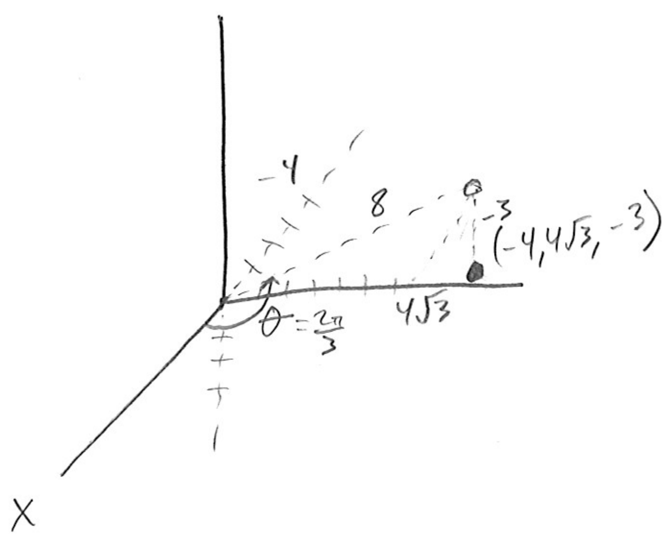
Usually...
Assume $r\geq 0$
and $0\leq\theta\leq 2\pi$
OR $-\pi\leq\theta\leq\pi$

Same tricks hold: $x^2+y^2=r^2$ and $\tan\theta = \frac{y}{x}$

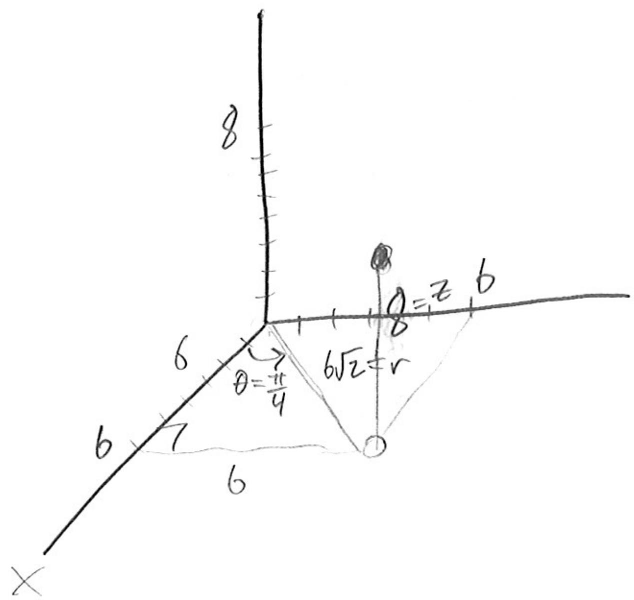
(Example 1a) Convert $\underline{c}(8, \frac{2\pi}{3}, -3)$ from cylindrical to Cartesian. Plot it in xyz space.

$$\underline{c}(\underset{r}{8}, \underset{\theta}{\tfrac{2\pi}{3}}, \underset{z}{-3}) = (\underbrace{8\cos\tfrac{2\pi}{3}}_{r\cos\theta}, \underbrace{8\sin\tfrac{2\pi}{3}}_{r\sin\theta}, \underbrace{-3}_{z})$$

$$= (-4, 4\sqrt{3}, -3)$$



(Example 1b) Convert $(6,6,8)$ from Cartesian to cylindrical. Plot it in xyz space.



$$x^2+y^2=r^2$$
$$6^2+6^2=r^2$$
$$2\cdot 6^2=r^2$$
$$\sqrt{2}\sqrt{6^2}=r$$
$$6\sqrt{2}=r$$

$$\tan\theta=\frac{y}{x}$$
$$\tan\theta=\frac{6}{6}$$
$$\tan\theta=1$$
$$\theta=\frac{\pi}{4}$$

$$z=z$$

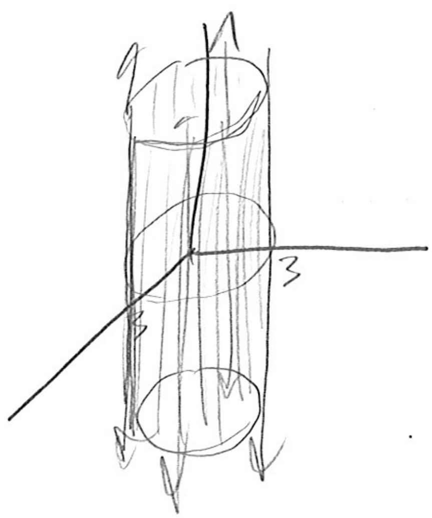
Thus $(6,6,8)=\underline{c}\left(6\sqrt{2},\frac{\pi}{4},8\right)$

(Example) Express the surface $x^2+y^2=9$ in terms of cylindrical coordinates. Then plot it in xyz space.

$r^2=9$

$\boxed{r=3}$

$\boxed{0 \le \theta \le 2\pi}$

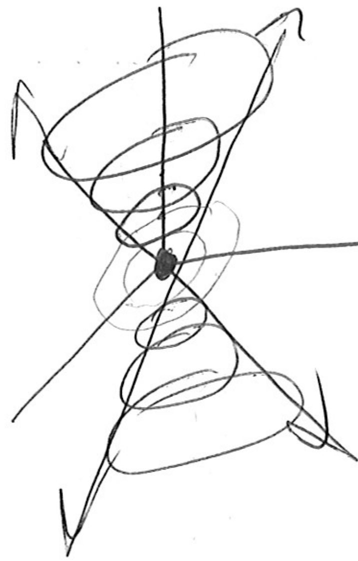


(Example) Express the surface $z^2=x^2+y^2$ in terms of cylindrical coordinates. Plot it.

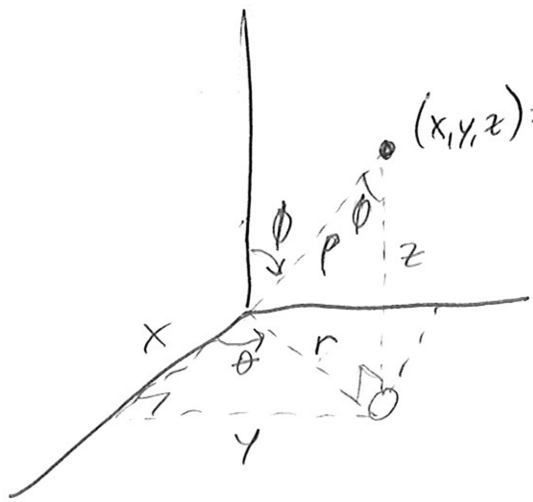
$$z^2 = r^2$$

$$z = \pm r$$

$$0 \leq \theta \leq 2\pi$$



Spherical Coordinates $\underline{s} : \mathbb{R}^3 \to \mathbb{R}^3$



"rho" "theta" "phi"

$$(x, y, z) = \underline{s}(\rho, \theta, \phi)$$

$$= (\underbrace{\rho \sin\phi}_{r} \cos\theta, \underbrace{\rho \sin\phi}_{r} \sin\theta, \underbrace{\rho \cos\phi}_{z})$$

r: from origin to shadow

ρ: from origin directly to point

ϕ: angle from positive z axis down to point

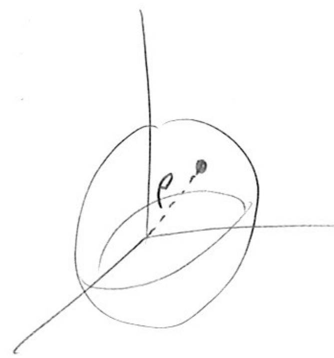
Usually, assume $\rho \geq 0$, $0 \leq \theta \leq 2\pi$ OR $-\pi \leq \theta \leq \pi$, $0 \leq \phi \leq \pi$

0 ↑ positive z axis; π ↑ negative z axis

FACTS:

$$x^2+y^2+z^2=\rho^2$$

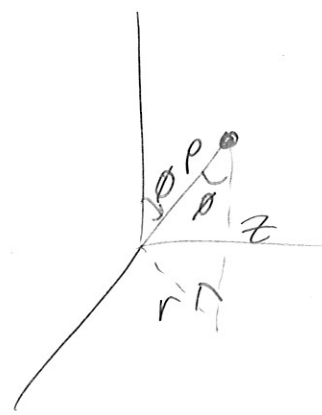
sphere of radius ρ



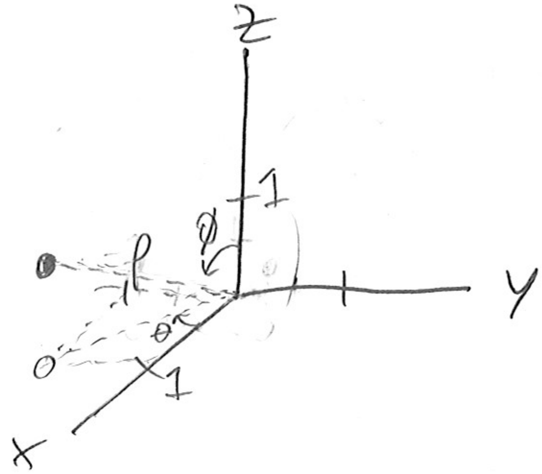
$$\tan\theta = \frac{y}{x}$$ (because polar)

$$\tan\phi = \frac{r}{z}$$ ← opposite, ← adjacent

$$\tan\phi = \frac{\sqrt{x^2+y^2}}{z}$$



Example 2a Convert $(1,-1,1)$ from Cartesian to spherical & plot.



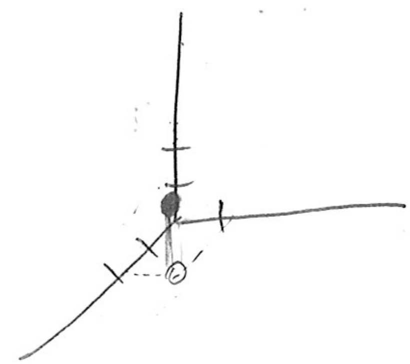
$\theta = -\frac{\pi}{4}$

$$x^2 + y^2 + z^2 = \rho^2$$
$$(1)^2 + (-1)^2 + (1)^2 = \rho^2$$
$$3 = \rho^2$$
$$\rho = \sqrt{3}$$

$$\tan\theta = \frac{y}{x}$$
$$= \frac{-1}{1}$$
$$\tan\theta = -1$$
$$\theta = -\pi/4$$

$$\tan\phi = \frac{\sqrt{x^2+y^2}}{z}$$
$$\tan\phi = \sqrt{2}$$
$$\phi = \text{Arctan}(\sqrt{2})$$
$$\approx 54.7^\circ$$

Example 2b Convert $s\left(3, \pi/6, \pi/4\right)$ from spherical to Cartesian. And plot.



$$= \left(3 \sin\tfrac{\pi}{4} \cos\tfrac{\pi}{6},\ 3 \sin\tfrac{\pi}{4} \sin\tfrac{\pi}{6},\ 3 \cos\tfrac{\pi}{4}\right)$$

$r \sin\phi \cos\theta \quad r \sin\phi \sin\theta \quad r \cos\phi$

$$= \left(3 \tfrac{\sqrt{2}}{2} \tfrac{\sqrt{3}}{2},\ 3 \tfrac{\sqrt{2}}{2} \tfrac{1}{2},\ 3 \tfrac{\sqrt{2}}{2}\right)$$

$$= \left(\frac{3\sqrt{6}}{4}, \frac{3\sqrt{2}}{4}, \frac{3\sqrt{2}}{2}\right)$$

$$\approx (1.8, 1.1, 2.1)$$