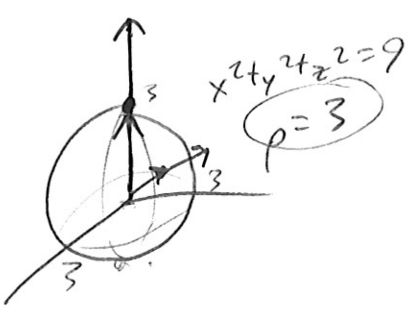
7.3 cont

(Example) Find a parametrization for the sphere centered at the origin with radius 3. Show that the vector (x_0, y_0, z_0) is normal to the sphere at each point (x_0, y_0, z_0) on the sphere. Then describe the plane tangent to the sphere at $(1, -2, 2)$ as an EQ of x, y, z.



$$\Phi(\theta, \phi)$$

$$= s(3, \theta, \phi)$$

$$\boxed{\begin{gathered}\Phi(\theta,\phi) = (3\sin\phi\cos\theta,\ 3\sin\phi\sin\theta,\ 3\cos\phi) \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi\end{gathered}}$$

To show normal vector:

$$\frac{\partial \Phi}{\partial \theta} = (-3\sin\phi\sin\theta,\ 3\sin\phi\cos\theta,\ 0)$$

$$\frac{\partial \Phi}{\partial \phi} = (3\cos\phi\cos\theta,\ 3\cos\phi\sin\theta,\ -3\sin\phi)$$

$$\begin{matrix}\text{Normal} \\ \text{vector}\end{matrix} = \frac{\partial \Phi}{\partial \theta} \times \frac{\partial \Phi}{\partial \phi} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin\phi\sin\theta & 3\sin\phi\cos\theta & 0 \\ 3\cos\phi\cos\theta & 3\cos\phi\sin\theta & -3\sin\phi \end{pmatrix}$$

$$= \Big(-9\sin^2\phi\cos\theta - 0,\ 0 - 9\sin^2\phi\sin\theta,$$

$$-9\sin\phi\cos\phi\sin^2\theta - 9\sin\phi\cos\phi\cos^2\theta\Big)$$

$$= \Big(-9\sin^2\phi\cos\theta,\ -9\sin^2\phi\sin\theta,$$

$$-9\sin\phi\cos\phi\,(\cancelto{1}{\sin^2\theta + \cos^2\theta})\Big)$$

$$= -3\sin\phi\,(3\sin\phi\cos\phi,\ 3\sin\phi\sin\theta,\ 3\cos\phi)$$

$$\cancel{= -3\sin\phi\ \underline{\Phi}(\theta,\theta}$$

$$\frac{\partial \Phi}{\partial \theta} \times \frac{\partial \Phi}{\partial \phi} = -3\sin\phi\ \underline{\Phi}(\theta,\phi)$$

So for (θ_0,ϕ_0) where $(x_0,y_0,z_0) = \underline{\Phi}(\theta_0,\phi_0)$, we've shown that $\underline{\Phi}(\theta_0,\phi_0) = (x_0,y_0,z_0)$ is normal to the sphere.

So the plane tangent to the sphere at $(1,-2,2)$ is: point & normal vector

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

$$1(x-1) - 2(y+2) + 2(z-2) = 0$$

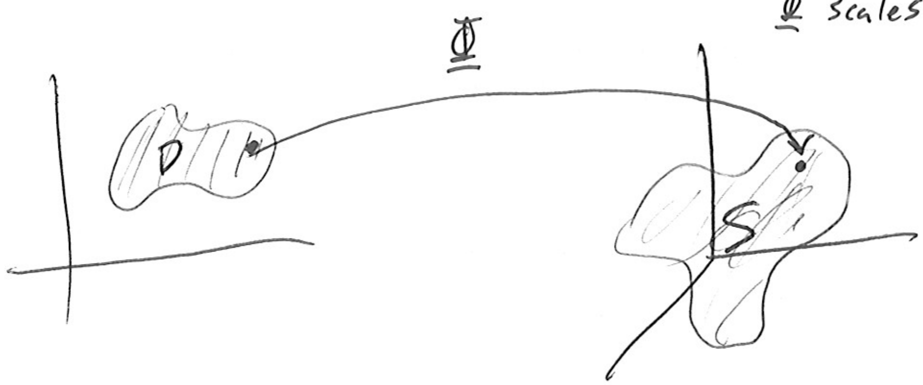
$$x - 1 - 2y - 4 + 2z - 4 = 0$$

$$\boxed{x - 2y + 2z = 9}$$

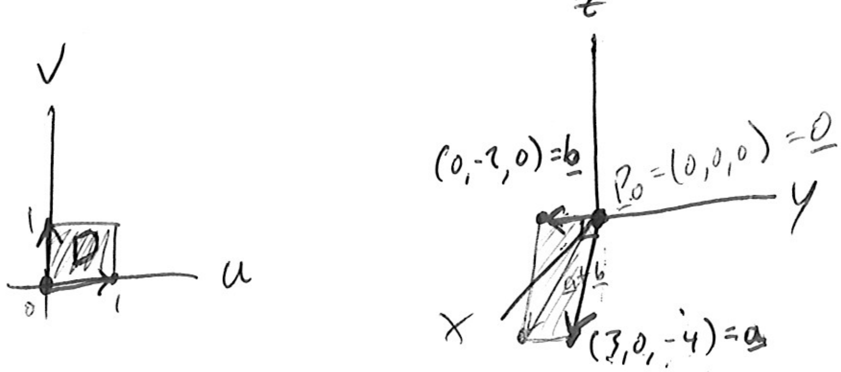
HW 7.3 1-3, 7-11

7.4 Area of a Surface

The area of a surface parameterized by $\underline{\Phi}$ with domain D is given by $\iint_D \underbrace{\left\| \frac{\partial \underline{\Phi}}{\partial u} \times \frac{\partial \underline{\Phi}}{\partial v} \right\|}_{\text{measures how } \underline{\Phi} \text{ scales areas}} dA$.



(Example) Verify this definition matches the area of the rectangle given by the vectors $(3,0,-4)$ and $(0,-2,0)$.



$\underline{\Phi}(u,v) = \cancel{\underline{P_0}} + u\underline{a} + v\underline{b}$ for $0 \le u,v \le 1$

$= (3u, -2v, -4u)$

$$\text{Area} = \iint_D \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| dA$$

$$\frac{\partial \Phi}{\partial u} = \underline{a} = (3, 0, -4)$$

$$\times \quad \frac{\partial \Phi}{\partial v} = \underline{b} = (0, -2, 0)$$

$$\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} = (8, 0, -6)$$

$$\left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| = \sqrt{64 + 0 + 36}$$

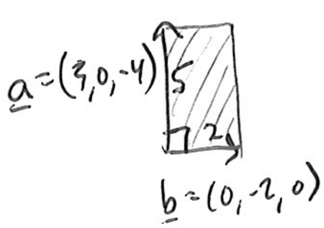
$$= \sqrt{100} = 10$$

$$\text{Area} = \int_0^1 \int_0^1 10 \, dv \, du$$

$$= \int_0^1 \left[10v \right]_0^1 du$$

$$= \int_0^1 10 \, du = \boxed{10}$$

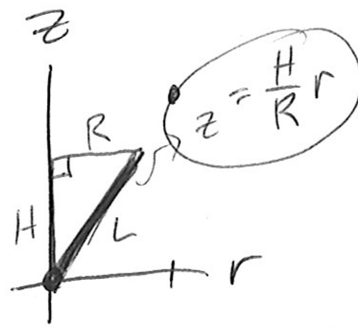
Compare with geometry:

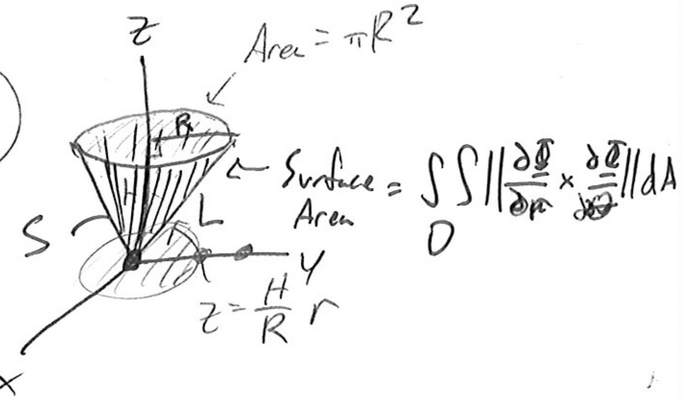


$$\text{Area} = (\text{base})(\text{height})$$

$$= (2)(5) = \boxed{10}$$

(Example 1) Show that the surface area of a cone with slant length L and radius R is given by $A = \pi R^2 + \pi R L$.





Using cylindrical coordinates...

$$\Phi(r, \theta) = c\left(r, \theta, \frac{H}{R} r\right)$$

$$\boxed{\begin{gathered} 0 \le \theta \le 2\pi \\ 0 \le r \le R \\ \Phi(r, \theta) = \left(r\cos\theta,\ r\sin\theta,\ \frac{H}{R} r\right) \end{gathered}}$$

$$\frac{\partial \Phi}{\partial r} = \left(\cos\theta,\ \sin\theta,\ \frac{H}{R}\right)$$

$$\times\ \frac{\partial \Phi}{\partial \theta} = \left(-r\sin\theta,\ r\cos\theta,\ 0\right)$$

$$\left(-\frac{H}{R} r\cos\theta,\ -\frac{H}{R} r\sin\theta,\ r\cos^2\theta + r\sin^2\theta\right)$$

$$= \left(-\frac{H}{R} r\cos\theta,\ -\frac{H}{R} r\sin\theta,\ r\right)$$

$$\left\|\frac{\partial\Phi}{\partial r}\times\frac{\partial\Phi}{\partial\theta}\right\| = \sqrt{\frac{H^2}{R^2}r^2\cancel{\cos^2\theta}\cancel{+}\cancel{\frac{H^2}{R^2}r^2\sin^2\theta}+r^2}$$

$$= r\sqrt{\frac{H^2}{R^2}+1}$$

$$\text{Surface area} = \iint_D \left\|\frac{\partial\Phi}{\partial r}\times\frac{\partial\Phi}{\partial\theta}\right\| dA$$

$$= \int_0^{2\pi}\left[\int_0^R r\sqrt{\frac{H^2}{R^2}+1}\,dr\right]d\theta$$

$$= \sqrt{\frac{H^2}{R^2}+1}\int_0^{2\pi}\left[\frac{1}{2}r^2\right]_0^R d\theta$$

$$= \sqrt{\frac{H^2}{R^2}+1}\int_0^{2\pi}\frac{1}{2}R^2\,d\theta$$

$$= \sqrt{\frac{H^2}{R^2}+1}\,\pi R^2$$

$$= \sqrt{H^2+R^2}\,\pi R$$

$$= \boxed{\pi R L}.$$

(More examples online.)

HW 7.4 3, 6-10

(7.4 extra examples)

(Example 2)

Show that the area of a helicoid parameterized by $\underline{\Phi}(r,\theta)=(r\cos\theta, r\sin\theta, \theta)$ from $0\le\theta\le 2\pi$, $0\le r\le 1$ is equal to $2\pi\int_0^1 \sqrt{r^2+1}\,dr$.

$$\text{Area} = \iint_D \left\| \frac{\partial\Phi}{\partial r}\times\frac{\partial\Phi}{\partial\theta}\right\| dA$$

$$\frac{\partial\Phi}{\partial r} = (\cos\theta, \sin\theta, 0)$$

$$\times\ \frac{\partial\Phi}{\partial\theta} = (-r\sin\theta, r\cos\theta, 1)$$

$$(\sin\theta, -\cos\theta, r\cos^2\theta + r\sin^2\theta)$$

$$= (\sin\theta, -\cos\theta, r)$$

$$\left\|\frac{\partial\Phi}{\partial r}\times\frac{\partial\Phi}{\partial\theta}\right\| = \sqrt{\sin^2\theta+\cos^2\theta+r^2}$$

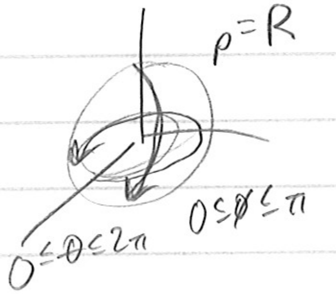
$$= \sqrt{1+r^2}$$

$$\text{Area} = \int_0^1\int_0^{2\pi}\sqrt{1+r^2}\,d\theta\,dr$$

$$= \int_0^1 \left[\sqrt{1+r^2}\,\theta\right]_0^{2\pi} dr$$

$$= \boxed{2\pi\int_0^1\sqrt{1+r^2}\,dr.}$$

(Example) Prove that the surface area of a sphere of radius R is given by $A = 4\pi R^2$.



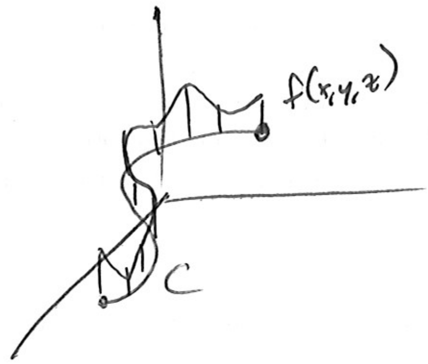
$$\Phi(\theta,\phi) = s(R,\theta,\phi)$$
$$= (R\sin\phi\cos\theta,\ R\sin\phi\sin\theta,\ R\cos\phi)$$

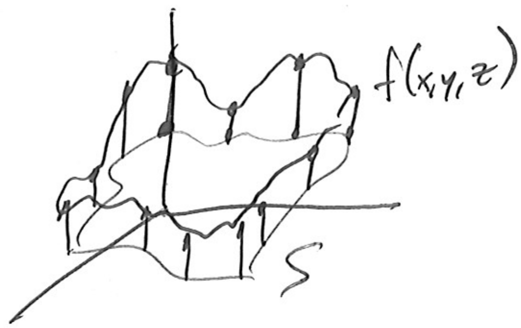
$$\frac{\partial\Phi}{\partial\theta} = (-R\sin\phi\sin\theta,\ R\sin\phi\cos\theta,\ 0)$$
$$\times\ \frac{\partial\Phi}{\partial\phi} = (R\cos\phi\cos\theta,\ R\cos\phi\sin\theta,\ -R\sin\phi)$$
$$(-R^2\sin^2\phi\cos\theta,\ -R^2\sin^2\phi\sin\theta,\ -R^2\sin\phi\cos\phi)$$

$$\left\|\frac{\partial\Phi}{\partial\theta}\times\frac{\partial\Phi}{\partial\phi}\right\| = \sqrt{R^4\sin^4\phi\cos^2\theta + R^4\sin^4\phi\sin^2\theta + R^4\sin^2\phi\cos^2\phi}$$
$$= \sqrt{R^4\sin^4\phi + R^4\sin^2\phi\cos^2\phi}$$
$$= \sqrt{R^4\sin^2\phi}$$
$$= R^2\sin\phi$$

$$\text{Area} = \int_0^{2\pi}\int_0^{\pi} R^2\sin\phi\, d\phi\, d\theta$$
$$= R^2\int_0^{2\pi}\left[-\cos\phi\right]_0^{\pi} d\theta$$
$$= R^2\int_0^{2\pi} -\cos\pi + \cos 0\, d\theta$$
$$= 2R^2\int_0^{2\pi} d\theta$$
$$= \boxed{4\pi R^2}.$$

7.5 Integrals of Scalar Functions over Surfaces





$$\text{Area} = \int_C f\, ds$$

$$= \int_{t=a}^{t=b} f(\underline{r}(t)) \left\| \frac{d\underline{r}}{dt} \right\| dt$$

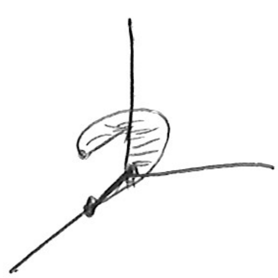
$$\left(\text{Length of } C \quad \int_{t=a}^{t=b} \left\| \frac{d\underline{r}}{dt} \right\| dt \right)$$

$$\text{Volume} = \iint_S f\, dS$$

$$= \iint_D f(\underline{\Phi}(u,v)) \left\| \frac{\partial \underline{\Phi}}{\partial u} \times \frac{\partial \underline{\Phi}}{\partial v} \right\| dA$$

$$\left(\text{Area of surface} \quad \iint_D \left\| \frac{\partial \underline{\Phi}}{\partial u} \times \frac{\partial \underline{\Phi}}{\partial v} \right\| dA \right)$$

(Example 1) Compute $\iint_S f\, dS$ where S is the helicoid parameterized by $\underline{\Phi}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$ from $0 \le \theta \le 2\pi$, $0 \le r \le 1$, and where $f(x,y,z) = \sqrt{x^2+y^2+1}$.



$$\iint_S f\,dS = \iint_D f(\Phi(r,\theta)) \left\| \frac{\partial \Phi}{\partial r} \times \frac{\partial \Phi}{\partial \theta} \right\| dA$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{x^2+y^2+1} \left\| \frac{\partial \Phi}{\partial r} \times \frac{\partial \Phi}{\partial \theta} \right\| dr\,d\theta$$

$$\frac{\partial \Phi}{\partial r} = (\cos\theta, \sin\theta, 0)$$

$$\times \quad \frac{\partial \Phi}{\partial \theta} = (-r\sin\theta, r\cos\theta, 1)$$

$$\frac{\partial \Phi}{\partial r} \times \frac{\partial \Phi}{\partial \theta} = (\sin\theta, -\cos\theta, r)$$

$$\left\| \frac{\partial \Phi}{\partial r} \times \frac{\partial \Phi}{\partial \theta} \right\| = \sqrt{\sin^2\theta + \cos^2\theta + r^2}$$

$$= \sqrt{1+r^2}$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{r^2+1} \cdot \sqrt{1+r^2}\, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^2+1 \, dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{3}r^3 + r\right]_0^1 d\theta$$

$$= \int_0^{2\pi} 4/3 \, d\theta$$

$$= \boxed{8/3\,\pi}$$