MATH 3142-001 — Spring 2016 — Dr. Clontz — Final Exam Part 1

Name: ______________________________

- Each problem is labeled with its worth toward the total grade of 100 points for this final exam.
- You do not need to show your work on these multiple-choice problems. No partial credit will be given.
- You may not use any notes/electronics on this portion of the exam.
- This part of the midterm is due after 30 minutes. Materials submitted late will be penalized by 50%.

For each of the following statements, choose if it is True or False.

1. (1 point) A continuous function on a closed bounded interval is integrable.

   A. True

   B. False

2. (1 point) Let $A \subseteq \mathbb{R}^n$ and $F : A \to \mathbb{R}^m$ be continuous. If $F(A)$ is sequentially compact, then A is sequentially compact.

   A. True

   B. False

3. (1 point) The sequence of points $\{\frac{1}{n}\}$ in $\mathbb{R}$ is Cauchy.

   A. True

   B. False

4. (1 point) Every function $f : \mathbb{R}^n \to \mathbb{R}$ with first-order partial derivatives is continuous.

   A. True

   B. False

5. (1 point) The exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is the unique function whose derivative is itself.

   A. True

   B. False

Choose the most appropriate response for each.

6. (1 point) Which of these is a closed subset of $\mathbb{R}$?

   A. $\{\frac{1}{2^n} : n \in \mathbb{N}\}$

   B. $\{0\} \cup \{\frac{1}{n+1} : n \in \mathbb{N}\}$

   C. $\mathbb{Q}$

7. (1 point) Which of these is a sequentially compact subset of $\mathbb{R}^2$?

   A. $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$

   B. $\{(x, y) \in \mathbb{R}^2 : y = 3x - 4\}$

   C. $\{(x, y) \in \mathbb{R}^2 : \max(|x|, |y|) = 1\}$

8. (1 point) Which of these is not a requirement for a metric $d : X^2 \to [0, \infty)$?

   A. $d(x, y) = d(y, x)$ for all $x, y \in X$

   B. $d(x, y)d(y, z)d(x, z) = 1$ for all $x, y, z \in X$

   C. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

9. (1 point) Let $f : \mathbb{R}^2 \to \mathbb{R}$. Which of these is equal to $\lim_{t \to 0} \dfrac{f(x + t, y + 2t) - f(x, y)}{t}$?

   A. $\langle (1, 2), \nabla f(x, y) \rangle$

   B. $f_x(x, y) + 2f_y(x, y)$

   C. $\frac{\partial^2 f}{\partial x \partial y}(x, 2y)$

10. (1 point) Which of these functions $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 0$, $f'(0) = 1$, $f''(x) = -f(x)$?

    A. $f(x) = \sin x$

    B. $f(x) = \cos x$

    C. $f(x) = \ln x$

MATH 3142-001 — Spring 2016 — Dr. Clontz — Final Exam Part 2

Name: ____________________

- Each problem is labeled with its worth toward the total grade of 100 points for this midterm.
- You must choose four of the eight problems to submit. These should be stapled to this cover sheet. Save the other four problems for your reference in Part 3.
- (Approved students may defer an extra problem to Part 3.)
- Each problem requires a rigorous proof. When in doubt, don't skip details. You may sketch pictures to help illustrate concepts, but the proof must still be valid without the use of illustrations.
- You may use your notes or textbook once Part 1 has been submitted. Electronics are still disallowed.
- This part of the midterm is due after 150 minutes. Materials submitted late will be penalized by 50%.

MATH 3142-001 — Spring 2016 — Dr. Clontz — Final Exam Part 3

Name: ______________________________

- Each problem is labeled with its worth toward the total grade of 100 points for this midterm.
- You must choose two of the four problems you didn't choose for Part 2 to submit by email.
- (Approved students who deferred a problem to Part 3 should submit three total by email.)
- Each problem requires a rigorous proof. When in doubt, don't skip details. You may sketch pictures to help illustrate concepts, but the proof must still be valid without the use of illustrations.
- You may use any notes you wish, and even collaborate with other students, as long as you submit your own work. **Plagiarism will be treated as a violation of academic honesty.**
- Solutions must be typeset using LaTeX. The professor will provide a template via email.
- This part of the midterm is due by email at 11:59pm on Friday, March 13. Materials submitted late will not be graded.

1. (15 points) Define $f : [0,2] \to \mathbb{R}$ by $f(x) = 3x$. Explicitly define a sequence of partitions $\{P_n\}$ of $[0,2]$, and then prove that this sequence is Archimedian for f on $[0,2]$.

2. (15 points) Compute the boundary of the subset $\mathbb{Q}^2$ of $\mathbb{R}^2$.

3. (15 points) Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^3+y^3} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is not continuous.

4. (15 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(\mathbf{x}) = \begin{cases} \|\mathbf{x}\| & \|\mathbf{x}\| \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & \|\mathbf{x}\| \in \mathbb{Q} \end{cases}.$$

Prove that f is continuous at $\mathbf{0}$.

5. (15 points) Let $d : X^2 \to [0, \infty)$ be a metric on X. Prove that the function $e : X^2 \to [0, \infty)$ defined by $e(x, y) = \min(d(x, y), 1)$ satisfies the triangle inequality for the below cases:

   Case $e(x, y) + e(y, z) < 1$**:**

   Case $e(x, y) + e(y, z) \geq 1$**:**

6. (15 points) Compute $f_y(0,0)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{x^2y}{x^3+y^3} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

7. (15 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous such that $f(\mathbf{0}) = 0$. Prove that for each vector $\mathbf{p} \in \mathbb{R}^n$, there exists another vector $\mathbf{x} \in \mathbb{R}^n$ such that $\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) = f(\mathbf{p})$.

8. (15 points) Recall that $\ln : (0, \infty) \to \mathbb{R}$ is defined by $\ln(x) = \int_1^x \frac{1}{t} dt$. Show that $f : (-\infty, 0) \to \mathbb{R}$ defined by $f(x) = \ln(-x)$ is the unique solution to the differential equation

$$\begin{cases} f'(x) = \frac{1}{x} & \forall x \in (-\infty, 0) \\ f(-1) = 0 \end{cases}.$$