MATH 3142 Notes — Spring 2016

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This document is a template for you to take notes in my MATH 3142 course. For your note check grade, you are required to complete all proofs/solutions for the problems specified. This template will be updated periodically throughout the course; you are responsible for updating your copy as the template is updated. See the syllabus for more details.

You should maintain your notes on Overleaf.com and provide me with a link so I can check on them. I'll give you notice before notes are "due"; when they are due I will download a copy myself from Overleaf.

This is not a replacement for the textbook for this course, *Advanced Calculus* by Patrick M. Fitzpatrick. Many proofs are outlined in that text, as well as all the relevant definitions and other results not included in these notes.

A proof is valid if and only if it uses concepts proven previously in the book. For example, you cannot prove a lemma in Chapter 6 using a theorem from Chapter 10, but using a proposition from Chapter 4 is allowed.

I hope you enjoy working through these results. Please email me with any questions.

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Chapter 6

Integration: Two Fundamental Theorems

6.1 Darboux Sums: Upper and Lower Integrals

Lemma (6.1)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and the numbers m, M have the property that

$$m \leq f(x) \leq M$$

for all x in $[a, b]$. Then, if P is a partition of the domain $[a, b]$,

$$m(b - a) \leq L(f, P) \text{ and } U(f, P) \leq M(b - a).$$

Proof. □

Lemma (6.2, The Refinement Lemma)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that P is a partition of its domain $[a, b]$. If $P^\star$ is a refinement of P, then

$$L(f, P) \leq L(f, P^\star) \text{ and } U(f, P^\star) \leq U(f, P).$$

Proof. □

Lemma (6.3)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that P_1, P_2 are partitions of its domain. Then $L(f, P_1) \leq U(f, P_2)$.

Proof. □

Lemma (6.4)**.** For a bounded function $f : [a, b] \to \mathbb{R}$,

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Proof. □

Exercise (2)**.** For an interval $[a, b]$ and a positive number δ, show that there is a partition $P = \{x_i : 0 \leq i \leq n\}$ of $[a, b]$ such that each partition interval $[x_i, x_{i+1}]$ of P has length less than δ.

Solution. □

Exercise (3)**.** Suppose that the bounded function $f : [a, b] \to \mathbb{R}$ has the property that for each rational number x in the interval $[a, b]$, $f(x) = 0$. Prove that

$$\underline{\int_a^b} f \leq 0 \leq \overline{\int_a^b} f.$$

Solution. □

Exercise (6)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function for which there is a partition P of $[a, b]$ with $L(f, P) = U(f, P)$. Prove that $f : [a, b] \to \mathbb{R}$ is constant.

Solution. □

6.2 The Archimedes-Riemann Theorem

Lemma (6.7)**.** For a bounded function $f : [a, b] \to \mathbb{R}$ and a partition P of $[a, b]$,

$$L(f, P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f, P).$$

Proof. □

Theorem (6.8, The Archimedes-Riemann Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[a, b]$ if and only if there is a sequence $\{P_n\}$ of partitions of the interval $[a, b]$ such that

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Moreover, for any such sequence of partitions,

$$\lim_{n\to\infty} L(f, P_n) = \int_a^b f = \lim_{n\to\infty} U(f, P_n).$$

Proof. □

Example (6.9)**.** Show that a monotonically increasing function $f : [a, b] \to \mathbb{R}$ is integrable.

Solution. □

Example (6.11)**.** Show that $\int_0^1 x^2 \, dx = \frac{1}{3}$.

Solution. □

Exercise (4)**.** Prove that for a natural number n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Then use this fact and the Archimedes-Riemann Theorem to show that $\int_a^b x\,dx = (b^2 - a^2)/2$.

Solution. □

Exercise (6b)**.** Use the Archimedes-Riemann Theorem to show that for $0 \leq a < b$,

$$\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}.$$

Solution. □

Exercise (9)**.** Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are integrable. Show that there is a sequence $\{P_n\}$ of partitions of $[a, b]$ that is an Archimedean sequence of partitions for f on $[a, b]$ and also an Archimedean sequence of partitions for g on $[a, b]$.

Solution. □

6.3 Additivity, Monotonicity, and Linearity

Theorem (6.12, Additivity over Intervals)**.** Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and let $c \in (a, b)$. Then f is integrable on $[a, c]$ and $[c, b]$, and furthermore

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. □

Theorem (6.13, Monotonicity of the Integral)**.** Suppose $f, g : [a, b] \to \mathbb{R}$ are integrable and that $f(x) \leq g(x)$ for all $x \in [a, b]$. Then

$$\int_a^b f \leq \int_a^b g.$$

Proof. □

Lemma (6.14)**.** Let $f, g : [a, b] \to \mathbb{R}$ be bounded and let P partition $[a, b]$. Then

$$L(f, P) + L(g, P) \leq L(f + g, P) \quad \text{and} \quad U(f + g, P) \leq U(f, P) + U(g, P).$$

Moreover, for any number α,

$$U(\alpha f, P) = \alpha U(f, P) \quad \text{and} \quad L(\alpha f, P) = \alpha L(f, P) \quad \text{if } \alpha \geq 0$$
$$U(\alpha f, P) = \alpha L(f, P) \quad \text{and} \quad L(\alpha f, P) = \alpha U(f, P) \quad \text{if } \alpha < 0.$$

Proof. □

Theorem (6.15, Linearity of the Integral)**.** Let $f, g : [a, b] \to \mathbb{R}$ be integrable. Then for any two numbers α, β, the function $\alpha f + \beta g : [a, b] \to \mathbb{R}$ is integrable and

$$\int_a^b [\alpha f + \beta g] = \alpha \int_a^b f + \beta \int_a^b g.$$

Proof. □

Exercise (1)**.** Suppose that the functions f, g, f^2, g^2, fg are integrable on $[a, b]$. Prove that $(f - g)^2$ is also integrable on $[a, b]$ and that $\int_a^b (f - g)^2 \geq 0$. Use this to prove that

$$\int_a^b fg \leq \frac{1}{2}\left[\int_a^b f^2 + \int_a^b g^2\right].$$

Solution. □

Exercise (4)**.** Suppose that S is a nonempty bounded set of numbers and that α is a number. Define αS to be the set $\{\alpha x : x \in S\}$. Prove that

$$\sup \alpha S = \alpha \sup S \quad \text{and} \quad \inf \alpha S = \alpha \inf S \quad \text{if } \alpha \geq 0$$

while

$$\sup \alpha S = \alpha \inf S \quad \text{and} \quad \inf \alpha S = \alpha \sup S \quad \text{if } \alpha < 0.$$

Solution. □

Exercise (6)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and let $a < c < b$. Prove that if f is integrable on both $[a, c], [c, b]$, then it is integrable on $[a, b]$.

Solution. □

6.4 Continuity and Integrability

Lemma (6.17)**.** Let the function $f : [a, b] \to \mathbb{R}$ be continuous let P partition its domain. Then there is a partition interval of P that contains two points u, v for which the following estimate holds:

$$0 \leq U(f, P) - L(f, P) \leq [f(v) - f(u)][b - a].$$

Proof. □

Theorem (6.18)**.** A continuous function on a closed bounded interval is integrable.

Proof. □

Theorem (6.19)**.** Supose $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$ and continuous on (a, b). Then f is integrable on $[a, b]$ and the value of $\int_a^b f$ does not depend on the values of f at the endpoints of $[a, b]$.

Proof. □

Exercise (1)**.** Determine whether each of the following statements is true or false, and justify your answer.

(a) If $f : [a, b] \to \mathbb{R}$ is integrable and $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

(b) If $f : [a, b] \to \mathbb{R}$ is integrable, then f is continuous.

(c) If $f : [a, b] \to \mathbb{R}$ is integrable and $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f \geq 0$.

(d) A continuous function $f : (a, b) \to \mathbb{R}$ defined on an open interval (a, b) is bounded.

(e) A continuous function $f : [a, b] \to \mathbb{R}$ defined on a closed interval $[a, b]$ is bounded.

Solution. (a)

(b)

(c)

(d)

(e)

□

Exercise (5)**.** Suppose that the continuous function $f : [a, b] \to \mathbb{R}$ has the property

$$\int_c^d f \leq 0 \quad \text{whenever } a \leq c < d \leq b.$$

Prove that $f(x) \leq 0$ for all $x \in [a, b]$. Is this true if we only require integrability of the function?

Solution. □

Exercise (6)**.** Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and that $f(x) \geq 0$ for all $x \in [0, 1]$. Prove that $\int_0^1 f > 0$ if and only if there is a point $x_0 \in [0, 1]$ at which $f(x_0) > 0$.

Solution. □

6.5 The First Fundamental Theorem: Integrating Derivatives

Lemma (6.21)**.** Suppose $f : [a, b] \to \mathbb{R}$ is integrable and that the number A has the property that for every P partitioning $[a, b]$,

$$L(f, P) \leq A \leq U(f, P).$$

Then

$$\int_a^b f = A.$$

Proof. □

Theorem (6.22, The First Fundamental Theorem: Integrating Derivatives)**.** Let $F : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Moreover, suppose that its derivative $F' : (a, b) \to \mathbb{R}$ is both continuous and bounded. Then

$$\int_a^b F'(x)\ dx = F(b) - F(a).$$

Proof. □

Exercise (1)**.** Let m, b be positive numbers. Find the value of $\int_0^1 mx + b\ dx$ in the following three ways:

(a) Using elementary geometry, interpreting the integral as an area.

(b) Using upper and lower Darboux sums based on regular partitions of the interval $[0, 1]$ and using the Archimedes-Riemann Theorem.

(c) Using the First Fundamental Theorem (Integrating Derivatives).

Solution. □

Exercise (5)**.** The monotonicity property of the integral implies that if the functions $g, h : [0, \infty) \to \mathbb{R}$ are continuous and $g(x) \leq h(x)$ for all $x \geq 0$, then

$$\int_0^x g \leq \int_0^x h \quad \text{for all } x \geq 0.$$

Use this and the First Fundamental Theorem to show that each of the following inequalities implies the next:

$$\cos x \leq 1 \quad \text{if } x \geq 0.$$

$$\sin x \leq x \quad \text{if } x \geq 0.$$

$$1 - \cos x \leq \frac{x^2}{2} \quad \text{if } x \geq 0.$$

$$x - \sin x \leq \frac{x^3}{6} \quad \text{if } x \geq 0.$$

$$x - \frac{x^3}{6} \leq \sin x \leq x \quad \text{if } x \geq 0.$$

(For this problem, you may assume that the sine and cosine functions are differentiable functions with the properties $\sin(0) = 0$, $\cos(0) = 1$, $\frac{d}{dx}[\sin(x)] = \cos(x)$, and $\frac{d}{dx}[\cos(x)] = -\sin(x)$.)

Solution. □

6.6 The Second Fundamental Theorem: Differentiating Integrals

Theorem (6.26, The Mean Value Theorem for Integrals)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then there is a point x_0 in the interval $[a, b]$ at which

$$\frac{1}{b-a} \int_a^b f = f(x_0).$$

Proof. □

Proposition (6.27)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is integrable. Define

$$F(x) = \int_a^x f \quad \text{for all } x \in [a, b].$$

Then the function $F : [a, b] \to \mathbb{R}$ is continuous.

Proof. □

Theorem (6.29, The Second Fundamental Theorem: Differentiating Integrals)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then

$$\frac{d}{dx}\left[\int_a^x\right] = f(x) \quad \text{for all } x \in (a, b).$$

Proof. □

Exercise (2b)**.** Suppose $f : [0, 2] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } 1 < x \leq 2 \end{cases}.$$

Define

$$F(x) = \int_a^x f(t)\, dt \quad \text{for all} x \in [a, b]$$

and find a formula for $F(x)$ which does not involve integrals.

Solution. □

Exercise (5)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Define

$$G(x) = \int_0^x (x-t)f(t)\,dt \quad \text{for all } x.$$

Prove that $G''(x) = f(x)$ for all x.

Solution. □

Exercise (12)**.** Suppose that $f, g : [a,b] \to \mathbb{R}$ are continuous and that α, β are real numbers. Define

$$H(x) = \int_a^x [\alpha f + \beta g] - \alpha \int_a^x [f] - \beta \int_a^x [g] \quad \text{for all } x \in [a,b].$$

Prove that $H(a) = 0$ and $H'(x) = 0$ for all $x \in (a,b)$. Use this fact and the Identity Criterion to give an alternate proof of Theorem 6.15 for continuous functions.

Solution. □

Chapter 10

The Euclidean Space $\mathbb{R}^n$

10.1 The Linear Structure of $\mathbb{R}^n$ and the Scalar Product

Proposition (10.2)**.** Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Then both of the following hold:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$

$$\langle \alpha\mathbf{u} + \beta\mathbf{w}, v \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{w}, \mathbf{v} \rangle$$

Proof. □

Lemma (10.4)**.** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{u}, \mathbf{v}$ are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Proof. □

Lemma (10.5)**.** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ where $\mathbf{v} \neq \mathbf{0}$, define $\lambda = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$ and $\mathbf{w} = \mathbf{u} - \lambda\mathbf{v}$. Then $\mathbf{v}, \mathbf{w}$ are orthogonal and $\mathbf{u} = \mathbf{w} + \lambda\mathbf{v}$.

Proof. □

Theorem (10.6, The Cauchy-Schwarz Inequality)**.** For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|.$$

Proof. □

Theorem (10.7, The Triangle Inequality)**.** For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Exercise (3)**.** Show that for $\mathbf{u} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$:

(a) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

(b) $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$.

Proof. □

Exercise (4)**.** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ verify the identity

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\langle \mathbf{u}, \mathbf{v} \rangle.$$

Solution. □

Exercise (9)**.** Let $\mathbf{u} \in \mathbb{R}^n$ and suppose $\|\mathbf{u}\| < 1$. Show that for $\mathbf{v} \in \mathbb{R}^n$, $\|\mathbf{v} - \mathbf{u}\| < 1 - \|\mathbf{u}\|$ implies $\|\mathbf{v}\| < 1$.

Solution. □

Exercise (10)**.** Let $\mathbf{u} \in \mathbb{R}^n$ and $r > 0$. Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are at a distance less than r from $\mathbf{u}$. Prove that if $0 \le t \le 1$, then the point $t\mathbf{v} + (1-t)]\mathbf{w}$ is also at a distance less than r from $\mathbf{u}$.

Solution. □

10.2 Convergence of Sequences in $\mathbb{R}^n$

Theorem (10.9, The Componentwise Convergence Criterion)**.** Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$. Then $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$ if and only if $\{p_i(\mathbf{u}_k)\}$ converges to $p_i(\mathbf{u})$ for all $1 \le i \le n$.

Proof. □

Theorem (10.10)**.** Let $\{\mathbf{u}_k\}$, $\{\mathbf{v}_k\}$ be sequences in $\mathbb{R}^n$ such that $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$ and $\{\mathbf{v}_k\}$ converges to $\mathbf{v}$. Then for any $\alpha, \beta \in \mathbb{R}$,

$$\lim_{k\to\infty} [\alpha\mathbf{u}_k + \beta\mathbf{v}_k] = \alpha\mathbf{u} + \beta\mathbf{v}.$$

Proof. □

Exercise (1)**.** Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$ that converges to $\mathbf{u}$. Prove the following for all $\mathbf{v} \in \mathbb{R}^n$:

$$\lim_{k\to\infty} \langle \mathbf{u}_k, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle.$$

Solution. □

Exercise (2)**.** Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^n$. Prove that if

$$\lim_{k\to\infty} \langle \mathbf{u}_k, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

holds for all $\mathbf{v} \in \mathbb{R}^n$, then $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$.

Solution. □

Exercise (5)**.** Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$ that converges to $\mathbf{u}$ where $\|\mathbf{u}\| = r > 0$. Prove that there is an index K where

$$\|\mathbf{u}_k\| > \frac{r}{2} \quad \text{if } k \geq K.$$

Solution. □

10.3 Open Sets and Closed Sets in $\mathbb{R}^n$

Example (10.11)**.** Let $a < b$ be in $\mathbb{R}$. Then $\operatorname{int}(a, b] = (a, b)$.

Proof. □

Example (10.12)**.** Let $\mathbb{Q} \subseteq \mathbb{R}$ be the set of rational real numbers. Then $\operatorname{int} \mathbb{Q} = \emptyset$.

Proof. □

Proposition (10.13)**.** Every open ball $B_r(\mathbf{u})$ in $\mathbb{R}^n$ is open.

Proof. □

Example (10.14)**.** Let $a < b$ be in $\mathbb{R}$. Then $[a, b]$ is closed.

Proof. □

Example (10.15)**.** The set

$$[-1, 1] \times [-1, 1] = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}$$

is closed in $\mathbb{R}^2$.

Proof. □

Theorem (10.16, The Complementing Characterization)**.** A subset $A \subseteq \mathbb{R}^n$ is open if and only if its complement $\mathbb{R}^n \setminus A$ is closed.

Proof. □

Proposition (10.17.i)**.** The union of a collection of open subsets of $\mathbb{R}^n$ is open.

Proof. □

Proposition (10.17.ii)**.** The intersection of a collection of closed subsets of $\mathbb{R}^n$ is closed.

Proof. □

Proposition (10.18.i)**.** The intersection of a finite collection of open subsets of $\mathbb{R}^n$ is open.

Proof. □

Proposition (10.18.ii)**.** The union of a finite collection of closed subsets of $\mathbb{R}^n$ is closed.

Proof. □

Proposition (10.19.i)**.** $A \subseteq \mathbb{R}^n$ is open if and only if $A \cap \operatorname{bd} A = \emptyset$.

Proof. □

Proposition (10.19.ii)**.** $A \subseteq \mathbb{R}^n$ is closed if and only if $\operatorname{bd} A \subseteq A$.

Proof. □

Exercise (2)**.** Determine which of the following subsets of $\mathbb{R}^2$ are open, closed, neither, or both.

(a) $\{(x, y) : x^2 > y\}$

(b) $\{(x, y) : x^2 + y^2 = 1\}$

(c) $\{(x, y) : x \text{ is rational}\}$

(d) $\{(x, y) : x \geq 0,\ y \geq 0\}$

Solution. (a)

(b)

(c)

(d)

□

Exercise (3)**.** Let $r > 0$ and $O = \{\mathbf{u} \in \mathbb{R}^n : \|\mathbf{u}\| > r\}$. Prove that O is open.

Solution. □

Exercise (7a)**.** Show that $A \subseteq \mathbb{R}^n$ is open if and only if

$$\mathbf{w} + A = \{\mathbf{w} + \mathbf{u} : \mathbf{u} \in A\}$$

is open for all $\mathbf{w} \in \mathbb{R}^n$.

Solution. □

Exercise (12)**.** For $A \subseteq \mathbb{R}^n$, denote its closure by

$$\operatorname{cl} A = \operatorname{int} A \cup \operatorname{bd} A.$$

Prove that $A \subseteq \operatorname{cl} A$. Then prove that $A = \operatorname{cl} A$ if and only if A is closed.

Solution. □

Chapter 11

Continuity, Compactness, and Connectedness

11.1 Continuous Functions and Mappings

Proposition (11.1)**.** For each $i \in \{1, \ldots, n\}$, the ith projection map $p_i : \mathbb{R}^n \to \mathbb{R}$ is continuous.

Proof. $\square$

Theorem (11.3)**.** Let $\mathbf{u} \in A \subseteq \mathbb{R}^n$ and $h, g : A \to \mathbb{R}$ be continuous at $\mathbf{u}$. Then for $\alpha, \beta \in \mathbb{R}$, the following functions are continuous at $\mathbf{u}$:

$$\alpha h + \beta g : A \to \mathbb{R} \qquad\qquad h \cdot g : A \to \mathbb{R}.$$

Also if $g(\mathbf{v}) \neq 0$ for all $\mathbf{v} \in A$, then the following function is also continuous at $\mathbf{u}$:

$$\frac{h}{g} : A \to \mathbb{R}.$$

Proof. $\square$

Theorem (11.5)**.** Let $\mathbf{u} \in A \subseteq \mathbb{R}^n$ and $G : A \to \mathbb{R}^m$ be continuous at $\mathbf{u}$. Also let $G(A) \subseteq B \subseteq \mathbb{R}^m$ and $H : B \to \mathbb{R}^k$ be continuous at $G(\mathbf{u})$. Then the composition

$$H \circ G : A \to \mathbb{R}^k$$

is continuous at $\mathbf{u}$.

Proof. $\square$

Theorem (11.9, The Componentwise Continuity Criterion)**.** Let $\mathbf{u} \in A \subseteq \mathbb{R}^n$ and $F : A \to \mathbb{R}^m$. Then F is continuous at $\mathbf{u}$ if and only if $F_i = p_i \circ F : A \to \mathbb{R}$ is continuous at $\mathbf{u}$ for each $i \in \{1, \ldots, n\}$.

Proof. □

Theorem (11.11, Exercise 12)**.** Let $\mathbf{u} \in A \subseteq \mathbb{R}^n$ and $F : A \to \mathbb{R}^m$. Then F is continuous at $\mathbf{u}$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{v} - \mathbf{u}\| < \delta$ implies $\|F(\mathbf{v}) - F(\mathbf{u})\| < \epsilon$.

Proof. □

Theorem (11.12)**.** Let $U \subseteq \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^m$. Then F is continuous if and only if $F^{-1}(V)$ is an open subset of $\mathbb{R}^n$ for every open $V \subseteq \mathbb{R}^m$.

Proof. □

Example (11.15)**.** Use corollary 11.13 and proposition 10.18.i to prove that $U = \{\mathbf{u} \in \mathbb{R}^n : a < \|\mathbf{u}\| < b\}$ is open. (You may assume $f(\mathbf{u}) = \|\mathbf{u}\|$ is continuous.)

Solution. □

Exercise (3)**.** Fix a point $\mathbf{v} \in \mathbb{R}^n$. Prove that $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v} \rangle$ is continuous.

Solution. □

Exercise (6)**.** Suppose $f, g : \mathbb{R}^n \to \mathbb{R}$ are continuous. Prove that $\{\mathbf{u} \in \mathbb{R}^n : f(\mathbf{u}) = g(\mathbf{u}) = 0\}$ is closed. (Hint: use corollary 11.13 and proposition 10.17.ii.)

Solution. □

Exercise (11)**.** Let $A \subseteq \mathbb{R}^n$. The characteristic function $\phi_A : \mathbb{R}^n \to \mathbb{R}$ for A is defined to be

$$\phi_A(\mathbf{u}) = \begin{cases} 1 & \text{if } \mathbf{u} \in A \\ 0 & \text{if } \mathbf{u} \notin A \end{cases}.$$

Prove that ϕ_A is continuous at points in $\operatorname{int} A$ and $\operatorname{ext} A$, but not continuous at points in $\operatorname{bd} A$.

Solution. □

11.2 Sequential Compactness, Extreme Values, and Uniform Continuity

Theorem (11.16)**.** Every sequentially compact subset of $\mathbb{R}^n$ is bounded and closed.

Proof. □

Theorem (11.17)**.** Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Proof. $\square$

Theorem (11.18, The Sequential Compactness Theorem)**.** A subset of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.

Proof. $\square$

Corollary (11.19)**.** The generalized rectangle $\prod_{i=1}^n [a_i, b_i] = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ is sequentially compact.

Proof. $\square$

Theorem (11.20)**.** Let $A \subseteq \mathbb{R}^n$ and suppose $F : A \to \mathbb{R}^m$ is continuous. If A is sequentially compact then $F(A)$ is also sequentially compact.

Proof. $\square$

Lemma (11.21)**.** Every nonempty seqeuntially compact subset of $\mathbb{R}$ has a maximum and minimum element.

Proof. $\square$

Theorem (11.22, The Extreme Value Theorem, Bolzano-Weierstrass Theorem)**.** Let $A \subseteq \mathbb{R}^n$ be nonempty sequentially compact and suppose $f : A \to \mathbb{R}$ is continuous. Then f attains a smallest and largest value.

Proof. $\square$

Theorem (11.24)**.** Let $A \subseteq \mathbb{R}^n$ be nonempty. A is sequentially compact if and only if every continuous $f : A \to \mathbb{R}$ attains a smallest and largest value (i.e. it has the Extreme Value Property).

Proof. $\square$

Theorem (11.25, Exercise 5)**.** Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be continuous. If A is sequentially compact then f is uniformly continuous.

Proof. $\square$

Theorem (11.27, Exercise 11)**.** Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. f is uniformly continuous if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{u} - \mathbf{v}\| < \delta$ implies $\|f(\mathbf{u}) - f(\mathbf{v})\| < \epsilon$.

Proof. $\square$

Exercise (3,4)**.** Recall that $B_r(\mathbf{u}) = \{\mathbf{v} : \|\mathbf{u} - \mathbf{v}\| < r\}$. Is $B_r(\mathbf{u})$ bounded? Sequentially compact?

Solution. □

Exercise (2)**.** Let $D_r(\mathbf{u}) = \{\mathbf{v} : \|\mathbf{u} - \mathbf{v}\| \leq r\}$. Prove $D_r(\mathbf{u})$ is sequentially compact.

Solution. □

Chapter 12

Metric Spaces

Definition. A pair (X, d) is called a *metric space* if X is a set and d is a function $d : X^2 \to [0, \infty)$ satisfying the following properties:

- Identity: $d(p, q) = 0$ if and only if $p = q$.
- Symmetry: $d(p, q) = d(q, p)$.
- Triangle Inequality: $d(p, q) \leq d(p, w) + d(w, q)$.

Theorem (12.2)**.** $dist(\mathbf{p}, \mathbf{q}) = \|\mathbf{q} - \mathbf{p}\|$ is a metric on $\mathbb{R}^n$.

Proof. □

Definition. Let (X, d) be a metric space. For $p \in X, r > 0$,

$$B_r(p) = \{q \in X : d(p, q) < r\}$$

is the open ball about p with radius r. For $A \subseteq X$,

- $\operatorname{int} A = \{q \in A : \exists r > 0(B_r(q) \subseteq A)\}$
- $\operatorname{ext} A = \{q \in A : \exists r > 0(B_r(q) \subseteq X \setminus A)\}$
- $\operatorname{bd} A = \{q \in A : \forall r > 0(B_r(q) \cap A \neq \emptyset \text{ and } B_r(q) \setminus A \neq \emptyset)\}$

Call A open in (X, d) if $A = \operatorname{int} A$. Note that these concepts match the definitions we gave for $\mathbb{R}^n$ using the metric $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{q} - \mathbf{p}\|$.

Theorem (12.8)**.** Let (X, d) be a metric space. Let $p \in X, r > 0$. Then $B_r(p)$ is open.

Proof. □

Definition. Let d be a metric on $\mathbb{R}^n$. We say d is *compatible with the usual topology on* $\mathbb{R}^n$ if the open sets determined by d are exactly the open sets determined by $dist$.

Example. $s : \mathbb{R}^n \to [0, \infty)$ defined by $s(\mathbf{u}, \mathbf{v}) = \max\{|p_i(\mathbf{v}) - p_i(\mathbf{u})| : 1 \le i \le n\}$ is a metric on $\mathbb{R}^n$.

Proof. □

Theorem. s is compatible with the usual topology on $\mathbb{R}^n$.

Proof. □

Example. $t : \mathbb{R}^n \to [0, \infty)$ defined by $t(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^n |p_i(\mathbf{v}) - p_i(\mathbf{u})|$ is a metric on $\mathbb{R}^n$.

Proof. □

Theorem. t is compatible with the usual topology on $\mathbb{R}^n$.

Proof. □

Definition. $d : X \to [0, \infty)$ defined by $d(p, q) = 1$ for $p \neq q$ and $d(p, p) = 0$ is called a *discrete metric* on X.

Theorem (12.4)**.** The discrete metric on X is a metric.

Proof. □

Theorem. The discrete metric on $\mathbb{R}^n$ is not compatible with the usual topology on $\mathbb{R}^n$. (Hint: show that every subset of a discrete metric space is open.)

Proof. □

Definition. Let $C([a, b], \mathbb{R})$ be the set of all continuous functions $f : [a, b] \to \mathbb{R}$, and for $f, g \in C([a, b], \mathbb{R})$ let $d(f, g) = \max\{|f(x) - g(x)| : x \in [a, b]\}$.

Theorem (12.3)**.** $d(f, g) = \max\{|f(x) - g(x)| : x \in [a, b]\}$ is a metric on $C([a, b], \mathbb{R})$.

Proof. □

Definition. Let $\{p_k\}$ denote a *sequence* in a metric space (X, d), i.e. a function from $\mathbb{N}$ to X.

Definition. We say the sequence $\{p_k\}$ *converges* to $p \in X$ when

$$\lim_{k \to \infty} d(p_k, p) = 0.$$

Definition. $C \subseteq X$ is said to be *closed* in the metric space (X, d) when for every sequence $\{p_k\}$ of points in C converging to $p \in X$, it follows that $p \in C$.

Example (12.11)**.** The set $\{f \in C([a, b], \mathbb{R}) : f(x) \ge 0\}$ is closed.

Proof. □

Theorem (12.12, The Complementing Characterization)**.** Let (X, d) be a metric space and $A \subseteq X$. Then A is open in (X, d) if and only if $X \setminus A$ is closed in (X, d).

Proof. $\square$

Definition. A sequence $\{p_k\}$ in a metric space (X, d) is called a *Cauchy sequence* when for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $k, l \geq N$ implies $d(p_k, p_l) < \epsilon$.

Proposition (12.15)**.** Every convergent sequence in a metric space is Cauchy.

Proof. $\square$

Lemma (9.3)**.** Every Cauchy sequence in $(\mathbb{R}, dist)$ is bounded.

Proof. $\square$

Theorem (9.4)**.** A sequence in $(\mathbb{R}, dist)$ is Cauchy if and only if it is convergent.

Proof. $\square$

Corollary (Example 12.16)**.** A sequence in $(\mathbb{R}^n, dist)$ is Cauchy if and only if it is convergent.

Proof. $\square$

Definition. A *complete metric space* is a metric space where every Cauchy sequence is convergent.

Chapter 13

Differentiating Functions of Several Variables

13.1 Limits

Definition. Let $A \subseteq \mathbb{R}^n$. We call $\mathbf{x}_* \in \mathbb{R}^n$ a *limit point* of A in the case that there exists a sequence in $A \setminus \{\mathbf{x}_*\}$ which converges to $\mathbf{x}_*$.

Definition. Let $A \subseteq \mathbb{R}^n$ have a limit point $x_* \in \mathbb{R}$, and $f : A \to \mathbb{R}$ be a function. Then we say the *limit of f as $\mathbf{x}$ approaches $\mathbf{x}_*$* is $L \in \mathbb{R}$, or

$$\lim_{\mathbf{x} \to \mathbf{x}_*} f(\mathbf{x}) = L$$

in the case that whenever $\{\mathbf{x}_k\}$ is a sequence of points in $A \setminus \{\mathbf{x}_*\}$ converging to $\mathbf{x}_*$, then $\{f(\mathbf{x}_k)\}$ is a sequence of real numbers which converges to L.

Theorem (13.3)**.** Let$A \subseteq \mathbb{R}^n$ and $\mathbf{x}_*$ be a limit point of A. Suppose the functions $f, g : A \to \mathbb{R}$ satsify

$$\lim_{\mathbf{x} \to \mathbf{x}_*} f(\mathbf{x}) = L_1 \quad \text{and} \quad \lim_{\mathbf{x} \to \mathbf{x}_*} g(\mathbf{x}) = L_2.$$

Then

$$\lim_{\mathbf{x} \to \mathbf{x}_*} [f(\mathbf{x}) + g(\mathbf{x})] = L_1 + L_2$$

and

$$\lim_{\mathbf{x} \to \mathbf{x}_*} [f(\mathbf{x})g(\mathbf{x})] = L_1 L_2.$$

And assuming $g(\mathbf{x}) \neq 0$ for $x \in A$ and $L_2 \neq 0$,

$$\lim_{\mathbf{x} \to \mathbf{x}_*} [f(\mathbf{x})/g(\mathbf{x})] = L_1/L_2.$$

Proof. □

Example (13.4)**.** The limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

does not exist.

Proof. □

Example (13.5)**.**

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0.$$

Proof. □

Exercise (4)**.** Let $m, n \in \mathbb{N}$. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{x^n y^m}{x^2+y^2}$$

exists if and only if $m + n > 2$.

Solution. □

Exercise (5)**.** Give an example of a subset $A \subseteq \mathbb{R}$ and point $x \in A$ such that x is not a limit point of A.

Solution. □

Exercise (12)**.** Show that $A \subseteq \mathbb{R}^n$ is closed if and only if it contains all its limit points.

Solution. □

13.2 Partial Derivatives

Definition. For each $1 \leq i \leq n$, let $\mathbf{e}_i \in \mathbb{R}^n$ satisfy $p_i(\mathbf{e}_i) = 1$ and $p_j(\mathbf{e}_i) = 0$ for $j \neq i$.

Definition. Let $\mathbf{x} \in U \subseteq \mathbb{R}^n$ with U open. For a function $f : U \to \mathbb{R}$, define its *first-order partial derivative with respect to its* i*th component at* $\mathbf{x}$ to be

$$\left[\frac{\partial}{\partial x_i} f\right](\mathbf{x}) = \frac{\partial f}{\partial x_i}(\mathbf{x}) = f_{x_i}(\mathbf{x}) = \lim_{t\to 0} \frac{f(\mathbf{x}+t\mathbf{e}_i) - f(\mathbf{x})}{t}$$

whenever the limit exists.

Definition. Let $U \subseteq \mathbb{R}^n$ be open. For a function $f : U \to \mathbb{R}$ such that $f_{x_i}(\mathbf{x})$ exists for all $\mathbf{x} \in U$, let $f_{x_i} : U \to \mathbb{R}^n$ be defined as its *first-order partial derivative with respect to its* i*th component.*

Definition. Let $U \subseteq \mathbb{R}^n$ be open. A function $f : U \to \mathbb{R}$ such that f_{x_i} exists for all $1 \leq i \leq n$ is said to have *first-order partial derivatives.*

Example (13.8*)**.** If $f : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$f(x, y, z) = xyz - 3xy^2$$

then $f_y : \mathbb{R}^3 \to \mathbb{R}$ satisfies

$$f_y(x, y, z) = xz - 6xy.$$

Proof. □

Example (13.9)**.** The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

has first-order partial derivatives, but is not continuous.

Proof. □

Definition. Let $U \subseteq \mathbb{R}^n$ be open. Then a function $f : U \to \mathbb{R}$ is *continuously differentiable* provided that $f_{x_i} : U \to \mathbb{R}$ exists and is continuous for $1 \leq i \leq n$.

Definition. Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^n$ have first-order partial derivatives. Then for $1 \leq i, j \leq n$ let

$$\frac{\partial^2 f}{\partial x_j \partial x_i} : U \to \mathbb{R}$$

be the partial derivative of $\partial f/\partial x_i : U \to \mathbb{R}$ with respect to its jth component. This is also denoted by $f_{x_i x_j}$. When $i = j$, this is also denoted by $\frac{\partial^2 f}{\partial x_i^2}$.

Definition. Let $U \subseteq \mathbb{R}^n$ be open. A function $f : U \to \mathbb{R}$ such that $f_{x_i x_j}$ exists for all $1 \leq i, j \leq n$ is said to have *second-order partial derivatives.*

Definition. Let $U \subseteq \mathbb{R}^n$ be open. A function $f : U \to \mathbb{R}$ such that $f_{x_i x_j}$ exists and is continuous for all $1 \leq i, j \leq n$ is said to have *continuous second-order partial derivatives.*

Lemma (13.11)**.** Let $U \subseteq \mathbb{R}^2$ be open and nonempty, and suppose $f : U \to \mathbb{R}$ has second-order partial derivatives. Then there are points $(x_1, y_1), (x_2, y_2) \in U$ such that $f_{xy}(x_1, y_1) = f_{yx}(x_2, y_2)$.

Proof. □

Theorem (13.10)**.** Let $U \subseteq \mathbb{R}^n$ be open and nonempty, and suppose $f : U \to \mathbb{R}$ has continuous second-order partial derivatives. Then for all $1 \leq i, j \leq n$, it follows that $f_{x_i x_j} = f_{x_j x_i}$.

Proof for n=2. □

Example (13.12, exercise 13)**.** The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} xy(x^2 - y^2)/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

has second-order partial derivatives, but

$$f_{xy}(0, 0) = -1 \quad \text{while} \quad f_{yx}(0, 0) = 1.$$

Proof. □

Exercise (4)**.** Prove that $g : \mathbb{R}^2 \to \mathbb{R}$ satisfying $|g(x, y)| \leq x^2 + y^2$ must have partial derivatives with respect to both x and y at the point $(0, 0)$.

Solution. □

13.3 The Mean Value Theorem and Directional Derivatives

Lemma (13.14, The Mean Value Lemma)**.** Let $U \subseteq \mathbb{R}^n$ be open and $1 \leq i \leq n$. Let $f : U \to \mathbb{R}$ be a function with a partial derivative with respect to its ith component at each point in U. Let $\mathbf{x} \in U$ and $a \in \mathbb{R}$ such that $\mathbf{x} + \theta a \mathbf{e}_i \in U$ for all $\theta \in [0, 1]$. Then there is some $\theta \in (0, 1)$ such that

$$f(\mathbf{x} + a\mathbf{e}_i) - f(\mathbf{x}) = a \frac{\partial f}{\partial x_i}(\mathbf{x} + \theta a \mathbf{e}_i).$$

Proof. □

Proposition (13.15, The Mean Value Proposition)**.** Let $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$. Let $f : B_r(\mathbf{x}) \to \mathbb{R}$ be a function with first-order partial derivatives. Then if $\mathbf{h} \in \mathbb{R}^n$ satisfies $\|\mathbf{h}\| < r$, then there are points $\mathbf{z}_i \in B_{\|\mathbf{h}\|}(\mathbf{x})$ for $1 \leq i \leq n$ satisfying

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \sum_{i=1}^{n} p_i(\mathbf{h}) \frac{\partial f}{\partial x_i}(\mathbf{z}_i).$$

Proof. □

Definition. Let $\mathbf{x} \in U \subseteq \mathbb{R}^n$ where U is open, let $f : U \to \mathbb{R}$ be a function, and let $\mathbf{p} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Define the *directional derivative* of f at $\mathbf{x}$ in the direction $\mathbf{p}$ by

$$\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{p}) - f(\mathbf{x})}{t}$$

whenever that limit exists.

Definition. Let $\mathbf{x} \in U \subseteq \mathbb{R}^n$ where U is open, and let $f : U \to \mathbb{R}$ be a function with first-order partial derivatives at $\mathbf{x}$. Define its *gradient* $\nabla f(\mathbf{x}) \in \mathbb{R}^n$ at $\mathbf{x}$ to satisfy $p_i(\nabla f(\mathbf{x})) = f_{x_i}(\mathbf{x})$ for all $1 \leq i \leq n$. If its gradient exists at every $\mathbf{x} \in U$, then let $\nabla f : U \to \mathbb{R}^n$ be the *gradient function* defined by evaluating the gradient at each point.

Theorem (13.16, The Directional Derivative Theorem)**.** Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be continuously differentiable. Then for each $\mathbf{x} \in U$ and $\mathbf{p} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, f has a directional derivative at $\mathbf{x}$ in the direction of $\mathbf{p}$ given by

$$\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) = \langle \mathbf{p}, \nabla f(\mathbf{x}) \rangle.$$

Proof. □

Theorem (13.17, The Mean Value Theorem)**.** Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be continuously differentiable. Let $\mathbf{x}, \mathbf{h} \in \mathbb{R}^n$ such that $\mathbf{x} + \theta\mathbf{h} \in U$ for all $\theta \in [0, 1]$. Then there is some $\theta \in (0, 1)$ such that

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \langle \mathbf{h}, \nabla f(\mathbf{x} + \theta\mathbf{h}) \rangle.$$

Proof. □

Corollary (13.18)**.** Let $\mathbf{x} \in U \subseteq \mathbb{R}^n$ with U open and let $f : U \to \mathbb{R}$ be continuously differentiable such that $\nabla f(\mathbf{x}) \neq \mathbf{0}$. Then the unit vector maximizing the value of the directional derivative of f at $\mathbf{x}$ is

$$\mathbf{p}_0 = \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}.$$

Proof. □

Theorem (13.20)**.** Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be continuously differentiable. Then f is continuous.

Proof. □

Exercise (4)**.** Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ has first-order partial derivatives and that $\mathbf{x}$ is a local minimizer for f, that is, there exists some $\epsilon > 0$ such that for all $\mathbf{y} \in B_\epsilon(\mathbf{x})$, $f(\mathbf{y}) \geq f(\mathbf{x})$. Prove that $\nabla f(\mathbf{x}) = \mathbf{0}$.

Solution. □

Chapter 0

Logrithmic, Exponential, and Trigonometric Functions

Assume all sequences in this chapter are indexed by the nonnegative integers, and that $0^0 = 1$.

0.1 Logrithms and Exponential Functions

Theorem (The Identity Criterion)**.** A differentiable function $g : U \to \mathbb{R}$ where U is an open interval of $\mathbb{R}$ is the constant function 0 if and only if

$$\begin{cases} g'(x) = 0 & \text{for all } x \in U \\ g(x_0) = 0 & \text{for some } x_0 \in U \end{cases}.$$

Proof. Proved in MATH 3141. □

Proposition (7.1)**.** Let $x_0 \in U$ for some open interval $U \subseteq \mathbb{R}$, and suppose $f : (a, b) \to \mathbb{R}$ is continuous. Then for any number y_0, the function $F : U \to \mathbb{R}$ defined by

$$F(x) = y_0 + \int_{x_0}^{x} f$$

is the unique funciton satisfying the conditions of the following differential equation:

$$\begin{cases} F'(x) = f(x) & \text{for all } x \in U \\ F(x_0) = y_0 \end{cases}.$$

Proof. □

Definition. Define the *natural logrithm* function $\ln : (0, \infty) \to \mathbb{R}$ by $\ln(x) = \int_1^x \frac{1}{t}\, dt$.

Theorem (5.1)**.** The natural logrithm function satisfies the following properties for all $a, b > 0$, $c \in \mathbb{R}$, and $r \in \mathbb{Q}$.

(a) $\ln(ab) = \ln(a) + \ln(b)$.

(b) $\ln(a^r) = r\ln(a)$.

(c) There exists $x > 0$ such that $\ln(x) = c$.

Proof. (a)

(b)

(c)

□

Proposition (pg. 120)**.** The natural logrithm $\ln : (0, \infty) \to \mathbb{R}$ has a differentiable inverse function $\exp : \mathbb{R} \to (0, \infty)$, called the *exponential* function.

Proof. □

Definition. Let the *natural number* $e \in \mathbb{R}$ be defined by $\exp(1)$. (Note $\ln(e) = \ln(\exp(1)) = 1$.)

Proposition (pg. 120)**.** The exponential function exp is the unique solution to the differential equation

$$\begin{cases} \frac{d}{dx}[\exp(x)] = \exp(x) & \text{for all } x \in \mathbb{R} \\ \exp(0) = 1 \end{cases}.$$

Proof. □

Definition. Define $a^x = \exp(x\ln(a))$ for $a > 0$ and all real numbers x. (In particular, $e^x = \exp(x\ln(e)) = \exp(x)$.)

Proposition (pg. 121)**.** The above definition is compatible with the previously given definition for exponentiation for rational numbers.

Proof. □

Proposition (5.2, 5.3)**.** Let $a > 0$ and $r \in \mathbb{R}$. Then

$$\frac{d}{dx}[a^x] = a^x \ln a \qquad \text{for all } x \in \mathbb{R}, \text{ and}$$

$$\frac{d}{dx}[x^r] = rx^{r-1} \qquad \text{for all } x > 0.$$

Proof. □

Theorem (5.4)**.** Let $c, k \in \mathbb{R}$. Then $F(x) = ce^{kx}$ is the unique solution to the differential equation

$$\begin{cases} F'(x) = kF(x) & \text{for all } x \in \mathbb{R} \\ F(0) = c \end{cases}.$$

Proof. □

0.2 Power Series

Definition. For a sequence of real numbers $\{a_k\}$, let $\{s_k\}$ be its *sequence of partial sums* defined by

$$s_n = \sum_{k=0}^{n} a_k.$$

Definition. Define the *series* of a sequence $\{a_k\}$ by

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left[\sum_{k=0}^{n} a_k \right]$$

whenever the limit exists. If so, the series is said to *converge*, and if not, the series is said to *diverge*.

Definition. Let $\{c_k\}$ be a sequence of real numbers, and let

$$D = \{x \in \mathbb{R} : \sum_{k=0}^{\infty} c_k x^k \text{ converges}\}.$$

Let $f : D \to \mathbb{R}$ defined by $f(x) = \sum_{k=0}^{\infty} c_k x^k$ be the *power series expansion* of $\{c_k\}$, and call D its *domain of convergence*.

In the interest of time, we will assume all given series converge on an appropriate domain (you did many such problems in your Calculus II course), as well as assume the following theorem without proof.

Theorem (9.41)**.** Let D be the domain of convergence of the power series expansion of $\{c_k\}$, and suppose $r > 0$ satisfies $(-r, r) \subseteq D$. Then $f : (-r, r) \to \mathbb{R}$ defined by $f(x) = \sum_{k=1}^{\infty} c_k x^k$ has derivatives of all orders satisfying

$$\frac{d^n}{dx^n}[f(x)] = \sum_{k=0}^{\infty} \frac{d^n}{dx^n}[c_k x^k].$$

Theorem.

$$\exp(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

Proof. □

0.3 Trigonometric Functions

Definition. Define the *cosine* function $\cos : \mathbb{R} \to \mathbb{R}$ by

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}.$$

Theorem (9.42)**.** The cosine function is a solution to the differential equation

$$\begin{cases} F''(x) = -F(x) & \text{for all } x \in \mathbb{R} \\ F(0) = 1 \\ F'(0) = 0 \end{cases}.$$

Proof. □

Lemma (5.5)**.** The unique solution to the differential equation

$$\begin{cases} z''(x) = -z(x) & \text{for all } x \in \mathbb{R} \\ z(0) = 0 \\ z'(0) = 0 \end{cases}$$

is the constant function valued at 0.

Proof. □

Corollary. $\cos(x)$ is the unique solution to the differential equation

$$\begin{cases} F''(x) = -F(x) & \text{for all } x \in \mathbb{R} \\ F(0) = 1 \\ F'(0) = 0 \end{cases}.$$

Proof. □

Definition. Let the *sine* function $\sin : \mathbb{R} \to \mathbb{R}$ be defined by $\sin(x) = -\frac{d}{dx}[\cos x]$.

Theorem. The sine function satisfies

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

and is the unique solution to the differential equation

$$\begin{cases} F''(x) = -F(x) & \text{for all } x \in \mathbb{R} \\ F(0) = 0 \\ F'(0) = 1 \end{cases}.$$

Proof. □

Theorem (5.6)**.** For all real numbers x, y:

(a) $[\sin x]^2 + [\cos x]^2 = 1$.

(b) $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.

(c) $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.

(d) $|\sin x| \leq 1$ and $|\cos x| \leq 1$.

Definition. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *periodic* with a *period* $T > 0$ if it satisfies

$$f(x+T) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

Theorem (5.7)**.** There exists a smallest $x > 0$ satisfying $\cos(x) = 0$.

Proof. □

Definition. Define the number $\pi \in \mathbb{R}$ such that $x = \pi/2$ is the smallest positive number satisfying $\cos(x) = 0$.

Theorem (5.8)**.** The sine and cosine functions are both periodic with period 2π.

Proof. □

Theorem (7.7, Integration by Substitution)**.** Let $f : [a,b] \to \mathbb{R}$ and $g : [c,d] \to \mathbb{R}$ be continuous, such that $g : (c,d) \to \mathbb{R}$ has a bounded continuous derivative and $g[(c,d)] \subseteq (a,b)$. Then

$$\int_c^d (f \circ g)g' = \int_{g(c)}^{g(d)} f.$$

Proof. □

Proposition (7.8, The Geometric Interpretation of π)**.**

$$\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}.$$

That is, π is the area of a circle with radius 1.

Proof. □

Midterm Part 3

Choose two of the below problems (which you did not choose for Part 2) and typeset your solutions. Delete the other three. Each will be worth 20/100 points towards your midterm grade for a total of 40/100 points.

Exercise (1)**.** Prove that if Q_n is a partition of $[a, b]$ refining the partition P_n of $[a, b]$ for each natural number n, and $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a, b]$, then $\{Q_n\}$ is also Archimedean.

Solution. □

Exercise (2)**.** Explain the error(s) in the following "proof", and then give a counterexample showing that the theorem is false.

Theorem: If $f : [0, 1] \to \mathbb{R}$ is integrable, then f is also continuous.

Proof: Since f is integrable, we may define $F : [0, 1] \to \mathbb{R}$ by $F(x) = \int_0^x f$. It follows that $F(x)$ is a differentiable function, because it is an antiderivative of f. Thus $\frac{d}{dx}[F(x)] = f(x)$ by the Second Fundamental Theorem of Calculus. Since the derivative of any differentiable function is continuous, we conclude f is continuous.

Solution. □

Exercise (3)**.** Recall that an **even** function satisfies the condition $f(x) = f(-x)$. Let $f : \mathbb{R} \to \mathbb{R}$ be an even continuous function. Prove that

$$\frac{d}{dx}\left[\int_{-x}^{x} f\right] = 2f(x).$$

(Hint: Corollary 6.30 says that $\frac{d}{dx}[\int_x^0 f] = -f(x)$.)

Solution. □

Exercise (4)**.** Prove the following theorem:

Let $\mathbf{x} \in \mathbb{R}^n$ and let $\{\mathbf{x}_k\}$ be a sequence of points in $\mathbb{R}^n$. If for every open set U containing $\mathbf{x}$, there is an index K such that $\mathbf{x}_k \in U$ for all $k \geq K$, then $\{\mathbf{x}_k\}$ converges to $\mathbf{x}$.

(Hint: $B_\epsilon(\mathbf{x})$ is open.)

Solution. □

Exercise (5)**.** Prove that any finite subset of $\mathbb{R}^n$ is closed.

(Hint: First prove that any singleton subset of $\mathbb{R}^n$ is closed.)

Solution. □

Final Exam Part 3

Delete any exercises you submitted for Part 2, and choose two others to delete as well. Typeset the solutions to the remaining exercises (so that you will have submitted six proofs in total for the final exam). Each will be worth 15/100 points towards your final exam grade.

Exercise (1)**.** Define $f : [0,2] \to \mathbb{R}$ by $f(x) = 3x$. Explicitly define a sequence of partitions $\{P_n\}$ of $[0,2]$, and then prove that this sequence is Archimedian for f on $[0,2]$.

Solution. $\square$

Exercise (2)**.** Compute the boundary of the subset $\mathbb{Q}^2$ of $\mathbb{R}^2$.

Solution. $\square$

Exercise (3)**.** Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^3+y^3} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is not continuous.

Solution. $\square$

Exercise (4)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(\mathbf{x}) = \begin{cases} \|\mathbf{x}\| & \|\mathbf{x}\| \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & \|\mathbf{x}\| \in \mathbb{Q} \end{cases}.$$

Prove that f is continuous at $\mathbf{0}$.

Solution. $\square$

Exercise (5)**.** Let $d : X^2 \to [0,\infty)$ be a metric on X. Prove that the function $e : X^2 \to [0,\infty)$ defined by $e(x,y) = \min(d(x,y),1)$ satisfies the triangle inequality for the below cases:

Solution. **Case** $e(x,y) + e(y,z) < 1$**:**

Case $e(x,y) + e(y,z) \geq 1$**:**

$\square$

Exercise (6)**.** Compute $f_y(0,0)$ where $f:\mathbb{R}^2\to\mathbb{R}$ is defined by

$$f(x,y)=\begin{cases}\frac{x^2y}{x^3+y^3} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}.$$

Solution. □

Exercise (7)**.** Let $f:\mathbb{R}^n\to\mathbb{R}$ be continuous such that $f(\mathbf{0})=0$. Prove that for each vector $\mathbf{p}\in\mathbb{R}^n$, there exists another vector $\mathbf{x}\in\mathbb{R}^n$ such that $\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x})=f(\mathbf{p})$.

Solution. □

Exercise (8)**.** Recall that $\ln:(0,\infty)\to\mathbb{R}$ is defined by $\ln(x)=\int_1^x\frac{1}{t}dt$. Show that $f:(-\infty,0)\to\mathbb{R}$ defined by $f(x)=\ln(-x)$ is the unique solution to the differential equation

$$\begin{cases}f'(x)=\frac{1}{x} & \forall x\in(-\infty,0)\\ f(-1)=0\end{cases}.$$

Solution. □