The Edge Unfolding of Generalized Pyramids

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# The Edge Unfolding of Generalized Pyramids

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A Thesis

Submitted to the

Auburn University Honors College

In Partial Fulfillment of the

Requirements for

University Honors Scholar

Auburn, Alabama

May 10, 2008

The Edge Unfolding of Generalized Pyramids

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## Thesis Abstract

## The Edge Unfolding of Generalized Pyramids

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Master of Science,
May 10, 2008

23 Typed Pages

Directed by Andras Bezdek

A long standing open problem in the field of discrete and computational geometry is the following: Can every convex polyhedron be cut along its edges and unfolded into a single, simple, nonoverlapping polygon? This paper studies a specific type of convex polyhedron, called a generalized pyramid, which is the convex hull of two parallel convex polygons such that the projection of one lays within the other, and two proposed cuttings for such generalized pyramids.

## Acknowledgments

The author would like to thank his research adviser, Dr. Andras Bezdek, for the guidance he provided in helping the author undergo his first research project. He also appreciates the support of the Auburn University Undergraduate Research Fellowship Program, and wishes to thank his Fellows in that program, and Dr. Leanne Lamke, the program adviser.

Most of all, he would like to thank his parents, Craig and Sherill, for their tremendous support throughout his college career.

Style manual or journal used Journal of Approximation Theory (together with the style known as "aums"). Bibliograpy follows van Leunen's *A Handbook for Scholars*.

Computer software used The document preparation package TEX (specifically LATEX) together with the departmental style-file `aums.sty`.

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# Chapter 1

## Introduction

A famous conjecture in combinatorial geometry is as follows:

**Conjecture 1.1.** *Every convex polyhedron's surface may be cut along its edges and unfolded flat into a single nonoverlapping simple polygon.*

This conjecture was perhaps first made in 1525 by the artist Albrecht Dürer. In his study of perspective in art, he investigated the unfoldings of various polyhedra [1]. However, this conjecture was first formally stated in a 1975 paper by Geoffrey C. Shephard [2]. A positive result has been returned for ”vertex unfoldings” of simplicial polyhedra (a polyhedron with only triangular faces), which allow for the resulting unfolded polyhedron to be a net of the polyhedron's faces without a continuous interior [4]. In addition, there exist nonconvex polyhedra, even nonconvex simplicial polyhedra, which cannot be cut along their edges and unfolded into a single nonoverlapping simple polygon [6]. However, the question about our more general conjecture for unfolding all convex polyhedra into a two-dimensional net with a continuous interior remains remains open. For a more detailed history on the problem, read [5].

Certainly, many unfoldings are familiar even to the layman. Perhaps the most basic would be that of a cube, whose unfolding is that of a Roman cross. For example, this unfolding is sometimes utilized in children's magazines and other print media in order to provide dice for a game. This ability to model three dimensional objects using flat surfaces such as paper or sheet metal gives us motivation into studying the unfoldings of various polyhedra.

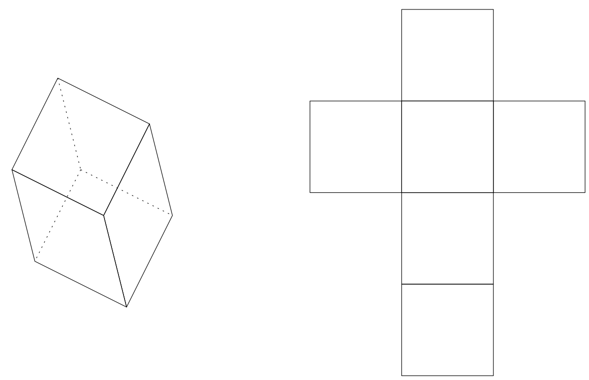

Figure 1.1: A cube and its unfolding.

## 1.1 Basic Definitions

Before we can discuss edge unfoldings, a bit of terminology should be clarified.

**Definition 1.1.** A *polygon* is the union of a finite number of line segments, or edges, in $\mathbb{R}^2$-space such that the intersection of any two segments is either empty or exactly a single endpoint of each edge, the entire union of edges is connected, and no point of the union is the subset of more than two edges. A polygon that has an edge endpoint which is a subset of only a single edge of the polygon is said to be *open*; otherwise, the polygon is said to be *closed*.

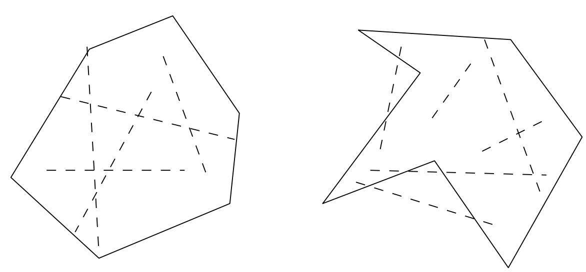

Figure 1.2: Two examples of polygons. The polygon on the left is convex; the one on the right is not.

In this paper, a polygon is assumed to be closed. Also, the term polygon also usually includes the space bounded by the edges of the polygon. The definition of a polyhedron is similar.

**Definition 1.2.** A *polyhedron* is the union of a finite number of polygonal faces in $\mathbb{R}^3$-space such that the intersection of any two faces is either empty or exactly a single edge of each polygon, the entire union of faces is connected, and no edge of the union is a subset of more than two faces. A polyhedron that has an edge which is subset of only a single face of the polyhedron is said to be *open*; otherwise, the polyhedron is said to be *closed*.[1]

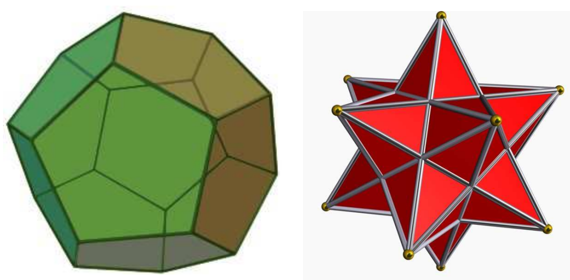

Figure 1.3: Two examples of polyhedra. The polyhedron on the left is convex; the one on the right is not.

In this paper, a polyhedron is assumed to be closed, unless otherwise stated. Also, a polyhedron is said to be *degenerate* if the union of some two faces of the polyhedron is a polygon. Polyhedra are assumed to not be degenerate in this paper, unless otherwise stated.

**Definition 1.3.** The space bounded by a polyhedron is said to be its *interior*.

[1] The left image in Figure 1.3 is licensed under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is available at `http://www.gnu.org/copyleft/fdl.html`. The right image in the same figure is courtesy Robert Webb's Great Stella software (`http://www.software3d.com/Stella.html`).

Sometimes the term polyhedron is used to describe the union of a polyhedron and its interior. In this case, the union of faces is denoted as the polyhedron's *surface*. Whether the term polyhedron includes its interior or not is left up to context.

**Definition 1.4.** A polygon is said to be *convex* if for any two points $x, y$ on the polygon, any point on the line segment connecting $x$ and $y$ is in the polygon. A polyhedron is said to be *convex* if for any two points $x, y$ on the polyhedron, any point on the line segment connecting $x$ and $y$ is either on the polyhedron's surface or its interior.

It is easy to see that the faces of a convex polyhedron are convex polygons.

**Definition 1.5.** The *angle defect* of a vertex $v$ in a polyhedron $P$ is equal to $2\pi$ minus the sum of the angles at the corners of the faces at the vertex $v$.

It is a well known fact that the angle defect of any vertex in a convex polyhedron is positive. If a vertex has an angle defect of zero, then that polyhedron is degenerate.

**Definition 1.6.** An *edge cutting* of a polyhedron $P$ is a collection $C$ of edges of $P$ such the removal of edges in $C$ from $P$ results in a connected surface $P - C$ that can be flattened into a plane without overlap.

The previously mentioned conjecture would guarantee that every convex polyhedron has an edge cutting. The resulting flattened polyhedron is known as an *edge unfolding* or sometimes a *polygonal net*. While it is possible to give more general definitions for *unfoldings* and *cuttings* that allow for more than just edges, in this paper those terms are assumed to refer to edge unfoldings and edge cuttings.

**Definition 1.7.** A collection of line segments is said to be *cyclic* if there is a subcollection of those segments $\{e_1,e_2,...,e_n = e_0\}$ such that the union of $e_i$ with $e_{i+1}$ is connected for all integers $i \in \mathbb{Z}_n$. Otherwise it is said to be *acyclic*.

**Lemma 1.1.** *Any edge cutting of a polyhedron $P$ is acyclic, and spans every nonboundary vertex of $P$ with a nonzero angle defect.*

*Proof.* This proof is from [7]. If the cutting contained a cycle, the resulting surface would not be connected. If some nonboundary vertex was not in the cutting, then the resulting surface could not be folded flat. $\square$

CHAPTER 2

GENERALIZED PYRAMIDS

While an answer to the general question alludes us, more progress can be made toward finding unfoldings for stricter classes of convex polyhedra.

## 2.1 Definition of a Generalized Pyramid

Certainly, finding an unfolding for a typical pyramid is a trivial matter. A valid cutting can by made by including a single lateral edge connecting to the top vertex to its base, and all but a single edge of the base. Taking this cue, we will extend the family of pyramids with the hope that we may adapt this method to unfold more general examples of convex polyhedra. We have denoted this extended family as *generalized pyramids*. From here on, we will often refer to pyramids as "standard" pyramids, to distinguish them from generalized pyramids.

Consider first the construction of a standard pyramid. A standard pyramid can be considered the convex hull of three or more nonlinear points in a plane with another point in three dimensional space, provided the perpendicular projection of the point lays within the convex hull of the planar points.[1]

In contrast, a generalized pyramid is the convex hull of two sets of points with the following restrictions:

1. The first set of points, which will a face we'll call the "pyramid base", must include three or more nonlinear points. The second set of points, which will form a face we'll call the "pyramid top", must contain at least one point.

[1]The image in Figure 2.1 is licensed under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is available at `http://www.gnu.org/copyleft/fdl.html`.)

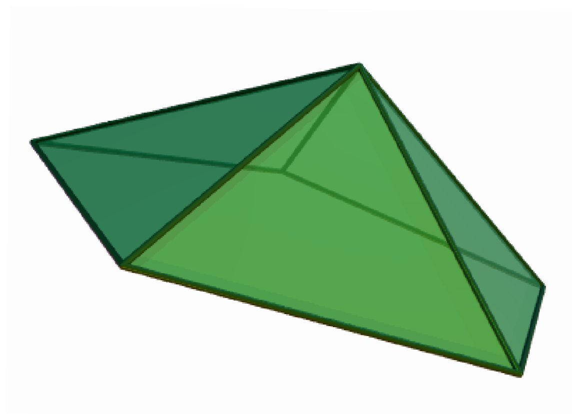

Figure 2.1: An example of a standard pyramid, the convex hull of a pentagon and a single point.

2. Any point in either set must lay in a plane containing every other point in that set.

3. The planes containing each set of points must be parallel.

4. The projection of the pyramid top must lay within the pyramid base.

As can easily be seen, a standard pyramid is simply the special case of a generalized pyramid with a single point in the set forming its pyramid top.

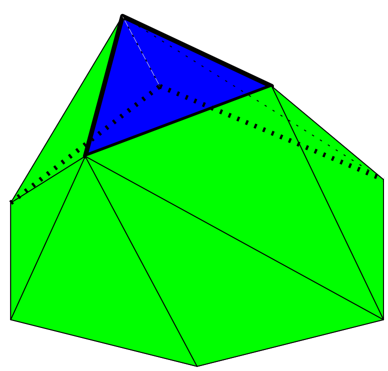

Figure 2.2: An example of a generalized pyramid, the convex hull of a hexagon and a triangle.

Certainly, two faces of the resulting generalized pyramid are evident. Each set of points forms either the pyramid top, which is some convex polygon, or the pyramid base, which is some nondegenerate polygon. The question remains how to draw the rest of the polyhedron. As a convex hull is the smallest polyhedron containing a set of points, it can be easily seen that every vertex of a generalized pyramid is a vertex of either the pyramid top or pyramid base. Thus, all that remains to be drawn is the system of edges connecting the top with the base.

### 2.2 Constructing a Generalized Pyramid

One way to find these edges is to rotate a plane around each edge of the pyramid base. When the rotated plane connects with a vertex or edge of the pyramid top, you draw edges between the ends of the base edge to that point (or to the ends of that edge). From this, we find that with the exception of the pyramid base and top, every face of a generalized pyramid is either a triangle or a trapezoid. From here on, I will refer to these faces as *lateral faces*, to distinguish them from the base and top of the pyramid. In addition, edges and vertices of the pyramid top will be called *top edges* and *vertices*, edges of the pyramid base will be called *base edges* and *vertices*, and all other edges will be called *lateral edges*.

An important observation is that the projection of a generalized pyramid into a plane parallel to its base provides all the information necessary to discuss its unfolding. This follows the fact that the pyramid top and base are parallel. The only information lost is the height between the top and base of the pyramid. Thus, often we will take advantage of this convenience and depict a "top-down" view of a generalized pyramid rather than an angled view.

In fact, one can recover the every lateral face of a generalized pyramid from the projection of the pyramid top and base into a place. This construction of a generalized pyramid from its two dimensional projection is simple; it follows directly from the plane-rotating

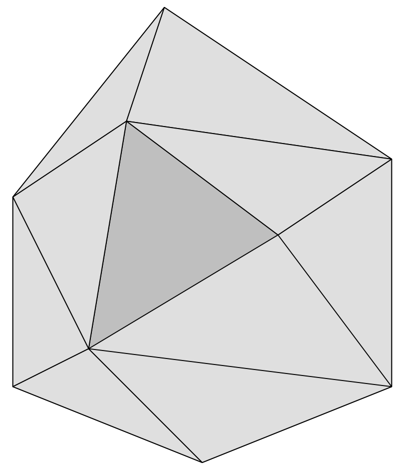

Figure 2.3: Top-down view of the generalized pyramid in Figure 2.2.

method used in three dimensions. For each edge of the pyramid base, find the vertex or edge of the pyramid top closest to it. The polygon containing that edge and point (or edges) is a lateral face of the generalized pyramid. Similarly, for each edge of the pyramid top, find the vertex or edge of the pyramid base farthest from it. The polygon containing that edge and point (or edges) is also a lateral face of the generalized pyramid.

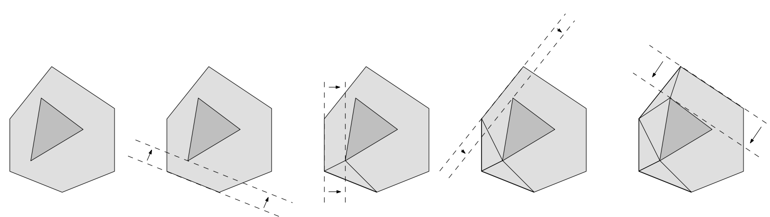

Figure 2.4: Constructing the generalized pyramid in Figure 2.2 from its two-dimensional projection.

## 2.3 Several Properties of a Generalized Pyramid

Following is an important property of the generalized pyramid for one of our unfoldings.

**Theorem 2.1.** *For a given generalized pyramid, there exists at least one lateral face containing a base edge that has no obtuse angle at that base edge.*

*Proof.* The proof of this proceeds by contradiction. Suppose there exists a generalized pyramid such that every lateral face F with a base edge has at least one obtuse angle at its base edge. There can only be one obtuse angle at each base vertex, otherwise the sum of the angles would exceed $\pi$ radians. This shows that no lateral face may have two obtuse angles at a base edge, since that would require the angle furthest clockwise of every other lateral face to be acute, but that would cause the base vertex furthest counterclockwise of F to have two obtuse angles, which defies our supposition.

So we conclude in this example that each lateral face can only have one obtuse angle at the base edge. Without loss of generality, we assume the angle furthest clockwise is acute, and the angle furthest counter-clockwise is obtuse. We label the lateral faces with an edge at the base, starting with an arbitrary face and moving counter-clockwise, as $F_0, ..., F_{n-1}$. We then denote the shortest distance between the pyramid top and the pyramid base at each face, that is, the altitudes of each face to be $d_0, ..., d_{n-1}$. We show that $d_{i+1} < d_i$. When $F_i$ and $F_{i+1}$ share an edge, we may draw them in two dimensions sharing this edge. Let $\beta_i$ be the obtuse angle of $F_i$ and $\alpha_{i+1}$ be the acute angle of $F_{i+1}$.

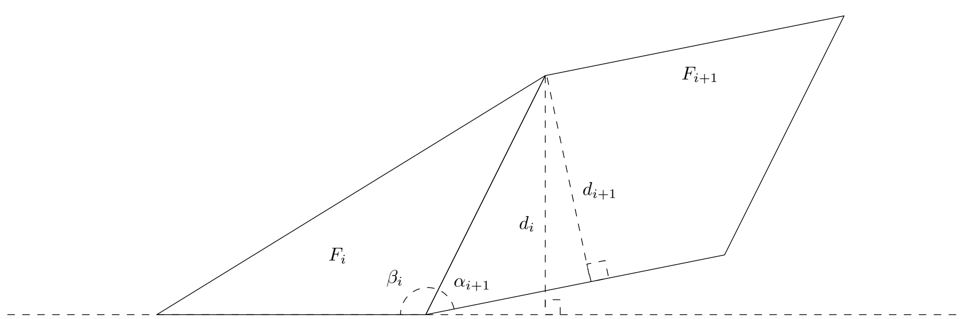


Figure 2.5: Representation of a pair of lateral faces that share an edge, such that both contain at least one obtuse angle at their respective base edges.

As can be seen from the image, $d_{i+1} < d_i$ must hold when the two lateral faces share an edge. In fact, if $c$ is the length of the shared edge, then $d_{i+1} = c \cdot \tan(\alpha_{i+1}) < c \cdot \tan(\pi - \beta_i) = d_i$.

In general, there may be some triangular lateral faces separating $F_i$ and $F_{i+1}$. This cannot affect our inequality, however. To see this, position $F_i$ on the Cartesian plane, so that the base edge runs through the x-axis. As the base is convex, it follows that a line running through the base edge of $F_{i+1}$ must have a positive slope. Certainly, the altitude of $F_{i+1}$, $d_{i+1}$, from its base edge must be less than the distance between the top vertex of $F_i$ and the base edge of $F_{i+1}$, which we will denote as $d^{\star}_{i+1}$ otherwise there would be a vertex closer to the base edge of $F_{i+1}$ than its current top vertex. In turn, $d^{\star}_{i+1}$ is less than the altitude of $F_i$, $d_i$, due to the positive slope of the base edge of $F_{i+1}$. Thus $d_{i+1} \leq d^{\star}_{i+1} < d_i$, and the inequality still holds.

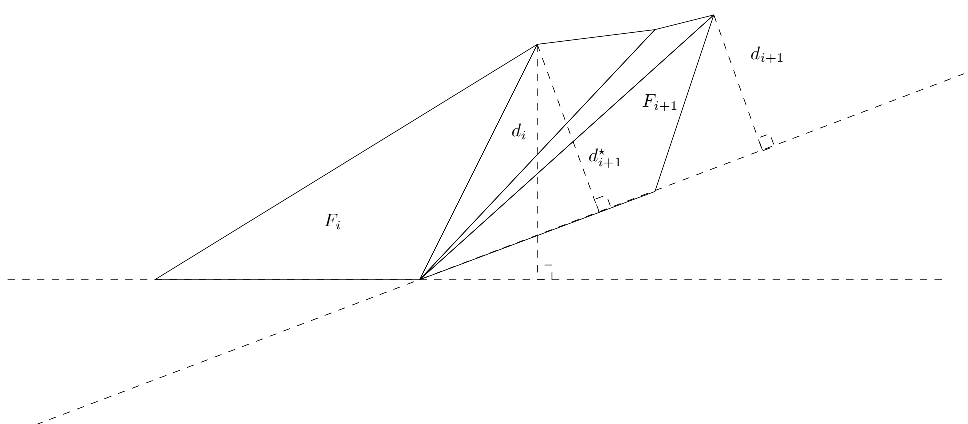


Figure 2.6: Representation of a pair of lateral faces such that both contain at least one obtuse angle at their respective base edges, that are separated by triangular lateral faces with no base edges.

Of course, this inequality results in a contradiction. If the altitudes of the lateral faces constantly shrunk as we move counter-clockwise around the generalized pyramid, we'd have

$d_0 < d_1 < ... < d_{n-1} < d_0$. Thus we have shown that there is at least once lateral face with two acute angles at its base edge. □

Since it is convenient to study the two-dimensional projection of a generalized pyramid, it is helpful to consider the difference between how faces appear in the projection, and the actual faces themselves. Certainly, if the height between the pyramid base and pyramid top is zero, the projected faces are exactly identical to the actual faces. But by increasing the distance between the top and base, the shape of the actual lateral faces warp from their counterparts in the projection.

**Theorem 2.2.** *Consider the projection of a generalized pyramid into a plane parallel to its base. Let $h$ be the height of the generalized pyramid.*

1. *The actual pyramid base and pyramid top are congruent to the projected base and top.*

2. *As $h$ increases, the actual angles of lateral faces that are at a top edge or base edge approach $\pi/2$.*

3. *As $h$ increases, the actual angles of lateral faces that are not at a top edge or base edge approach zero.*

4. *As $h$ increases, the angle defect at a vertex approaches the supplement of the angle of the pyramid top or pyramid base at that vertex.*

*Proof.* As $h$ increases, the altitude of each lateral face (from the base edge or top edge in that lateral face) increases. Let $h^*$ be the height of this altitude.

1. As the pyramid base and pyramid top lay in parallel planes, their shape is not affected in a projection into another parallel plane.

2. The distance between the angle and the base of the altitude of the face stays constant regardless of h. Let $d$ be this distance. Then, the measure of such an angle is equal to $\tan^{-1}(h^*/d)$, the limit of which as $h^*$ approaches infinity, is $\pi/2$.

3. This is a direct consequence of (2). Such an angle is the third angle of a triangle containing two angles which are at a top edge or base edge. As the sum of the angles of any triangle is $\pi$, it follows that as $h^*$ approaches infinity, this angle must approach 0.

4. The angle defect at a given vertex of a generalized pyramid is equal to $2\pi - (\sum \alpha_i + \beta_1 + \beta_2 + \gamma)$, where $\alpha_i$ is an angle of a lateral face not including a base edge or top edge at that vertex, $\beta_i$ is an angle of a lateral face including a base edge or top edge at that vertex, and $\gamma$ is the angle of the pyramid top or pyramid base at that vertex. Knowing that

   (a) $\lim_{h^* \to \infty} \sum \alpha_i = 0$

   (b) $\lim_{h^* \to \infty} \beta_i = \pi/2$

   (c) $\lim_{h^* \to \infty} \gamma = \gamma$

   we conclude that $\lim_{h^* \to \infty} 2\pi - (\sum \alpha_i + \beta_1 + \beta_2 + \gamma) = \pi - \gamma$.

□

**Lemma 2.1.** *All angles which appear acute in the projection of a generalized pyramid, are actually acute.*

*Proof.* Any angle in a generalized pyramid is either an angle of the pyramid top, pyramid base, or a lateral face. If it is of the pyramid top or base, it is precisely the same measure as its actual measure. Otherwise, it must be on a lateral face. If it's at a base or top edge, its measure approaches $\pi/2$ as the height increases, so it must remain acute for any height. Otherwise, its measure approaches 0 as the height increases, and it also remains acute. □

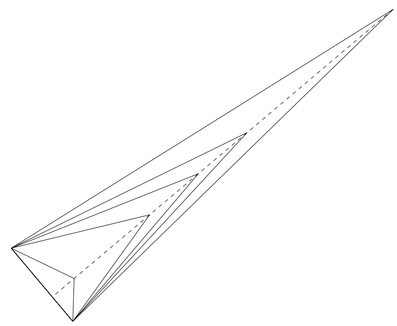

Figure 2.7: Representation of a triangle with increasing height. The angles at the base, which corresponds to a base edge or top edge of a generalized pyramid, approach $\pi/2$ radians, and the other angle approaches zero.

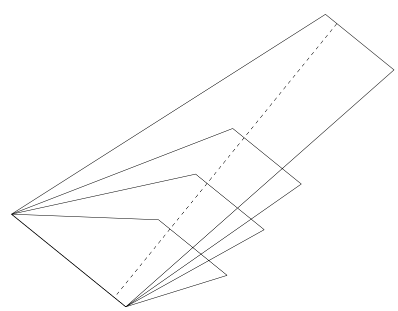

Figure 2.8: Representation of a trapezoid with increasing height. Each angle corresponds to an angle at a base edge or top edge, and approaches $\pi/2$ radians.

## Chapter 3

## Edge Unfoldings for Partial Generalized Pyramids

Here we will describe two different edge unfoldings, for partial generalized pyramids. The first unfolding is for a generalized pyramid without its pyramid base.

### 3.1 The Blossom Unfolding

The first unfolding is denoted a "blossom" unfolding after the shape of a flower. The pyramid top corresponds to the center of the bloom, and the lateral faces are the pedals.

**Theorem 3.1.** *Let $U$ be a collection of edges of a generalized pyramid $P$, where $U$ includes exactly one edge incident to each vertex of the pyramid top. The union of elements in $U$ is an edge cutting for the open polyhedron $P^*$ formed from the removal of the pyramid base from $P$.*

*Proof.* Consider each top vertex of the unfolded $P^*$. Certainly, there can be no local overlap of the faces neighboring each vertex.

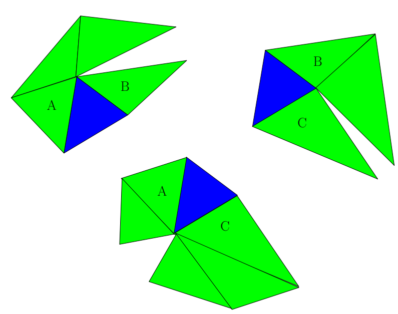


Figure 3.1: Local unfoldings for each top vertex of the generalized pyramid in Figure 2.3.

To show that the unfolding of $P^*$ has no self overlap, we paste corresponding sections of the local unfoldings over each other. The pyramid top and the lateral faces with top edges cannot self-overlap, and the faces between each lateral edge with a top edge cannot overlap with each other due to there being no overlap with the vertex-local unfoldings. It remains to be shown that a face between two lateral faces with top edges cannot overlap a face between two other lateral faces with top edges.

Observing that the sum of the angles of lateral faces at any base vertex is less then *pi*, we can partition the plane containing the unfolding into $n$ disjoint sectors, where $n$ is the number of top edges in the generalized pyramid, as seen in Figure 3.2. The faces between lateral faces with top edges must lie within these sectors, thus guaranteeing no overlap.

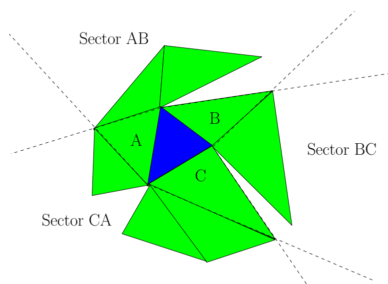


Figure 3.2: Blossom unfolding for the partial generalized pyramid in Figure 2.3.

□

This unfolding can be adapted to create an unfolding for a generalized pyramid with its top removed rather than its base. The proof is similar.

### 3.2 The Band Unfolding

The second unfolding is a different valid unfolding for a generalized pyramid without its pyramid top.

**Theorem 3.2.** *Let $U$ be a collection of edges of a generalized pyramid $P$, where $U$ includes exactly one lateral edge, and every base edge except for one. The union of elements in $U$*

*is an edge cutting for the open polyhedron $P^*$ formed from the removal of the pyramid top from $P$.*

To prove this, we need to show a property of the top edges in the resulting polygonal net.

**Theorem 3.3.** *The angle defect at a top vertex of a generalized pyramid is less than $\pi - \alpha$ where $\alpha$ is the angle of the pyramid top at that vertex.*

*Proof.* For a top vertex, take the angle bisector of the top face's angle at that vertex, in the projection, and draw an edge from the top vertex to a base edge along that bisector.

One may draw the projection of a partial generalized pyramid with the same pyramid top, and to lateral trapezoids sharing that edge. For any height, the angle defect of the original generalized pyramid at this vertex is less than or equal to the angle defect of a new generalized pyramid with this projection at this corresponding vertex. We now show that this new angle defect is less than $\pi - \alpha$.

The measure of the two lateral angles is $\pi - \frac{\alpha}{2}$. As $0 < \alpha < \pi$, we know $\frac{\pi}{2} < \pi - \frac{\alpha}{2} < \pi$. We draw two new lateral edges from the top vertex to the base edge, so that the outside lateral angles are right angles, and the middle two are acute. ($0 < \frac{\pi}{2} - \frac{\alpha}{2} < \frac{\pi}{2}$.)

As the height increases for a generalized pyramid corresponding for this projection, the top angle is constant and the outside lateral angles remain constant (as they are right angles), while the center lateral angles strictly decrease towards zero. If $\gamma$ is the measure of these central lateral angles, then the angle defect of this generalized pyramid is $2\pi - \alpha - 2(\frac{\pi}{2}) - 2\gamma = \pi - \alpha - 2\gamma < \pi - \alpha$. □

Figure 3.3 shows a portion of the proposed blossom unfolding. Note that with the removal of the pyramid top, the angle between the top edges is less than $\pi$.

*Proof. (for Theorem 3.2)* To show that lateral band unfolding has no overlap, consider the unfolded lateral faces. This results in a band of lateral faces which cannot self overlap if

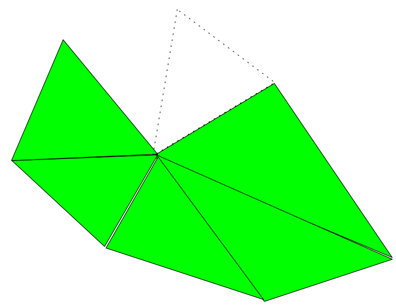

Figure 3.3: Portion of the band unfolding of the generalized pyramid in Figure 2.3.

their top edges do not self-overlap. However, we have just shown that the angle defect at each top vertex is less than $\pi - \alpha$, so the angle between each top edge is between $\alpha$ and $\pi$.

The motion from the original top polygon into the unfolded polygonal chain of top edges is strictly expansive as described in [3]. Increasing the angle between two edges is analogous to increasing the length of a strut between the two other vertices incident to the edges at that angle, provided that the angle is not increased to more than $\pi$ radians as we have shown. Thus, there can be no overlap in the resulting polygonal chain of top edges in the unfolding.

We note that the sum of lateral face angles at every base vertex is less than $\pi$ radians. Thus, a line passing through any base edge cannot intersect any other portion of the unfolded lateral faces. Thus, the pyramid base may be identified at any base edge of the lateral faces in completed the unfolded $P^*$, without overlap. $\square$

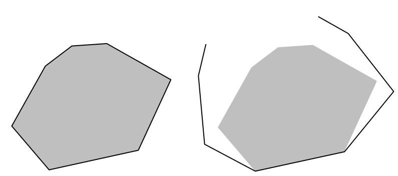

Figure 3.4: Unfolded top edges of some pyramid top.

Figure 3.4 illustrates the expansive motion of the unfolding of the top edges of a pyramid top. As the angle defect is less than $\pi - \alpha$ for each vertex, the resulting angle is always less than $\pi$, preventing an overlap. Note that nothing has been said to show that the original pyramid top can be positioned to avoid overlap with its unfolded edges.

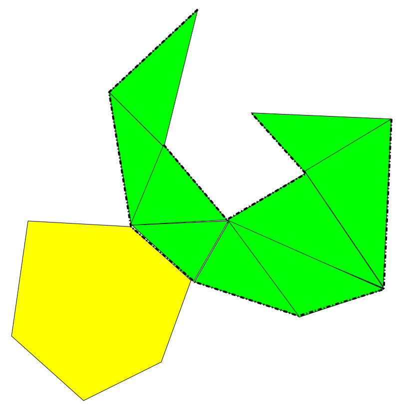

Figure 3.5: Band unfolding of the generalized pyramid in Figure 2.3.

Figure 3.5 depicts the band unfolding of a partial generalized pyramid. The accented edges represent the top and base edges of the lateral faces.

# Chapter 4

## Completing the Blossom and Band Unfoldings

Both the blossom and band unfoldings have been shown for partial generalized pyramids, excluding the base and top respectively. By attaching the missing face to the resulting unfolding, they can be extended for the whole pyramid.

### 4.1 Attaching the Pyramid Base to the Blossom Unfolding

A simple way to find a way to attach the pyramid base to the partial blossom unfolding would be to demonstrate a base edge with the property that a line passing through it does not intersect the rest of the partial unfolding. From Theorem 2.1 we've seen that at least one lateral face of any generalized pyramid has the property that it contains a base edge with no obtuse angle at that edge.

**Lemma 4.1.** *A line passing through a base edge of the blossom unfolding with two acute angles cannot intersect the lateral faces neighboring the lateral face containing that base edge.*

*Proof.* The sum of the angles of lateral faces at any vertex of a generalized pyramid is less than $\pi$, so if there is no cut made at either lateral edge, the line will not intersect the either neighboring face.

If a cut is made at either lateral edge, the neighboring face or faces will be rotated away from the base edge in the unfolding, so there still can be no overlap. □

While this argument isn't strong enough to guarantee that the line cannot intersect another lateral face, it does seem to suggest the following conjecture.

**Conjecture 4.1.** *A line passing through a base edge of the blossom unfolding with two acute angles cannot intersect any other part of the unfolding.*

Should this conjecture be true, then the base can be attached to the folding at this edge without overlap, giving us a completed blossom unfolding for any generalized pyramid. Its edge cutting would be the edge cutting described above unioned with every edge of the base besides this one.

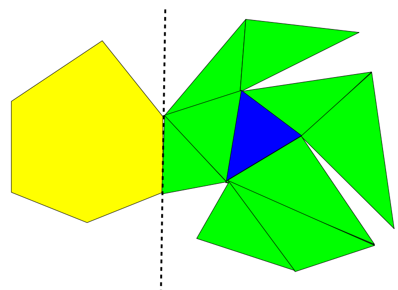

Figure 4.1: Proposed completed blossom unfolding.

### 4.2 Attaching the Pyramid Top to the Band Unfolding

The band unfolding suffers from a similar setback; it is not apparent how to reattach the pyramid top to this unfolding. In fact, one cannot simply attach it to any top edge, even for the simple example of the frustum of a regular tetrahedron. In this case, the pyramid top may only be identified to the middle top edge without causing an overlap. Similarly, the pyramid top of the frustum of a square pyramid must be attached to one of the middle two top edges to prevent an overlap.

This leads us to the following conjecture.

**Conjecture 4.2.** *If a generalized pyramid has $n$ top edges, the band unfolding may be completed by attaching the pyramid top to the $\frac{n}{2}^{th}$ or $(\frac{n}{2}+1)^{th}$ top edge if $n$ is even, or the $\frac{n+1}{2}^{th}$ top edge if $n$ is odd.*

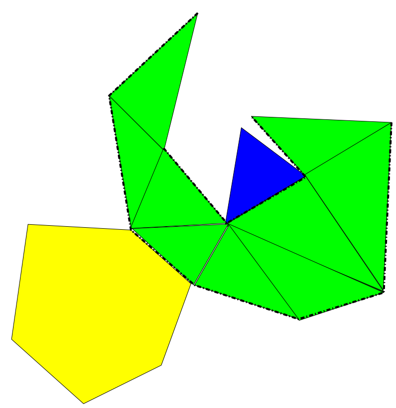

Figure 4.2: Proposed completed band unfolding.

# Bibliography

[1] A. Dürer. "The painter's manual: A manual of measurement of lines, areas, and solids by means of compass and ruler assembled by Albrecht Drer for the use of all lovers of art with appropriate illustrations arranged to be printed in the year MDXXV". New York: Abaris Books, 1977, 1525, English translation by Walter L. Strauss of "Unterweysung der Messung mit dem Zirkel un Richtscheyt in Linien Ebnen uhnd Gantzen Corporen".

[2] G. C. Shephard. Convex polytopes with convex nets. Math. Proc. Camb. Phil. Soc., 78:389-403, 1975.

[3] R. Connelly, E. D. Demaine, and G. Rote, Straightening Polygonal Arcs and Convexifying Polygonal Cycles, in Proceedings of the 41st Annual Symposium on Foundations of Computer Science (FOCS 2000), Redondo Beach, California, November 12-14, 2000, pages 432-442.

[4] E. D. Demaine, D. Eppstein, J. Erickson, G. W. Hart, and J. O'Rourke. Vertex-Unfoldings of Simplicial Manifolds. Technical Report 072, Smith College, October 2001.

[5] J. O'Rourke. "Folding and unfolding in computational geometry". Proc. 1998 Japan Conf. Discrete Comput. Geom., volume 1763 of Lecture Notes Comput. Sci., pages 258-266. Springer-Verlag, 2000.

[6] M. Bern, E. D. Demaine, D. Eppstein, and E. Kuo, Ununfoldable Polyhedra, Technical Report CS-99-04, Department of Computer Science, University of Waterloo, August 1999.

[7] M. Bern, E. D. Demaine, D. Eppstein, E. Kuo, A. Mantler, and J. Snoeyink, Ununfoldable Polyhedra with Convex Faces, Computational Geometry: Theory and Applications, volume 24, number 2, February 2003, pages 51-62. Special issue of selected papers from the 4th CGC Workshop on Computational Geometry, 1999.