Dynamic Time Warping

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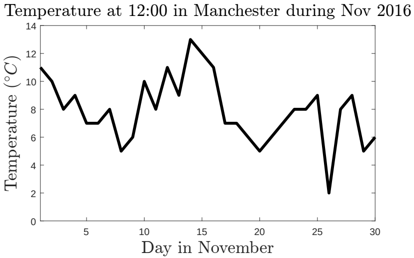

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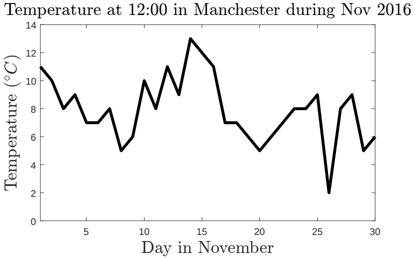

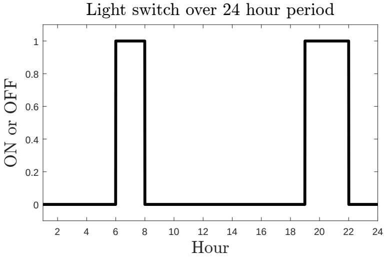

What is a distance measure?

Consider two time series $\mathbf{x} = \{x_i\}_{i=1}^{m}$ and $\mathbf{y} = \{y_i\}_{i=1}^{n}$. A distance/ similarity measure $d(\mathbf{x}, \mathbf{y})$ gives the 'distance/ similarity' between the time series $\mathbf{x}$ and $\mathbf{y}$. Some examples include:

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- Threshold based distance measures

Example data

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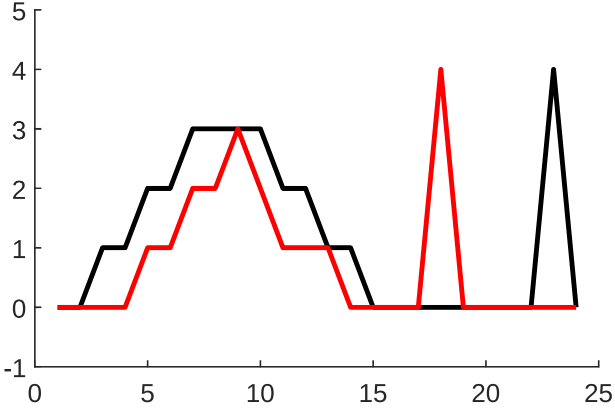

Euclidean distance

Defined by $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=0}^{n} d(x_i, y_i)^2}$ where $d(x_i, y_i) = |x_i - y_i|$ or $d(x_i, y_i) = \sqrt{(x_i - y_i)^2}$.

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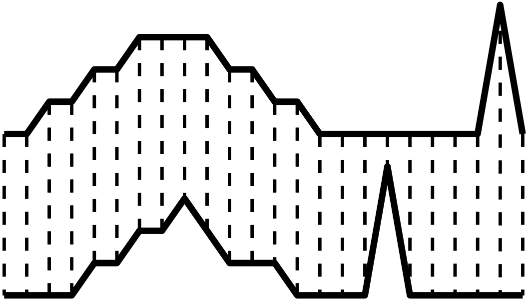

Dynamic Time Warping

First proposed by Berndt and Clifford (1994). Start by constructing $n \times m$ matrix D, where $D_{i,j} = d(x_i, y_j)$ where $d(x_i, y_i) = |x_i - y_i|$ or $d(x_i, y_i) = \sqrt{(x_i - y_i)^2}$.

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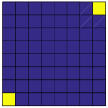
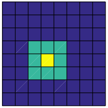
3. Monotonicity: if $w_i = (a, b)$ then $w_{i-1} = (a', b')$ where $a - a' \geq 0$ and $b - b' \geq 0$.

Cont.

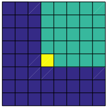
Boundary



Continuity

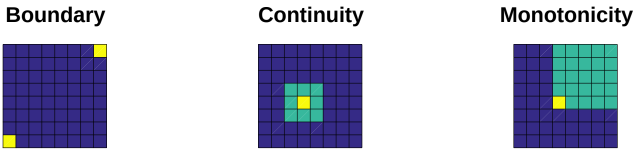


Monotonicity



We want $d_{DTW}(\mathbf{x}, \mathbf{y}) = \min \sum_{i=1}^{k} D(w_i)$.

Cont.



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In practise we use dynamic programming to prevent construction of the whole matrix. Define $\gamma(i, j)$ to be the cumulative distance then

$$\gamma(i, j) = d(x_i, y_j) + \min\{\gamma(i-1, j-1), \gamma(i-1, j), \gamma(i, j-1)\}.$$

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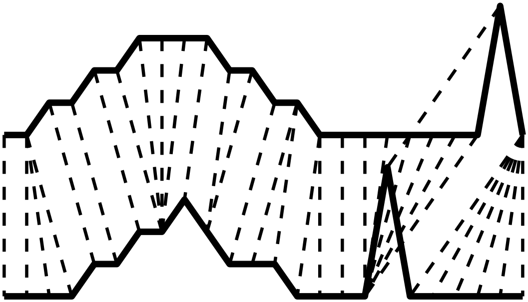
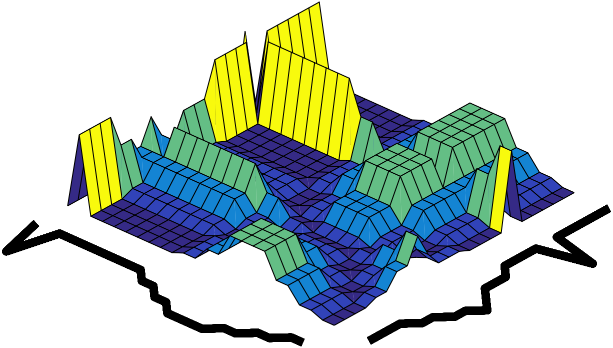


3D graphical view of Dynamic Time Warping



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- Classification - Given clusters $DB_1, \cdots, DB_k$, which cluster does timeseries $\mathbf{x}$ belong to?
- Query search - Given large time series $\mathbf{x}$ and small query time series $\mathbf{q}$ and distance d, find subsequences of $\mathbf{x}$ which are less than d away from $\mathbf{q}$.

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Global Constraints

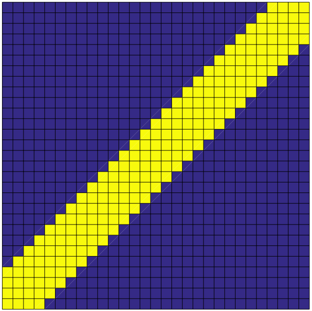

Figure: Sakoe-Chimba band

Global Constraints

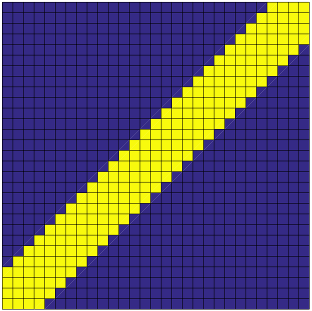

Figure: Sakoe-Chimba band

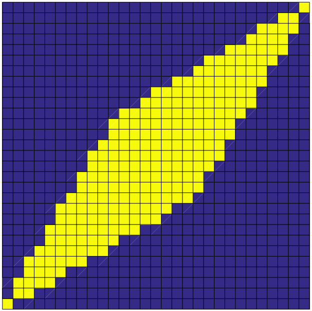

Figure: Itakura parallelogram

Lower Bounds

Many lower bounds have been developed. These are cheaper to compute but prone off unnecessary DTW distance measures. Two most popular are LB_{KIM} and LB_{KEOGH}.

LB_{kimFL} is the simplest and of $\mathcal{O}(1)$. It consists of finding the Eulidean distance of the first and last point in time series. LB_{kim} is of $\mathcal{O}(n)$ and consists of distance between maximum and minimum values of time series.

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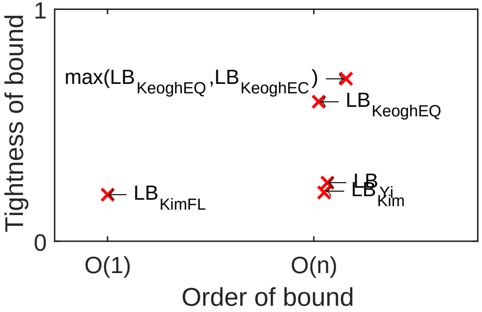

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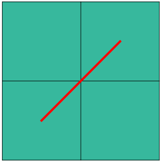

Figure: 1/4

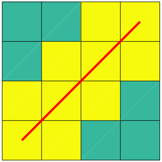

Figure: 1/2

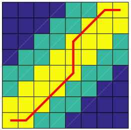

Figure: 1/1

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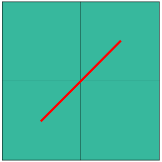

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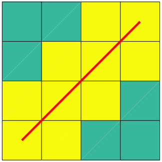

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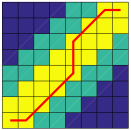
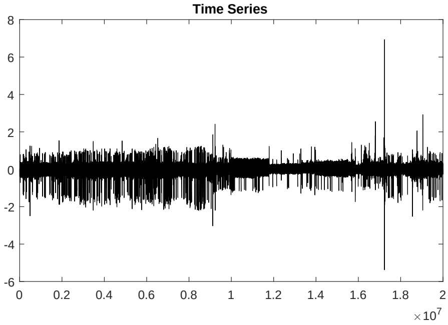


Figure: 1/1

Doesnt always find optimal solution.

Query search example

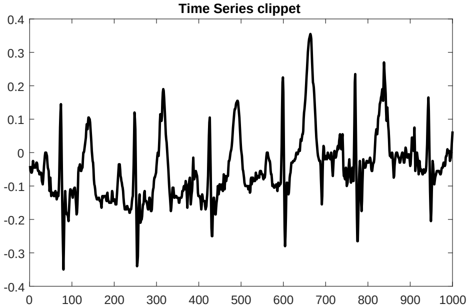
Consider the following time series.



This is an electrocardiogram of an 84-year old female. Recording is 22 hours and 23 minutes long containing 110,087 beats. End cropped for visual purposes.

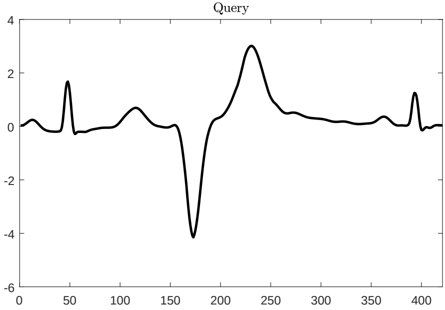
Time series: $\mathbf{x}$

250 samples per second giving a total length of 20,145,000. Here is a small 4s clippet (1000 data points).



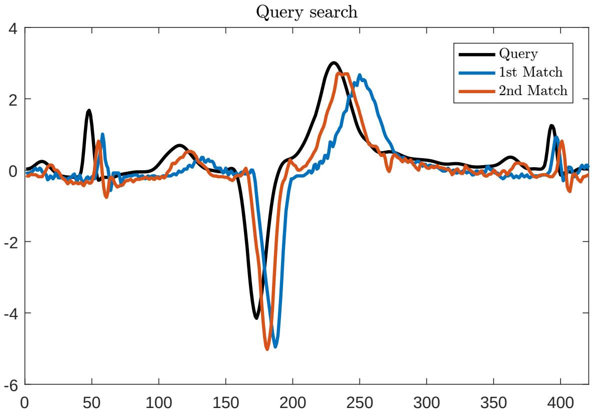
Query: $\mathbf{q}$

Premature venticular contraction. These are premature heartbeats originating from the ventricles of the heart, possibly due to high blood pressure or heart disease. Query containing 421 data points, about 1.7 seconds.



Results

Find the following two closest matches.



References

- DTW : Berndt and Clifford (1994)
- Sakoe-Chimba band: Sakoe and Chimba (1978)
- Itakura parallelogram: Itakura (1975)
- Bounds and UCR suite: Rakthanmanon et al. (2012)
- FastDTW: Salvador and Chan (2007)
- Last worked example:
  `http://www.cs.ucr.edu/$\sim$eamonn/UCRsuite.html`