Generating Homogeneous Poisson Processes

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We present an overview of existing methods to generate pseudorandom numbers from homogeneous Poisson processes. We provide three well-known definitions of the homogeneous Poisson process, present results that form the basis of various existing generation algorithms, and provide algorithm listings. With the intent of guiding users seeking an appropriate algorithm for a given setting, we note the computationally burdensome operations within each algorithm. Our treatment includes one-dimensional and two-dimensional homogeneous Poisson processes.

Key words: statistics; simulation; random process generation; Poisson processes.

Recall that a *counting process* $\{N_t, t \geq 0\}$ is a stochastic process defined on a sample space Ω such that for each $\omega \in \Omega$, the function $N_t(\omega)$ is a "realization" of the number of "events" happening in the interval $(0, t]$, with $N_0(\omega) = 0$. By this definition, $N_t(\omega)$ is automatically integer valued, non-decreasing, and right-continuous for each ω. A homogeneous Poisson process is a type of counting process that is characterized as follows.

Definition 1. *A counting process $\{N_t, t \geq 0\}$ is called a homogeneous Poisson process if:*

(i) $\forall t, s \geq 0$, *and* $0 \leq u \leq t$, $N_{t+s} - N_t$ *is independent of* N_u*;*

(ii) $\forall t, s \geq 0, Pr\{N_{t+s} - N_t \geq 2\} = o(s)$*; and*

(iii) $\forall t, s \geq 0, Pr\{N_{t+s} - N_t = 1\} = \lambda s + o(s)$*, where* $\lambda > 0$.

Definition 1 has been adopted from Ross [11, Chapter 2], with the notation $o(h)$ used in the usual sense — to denote a function $v(h)$ that satisfies $\lim_{h \to 0} v(h)/h = 0$ [14, pp. 1]. The constant $\lambda > 0$ appearing in Definition 1 is called the rate of the Poisson process. Stipulation (i) ensures that the resulting process has *independent increments*, i.e., the number of events happening in disjoint time intervals is independent. Likewise, the stipulations (ii) and (iii) ensure that the process has *stationary increments*, i.e., the distribution of the number of events happening in an interval depends only on the length of the interval. Definition 1 implies that the time between events are independent and identically distributed (i. i. d.) exponential random variables with mean $1/\lambda$ [11, pp. 64]. Moreover, the number of events in any fixed interval of time with length t has a Poisson distribution with mean λt. The latter fact motivates an alternative equivalent definition of a Poisson process.

Definition 2. *A counting process $\{N_t, t \geq 0\}$ is called a homogeneous Poisson process if:*

(i) $\forall t, s \geq 0$, *and* $0 \leq u \leq t$, $N_{t+s} - N_t$ *is independent of* N_u*; and*

(ii) $\forall t, s \geq 0, Pr\{N_{t+s} - N_t = j\} = \exp\{-\lambda s\} \frac{(\lambda s)^j}{j!}, \quad j = 0, 1, \ldots.$

That Definitions 1 and 2 are equivalent follows in a rather straightforward manner (e.g., see [11, pp. 61] for a proof).

A remarkable, completely qualitative definition of a homogeneous Poisson process is provided by Çinlar [2, pp. 71].

Definition 3. *A counting process $\{N_t, t \geq 0\}$ is called a homogeneous Poisson process if:*

(i) $\forall t \geq 0$ *and almost all* ω, *the mapping* $t \rightarrow N_t(\omega)$ *has jumps of unit magnitude only; i.e.,* $N_t(\omega) - \lim_{s \uparrow t} N_s(\omega) = 0$ *or* 1 $\forall t \geq 0$ *and for almost all* ω*;*

(ii) $\forall t, s \geq 0$, *and* $0 \leq u \leq t$, $N_{t+s} - N_t$ *is independent of* N_u*; and*

(iii) $\forall t, s \geq 0$, *the distribution of* $N_{t+s} - N_t$ *does not depend on* t.

(Çinlar [2, pp. 71] demonstrates that the above assumptions are sufficient to ensure that assumptions (ii) and (iii) of Definition 1 hold.)

In what follows, we discuss existing methods to generate pseudorandom numbers from a homogeneous Poisson process. By this we mean generating, on a digital computer, a realization of event times $\{t_i, i = 1, 2, \ldots\}$ from a homogeneous Poisson process that is specified through its rate λ. (For a more general definition of a Poisson process and its generation on a digital computer, see "Generating Nonhomogeneous Poisson Processes.")

1 Generation on the Nonnegative Real Line

Generating random variates from a homogeneous Poisson process is straightforward upon noting that the inter-event times in such a process are i. i. d. exponential random variables with mean $1/\lambda$. This is owing to the fact that exponential random variates can be generated with ease through the cdf-inverse method [1, 4]. Algorithm 1 thus obtains the required Poisson process realization by independently generating exponential random variates and then summing them.

Algorithm 1

(0) Initialize $t = 0$.

(1) Generate $u \sim U(0, 1)$.

(2) Set $t \leftarrow t - \frac{1}{\lambda}\log(u)$.

(3) Deliver t.

(4) Go to Step (1).

(Standard packages use more elaborate techniques that avoid the use of the "log" function in Step (2).) Another straightforward method to generate such realizations arises from a well known property of a homogeneous Poisson process. Conditioned on the number of events $N(t_0) = n$ that occur in the fixed interval $[0, t_0]$, the event times $T_1, T_2, \ldots, T_n$ of a homogeneous Poisson process are distributed as order statistics from a uniform distribution supported on $[0, t_0]$ [7]. (See "Generating Nonhomogeneous Poisson Processes" for a more general result and a proof.) This property gives rise to an algorithm to generate events from a homogeneous Poisson process confined to the fixed interval $[0, t_0]$. We list this as Algorithm 2.

Algorithm 2

(1) Generate $n \sim \text{Poisson}(\lambda t)$.

(2) If $n = 0$ then exit. Otherwise independently generate $u_1 \sim U(0,1), u_2 \sim U(0,1), \ldots, u_n \sim U(0,1)$, and sort (in increasing order) to obtain $u_{(1)}, u_{(2)}, \ldots, u_{(n)}$.

(3) Set $t_i \leftarrow t_0 u_{(i)}, i = 1, 2, \ldots, n$.

(4) Deliver $t_i, i = 1, 2, \ldots, n$.

In Step (1), the Poisson random variate n can be generated using any number of available methods, most notably table-based methods [3, 5, 9, 15] and cdf-inverse adaptations [6]. See [4] for a complete treatment. The sorting operation in Step (2) is $O(n \log n)$ and so the efficiency of Algorithm 2 depends on the magnitudes of t_0 and λ.

2 Generation on the Plane

A counting process $\{N(t), t \geq 0\}$ is said to constitute a two-dimensional homogeneous Poisson process on $C \subseteq \mathbb{R}^2$ with rate $\lambda \geq 0$ if:

(i) the number of events in a region $R \subseteq C$ is Poisson distributed with parameter $\Lambda(R) = \int_R \lambda \, dV = \lambda A(R)$, where $A(R)$ is the volume of the region R;

(ii) the number of events occurring in any finite set of nonoverlapping regions are mutually independent.

(For a more general definition of a Poisson process on a plane, and corresponding generation methods on a digital computer, see "Generating Nonhomogeneous Poisson Processes.")

Generation from a two-dimensional homogeneous Poisson process with rate λ on a circle of radius $r > 0$ turns out to be easy based on the following result by Lewis and Shedler [8]. (See also Rubinstein and Kroese [13, pp. 70].)

Theorem 1. *[Lewis and Shedler, 1979] Suppose $(R_1, \theta_1), (R_2, \theta_2), \ldots, (R_N, \theta_N)$ are the polar coordinates of $N > 0$ events from a homogeneous Poisson process on the circle $C \equiv \{(x, y) : x^2 + y^2 \leq r^2\}$. Then, conditional on $N = n$, the ordered event radii $R_{(1)}, R_{(2)}, \ldots, R_{(n)}$ are order statistics from the density $f(z) = 2z/r^2, z \in [0, r]$. Furthermore, $\theta_1, \theta_2, \ldots, \theta_n$ are i. i. d. uniform on $[0, 2\pi]$ and independent of $R_1, R_2, \ldots, R_n$.*

The resulting algorithm for generating events from a homogenous Poisson process on a circle of radius r thus involves first generating the number of events $N = n$ from a Poisson distribution with parameter $\pi r^2 \lambda$, then generating the radii $R_{(1)}, R_{(2)}, \ldots, R_{(n)}$ as order statistics from the density $f(z)$, and then finally generating the angles $\theta_1, \theta_2, \ldots, \theta_n$ uniformly (and independently) from $[0, 2\pi]$.

Algorithm 3

(0) Generate $n \sim \text{Poisson}(\pi r^2 \lambda)$. If $n = 0$ then exit. Otherwise generate $u_1 \sim U(0,1), u_2 \sim U(0,1), \ldots, u_n \sim U(0,1)$ independently.

(1) Set $R_1 \leftarrow r\sqrt{u_1}, R_2 \leftarrow r\sqrt{u_2}, \ldots, R_n \leftarrow r\sqrt{u_n}$.

(2) Sort $R_1, R_2, \ldots, R_n$ in ascending order to obtain $R_{(1)}, R_{(2)}, \ldots, R_{(n)}$.

(3) Generate $u_{n+1} \sim U(0,1), u_{n+2} \sim U(0,1), \ldots, u_{2n} \sim U(0,1)$ independently.

(4) Set $\theta_1 \leftarrow 2\pi u_{n+1}, \theta_2 \leftarrow 2\pi u_{n+2}, \ldots, \theta_n \leftarrow 2\pi u_{2n}$.

(5) Deliver $(R_{(1)}, \theta_1), (R_{(2)}, \theta_2), \ldots, (R_{(n)}, \theta_n)$.

(In the above algorithm, Step (2) is not required if there is no notion of ordering that is implicit in the abcissa and ordinate.)

Ross [12] gives a method to extend Algorithm 3 to the context of more irregular regions. For instance, suppose we want to generate events from a homogeneous Poisson process in the region bounded by the first quadrant, the positive valued function $f(x)$, and the line $x = T$, i.e., the region of interest $C \equiv \{(x, y) : x \geq 0, y \geq 0, y \leq f(x), x \leq T\}$. Ross [12] shows that the order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(N)}$ of the x-coordinates of the events are such that $\int_{X_{(i-1)}}^{X_{(i)}} f(x)\, dx, X_{(0)} = 0, i = 1, 2, \ldots,$ are i. i. d. exponential with mean $1/\lambda$. The corresponding ith y-coordinate Y_i is uniformly distributed between 0 and $f(X_{(i)})$. This gives rise to an algorithm that is straightforward, at least in principle.

Algorithm 4

(0) Initialize $j = 1$, $n = 0$, $w = 0$, $x_0 = 0$.

(1) Generate $u_j \sim U(0,1)$ independently.

(2) Set $w_j \leftarrow -(1/\lambda)\ln(1 - u_j)$.

(3) Set $w \leftarrow w + w_j$.

(4) If $w \leq \int_0^T f(x)\,dx$ then set $n \leftarrow n + 1$ and go to Step (1).

(5) If $n = 0$ then exit. Otherwise solve for $x_{(i)}, i = 1, 2, \ldots, n$ satisfying $\int_{x_{(i-1)}}^{x_{(i)}} f(x)\,dx = w_i$.

(6) Generate $u_{n+1} \sim U(0,1), u_{n+2} \sim U(0,1), \ldots, u_{2n} \sim U(0,1)$ independently.

(7) Set $y_i \leftarrow u_{n+i} f(x_i)$.

(8) Deliver $(x_{(i)}, y_i), i = 1, 2, \ldots, n$.

The efficiency of Algorithm 4 depends critically on the inversion in Step (5). Theorem 1 can be generalized even further to give a result that is the multidimensional analogue of the order statistics property discussed in Section 1. Specifically, consider a two-dimensional homogeneous Poisson process defined on an arbitrary region C. Then it is seen somewhat easily that the location (X, Y) of an event on the plane, conditional on the number of events $N = n$, is uniformly distributed over the region C [10, pp. 359]. This then gives rise to a very simple algorithm that is analogous to Algorithm 2, and generalizes Algorithm 3.

Algorithm 5

(1) Generate $n \sim \text{Poisson}(\lambda A(C))$.

(2) If $n = 0$ then exit. Otherwise independently generate n points $(x_i, y_i), i = 1, 2, \ldots, n$ that are uniformly distributed in C.

(3) Deliver $(x_i, y_i), i = 1, 2, \ldots, n$.

If the region C is irregular, Step 2 may not be straightforward. In such a case, a two-dimensional homogeneous process can be generated on any convenient region $\overline{C}$ (e.g., a rectangle) that covers C, and the events falling within C can be delivered as a realization of the required process.