MATH 132 Numerical Analysis II
Final Project

Biological Pattern:

$\frac{\partial u}{\partial t} = D\Delta u + f(u, v, x, y) \quad \forall x \in \Omega \qquad \Omega \in (0,1)\times(0,1)$

$\frac{\partial v}{\partial t} = D\Delta v + g(u, v, x, y). \quad \forall x \in \Omega$

u, v: Concentration $\qquad f, g$: Chemical Reaction between u & v.

D: Diffusivity

$u(x,y,0)$, $v(x,y,0)$: Initial Concentration Profile Known

Boundary Value Problem:

$u(x+1, y, t) = u(x, y, t) \qquad u(x, y+1, t) = u(x, y, t)$

$v(x+1, y, t) = v(x, y, t) \qquad v(x, y+1, t) = v(x, y, t)$

Laplacian

$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$

Semi-Implicit Scheme:

$$\frac{u^{n+1} - u^n}{\Delta t} = D\Delta u^{n+1} + F(u^n, v^n)$$

$$\frac{v^{n+1} - v^n}{\Delta t} = D\Delta v^{n+1} + G(u^n, v^n)$$

$\Delta x = \frac{1}{N-1}$ in both direction

(1) Justification: The diffusive term is treated implicitly because $\Delta u / \Delta v$ is laplacian which is non-linear and second order.

(2) For treating non-linear reactive term explicitly.

Advantage: Computationally Cheap. (No Root Find).

Disadvantage: Not Stable with big step sizes.

$\frac{V^{n+1}-V^n}{\Delta t}$ LTE derivation will be very similar

3) $\frac{u^{n+1}-u^n}{\Delta t} = DAu^{n+1} + F(u^n, v^n)$

$$LTE(h) = \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} - \left[D\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2}\right) + f(u,v,x,y)\right]$$

$$= \frac{u(x_i,y_i,t_{n+1}) - u(x_i,y_i,t_n)}{\Delta t} - D\left[\frac{u(x_{i+1},y_i,t_n) - 2u(x_i,y_i,t_n) + u(x_{i-1},y_i,t_n)}{\Delta x^2} + \frac{u(x_i,y_{i+1},t_n) - 2u(x_i,y_i,t_n) + u(x_i,y_{i-1},t_n)}{\Delta y^2}\right] - f(u,v,x,y)$$

$$= \frac{\cancel{u(x_i,y_i,t_n)} + \cancel{\Delta t}\frac{\partial u}{\partial t} + O(\Delta t^2) - \cancel{u(x_i,y_i,t_n)}}{\Delta t} - D\left[\frac{\cancel{u(x_i,y_i,t_n)} + \cancel{\Delta x\frac{\partial u}{\partial x}} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \cancel{\frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}} + O(\Delta x^4) - 2u(x_i,y_i,t_n) + \cancel{u(x_i,y_i,t_n)} - \cancel{\Delta x\frac{\partial u}{\partial x}} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} - \cancel{\frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}} + O(\Delta x^4)}{\Delta x^2}\right.$$

$$\left. + \frac{\cancel{u(x_i,y_i,t_n)} + \cancel{\Delta y\frac{\partial u}{\partial y}} + \frac{\Delta y^2}{2}\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\Delta y^3}{6}\frac{\partial^3 u}{\partial y^3}} + O(\Delta y^4) - 2u(x_i,y_i,t_n) + \cancel{u(x_i,y_i,t_n)} - \cancel{\Delta y\frac{\partial u}{\partial y}} + \frac{\Delta y^2}{2}\frac{\partial^2 u}{\partial y^2} - \cancel{\frac{\Delta y^3}{6}\frac{\partial^3 u}{\partial y^3}} + O(\Delta y^4)}{\Delta y^2}\right] - f(u,v,x,y)$$

$$= \frac{\partial u}{\partial t} + O(\Delta t) - D\left[\frac{1}{2}\frac{\partial^2 u}{\partial x^2} + O(\Delta x^2) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + O(\Delta x^2) + \frac{1}{2}\frac{\partial^2 u}{\partial y^2} + O(\Delta y^2) + \frac{1}{2}\frac{\partial^2 u}{\partial y^2} + O(\Delta y^2)\right] - f(u,v,x,y)$$

$$= \underbrace{\frac{\partial u}{\partial t}} + O(\Delta t) - D\left[\underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}} + \underbrace{2O(\Delta x^2) + 2O(\Delta y^2)}\right] - f(u,v,x,y)$$

$$= \cancel{DAu} + \cancel{f(u,v,x,y)} + O(\Delta t) - \cancel{DAu} + O(\Delta x^2) - \cancel{f(u,v,x,y)}$$

$$= O(\Delta t) + O(\Delta x^2) \longrightarrow \boxed{O(\Delta t + \Delta x^2)}$$

(4) Test Problem.

$$u - \Delta u = \sin^2(\pi x)\sin^2(\pi y) - 2\pi^2\left[\cos(2\pi x)\sin^2(\pi y) + \sin^2(\pi x)\cos(2\pi y)\right] \quad \forall x \in \Omega$$

with PBC.

(a) Linear System:

$$Mu = B \quad \rightarrow \quad I - \frac{1}{\Delta x^2}A = M \quad \rightarrow \quad \boxed{\left(I - \frac{1}{\Delta x^2}A\right)u = B}$$

(b) Verify $u_{true}(x,y) = \sin^2(\pi x)\sin^2(\pi y)$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} = 2\pi \sin(\pi x)\cos(\pi x)\sin^2(\pi y)$$

$$\frac{\partial u}{\partial y} = 2\pi \sin(\pi y)\cos(\pi y)\sin^2(\pi x)$$

$$\frac{\partial^2 u}{\partial x^2} = 2\pi^2 \sin^2(\pi y)\left(\cos^2(\pi x) - \sin^2(\pi x)\right)$$

$$\frac{\partial^2 u}{\partial y^2} = 2\pi^2 \sin^2(\pi x)\left(\cos^2(\pi y) - \sin^2(\pi y)\right)$$

$u_{true} - \Delta u_{true}$

$$= \sin^2(\pi x)\sin^2(\pi y) - \left[2\pi^2\sin^2(\pi y)\left(\cos^2(\pi x) - \sin^2(\pi x)\right) + 2\pi^2\sin^2(\pi x)\left(\cos^2(\pi y) - \sin^2(\pi y)\right)\right]$$

$$= \boxed{\sin^2(\pi x)\sin^2(\pi y) - 2\pi^2\left[\sin^2(\pi y)\cos(2\pi x) + \sin^2(\pi x)\cos(2\pi y)\right]}$$

(c)(d) MATLAB File, the matrix looks right in "Problem_4_d"

(e) MATLAB File.

	Error
N=5	0.3056
N=10	0.1114
N=15	0.0496

↑ N ↓ Err

The implementation looks correct

```
>> A

A =                                Laplacian: A (25 X 25)

   -64    16     0    16     0    16     0     0     0     0     0     0     0     0     0    16     0     0     0     0     0     0     0     0     0
    16   -64    16     0     0     0    16     0     0     0     0     0     0     0     0     0    16     0     0     0     0     0     0     0     0
     0    16   -64    16     0     0     0    16     0     0     0     0     0     0     0     0     0    16     0     0     0     0     0     0     0
     0     0    16   -64    16     0     0     0    16     0     0     0     0     0     0     0     0     0    16     0     0     0     0     0     0
     0    16     0    16   -64     0     0     0     0    16     0     0     0     0     0     0     0     0     0    16     0     0     0     0     0
    16     0     0     0     0   -64    16     0    16     0    16     0     0     0     0     0     0     0     0     0     0     0     0     0     0
     0    16     0     0     0    16   -64    16     0     0     0    16     0     0     0     0     0     0     0     0     0     0     0     0     0
     0     0    16     0     0     0    16   -64    16     0     0     0    16     0     0     0     0     0     0     0     0     0     0     0     0
     0     0     0    16     0     0     0    16   -64    16     0     0     0    16     0     0     0     0     0     0     0     0     0     0     0
     0     0     0     0    16     0    16     0    16   -64     0     0     0     0    16     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0    16     0     0     0     0   -64    16     0    16     0    16     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0    16     0     0     0    16   -64    16     0     0     0    16     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0    16     0     0     0    16   -64    16     0     0     0    16     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0    16     0     0     0    16   -64    16     0     0     0    16     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0    16     0    16     0    16   -64     0     0     0     0    16     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0    16     0     0     0     0   -64    16     0    16     0    16     0     0     0     0
     0     0     0     0     0     0     0     0     0     0     0    16     0     0     0    16   -64    16     0     0     0    16     0     0     0
     0     0     0     0     0     0     0     0     0     0     0     0    16     0     0     0    16   -64    16     0     0     0    16     0     0
     0     0     0     0     0     0     0     0     0     0     0     0     0    16     0     0     0    16   -64    16     0     0     0    16     0
     0     0     0     0     0     0     0     0     0     0     0     0     0     0    16     0    16     0    16   -64     0     0     0     0    16
     0     0     0     0     0    16     0     0     0     0     0     0     0     0     0    16     0     0     0     0   -64    16     0    16     0
     0     0     0     0     0     0    16     0     0     0     0     0     0     0     0     0    16     0     0     0    16   -64    16     0     0
     0     0     0     0     0     0     0    16     0     0     0     0     0     0     0     0     0    16     0     0     0    16   -64    16     0
     0     0     0     0     0     0     0     0    16     0     0     0     0     0     0     0     0     0    16     0     0     0    16   -64    16
     0     0     0     0     0     0     0     0     0    16     0     0     0     0     0     0     0     0     0    16     0    16     0    16   -64
```

$D = 10^{-5}$

(5) Reformulate: $Mu^{n+1} = R_u(u^n, v^n)$

$Mv^{n+1} = R_v(u^n, v^n)$

$$\frac{u^{n+1} - u^n}{\Delta t} = DAu^{n+1} + F(u^n, v^n)$$

$$u^{n+1} - u^n = \Delta t DAu^{n+1} + \Delta t F(u^n, v^n)$$

$$u^{n+1} - \Delta t DAu^{n+1} = u^n + \Delta t F(u^n, v^n)$$

$$u^{n+1}(I - \Delta t DA) = u^n + \Delta t F(u^n, v^n)$$

$$Mu^{n+1} = u^n + \Delta t F(u^n, v^n)$$

$$\boxed{\begin{array}{l} Mu^{n+1} = R_u(u^n, v^n) \\ \text{Explicit: } R_u(u^n, v^n) = u^n + \Delta t F(u^n, v^n),\ M = I - \Delta t DA \end{array}}$$

$$\frac{v^{n+1} - v^n}{\Delta t} = DAv^{n+1} + G(u^n, v^n)$$

$$v^{n+1} - v^n = \Delta t DAv^{n+1} + \Delta t G(u^n, v^n)$$

$$v^{n+1} - \Delta t DAv^{n+1} = v^n + \Delta t G(u^n, v^n)$$

$$v^{n+1}(I - \Delta t DA) = v^n + \Delta t G(u^n, v^n)$$

$$\boxed{\begin{array}{l} Mv^{n+1} = R_v(u^n, v^n) \\ \text{Explicit: } R_v(u^n, v^n) = v^n + \Delta t G(u^n, v^n),\ M = I - \Delta t DA \end{array}}$$

6) $f(u,v) = g(u,v) = 0$

a) If the problem is linear, we can analyze the convergency of the implicit scheme by using Lax's theorem

b) $R_u = u^n + \Delta t \cdot 0 = u^n$

$R_v = v^n + \Delta t \cdot 0 = v^n$

$$\boxed{\begin{cases} R_u = u^n \\ R_v = v^n \end{cases} \quad \leftarrow \text{Decoupled System}}$$

☆ c) Detailed Eigenvalue of M is in the "Problem_6C.png" In MATLAB File.

We proved that the eigenvalue of M greater than 1

$\forall\ |\lambda_i| > 1$

$$M u^{n+1} = R_u = u^n$$

$$(I - \Delta t A) u^{n+1} = u^n$$

$$(I - \Delta t A) u^n = u^0$$

$$u^n = [(I - \Delta t A)^{-1}]^n u^0$$

The method is stable if $\max|\mathrm{eig}(I - \Delta t A)^{-1}| \leq 1$

In MATLAB File.

It's found out that $\mathrm{Max}|\mathrm{eig}(I - \Delta t A)^{-1}| = 1$.

$\leftrightarrow$ Method is unconditionally stable for the linear problem

d) In MATLAB File

```
ei =
```

N = 5

```
   1.000080000000000
   1.000000000000000
   1.000060000000000
   1.000060000000000
   1.000040001253000
   1.000040001253000
   1.000039997494003
   1.000039999999999
   1.000039999999999
   1.000020000000000
   1.000020000000000
   1.000060000000000
   1.000060000000000
   1.000060000000001
   1.000059999999999
   1.000020000033286
   1.000019999966714
   1.000020000000000
   1.000020000000000
   1.000039999958774
   1.000040000041228
   1.000039999999999
   1.000040000000001
   1.000040000000000
   1.000040000000000
```

(e) In MATLAB file,

☆ The eigenvalue details is in "Problem_6e_#.png"

(7) Non-linear Problem

$$\frac{du}{dt} = f(u,v) \rightarrow \frac{u^{n+1} - u^n}{\Delta t} = F(u^n, v^n)$$

$$\frac{dv}{dt} = g(u,v) \rightarrow \frac{v^{n+1} - v^n}{\Delta t} = G(u^n, v^n)$$

$$u = \tilde{u} + \varepsilon, \quad v = \tilde{v} + \varepsilon$$

$$f(u,v) = f(\tilde{u},\tilde{v}) + \varepsilon \frac{\partial f}{\partial u}(\tilde{u},\tilde{v}) + \varepsilon \frac{\partial f}{\partial v}(\tilde{u},\tilde{v}) + O(\varepsilon^2)$$

$$g(u,v) = g(\tilde{u},\tilde{v}) + \varepsilon \frac{\partial g}{\partial u}(\tilde{u},\tilde{v}) + \varepsilon \frac{\partial g}{\partial v}(\tilde{u},\tilde{v}) + O(\varepsilon^2)$$

$$\boxed{\begin{array}{l}\text{The non-linear problem is stable } \forall\, |\lambda| \geq 1 \\ \text{where } \lambda = \max\left(\left|\frac{\partial f}{\partial u}(\tilde{u},\tilde{v})\right|, \left|\frac{\partial f}{\partial v}(\tilde{u},\tilde{v})\right|, \left|\frac{\partial g}{\partial u}(\tilde{u},\tilde{v})\right|, \left|\frac{\partial g}{\partial v}(\tilde{u},\tilde{v})\right| \right)\end{array}}$$

(8) In MATLAB file.

(9) $f(u,v) = -uv^2 + F(1-u)$

$g(u,v) = uv^2 - (F+k)v$

(a) $F = 0.09$, $k = 0.059$

☆ Uniform Initial Condition: "Problem_9a_1.png"

☆ Random Initial Condition: "Problem_9a_2.png"

☆ Localized Initial Condition: "Problem_9a_3.png"

```
1.001171572875254
1.002585786437627
1.002585786437627
1.004000000000000
1.004000000000000
1.004000000000000
1.005414213562373
1.005414213562374
1.006828427124746
```

N = 3
D = 10e-3

```
1.000000000000000
1.000400000000002
1.000600000000001
1.000600000000001
1.000500000000000
1.000500000000000
1.000199999999999
1.000200000000000
1.000299999999998
1.000300000000000
1.000300000000000
1.000600000000000
1.000600000000000
1.000499999999999
1.000500000000000
1.000300000000000
```

N = 4
D = 10e-4

N = 6
D = 10e-6

```
1.000007236067979
1.000000000000002
1.000007236067980
1.000004000000000
1.000001999999997
1.000002000000000
1.000007236067981
1.000007236067978
1.000005618033993
1.000005618033989
1.000001381966011
1.000001381966013
1.000001381966010
1.000001381966012
1.000005618033988
1.000005618033991
1.000002763932019
1.000002763932024
1.000002763932025
1.000002763932022
1.000003618033990
1.000003381966009
1.000003381966012
1.000003381966010
1.000003381966011
1.000003618033988
1.000003618033990
1.000003618033989
1.000005000000003
1.000005000000002
1.000005000000001
1.000005000000001
1.000005000000000
1.000005000000000
1.000004999999998
1.000004999999999
```

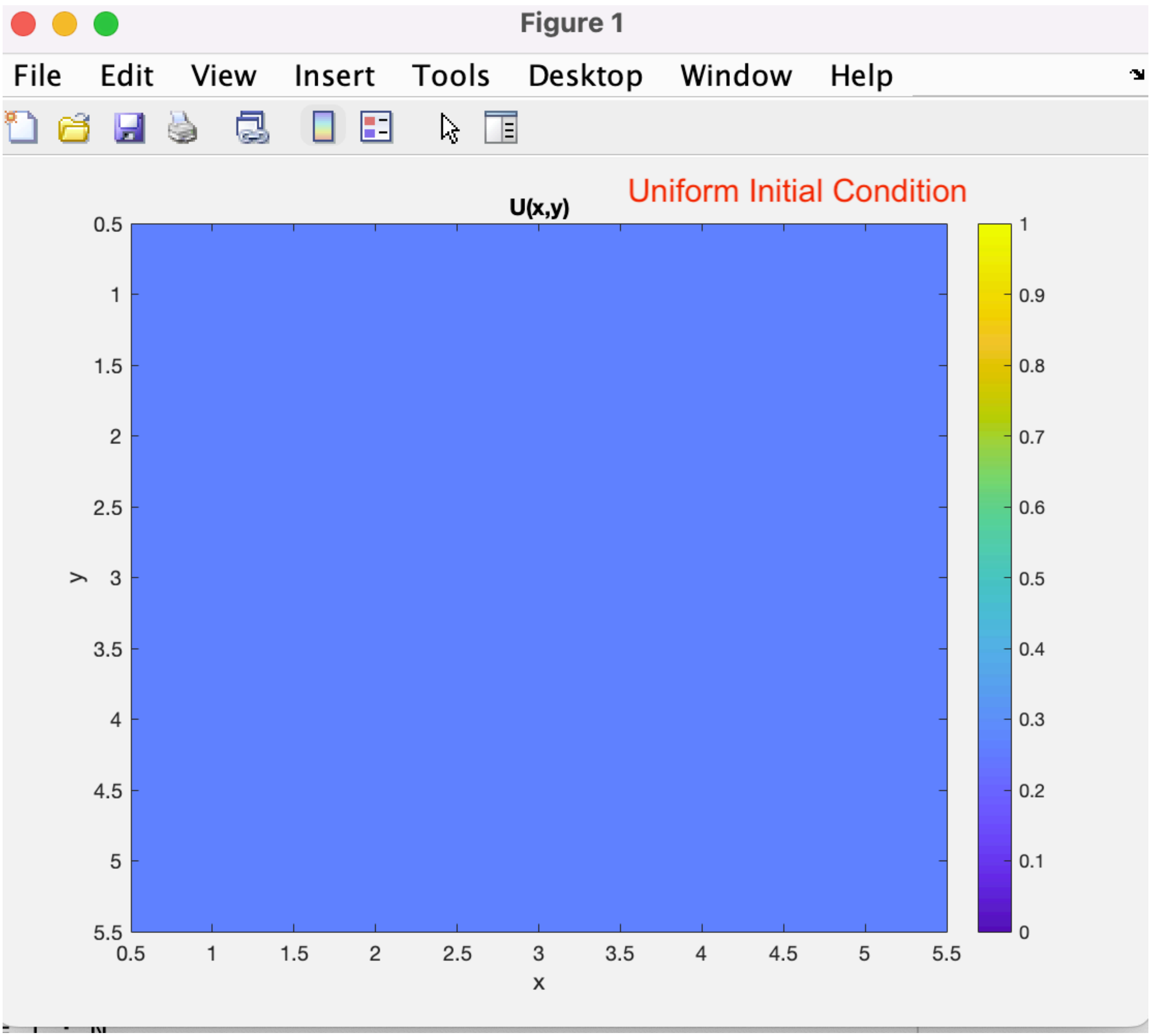
Figure 1
File Edit View Insert Tools Desktop Window Help
Uniform Initial Condition
U(x,y)
y
x
0.5 1 1.5 2 2.5 3 3.5 4 4.5 5 5.5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

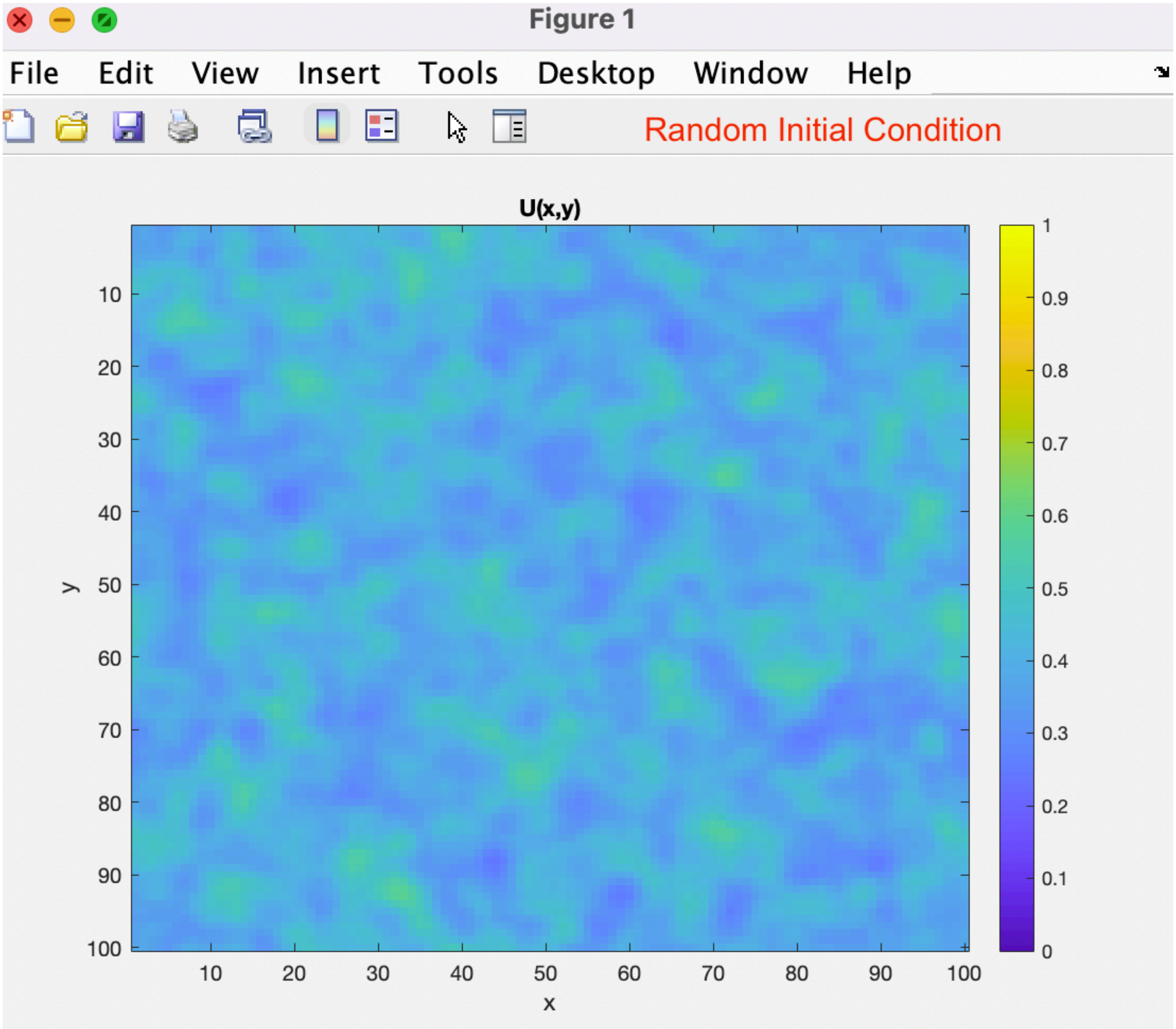
Figure 1
File Edit View Insert Tools Desktop Window Help
Random Initial Condition
U(x,y)
y
x
10 20 30 40 50 60 70 80 90 100
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

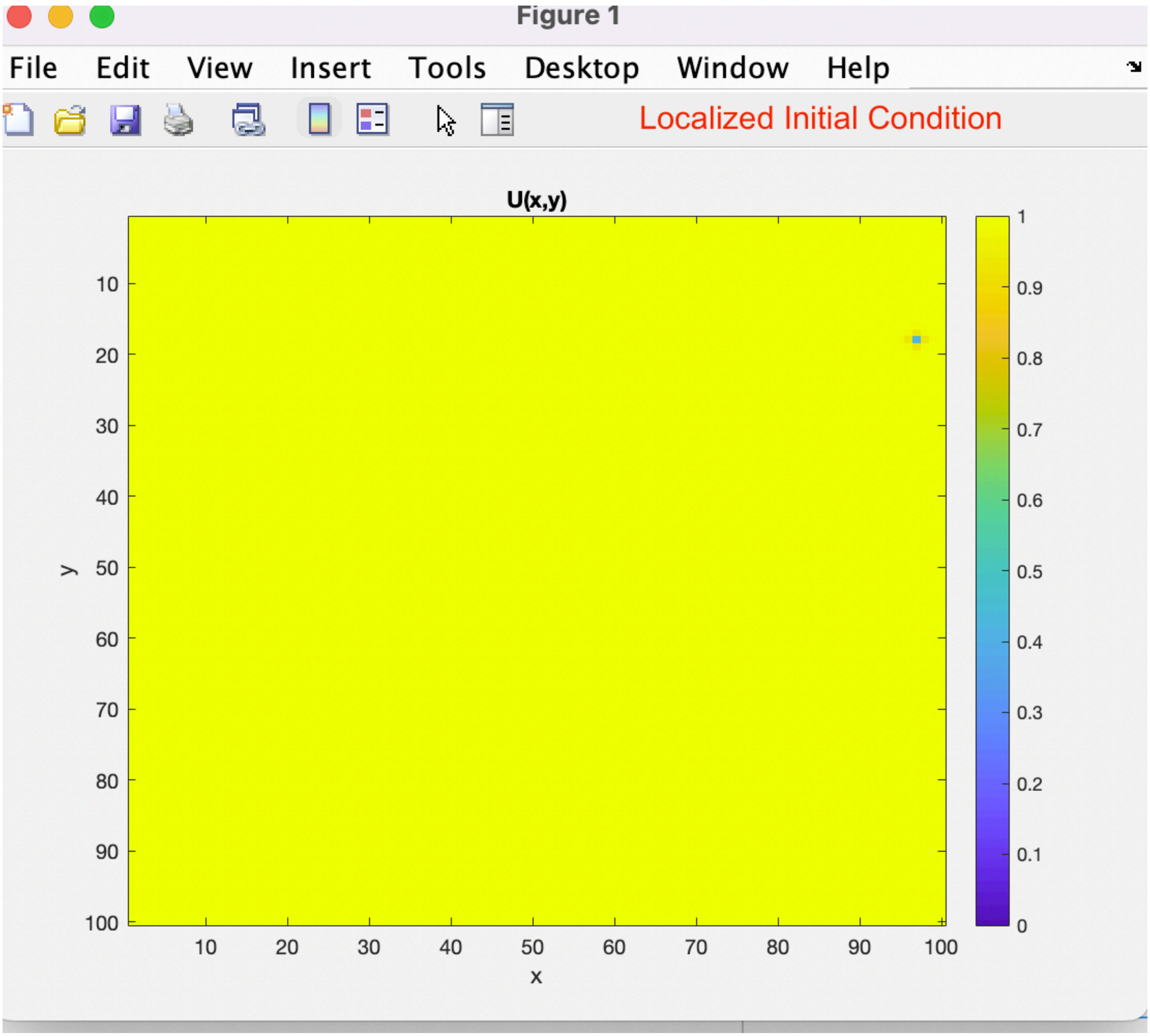
Figure 1
File Edit View Insert Tools Desktop Window Help
Localized Initial Condition
U(x,y)
y
x
10 20 30 40 50 60 70 80 90 100
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

b) $F = 0.029$, $k = 0.059$ → Plot In MATLAB

Resource : experiments.withgoogle.com/gray-scott-simulation

c) $F = 0.046$, $k = 0.022$ → Plot In MATLAB

Correction: A video is submitted

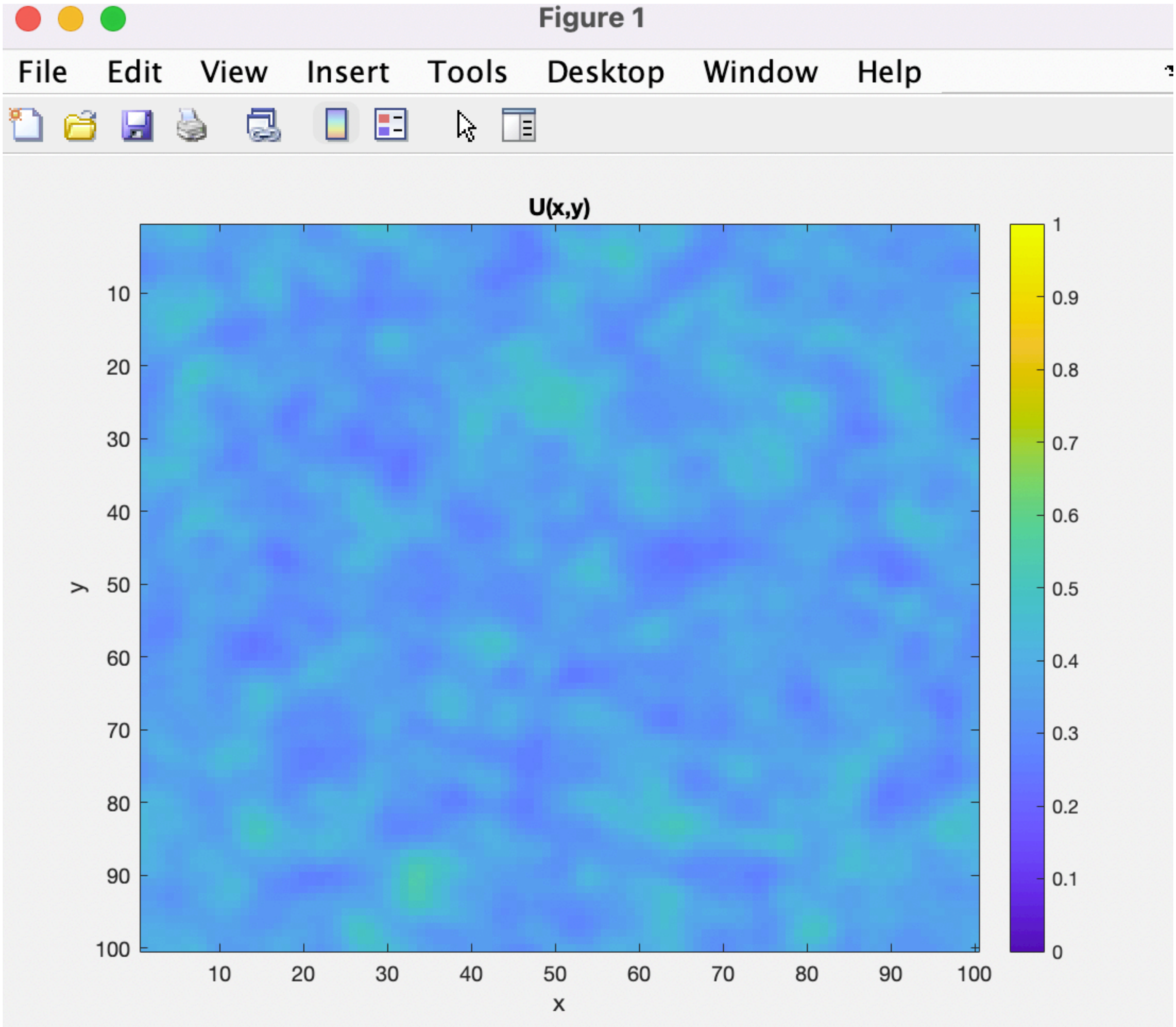
Figure 1
File
Edit
View
Insert
Tools
Desktop
Window
Help
U(x,y)
10
20
30
40
50
60
70
80
90
100
y
x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0