

MATHEMATICS



Study Notes

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January 2, 2021

Abstract

This document pertains to mathematical concepts which I have studied. It includes my notes, work, and solutions when dealing with various problems. The source code for this document can be found on my *GitHub* at any time.

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1 Definitions

1.1 Binomial Theorems

Definition 1

$$(a + b)^2 = a^2 + 2ab + b^2 \quad (1)$$

$$(a - b)^2 = a^2 - 2ab + b^2 \quad (2)$$

$$(a + b)(a - b) = a^2 - b^2 \quad (3)$$

For higher exponentiations:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \quad (4)$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 \quad (5)$$

$$(-a - b)^3 = -a^3 - 3a^2b - 3ab^2 - b^3 \quad (6)$$

Binomial Formula:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \quad (7)$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad (8)$$

1.2 Fractions

Definition 2

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \quad (9)$$

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b} \quad (10)$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \quad (11)$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd} \quad (12)$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad-bc}{bd} \quad (13)$$

Inverse:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \quad (14)$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c} \quad (15)$$

$$\frac{a \cdot \frac{b}{c}}{\frac{d}{e}} = \frac{\frac{a \cdot b}{c}}{\frac{d}{e}} = \frac{a \cdot b}{c} \cdot \frac{e}{d} = \frac{(a \cdot b) \cdot e}{c \cdot d} \quad (16)$$

$$\frac{ab}{c} = \frac{a}{c} \cdot b \quad (17)$$

$$\frac{a}{b} = \frac{1}{b} \cdot a \quad (18)$$

$$\frac{a \div b}{c} = \frac{a}{c} \div b \quad (19)$$

1.3 Parentheses Rules

Definition 3

$$+(a + b) = a + b \quad (20)$$

$$+(-a - b) = -a - b \quad (21)$$

$$-(a - b) = -a + b \quad (22)$$

$$-(-a + b) = +a - b \quad (23)$$

$$-(a + b) = -a - b \quad (24)$$

Associative properties:

$$(a + b) + c = a + (b + c) \quad (25)$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad (26)$$

Distributive properties:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad (27)$$

$$(a + b) \cdot c = (a \cdot c) + (b \cdot c) \quad (28)$$

Commutative properties:

$$a + b = b + a \quad (29)$$

$$a \cdot b = b \cdot a \quad (30)$$

1.4 Multiply with -1

Definition 4 *Mathematical operators may be swapped by multiplying with -1, because the result does not change.*

$$a + b = c \leftrightarrow -1 \cdot (-a - b) = c \quad (31)$$

Example:

$$(a - b)^2 = (b - a)^2 \quad (32)$$

1.5 Square Roots

Definition 5

$$\sqrt[n]{a} = a \quad (33)$$

$$\sqrt[n]{a} = \sqrt{a} \quad (34)$$

$$\sqrt{a^2} = a \quad (35)$$

$$(\sqrt{a})^2 = a \quad (36)$$

$$\frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}} = \frac{1}{n} \quad (37)$$

$$\sqrt{n} \cdot \sqrt{n} = n \quad (38)$$

Addition:

$$a \sqrt[n]{x} + b \sqrt[n]{x} = (a + b) \sqrt[n]{x} \quad (39)$$

Subtraction:

$$a \sqrt[n]{x} - b \sqrt[n]{x} = (a - b) \sqrt[n]{x} \quad (40)$$

Multiplication:

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b} \quad (41)$$

Division:

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad (42)$$

Root exponentiation:

$$\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m} \quad (43)$$

Root extraction:

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[m \cdot n]{a} \quad (44)$$

Transforming roots into exponents:

$$\sqrt[n]{a} = a \cdot \frac{1}{n} \quad (45)$$

$$\sqrt{a} = a \cdot \frac{1}{2} \quad (46)$$

$$\sqrt[n]{a^m} = a^{\frac{m}{n}} \quad (47)$$

1.6 Exponentiation

Definition 6

$$x^n \cdot x^b = x^{n+b} \quad (48)$$

$$x^n \div x^b = \frac{x^n}{x^b} = x^{n-b} \quad (49)$$

$$\left(x^a\right)^b = x^{a \cdot b} \quad (50)$$

$$a^n \cdot a^b = (a \cdot b)^n \quad (51)$$

$$a^n \div b^n = \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n \quad (52)$$

$$x^0 = 1 \quad (53)$$

$$x^1 = x \quad (54)$$

$$x^{-n} = \frac{1}{x^n} \quad (55)$$

$$\frac{1}{x} = x^{-1} \quad (56)$$

$$x^{\frac{1}{n}} = \sqrt[n]{x} \quad (57)$$

Disclaimer (for 56): If n is even, then x must be > 0 !

$$x^{\frac{m}{n}} = \sqrt[n]{m} \quad (58)$$

$$x^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{x^m}} \quad (59)$$

Addition:

$$ax^n + bx^n = (a + b)x^n \quad (60)$$

Subtraction:

$$ax^n - bx^n = (a - b)x^n \quad (61)$$

Transform a single root into a exponent:

$$\sqrt{a} = (a)^{\frac{1}{2}} \cdots \sqrt[3]{a} = (a)^{\frac{1}{3}} \cdots \quad (62)$$

2 Calculus

2.1 Simplifying Algebraic Terms

Exercise 1:

$$\begin{aligned} & [12x + 5x \cdot 2 - (10x - 8x)] + 18x \div 3 \\ \Rightarrow & [12x + 5x \cdot 2 - 10x + 8x] + 18x \div 3 && \left. \begin{array}{l} \text{Apply parentheses rule (22)} \\ \text{Subtract } -10x + 8 \end{array} \right\} \\ \Rightarrow & [12x + 5x \cdot 2 - 2x] + 18x \div 3 && \left. \begin{array}{l} \text{Multiply } 5x \cdot 2 \end{array} \right\} \\ \Rightarrow & 20x + 18x \div 3 \\ \Rightarrow & 20x + 6x \\ \Rightarrow & 26x \end{aligned}$$

Exercise 2:

$$\begin{aligned} & x - ((x - 4) - (14 + 2x)) + 1 \\ \Leftrightarrow & x - (x - 4 - 14 - 2x) + 1 && \left. \begin{array}{l} \text{Apply parentheses rule (24)} \\ (22) \end{array} \right\} \\ \Leftrightarrow & x - x + 4 + 14 + 2x + 1 \\ \Leftrightarrow & 19 + 2x \\ \Leftrightarrow & 2x + 19 && \left. \begin{array}{l} (29) \end{array} \right\} \end{aligned}$$

2.2 Square Root Equations

Exercise 1:

$$\begin{array}{lcl}
 2x + \sqrt{x^2 + 9} = 4x + 3 & \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} & \begin{array}{l} -2x \\ \\ ()^2 \\ \text{Apply first binomial rule (1)} \\ \text{Summarize} \\ -x^2 - 9 \end{array} \\
 \sqrt{x^2 + 9} = 2x + 3 & & \\
 x^2 + 9 = (2x + 3)^2 & & \\
 x^2 + 9 = 4x^2 + 2x \cdot 2 \cdot 3 + 9 & & \\
 x^2 + 9 = 4x^2 + 12x + 9 & & \\
 0 = 3x^2 + 12x & & \\
 0 = \underbrace{x}_{x_1} \underbrace{(3x + 12)}_{x_2} & & \\
 \Rightarrow x_1 = 0 \quad | \quad \text{Verify for } x_1: \quad 2 \cdot 0 + \sqrt{0^2 + 9} = 3 & &
 \end{array}$$

$$\Rightarrow x_2 = 3x + 12 = 0 \quad \Leftrightarrow \quad x_2 = -4 \quad | \quad \text{Verify for } x_2: \quad 4 \cdot (-4) + 3 = -13$$

$$\Rightarrow 3 = -13$$

Q.E.D

Exercise 2:

$$\begin{array}{lcl}
 \sqrt{x+2} + \sqrt{x-1} = 3 & \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} & \begin{array}{l} ()^2 \\ \\ \text{Apply first binomial rule (1)} \\ \text{Square, and apply rule (35)} \\ x + 2 + x - 1 \\ -2x - 1 \\ \div 2 \\ \text{Apply third binomial rule (3)} \\ ()^2 \\ \text{No binomial rule applies for } (-x + 4)^2 \\ -x^2 - 1x + 2 \\ +9x \\ \div 9 \end{array} \\
 (\sqrt{x+2} + \sqrt{x-1})^2 = 3^2 & & \\
 \sqrt{x+2}^2 + 2 \cdot \sqrt{x+2} \cdot \sqrt{x-1} + \sqrt{x-1}^2 = 9 & & \\
 x + 2 + 2\sqrt{(x+2) \cdot (x-1)} + x - 1 = 9 & & \\
 2x + 1 + 2\sqrt{(x+2) \cdot (x-1)} = 9 & & \\
 2\sqrt{(x+2) \cdot (x-1)} = -2x + 8 & & \\
 \sqrt{(x+2) \cdot (x-1)} = -x + 4 & & \\
 \sqrt{x^2 - 1x + 2x - 2} = -x + 4 & & \\
 x^2 - 1x + 2x - 2 = (-x + 4)^2 & & \\
 x^2 + 1x - 2 = x^2 - 2 \cdot 4x + 16 & & \\
 0 = -9x + 18 & & \\
 9x = 18 & & \\
 x = 2 & &
 \end{array}$$

Exercise 3:

$$\begin{aligned}
 \sqrt{-3x-1-\sqrt{4x+5}} &= 1 & ()^2 \\
 -3x-1-\sqrt{4x+5} &= 1 & +3x+1 \\
 -\sqrt{4x+5} &= 3x+2 & ()^2 \\
 4x+5 &= (3x+2)^2 & \text{Apply first binomial rule (1)} \\
 4x+5 &= 9x^2+2\cdot 3x\cdot 2+4 \\
 4x+5 &= 9x^2+12x+4 & -4x-5 \\
 0 &= 9x^2+8x-1 & \div 9 \\
 0 &= x^2+\frac{8}{9}\cdot x-\frac{1}{9} & \text{Apply PQ formula} \\
 x_{1,2} &= -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} \\
 x_1 &\Leftrightarrow \sqrt{-\frac{11}{3}} = 1 \\
 x_2 &\Leftrightarrow 1 = 1 \\
 \Rightarrow L &= \{-1\}
 \end{aligned}$$

Exercise 4:

$$\begin{aligned}
 x &= \sqrt{x+20} & x^2 \\
 x^2 &= x+20 & -x-20 \\
 \Leftrightarrow x^2-x-20 &= 0 \\
 \Leftrightarrow (x-5)(x+4) &= 0 \\
 \Rightarrow x=5 \quad \wedge x=-4 \\
 \Rightarrow \sqrt{5+20}=5 \quad | \quad \sqrt{-4+20}=\sqrt{16}=4 \\
 \Rightarrow L &= \{5\}
 \end{aligned}$$

2.3 Changing the Subject of an Equation

Exercise 1:

$$\begin{aligned}
 & \frac{6x+7}{9} - \frac{10x+7}{18} = \frac{9x+5}{14} - \frac{9x-16}{20} && \text{Find lowest common denominator.} \\
 \Leftrightarrow & \frac{6x+7}{2 \cdot 9} - \frac{10x+7}{18} = \frac{9x+5}{14 \cdot 10} - \frac{9x-16}{20 \cdot 7} \\
 \Leftrightarrow & \frac{12+14}{18} - \frac{10x+7}{18} = \frac{90x+50}{140} - \frac{63x-112}{140} \\
 \Leftrightarrow & \frac{12x+14-(10x+7)}{18} = \frac{90x+50-(63x-112)}{140} \\
 \Leftrightarrow & \frac{12x+14-10x-7}{18} = \frac{90x+50-63x+112}{140} \\
 & \Leftrightarrow \frac{2x+7}{18} = \frac{27x+162}{140} && \cdot 18 \\
 & \Leftrightarrow 2x+7 = \frac{(27x+162) \cdot 18}{140} && \text{Reduce fraction with 2.} \\
 & \Leftrightarrow 2x+7 = \frac{(27x+162)9}{70} && \cdot 70 \\
 \Leftrightarrow & 140x+490 = 9(27x+162) \\
 \Leftrightarrow & 140x+490 = 243x+1458 && -140x \\
 \Leftrightarrow & 490 = 103x+1458 && -1458 \\
 \Leftrightarrow & -968 = 103x && \text{Apply (29)} \\
 \Leftrightarrow & 103x = -968 && \div 103 \\
 \Leftrightarrow & x = -\frac{968}{103}
 \end{aligned}$$

Exercise 2:

$$\begin{aligned}
 \frac{3x-9}{6x-1} &= \frac{4x-16}{8x-5} && \cdot (8x-5) \cdot (6x-1) \\
 \Leftrightarrow (3x-9) \cdot (8x-5) &= (4x-16) \cdot (6x-1) \\
 \Leftrightarrow 24x^2 - 15x - 72x + 45 &= 24x^2 - 4x - 96x + 16 && -24x^2 \\
 \Leftrightarrow -15x - 72x + 45 &= -4x - 96x + 16 && -45 \\
 \Leftrightarrow -15x - 72x &= -4x - 96x - 29 \\
 \Leftrightarrow -87x &= -100x - 29 && +100x \\
 \Leftrightarrow 13x &= -29 && \div 13 \\
 \Leftrightarrow x &= -\frac{29}{13}
 \end{aligned}$$

$$\begin{aligned}
 x^2 + (x-2)^2 &= 10 \\
 \Leftrightarrow x^2 + (x-2)(x-2) &= 10 \\
 \Leftrightarrow x^2 + x^2 - 2x - 2x + 4 &= 10 \\
 \Leftrightarrow 2x^2 - 4x + 4 &= 10 && -10 \\
 \Leftrightarrow 2x^2 - 4x - 6 &= 0 && \div 2 \\
 \Leftrightarrow x^2 - 2x - 3 &= 0 && \text{Apply PQ formula.} \\
 \Leftrightarrow x_1 = 3 \quad | \quad x_2 = -1 \\
 \Rightarrow L &= \{3\}
 \end{aligned}$$

2.4 Exponential Equations

Exercise 1:

$$\begin{aligned}
 & \frac{(2^{-4})^{-5} \cdot 2^{17}}{(2^{-3})^{-6} \cdot (2^{-4})^3} && \text{Apply (50).} \\
 &= \frac{2^{(-4) \cdot (-5)} \cdot 2^{17}}{2^{(-3) \cdot (-6)} \cdot 2^{(-4) \cdot 3}} \\
 &= \frac{2^{20} \cdot 2^{17}}{2^{18} \cdot 2^{-12}} && \text{Apply (48) and (49)} \\
 &= 2^{20+17-18-(-12)} \\
 &= 20^{20+17-18+12} \\
 &= 20^{31}
 \end{aligned}$$

Exercise 2:

$$\begin{aligned}
 & \frac{3^7 \cdot (3^{-2})^3}{3^{-4} \cdot 3^7} \div \frac{(3^4)^{-3}}{(3^{-2})^{-6}} && \text{Apply rule (50)} \\
 &= \frac{3^7 \cdot 3^{(-2) \cdot 3}}{3^{-4} \cdot 3^7} \div \frac{3^{4 \cdot (-3)}}{3^{(-2) \cdot (-6)}} && \text{Apply rule (14) and (48)} \\
 &= \frac{3^7 \cdot 3^{-6}}{3^{-4+7}} \cdot \frac{3^{12}}{3^{-12}} \\
 &= \frac{3^7 \cdot 3^{-6} \cdot 3^{12}}{3^3 \cdot 3^{-12}} && \text{Apply (48) and (49)} \\
 &= 3^{7-6+12-3-(-12)} \\
 &= 3^{22}
 \end{aligned}$$

Exercise 3:

$$\begin{aligned}
 & \frac{12x^{-2}y^3}{8z^2} \cdot \frac{4y^{-2}z}{3x^{-5}} \div \frac{6z^{-3}}{2y^{-4}z} && \text{Apply rule (14).} \\
 &= \frac{12x^{-2}y^3}{8z^2} \cdot \frac{4y^{-2}z}{3x^{-5}} \cdot \frac{2y^{-4}z}{6z^{-3}} && \text{Reduce } \frac{12 \cdot 4 \cdot 2}{8 \cdot 3 \cdot 6} \text{ and apply (48) and (49).} \\
 &= \frac{2}{3} x^{-2-(-5)} \cdot y^{3-2-4} \cdot z^{1+1-2-(-3)} \\
 &= \frac{2}{3} x^3 y^{-3} z^3 && \text{Apply rule (55).} \\
 &= \frac{2}{3} \frac{x^3 z^3}{y^3}
 \end{aligned}$$

Exercise 4:

$$\begin{aligned}
& \frac{2^4 \cdot x^5 \cdot y^7 \cdot z^8}{8 \cdot x^2 \cdot y^5 \cdot z^{10}} \div \frac{2 \cdot x^2 \cdot y^5 \cdot z^8}{16 \cdot x^4 \cdot y^3 \cdot z^5} && \text{Apply rule (14).} \\
& = \frac{2^4 \cdot x^5 \cdot y^7 \cdot z^8}{8 \cdot x^2 \cdot y^5 \cdot z^{10}} \cdot \frac{16 \cdot x^4 \cdot y^3 \cdot z^5}{2 \cdot x^2 \cdot y^5 \cdot z^8} && \text{Reduce } \frac{2}{8} \cdot \frac{16}{2} \text{ and apply (48).} \\
& = \frac{2}{1} \cdot \frac{x^{5+4} \cdot y^{7+3} \cdot z^{8+5}}{x^{2+2} \cdot y^{5+5} \cdot z^{10+8}} \\
& = \frac{2}{1} \cdot \frac{x^9 \cdot y^{10} \cdot z^{13}}{x^4 \cdot y^{10} \cdot z^{18}} && \text{Apply rule (49).} \\
& = 2 \cdot x^5 \cdot z^{-5} && \text{Apply rule (55).} \\
& = \frac{2 \cdot x^5}{z^5}
\end{aligned}$$

Exercise 5:

$$\begin{aligned}
& \frac{4x^{2-m}y^{3-m}}{7z^{m-n}} \div \frac{5z^{m+n}x^{3-m}}{14y^{1-2m}} && \text{Apply rule (14).} \\
& = \frac{4x^{2-m}y^{3-m}}{7z^{m-n}} \cdot \frac{14y^{1-2m}}{5z^{m+n}x^{3-m}} && \text{Reduce } \frac{4}{7} \cdot \frac{14}{5} \text{ and apply (48) and (49).} \\
& = \frac{8}{5} \cdot \frac{x^{2-m-(3-m)} \cdot y^{3m+1-2m}}{z^{m-n+m+n}} \\
& = \frac{8}{5} \cdot \frac{x^{-1} \cdot y^{m+1}}{z^{2m}} && \text{Apply rule (55).} \\
& = \frac{8}{5} \frac{y^{m+1}}{x \cdot z^{2m}}
\end{aligned}$$

2.5 Square Root Equations

Exercise 1:

$$\begin{aligned}
 a^{\frac{2}{5}} \cdot \sqrt[5]{a^3} & \quad \text{Apply rule (46).} \\
 a^{\frac{2}{5}} \cdot (a^3)^{\frac{1}{5}} & \quad \text{Apply rule (50).} \\
 = a^{\frac{2}{5}} \cdot a^{\frac{3}{5}} & \quad \text{Apply rule (48).} \\
 = a^{\frac{2}{5} + \frac{3}{5}} & \\
 = a^1 & \quad \text{Apply rule (54).} \\
 = a &
 \end{aligned}$$

Exercise 2:

$$\begin{aligned}
 \sqrt{a^3 a^2} \div \left(a \sqrt{a^{-3} \sqrt{a^{-1}}} \right) & \quad \text{Apply rule (47).} \\
 = \sqrt{a^1 \cdot a^{\frac{2}{3}}} \div \left(a^1 \cdot \sqrt{a^{-3} \cdot a^{-\frac{1}{2}}} \right) & \quad \text{Apply rule (48) and (62).} \\
 = \left(a^{1+\frac{2}{3}} \right)^{\frac{1}{2}} \div \left(a^1 \cdot \left(a^{-3-\frac{1}{2}} \right)^{\frac{1}{2}} \right) & \quad \text{Apply rule (50).} \\
 = a^{\frac{5}{3} \cdot \frac{1}{2}} \div \left(a^1 \cdot a^{-\frac{7}{2} \cdot \frac{1}{2}} \right) & \quad \text{Simplify and apply rule (48).} \\
 = a^{\frac{5}{6}} \div a^{1-\frac{7}{4}} & \\
 = a^{\frac{5}{6}} \div a^{-\frac{3}{4}} & \quad \text{Apply rule (49).} \\
 = a^{\frac{5}{6} - (-\frac{3}{4})} & \\
 = a^{\frac{19}{12}} &
 \end{aligned}$$

Exercise 3:

$$\begin{aligned}
 \sqrt[3]{(a^2)^{-5} \cdot \sqrt[4]{a^{16}}} & \quad \text{Apply rule (62), (50), and (47).} \\
 = \left(a^{-10} \cdot a^{\frac{16}{4}} \right)^{\frac{1}{3}} & \\
 = \left(a^{-10} \cdot a^4 \right)^{\frac{1}{3}} & \quad \text{Apply rule (48).} \\
 = \left(a^{-10+4} \right)^{\frac{1}{3}} & \\
 = \left(a^{-6} \right)^{\frac{1}{3}} & \quad \text{Apply rule (50) and (55).} \\
 = a^{-2} = \frac{1}{a^2} &
 \end{aligned}$$

Exercise 4:

$$\begin{aligned}
 & \frac{1}{\sqrt[3]{a\sqrt[5]{a^{-20}}}} && \text{Apply rule (62) and (47).} \\
 & = \frac{1}{\left(a^1 \cdot a^{-\frac{20}{5}}\right)^{\frac{1}{3}}} && \frac{-20}{5} \text{ equals } -4. \\
 & = \frac{1}{\left(a^1 \cdot a^{-4}\right)^{\frac{1}{3}}} && \text{Apply rule (48).} \\
 & = \frac{1}{\left(a^{1-4}\right)^{\frac{1}{3}}} = \frac{1}{\left(a^{-3}\right)^{\frac{1}{3}}} && \text{Apply rule (50).} \\
 & = \frac{1}{a^{-1}} = a^1 = a
 \end{aligned}$$

Exercise 5:

$$\begin{aligned}
 & \sqrt[3]{a\sqrt{a}} \div \sqrt{a^{-3}\sqrt[4]{a^6}} && \text{Apply rule (62) and (47).} \\
 & = \left(a^1 \cdot a^{\frac{1}{2}}\right)^{\frac{1}{3}} \div \left(a^{-3} \cdot a^{\frac{6}{4}}\right)^{\frac{1}{2}} && \text{eq-50} \\
 & = \left(a^{\frac{1}{3}} \cdot a^{\frac{1}{6}}\right) \div \left(a^{-\frac{3}{2}} \cdot a^{\frac{3}{4}}\right) && \text{Apply rule (48).} \\
 & = \left(a^{\frac{1}{3}+\frac{1}{6}}\right) \div \left(a^{\frac{-3}{2}+\frac{3}{4}}\right) = a^{\frac{1}{2}} \div a^{\frac{-3}{4}} && \text{Apply rule (49).} \\
 & = a^{\frac{1}{2}-\left(\frac{-3}{4}\right)} = a^{\frac{1}{2}+\frac{3}{4}} \\
 & = a^{\frac{5}{4}}
 \end{aligned}$$

3 Inequalities

Coming up...

4 Ring Theory

On hold...

5 Algorithms

Coming soon. . .