A Sufficiency Condition for Discrete Gabor Frames

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Chapter 1

The Discrete Gabor Transform

1.1 The Discrete Fourier Transform

Definition 1.1. Let $E = \{\vec{v_0}, \vec{v_1}, ..., \vec{v_{L-1}}\} \subseteq \mathbb{C}^L$ where $\vec{v_j}(k) = 1$ when $j = k$ and zero otherwise. We call E the *standard orthonormal basis* for $\mathbb{C}^L$.

Definition 1.2. Let $F = \left\{\vec{f_0}, \vec{f_1}, ..., \vec{f_{L-1}}\right\} \subseteq \mathbb{C}^L$ where $\vec{f_k}(j) = e^{\frac{2\pi ijk}{L}}$ for $0 \leq j \leq L-1$. Note that F is an orthogonal basis for $\mathbb{C}^L$. We call it the *Fourier basis.* Also note the L-periodic extension of $\vec{f_k}$ is a periodic vector with period $\frac{L}{k}$ when $\frac{L}{k} \in \mathbb{N}$. We define its *frequency* to be $\frac{k}{L}$.

Remark: While there are many parallels between discrete and continuous Fourier analysis, the discrete Fourier basis vectors differ from their continuous counterparts when it comes to periodicity. The Fourier functions $\left\{f_k(t) = \frac{e^{2\pi ikt}}{\sqrt{2}}\right\}_{k\in\mathbb{Z}}$, which form an orthonormal basis for $L^2([-1,1])$, are periodic for all $k \in \mathbb{Z}$. On the other hand, the discrete Fourier vectors $F = \left\{\vec{f}_k\right\}_{k=0}^{L-1}$, defined in Defintion 1.2, are periodic only when $\frac{L}{k} \in \mathbb{N}$.

Definition 1.3. The *discrete Fourier transform (DFT)* is the operator $\mathcal{F} : \mathbb{C}^L \to \mathbb{C}^L$ given by

$$\mathcal{F}(\vec{z})(k) = \sum_{j=0}^{L-1} \vec{z}(j) e^{\frac{-2\pi ikj}{L}}$$

for $0 \leq k \leq L-1$. We call $\{|\widehat{z}(k)| : 0 \leq k \leq L-1\}$ the *spectrum* of $\vec{z}$.

Note that $\mathcal{F}$ is the operator corresponding to left multiplication by a change-of-basis matrix. That is, for all $\vec{z} \in \mathbb{C}^L$, let $\vec{z}_E$ and $\vec{z}_F$ denote the E and F basis representations for $\vec{z}$, respectively. Then, $\mathcal{F}(\vec{z}_E) = \vec{z}_F$. When properly scaled (multiplied by $\frac{1}{\sqrt{L}}$), the discrete Fourier transform is a unitary operator. The *DFT* has an inverse, called the *inverse Fourier transform (IDFT)*, which is a change of basis operator from F to E.

Definition 1.4. The *inverse discrete Fourier transform (IDFT)* is the operator $\mathcal{F}^{-1} : \mathbb{C}^L \to \mathbb{C}^L$ given by

$$\mathcal{F}^{-1}(\vec{z})(k) = \frac{1}{L} \sum_{j=0}^{L-1} \vec{z}(j) e^{\frac{2\pi ikj}{L}}$$

for $0 \leq k \leq L-1$. Note that $\mathcal{F}^{-1}\mathcal{F}(\vec{z}) = \vec{z}$.

In the context of signal processing, the *DFT* is often characterized as decomposing a signal into its constituent frequencies. It does so by finding the vector's coefficients when represented under the Fourier basis. This notion is formalized with the Fourier inversion theorem.

Theorem 1.1. *[3, Theorem 2.6] (Fourier inversion theorem) Let $\vec{z} \in \mathbb{C}^L$ and let $\mathcal{F}(\vec{z}) = \widehat{z}$ be the discrete Fourier transform of $\vec{z}$. Then*

$$\vec{z}(j) = \frac{1}{L}\sum_{k=0}^{L-1} \widehat{z}(k) e^{\frac{2\pi i k j}{L}} \tag{1.1}$$

for $j \in \{0, 1, 2, ..., L-1\}$.

An important and related result is *Parseval's theorem*, which states the norm squared of the Fourier transform of a vector is always the norm squared of the vector times its length.

Theorem 1.2. *[3, Theorem 2.6] (Parseval's Theorem) Let $\vec{z} \in \mathbb{C}^L$ and let $\mathcal{F}(\vec{z})$ be its Fourier transform. Then,*

$$\sum_{k=0}^{L-1} |\vec{z}(k)|^2 = \frac{1}{L}\sum_{k=0}^{L-1} |(\mathcal{F}(\vec{z}))(j)|^2 .$$

When our vector $\vec{z}$ is sampled from a continuous time function $f(t) \in L^2[0,T]$, the frequency associated with its entries are dependent on the *sampling rate* and length of the sampling time. That is, let $f(t) \in L^2[0,T]$ be a continuous time function, let $\gamma \in \mathbb{N}$ and let $\vec{z} \in \mathbb{C}^{T\gamma}$ be defined as $\vec{z}(j) = f(\frac{j}{\gamma})$ for $0 \leq j \leq T\gamma - 1$. We call γ our *sampling rate* and $\vec{z}$ our *sampled signal vector*. Let $|\widehat{z}(k)|$ be the spectrum of $\vec{z}$. By the symmetry of the Fourier transfrom (6, pg. 122), we always have $|\widehat{z}(k)| = |\widehat{z}(T\gamma - k)|$ for all $0 \leq k \leq T\gamma - 1$. By the Nyquist-Shannon Sampling Theorem (9, pg. 29) and symmetry of the Fourier transform, $|\widehat{z}(k)|$ quantifies frequency

$k\frac{\gamma}{2}\frac{2}{\gamma T} = \frac{k}{T}$ in the continuous-time signal $f(t)$.

While the Fourier transform decomposes a signal into its constituent frequencies, it provides no temporal information on these frequencies. To extract time-frequency information, we introduce an operator called the *short-time Fourier transform* and discuss its connection to Gabor frames.

1.2 The Discrete Short-Time Fourier Transform

A technique to extract time-frequency information from a signal is to break up the signal into short temporal segments and apply the Fourier transform to each of these segments. This results in frequency information localized to temporal regions. This is the essence of the *Short-time Fourier transform.* The temporal segments are extracted using a window vector that is positive in a interval and zero elsewhere. Note it's a transformation of a one dimensional vector into a two dimensional time by frequency matrix.

Definition 1.5. Let $\vec{w} \in \mathbb{C}^L$ be a non-negative vector which is supported on a discrete subinterval of $\{0, 1, 2, ..., L-1\}$. We will call $\vec{w}$ the *window vector.* The *discrete short-time Fourier transform* associated with window vector $\vec{w}$ is the mapping $DSTFT : \mathbb{C}^L \to \mathbb{C}^{L\times L}$ defined as

$$DSTFT(\vec{z}) = M(m, n)$$

where

$$M(m,n) = \sum_{k=0}^{L-1} \vec{z}(k) e^{\frac{-2\pi ikm}{L}} \vec{w}(k-n)$$

for $0 \leq m \leq L-1$ and $0 \leq n \leq L-1$.

From the definition, we see the DSTFT is simply the DFT applied to segments of the signal $\vec{z}$ multiplied to window vector $\vec{w}(k-n)$ as n ranges from 0 to $L-1$. Since $\vec{w}(k-n)$ is zero outside an interval, we are applying the DFT to small temporally localized segments of $\vec{z}$, extracting a time by frequency profile. Note m is the frequency index, while n is the temporal index.

While the DFT represents signals as superpositions of their frequency components, we want to use the DSTFT to represent signals as a superposition of time-by-frequency components. That is, let $\vec{z} \in \mathbb{C}^L$ and let $M(m,n)$ be the DSTFT of $\vec{z}$. Then

$$\vec{z} = K \sum_{n=0}^{L-1} \sum_{m=0}^{L-1} M(m,n) e^{\frac{2\pi ikm}{L}} \vec{w}(k-n) \tag{1.2}$$

where K is some constant. Here, the DSTFT decomposes a signal into time-frequency components, where $M(m,n)$ quantifies the $\frac{L}{m}$ frequency component of the signal localized around time n. Unlike the Fourier basis, note the set of vectors

$$\left\{ e^{\frac{2\pi ikm}{L}} \vec{w}(k-n) \right\}_{n,m}$$

cannot be a basis for $\mathbb{C}^L$ as there are L^2 elements in the set. But it does usually form a *frame*, which we introduce in the next section.

1.3 Frames in Finite Dimensional Space

Definition 1.6. A sequence of vectors $\left\{\vec{f}_k\right\}_{k\in I} \subseteq \mathbb{C}^L$ is a *frame* in $\mathbb{C}^L$ if there exists constants $A, B > 0$ such that $\forall \vec{f} \in \mathbb{C}^L$

$$A\left\|\vec{f}\right\|^2 \leq \sum_{k\in I}\left|\left\langle \vec{f}, \vec{f}_k\right\rangle\right|^2 \leq B\left\|\vec{f}\right\|^2.$$

We call A and B the *frame bounds* of $\left\{\vec{f}_k\right\}_{k\in I}$.

We say a sequence of vectors is *complete* if it spans a vector space. A frame in $\mathbb{C}^L$ is always complete in $\mathbb{C}^L$ but a complete sequence of vectors is not necessarily a frame. For example, consider the sequence $S = \{(1,0),(0,1)\} \cup \{e_k\}_{k\in\mathbb{N}}$ where $e_k = (0, \frac{1}{\sqrt{k}})$ in $\mathbb{C}^2$. While S spans $\mathbb{C}^2$,

$$\sum_{k=1}^{\infty} \|\langle e_k, (0,1)\rangle\|^2 = \infty$$

and therefore is not a frame. Hence, completeness is a necessary but not sufficient condition for a sequence to be a frame. However, in the special case where the sequence is finite, completeness is necessary and sufficient to being a frame.

Theorem 1.3. *[2, Corollary 1.1.3] Let* $m \in \mathbb{N}$*. Then* $\left\{\vec{f}_k\right\}_{k=1}^{m}$ *is a frame for* $\mathbb{C}^L$ *if and only if span* $\left\{\vec{f}_k\right\}_{k=1} = \mathbb{C}^L$.

The above theorem implies frames are generalizations of bases. All bases are frames but not all frames are bases. Since a frame $\left\{\vec{f}_k\right\}_{k=1}^{m}$ spans $\mathbb{C}^L$, every vector in $\mathbb{C}^L$ can be represented (not necessarily uniquely) as a linear combination of $\left\{\vec{f}_k\right\}_{k=1}^{m}$. We formalize this notion by defining *frame operators.*

Definition 1.7. Let $\left\{\vec{f}_k\right\}_{k=1}^{m}$ be a frame for $\mathbb{C}^L$. The *synthesis operator* associated with $\left\{\vec{f}_k\right\}_{k=1}^{m}$ is $T : \mathbb{C}^m \to \mathbb{C}^L$ where

$$T\left\{c_k\right\}_{k=1}^{m} = \sum_{k=1}^{m} c_k \vec{f}_k.$$

Definition 1.8. Let $\left\{\vec{f}_k\right\}_{k=1}^{m}$ be a frame for $\mathbb{C}^L$. The *analysis operator* associated with $\left\{\vec{f}_k\right\}_{k=1}^{m}$ is $T^* : \mathbb{C}^L \to \mathbb{C}^m$ where

$$T^*(\vec{f}) = \left\{\left\langle \vec{f}, \vec{f}_k \right\rangle\right\}_{k=1}^{m}.$$

Definition 1.9. Let $\left\{\vec{f}_k\right\}_{k=1}^{m}$ be a frame for $\mathbb{C}^L$. The *frame operator* associated with $\left\{\vec{f}_k\right\}_{k=1}^{m}$ is $S : \mathbb{C}^L \to \mathbb{C}^L$ by $S(\vec{f}) = TT^*\vec{f}$ or more explicitly

$$S(f) = \sum_{k=1}^{m} \left\langle \vec{f}, \vec{f}_k \right\rangle f_k.$$

Theorem 1.4. *[2, Theorem 1.1.5] The frame operator is invertible, positive, and self-adjoint. Furthermore, the sequence of vectors* $\left\{S^{-1}\vec{f}_k\right\}_{k=1}^{m}$ *is also a frame, called the canonical dual frame. An important relationship between a frame and its canonical dual is* $\forall \vec{f} \in \mathbb{C}^L$

$$\vec{f} = \sum_{k=1}^{m} \left\langle \vec{f}, S^{-1}\vec{f_k} \right\rangle \vec{f_k} = \sum_{k=1}^{m} \left\langle \vec{f}, \vec{f_k} \right\rangle S^{-1}\vec{f_k} \tag{1.3}$$

The frame operator and its inverse provides a representation of any vector as a linear combination of the frame, or a *frame expansion*. An important property of the canonical dual frame is that it offers a minimal representation of a given vector. We formalize this with the following theorem.

Theorem 1.5. *[2, Theorem 1.1.5] Let* $\left\{\vec{f_k}\right\}_{k=1}^{m}$ *be a frame for* $\mathbb{C}^L$*. Then,* $\forall f \in \mathbb{C}^L$ *with representation* $f = \sum_{k=1}^{m} c_k f_k$*,*

$$\sum_{k=1}^{m} |c_k|^2 = \sum_{k=1}^{m} \left|\left\langle f, S^{-1} f_k \right\rangle\right|^2 + \sum_{k=1}^{m} \left|c_k - \left\langle f, S^{-1} f_k \right\rangle\right|^2 .$$

In other words, $\forall \vec{f} \in \mathbb{C}^L$, among all scalar sequences $\{c_k\}_{k=1}^{m}$ such that $\vec{f} = \sum_{k=1}^{m} c_k \vec{f_k}$, the sequence with the minimal ℓ^2-norm is $\left\{\left\langle \vec{f}, S^{-1}\vec{f_k} \right\rangle\right\}_{k=1}^{m}$.

We now discuss a class of frames called *Gabor frames*, which are useful in time-frequency analysis.

1.4 Gabor Frames

Definition 1.10. Let $T_j : \mathbb{C}^L \to \mathbb{C}^L$ by $(T_j(\vec{f}))(t) = \vec{f}((t-j)(\text{mod L}))$ for all $0 \le t \le L-1$. We call T_j the *translation-by-j operator.*

Recall that $t(\text{mod L}) = r$ means there exists $s \in \mathbb{Z}$ such that $sL + t = r$ where $r \in \{0, 1, .., L-1\}$. We apply mod L to the indices to ensure T_j is a well-defined operator. When indices fall out the range of $\{0, 1, .., L-1\}$, we mod them back in. This is can also be called *circular translation.*

Definition 1.11. Let $E_k : \mathbb{C}^L \to \mathbb{C}^L$ by $(E_k(\vec{f}))(t) = e^{2\pi ikt}\vec{f}(t)$ where $0 \leq t \leq L-1$. We call E_k the *modulation-by-k operator.*

Definition 1.12. Let $\vec{g} \in \mathbb{C}^L$, $N \in \mathbb{N}$, and $M \in \mathbb{N}$ such that $N\,|L$ and $M\,|L$. We call $\vec{g}$ the *generating vector.* Let $a = \frac{L}{N}$ and $b = \frac{1}{M}$. The *Gabor system generated by* $\vec{g}$ is the collection of vectors of the form

$$\{E_{mb}T_{na}\vec{g}\}_{m=0,n=0}^{M-1,N-1}$$

where

$$E_{mb}T_{na}\vec{g}(k) = e^{2\pi imbk}\vec{g}((k-na)(\text{mod L}))$$

for $k \in 0, 1, 2, ..., L-1$.

We call $a = \frac{L}{N}$ the *temporal sampling rate* and $b = \frac{1}{M}$ the *frequency sampling rate.* Note that when $N = M = L$, the Gabor system are the time-frequency vectors in the short-time Fourier transform representation given in Equation 1.2 using window vector $\vec{g}$.

A Gabor system can be thought of as a sampling of the short-time Fourier transform's time-frequency vectors in the following sense: Rather than take all L^2 time-frequency vectors, we take regular samples across both temporal and frequency dimensions determined by a and b, respectively. Note the following change of terminology: The *window vector* in the short-time Fourier transform is called the *generating vector* in a Gabor system.

This time-frequency decomposition of signals is the motivation behind Gabor systems. But note that not all Gabor systems span $\mathbb{C}^L$. An extreme case is when $M = N = 1$ where our Gabor system consists of just the generating vector. When a Gabor system does not span the space, we cannot represent all vectors as a linear combination of vectors in the Gabor system. At best, we approximate the vector as an orthogonal projection onto the span of the Gabor system. As such, we will not be able to recover the original vector from the Gabor system coefficients. Therefore, a spanning Gabor system, called a Gabor frame, is of great interest.

Definition 1.13. A Gabor system $\{E_{mb}T_{na}\vec{g}\} \subseteq \mathbb{C}^L$ that is also a frame in $\mathbb{C}^L$ is a *Gabor frame.*

Theorem 1.6. *Let $\{E_{mb}T_{na}\vec{g}\}$ be a Gabor system in $\mathbb{C}^L$. Then the following holds:*

(i) If $ab > 1$, then $\{E_{mb}T_{na}\vec{g}\}$ cannot be a frame.

(ii) If $\{E_{mb}T_{na}\vec{g}\}$ spans $\mathbb{C}^L$ and $ab = 1$, then it is a basis.

(iii) If $\{E_{mb}T_{na}\vec{g}\}$ spans $\mathbb{C}^L$ and $ab < 1$, then it is a frame but not a basis.

The proof follows immediately by the fact there are NM vectors in $\{E_{mb}T_{na}\vec{g}\}$.

Definition 1.14. Let $\vec{g} \in \mathbb{C}^L$ such that for $\vec{g} > 0$ on a discrete subinterval $\{a, a+1, ..., b-1, b\} \subseteq \{0, 1, ..., L-1\}$ and zero elsewhere. We call $\{a, a+1, ..., b-1, b\}$ the *support* of $\vec{g}$.

1.4.1 A Sufficient Condition For a Gabor System Forming a Frame

There are three parameters for constructing a Gabor frame in $\mathbb{C}^L$: The number of time bins $N = \frac{L}{a}$ (where a is the temporal sampling rate), the number of frequency bins $M = \frac{1}{b}$ (where b is the frequency sampling rate), the generating (window) vector $\vec{g}$ and the length of its support I. We now prove a condition which ensures a Gabor system generates a Gabor frame. We first show a relation between the Gabor frame bounds and the generating vector.

Proposition 1.7. *Let $\vec{g}$ be a real-valued vector in $\mathbb{C}^L$. If $\{E_{mb}T_{na}\vec{g}\}$ is a Gabor frame with bounds A and B, then*

$$bA \leq \sum_{n=0}^{N-1} |\vec{g}(l - na)|^2 \leq bB \tag{1.4}$$

for all $0 \leq l \leq L - 1$.

Proof. We prove both bounds by contradiction. Suppose there exists k such that $\sum_{n=0}^{N-1} |\vec{g}(k - na)|^2 > bB$. We let $\vec{f}$ be $f(j) = 1$ when $j = k$ and $f(j) = 0$

otherwise. Then note

$$\begin{aligned}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\left|\langle f, E_{mb}T_{na}\vec{g}\rangle\right|^2 &= \sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\left|\sum_{j=0}^{L-1} f(j)E_{mb}T_{na}\vec{g}(j)\right|^2\\ &= \sum_{n=0}^{N-1}\sum_{m=0}^{M-1}|f(k)|^2\left|e^{2\pi imbk}\right|^2|\vec{g}(k-na)|^2\\ &= \sum_{n=0}^{N-1}\sum_{m=0}^{M-1}|\vec{g}(k-na)|^2\\ &= \sum_{n=0}^{N-1} M\,|\vec{g}(k-na)|^2\\ &= \sum_{n=0}^{N-1}\frac{1}{b}|\vec{g}(k-na)|^2 > \frac{1}{b}bB = B\end{aligned}$$

a contradiction. The same argument follows in proving the lower bound. □

We now show a sufficient condition between $a = \frac{L}{N}$ and the length of support of $\vec{g}$ such that equation 1.4 is satisfied.

Lemma 1.8. *Let $\vec{g} \in \mathbb{C}^L$ with support greater than $a = \frac{L}{N}$. Then there exists constants $0 < C \leq D$ such that*

$$C \leq \sum_{n=0}^{N-1}\left|\vec{g}((l-na)(mod\ L))\right|^2 \leq D$$

for all $l \in \mathbb{Z}$.

Proof. Since we are dealing with a finite sum the upper bound D is automatically satisfied. To prove the lower bound, suppose (WLOG) the support of $\vec{g}$ is $0, 1, 2, ..., a-1$. Then, $\forall l \in \{0, 1, 2, ..., L-1\}$, let $l' = l(\text{mod a})$. Then,

$\exists n' \in \{0, 1, 2, ..., n-1\}$ and $\exists l' \in \{0, 1, 2, ..., a-1\}$ such that $l' + n'a = l$. Then note $g(l - n'a) = g(l') \neq 0$ since l' is in the support of $\vec{g}$. Therefore

$$\sum_{n=0}^{N-1} |\vec{g}(l - na)|^2 \geq |\vec{g}(l - n'a)|^2 > 0$$

for all $l \in \{0, 1, 2, ..., L-1\}$. Taking $C = \mathrm{Min}\{|\vec{g}(l)| : \vec{g} \neq 0\}$ and $D = \sum_{n=0}^{N-1} |\vec{g}(l - na)|^2$, we have

$$C \leq \sum_{n=0}^{N-1} |\vec{g}(l - na)|^2 \leq D$$

$\square$

Lemma 1.9. *Let $\vec{f}, \vec{g} \in \mathbb{C}^L$. Let $\frac{L}{N} = a$. Let $\frac{1}{M} = b$. Let $n \in \{0, 1, 2, ..., N-1\}$ be fixed. We define $\vec{F} \in \mathbb{C}^M$ by*

$$\vec{F}(m) = \sum_{l'=0}^{\frac{L}{M}-1} \vec{f}(m - Ml')\vec{g}(m - Ml' - na)$$

for $m \in \{0, 1, 2, ..., M-1\}$. Then $\vec{F}(m) = \vec{F}(m+M)$ for all $m \in \{0, 1, 2, ..., M-1\}$

Proof. Note for $\vec{F}$ to be defined, it's assumed that $\vec{f}$ and $\vec{g}$ are both L-periodically extended. That is, $\vec{f}(k) = \vec{f}(k(\text{mod L}))$ and $\vec{g}(k) = \vec{g}(k(\text{mod L}))$

for all $k \in \mathbb{Z}$. Then, note that

$$\begin{aligned}\vec{F}(m+M) &= \sum_{l'=0}^{\frac{L}{M}-1} \vec{f}(m-Ml'-M)\vec{g}(m-Ml'-M-na) \\ &= \sum_{l'=0}^{\frac{L}{M}-1} \vec{f}(m-M(l'+1))\vec{g}(m-M(l'+1)-na)\end{aligned}$$

Re-indexing with $l''=l'+1$, we get

$$\vec{F}(m+M) = \sum_{l''=1}^{\frac{L}{M}} \vec{f}(m-Ml'')\vec{g}(m-Ml''-na)$$

But since $\vec{f}(m-M(\frac{L}{M})) = \vec{f}(m-L) = \vec{f}(m)$ and $\vec{f}(m-M(\frac{L}{M})) = \vec{f}(m-L) = \vec{f}(m)$, we get

$$\vec{F}(m+M) = \sum_{l''=0}^{\frac{L}{M}-1} \vec{f}(m-Ml'')\vec{g}(m-Ml''-na) = \vec{F}(m)$$

□

We next prove another lemma showing an intermediate result when assuming a certain condition satisfied between M and $\vec{g}$.

Lemma 1.10. *Let $\vec{g} \in \mathbb{C}^L$ be a real-valued vector with support length less than or equal to M. Let $\vec{f}$ be any vector in $\mathbb{C}^L$. Then*

$$\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} |\langle f, E_{mb}T_{na}\vec{g}\rangle|^2 = M\sum_{l=0}^{L-1}\left|\vec{f}(l)\right|^2 \sum_{n=0}^{N-1} |\vec{g}(l-na)|^2 .$$

Proof.

$$
\begin{aligned}
\langle f, E_{mb}T_{na}\vec{g}\rangle &= \sum_{l=0}^{L-1} \vec{f}(l)\vec{g}(l-na)e^{-2\pi imbl} \\
&= \sum_{l'=0}^{\frac{L}{M}-1}\sum_{m'=0}^{M-1} \vec{f}(m'-Ml')\vec{g}(m'-Ml'-na)e^{-\frac{2\pi im(m'-Ml')}{M}} \\
&= \sum_{m'=0}^{M-1}\sum_{l'=0}^{\frac{L}{M}-1} \vec{f}(m'-Ml')\vec{g}(m'-Ml'-na)e^{-\frac{2\pi imm'}{M}}
\end{aligned}
$$

where we split the sum indexed by l into two nested sums indexed by m' and l'. Note then, the outer sum is the Fourier transform in $\mathbb{C}^M$ of the vector $\vec{F_n}(m') = \sum_{l'=0}^{\frac{L}{M}-1} \vec{f}(m'-Ml')\vec{g}(m'-Ml'-na)$. In other words, $\frac{1}{M}\langle f, E_{mb}T_{na}\vec{g}\rangle$ is the *mth* Fourier coefficient of $\vec{F_n} \in \mathbb{C}^M$. Recognizing this, we invoke Parseval's identity to get

$$
\begin{aligned}
\sum_{m=0}^{M-1} |\langle f, E_{mb}T_{na}\vec{g}\rangle|^2 &= M\left\|\vec{F_n}\right\|^2 \\
\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} |\langle f, E_{mb}T_{na}\vec{g}\rangle|^2 &= M\sum_{n=0}^{N-1}\left\|\vec{F_n}\right\|^2 \\
&= M\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\overline{\vec{F_n}(m)}\vec{F_n}(m).
\end{aligned}
$$

We substitute $\vec{F_n}(m) = \sum_{l'=0}^{\frac{L}{M}-1} \vec{f}(m-Ml')\vec{g}(m-Ml'-na)$ and use lemma

1.9 to get

$$M\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\overline{\vec{F_n}(m)}\vec{F_n}(m) = M\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\left(\sum_{l'=0}^{\frac{L}{M}-1}\overline{\vec{f}(m-Ml')\vec{g}(m-Ml'-na)}\right)\vec{F_n}(m-Ml').$$

We combine the nested sums with indices m and l' a single sum indexed by l. Then, we substitute $\vec{F_n}(l) = \sum_{l'=0}^{\frac{L}{M}-1}\vec{f}(l-Ml')\vec{g}(l-Ml'-na)$ back in to get

$$M\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\left(\sum_{l'=0}^{\frac{L}{M}-1}\overline{\vec{f}(m-Ml')\vec{g}(m-Ml'-na)}\right)\vec{F_n}(m-Ml')$$

$$= M\sum_{n=0}^{N-1}\sum_{l=0}^{L-1}\overline{\vec{f}(l)}\vec{g}(l-na)\vec{F_n}(l)$$
$$= \sum_{n=0}^{N-1}\sum_{l=0}^{L-1}\overline{\vec{f}(l)}\vec{g}(m-na)\sum_{l'=0}^{\frac{L}{M}-1}\vec{f}(l-Ml')\vec{g}(l-Ml'-na).$$

We switch the order of the two inner most sums to get

$$\sum_{n=0}^{N-1}\sum_{l=0}^{L-1}\overline{\vec{f}(l)}\vec{g}(m-na)\sum_{l'=0}^{\frac{L}{M}-1}\vec{f}(l-Ml')\vec{g}(l-Ml'-na)$$

$$= M\sum_{l=0}^{L}\sum_{l'=0}^{\frac{L}{M}-1}\vec{f}(l-Ml')\overline{\vec{f}(l)}\sum_{n=0}^{N-1}\vec{g}(l-na)\overline{\vec{g}(l-Ml'-na)}.$$

We then split the sum indexed by l' into the cases $l' = 0$ and $l' > 0$ to get

$$M \sum_{l=0}^{L} \sum_{l'=0}^{\frac{L}{M}-1} \vec{f}(l - Ml')\overline{\vec{f}(l)} \sum_{n=0}^{N-1} \vec{g}(l - na)\overline{\vec{g}(l - Ml' - na)} =$$

$$M \sum_{l=0}^{L-1} \left|\overline{\vec{f}(l)}\right|^2 \sum_{n=0}^{N-1} |\vec{g}(l - na)|^2 +$$

$$M \sum_{l=0}^{L} \sum_{l'=1}^{\frac{L}{M}-1} \vec{f}(l - Ml')\overline{\vec{f}(l)} \sum_{n=0}^{N-1} \vec{g}(l - na)\overline{\vec{g}(l - Ml' - na)}.$$

Since $\vec{g}$ has support on an interval with length at most M, we have that

$$\vec{g}(l - na)\vec{g}(l - Ml' - na) = 0$$

for all $n \in \{0, 1, ..., N - 1\}$, $l' \in \left\{1, 2, ..., \frac{L}{M} - 1\right\}$ and $l \in \{0, 1, ..., L - 1 - 1\}$. Therefore, the second sum is zero and we have that

$$\sum_{n=0}^{N-1} \sum_{m=0}^{M-1} |\langle f, E_{mb}T_{na}\vec{g}\rangle|^2 = M \sum_{l=0}^{L-1} \left|\vec{f}(l)\right|^2 \sum_{n=0}^{N-1} |\vec{g}(l - na)|^2 .$$

□

We use the preceeding lemmas to derive our main result.

Theorem 1.11. *Suppose $\vec{g} \in \mathbb{C}^L$ has support on an interval of length at most M and at least $\frac{L}{N}$. Then $\{E_{mb}T_{na}\vec{g}\}$ is a frame for $\mathbb{C}^L$. The frame operator*

$S : \mathbb{C}^L \to \mathbb{C}^L$ and its inverse $S^{-1} : \mathbb{C}^L \to \mathbb{C}^L$ are given by

$$(S\vec{f})(l) = M\vec{G}(l)\vec{f}(l)$$

and

$$(S^{-1}\vec{f})(l) = \frac{1}{M\vec{G}(l)}\vec{f}(l)$$

where $\vec{G}(l) = \sum_{n=0}^{N-1} |\vec{g}(l-na)|^2 \in \mathbb{C}^L$ for $0 \le l \le L-1$.

A clarification on the notation above. $\vec{G}$ and $\vec{f}$ are both vectors in $\mathbb{C}^L$ while M (number of frequency bins) is a positive integer . $\vec{G}\vec{f}$ denotes entry-wise multiplication of two vectors.

Proof. By Lemma 1.10, we have for all $\vec{f} \in \mathbb{C}^L$,

$$\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} |\langle f, E_{mb}T_{na}\vec{g}\rangle|^2 = M\sum_{l=0}^{L-1}\left|\vec{f}(l)\right|^2 \sum_{n=0}^{N-1} |\vec{g}(l-na)|^2 .$$

By Lemma 1.8, we have $bA \le \sum_{n=0}^{N-1} |\vec{g}(l-na)|^2 \le bB$ for some $A, B > 0$. Then we have

$$A \le \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} |\langle f, E_{mb}T_{na}\vec{g}\rangle|^2 \le B.$$

That is, $\{E_{mb}T_{na}\vec{g}\}$ is a frame with bounds A and B. To derive the frame operator, note for all $\vec{f} \in \mathbb{C}^L$, we have

$$\begin{aligned}
\langle Sf, f\rangle &= \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} |\langle f, E_{mb}T_{na}\vec{g}\rangle|^2 \\
&= M\sum_{l=0}^{L-1}\left|\vec{f}(l)\right|^2 \sum_{n=0}^{N-1} |\vec{g}(l-na)|^2 \\
&= M\sum_{l=0}^{L-1}\left|\vec{f}(l)\right|^2 \vec{G}(l) \\
&= \left\langle M\vec{G}\vec{f}, \vec{f}\right\rangle
\end{aligned}$$

where we again use Lemma 1.8. The inverse frame operator is derived similarly. □

The significance of Theorem 1.11 is it provides a sufficient condition involving M, N and the length of support of $\vec{g}$ to ensure a Gabor frame, allowing for complete recovery of the original signal with the Gabor coefficients. *Note this isn't a necessary condition and there may exist Gabor frames which violate these conditions.* Also note this means the vectors in the short-time Fourier transform ($N = M = L$) form Gabor frames for all windows with support between $\frac{L}{N} = 1$ and $M = L$. We now introduce the discrete Gabor transform.

1.4.2 The Discrete Gabor Transform

Definition 1.15. Let $\{E_{mb}T_{na}\vec{g}\}$ be a Gabor frame. The *discrete Gabor transform* $\mathcal{G} : \mathbb{C}^L \to \mathbb{C}^M \times \mathbb{C}^N$ is

$$\mathcal{G}_f(m,n) = \left\langle \vec{f}, E_{mb}T_{na}\vec{g} \right\rangle$$

where $0 \leq n \leq N-1$ and $0 \leq m \leq M-1$. We call the $M \times N$ complex-valued matrix, $\mathcal{G}_f(m,n)$, the *Gabor matrix* of $\vec{f}$.

Note the discrete Gabor transform is just the analysis operator applied to a signal. Rather than store the coefficients as a vector, the DGT stores it as a time-by-frequency matrix.

We now discuss the issue of reconstructing the original vector from the Gabor matrix. Recall from Theorem 1.4, that reconstruction from the Gabor matrix is possible, as long as it was created by a Gabor frame. Namely, if $\vec{f}$ was the original signal vector and $G(m,n)$ is our $M \times N$ Gabor matrix created by the Gabor frame $\{E_{mb}T_{na}\vec{g}\}$, then

$$\vec{f} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} G(m,n) S^{-1}(E_{mb}T_{na}\vec{g})$$

where S^{-1} is the inverse frame operator. That is, we can reconstruct $\vec{f}$ using our Gabor matrix and the dual Gabor frame $\{S^{-1}(E_{mb}T_{na}\vec{g})\}$. In a special case , when our Gabor frame satisfies the conditions in Theorem 1.11, we have an explicit form for the inverse frame operator. Namely, $S^{-1}(\vec{f}) = \frac{\vec{f}}{M\vec{G}}$

where M is frequency bins, $\vec{G} \in \mathbb{C}^L$ is

$$\vec{G}(l) = \sum_{n=0}^{N-1} |\vec{g}(l - na)|^2$$

for $0 \leq l \leq L-1$ and $\frac{\vec{f}}{\vec{G}}$ denotes point-wise division between two vectors.

A useful fact for the construction of the dual Gabor frame $\{S^{-1}E_{mb}T_{na}\vec{g}\}$, is that the frame operator (and its inverse) commutes with the translation and modulation operator:

Lemma 1.12. *[2, Lemma 9.3.1] Suppose S is the frame operator and S^{-1} is its inverse. Then, $SE_{mb}T_{na} = E_{mb}T_{na}S$ and $S^{-1}E_{mb}T_{na} = E_{mb}T_{na}S^{-1}$.*

The above lemma simplifies the calculation of the canonical dual frame. Rather than apply S^{-1} to all $M \times N$ Gabor frame elements, we can simply take translations and modulations of $S^{-1}\vec{g}$ to generate the canonical dual frame. Nevertheless this still requires the inversion of the frame operator S. In our special case where a Gabor frame satisfies the conditions in Theorem 1.11, calcuation of the canonical dual frame still requires computing vector $\vec{G}(l) = \sum_{n=0}^{N-1} |\vec{g}(l - na)|^2$. One way to avoid these computations is with tight Gabor frames.

1.4.3 Tight Gabor Frames

Definition 1.16. A frame $\left\{\vec{f}_k\right\}_{k=1}^{K}$ is tight if there exists constant $A > 0$ such that

$$\sum_{k=1}^{K} \left|\left\langle \vec{f}, \vec{f}_k \right\rangle\right|^2 = A \left\|\vec{f}\right\|^2$$

for all $\vec{f} \in \mathbb{C}^L$.

The frame operator of a tight frame with frame bound A is $S = AI$ where I is the identity operator and $S^{-1} = \frac{1}{A}I$. With a tight frame, its canonical dual is simply a scalar multiple of it. Using Theorem 1.11, we now present a sufficient condition for generating a tight Gabor frames.

Corollary 1.13. *Let $\vec{h} \in \mathbb{C}^L$ be a vector with support length at most M and at least $\frac{L}{N}$ such that*

$$\sum_{n=0}^{N-1} \vec{h}(l + na) = 1$$

for all $l \in \{0, 1, 2, ..., L-1\}$. Again, we are using circular (mod L) translation. Let $\vec{g} \in \mathbb{C}^L$ be

$$\vec{g}(l) = \sqrt{\frac{h(l)}{M}}$$

for all $l \in \{0, 1, 2, ..., L-1\}$. Then $\{E_{mb}T_{na}\vec{g}\}$ is a tight frame with frame bound $A = 1$.

Proof. Note that for all $l \in \{0, 1, ..., L-1\}$,

$$\begin{aligned}\vec{G}(l) &= \sum_{n=0}^{N-1} |\vec{g}(l-na)|^2 \\ &= \frac{1}{M} \sum_{n=0}^{N-1} \vec{h}(l+na) \\ &= \frac{1}{M}.\end{aligned}$$

To clarify, the above means $\vec{G}$ is a constant vector with value $\vec{G}(l) = \frac{1}{M}$ for all $l \in \{0, 1, ..., L-1\}$. Then by Theorem 1.11, the frame operator is $S(\vec{f}) = M\frac{1}{M}\vec{f} = \vec{f}$, the identity operator, making $\{E_{mb}T_{na}\vec{g}\}$a tight frame bound with $A = 1$. □

When a frame operator is the identity, the frame is also its own canonical dual frame, allowing for reconstruction without computing $\{S^{-1}E_{mb}T_{na}\vec{g}\}$.

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