CMSC 303 Introduction to Theory of Computation, VCU
Assignment: 6
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Total marks: 59 marks + 6 marks bonus for typing your solutions in LaTeX.

Unless otherwise noted, the alphabet for all questions below is assumed to be $\Sigma = \{0, 1\}$. This assignment will get you primarily to practice reductions in the context of decidability.

1. [10 marks] We begin with some mathematics regarding uncountability. Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ denote the set of natural numbers.

   (a) [5 marks] Prove that the set of integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3\ldots\}$ has the same size as $\mathbb{N}$ by giving a bijection between $\mathbb{Z}$ and $\mathbb{N}$.

   Proof: By bijection, we create a function $f : N \mapsto Z$ such that all even numbers in N map into a some even number in Z and odd numbers in N map to negative numbers in Z.

   $$f(x) = \begin{cases} -\dfrac{x+1}{2} & \text{if } x \text{ is odd} \\ \dfrac{x}{2} & \text{if } x \text{ is even} \end{cases}$$

   (b) [5 marks] Let B denote the set of all infinite sequences over $\{0, 1\}$. Show that B is uncountable using a proof by diagonalization.

   Proof: We can prove this by contradiction.

   Let's begin by only concern ourselves with very the large (infinite) strings. We assume $\exists$ list L of all these strings.

   $$\begin{aligned} L = &011001100\ldots \\ &100110011\ldots \\ &100100100\ldots \\ &111111111\ldots \\ &\ldots\ldots\ldots \end{aligned} \tag{1}$$

   We can construct a string $x \in B$ which is not in L by taking the ith symbol of x to be the opposite symbol from the ith entry of L. Thus, contradiction.

2. [9 marks] We next move to a warmup question regarding reductions.

   (a) [2 marks] Intuitively, what does the notation $A \leq B$ mean for problems A and B?
   $A \leq B$ means that B is harder than A (or equally hard).

   (b) [2 marks] What is a mapping reduction $A \leq_m B$ from language A to language B? Give both a formal definition, and a brief intuitive explanation in your own words.

A mapping reduction gives us a way to handle deciding whether a problem is decidable or not from knowing that some other problem is decidable or not.

So for example with an example of adding and multiplying. Multiplying $\leq$ adding because multiplying can be achieved by simply using adding. Thus, adding is a more powerful construct.

With this, we can say that if A is not decidable, B is not decidable either because B is harder than A. As well as, if B is decidable, A is also decidable, because A is not as hard as B.

(c) [2 marks] What is a computable function? Give both a formal definition, and a brief intuitive explanation in your own words.

(d) [3 marks] Suppose $A \leq_m B$ for languages A and B. Please answer each of the following with a brief explanation.

    i. If B is decidable, is A decidable?
       A is decidable because B is 'harder', yet is decidable.

    ii. If A is undecidable, is B undecidable?
       B is undecidable because B is 'harder' than A and A is not decidable.

    iii. If B is undecidable, is A undecidable?
       It is unknown whether A is decidable or not, because while B is not decidable, it is also said to be harder than A.

3. [40 marks] Prove using reductions that the following languages are undecidable.

(a) [8 marks] $L = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \Sigma^*\}$.

(b) [8 marks] $L = \{\langle M \rangle \mid M \text{ is a TM and } \{000, 111\} \subseteq L(M)\}$.

(c) [8 marks] $L = \{\langle M \rangle \mid M \text{ is a TM which accepts all strings of even parity}\}$. (Recall the *parity* of a string $x \in \{0, 1\}$ is the number of 1's in x.)

(d) [8 marks] $L = \{\langle M \rangle \mid M \text{ is a TM that accepts } w^R \text{ whenever it accepts } w\}$. Recall here that w^R is the string w written in reverse, i.e. $011^R = 110$.

(e) [8 marks] Consider the problem of determining whether a TM M on an input w ever attempts to move its head left when its head is on the left-most tape cell. Formulate this problem as a language and show that it is undecidable.