CMSC 303 Introduction to Theory of Computation, VCU
Spring 2017, Assignment 6
Due: Thursday, April 14, 2017 in class

Total marks: 59 marks + 6 marks bonus for typing your solutions in LaTeX.

Unless otherwise noted, the alphabet for all questions below is assumed to be $\Sigma = \{0, 1\}$. This assignment will get you primarily to practice reductions in the context of decidability.

1. [10 marks] We begin with some mathematics regarding uncountability. Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ denote the set of natural numbers.

   (a) [5 marks] Prove that the set of integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots\}$ has the same size as $\mathbb{N}$ by giving a bijection between $\mathbb{Z}$ and $\mathbb{N}$.

   (b) [5 marks] Let B denote the set of all infinite sequences over $\{0, 1\}$. Show that B is uncountable using a proof by diagonalization.

2. [9 marks] We next move to a warmup question regarding reductions.

   (a) [2 marks] Intuitively, what does the notation $A \leq B$ mean for problems A and B?

   (b) [2 marks] What is a mapping reduction $A \leq_m B$ from language A to language B? Give both a formal definition, and a brief intuitive explanation in your own words.

   (c) [2 marks] What is a computable function? Give both a formal definition, and a brief intuitive explanation in your own words.

   (d) [3 marks] Suppose $A \leq_m B$ for languages A and B. Please answer each of the following with a brief explanation.

      i. If B is decidable, is A decidable?

      ii. If A is undecidable, is B undecidable?

      iii. If B is undecidable, is A undecidable?

3. [40 marks] Prove using reductions that the following languages are undecidable.

   (a) [8 marks] $L = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \Sigma^*\}$.

   (b) [8 marks] $L = \{\langle M \rangle \mid M \text{ is a TM and } \{000, 111\} \subseteq L(M)\}$.

   (c) [8 marks] $L = \{\langle M \rangle \mid M \text{ is a TM which accepts all strings of even parity}\}$. (Recall the *parity* of a string $x \in \{0, 1\}$ is the number of 1's in x.)

   (d) [8 marks] $L = \big\{\langle M \rangle \mid M \text{ is a TM that accepts } w^R \text{ whenever it accepts } w\big\}$. Recall here that w^R is the string w written in reverse, i.e. $011^R = 110$.

   (e) [8 marks] Consider the problem of determining whether a TM M on an input w ever attempts to move its head left when its head is on the left-most tape cell. Formulate this problem as a language and show that it is undecidable.