Steven Maharaj 695281 Assignment 2, Question 3 MAST90125: Bayesian Statistical Learning

Due: Friday 20 September 2019

There are places in this assignment where R code will be required. Therefore set the random seed so assignment is reproducible.

```
set.seed(695281)  #Please change random seed to your student id number.
library(dplyr)
```

```
##
## Attaching package: 'dplyr'

## The following objects are masked from 'package:stats':
##
##     filter, lag

## The following objects are masked from 'package:base':
##
##     intersect, setdiff, setequal, union
```

```
library(ggplot2)
library(tidyr)
library(TruncatedNormal)
```

Question Three (18 marks)

A group of 453 Bangladeshi women in 5 districts were asked about contraceptive use. The response variable *use* is an indicator for contraceptive use (coded N for no and Y for yes). Other covariates of interest are categorical variables for geographical location *district* (5 levels), and *urban* (2 levels), and number of living children *livch* (4 levels), and the continuous covariate for standardised age *age*.A random intercept for the district was suggested. This suggested the following model should be fitted,

$$\boldsymbol{\theta} = \mathbf{Zu} + \mathbf{X}\beta,$$

where $\boldsymbol{\theta}$ is a link function, $\mathbf{Z}$ is an indicator variable for district, $\mathbf{u}$ is a random intercept with prior $p(\mathbf{u}) = \mathcal{N}(\mathbf{0}, \sigma_u^2\mathbf{I})$, and $\mathbf{X}$ is a design matrix for fixed effects $\boldsymbol{\beta}$, where $\boldsymbol{\beta}$ includes the coefficients for the intercept, urban status, living children, and age.

Data can be downloaded from LMS as `Contraceptionsubset.csv`.

a) Fit a generalised linear mixed model assuming a logistic link using `Stan`. The R and stan code below covers the following steps.

- Importing the data.
- Constructing design matrices.
- Provides code to go into the stan file.
- Running stan in R. This assumes your stan file is called *logitmm.stan*, and that you will run the sampler for 2000 iterations and 4 chains.

Note that provided code assumes everything required is located in your working directory in R.

```r
#Step one: Importing data,  constructing design matrices and calculating matrix dimensions.
dataX= read.csv("Contraceptionsubset.csv",header=TRUE)

n<-dim(dataX)[1]
Z    = table(1:n,dataX$district)     #incidence matrix for district
Q    = dim(Z)[2]
D1   = table(1:n,dataX$livch) #Dummy indicator for living children
D2   = table(1:n,dataX$urban) #Dummy indicator for urban status

#fixed effect design matrix
X    = cbind(rep(1,n),dataX$age,D1[,-1],D2[,-1])
P    = dim(X)[2]
y    = rep(0,n)
y[dataX$use %in% 'Y'] = 1
```

An example stan file.

```stan
//
// This Stan program defines a logistic mixed model
//
// Learn more about model development with Stan at:
//
//    http://mc-stan.org/users/interfaces/rstan.html
//    https://github.com/stan-dev/rstan/wiki/RStan-Getting-Started
//

data {
  int<lower=0> n;   //number of observations
  int<lower=0> Q;   //number of random effect levels
  int<lower=0> P;   //number of fixed effect levels
  int y[n];      //response vector
  matrix[n,Q] Z;     //indicator matrix for random effect levels
  matrix[n,P] X;     //design matrix for fixed effects
}

// The parameters accepted by the model.
// accepts three sets of parameters 'beta', 'u' and 'sigma'.
parameters {
  vector[P] beta; //vector of fixed effects of length P.
  vector[Q] u; //vector of random effects of length Q.
  real<lower=0> sigma; //random effect standard deviation
}


// The model to be estimated. We model the output
// 'y' to be bernoulli with logit link function,
// and assume a i.i.d. normal prior for u.
model {
  u ~ normal(0,sigma);                //prior for random effects.
  y ~ bernoulli_logit(X*beta+ Z*u);  //likelihood

}
```

```r
library(rstan)
```

```
## Loading required package: StanHeaders

## rstan (Version 2.19.2, GitRev: 2e1f913d3ca3)

## For execution on a local, multicore CPU with excess RAM we recommend calling
## options(mc.cores = parallel::detectCores()).
## To avoid recompilation of unchanged Stan programs, we recommend calling
```

```
## rstan_options(auto_write = TRUE)
##
## Attaching package: 'rstan'
## The following object is masked from 'package:tidyr':
##
##     extract
```

```r
logistic.mm <-stan(file="logitmm.stan",data=c('Z','X','y','n','P','Q'),iter=2000,chains=4)
```

```
##
## SAMPLING FOR MODEL 'logitmm' NOW (CHAIN 1).
## Chain 1:
## Chain 1: Gradient evaluation took 0.000104 seconds
## Chain 1: 1000 transitions using 10 leapfrog steps per transition would take 1.04 seconds.
## Chain 1: Adjust your expectations accordingly!
## Chain 1:
## Chain 1:
## Chain 1: Iteration:    1 / 2000 [  0%]  (Warmup)
## Chain 1: Iteration:  200 / 2000 [ 10%]  (Warmup)
## Chain 1: Iteration:  400 / 2000 [ 20%]  (Warmup)
## Chain 1: Iteration:  600 / 2000 [ 30%]  (Warmup)
## Chain 1: Iteration:  800 / 2000 [ 40%]  (Warmup)
## Chain 1: Iteration: 1000 / 2000 [ 50%]  (Warmup)
## Chain 1: Iteration: 1001 / 2000 [ 50%]  (Sampling)
## Chain 1: Iteration: 1200 / 2000 [ 60%]  (Sampling)
## Chain 1: Iteration: 1400 / 2000 [ 70%]  (Sampling)
## Chain 1: Iteration: 1600 / 2000 [ 80%]  (Sampling)
## Chain 1: Iteration: 1800 / 2000 [ 90%]  (Sampling)
## Chain 1: Iteration: 2000 / 2000 [100%]  (Sampling)
## Chain 1:
## Chain 1:  Elapsed Time: 1.53192 seconds (Warm-up)
## Chain 1:                1.20015 seconds (Sampling)
## Chain 1:                2.73207 seconds (Total)
## Chain 1:
##
## SAMPLING FOR MODEL 'logitmm' NOW (CHAIN 2).
## Chain 2:
## Chain 2: Gradient evaluation took 4.3e-05 seconds
## Chain 2: 1000 transitions using 10 leapfrog steps per transition would take 0.43 seconds.
## Chain 2: Adjust your expectations accordingly!
## Chain 2:
## Chain 2:
## Chain 2: Iteration:    1 / 2000 [  0%]  (Warmup)
## Chain 2: Iteration:  200 / 2000 [ 10%]  (Warmup)
## Chain 2: Iteration:  400 / 2000 [ 20%]  (Warmup)
## Chain 2: Iteration:  600 / 2000 [ 30%]  (Warmup)
## Chain 2: Iteration:  800 / 2000 [ 40%]  (Warmup)
## Chain 2: Iteration: 1000 / 2000 [ 50%]  (Warmup)
## Chain 2: Iteration: 1001 / 2000 [ 50%]  (Sampling)
## Chain 2: Iteration: 1200 / 2000 [ 60%]  (Sampling)
## Chain 2: Iteration: 1400 / 2000 [ 70%]  (Sampling)
## Chain 2: Iteration: 1600 / 2000 [ 80%]  (Sampling)
## Chain 2: Iteration: 1800 / 2000 [ 90%]  (Sampling)
## Chain 2: Iteration: 2000 / 2000 [100%]  (Sampling)
```

```
## Chain 2:
## Chain 2:  Elapsed Time: 1.61333 seconds (Warm-up)
## Chain 2:                1.25627 seconds (Sampling)
## Chain 2:                2.8696 seconds (Total)
## Chain 2:
##
## SAMPLING FOR MODEL 'logitmm' NOW (CHAIN 3).
## Chain 3:
## Chain 3: Gradient evaluation took 4.9e-05 seconds
## Chain 3: 1000 transitions using 10 leapfrog steps per transition would take 0.49 seconds.
## Chain 3: Adjust your expectations accordingly!
## Chain 3:
## Chain 3:
## Chain 3: Iteration:    1 / 2000 [  0%]  (Warmup)
## Chain 3: Iteration:  200 / 2000 [ 10%]  (Warmup)
## Chain 3: Iteration:  400 / 2000 [ 20%]  (Warmup)
## Chain 3: Iteration:  600 / 2000 [ 30%]  (Warmup)
## Chain 3: Iteration:  800 / 2000 [ 40%]  (Warmup)
## Chain 3: Iteration: 1000 / 2000 [ 50%]  (Warmup)
## Chain 3: Iteration: 1001 / 2000 [ 50%]  (Sampling)
## Chain 3: Iteration: 1200 / 2000 [ 60%]  (Sampling)
## Chain 3: Iteration: 1400 / 2000 [ 70%]  (Sampling)
## Chain 3: Iteration: 1600 / 2000 [ 80%]  (Sampling)
## Chain 3: Iteration: 1800 / 2000 [ 90%]  (Sampling)
## Chain 3: Iteration: 2000 / 2000 [100%]  (Sampling)
## Chain 3:
## Chain 3:  Elapsed Time: 1.50747 seconds (Warm-up)
## Chain 3:                1.17569 seconds (Sampling)
## Chain 3:                2.68315 seconds (Total)
## Chain 3:
##
## SAMPLING FOR MODEL 'logitmm' NOW (CHAIN 4).
## Chain 4:
## Chain 4: Gradient evaluation took 4.3e-05 seconds
## Chain 4: 1000 transitions using 10 leapfrog steps per transition would take 0.43 seconds.
## Chain 4: Adjust your expectations accordingly!
## Chain 4:
## Chain 4:
## Chain 4: Iteration:    1 / 2000 [  0%]  (Warmup)
## Chain 4: Iteration:  200 / 2000 [ 10%]  (Warmup)
## Chain 4: Iteration:  400 / 2000 [ 20%]  (Warmup)
## Chain 4: Iteration:  600 / 2000 [ 30%]  (Warmup)
## Chain 4: Iteration:  800 / 2000 [ 40%]  (Warmup)
## Chain 4: Iteration: 1000 / 2000 [ 50%]  (Warmup)
## Chain 4: Iteration: 1001 / 2000 [ 50%]  (Sampling)
## Chain 4: Iteration: 1200 / 2000 [ 60%]  (Sampling)
## Chain 4: Iteration: 1400 / 2000 [ 70%]  (Sampling)
## Chain 4: Iteration: 1600 / 2000 [ 80%]  (Sampling)
## Chain 4: Iteration: 1800 / 2000 [ 90%]  (Sampling)
## Chain 4: Iteration: 2000 / 2000 [100%]  (Sampling)
## Chain 4:
## Chain 4:  Elapsed Time: 2.12798 seconds (Warm-up)
## Chain 4:                1.21135 seconds (Sampling)
## Chain 4:                3.33933 seconds (Total)
```

```
## Chain 4:
```

```r
print(logistic.mm)
```

```
## Inference for Stan model: logitmm.
## 4 chains, each with iter=2000; warmup=1000; thin=1;
## post-warmup draws per chain=1000, total post-warmup draws=4000.
##
##            mean se_mean   sd    2.5%     25%     50%     75%   97.5% n_eff
## beta[1]   -2.01    0.02 0.59   -3.21   -2.36   -2.00   -1.66   -0.81  1126
## beta[2]   -0.04    0.00 0.02   -0.08   -0.05   -0.04   -0.03   -0.01  2301
## beta[3]    1.23    0.01 0.34    0.57    1.01    1.23    1.46    1.89  2432
## beta[4]    1.45    0.01 0.36    0.74    1.20    1.44    1.69    2.15  2234
## beta[5]    1.79    0.01 0.38    1.07    1.54    1.78    2.04    2.54  1610
## beta[6]    1.22    0.00 0.26    0.72    1.04    1.22    1.39    1.74  2732
## u[1]      -1.06    0.02 0.56   -2.27   -1.37   -1.04   -0.72   -0.06  1082
## u[2]      -0.25    0.02 0.57   -1.41   -0.57   -0.26    0.08    0.83  1239
## u[3]       0.43    0.02 0.56   -0.76    0.12    0.43    0.74    1.50  1055
## u[4]       0.18    0.02 0.56   -0.95   -0.13    0.18    0.51    1.28  1078
## u[5]       0.69    0.02 0.56   -0.44    0.37    0.67    1.00    1.82  1197
## sigma      1.10    0.02 0.70    0.41    0.70    0.93    1.27    2.86  1480
## lp__    -274.34    0.07 2.51 -280.05 -275.89 -274.04 -272.49 -270.41  1246
##         Rhat
## beta[1]    1
## beta[2]    1
## beta[3]    1
## beta[4]    1
## beta[5]    1
## beta[6]    1
## u[1]       1
## u[2]       1
## u[3]       1
## u[4]       1
## u[5]       1
## sigma      1
## lp__       1
##
## Samples were drawn using NUTS(diag_e) at Wed Sep 25 12:08:50 2019.
## For each parameter, n_eff is a crude measure of effective sample size,
## and Rhat is the potential scale reduction factor on split chains (at
## convergence, Rhat=1).
```

Note that in Stan, defaults for burn-in (warm-up) is one half of all iterations in stan, and no thinning. Note the code is written using the stan file and csv is in your working directory. Use the `print` function to report posterior means, standard deviations, 95 % central credible intervals and state from the output whether you believe the chains have converged. Also report the reference categories for *urban* and *livch*.

Reporting for PART A:

posterior means, standard deviations can be found in the above table. The lower limit of the 95 % central credible intervals given by the column labeled "2.5%" while the upperer limit of the 95 % central credible intervals given by the column labeled "97.5%".

```r
print(summary(logistic.mm)$summary)
```

```
##                 mean      se_mean         sd        2.5%           25%
## beta[1]   -2.01030050 0.0177090209 0.59412704   -3.2126422   -2.35819525
```

```
## beta[2]    -0.04214017 0.0003647766 0.01749937    -0.0771671    -0.05338352
## beta[3]     1.23207517 0.0068231905 0.33646783     0.5659346     1.01390920
## beta[4]     1.44766009 0.0076461800 0.36137498     0.7375624     1.20459896
## beta[5]     1.78929396 0.0093728704 0.37613535     1.0743897     1.53649493
## beta[6]     1.21639026 0.0049852543 0.26059271     0.7185883     1.03977246
## u[1]       -1.06275576 0.0171555132 0.56436421    -2.2699659    -1.37142413
## u[2]       -0.25489581 0.0161806806 0.56952451    -1.4074344    -0.57137738
## u[3]        0.42541069 0.0173308629 0.56299322    -0.7568421     0.12006699
## u[4]        0.18475245 0.0169065649 0.55506518    -0.9452371    -0.12563814
## u[5]        0.69203489 0.0160977691 0.55701986    -0.4360102     0.36515485
## sigma       1.10279222 0.0181485968 0.69807840     0.4141299     0.69594181
## lp__     -274.33957141 0.0710899545 2.50946526 -280.0506863 -275.89450998
##                   50%           75%         97.5%    n_eff      Rhat
## beta[1]    -2.00071542   -1.66048095 -8.137643e-01 1125.562 1.0028517
## beta[2]    -0.04190229   -0.03039232 -9.684808e-03 2301.393 1.0008164
## beta[3]     1.22785036    1.45582991  1.893473e+00 2431.712 1.0003286
## beta[4]     1.44472312    1.69041346  2.151146e+00 2233.712 1.0022645
## beta[5]     1.77969771    2.03909874  2.538047e+00 1610.435 1.0021622
## beta[6]     1.21564004    1.38717360  1.744403e+00 2732.435 1.0020165
## u[1]       -1.03749524   -0.72160839 -6.331785e-02 1082.210 1.0035870
## u[2]       -0.25724419    0.07818563  8.251233e-01 1238.886 1.0028778
## u[3]        0.43126008    0.74223850  1.496745e+00 1055.276 1.0032488
## u[4]        0.17671158    0.50561014  1.278456e+00 1077.897 1.0029502
## u[5]        0.66847636    1.00332512  1.815226e+00 1197.319 1.0031387
## sigma       0.93149252    1.27266697  2.858592e+00 1479.525 1.0007791
## lp__     -274.04470478 -272.48530390 -2.704118e+02 1246.080 0.9999782
```

Using the Gelman-Rubin diagnostic (Rhat) only 6 out of the 12 parameters These were beta[2],beta[3],beta[4],beta[5],beta[6],sigm had an Rhat value close to 1. All u parameters and beta[1] do not appear to converge thus the chain did not converge.

For *urban* and *livch* the reference categories are "N" and "0" respectivley. This is infered from the way the DataFrame X was constructed. `X = cbind(rep(1,n),dataX$age,D1[,-1],D2[,-1])`

b) An alternative to the logit link when analysing binary data is the probit. The probit link is defined as,

$$\begin{aligned} y_i &= \begin{cases} 1 & \text{if } z_i \geq 0 \\ 0 & \text{if } z_i < 0 \end{cases} \\ z_i &= \mathbf{x}_i'\boldsymbol{\beta} + \epsilon_i, \quad \epsilon \sim \mathcal{N}(0,1). \end{aligned}$$

In lecture 14, we showed how by letting z_i be normal, probit regression can be fitted using a Gibbs sampler, but to do so, it requires the ability to sample from a truncated normal defined on either $(-\infty, 0)$ (if $y_i = 0$) or $(0, \infty)$ (if $y_i = 1$). Check by comparing the empirical and the true density that a modified version of the inverse cdf method can be used to produce draws from a truncated normal. Do this for the case where $x \in (0, \infty)$ and $x \in (-\infty, 0)$ with parameters $\mu = 0.5$ and $\sigma = 1$.

Hints: If y is drawn from a truncated normal with lower bound a, upper bound b and parameters μ, σ^2 then then $p(y|\mu, \sigma^2, a, b)$ is

$$\frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2}}{\int_{-\infty}^{b} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2} dy - \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2} dy},$$

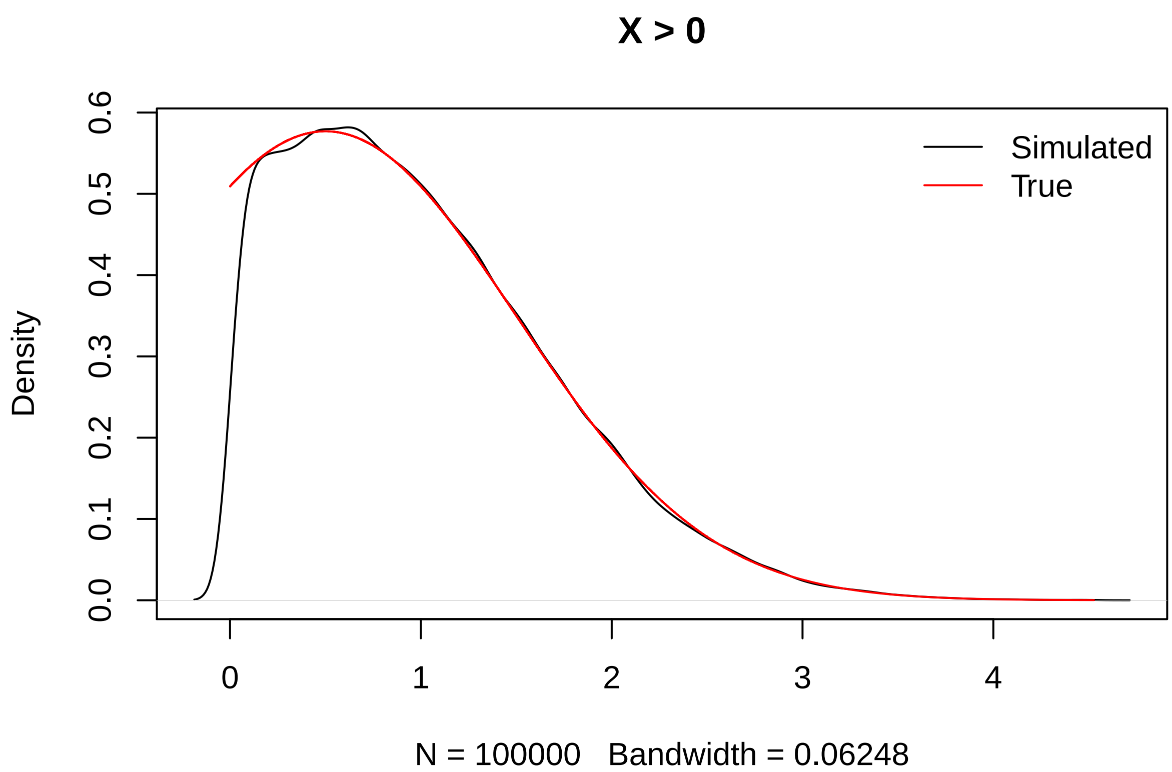
which in `R` means the truncated normal density can be written as `dnorm(x,mean=mu,sd=sigma)/(pnorm(b,mean=mu,sd=sigma)-pnorm(a,mean=mu,sd=sigma))`

The inverse cdf method involves drawing v from $U(0,1)$ so that $x \sim p(x)$ can be found solving $x = F^{-1}(x)$, where F is the cdf. If the only change compared to drawing from a normal distribution is truncation, think about what happens to the bounds of the uniform distribution.

Answer: Part B

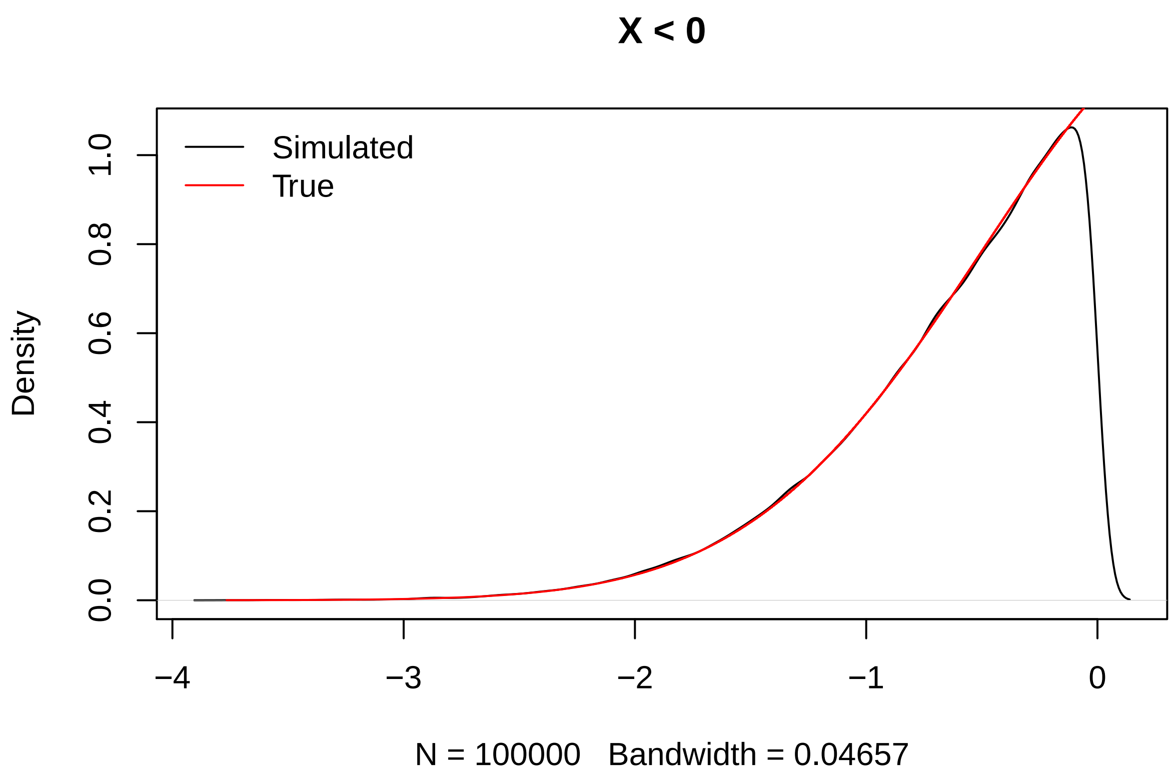
Case $x \in (0, \infty)$

```r
rtn <- function(n,b,a,mu,Sigma){
  u <- runif(n)
  g <- pnorm((b-mu)/Sigma) - pnorm((a-mu)/Sigma)
  x <-qnorm((g) * u + pnorm((a-mu)/Sigma))*Sigma + mu
}



x<- rtn(n=100000,b=100,a=0,mu = 0.5,Sigma = 1)

mu = 0.5
Sigma = 1
b=100
a=0


x<-sort(x)
true_den <- dnorm((x-mu)/Sigma)/(pnorm((b-mu)/Sigma) - pnorm((a-mu)/Sigma))

plot(density(x),col = 1,main="X > 0")
lines(x,true_den,col = 2,type = "l")
legend('topright',legend=c('Simulated','True'),col=1:2,lty=1,bty='n')
```



Case $x \in (-\infty, 0)$

```
x<- rtn(n=100000,b=0,a=-100,mu = 0.5,Sigma = 1)

mu = 0.5
Sigma = 1
b=0
a=-100


x<-sort(x)
true_den <- dnorm((x-mu)/Sigma)/(pnorm((b-mu)/Sigma) - pnorm((a-mu)/Sigma))

plot(density(x),col = 1,main="X < 0")
lines(x,true_den,col = 2,type = "l")
legend('topleft',legend=c('Simulated','True'),col=1:2,lty=1,bty='n')
```



c) Implement a Gibbs sampler to fit the same mixed model as fitted in Stan in a), but now with a probit link. As before, fit 4 chains, each running for 2000 iterations, with the first 1000 iterations discarded as burn-in. Perform graphical convergence checks and Gelman-Rubin diagnostics. Report posterior means, standard deviations and 95 % central credible intervals for $\sigma, \boldsymbol{\beta}, \mathbf{u}$ by combining chains.

d) For the co-efficients $\boldsymbol{\beta}$, $\mathbf{u}$, calculate the mean of the ratio of the posterior means $\boldsymbol{\beta}_{i,\text{logit}}/\boldsymbol{\beta}_{i,\text{probit}}, \mathbf{u}_{i,\text{logit}}/\mathbf{u}_{i,\text{probit}}$ obtained when fitting the logistic mixed model and the probit mixed model. To do this, you will need to apply the `extract` function to the stan model object. Once calculated, multiply the iterations obtained assuming a probit link by this constant and compare to the iterations obtained assuming a logit link.

e) The logistic link can be written in the same way as the probit link, but instead of $e_i \sim \mathcal{N}(0, 1)$, the error term is $e_i \sim \text{Logistic}(0, 1)$. By evaluating the standard normal and logistic inverse cdfs and superimposing the line $y = mx$ where m is the posterior ratio, do you think the results in d) were surprising.