Active Calculus

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Chapter 1

Multivariable and Vector Functions

1.1 Functions of Several Variables and Three Dimensional Space

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is the difference between a left-hand system and a right-hand system? Why is there a difference?
- What is a function of several variables?
- What do we mean by the domain of a function of several variables?
- How do we find the distance between two points in $\mathbb{R}^3$?
- What is the equation of a sphere in $\mathbb{R}^3$?
- What is a trace of a function of two variables? What does a trace tell us about a function?
- What is a level curve of a function of two variables? What does a level curve tell us about a function?

Introduction

Throughout our mathematical careers we have studied functions of a single variable. We defined a function of one variable as a rule that assigns exactly one output to each input. We analyze these functions by looking at their graphs, calculating limits, differentiating, and integrating. In this and subsequent sections, we study functions whose input is defined in terms of more than one variable, and then analyze these functions by looking at their graphs, calculating limits, differentiating, and integrating. We will see that many of the ideas from single variable calculus translate well to functions of several variables, but we will have to make some adjustments as well.

Preview Activity 1.1. When most people buy a large ticket item like a car or a house, they have to take out a loan to make the purchase. The loan is paid back in monthly installments until the entire amount of the loan, plus interest, is paid. The monthly payment that the borrower has to make depends on the amount P of money borrowed (called the principal), the duration t of the loan in years, and the interest rate r. For example, if we borrow $18,000 to buy a car, the monthly payment M that we need to make to pay off the loan is given by the formula

$$M = \frac{1500r}{1 - \frac{1}{\left(1+\frac{r}{12}\right)^{12t}}}.$$

The variables r and t are independent of each other, so using functional notation we write

$$M(r,t) = \frac{1500r}{1 - \frac{1}{\left(1+\frac{r}{12}\right)^{12t}}}.$$

(a) Find the monthly payments on this loan if the interest rate is 6% and the duration of the loan is 5 years.

(b) Evaluate $M(0.05, 4)$ Explain in words what this calculation represents.

(c) Now consider only loans where the interest rate is 5%. Calculate the monthly payments as indicated in Table 1.1. Round payments to the nearest penny.

Duration (in years)	2	3	4	5	6
Monthly payments (dollars)					

Table 1.1: Monthly payments at an interest rate of 5%.

(d) Now consider only loans where the duration is 3 years. Calculate the monthly payments as indicated in Table 1.2. Round payments to the nearest penny.

Interest rate	0.03	0.05	0.07	0.09	0.11
Monthly payments (dollars)					

Table 1.2: Monthly payments over three years.

(e) Describe as best you can the combination of interest rates and durations of loans that result in a monthly payment of $200.

⋈

Functions of Several Variables

Suppose we launch a projectile, using a golf club, a cannon, or some other device, from ground level. Under ideal conditions (ignoring wind resistance, spin, or any other forces except the force of gravity) the horizontal distance the object will travel depends on the initial velocity x the object is given, and the angle y at which it is launched. If we let f represent the horizontal distance the object travels (its range), then f is a function of the two variables x and y, and we represent f in functional notation by

$$f(x, y) = \frac{x^2 \sin(2y)}{g},$$

where g is the acceleration due to gravity.[1]

Definition 1.1. A **function** f **of two independent variables** is a rule that assigns to each ordered pair (x, y) in some set D exactly one real number $f(x, y)$.

There is, of course, no reason to restrict ourselves to functions of only two variables - we can use any number of variables we like. For example,

$$f(x, y, z) = x^2 - 2xz + \cos(y)$$

defines f as a function of the three variables x, y, and z. In general, a function of n independent variables is a rule that assigns to an ordered n-tuple $x_1, x_2, \ldots, x_n)$ in some set D exactly one real number.

As with functions of a single variable, it is important to understand the set of inputs for which the function is defined.

Definition 1.2. The **domain** of a function f is the set of inputs at which the function is defined.

Activity 1.1.

Identify the domains of the following functions. Draw a picture of each domain in the plane.

(a) $f(x, y) = x^2 + y^2$

(b) $f(x, y) = \sqrt{x^2 + y^2}$

(c) $Q(x, y) = \frac{x+y}{x^2-y^2}$

(d) $s(x, y) = \frac{1}{\sqrt{1-xy^2}}$

◁

[1] We will derive this equation later in the course.

Graphs of Functions of Two Variables

One of the techniques we used in plotting graphs of functions of one variable was to create a table of values. We can do the same for functions of two variables, except that our tables will have to allow us to keep track of both input variables. We can do this with a 2-dimensional table, where we list the x-values down the first column and the y-values along the first row. Let f be the function defined by $f(x,y) = \frac{x^2 \sin(2y)}{g}$ that gives the range of a projectile as a function of the initial velocity x and launch angle y of the projectile. The value $f(x,y)$ is then displayed in the spot where the x row intersects the y column, as shown in Table 1.3 (where we measure x in feet and y in radians).

	$\frac{2\pi}{20}$	$\frac{3\pi}{20}$	$\frac{4\pi}{20}$	$\frac{5\pi}{20}$	$\frac{6\pi}{20}$	$\frac{7\pi}{20}$	$\frac{8\pi}{20}$	$\frac{9\pi}{20}$
25	11.480	15.801	18.575	19.531	18.575	15.801	11.480	6.0356
50	45.921	63.205	74.301	78.125	74.301	63.205	45.921	24.142
75	103.322	142.210	167.178	175.781	167.178	142.210	103.322	54.319
100	183.683	252.818	297.205	312.500	297.205	252.818	183.683	96.568
125	287.005	395.028	464.383	488.281	464.383	395.028	287.005	150.887
150	413.287	568.840	668.712	703.125	668.712	568.840	413.287	217.278
175	562.529	774.255	910.191	957.031	910.191	774.255	562.529	295.739
200	734.732	1011.271	1188.821	1250.000	1188.821	1011.271	734.732	386.271
225	929.895	1279.890	1504.601	1582.031	1504.601	1279.890	929.895	488.875
250								

Table 1.3: Values of $f(x,y) = \frac{x^2 \sin(2y)}{g}$.

Activity 1.2.

Complete the last row of Table **??**.

$\triangleleft$

If f is a function of a single variable x, then we defined the graph of f to be the set of points of the form $(x, f(x))$, where x is in the domain of f. We then plotted these points using the coordinate axes in order to visualize the graph. We can do a similar thing with functions of several variables. Table 1.3 identifies points of the form $(x, y, f(x,y))$, and we define the graph of f to be the set of these points.

Definition 1.3. The **graph** of a function $f = f(x,y)$ is the set of points of the form $(x, y, f(x,y))$, where the point (x,y) is in the domain of f.

Points in the form $(x, y, f(x,y))$ are in three dimensions, so plotting these points takes a bit more work than graphs of functions in two dimensions. To plot these three-dimensional points, we need to set up a coordinate system with three mutually perpendicular axes – the x-axis, the

y-axis, and the z-axis (called the *coordinate axes*. There are essentially two different ways we could set up a 3D coordinate system – see Figures 1.1 and 1.2, so before we can proceed, we need to establish a convention. The world is biased toward right handed people, so we will adopt a right-

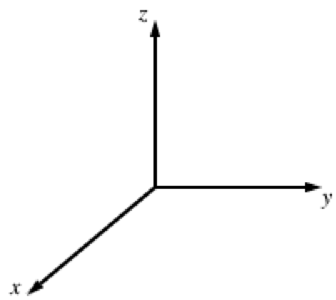

Figure 1.1: A right hand system

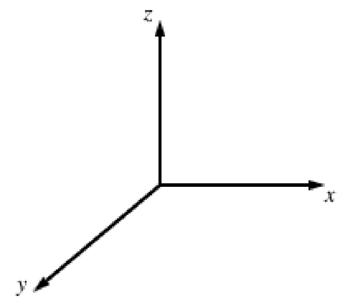

Figure 1.2: A left hand system

hand system as illustrated in Figure 1.1. To see the difference, think about a socket wrench – a tool for driving screws. It consists of a handle (the wrench) and a socket. The socket is placed on the screw and attached to the wrench. As the handle of the wrench is turned, the socket provides a force that drives the screw. If you point the index finger of your right hand in the direction of the force you apply to the handle of the wrench, and point your middle finger in the direction of the handle toward the socket, then your thumb will point in the direction in which the wrench is driving the screw. If you use your left hand instead, then your thumb will point in the opposite direction. So the socket wrench is set up to drive right-handed screws. This is exactly the 3D coordinate system we want to adopt. In a right hand system, if we point the index finger of your right hand in the direction of the positive x-axis and your middle finger in the direction of the positive y-axis, then your thumb will point in the direction of the positive z-axis. A left hand system and a right hand system have different orientations – we need pick one as a standard so that the convention of orientation is understood by everyone. We will always use a right-hand system.

Now that we have established a convention for a right hand system, we can draw a graph of our range function. Note that the function f is continuous in both variables, so when we plot these points in the right hand coordinate system, we can connect them all up to form a surface in 3-space. The graph of our range function f is shown in Figure 1.3. There are many graphing tools available for drawing three-dimensional surfaces.[2] Since we will be able to visualize graphs of functions of two variables, but not functions of more than two variables, we will primarily deal with functions of two variables in this course. It is important to note, however, that the techniques we develop apply to functions of any number of variables. We will work extensively with functions of several variables in subsequent sections.

Notation: We denote the set of all ordered pairs of real numbers in the plane as $\mathbb{R}^2$ (two copies of the real number system) and the set of all ordered triples of real numbers (making up three-space)

[2] e.g., Wolfram Alpha and http://web.monroecc.edu/manila/webfiles/calcNSF/JavaCode/CalcPlot3D.htm

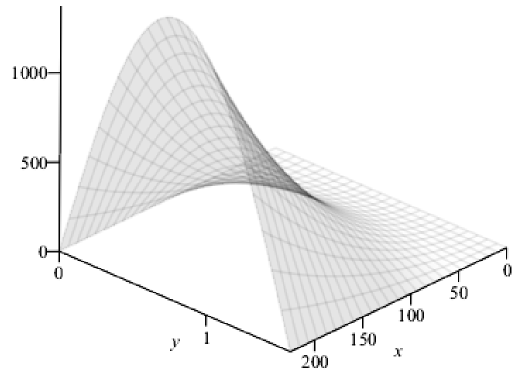

Figure 1.3: The range surface.

as $\mathbb{R}^3$.

Graphs of Some Standard Equations in Three-Space

In addition to graphing functions, we will also want to understand graphs of some simple equations in three dimensions. For example, in two dimensions, the graphs of the equations $x = a$ and $y = b$, where a and b are constants, are lines parallel to the coordinate axes. In the next activity we consider the three-dimensional analogs.

Activity 1.3.

(a) Consider the set of points (x, y, z) that satisfy the equation $x = 2$. Describe this set as best you can.

(b) Consider the set of points (x, y, z) that satisfy the equation $y = -1$. Describe this set as best you can.

(c) Consider the set of points (x, y, z) that satisfy the equation $z = 0$. Describe this set as best you can.

◁

Activity 1.3 shows that the equations where one independent variable is constant lead to planes parallel to one made by a pair of the coordinate axes. When we make the constant 0, we get the *coordinate planes*. The xy-plane satisfies $z = 0$, the xz-plane satisfies $y = 0$, and the yz-plane satisfies $z = 0$ (see Figure 1.4).

On a similar note, we defined a circle in two-dimensional space as the set of all points equidistant from a fixed point. In three-dimensional space we call the set of all points equidistant from a fixed point a *sphere*. To find the equation of a sphere, we need to understand how to calculate the

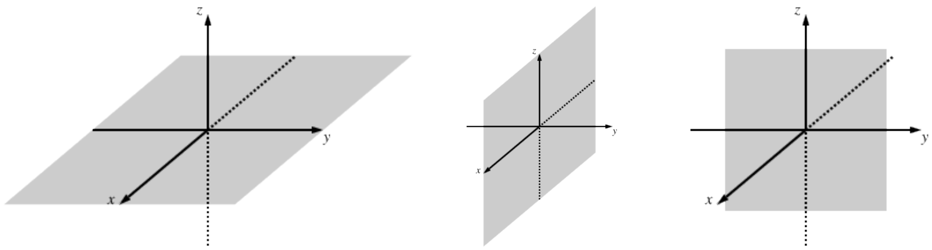

Figure 1.4: The coordinate planes.

distance between two points in three-space.

Activity 1.4.

Let $P = (x_0, y_0, z_0)$ and $Q = (x_1, y_1, z_1)$ be two points in $\mathbb{R}^3$. These two points form opposite vertices of a rectangle whose sides are planes parallel to the coordinate planes as illustrated in Figure 1.5, and the distance between P and Q is the length of the diagonal shown in Figure 1.5.

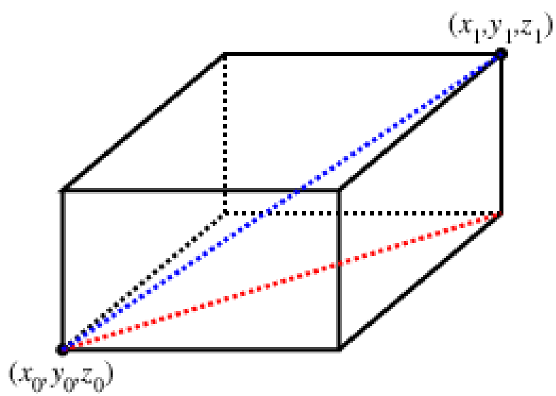

Figure 1.5: The distance formula in $\mathbb{R}^3$.

(a) Consider the right triangle in the base of the rectangle whose hypotenuse is shown as a dashed line in Figure 1.5. What are the vertices of this triangle? Since this right triangle lies in a plane, we can use the Pythagorean Theorem to find a formula for the length of the hypotenuse of this triangle. Find such a formula, which will be in terms of x_0, y_0, x_1, and y_1.

(b) Now notice that the triangle whose hypotenuse is the segment connecting the points P and Q with a leg as the hypotenuse of the triangle found in part (a) lies entirely in a plane, so we can again use the Pythagorean Theorem to find the length of its hypotenuse. Explain why the length of this hypotenuse is

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.$$

This is the formula for the distance between the points P and Q.

$\triangleleft$

The formula in Activity 1.4 is one you should remember.

The distance between points $P = (x_0, y_0, z_0)$ and $Q = (x_1, y_1, z_1)$ (denoted as $|PQ|$) in $\mathbb{R}^3$ is given by the formula

$$|PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.$$

Equation (2) can be used to derive the formula for a sphere, centered at a point (x_0, y_0, z_0) with radius r. Since the distance from any point (x, y, z) on such a sphere to the point (x_0, y_0, z_0) is r, the point (x, y, z) will satisfy the equation

$$\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = r$$

or

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

This is a result that you should remember.

The equation of a sphere with center (x_0, y_0, z_0) and radius r is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

Activity 1.5.

Find the equation of the sphere centered at the point $(2, 1, 3)$ if the point $(-1, 0, -1)$ lies on the sphere.

$\triangleleft$

Traces

When we study functions of several variables we are often interested in how each individual variable affects the function in and of itself. In Preview Activity 1.1 we saw that the monthly payment on an $18,000 loan depends on the interest rate and the duration of the loan. However, if we fix the interest rate, the monthly payment depends only on the duration of the loan, and if we set the duration the payment depends only on the interest rate. This idea of keeping one variable constant while we allow the other to change will be an important tool for us when studying functions of several variables.

As another example, consider the range function f defined by

$$f(x, y) = \frac{x^2 \sin(2y)}{g}$$

where x is the initial velocity of an object in feet per second, y is the launch angle in radians, and g is the acceleration due to gravity (32 feet per second squared). If we hold the launch angle constant at $\frac{2\pi}{5}$, we can consider f as a function of the initial velocity alone. In this case we have

$$f(x) = \frac{x^2}{32} \sin\left(\frac{4\pi}{5}\right).$$

We can plot this curve on the surface by tracing out the points on the surface when $y = \frac{2\pi}{5}$, as shown in Figure 1.6. The graph and the formula clearly show that f is quadratic in the x-direction. More descriptively, as we increase the launch velocity while keeping the launch angle constant, the range increases as the square of the initial velocity. Similarly, if we fix the initial velocity at 150 feet per second, we can consider the range as a function of the launch angle only. In this case we have

$$f(y) = \frac{150^2 \sin(2y)}{32}.$$

We can again plot this curve on the surface by tracing out the points on the surface when $x = 150$, as shown in Figure 1.7. The graph and the formula clearly show that f is sinusoidal in the y-direction. More descriptively, as we increase the launch angle while keeping the initial velocity constant, the range is proportional to the sine of twice the launch angle.

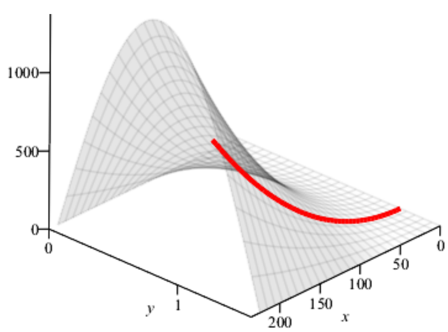

Figure 1.6: The trace with $y = \frac{2\pi}{5}$.

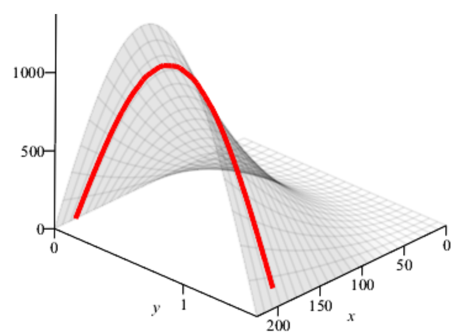

Figure 1.7: The trace with $x = 150$.

The curves we define when we fixed one of the independent variables in our two variable function are called *traces*.

Definition 1.4. A **trace** of a function f of two independent variables x and y is a curve of the form $z = f(c, y)$ or $z = f(x, c)$, where c is a constant.

Activity 1.6.

We can visualize these traces in a table of values as well.

(a) Identify the $y = \frac{2\pi}{5}$ trace for our range function by highlighting the appropriate cells in Table 1.3.

(b) Identify the $x = 150$ trace for our range function by highlighting the appropriate cells in Table 1.3.

$\triangleleft$

Identifying and drawing the graphs of functions of two variables can be difficult, and traces can help us with this task.

Activity 1.7.

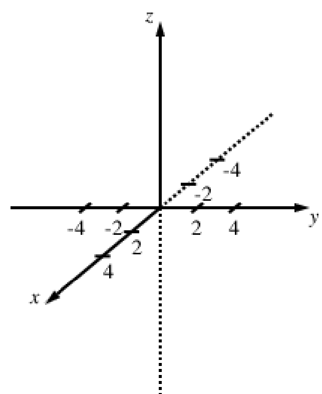

Figure 1.8: Coordinate axes to sketch traces.

(a) Identify the traces in the x direction (keeping y constant). Draw the $y = -4$, $y = -2$, $y = 0$, $y = 2$, and $y = 4$ traces in 3-dimensional coordinate system in Figure 1.8.

(b) Identify the traces in the y direction (keeping x constant). Draw the $x = -4$, $x = -2$, $x = 0$, $x = 2$, and $x = 4$ traces in 3-dimensional coordinate system in Figure 1.8.

(c) Describe the surface defined by the function f.

$\triangleleft$

Contour Maps and Level Curves

We have all seen topographic maps like the one of the Porcupine Mountains in the upper peninsula of Michigan as shown in Figure 1.9.[3] The curves on these maps show the regions of constant altitude. The contours also depict changes in altitude – contours that are close together signify steep ascents or descents. Hikers can use these maps to decide on trials to hike – easy ones to

[3]Michigan Department of Natural Resources, https://www.michigan.gov/dnr/0,4570,7-153-10369_46675_58093---,00.html

challenging ones. So contour maps tell us a lot about three-dimensional surfaces. Mathematically, if $f(x, y)$ represents the altitude at the point (x, y), then each contour is the graph of an equation of the form $f(x, y) = k$, for some constant k.

Activity 1.8.

On the topographical map of the Porcupine Mountains in Figure 1.9,

(a) identify the highest and lowest points you can find;

(b) determine a path of steepest ascent that leads to the highest point;

(c) determine the least steep path that leads to the highest point.

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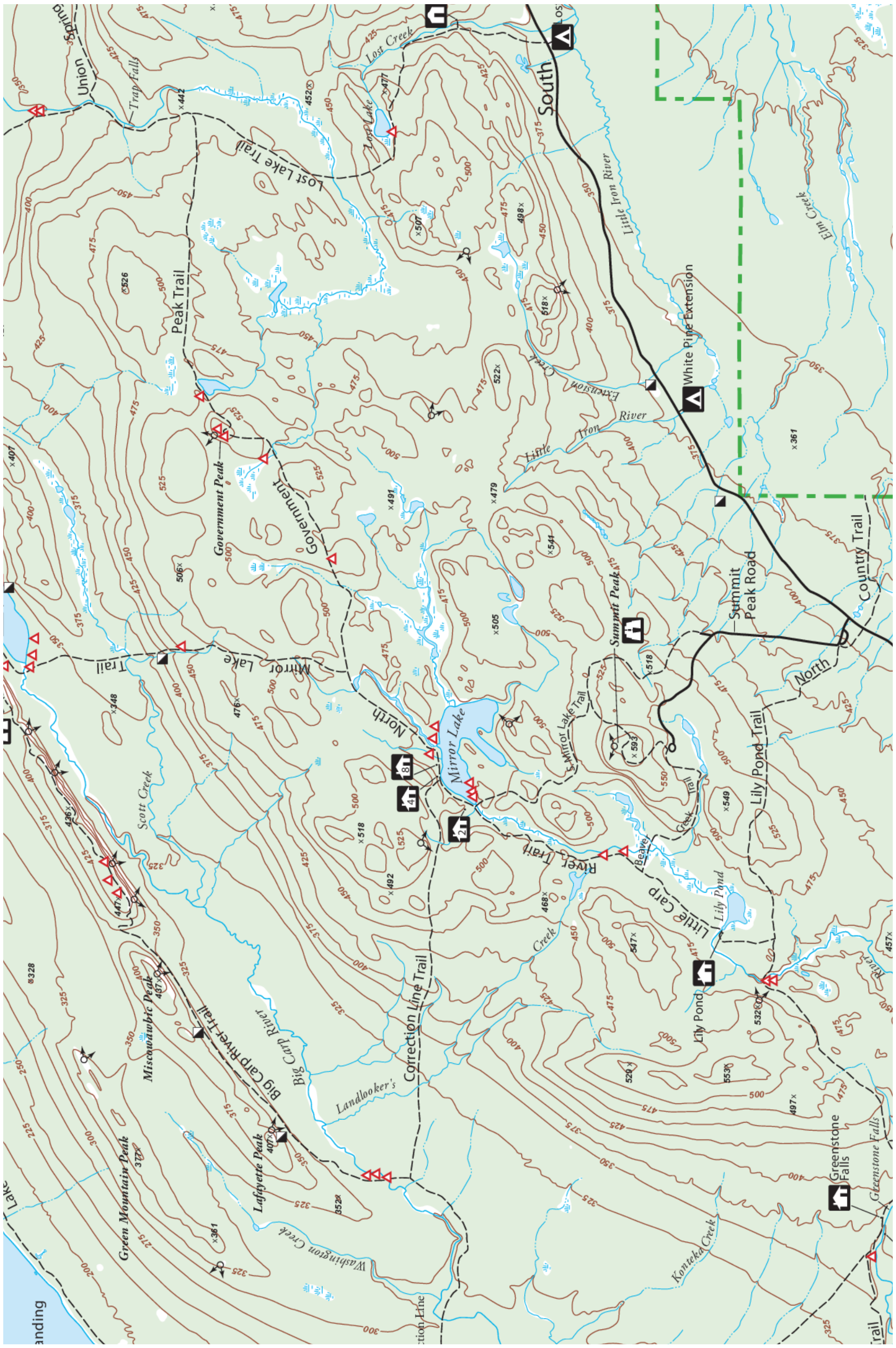

Figure 1.9: Contour map of the Porcupine Mountains.

Definition 1.5. A **level curve (or contour)** of a function f of two independent variables x and y is a curve of the form $k = f(x, y)$, where k is a constant.

These topographical maps can be used to create a three-dimensional surface from the two-dimensional contours or level curves. For example, level curves of our range function $f(x, y) = \frac{x^2 \sin(2y)}{32}$ when plotted in the xy-plane are shown in Figure 1.10. If we plot these contours at their respective heights, then we get a picture of the surface, as illustrated in Figure 1.11.

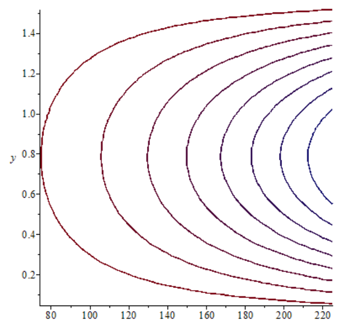

Figure 1.10: Several level curves.

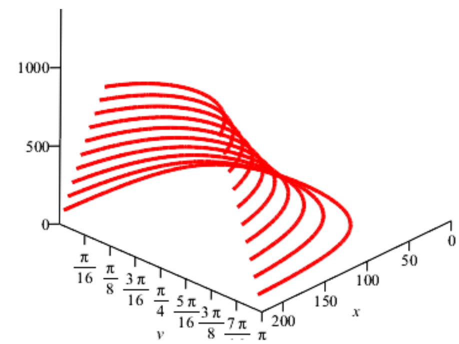

Figure 1.11: Level curves at the appropriate height.

The use of level curves and traces can help us construct the graph of a function of two variables.

Activity 1.9.

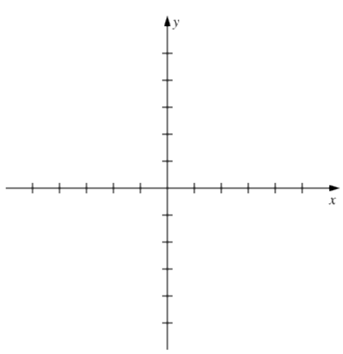

Figure 1.12: Level curves for $f(x, y) = x^2 + y^2$.

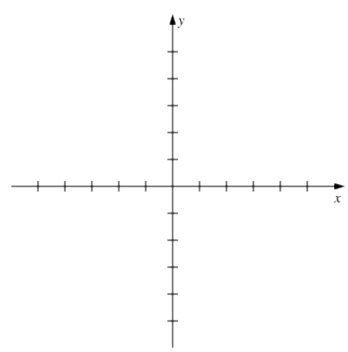

Figure 1.13: Level curves for $g(x, y) = \sqrt{x^2 + y^2}$.

(a) Let $f(x, y) = x^2 + y^2$. Draw the level curves $f(x, y) = k$ for $k = 1$, $k = 2$, $k = 3$, and $k = 4$ on the axes given in Figure 1.12. (You decide on the scaling of the axes.) Explain what the surface defined by f looks like.

(b) Let $g(x,y) = \sqrt{x^2+y^2}$. Draw the level curves $g(x,y) = k$ for $k = 1$, $k = 2$, $k = 3$, and $k = 4$ on the axes given in Figure 1.13. (You decide on the scaling of the axes.) Explain what the surface defined by g looks like.

(c) Compare and contrast the graphs of f and g. How are they alike? How are they different? Use traces for each function to help answer these questions.

◁

Activity 1.10.

The Ideal Gas Law $PV = RT$ relates the pressure (P, in pascals), temperature (T, in Kelvin), and volume (V, in cubic meters) of 1 mole of a gas ($R = 8.314 \, \frac{\text{J}}{\text{mol K}}$ is the universal gas constant), and describes the behavior of gases that do not liquefy easily, such as oxygen and hydrogen. We can solve the ideal gas law for the volume and treat the volume as a function of the pressure and temperature:

$$V(P,T) = \frac{8.314T}{P}.$$

(a) Explain in detail what the trace of V with $P = 1000$ tells us.

(b) Explain in detail what the trace of V with $T = 5$ tells us.

(c) Explain in detail what the level curve $V = 0.5$ tells us.

◁

The traces and level curves of a function of two variables are curves in space. In order to understand these traces and level curves better, we will focus on vectors and vector-valued functions in the next few sections and return to our study of functions of several variables once we have those tools in our mathematical toolbox.

Summary

In this section, we encountered the following important ideas:

- A three-dimensional system has two possible orientations. In a right hand system, if we point the index finger of our right hand in the direction of the positive x-axis and our middle finger in the direction of the positive y-axis, then our thumb will point in the direction of the positive z-axis. A left hand system has a different orientation and we need pick one as a standard so that the convention of orientation is understood by everyone.
- A function f of several variables is a rule that assigns a unique number to a collection of independent inputs.
- The domain of a function of several variables is the set of all inputs for which the function is defined.

- The distance between points $P = (x_0, y_0, z_0)$ and $Q = (x_1, y_1, z_1)$ (denoted as $|PQ|$) in $\mathbb{R}^3$ is given by the formula

$$|PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.$$

- The equation of a sphere with center (x_0, y_0, z_0) and radius r is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

- A trace of a function f of two independent variables x and y is a curve of the form $z = f(c, y)$ or $z = f(x, c)$, where c is a constant. A trace tells us how the function depends on a single independent variable if we treat the other independent variable as a constant.

- A level curve of a function f of two independent variables x and y is a curve of the form $k = f(x, y)$, where k is a constant. A level curve describes the set of inputs that lead to a specific output of the function.

1.2 Vectors

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a vector?
- What does it mean for two vectors to be equal?
- How do we add two vectors together?
- How do we multiply a vector by a scalar?
- How do we determine the magnitude of a vector?
- What is a unit vector? How do we find a unit vector in the direction of a given vector?

Introduction

If we are at a point on the graph of a function of one variable, there are only two directions we can move – the positive or negative x-direction. If we are at a point on a surface, we can move in any direction. So it will be important for us to have a way to indicate direction, and we will do this with vectors.

Preview Activity 1.2. Bilbo and Frodo are walking in the Shire. See the Shire map in Figure 1.14.[4] As Bilbo and Frodo walk, record their movements in a pair $\langle x, y\rangle$ (we will call this pair a *vector*), where x is the horizontal distance (in miles) they move (with east as the positive direction) and y as the vertical distance (in miles) they move (with north as the positive direction). Assume there are bridges and other devices along the paths Bilbo and Frodo take so that they can make the trips indicated. Use the legend to approximate the following as best you can.

(a) What is the vector $\mathbf{v}_1 = \langle x, y\rangle$ that describes Bilbo and Frodo's movement if they walk from Hobbiton to Little Delving? Explain how you found this vector. How many miles did they walk? Measure the number of miles walked directly and then explain how to calculate the number of miles walked in terms of x and y.

(b) What is the vector $\mathbf{v}_2 = \langle x, y\rangle$ that describes Bilbo and Frodo's movement if they walk from Hobbiton to Quarry? How many miles did they walk, in terms of x and y?

(c) What is the vector $\mathbf{v}_3 = \langle x, y\rangle$ that describes Bilbo and Frodo's movement if they walk from Little Delving to Quarry? What relationship do you see between the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$? Explain why this relationship should hold.

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[4] http://www.shirepost.com/ShireMapLarge.html

Figure 1.14: Shire map.

Representations of Vectors

Preview Activity 1.2 shows how we can record directions and length using ordered pairs of numbers. There are many other types of objects that possess these two attributes of length and direction, e.g., forces and velocity, and we will study them mathematically by defining the concept of *vector*. Vectors are algebraic objects that are possess the attributes of magnitude (length or norm) and direction. Our definition of a vector is the following.

Definition 1.6. A **vector** is any object that has a magnitude and direction.

We can represent a vector geometrically as a directed line segment, with the magnitude as the length of the segment and an arrowhead indicating direction, as shown in Figure 1.15.

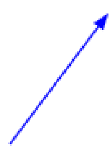

Figure 1.15: A vector

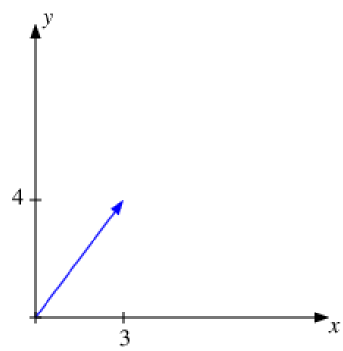

Figure 1.16: A vector at the origin

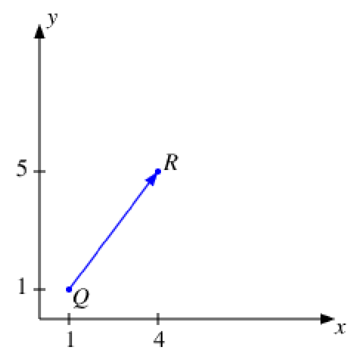

Figure 1.17: Vector between two points

According to our definition, a vector possesses the attributes of length (magnitude) and direction, but position is not mentioned. So any two vectors that have the same magnitude and direction are equal.

Two vectors are equal if the have the same magnitude and direction.

In other words, vectors are position independent. That means that we can position our vector in the plane and identify it in different ways. For one, we can place the tip of our vector (the point from which the vector originates) at the origin and the tail (the terminal point of the vector) will wind up at some point (x, y), as illustrated in Figure 1.16 with the vector that has a horizontal displacement of 3 units and a vertical displacement of 4 units. A vector like this with its initial point at the origin is said to be in *standard position*. We can then just represent the vector as the point at the tip. The fact that this vector is represented by the directed line segment from the origin to the point (3,4) is indicated by using the notation $\langle 3, 4\rangle$. If O is the origin and P is the point $(3, 4)$, we will also denote this directed line segment as $\overrightarrow{OP}$ – so

$$\overrightarrow{OP} = \langle 3, 4\rangle.$$

It is also standard to use boldface letters to represent vectors, like $\mathbf{v} = \langle 3, 4\rangle$ to distinguish them from scalars. The entries of the vector are called its *components* – so in the vector $\langle 3, 4\rangle$ the 3 is the x component and the 4 is the y component. (We use the term *component* and the pointed brackets $\langle\ ,\ \rangle$ to distinguish a vector from a point (,) and its coordinates.)

We can also use this geometric representation to define a vector between any two points. For example, consider the vector $\overrightarrow{QR}$ shown in Figure 1.17 from the point $Q = (1, 1)$ to the point $R = (4, 5)$. Note that to get from Q to R we add 3 to the x-coordinate of Q and 4 to the y-coordinate of Q. So

$$\overrightarrow{QR} = \langle 3, 4\rangle.$$

In general, the vector $\overrightarrow{QR}$ from the point $Q = (q_1, q_2)$ to $R = (r_1, r_2)$ is

$$\overrightarrow{QR} = \langle r_1 - q_1, r_2 - q_2\rangle.$$

While we will often illustrate vectors in the plane so that it is easy to draw pictures, from a mathematical standpoint there is no reason to stop there – we can define vectors in any dimensional space. An arbitrary vector $\mathbf{v}$ in n-dimensional space $\mathbb{R}^n$ has n components and may be represented as

$$\mathbf{v} = \langle v_1, v_2, v_3, \dots, v_n\rangle.$$

The next activity will help us to become accustomed to vectors and operations on vectors in three dimensions.

Activity 1.11.

As a class determine a coordinatization of your classroom, agreeing on some convenient set of axes (e.g., an intersection of walls and floor) and some units in the x, y, and z directions (e.g., using lengths of sides of floor, ceiling, or wall tiles). Let O be the origin of your coordinate system. Then agree on three points, A, B, and C in the room Complete the following.

(a) Determine the coordinates of the point A.

(b) Determine the coordinates of the point B.

(c) Determine the coordinates of the point C.

(d) Determine the components of the indicated vectors.

(i) $\overrightarrow{OA}$ (ii) $\overrightarrow{OB}$ (iii) $\overrightarrow{OC}$ (iv) $\overrightarrow{AB}$ (v) $\overrightarrow{AC}$ (vi) $\overrightarrow{BC}$

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Equality of Vectors

We discussed earlier that location is not mentioned as part of the definition of a vector, so any two vectors that have the same magnitude and direction are equal. It will be helpful to have an algebraic way to determine when this happens. That is, if we know the components of two vectors $\mathbf{u}$ and $\mathbf{v}$, we will want to be able to determine algebraically when $\mathbf{u}$ and $\mathbf{v}$ are equal. There is an obvious set of conditions that we use.

Activity 1.12.

(a) In terms of the components, how can we tell if two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ in $\mathbb{R}^2$ are equal? Explain your reasoning.

(b) In terms of the components, how can we tell if two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$ are equal? Explain your reasoning.

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Operations on Vectors

Vectors are not numbers, but we can represent them with components that are real numbers. As such, we might wonder if it is possible to add two vectors together, multiply two vectors, or combine vectors in any other ways. In this section we will study two operations on vectors – addition and multiplication by scalars. To begin we investigate a natural way to add two vectors together.

Activity 1.13.

Let $\mathbf{u} = \langle 2, 3 \rangle$, $\mathbf{v} = \langle -1, 4 \rangle$.

(a) Would it seem reasonable to you to define a vector sum $\mathbf{u}+\mathbf{v}$? If yes, how do you think the sum should be defined? If no, explain why.

(b) In general, how do you think a vector sum $\mathbf{a}+\mathbf{b}$ of vectors $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ in $\mathbb{R}^2$ should be defined? Write a formal definition of a vector sum based on your intuition.

(c) In general, how do you think a vector sum $\mathbf{a}+\mathbf{b}$ of vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ in $\mathbb{R}^3$ should be defined? Write a formal definition of a vector sum based on your intuition.

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There is also a natural way to multiply a vector by a scalar (a real number).

Activity 1.14.

Let $\mathbf{v} = \langle -1, 4 \rangle$.

(a) Would it seem reasonable to you to define the scalar multiple $\frac{1}{2}\mathbf{v}$? If yes, how do you think this scalar multiple should be defined? If no, explain why.

(b) In general, how do you think a scalar multiple of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ in $\mathbb{R}^2$ by a scalar c should be defined? Write a formal definition of a scalar multiple of a vector based on your intuition.

(c) In general, how do you think a scalar multiple of a vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ in $\mathbb{R}^3$ by a scalar c should be defined? Write a formal definition of a scalar multiple of a vector based on your intuition.

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We can add vectors and multiply vectors by scalars, so we can add together scalar multiples of vectors. For completeness, we define vector subtraction as adding a scalar multiple:

$$\mathbf{v} - \mathbf{u} = \mathbf{v} + (-1)\mathbf{u}.$$

Activity 1.15.

Let $\mathbf{v} = \langle 1, -2 \rangle$, $\mathbf{u} = \langle 0, 4 \rangle$, and $\mathbf{w} = \langle -5, 7 \rangle$.

(a) Determine the components of the vector $2\mathbf{v} - 3\mathbf{u}$.

(b) Determine the components of the vector $\mathbf{v} + 2\mathbf{u} - 7\mathbf{w}$.

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There is an alternate representation that you will often see for a vector. If we let $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$, then we can write the vector $\langle a, b \rangle$ in $\mathbb{R}^2$ as

$$\langle a, b \rangle = a\langle 1, 0 \rangle + b\langle 0, 1 \rangle = a\mathbf{i} + b\mathbf{j}.$$

So for example,

$$\langle 2, -3 \rangle = 2\mathbf{i} - 3\mathbf{j}.$$

If we define $\mathbf{i} = \langle 1, 0, 0 \rangle$ and $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ in $\mathbb{R}^3$, then we can write the vector $\langle a, b, c \rangle$ in $\mathbb{R}^3$ as

$$\langle a, b, c \rangle = a\langle 1, 0, 0 \rangle + b\langle 0, 1, 0 \rangle + c\langle 0, 0, 1 \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

These vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are called the *standard unit vectors* (unit vectors have length 1), and are quite important in the physical sciences.

Properties of Vector Operations

We know that the scalar sum $1 + 2$ is equal to the scalar sum $2 + 1$. This is called the *commutative* property of scalar addition. Any time we define operations on objects (like addition of vectors) we usually want to know what kind of properties this operation has. For example, is addition of vectors a commutative operation? To answer this question we take two *arbitrary* vectors $\mathbf{v}$ and $\mathbf{u}$ and add them together and see what happens. Let $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{u} = \langle u_1, u_2 \rangle$. Now we use the fact that v_1, v_2, u_1, and u_2 are scalars, and that the addition of scalars is commutative to see that

$$\mathbf{v} + \mathbf{u} = \langle v_1, v_2 \rangle + \langle u_1, u_2 \rangle = \langle v_1 + u_1, v_2 + u_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \mathbf{u} + \mathbf{v}.$$

So the vector sum is a commutative operation. Similar arguments can be used to show the following properties of vector addition and multiplication by scalars.

Let $\mathbf{v}$, $\mathbf{u}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let a and b be scalars. Then

1. $\mathbf{v}+\mathbf{u}=\mathbf{u}+\mathbf{v}$
2. $(\mathbf{v}+\mathbf{u})+\mathbf{w}=\mathbf{v}+(\mathbf{u}+\mathbf{w})$
3. The vector $\mathbf{z}=\langle 0,0,\ldots,0\rangle$ has the property that $\mathbf{v}+\mathbf{z}=\mathbf{v}$. The vector $\mathbf{z}$ is called the **zero vector**.
4. $(-1)\mathbf{v}+\mathbf{v}=\mathbf{z}$. The vector $(-1)\mathbf{v}=-\mathbf{v}$ is called the **additive inverse** of the vector $\mathbf{v}$.
5. $(a+b)\mathbf{v}=a\mathbf{v}+b\mathbf{v}$
6. $a(\mathbf{v}+\mathbf{u})=a\mathbf{v}+a\mathbf{u}$
7. $(ab)\mathbf{v}=a(b\mathbf{v})$
8. $1\mathbf{v}=\mathbf{v}$.

We verified the first property for vectors in $\mathbb{R}^2$ and we will assume that the rest of the properties hold.

Geometric Interpretation of Vector Operations

There is a geometric interpretation of the sum of two vectors and scalar multiples that allows us to visualize these operations that we will explore next.

Let $\mathbf{v}=\langle 4,6\rangle$ and $\mathbf{u}=\langle 3,-2\rangle$. Then $\mathbf{w}=\mathbf{v}+\mathbf{u}=\langle 7,4\rangle$ as shown in Figure 1.18.

One way to interpret this sum is to create the parallelogram determined by $\mathbf{v}$ and $\mathbf{u}$ as shown in Figure 1.19. Notice that the coordinates of $\mathbf{v}+\mathbf{u}$ are given by the point on the parallelogram opposite the origin. Note also that if we place the tip of $\mathbf{u}$ at the tail of $\mathbf{v}$ (recall that vectors are position independent), then the vector $\mathbf{v}+\mathbf{u}$ is the vector from the tip of $\mathbf{v}$ to the tail of $\mathbf{u}$.

In a similar way we can geometrically represent a scalar multiple of a vector.

Activity 1.16.

Let $\mathbf{v}=\langle 1,2\rangle$.

(a) Draw geometric representations of the vectors $2\mathbf{v}$, $\frac{1}{2}\mathbf{v}$, $(-1)\mathbf{v}$, and $(-3)\mathbf{v}$ on the same set of axes.

(b) Explain as best you can how the vector $c\mathbf{v}$ is related to $\mathbf{v}$ for any scalar c.

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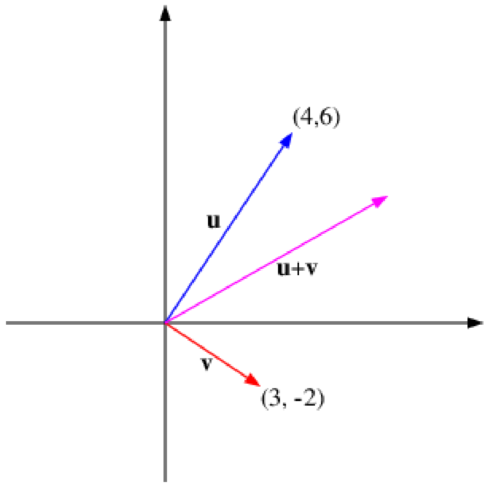

Figure 1.18: A vector sum

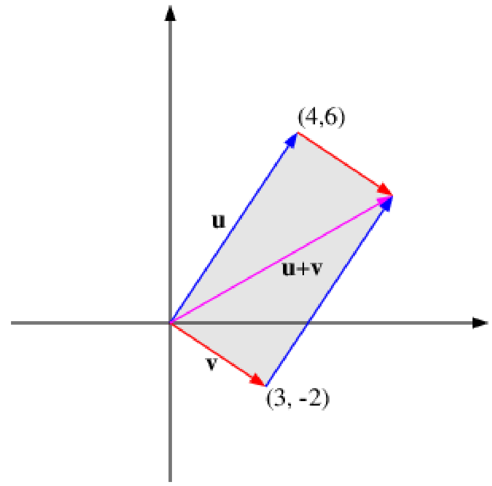

Figure 1.19: A vector sum in a parallelogram

The Magnitude of a Vector

Vectors have both direction and magnitude (or length). We will see how to formally calculate the magnitude of a vector in this section. Since a vector $\mathbf{v}$ can be represented by a directed line segment, we can use the distance formula to calculate the length of the segment. This length is the *magnitude* of the vector $\mathbf{v}$ and is denoted as $|\mathbf{v}|$.

Activity 1.17.

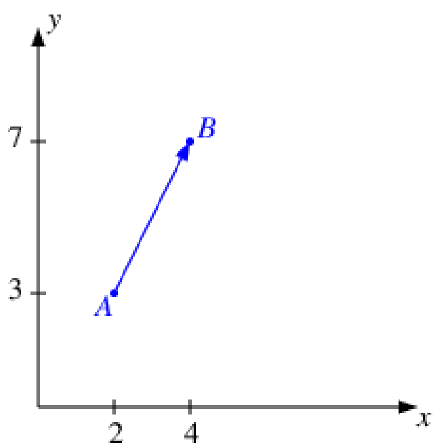

Figure 1.20: The magnitude of a specific vector

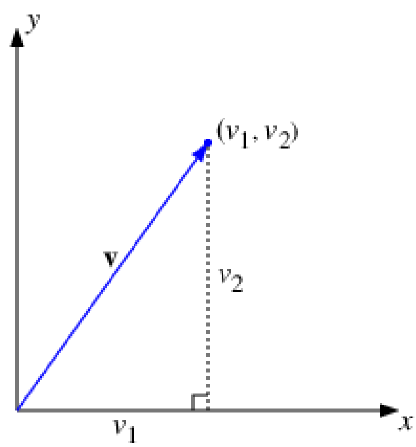

Figure 1.21: The magnitude of an arbitrary vector

(a) Let $A = (2, 3)$ and $B = (4, 7)$ as shown in Figure 1.20. Calculate $|\overrightarrow{AB}|$.

(b) Let $\mathbf{v} = \langle v_1, v_2 \rangle$ be the vector in $\mathbb{R}^2$ with components v_1 and v_2 as shown in Figure 1.21. Use the distance formula to find a general formula for $|\mathbf{v}|$.

(c) Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be a vector in $\mathbb{R}^3$. Use the distance formula to find a general formula for $|\mathbf{v}|$.

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As we will see later, unit vectors will be important for defining the direction of a vector and projections of vectors.

Definition 1.7. A **unit vector** is a vector with magnitude or length 1.

Activity 1.18.

(a) Let $\mathbf{v} = \langle 1, 2 \rangle$ in $\mathbb{R}^2$. What is the length of $\mathbf{v}$? Use the length of $\mathbf{v}$ to find a unit vector that is parallel to $\mathbf{v}$.

(b) Let $\mathbf{v}$ be an arbitrary nonzero vector in $\mathbb{R}^3$. Write a general formula for a unit vector that is parallel to $\mathbf{v}$.

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Summary

In this section, we encountered the following important ideas:

- A vector is any object that possesses the attributes of magnitude and direction. Examples of vector quantities are forces, velocity, and acceleration.
- Two vectors are equal if the have the same direction and magnitude. Notice that position is not mentioned here so that a vector is independent of its position.
- If $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ are two vectors in $\mathbb{R}^n$, then their vector sum is the vector
$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n \rangle.$$
- If $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ is a vector in $\mathbb{R}^n$ and c is a scalar, then the scalar multiple c of the vector $\mathbf{u}$ is the vector
$$c\mathbf{u} = \langle cu_1, cu_2, \ldots, cu_n \rangle.$$
- The magnitude of the vector $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ in $\mathbb{R}^n$ is the scalar
$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$
- A vector $\mathbf{u}$ is a unit vector if $|\mathbf{u}| = 1$. If $\mathbf{v}$ is a nonzero vector, then the vector $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector with the same direction as $\mathbf{v}$.

1.3 The Dot Product

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How and when is the dot product of two vectors defined?
- What are two things that the dot product can tell us?
- How can we tell if two vectors in $\mathbb{R}^n$ are perpendicular?
- How do we find the projection of one vector onto another?

Preview Activity 1.3. Let us return to Bilbo and Frodo walking in the Shire (see the Shire map in Figure 1.14). Let $\mathbf{v}_1 = \langle a, b \rangle$ be the vector representing the displacement from Hobbiton to Little Delving, and let $\mathbf{v}_2 = \langle c, d \rangle$ be the vector representing the displacement from Hobbiton to Quarry.

(a) Use a protractor, or some other appropriate tool, to approximate the angle between the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Draw a triangle with Hobbiton, Little Delving, and Quarry as the vertices.

i. What vector, in terms of a, b, c, and d, is the vector representing the displacement from Little Delving to Quarry?

ii. Let α be the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$. Use the Law of Cosines (look it up if you don't remember it) to write an equation that relates α, a, b, c, and d. Then simplify and solve this equation (this will require some algebra!) to show that

$$\cos(\alpha) = \frac{ac + bd}{\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}}. \tag{1.1}$$

iii. Assume that $\mathbf{v}_1 = \langle -18, 3.75 \rangle$ and $\mathbf{v}_2 = \langle 19.5, 7 \rangle$. Use these vectors and equation (1.1) to calculate the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$. How does this compare to the angle you measured in part (a)?

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Introduction

In physics, work (or energy) is a measure of the cumulative effect of applying a force on an object over a distance. If we apply a constant force of one Newton to move an object in a given direction for one meter, the resulting work done is defined to be one Joule or Newton-meter. Consequently, if a constant force of F Newtons is exerted to move an object along a straight line path a distance of x meters, the work done is Fx Newton-meters (or Joules). Similarly, if we measure force in pounds and distance in feet, the corresponding unit of work is pound-feet. This all assumes that 100 percent of the force is being applied to move the object. If that is not the case, then we need to apply some trigonometry to determine the actual amount of force applied to the object. Suppose

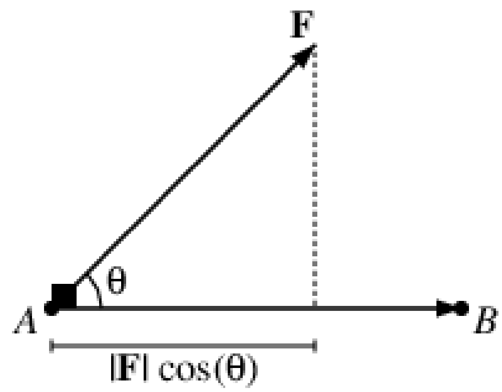

Figure 1.22: Calculating work

a force $\mathbf{F}$ is being exerted to move an object along a path from a point A to a point B, where $\mathbf{F}$ is applied at an angle θ to $\mathbf{v} = \overrightarrow{AB}$ as illustrated in Figure 1.22. The force applied parallel to the motion is what causes the object to move, while the force that is perpendicular to the object is wasted. The magnitude of the force applied parallel to the object is the component of the force in the direction of $\mathbf{v}$ or $|\mathbf{F}|\cos(\theta)$. Consequently the work W done by the force $\mathbf{F}$ to move the object from point A to point B is

$$W = |\mathbf{F}|\cos(\theta)|\mathbf{v}|.$$

Activity 1.19.

Determine the work done by a 25 pound force to pull an object 10 feet if the force acts at a 30° angle from the direction of motion of the object.

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The Dot Product

The quantity $|\mathbf{F}|\ |\mathbf{v}|\cos(\theta)$ that represents the work done in moving an object by applying a force $\mathbf{F}$ at an angle θ over a distance determined by a vector $\mathbf{v}$ can be applied to any two vectors. This expression is special and we give it a name, the *dot product* of vector $\mathbf{F}$ with vector $\mathbf{v}$.

Definition 1.8. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$ and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, with $0 \le \theta \le \pi$. The **dot product**, $\mathbf{u} \cdot \mathbf{v}$, of $\mathbf{u}$ and $\mathbf{v}$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\ |\mathbf{v}|\cos(\theta).$$

A note on the angle θ in the dot product. Given two vectors $\mathbf{u}$ and $\mathbf{v}$, there are two angles that these vectors create as depicted in Figure 1.23. We will define the angle between the vectors to be the smaller of these two angles, and that is why we insist that θ be between 0 and π in our definition.

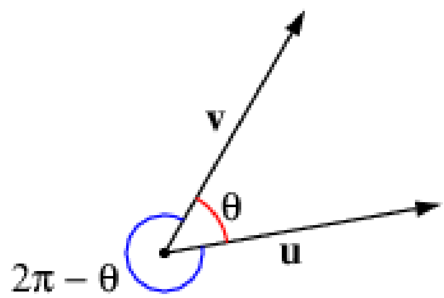

Figure 1.23: The angle between vectors $\mathbf{u}$ and $\mathbf{v}$.

With the dot product as defined, then the work W done by a force $\mathbf{F}$ applied at an angle θ over a distance determined by a vector $\mathbf{v}$ can be written as

$$W = |\mathbf{F}| \cos(\theta) |\mathbf{v}| = |\mathbf{F}| \frac{\mathbf{F} \cdot \mathbf{v}}{|\mathbf{F}|\ |\mathbf{v}|}\ |\mathbf{v}| = \mathbf{F} \cdot \mathbf{v}.$$

So the dot product allows us to calculate work done by a force.

The definition of the dot product is natural when we think of forces and work done, but doesn't lend itself well to computation. In Activity 1.19 we were given the angle at which the force was acting, which allowed us to directly calculate the component of the force in the direction of motion. In general we may not know this angle, so we would like a different method for calculating this component that does not involve our directly measuring the angle. We address this problem in the next section.

The Angle Between Two Vectors

In Preview Activity 1.3 we saw how the Law of Cosines can be used to find the angle between two specific vectors. We now apply that law to determine a general formula for finding angles between vectors. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$ and θ the angle between them. To calculate $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\ |\mathbf{v}| \cos(\theta)$, we apply the Law of Cosines to the triangle shown in Figure 1.24 to obtain

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\ |\mathbf{v}|\ \cos(\theta). \tag{1.2}$$

So

$$\mathbf{u} \cdot \mathbf{v} = \frac{|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{u} - \mathbf{v}|^2}{2}. \tag{1.3}$$

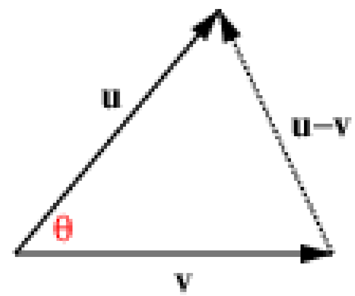

Figure 1.24: Calculating $\mathbf{u} \cdot \mathbf{v}$.

Let us consider the numerator of the right side of equation (1.3) in $\mathbb{R}^2$ with $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$. Then $\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle$ and

$$\begin{aligned} |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{u} - \mathbf{v}|^2 &= \left(u_1^2 + u_2^2\right) + \left(v_1^2 + v_2^2\right) - \left((u_1 - v_1)^2 + (u_2 - v_2)^2\right) \\ &= \left(u_1^2 + u_2^2\right) + \left(v_1^2 + v_2^2\right) - \left(u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2\right) \\ &= 2u_1v_2 + 2u_2v_2. \end{aligned}$$

Substituting into equation (1.3) shows that

$$\mathbf{u} \cdot \mathbf{v} = u_1v_2 + u_2v_2.$$

So the dot product of vectors $\mathbf{u}$ and $\mathbf{v}$ can be found by adding the products of like components. While we made this argument in $\mathbb{R}^2$, it works in any dimension, giving us the following alternate definition which does not require us to know the angle between the vectors.

> The dot product of vectors $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ in $\mathbb{R}^n$ is the scalar
>
> $$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{i=1}^{n} u_iv_i.$$

With this formulation of the dot product in mind, we can rewrite the norm of the vector $\mathbf{u}$ as

$$|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}.$$

In addition, we have seen that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\ |\mathbf{v}|\ \cos(\theta).$$

Solving for $\cos(\theta)$ gives us a straightforward computational method for calculating the angle θ between $\mathbf{u}$ and $\mathbf{v}$:

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\ |\mathbf{v}|},$$

provided that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors.

IMPORTANT NOTE: The dot product is only defined between two vectors with the *same number of components*.

Activity 1.20.

Find each of the following.

(a) $\langle 1, 2, -3\rangle \cdot \langle 4, -2, 0\rangle$.

(b) $\langle 0, 3, -2, 1\rangle \cdot \langle 5, -6, 0, 4\rangle$

(c) The angle between the vectors $\mathbf{u} = \langle 1, 2\rangle$ and $\mathbf{v} = \langle 4, -1\rangle$ to the nearest tenth of a degree.

$\lhd$

Properties of the Dot Product

The dot product of two vectors has some useful properties that are not difficult to verify from the alternate definition and properties of matrix multiplication.

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$. Then

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (the dot product is *commutative*)

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$ (the dot product *distributes over vector addition*)

(c) If c is an arbitrary constant, then $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$

(d) $\mathbf{u} \cdot \mathbf{u} \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

The verifications of these properties are straightforward and we leave them to the reader.

Activity 1.21.

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^5$ with $\mathbf{u} \cdot \mathbf{v} = -1$, $|\mathbf{u}| = 2$ and $|\mathbf{v}| = 3$. Use the properties of the dot product to find each of the following.

(a) $\mathbf{u} \cdot 2\mathbf{v}$

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{v}$

(c) $(2\mathbf{u} + 4\mathbf{v}) \cdot (\mathbf{u} - 7\mathbf{v})$

$\lhd$

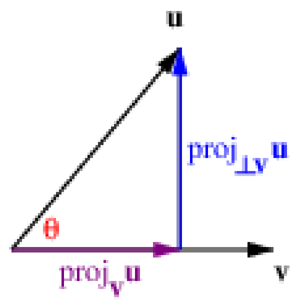

Figure 1.25: The projection of $\mathbf{u}$ onto $\mathbf{v}$.

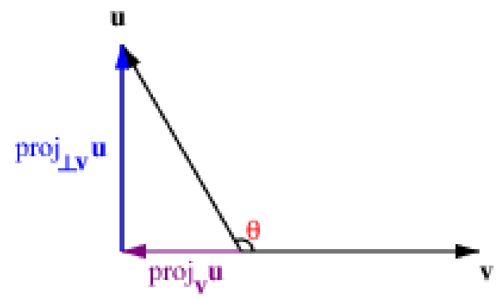

Figure 1.26: $\text{proj}_{\mathbf{v}}\mathbf{u}$ when $\theta > 90°$.

Projections

Let us return to the situation described in the introduction, where we wanted to find the component of a force acting in a given direction. Suppose we have two vectors, $\mathbf{u}$ and $\mathbf{v}$. If we think of $\mathbf{u}$ as a force and $\mathbf{v}$ as the direction in which the force is moving an object, then we would be interested in the force that the vector $\mathbf{u}$ produces in the direction of the vector $\mathbf{v}$. We can decompose the vector $\mathbf{u}$ into two component vectors; one in the direction (or its opposite) of the vector $\mathbf{v}$ and the other perpendicular to $\mathbf{v}$ as illustrated in Figure 1.25. The vector in the direction (or its opposite) of $\mathbf{v}$ is called the *projection of* $\mathbf{u}$ *in the direction of* $\mathbf{v}$, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, and the other is the *projection of* $\mathbf{u}$ *perpendicular to* $\mathbf{v}$, denoted $\text{proj}_{\perp\mathbf{v}}\mathbf{u}$. As we did with the force vectors earlier, we can find these projections with some trigonometry. To find the vector $\text{proj}_{\mathbf{v}}\mathbf{u}$, we determine its magnitude and direction, then multiply by a unit vector in the direction of $\mathbf{v}$. Notice that whether $\text{proj}_{\mathbf{v}}\mathbf{u}$ is in the direction of $\mathbf{v}$ or opposite of $\mathbf{v}$ is determined by the value of $\cos(\theta)$. Figure 1.26 illustrates the case where θ exceeds $90°$ – in this case $\text{proj}_{\mathbf{v}}\mathbf{u}$ is in the direction opposite of $\mathbf{v}$ (and $\cos(\theta) < 0$). In either case, since $\cos(180° - \theta) = -\cos(\theta)$ we have

$$|\text{proj}_{\mathbf{v}}\mathbf{u}| = |\mathbf{u}||\cos(\theta)|.$$

So to find $\text{proj}_{\mathbf{v}}\mathbf{u}$ we multiply its magnitude by a unit vector in the direction of $\mathbf{v}$ and account for direction. What results is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos(\theta)\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}. \tag{1.4}$$

The absolute value of the scalar $\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$ is the magnitude of the projection vector, and we call this scalar the *component* of $\mathbf{u}$ in the direction of $\mathbf{v}$. To summarize:

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$.

1. The component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is the scalar
$$\text{comp}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$$
2. The projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector
$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}.$$

We can also write the projection of $\mathbf{u}$ onto $\mathbf{v}$ as
$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}.$$
Since
$$\mathbf{u} = \text{proj}_{\mathbf{v}}\mathbf{u} + \text{proj}_{\perp\mathbf{v}}\mathbf{u},$$
it follows that
$$\text{proj}_{\perp\mathbf{v}}\mathbf{u} = \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}.$$

Activity 1.22.

Let $\mathbf{u} = \langle 5, 8 \rangle$ and $\mathbf{v} = \langle 6, -10 \rangle$. Find $\text{comp}_{\mathbf{v}}\mathbf{u}$, $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\text{proj}_{\perp\mathbf{v}}\mathbf{u}$ and draw a picture to illustrate.

$\triangleleft$

The Dot Product and Perpendicularity

Finding optimal solutions to systems is an important problem in applied mathematics. For example, finding the point in 3-space that is closest to a plane is a situation where we will need to employ perpendicular vectors.

Activity 1.23.

We investigate the situation when two vectors in $\mathbb{R}^n$ are perpendicular in this activity.[5]

(a) Suppose the angle between vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is $90°$. What is $\mathbf{u} \cdot \mathbf{v}$? Explain.

(b) Now suppose $\mathbf{u} \cdot \mathbf{v} = 0$. What, then, is the angle between $\mathbf{u}$ and $\mathbf{v}$? Explain.

(c) Use the results from (a) and (b) to explain the following statement.

[5]There are situations (which we won't encounter in this class) where objects other than vectors in $\mathbb{R}^n$ can be considered to be perpendicular. We introduce the word *orthogonal* in these cases to be a word that also means perpendicular.

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Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **perpendicular** (to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.

Activity 1.24.

Find a vector orthogonal to $\mathbf{u} = \langle 0, 3, -2 \rangle$. How many such vectors are there?

◁

Summary

In this section, we encountered the following important ideas:

- The dot product of two vectors $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$, both in $\mathbb{R}^n$, is the scalar

  $$|\mathbf{u}|\,|\mathbf{v}|\cos(\theta),$$

  where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. We can also calculate the dot product by

  $$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i.$$

- We can use the dot product to calculate the work done by a force, the angle between two vectors, and also to find the projection of one vector onto another.
- Two vectors are orthogonal if the angle between them is 90°. In other words, vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.
- The projection of a vector $\mathbf{u}$ in $\mathbb{R}^n$ onto a vector $\mathbf{v}$ in $\mathbb{R}^n$ is the vector

  $$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}.$$

1.4 The Cross Product

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How and when is the cross product of two vectors defined?
- What are two things that the cross product can tell us?

Preview Activity 1.4. A wrench is a tool for driving bolts or screws. It consists of a handle (the wrench) and an end that fits around a screw head as shown in Figure **??**. The wrench is situated on the screw and, as force is applied to the handle of the wrench, the wrench provides a force that drives the screw. The force that is applied to the screw is a vector, called *torque*, which depends on the force applied to the handle and the length of the handle. A longer handle with the same applied force creates more torque. The torque acts in a direction that is perpendicular to both the handle and the direction that the force is applied to the handle in that plane. If we consider the force applied as a vector $\mathbf{F}$ and the handle of the wrench as a vector $\mathbf{v}$, then the magnitude of the torque will be the product of the magnitude of the force applied perpendicular to the handle and the length of the handle. The torque is an example of another "product" of vectors, called the *cross product*. In this preview activity we derive a formula for the torque.

(a) Let $\mathbf{F}$ be the force applied to the wrench to turn the screw, and assume that the force is acting at an angle θ to the handle, as depicted in Figure 1.27. Determine the force that is acting perpendicular to the wrench handle. Write your answer in terms of the vectors $\mathbf{F}$ and $\mathbf{v}$, and the angle θ.

(b) Determine the magnitude of the force that is acting perpendicular to the wrench handle. Write your answer in terms of the vector $\mathbf{F}$ and the angle θ.

(c) Find the magnitude of the torque that the force $\mathbf{F}$ applied to the wrench imposes on the screw. Write your answer in terms of the vectors $\mathbf{F}$ and $\mathbf{v}$, and the angle θ.

(d) Let $\mathbf{T}$ be the torque whose magnitude we calculated in part (c). The torque acts in a direction perpendicular to both $\mathbf{F}$ and $\mathbf{v}$, but there are two such directions. Do $\mathbf{F}$, $\mathbf{v}$, and $\mathbf{T}$ (in that order) form a right or left hand system? Explain.

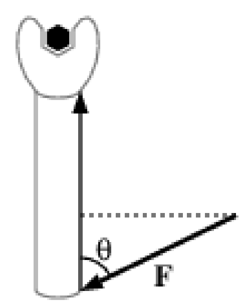

Figure 1.27: A wrench.

⋈

Introduction

In Preview Activity 1.4 we were introduced to the idea of two vectors – the handle of a wrench and the force applied to the wrench – producing a new force, *torque*, that acts to turn a screw in a direction that is perpendicular to both the wrench handle and the force applied to the handle to turn the screw. This idea is useful in a variety of situations, and will provide a method (called the *cross product*) for finding a vector in $\mathbb{R}^3$ that is perpendicular to two other vectors in $\mathbb{R}^3$, which will allow us to find equations for tangent planes to surfaces and understand the concept of local linearity of functions of two variables.

The Cross Product of Vectors in $\mathbb{R}^3$

As mentioned in Preview Activity 1.4, torque is a result of two applied vectors $\mathbf{u}$ and $\mathbf{v}$, and it acts in a direction perpendicular to both vectors. The Preview Activity showed us how to find the magnitude of this torque vector, so all we need to define the actual torque vector is a unit vector that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ in the correct direction. One issue that we have to address is that there are two different directions perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. Again, in our Preview Activity we saw that the vectors $\mathbf{F}$, $\mathbf{v}$, and the torque (in that order) formed a right hand system. The torque is an example of what we call the *cross product* of two vectors in $\mathbb{R}^3$.

Definition 1.9. The **cross product** of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ is the vector

$$\mathbf{u} \times \mathbf{v} = |\mathbf{u}|\ |\mathbf{v}|\ \sin(\theta)\ \mathbf{n},$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ and $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ so that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{n}$ in that order form a right hand system.

IMPORTANT NOTE 1: The cross product is only defined for vectors in $\mathbb{R}^3$ because there is no unique vector $\mathbf{n}$ in any other dimension.

IMPORTANT NOTE 2: Since the cross product of $\mathbf{u}$ and $\mathbf{v}$ is a scalar multiple of the vector $\mathbf{n}$ in the definition, the cross product of $\mathbf{u}$ and $\mathbf{v}$ is a vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ so that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{n}$ (in that order) form a right hand system. This will be important to use later when we want to find the equations of planes.

Activity 1.25.

(a) What are two unit vectors that are orthogonal to both $\mathbf{i}$ and $\mathbf{j}$?

(b) What is $\mathbf{i} \times \mathbf{j}$? Explain.

(c) Determine each of

$$\mathbf{i} \times \mathbf{k} \quad \mathbf{j} \times \mathbf{i} \quad \mathbf{j} \times \mathbf{k} \quad \mathbf{k} \times \mathbf{i} \quad \mathbf{k} \times \mathbf{j}$$

(d) Let $\mathbf{v}$ be a vector in $\mathbb{R}^3$. What is $\mathbf{v} \times \mathbf{v}$?

(e) Let $\mathbf{u}$ and $\mathbf{v}$ be parallel vectors in $\mathbb{R}^3$. What is $\mathbf{u} \times \mathbf{v}$? Explain.

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While we can calculate some simple cross products with the definition, it is generally an inefficient tool to use in computation. Fortunately, there is a straightforward formula for calculating cross products that we will soon see.

Properties of Cross Products

Any time we have an operation like a product, it is natural to ask if the operation satisfies certain properties.

Activity 1.26.

(a) Is the cross product commutative? That is, is $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$ for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$? If yes, explain why. If no, how are $\mathbf{v} \times \mathbf{u}$ and $\mathbf{u} \times \mathbf{v}$ related? Why? (Hint: Think about how $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ are related.)

(b) Calculate $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$ and $\mathbf{i} \times (\mathbf{i} \times \mathbf{j})$.

(c) Is the cross product associative? That is, is $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ for any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$?

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Activity 1.26 shows that not all of the standard properties of operations hold for the cross product. For example, the cross product is not commutative – rather

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \tag{1.5}$$

for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$. However, the cross product does satisfy some of the standard properties of operations. For example, it is true that if c is any scalar, then

$$(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}).$$

To see why, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$, θ the angle between $\mathbf{u}$ and $\mathbf{v}$, and let $\mathbf{n}$ be a unit vector so that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{n}$ form a right hand system. Now let c be a scalar. If $c > 0$, then $c\mathbf{u}$ is in the same direction as $\mathbf{u}$, and $c\mathbf{u}$, $\mathbf{v}$, and $\mathbf{n}$ also form a right hand system. Therefore,

$$(c\mathbf{u}) \times \mathbf{v} = |c\mathbf{u}|\ |\mathbf{v}|\ \sin(\theta)\ \mathbf{n} = c|\mathbf{u}|\ |\mathbf{v}|\ \sin(\theta)\ \mathbf{n} = c(\mathbf{u} \times \mathbf{v}).$$

If $c < 0$, then $c\mathbf{u}$, $\mathbf{v}$, and $-\mathbf{n}$ form a right hand system. In this case the angle between $c\mathbf{u}$ and $\mathbf{v}$ is $\pi - \theta$, but from trigonometry we know that $\sin(\pi - \theta) = \sin(\theta)$. So

$$(c\mathbf{u}) \times \mathbf{v} = |c\mathbf{u}|\ |\mathbf{v}|\ \sin(\pi - \theta)\ (-\mathbf{n}) = (-|c|)|\mathbf{u}|\ |\mathbf{v}|\ \sin(\theta)\ \mathbf{n} = c(\mathbf{u} \times \mathbf{v}).$$

If $c = 0$, then $(c\mathbf{u}) \times \mathbf{v} = \mathbf{0} = c(\mathbf{u} \times \mathbf{v})$. Thus, in every case we have

$$(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}). \tag{1.6}$$

Activity 1.27.

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$ and let c be a scalar. Use (1.5) and (1.6) to show that

$$\mathbf{u} \times (c\mathbf{v}) = c(\mathbf{u} \times \mathbf{v}).$$

$\lhd$

Below we state some important properties that the cross product satisfies.

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^3$, and let c be a scalar. Then

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

We have already argued for properties (1) and (2). Verifying that the cross product distributes over addition (as in properties (3) and (4)) is a bit more difficult than what we have done so far, but we will be able to verify these properties shortly.

The Area of a Parallelogram

There is a standard geometric interpretation of the norm of the cross product of two vectors. If the vectors $\mathbf{u}$ and $\mathbf{v}$ are not parallel, then they form the sides of a parallelogram as illustrated in Figure 1.28.

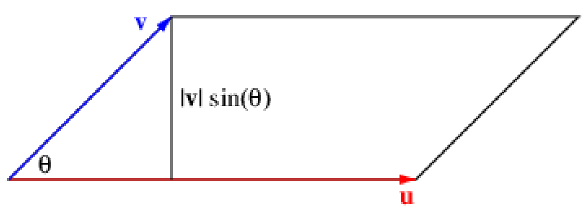

Figure 1.28: The parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$

If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the height of the parallelogram is $|\mathbf{v}|\sin(\theta)$ and the area of the parallelogram is the length of the base times the height or

$$|\mathbf{u}|\ |\mathbf{v}|\ \sin(\theta).$$

Since the normal vector $\mathbf{n}$ in the definition of the cross product is a unit vector, we also have that

$$|\mathbf{u} \times \mathbf{v}| = ||\mathbf{u}|\ |\mathbf{v}|\ \sin(\theta)\ \mathbf{n}| = |\mathbf{u}|\ |\mathbf{v}|\ |\sin(\theta)|\ |\mathbf{n}| = |\mathbf{u}|\ |\mathbf{v}|\ \sin(\theta).$$

So $|\mathbf{u} \times \mathbf{v}|$ gives the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

The length $|\mathbf{u} \times \mathbf{v}|$ of the cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ gives the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

Activity 1.28.

A parallelogram in $\mathbb{R}^3$ has vertices $(1,0,1)$, $(0,0,1)$, $(2,1,0)$, and $(1,1,0)$. Find the area of this parallelogram.

$\lhd$

The Triple Scalar Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^3$ that form a right hand system. Since $\mathbf{u} \times \mathbf{v}$ is a vector in $\mathbb{R}^3$, we can form the dot product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$. The result is a scalar called the *triple scalar product* and has an interesting geometric interpretation. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are not coplanar, then they form a three-dimensional solid called a *parallelepiped* as illustrated in Figure 1.29. The volume of this

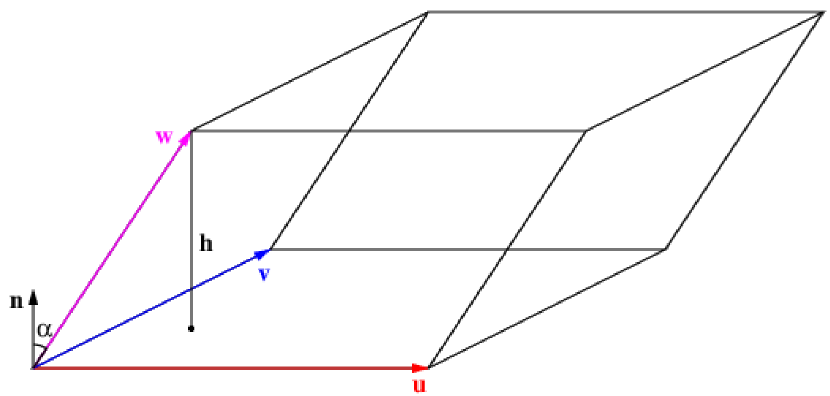

Figure 1.29: The parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$

parallelepiped is found by multiplying the area A of its base (the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$) and its height h. We have just seen that the area A is equal to $|\mathbf{u} \times \mathbf{v}|$. It is also the case that $\mathbf{u} \times \mathbf{v}$ is a vector in the direction of $\mathbf{n}$ in Figure 1.29, since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as shown form a right hand system. If α is the angle between $\mathbf{n}$ and $\mathbf{w}$ as shown, then

$$h = |\mathbf{w}| \cos(\alpha)$$

and the area of the parallelepiped is

$$|\mathbf{u} \times \mathbf{v}|h = |\mathbf{u} \times \mathbf{v}|\,|\mathbf{w}| \cos(\alpha).$$

This is the definition of

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w},$$

so the triple scalar product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ gives the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, provided $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form a right hand system. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form a left hand system, then $\mathbf{u} \times \mathbf{v}$ points in the opposite direction and $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is the opposite of the volume of the parallelepiped. Since the dot product is commutative, we also have that

$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

and both give the volume of the parallelepiped defined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

The triple scalar product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ gives the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, provided $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form a right hand system.

One important consequence of this result is that when $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$ that form a right hand system, then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) \tag{1.7}$$

since all three scalar triple products give the volume of the same parallelepiped. The commutativity of the dot product and (1.7) then give us that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}. \tag{1.8}$$

Hence, we can interchange the dot and cross products in a scalar triple product.

Why the Cross Product Distributes Over Vector Addition

With the tools we have developed, we can now show that the cross product distributes over vector addition. That is, let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^3$. We will show that

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}).$$

Our method will be to show that the vector

$$\mathbf{c} = \mathbf{u} \times (\mathbf{v} + \mathbf{w}) - (\mathbf{u} \times \mathbf{v}) - (\mathbf{u} \times \mathbf{w})$$

is the zero vector. To show that $\mathbf{c} = \mathbf{0}$, we show that $\mathbf{c} \cdot \mathbf{c} = 0$. Now the distributive property of the dot product and (1.8) give us

$$\begin{aligned}
\mathbf{c} \cdot \mathbf{c} &= \mathbf{c} \cdot [\mathbf{u} \times (\mathbf{v} + \mathbf{w}) - (\mathbf{u} \times \mathbf{v}) - (\mathbf{u} \times \mathbf{w})] \\
&= \mathbf{c} \cdot [\mathbf{u} \times (\mathbf{v} + \mathbf{w})] - \mathbf{c} \cdot (\mathbf{u} \times \mathbf{v}) - \mathbf{c} \cdot (\mathbf{u} \times \mathbf{w}) \\
&= (\mathbf{c} \times \mathbf{u}) \cdot (\mathbf{v} + \mathbf{w}) - (\mathbf{c} \times \mathbf{u}) \cdot \mathbf{v} - (\mathbf{c} \times \mathbf{u}) \cdot \mathbf{w} \\
&= [(\mathbf{c} \times \mathbf{u}) \cdot \mathbf{v} + (\mathbf{c} \times \mathbf{u}) \cdot \mathbf{w}] - (\mathbf{c} \times \mathbf{u}) \cdot \mathbf{v} - (\mathbf{c} \times \mathbf{u}) \cdot \mathbf{w} \\
&= 0.
\end{aligned}$$

Now $\mathbf{c} \cdot \mathbf{c} = 0$ implies $\mathbf{c} = \mathbf{0}$, so we have

$$\mathbf{0} = \mathbf{c} = \mathbf{u} \times (\mathbf{v} + \mathbf{w}) - (\mathbf{u} \times \mathbf{v}) - (\mathbf{u} \times \mathbf{w})$$

and

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}).$$

The properties of the cross product will now allow us to find a computationally efficient formula for computing cross products in terms of the components of the vectors. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= (u_1v_1)(\mathbf{i} \times \mathbf{i}) + (u_1v_2)(\mathbf{i} \times \mathbf{j}) + (u_1v_3)(\mathbf{i} \times \mathbf{k}) + (u_2v_1)(\mathbf{j} \times \mathbf{i}) + (u_2v_2)(\mathbf{j} \times \mathbf{j}) + (u_2v_3)(\mathbf{j} \times \mathbf{k}) \\
&\qquad + (u_3v_1)(\mathbf{k} \times \mathbf{i}) + (u_3v_2)(\mathbf{k} \times \mathbf{j}) + (u_3v_3)(\mathbf{k} \times \mathbf{k}) \\
&= (u_1v_2)\mathbf{k} - (u_1v_3)\mathbf{j} - (u_2v_1)\mathbf{k} + (u_2v_3)\mathbf{i} + (u_3v_1)\mathbf{j} - (u_3v_2)\mathbf{i} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.
\end{aligned}$$

This gives the formula for easily calculating cross products of vectors in $\mathbb{R}^3$:

$$(u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.\text{[6]} \quad (1.9)$$

Activity 1.29.

Use the formula (1.9) to calculate the following.

(a) $\langle 2, -1, 0\rangle \times \langle 0, 1, 3\rangle$.

(b) $\langle 2, -1, 0\rangle \times \langle -4, 2, 0\rangle$. Why should you have expected this result?

(c) The volume of the parallelepiped determined by the vectors $\langle 2, -1, 0\rangle$, $\langle 0, 1, 3\rangle$, and $\langle 1, 1, 1\rangle$.

$\triangleleft$

Summary

In this section, we encountered the following important ideas:

- The cross product is defined *only* for vectors in $\mathbb{R}^3$. The cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ is the vector

$$\mathbf{u} \times \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\sin(\theta)\,\mathbf{n},$$

[6]If you are familiar with determinants, you can remember this formula for the cross product as the determinant of the matrix $\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$.

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ and $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ so that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{n}$ in that order form a right hand system. We can calculate the cross product of two vectors as

$$(u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

- The cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ is a vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. The magnitude $|\mathbf{u}\times\mathbf{v}|$ of the cross product of the vectors $\mathbf{u}$ and $\mathbf{v}$ gives the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$. Also, when $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form a right hand system, the scalar triple product $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$ gives the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

1.5 Lines and Planes in Space

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a line?
- What is the vector form of a line in $\mathbb{R}^2$. In $\mathbb{R}^3$? Why?
- What is a plane?
- How do we find the equation of a plane through three given non-collinear points?

Preview Activity 1.5. We are familiar with equations of lines in the plane in the form $y = mx + b$, where m is the slope of the line and $(0, b)$ is the y-intercept. In this activity we suggest a more flexible way of representing lines that we can use not only in the plane, but in higher dimensions as well.

(a) We begin by looking at an example. Consider the line through the point $(2, -1)$ with slope $\frac{2}{3}$.

i. Suppose we increase x by 1 from the point $(2, -1)$. How does the y-value change? What is the point on the line with x-coordinate 3?

ii. Suppose we decrease x by 3.25 from the point $(2, -1)$. How does the y-value change? What is the point on the line with x-coordinate -1.25?

iii. Suppose we increase x by some arbitrary value $3t$ from the point $(2, -1)$. How does the y-value change? What is the point on the line with x-coordinate $2 + 3t$?

iv. Let us think about the slope as a vector whose y-component divided by the x-component is the slope of the line. For our line we might use the vector $\langle 3, 2\rangle$. This vector $\langle 3, 2\rangle$ describes the direction of the line. Explain why the terminal points of the vectors $\mathbf{r}(t)$, where

$$\mathbf{r}(t) = \langle 2, -1\rangle + \langle 3, 2\rangle t$$

trace out the graph of the line through the point $(2, -1)$ with slope $\frac{2}{3}$.

(b) Now we extend this approach to $\mathbb{R}^3$ and consider a second example. Let $\mathcal{L}$ be the line in $\mathbb{R}^3$ through the point $(1, 0, 2)$ in the direction of the vector $\langle 2, -1, 4\rangle$.

i. Find the coordinates of three distinct points on this line. Explain your process.

ii. Find a vector, depending on a variable t, whose terminal points trace out the line $\mathcal{L}$ in $\mathbb{R}^3$

$\bowtie$

Introduction

In single variable calculus we learned that a differentiable function is locally linear. In other words, if we zoom in on the graph of a differentiable function at a point, the graph looks like the tangent line to the function at that point. We will soon study curves in space and the differentiable curves will be locally linear. It will therefore be useful to us to understand lines in space. In addition, we will study functions of two variables. As we will see, a function of two variables will be locally linear at a point if the surface defined by the function looks like a plane (the tangent plane) as we zoom in on the graph. Consequently, it will be important for us to understand planes in space. In this section we will study lines and planes in space.

Lines in Space

In two-dimensional space, we have defined a non-vertical line to be a set of points satisfying an equation of the form

$$y = mx + b,$$

for some constants m and b. The point $(0, b)$ is the y-intercept and anchors the line to a spot, while the value of m (the slope) tells us how the dependent variable changes for every one unit increase in the independent variable. As an alternative, we can think of the slope as the vector $\langle 1, m\rangle$ telling us the direction of the line. So we can identify a line in space by fixing a point P and a direction $\mathbf{v}$.

Definition 1.10. A **line** in space is the set of terminal points of vectors emanating from a given point that are parallel to a fixed vector.

Important Note! In $\mathbb{R}^n$, with $n > 2$, we have more than just two variables, so there is no concept of slope. Instead, we have the notion of direction, which is determined by a vector. The fixed vector in our definition of a line is what determines the direction of the line.

The fixed vector in our definition is called a *direction vector* for the line. As we saw in our Preview Activity, to find an equation for a line through the point P in the direction of the vector $\mathbf{v}$, note that any vector parallel to $\mathbf{v}$ will have the form $t\mathbf{v}$ for some scalar t. So any vector emanating from the point P in a direction parallel to the vector $\mathbf{v}$ will be of the form

$$\overrightarrow{OP} + \mathbf{v}t \tag{1.10}$$

for some scalar t (where O is the origin).

Activity 1.30.

Run the animation in Figure 1.30. Pick a frame and identify the vector $\overrightarrow{OP}$ and the vector $t\mathbf{v}$ as in equation (1.10).

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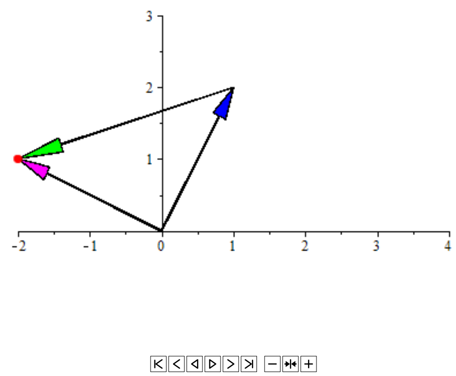

Figure 1.30: A line in 2-space.

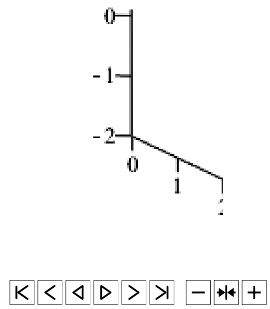

Figure 1.31: A line in 3-space.

The terminal points of the vectors of the form in (1.10) define a linear function $\mathbf{r}$ in space of the form

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} \tag{1.11}$$

where $\mathbf{r}_0$ is the vector $\overrightarrow{OP}$ from the origin to a fixed point P on the line and $\mathbf{v}$ is a vector in the direction of the line. Note that this construction can be made in any dimension.

> The **vector form** of a line through the point P in the direction of the vector $\mathbf{v}$ is
>
> $$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v},$$
>
> where $\mathbf{r}_0$ is the vector from the origin to the point P.

Activity 1.31.

Let $P_1 = (1, 2, -1)$ and $P_2 = (-2, 1, -2)$. Let $\mathcal{L}$ be the line in $\mathbb{R}^3$ through P_1 and P_2. An animation of this line is shown in Figure 1.31.

(a) Find a direction vector for the line $\mathcal{L}$.

(b) Find a vector equation as in (1.11) of $\mathcal{L}$.

◁

The Parametric Equations of a Line

The vector form $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ in (1.11) of a line describes a line as the set of terminal points of the vectors $\mathbf{r}(t)$. If we let

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, \quad \mathbf{r}_0 = \langle x_0, y_0, z_0\rangle, \quad \text{and} \quad \mathbf{v} = \langle a, b, c\rangle,$$

then we can equate the components on both sides of $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ to obtain the *parametric equations*

$$x(t) = x_0 + at, \qquad y(t) = y_0 + bt, \qquad z(t) = z_0 + ct.$$

of the line that describe the coordinates of the points on the line. The variable t represents an arbitrary scalar and is called a *parameter*.

The **parametric equations** for a line through the point $P = (x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c\rangle$ are

$$x(t) = x_0 + at, \qquad y(t) = y_0 + bt, \qquad z(t) = z_0 + ct.$$

Activity 1.32.

Let $P_1 = (1, 2, -1)$ and $P_2 = (-2, 1, -2)$, and let $\mathcal{L}$ be the line in $\mathbb{R}^3$ through P_1 and P_2 as in Activity 1.31.

(a) Find the parametric equations of the line $\mathcal{L}$.

(b) Does the point $(1, 2, 1)$ lie on $\mathcal{L}$? Explain.

◁

The Symmetric Equations of a Line

If we assume that a, b, and c are not zero in our parametric equations of a line, then we can solve each of the three equations for t and obtain the *symmetric equations* of a line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

The **symmetric equations** for a line through the point $P = (x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c\rangle$ are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

provided that a, b, and c are not zero.

Activity 1.33.

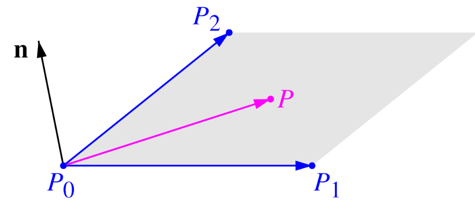

Figure 1.32: A plane determined by three points

Let $P_1 = (1, 2, -1)$ and $P_2 = (-2, 1, -2)$, and let $\mathcal{L}$ be the line in $\mathbb{R}^3$ through P_1 and P_2 as in Activity 1.31.

(a) Find the symmetric equations of the line $\mathcal{L}$.

(b) What will the symmetric equations look like if $a = 0$? Explain.

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The vector and parametric forms of a line allow us to easily describe line segments in space.

Activity 1.34.

Let $P_1 = (1, 2, -1)$ and $P_2 = (-2, 1, -2)$, and let $\mathcal{L}$ be the line in $\mathbb{R}^3$ through P_1 and P_2 as in Activity 1.31.

(a) What values of the parameter t describe the entire line $\mathcal{L}$?

(b) What value of the parameter t makes $(x(t), y(t), z(t)) = P_1$?

(c) What value of the parameter t makes $(x(t), y(t), z(t)) = P_2$?

(d) What restrictions on the parameter t describe the line segment between the points P_1 and P_2?

$\triangleleft$

Planes in Space

Two distinct points in space determine a line and three non-collinear points in space determine a plane. Consider three points P_0, P_1, and P_2 in space, not all lying on the same line as shown in Figure 1.32. We will see how to determine the equation of the plane p containing these points.

Activity 1.35.

Notice that the vectors $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$ both lie in the plane p.

(a) What vector $\mathbf{n}$ do we know that is perpendicular to both $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$?

(b) Let P be any point the plane p. What relationship will the vector $\overrightarrow{P_0P}$ have to $\mathbf{n}$?

(c) Explain why the equation

$$\overrightarrow{P_0P} \cdot \mathbf{n} = 0. \tag{1.12}$$

describes all points in the plane p.

◁

Equation (1.12) is the *vector form* of the equation of a plane. This equation also tells us that a plane is the set of terminal points of vectors from a fixed point that are perpendicular to the vector $\mathbf{n}$ in 1.12.

Definition 1.11. A **plane** in space is the set of all terminal points of vectors emanating from a given point perpendicular to a fixed vector.

The fixed vector in our definition of a plane is called a *normal vector* to the plane (the vector $\mathbf{n}$ is a normal vector in (1.12).

If we let $P_0 = (x_0, y_0, z_0)$, $P = (x, y, z)$, and $\mathbf{n} = \langle a, b, c \rangle$, then (1.12) becomes

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0.$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{1.13}$$

Equation (1.13) is called the *scalar equation* of the plane containing P_0 and with normal vector $\mathbf{n}$.

The **scalar equation** of the plane with normal vector $\mathbf{n} = \langle a, b, c \rangle$ containing the point $P_0 = (x_0, y_0, z_0)$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Activity 1.36.

Let $P_0 = (1, 2, -1)$, $P_1 = (1, 0, -1)$, and $P_2 = (0, 1, 3)$ and let p be the plane containing P_0, P_1, and P_2.

(a) Determine the components of the vectors $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$.

(b) Find a normal vector $\mathbf{n}$ to the plane p.

(c) Find the scalar equation of the plane p.

◁

We can use what we know about vectors and projections to find the distance from a point to a plane.

Activity 1.37.

Let p be the plane with equation $z = -4x + 3y + 4$, whose graph is shown in Figure 1.33. (Note that we can always find an equation of this form by expanding the scalar equation and collecting the constant terms). Let $Q = (4, -1, 8)$.

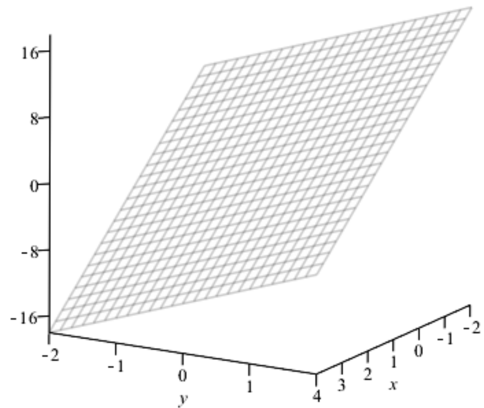

Figure 1.33: Graph of $z = -4x + 3y + 4$.

(a) Show that Q does not lie in the plane p.

(b) Find a normal vector $\mathbf{n}$ to the plane p.

(c) Find the coordinates of a point P in p.

(d) Find the components of $\overrightarrow{PQ}$. Draw a picture in Figure 1.33 to illustrate the objects found so far.

(e) Explain why $|\text{comp}_{\mathbf{n}}\overrightarrow{PQ}|$ gives the distance from the point Q to the plane p. Find this distance.

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Summary

In this section, we encountered the following important ideas:

- A line in space is the set of terminal points of vectors emanating from a given point that are parallel to a fixed vector.
- The terminal points of the vectors of the form $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ define a linear function $\mathbf{r}$ in space through the terminal point of the vector $\mathbf{r}_0$ in the direction of the vector $\mathbf{v}$.
- A plane in space is the set of all terminal points of vectors emanating from a given point perpendicular to a fixed vector.
- If P_1, P_2, and P_3 are non-collinear points in space, the vectors $\overrightarrow{P_1P_2}$ and and $\overrightarrow{P_1P_3}$ are vectors in the plane and the vector $\mathbf{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is a normal vector to the plane. So any point P in the plane satisfies the equation $\overrightarrow{PP_1} \cdot \mathbf{n} = 0$.

1.6 Vector-Valued Functions

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a vector-valued function? What do we mean by the graph of a vector-valued function?
- What is a parameterization of a curve in $\mathbb{R}^2$? In $\mathbb{R}^3$? What are two things that a parameterization of a curve can tell us?

Preview Activity 1.6. In this activity we consider how we might use vectors to define a curve in space.

(a) Draw the vectors $\langle \cos(0), \sin(0) \rangle$, $\left\langle \cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right) \right\rangle$, $\langle \cos(\pi), \sin(\pi) \rangle$, and $\left\langle \cos\left(\frac{3\pi}{2}\right), \sin\left(\frac{3\pi}{2}\right) \right\rangle$ with their initial points at the origin.

(b) Now draw the vectors $\left\langle \cos\left(\frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}\right) \right\rangle$, $\left\langle \cos\left(3\frac{\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right) \right\rangle$, $\left\langle \cos\left(\frac{5\pi}{4}\right), \sin\left(\frac{5\pi}{4}\right) \right\rangle$, and $\left\langle \cos\left(\frac{7\pi}{4}\right), \sin\left(\frac{7\pi}{4}\right) \right\rangle$ on the same axes with their initial points at the origin.

(c) Based on the pictures from parts (a) and (b), sketch a graph of the set of *terminal* points of all of the vectors of the form $\langle \cos(t), \sin(t) \rangle$, where t assumes values from 0 to 2π. What is the resulting figure? Why?

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Introduction

Traces and contours of functions of two variables are curves in space. Our objective in this section is to find a way to describe and draw curves in two and three space. Since a curve is a one-dimensional object, we will need to be able to express the coordinates of points on a curve in terms of a single variable. A vector is a perfect vehicle for this – we can use vectors based at the origin to identify points in space, and connect the terminal points of these vectors to draw a curve in space. The upshot is that we will introduce a new type of function, one whose output is a vector instead of a scalar. This approach will allow us to draw an incredible variety of graphs in 2- and 3-space, as well as identify curves in n-space for any n.

Vector-Valued Functions

Consider the graph shown in Figure 1.34. As we did in Preview Activity 1.6 we can think of a point on this graph as defining a vector from the origin to the point. As the point travels along the graph, the vectors change. An animation is provided in Figure 1.35.

So we can think of the curve as a collection of terminal points of vectors emanating from the origin. We then think of a point traveling along this curve as a function of time t, and define a function $\mathbf{r}$ whose input is the variable t and whose output is the vector from the origin to the point on the curve at time t. The terminal points of the vector outputs of $\mathbf{r}$ then trace out the curve in space. From this perspective, the x, y, and z coordinates of the point are functions of time, t – that is

$$x = x(t), \quad y = y(t), \quad \text{and} \qquad z = z(t)$$

and we have three functions that represent the curve. The new variable t is called a *parameter* and the equations $x = x(t)$, $y = y(t)$, and $z = z(t)$ are called *parametric equations* (or a *parameterization* of the curve). The function $\mathbf{r}$ whose output is the vector from the origin to a point on the curve is defined as

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

Note that the input of $\mathbf{r}$ is the parameter t and the corresponding output is vector $\langle x(t), y(t), z(t) \rangle$. Such a function is called a *vector-valued function.*

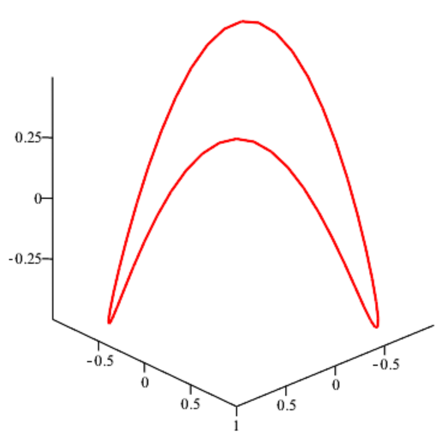

Figure 1.34: The graph of a curve in space.

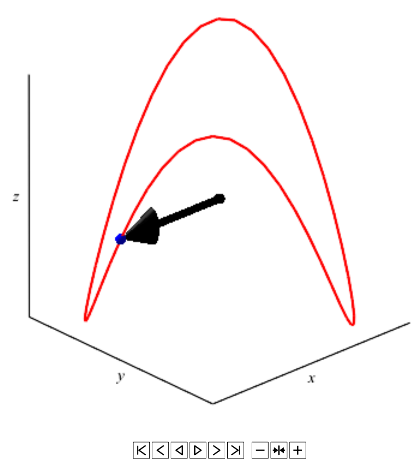

Figure 1.35: The vector-valued function.

Definition 1.12. A **vector-valued function** is a function whose input is a real parameter t and whose output is a vector depending on t. The graph of a vector-valued function is the set of all terminal points of the output vectors with their initial points at the origin.

Parametric equations for a curve are equations of the form

$$x = x(t), \quad y = y(t), \quad \text{and} \quad z = z(t)$$

that describe the (x, y, z) coordinates of a point on a curve.

Every set of parametric equations determines a vector-valued function of the form

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle,$$

and every vector-valued function defines a set of parametric equations for a curve.

As a reminder, in an earlier section we determined the parametric equations of a line in space using a point and a direction vector. So this idea of vector-valued functions and parametric equations is one we have already encountered. As another example, the graph in Figure 1.34 has the parametric equations

$$x(t) = \cos(t), \quad y(t) = \sin(t), \quad \text{and} \quad z(t) = \cos(t)\sin(t).$$

Represented as a vector-valued function $\mathbf{r}$ our curve is the graph of

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), \cos(t)\sin(t) \rangle.$$

We can define a curve in any dimension using a vector-valued function, so this approach is very valuable.

Activity 1.38.

The same curve can be represented with different parameterizations. Use your calculator,[7] Wolfram|Alpha, or some other graphing device[8] to draw the following. Compare and contrast the graphs – explain how they are alike and how they are different.

(a) $\mathbf{r}(t) = \langle \sin(t), \cos(t) \rangle$

(b) $\mathbf{r}(t) = \langle \sin(2t), \cos(2t) \rangle$

(c) $\mathbf{r}(t) = \langle \cos(t + \pi), \sin(t + \pi) \rangle$

[7] If you have a graphing calculator you can draw graphs of two-dimensional vector-valued functions using the parametric mode (often found in the MODE menu).

[8] e.g., http://webspace.ship.edu/msrenault/ggb/parametric_grapher.html

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The examples in Activity 1.38 illustrate that a parameterization allows us to look not only at the graph, but at the direction in which the graph is traversed as t changes. In the different parameterizations of the circle in Activity 1.38, we see that we can start at different points and move around the circle in either direction. In this way, describing curves parametrically allows us to not only indicate the curve itself, but also describe the motion along the curve.

Activity 1.39.

Find a vector-valued function $\mathbf{r}$ that describes a point traveling along the unit circle so that at time $t = 0$ the point is at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and travels clockwise along the circle as t increases.

◁

Using parametric equations to define vector-valued functions in two dimensions is much more versatile than just defining y as a function of x. In fact, if $y = f(x)$ is a function of x, then we can parameterize the graph of f by

$$\mathbf{r}(t) = \langle t, f(t)\rangle.$$

So every curve we have worked with until now can be described parametrically. In addition, we saw in Preview Activity 1.6 and Activity 1.38 that we can use vector-valued functions to draw graphs of curves in the plane that do not define y as a function of x (or x as a function of y).[9]

Activity 1.40.

We can draw some very interesting curves using vector-valued functions. Graph each of the following using an appropriate graphing tool.[10]

(a) $\mathbf{r}(t) = \langle t\cos(t), t\sin(t)\rangle$

(b) $\mathbf{r}(t) = \langle \sin(t)\cos(t), t\sin(t)\rangle$

(c) $\mathbf{r}(t) = \langle t^2\sin(t)\cos(t), 0.9t\cos(t^2), \sin(t)\rangle$

(d) Play around and find your own crazy parametric equation to share with your classmates.

◁

As we have discussed, the traces and level curves of a function are curves in space, and we can now find parameterizations for them.

Activity 1.41.

Consider the paraboloid defined by $f(x, y) = x^2 + y^2$.

[9]As an aside, vector-valued functions make it easy to draw the inverse of an invertible function in two dimensions. To see how, if $y = f(x)$ defines an invertible function, then we can parameterize this function by $\mathbf{r}(t) = \langle t, f(t)\rangle$. Since the inverse function just reverses the role of input and output, a parameterization for f^{-1} is $\langle f(t), t\rangle$.

[10]e.g., the 2D grapher at `http://webspace.ship.edu/msrenault/ggb/parametric_grapher.html`, or for 3D graphs Wolfram|Alpha, an on-line 3D grapher like `http://www.math.uri.edu/~bkaskosz/flashmo/parcur/`, or some other device

(a) Find a parameterization for the $x = 2$ trace of f. What type of curve does this trace describe? (Hint: A trace of f lies on the surface defined by f, so the function f describes the z-coordinate of a point on a trace.)

(b) Find a parameterization for the $y = -1$ trace of f. What type of curve does this trace describe?

(c) Find a parameterization for the level curve $f(x, y) = 25$. What type of curve does this trace describe?

$\triangleleft$

Summary

In this section, we encountered the following important ideas:

- A vector-valued function is a function whose input is a real parameter t and whose output is a vector depending on t. The graph of a vector-valued function is the set of all terminal points of the output vectors with their initial points at the origin.
- Every vector-valued function provides a parameterization of a curve. In $\mathbb{R}^2$ a parameterization of a curve is a pair of equations $x = x(t)$ and $y = y(t)$ that describes the coordinates of a point (x, y) on the curve in terms of a parameter t. In $\mathbb{R}^3$ a parameterization of a curve is a set of three equations $x = x(t)$, $y = y(t)$, and $z = z(t)$ that describes the coordinates of a point (x, y, z) on the curve in terms of a parameter t.
- If we think of the parameter in a parameterization as representing time, then a parameterization of a curve not only describes the curve, but also a direction of motion along the curve and the rate at which a point travels along the curve as a function of time.

1.7 Derivatives and Integrals of Vector-Valued Functions

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What do we mean by the derivative of a vector-valued function?
- How do we calculate the derivative of a vector-valued function? Why?
- What are two things that the derivative of a vector-valued function tells us?
- What do we mean by the integral of a vector-valued function?
- How do we calculate the integral of a vector-valued function? Why?
- How do we describe the motion of a projectile if the only force acting on the object is the acceleration g due to gravity?

Preview Activity 1.7. In this section we will discuss the derivative of a vector-valued function.

(a) Review the definition of the derivative of a function of a single variable from single variable calculus and state the definition here. If a function $f = f(t)$ represents the position of a moving object at time t, what does $f'(t)$ tell us about the object?

(b) Let $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(2t)\mathbf{j}$ describe the path traveled by an object at time t. Use some appropriate technology to draw the graph of this vector-valued function, and then locate and identify the point on the graph when $t = \pi$.

(c) Recall that the derivative of a sum is the sum of the derivatives. With this idea in mind, what do you expect the derivative of $\mathbf{r}$ to be?

(d) Use your result from part (c) to find $\mathbf{r}'(\pi)$. Draw a picture of this vector $\mathbf{r}'(\pi)$ from the point on the graph when $t = \pi$ on your graph from (b) and explain what you think $\mathbf{r}'(\pi)$ tells us about the object.

$\bowtie$

Introduction

A vector-valued function $\mathbf{r}$ defines a curve in space as the collection of terminal points of the vectors $\mathbf{r}(t)$. If the curve is smooth, it is natural to ask if it has a derivative. Once we discuss derivatives, we might as well talk about integrals. Derivatives and integrals of vector-valued functions are the subject of this section.

The Derivative

The definition of the derivative of a function from single variable calculus can also be applied to vector-valued functions and curves in space.

Definition 1.13. The **derivative** of a vector-valued function $\mathbf{r}$ is defined as

$$\frac{d}{dt}\mathbf{r}(t) = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

for those values of t at which the limit exists.

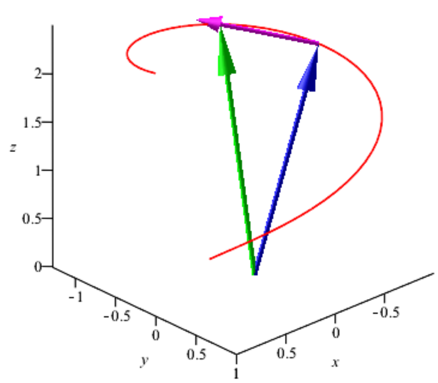

Figure 1.36: A difference quotient.

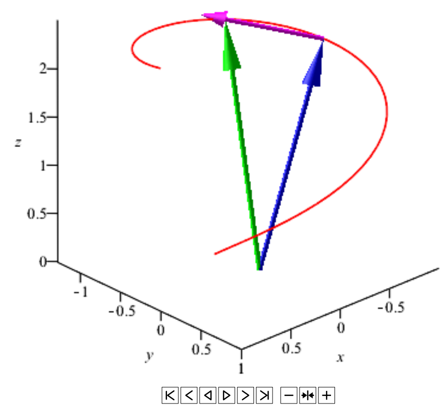

Figure 1.37: The difference quotients.

Activity 1.42.

Let's examine how we can interpret the derivative $\mathbf{r}'(t)$. Let $\mathbf{r}$ be the vector-valued function whose graph is shown in Figure **??**, and let h be a scalar. The vector $\mathbf{r}(t)$ is the blue vector in Figure 1.36 and $\mathbf{r}(t+h)$ is the green vector.

(a) What kind of an object is $\mathbf{r}(t+h) - \mathbf{r}(t)$? A vector or a scalar? What does this object look like in Figure 1.36?

(b) What kind of an object is $\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$? A vector or a scalar? Identify what this object might be in Figure **??**.

(c) What words would you use to describe the vector $\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$? (Think about how we described the difference quotient $\frac{f(x+h)-f(x)}{h}$ for a function of a single variable, and remember that there is no concept of slope in three dimensions.)

(d) Figure 1.37 presents an animation of the vectors $\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ as we let $h \to 0$. Run this animation and describe what you see. What words would you use to describe the vector

$$\lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}?$$

(Think about how we described the limit of the difference quotient $\frac{f(x+h)-f(x)}{h}$ for a function of a single variable, and remember that there is no concept of slope in three dimensions.)

◁

As Activity 1.42 indicates, if $\mathbf{r}(t)$ represents the position of an object at time t, then $\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ represents an average change in the position of the object over the interval $[t, t+h]$, and so

$$\mathbf{v}(t) = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$

is the (instantaneous) velocity of the object at time t, for those values of t for which the limit exists. We also saw that we can interpret the derivative $\mathbf{r}'(t)$ as the direction vector of the line tangent to the graph of $\mathbf{r}$ at the input t. Similarly,

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \lim_{h\to 0} \frac{\mathbf{v}(t+h)-\mathbf{v}(t)}{h}$$

is the instantaneous change in position (or acceleration) of the object at time t, for those values of t for which the limits exists. Note that both the velocity and acceleration are *vector quantities* – they have magnitude and direction. The magnitude of the velocity vector, $\mid \mathbf{v}(t) \mid$ is the *speed* of the object at time t.

Calculating Derivatives

As we saw in single variable calculus, calculating derivatives from the definition is usually quite difficult. Fortunately, properties of the limit make is straightforward to calculate the derivative of a vector-valued function. To see why, recall that the limit of a sum is the sum of the limits, and that we can factor constants out of limits. So, as we did in Preview Activity 1.7, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} \\
&= \lim_{h\to 0} \frac{[x(t+h)-x(t)]\mathbf{i} + [y(t+h)-y(t)]\mathbf{j} + [z(t+h)-z(t)]\mathbf{k}}{h} \\
&= \left(\lim_{h\to 0} \frac{x(t+h)-x(t)}{h}\right)\mathbf{i} + \left(\lim_{h\to 0} \frac{y(t+h)-y(t)}{h}\right)\mathbf{j} + \left(\lim_{h\to 0} \frac{z(t+h)-z(t)}{h}\right)h \\
&= x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.
\end{aligned}$$

Thus, we can calculate the derivative of a vector-valued function by differentiating each of its components.

If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then

$$\frac{d}{dt}\mathbf{r}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

for those values of t at which x, y, and z are differentiable.

Activity 1.43.

Find $\mathbf{r}'(t)$ if $\mathbf{r}(t) = \langle \cos(t), t\sin(t), \ln(t) \rangle$.

◁

We derived many differentiation rules in single variable calculus that transfer to vector-valued functions and products of scalar functions and vector-valued functions. These are summarized in the following theorem.

Let f be a differentiable real valued function of a real variable t and let $\mathbf{r}$ and $\mathbf{s}$ be differentiable vector-valued functions of the parameter t. Then

1. $\dfrac{d}{dt}\left(\mathbf{r}(t) + \mathbf{s}(t)\right) = \mathbf{r}'(t) + \mathbf{s}'(t)$
2. $\dfrac{d}{dt}f(t)\mathbf{r}(t) = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$
3. $\dfrac{d}{dt}\left(\mathbf{r}(t) \cdot \mathbf{s}(t)\right) = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}(t)$
4. $\dfrac{d}{dt}\left(\mathbf{r}(t) \times \mathbf{s}(t)\right) = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t)$
5. $\dfrac{d}{dt}\mathbf{r}(f(t)) = f'(t)\mathbf{r}'(f(t))$.

Be careful about these properties. Note that $\mathbf{r}(t) \cdot \mathbf{s}(t)$ is a scalar function while $\mathbf{r}(t) \times \mathbf{s}(t)$ is a vector-valued function.

To get a flavor for how to verify these properties, consider property (4). Let $\mathbf{r}(t) = \langle x_r(t), y_r(t), z_r(t) \rangle$

and $\mathbf{s}(t) = \langle x_s(t), y_s(t), z_s(t)\rangle$. Then

$$\begin{aligned}
\frac{d}{dt}\left(\mathbf{r}(t)\times\mathbf{s}(t)\right) &= \frac{d}{dt}\big([y_r(t)z_s(t) - y_s(t)z_r(t)]\mathbf{i} + [x_s(t)z_r(t) - x_r(t)z_s(t)]\mathbf{j} \\
&\quad + [x_r(t)y_s(t) - x_s(t)y_r(t)]\mathbf{k}\big) \\
&= \left(\frac{d}{dt}[y_r(t)z_s(t) - y_s(t)z_r(t)]\right)\mathbf{i} + \left(\frac{d}{dt}[x_s(t)z_r(t) - x_r(t)z_s(t)]\right)\mathbf{j} \\
&\quad + \left(\frac{d}{dt}[x_r(t)y_s(t) - x_s(t)y_r(t)]\right)\mathbf{k} \\
&= [(y_r(t)z_s'(t) + y_r'(t)z_s(t)) - (y_s(t)z_r'(t) + y_s'(t)z_r(t)]\mathbf{i} \\
&\quad + [(x_s(t)z_r'(t) + x_s'(t)z_r(t)) - (x_r(t)z_s'(t) + x_r'(t)z_s(t)]\mathbf{j} \\
&\quad + [(x_r(t)y_s'(t) + x_r'(t)y_s(t)) - (x_s(t)y_r'(t) + x_s'(t)y_r(t))]\mathbf{k} \\
&= \big([y_r'(t)z_s(t) - y_s(t)z_r'(t)]\mathbf{i} + [x_s(t)z_r'(t) - x_r'(t)z_s(t)]\mathbf{j} + [x_r'(t)y_s(t) - x_s(t)y_r'(t)]\mathbf{k}\big) \\
&\quad + \big([y_r(t)z_s'(t) - y_s'(t)z_r(t)]\mathbf{i} + [x_s'(t)z_r(t) - x_r(t)z_s'(t)]\mathbf{j} \\
&\quad + [x_r(t)y_s'(t) - x_s'(t)y_r(t)]\mathbf{k}\big) \\
&= \mathbf{r}'(t)\times\mathbf{s}(t) + \mathbf{r}(t)\times\mathbf{s}'(t).
\end{aligned}$$

The verifications of the remainder of the properties are left to the reader.

Activity 1.44.

Use the properties of differentiation of vector-valued functions to find the following derivatives.

(a) $\dfrac{d}{dt}\sin(t)\langle 2t, t^2, \arctan(t)\rangle$

(b) $\dfrac{d}{dt}\left(\mathbf{r}(2^t)\right)$, where $\mathbf{r}(t) = \langle t+2, \ln(t), 1\rangle$.

$\triangleleft$

A smooth curve in 3-space will be locally linear. Just as in single variable calculus we can investigate the tangent line to such a curve.

Activity 1.45.

Let

$$\mathbf{r}(t) = \cos(t)\mathbf{i} - \sin(t)\mathbf{j} + t\mathbf{k}.[11]$$

(a) Find a direction vector for the line tangent to the graph of $\mathbf{r}$ at the point where $t = \pi$.

(b) Find the parametric equations of the line tangent to the graph of $\mathbf{r}$ when $t = \pi$.

[11] You can draw the graph with Wolfram Alpha, http://www.math.uri.edu/~bkaskosz/flashmo/parcur/, or some other appropriate device.

◁

A smooth surface in 3-space will be locally linear. That means that the surface will look like a plane (its tangent plane) as we zoom in on the graph and we can investigate the tangent plane to such a surface.

Activity 1.46.

In this activity we determine the equation of a plane tangent to the surface defined by $f(x,y) = x^2 + y^2$ at the point $(3,4,5)$.

(a) Find a parameterization for the $x = 3$ trace to f. What is a direction vector for the line tangent to this trace at the point $(3,4,5)$?

(b) Find a parameterization for the $y = 4$ trace to f. What is a direction vector for the line tangent to this trace at the point $(3,4,5)$?

(c) The direction vectors in parts (a) and (b) form a plane containing the point $(3,4,5)$. What is a normal vector for this plane? Find the equation of this plane. Use appropriate technology to draw the graph of f and the plane you determined on the same set of axes. What do you notice? (This plane is the tangent plane to f at the point $(3,4,5)$. We will discuss tangent planes in more detail in a later section.)

◁

Integrating a Vector Valued Function

Recall from single variable calculus that an antiderivative of a function f of the independent variable x is a function F so that $F'(x) = f(x)$. We then defined the indefinite integral $\int f(x)\,dx$ to be the collection of all antiderivatives of f. We can do the same for vector-valued functions.

Definition 1.14. An **antiderivative** of a vector-valued function $\mathbf{r}$ is a vector-valued function $\mathbf{R}$ such that

$$\mathbf{R}'(t) = \mathbf{r}(t).$$

The **indefinite integral** $\int \mathbf{r}(t)\,dt$ of a vector-valued function $\mathbf{r}$ is the set of all antiderivatives of $\mathbf{r}$.

The same reasoning that allows us to differentiate a vector-valued function by differentiating its components applies to integrating as well. Recall that the integral of a sum is the sum of the integrals and we can factor constants out of integrals. So if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then

$$\begin{aligned}\int \mathbf{r}(t)\,dt &= \int (x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k})\;dt \\ &= \left(\int x(t)\,dt\right)\mathbf{i} + \left(\int y(t)\,dt\right)\mathbf{j} + \left(\int z(t)\,dt\right)\mathbf{k}.\end{aligned}$$

So we can integrate a vector-valued function by integrating its components.

If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then

$$\int \mathbf{r}(t)\, dt = \left(\int x(t)\, dt\right)\mathbf{i} + \left(\int y(t)\, dt\right)\mathbf{j} + \left(\int z(t)\, dt\right)\mathbf{k}.$$

Activity 1.47.

Let $\mathbf{r}(t) = \left\langle \cos(t), \frac{1}{t+1}, te^t \right\rangle$.

(a) Find $\int \mathbf{r}(t)\, dt$

(b) Find $\int_0^1 \mathbf{r}(t)\, dt$. Explain why what you have done makes sense. (Hint: Recall some important theorem from single variable calculus.)

$\triangleleft$

Now we combine what we have learned about differentiating and integrating vector-valued functions in the following activity.

Activity 1.48.

Suppose an object is at the point $(1.5, -1, 0)$ at time $t = 0$ and has velocity

$$\mathbf{v}(t) = (-2\sin 2t)\mathbf{i} + 2\cos(t)\mathbf{j} + \left(\frac{t}{1+t}\right)\mathbf{k}.$$

A graph of the position of the object for times t in $[-0.5, 3]$ is shown in Figure 1.38.

(a) Find the acceleration of the object at time t.

(b) Find the position of the object at time t.

(c) Approximate and then draw a picture of the position, velocity, and acceleration vectors of the object at time $t = 1$.

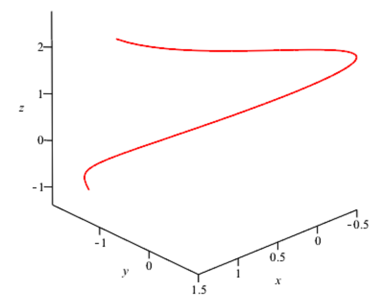

Figure 1.38: The position graph.

$\triangleleft$

Projectile Motion

Preview Activity 1.8. When computers first came out, there was a QBasic game, Gorillas, in which gorillas threw banana grenades at each other attempting to blow each other up.[12] Typically, a

[12] A on-line version can be found at http://www.online-games-zone.com/pages/classic/gorillas.php. No endorsement of this site or the advertisements is implied.

player uses trial and error to determine the correct velocity and launch angle to destroy his/her opponent. Play a few rounds of this game and come up with a strategy to win the game.

$\bowtie$

The Gorillas game illustrated in Preview Activity 1.8 provides the basic idea behind all projective motion. Given an initial speed and launch angle of a projectile, we want to determine where the projectile will land. Applications of this idea appear in many places, including sports (e.g., golf, archery, shotput), military (e.g., artillery and missiles), firefighting, etc. In these applications (and in our Gorillas game) we can use our knowledge of vector-valued functions to determine the path traveled by an object that is launched from a given position at a given angle from the horizontal with a given initial velocity. Once we have determined this path, we can apply the results in many situations, including finding the correct initial conditions for our gorilla to hit its target. This will be nothing but a straightforward, and FUN, application of MATHEMATICS!

Assume we are firing a projectile from a launcher and the only force acting on the fired object is the force of gravity pulling down on the object. (So we assume no effect due to spin or wind resistance.) With these assumptions, the motion of the object will be planar, so we can also assume that the motion is in two dimensional space. Suppose we launch the object from an initial position (x_0, y_0) at an angle θ with the positive x-axis as illustrated in Figure 1.39, and that we fire the object with an initial speed of $v_0 = |\mathbf{v}(0)|$, where $\mathbf{v}(t)$ is the velocity vector of the object at time t. Assume g is the constant acceleration force due to gravity (a positive constant), acting to pull the fired object down (the negative y direction) to the ground. Since gravity acts only in the downward direction, there is no external force acting on the object to move it in the x direction.

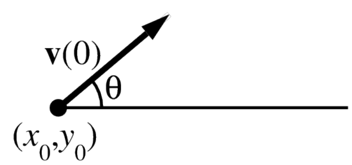

Figure 1.39: Projectile motion.

The following activities will help us determine the path traveled by the launched object.

Activity 1.49.

Explain why the acceleration vector $\mathbf{a}$ of the fired object is $\mathbf{a} = \langle 0, -g\rangle$.

$\triangleleft$

Ultimately, we want to find the position of the object as it travels through space so that we can determine what initial velocity and launch angle to provide so that we hit our target.

Activity 1.50.

Let $\mathbf{a} = \langle 0, -g\rangle$ be the acceleration vector from Activity 1.49.

(a) Find all velocity vectors for the given acceleration vector $\mathbf{a}$.

(b) Now we find the specific velocity vector function $\mathbf{v}(t)$ for our situation. Use the fact that $v_0 = |\mathbf{v}(0)|$ and that the object is launched at an angle θ from the horizontal to determine the components of the vector $\mathbf{v}(0)$ (your vector will be in terms of v_0 and the cosine and sine of θ). Then find $\mathbf{v}(t)$ at any time t.

◁

Now we can find the position vector for the object at any time.

Activity 1.51.

Let $\mathbf{v}(t)$ be the velocity vector found in Activity 1.50.

(a) Find all possible position vectors for the velocity vector $\mathbf{v}(t)$.

(b) Let $\mathbf{r}(t)$ denote the position vector function for our projectile. Use the fact that the object is fired from the position (x_0, y_0) to explain why

$$\mathbf{r}(t) = \left\langle v_0 \cos(\theta)t + x_0, -g\frac{t^2}{2} + v_0 \sin(\theta)t + y_0 \right\rangle.$$

◁

We summarize the result of Activities 1.48 through 1.50.

If an object is launched from a point (x_0, y_0) with initial velocity v_0 at an angle θ with the horizontal, then the position of the object at time t is given by

$$\mathbf{r}(t) = \left\langle v_0 \cos(\theta)t + x_0, -g\frac{t^2}{2} + v_0 \sin(\theta)t + y_0 \right\rangle.$$

This assumes that the only force acting on the object is the acceleration g due to gravity.

Activity 1.52.

Suppose our gorilla is perched on a building at a height of 300 feet, and her opponent is at the top of a building 500 feet away at a height of 200 feet. Use $g = 32$ feet per second per second.

(a) If our gorilla throws a banana with initial speed of 85 feet per second with a launch angle of $60°$ (assuming no buildings are in the way), will the banana hit her opponent?

(b) Is it possible for our gorilla to hit her opponent with the banana if she can provide an initial speed for her banana of 85 feet per second. If yes, at what angle should she release her banana? If no, why not?

◁

Summary

In this section, we encountered the following important ideas:

- If $\mathbf{r}$ is a vector-valued function, then the derivative of $\mathbf{r}$ is defined as

$$\frac{d}{dt}\mathbf{r}(t) = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

for those values of t at which the limit exists.

- Since the limit of a sum is the sum of the limits, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then

$$\frac{d}{dt}\mathbf{r}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

for those values of t at which x, y, and z are differentiable.

- Two things that the derivative $\mathbf{r}'(t)$ of the vector-valued function $\mathbf{r}$ tells us are:

  1. a direction vector for the line tangent to the graph of $\mathbf{r}$ at the point $\mathbf{r}(t)$,
  2. the instantaneous velocity of an object traveling along the graph defined by $\mathbf{r}(t)$ at time t.

- The indefinite integral $\int \mathbf{r}(t)\ dt$ of a vector-valued function $\mathbf{r}$ is the set of all antiderivatives of $\mathbf{r}$, where an antiderivative of $\mathbf{r}$ is a vector-valued function $\mathbf{R}$ such that $\mathbf{R}'(t) = \mathbf{r}(t)$.

- Since the integral of a sum is the sum of integrals, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then

$$\int \mathbf{r}(t)\ dt = \left(\int x(t)\ dt\right)\mathbf{i} + \left(\int y(t)\ dt\right)\mathbf{j} + \left(\int z(t)\ dt\right)\mathbf{k}.$$

- If an object is launched from a point (x_0, y_0) with initial velocity v_0 at an angle θ with the horizontal, then the position of the object at time t is given by

$$\mathbf{r}(t) = \left\langle v_0\cos(\theta)t + x_0, -g\frac{t^2}{2} + v_0\sin(\theta)t + y_0 \right\rangle.$$

This assumes that the only force acting on the object is the acceleration g due to gravity.

1.8 Arc Length and Curvature

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How can a definite integral be used to measure the length of a curve in 2- or 3-space?
- Why is arc length useful as a parameter?
- What is the curvature of a curve?

Preview Activity 1.9. We have used the integration process in many situations, to calculate area, volume, mass, work, among others. In this section we apply the process to calculate the length of a curve in 3-space. This process will also allow us to find the length of a curve in 2-space that is defined parametrically. We motivate the ideas with an example. Consider the smooth curve in 3-space defined by the vector-valued function

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle = \langle \cos(t), \sin(t), t\rangle$$

for t in the interval $[0, 2\pi]$. A picture of the graph of $\mathbf{r}$ is shown in Figure 1.40.

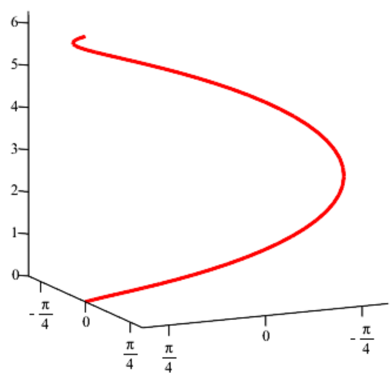

Figure 1.40: The graph of $\mathbf{r}$.

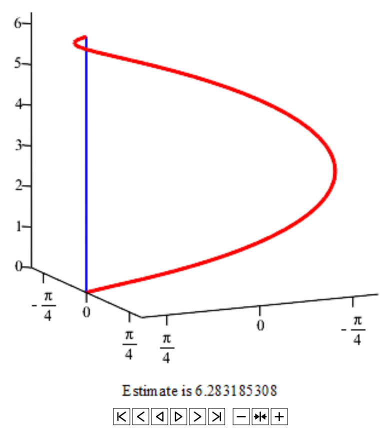

Figure 1.41: Approximating the length of the curve.

(a) The animation in Figure 1.41 illustrates the process of approximating the length of the curve defined by $\mathbf{r}(t)$ on the interval $[0, 2\pi]$. Run the animation and explain what is happening.

(b) Partition the interval $[0, 2\pi]$ into n subintervals of equal length and let $0 = t_0 < t_1 < t_2, \cdots < tn = b$ be the endpoints of the subintervals. Write a formula to approximate the length of the curve on the ith subinterval $[t_{i-1}, t_i]$. (Hint: the animation in Figure **??** implies that you should use the length of an appropriate line segment.)

(c) Use your formula in part (b) to write a sum that adds all of the approximations to the lengths on each subinterval.

(d) What do we need to do with the sum in part (c) in order to obtain the exact value of the length of the graph of $\mathbf{r}(t)$ on the interval $[0, 2\pi]$?

$\bowtie$

Introduction

In this section, we will use the integration process to find the length of a parametrically defined curve in 3-space, which will also show us how to find the length of a parametrically defined curve in 2-space. Once we have defined arc length, we use arc length as a parameter to define curvature.

Arc Length

Consider a vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ that defines a smooth curve in 3-space. Preview Activity 1.9 and the animation in Figure 1.41 show that to approximate the length of the curve defined by $\mathbf{r}(t)$ as the values of t run over an interval $[a, b]$, we partition the interval $[a, b]$ into n subintervals of equal length Δt, with $a = t_0 < t_1 < \cdots < t_n = b$ as the endpoints of the subintervals. On each subinterval we approximate the length of the curve by the length of the line segment connecting the endpoints. The points on the curve corresponding to $t = t_{i-1}$ and $t = t_i$ are $(x(t_{i-1}), y(t_{i-1}), z(t_{i-1}))$ and $(x(t_i), y(t_i), z(t_i))$, respectively, so the length of the line segment connecting these points is

$$\sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2 + (z(t_i) - z(t_{i-1}))^2}.$$

Now we add all of these approximations together to obtain an approximation to the length L of the curve:

$$L \approx \sum_{i=1}^{n} \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2 + (z(t_i) - z(t_{i-1}))^2}.$$

We now want to take the limit of this sum as n goes to infinity, but in its present form it might be difficult to see how. A little rewriting should make this more clear:

$$\begin{aligned}
L &\approx \sum_{i=1}^{n} \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2 + (z(t_i) - z(t_{i-1}))^2} \\
&= \sum_{i=1}^{n} \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2 + (z(t_i) - z(t_{i-1}))^2}\,\frac{\Delta t}{\Delta t} \\
&= \sum_{i=1}^{n} \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2 + (z(t_i) - z(t_{i-1}))^2}\,\frac{\Delta t}{\sqrt{(\Delta t)^2}} \\
&= \sum_{i=1}^{n} \sqrt{[(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2 + (z(t_i) - z(t_{i-1})^2]\,\frac{1}{(\Delta t)^2}}\,\Delta t \\
&= \sum_{i=1}^{n} \sqrt{\left(\frac{x(t_i) - x(t_{i-1})}{\Delta t}\right)^2 + \left(\frac{y(t_i) - y(t_{i-1})}{\Delta t}\right)^2 + \left(\frac{z(t_i) - z(t_{i-1})}{\Delta t}\right)^2}\,\Delta t.
\end{aligned}$$

Recall that as $n \to \infty$ we also have $\Delta t \to 0$. Since

$$\lim_{\Delta t \to 0} \frac{x(t_i) - x(t_{i-1})}{\Delta t} = x'(t), \quad \lim_{\Delta t \to 0} \frac{y(t_i) - y(t_{i-1})}{\Delta t} = y'(t), \quad \text{and} \quad \lim_{\Delta t \to 0} \frac{z(t_i) - z(t_{i-1})}{\Delta t} = z'(t)$$

we see that

$$\begin{aligned}
L &= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\left(\frac{x(t_i) - x(t_{i-1})}{\Delta t}\right)^2 + \left(\frac{y(t_i) - y(t_{i-1})}{\Delta t}\right)^2 + \left(\frac{z(t_i) - z(t_{i-1})}{\Delta t}\right)^2}\,\Delta t \\
&= \lim_{\Delta t \to 0} \sum_{i=1}^{n} \sqrt{\left(\frac{x(t_i) - x(t_{i-1})}{\Delta t}\right)^2 + \left(\frac{y(t_i) - y(t_{i-1})}{\Delta t}\right)^2 + \left(\frac{z(t_i) - z(t_{i-1})}{\Delta t}\right)^2}\,\Delta t \\
&= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt. \qquad (1.14)
\end{aligned}$$

Note that

$$|\mathbf{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2},$$

so we can rewrite (1.14) in a more succinct form as follows.

> If $\mathbf{r}(t)$ defines a smooth curve C on an interval $[a, b]$, then the length L of C is given by
>
> $$L = \int_a^b |\mathbf{r}'(t)|\, dt. \qquad (1.15)$$

Note that formula (1.15) applies to curves in any dimensional space.

Activity 1.53.

Use (1.15) to calculate the circumference of a circle of radius r.

◁

Activity 1.54.

Find the exact length of the spiral defined by $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ on the interval $[0, 2\pi]$. (Hint: Use Wolfram|Alpha or a table of integrals.)

◁

We can adapt the arc length formula to curves in 2-space that define y as a function of x as the following activity shows.

Activity 1.55.

Let $y = f(x)$ define a smooth curve in 2-space. Parameterize this curve and use (1.15) to show that the length of the curve define by f on an interval $[a, b]$ is

$$\int_a^b \sqrt{1 + [f'(t)]^2}\, dt.$$

◁

The calculation of arc length is useful, but arc length can be even more important when used as a parameter, as we see in the following sections.

Parameterizing With Respect To Arc Length

If we let s be the length of a curve from some time t_0 to time t, then we have

$$s(t) = \int_{t_0}^{t} \sqrt{(x'(w))^2 + (y'(w))^2 + (z'(w))^2}\, dw,$$

and s is a function of t. Moreover, the Fundamental Theorem of Calculus shows us that

$$\frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = |\mathbf{r}'(t)|. \tag{1.16}$$

So if $\mathbf{r}'(t)$ is never 0, then $s'(t) > 0$ for all t and s is always increasing. (This should seem reasonable, since the curve length increases as t increases, unless we stop.) In 2-space (using $z(t) = 0$) the argument gives us

$$\frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2} = |\mathbf{r}'(t)|.$$

As we will shortly see, there is some use to being able to parameterize curves in terms of their arc length instead of just time. Think of this as akin to determining your position in your car if you know your odometer reading and the path of the road on which you are traveling.

Example 1.1. As an example of an arc length parameterization, consider a circle of radius a in 2-space centered at the origin. We know that we can parameterize this circle as

$$x(t) = a\cos(t) \quad \text{and} \quad y(t) = a\sin(t).$$

If we think of t as an angle in radians measured from the positive x-axis, then $s(t) = at$, and so $t = \frac{s}{a}$. This give us a parameterization of the unit circle in terms of arc length as

$$x(s) = a\cos\left(\frac{s}{a}\right) \quad \text{and} \quad y(t) = a\sin\left(\frac{s}{a}\right). \tag{1.17}$$

In this case finding a formula for s in terms of t was easy because we know how to find the length of an arc on a circle. If we didn't know that , we could also have found the relationship between s and t by using formula (1.15):

$$\begin{aligned} s(t) &= \int_0^t \sqrt{(x'(w))^2 + (y'(w))^2}\, dw \\ &= \int_0^t \sqrt{a^2[-\sin(t)^2 + (\cos(t)^2]}\, dw \\ &= \int_0^t a\, dw \\ &= at. \end{aligned}$$

Activity 1.56.

In this activity we parameterize a line in 2-space terms of arc length. Consider the line with parametric equations

$$x(t) = x_0 + at \quad \text{and} \quad y(t) = y_0 + bt.$$

(a) To write t in terms of s, evaluate the integral

$$s(t) = \int_0^t \sqrt{(x'(w))^2 + (y'(w))^2}\, dw$$

to determine the length of the line from time 0 to time t.

(b) Use the formula from (a) for s in terms of t to write t in terms of s. Then explain why a parameterize of the line in terms of arc length is

$$x(s) = x_0 + \frac{a}{\sqrt{a^2 + b^2}}s \quad \text{and} \quad y(s) = y_0 + \frac{b}{\sqrt{a^2 + b^2}}s. \tag{1.18}$$

◁

A little more complicated example is the following.

Example 1.2. Let us parameterize the curve defined by

$$\mathbf{r}(t) = \left\langle t^2, \frac{8}{3}t^{3/2}, 4t \right\rangle$$

for $t \geq 0$ in terms of arc length. To write t in terms of s we find s in terms of t:

$$\begin{aligned}
s(t) &= \int_0^t \sqrt{(x'(w))^2 + (y'(w))^2 + (z'(w))^2}\, dw \\
&= \int_0^t \sqrt{(2w)^2 + (4w^{1/2})^2 + (4)^2}\, dw \\
&= \int_0^t \sqrt{4w^2 + 16w + 16}\, dw \\
&= 2\int_0^t \sqrt{(w+2)^2}\, dw \\
&= 2\int_0^t w + 2\, dw \\
&= \left(w^2 + 4w\right)\Big|_0^t \\
&= t^2 + 4t.
\end{aligned}$$

Since $t \geq 0$, we can solve the equation $s = t^2 + 4t$ (or $t^2 + 4t - s = 0$) for t to obtain $t = \frac{-4+\sqrt{16+4s}}{2} = -2 + \sqrt{4+s}$. So we can parameterize our curve in terms of arc length by

$$\mathbf{r}(s) = \left\langle \left(-2+\sqrt{4+s}\right)^2, \frac{8}{3}\left(-2+\sqrt{4+s}\right)^{3/2}, 4\left(-2+\sqrt{4+s}\right) \right\rangle.$$

Example 1.2 points out a general method. If we can evaluate the integral for arc length in terms of t to write $s = f(t)$, then suitable restrictions on t can make f invertible and we can write $t = f^{-1}(s)$. Substituting for the parameter t in a parameterization gives us a parameterization in terms of arc length. Of course, evaluating an arc length integral can be complicated, and finding a formula for the inverse of a function can also be difficult, so while this process is theoretically possible, it is not always practical to parameterize a curve in terms of arc length. However, the existence of such a parameterization is important, as we see in the next section.

Curvature

For a smooth space curve, the *curvature* measures how fast the curve is bending or changing direction at a given point. For example, a line should have 0 curvature everywhere, while a circle (which is bending the same at every point) should have constant curvature, and circles with larger radii should have smaller curvature.

A natural place to start to think about measuring curvature is to determine how fast the tangent vector is changing, or $|\mathbf{v}'(t)|$, as illustrated in Figure 1.42. However, $\mathbf{v}'(t)$ is a function of time, and curvature should not depend on how fast we are traveling along the curve. How the curve is bending should be independent of how fast we travel along the curve. Think about driving a car around a curve – the curve is the same regardless of how fast we drive, but you can feel the effect of the curve more if you drive through it faster. We need a way to measure curvature that does not depend on speed or parameterization. A next logical step is to try to remove the dependence on speed by making the speed constant, that is normalizing the velocity vector and consider using the *unit tangent vector*

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}.$$

We might then try to use $|\mathbf{T}'(t)|$ as a measure of curvature. But this, too, has its problems. To see why, note that $\mathbf{r}_1(t) = \langle t, t^2 \rangle$ and $\mathbf{r}_2(t) = \langle 2t, 4t^2 \rangle$ are two parameterizations of the parabola $y = x^2$. Their respective velocity vectors are $\mathbf{v}_1(t) = \langle 1, 2t \rangle$ and $\mathbf{v}_2(t) = \langle 2, 8t \rangle$ and the respective unit tangent vectors are $\mathbf{T}_1(t) = \frac{1}{\sqrt{1+4t^2}}\langle 1, 2t \rangle$ and $\mathbf{T}_2(t) = \frac{1}{\sqrt{1+16t^2}}\langle 1, 4t \rangle$. It follows that

$$\left|\mathbf{T}_1'(t)\right| = \frac{2}{1+4t^2} \quad \text{and} \quad \left|\mathbf{T}_2'(t)\right| = \frac{4}{1+16t^2}.$$

Now the point $(1,1)$ is the terminal point of $\mathbf{r}_1(1)$ and $\mathbf{r}_2\left(\frac{1}{2}\right)$, so we should expect the curvature of this curve to be the same whether we use the parameterization $\mathbf{r}_1$ or $\mathbf{r}_2$. But

$$\frac{2}{1+4(1)^2} = \frac{2}{5} \quad \text{and} \quad \frac{4}{1+16\left(\frac{1}{2}\right)^2} = \frac{4}{5}.$$

So the value of $|\mathbf{T}'(t)|$ at a point depends on the parameterization we use. In other words, we still haven't removed the dependence on how fast we travel along a curve. That is, $|\mathbf{T}'(t)|$ still depends on the parameterization we choose.

To remove the dependence of our computations on the parameterization, we need to take a different approach – one that relies only on the geometric properties of the curve. A perfect method for this it to use the arc length parameterization discussed in the previous section. Even if we have different parameterizations of a curve, we travel the same distance when we move along the curve from point P to point Q regardless of how fast we go. In other words, arc length is independent of the parameterization, so arc length is the kind of parameter we want.

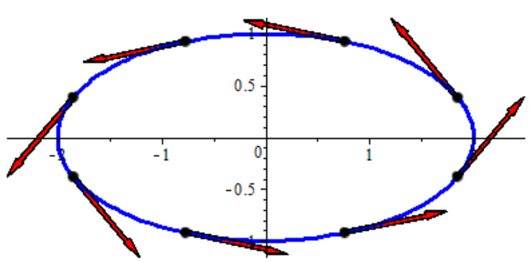

Figure 1.42: Tangent vectors to an ellipse.

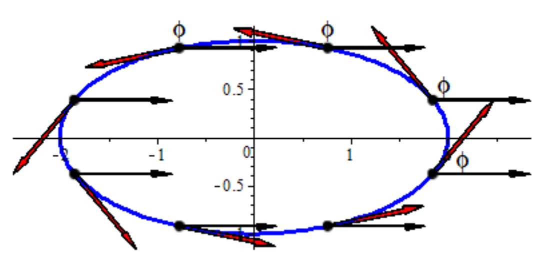

Figure 1.43: Angles of tangent vectors.

We still want to measure the change in the unit tangent vectors, but the key is to parameterize the tangent vectors in terms of arc length. The magnitude of the change in the unit tangent vector with respect to arc length will tell us how fast the unit tangent vector is changing as the arc length changes, giving us a measure of the bend in the curve that does not depend on the parameterization. This is how we define curvature.

Definition 1.15. If C is a smooth curve in 2-space or 3-space and s is an arc length parameter for C, then the **curvature**, κ,[13] of C is

$$\kappa = \kappa(s) = \left|\frac{d\mathbf{T}}{ds}\right|.$$

Example 1.3. We should expect that the curvature of a line is 0 everywhere. To show that our definition of curvature measures this correctly in 2-space, recall that (1.18) gives us the arc length parameterization

$$x(s) = x_0 + \frac{a}{\sqrt{a^2+b^2}}s \quad \text{and} \quad y(s) = y_0 + \frac{b}{\sqrt{a^2+b^2}}s$$

of a line. So

$$\mathbf{T}(s) = \left\langle \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right\rangle$$

and, since $\mathbf{T}(s)$ is constant we have

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = 0$$

as expected.

Activity 1.57.

Recall from (1.17) that an arc length parameterization of a circle in 2-space of radius a centered at the origin is (from (1.17))

$$x(s) = a\cos\left(\frac{s}{a}\right) \quad \text{and} \quad y(t) = a\sin\left(\frac{s}{a}\right).$$

Show that the curvature of this circle is the constant $\frac{1}{a}$. Conclude that smaller circles have larger curvature. Does this make sense?

◁

These calculations of curvature that we have made depend on our ability to parameterize our curves in terms of arc length. As we saw, this usually involves finding arc length s in terms of the parameter t in the form $s = f(t)$, and then using the inverse of f. This is not practical in general, so it would be useful to have a formulation of curvature that can be used with a given (non-arc length) parameterization. The Chain Rule and equation (1.16) will give us such a formulation.

Suppose $\mathbf{r}(t)$ is a smooth vector-valued function. Then the curvature of the curve defined by $\mathbf{r}$ is

$$\begin{aligned}\kappa &= \left|\frac{d\mathbf{T}}{ds}\right| \\ &= \left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right| \\ &= \frac{\left|\frac{d\mathbf{T}}{dt}\right|}{\left|\frac{ds}{dt}\right|} \\ &= \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.\end{aligned}$$

This last formula allows us to use any parameterization of a curve to calculate it's curvature. There is another useful formula, given below, whose derivation is left for the exercises.

If $\mathbf{r}(t)$ is a vector-valued function defining a smooth curve C in 2-space or 3-space, and if $\mathbf{r}'(t)$ is not zero and if $\mathbf{r}''(t)$ exists, then the curvature κ of C satisfies

- $\kappa = \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$
- $\kappa = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.

Activity 1.58.

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has parameterization

$$\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle.$$

Find the curvature of the ellipse. Assuming $0 < b < a$, at what points is the curvature the greatest and at what points is the curvature the smallest? Does this agree with your intuition?

◁

Of course, we can calculate curvature in 3-space.

Activity 1.59.

Find the curvature of the helix with parameterization $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$.

◁

We can interpret curvature in 2-space in an interesting way. Let $\mathbf{r}(t)$ define a smooth curve in $\mathbb{R}^2$ and let ϕ be the angle that the tangent vector $\mathbf{r}'(t)$ makes with the vector $\mathbf{i}$, as shown in Figure 1.43. If $\mathbf{T}$ is a unit tangent vector to the curve where ϕ is the angle between $\mathbf{T}$ and $\mathbf{i}$, then we can write $\mathbf{T}$ as

$$\mathbf{T} = \langle\cos(\phi), \sin(\phi)\rangle.$$

As a result, we have

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{d\phi}\frac{d\phi}{ds},$$

so

$$\begin{aligned}
\kappa &= \left|\frac{d\mathbf{T}}{ds}\right| \\
&= \left|\frac{d\mathbf{T}}{d\phi}\right| \left|\frac{d\phi}{ds}\right| \\
&= |\langle -\sin(\phi), \cos(\phi)| \left|\frac{d\phi}{ds}\right| \\
&= \sqrt{\sin^2(\phi)+\cos^2(\phi)}\left|\frac{d\phi}{ds}\right| \\
&= \left|\frac{d\phi}{ds}\right|.
\end{aligned}$$

So the curvature of a curve in 2-space is also a measure of how the angle between the tangent vectors and the positive x-axis is changing as we move along the curve.

Summary

In this section, we encountered the following important ideas:

- The integration process shows that the length L of a smooth curve defined by $\mathbf{r}(t)$ on an interval $[a, b]$ is
$$L = \int_a^b |\mathbf{r}'(t)|\, dt.$$
- Arc length is useful as a parameter because when we parameterize with respect to arc length, we eliminate the role of speed in our calculation of curvature and the result is a measure that depends only on the geometry of the curve and not on the parameterization of the curve.
- We define the curvature κ of a curve in 2- or 3-space to be the rate of change of the unti tangent vector with respect to arc length, or
$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

Chapter 2

Derivatives of Multivariable Functions

2.1 Limits

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What do we mean by the limit of a function f of two variables at a point in the domain of the function?
- What techniques can we use to show that a function of two variables does not have a limit at a point (a, b) in its domain?
- What does it mean for a function f of two variables to be continuous at a point in its domain?

Preview Activity 2.1. In this section we study limits of functions of several variables, with a focus on limits of functions of two variables.

(a) Review the definition of the limit of a function f of a single variable x from first semester calculus and correctly complete the following sentence.

A function $f = f(x)$ has a limit at the point where $x = a$ if ... .

(b) Now let f be the function of the variables x and y defined by $f(x, y) = \frac{x^2+y^2}{x+y}$.

i. Let y be held constant at $y = 1$. Write a formula for f then as a function of x alone on this trace. Does this single variable function f have a limit at $x = 2$? If yes, explain why and find the limit. If no, explain why not.

ii. Now let y vary and hold x constant at $x = 0$. Write a formula for f then as a function of y alone on this trace. Does this single variable function f have a limit at $y = 0$? If yes, explain why and find the limit. If no, explain why not.

(c) Now let f be any function of two variables x and y. Use the idea of limit as reviewed in part (a) to complete the following statement of what it means for $f = f(x, y)$ to have a limit L at a point $(x, y) = (a, b)$.

> A function f of the two variables x and y has a limit L at the point where $(x, y) = (a, b)$ if . . .

$\bowtie$

Introduction

In single variable calculus we studied the notion of limit, a critical concept which formed the basis for the derivative and the definite integral. In this section we will begin to understand how the concept of limit for functions of two variables is similar to what we encountered for functions of a single variable and, more importantly, how it is quite different. The limit will again be the fundamental idea in multivariable calculus, and we will use this notion of limits of functions of several variables to define the important concepts of partial derivatives and directional derivatives in the next few sections.

In Preview Activity 2.1 we recalled the notion of limit from single variable calculus. The same idea holds for functions of several variables as we will investigate in this section.

Limits of Functions of Two Variables

For the sake of discussion, we will focus on functions of two variables. However, all the ideas we establish here are valid for functions of any number of variables. As we did in Preview Activity 2.1, we describe what it means for a function of two variables to have a limit at a point.

Definition 2.1. A function $f = f(x, y)$ of the independent variables x and y has a **limit** L at the point where $(x, y) = (a, b)$ if we can make all of the values of $f(x, y)$ as close as we want to L by choosing (x, y) as close to (a, b) (but not equal to (a, b)) as we need.

We use the same notation as in single variable calculus to denote a limit, writing

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

to mean that f has a limit L at the point where $(x, y) = (a, b)$.

Investigating limits for multivariable functions is quite a bit more complicated than for single variable functions. In single variable calculus, to find a limit we only had to determine the behavior of our functions from two directions (from the left of the base point and from the right). For a function of two independent variables, there are *infinitely many* different directions from which we can approach a point (a, b), and we need to determine the behavior of a function from any of those directions. We will consider a few examples to highlight the main ideas.

Activity 2.1.

Consider the function f defined by $f(x, y) = \frac{2xy}{x^2+y^2}$.

(a) Is f defined at $(0, 0)$? Explain. Could f have a limit at $(0, 0)$? Explain.

(b) Calculate values of f at several points close to $(0, 0)$. Based on your calculations, do you think f has a limit at $(0, 0)$? Explain.

(c) The graph of f on the domain $[-1, 1]$ by $[-1, 1]$ is shown in Figure 2.1. Based on this

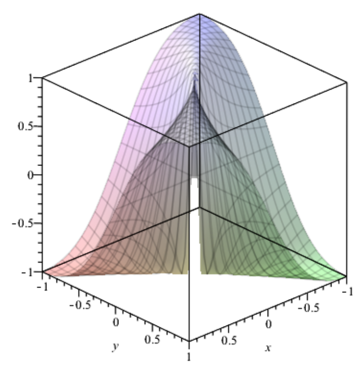

Figure 2.1: A graph of $f(x, y) = \frac{2xy}{x^2+y^2}$

graph, do you think f has a limit at $(0, 0)$? Explain.

(d) To explicitly determine if f has a limit at $(0, 0)$ we can algebraically consider what happens to the values $f(x, y)$ as (x, y) approaches $(0, 0)$ from different directions. For example, let's consider what happens to the function values of f as (x, y) approaches $(0, 0)$ along the line $x = 0$. To do this, substitute 0 for x in the rule for f and explain if the resulting curve has a limit at $y = 0$. If so, what is the limit? Explain why, if f has a limit at $(0, 0)$, it must be this limit that we found along the line $x = 0$.

(e) Now try a different path. Does f have a limit at $(0, 0)$ along the path $y = x$? If so, what is the limit? Explain.

(f) Does f have a limit at $(0, 0)$? Explain.

◁

Activity 2.1 shows that $f(x, y) = \frac{2xy}{x^2+y^2}$ approaches different values as the input gets close to $(0, 0)$ along different paths, A function cannot have two different limits at the same point, so since the values of f approach different numbers along different paths, f does not have a limit at $(0, 0)$.

This is a general rule we can apply.

> If a function has different limits at some point (a, b) along different paths containing (a, b), then the function has no limit at the point (a, b).

IMPORTANT NOTE: We can use this theorem to determine if a function does not have a limit at a point, but not to determine that a function has a limit at a point.

Activity 2.2.

As a second example, let $f(x, y) = \frac{x}{\sqrt{x^2+y^2}}$. Determine if f has a limit at $(0, 0)$. Hint: Consider the behavior of f along the simplest paths that contain $(0, 0)$. A graph of f is shown in Figure 2.2.

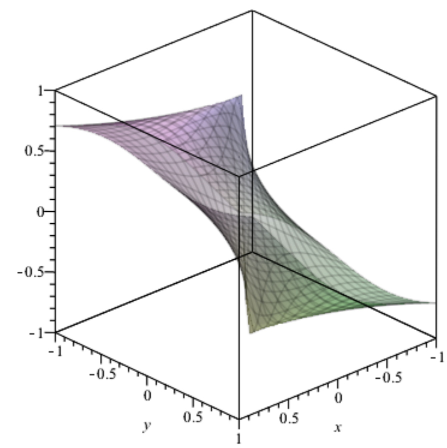

Figure 2.2: A graph of $f(x, y) = \frac{x}{\sqrt{x^2+y^2}}$

◁

We must be **VERY CAREFUL** when drawing conclusions about limits of functions along different paths. For example, consider the function f defined by

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}.$$

Suppose we consider the limit of f at $(0, 0)$ by looking at the behavior of f along linear paths that contain the origin. In other words, consider the values of f along the graphs of $y = mx$ for any value of $m \neq 0$, that is functions of the form

$$f(x, mx) = \frac{2x^2(mx)}{x^4 + (mx)^2} = \frac{2mx^3}{x^4 + m^2x^2} = \frac{2xm}{x^2 + m^2}.$$

Now

$$\lim_{x \to 0} \frac{2xm}{x^2 + m^2} = 0,$$

so along all of these linear paths f has a limit of 0 at $(0, 0)$. We might then be tempted to say that f has a limit of 0 at $(0, 0)$. However, there are still infinitely many other paths we can take to approach the origin. Suppose we approach the origin along the parabola $y = x^2$. When $y = x^2$ the corresponding values of f are

$$f(x, x^2) = \frac{2x^2x^2}{x^4 + (x^2)^2} = \frac{2x^4}{2x^4} = 1$$

if $x \neq 0$. This would imply that the limit of f at $(0, 0)$ is 1. Since we have two different limits along different paths, we conclude that f has no limit at $(0, 0)$.

IMPORTANT NOTE AGAIN: In order to show that a function f has a limit at a point (a, b), we need to consider the behavior of f using *all possible ways* that (x, y) can approach (a, b). In general, this is a very difficult problem.

Lest we be misled by the above examples, there are many functions that have limits at points at which they are not defined. For example, let

$$f(x, y) = \frac{x^2y^2}{x^2 + y^2}.$$

To see that this function has a limit at $(0, 0)$ we have to do more than just consider the behavior of f along a variety of paths. A different type of argument is needed to prove the existence of a limit.

Note that if either x or y is 0, then $f(x, y) = 0$. Therefore, if f has a limit at $(0, 0)$, then the limit must be 0. Now, for any value of y we have

$$y^2 \leq x^2 + y^2$$

so

$$\frac{y^2}{x^2 + y^2} \leq 1.$$

Then

$$f(x, y) = \frac{x^2y^2}{x^2 + y^2} = x^2 \left(\frac{y^2}{x^2 + y^2} \right) \leq x^2.$$

So as x gets arbitrarily close to 0 along any path, so does $f(x, y)$. Therefore,

$$\lim_{(x,y) \to (0,0)} \frac{x^2y^2}{x^2 + y^2} = 0.$$

The limit for multivariable functions has the same properties as the limit for single variable functions, stated here (without proof) for functions of two variables.

Let f and g be functions of the independent variables x and y so that $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y)$ both exist. Then

1. $\lim\limits_{(x,y)\to(a,b)} cf(x,y) = c \lim\limits_{(x,y)\to(a,b)} f(x,y)$ for any scalar c,
2. $\lim\limits_{(x,y)\to(a,b)} [f(x,y) + g(x,y)] = \lim\limits_{(x,y)\to(a,b)} f(x,y) + \lim\limits_{(x,y)\to(a,b)} g(x,y)$,
3. $\lim\limits_{(x,y)\to(a,b)} [f(x,y) - g(x,y)] = \lim\limits_{(x,y)\to(a,b)} f(x,y) - \lim\limits_{(x,y)\to(a,b)} g(x,y)$,
4. $\lim\limits_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = \lim\limits_{(x,y)\to(a,b)} f(x,y) \lim\limits_{(x,y)\to(a,b)} g(x,y)$,
5. $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$ if $\lim\limits_{(x,y)\to(a,b)} g(x,y) \neq 0$.

We can use these properties and results from single variable calculus to verify that some limits exist. For example, the function f defined by $f(x,y) = x^2y^3$ has a limit at every point. We can explain this since the polynomial x^2 has a limit everywhere as does the polynomial y^3. Since the limit of the product is the product of the limits, we can conclude that x^2y^3 has a limit for any (x,y).

Continuity

We defined a function f of a single variable x to be continuous at $x = a$ if three conditions were satisfied:

$$f(a) \text{ exists}, \quad \lim_{x\to a} f(x) \text{ exists}, \quad \text{and} \quad \lim_{x\to a} f(x) = f(a).$$

Since we have the concept of limit for multivariable functions, we can define continuity in the same way.

Definition 2.2. A function f of the independent variables x and y is **continuous** at the point (a,b) if

1. f is defined at the point (a,b),
2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists,
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Just as with single variable functions, continuity has certain properties that are based on the properties of limits.

Let f and g be functions of the independent variables x and y so that f and g are continuous at the point (a, b). Then

1. cf is continuous at (a, b) for any scalar c,
2. $f + g$ is continuous at (a, b),
3. $f - g$ is continuous at (a, b),
4. fg is continuous at (a, b),
5. $\frac{f}{g}$ is continuous at (a, b) if $g(a, b) \neq 0$.

With these properties we can use results from single variable calculus to decide about continuity of multivariable functions. For example, we know $g(x) = x^2$ and $h(y) = y^3$ are continuous single variable functions, so their product $f(x, y) = x^2y^3$ is a continuous multivariable function.

Summary

In this section, we encountered the following important ideas:

- A function $f = f(x, y)$ has a limit L at a point (a, b) if we can make all of the values of $f(x, y)$ as close to L as we want by choosing (x, y) as close to (a, b) (but not equal to (a, b)) as we need.
- In order for a function $f = f(x, y)$ to have a limit L at a point (a, b) in its domain, all of the values of f have to be getting as close to L as we want no matter what path the points (x, y) take towards (a, b). So if we can find a path containing the point (a, b) along which f does not have a limit at (a, b), or if we can find two different paths containing (a, b) so that along these paths f has different limits at (a, b), then f cannot have a limit at (a, b).
- A function $f = f(x, y)$ is continuous at a point (a, b) in its domain if f has a limit at (a, b) and

$$f(a, b) = \lim_{(x,y)\to(a,b)} f(x, y).$$

2.2 First Order Partial Derivatives

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How are the first order partial derivatives of a function f of the independent variables x and y defined?
- What do the first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of a function f of the independent variables x and y tell us about f?

Preview Activity 2.2. An object in motion can do work. As an example, a wrecking ball is effective because a massive object moving at a large velocity possesses a significant amount of energy that can be imparted on collision. The energy that an object possesses due to its motion is called *kinetic energy*. Kinetic energy is defined as the work required to accelerate a body of a given mass m from rest to its current velocity v, so kinetic energy is a function of the two variables m and v. More specifically, the kinetic energy f of an object of mass m and velocity v is given by

$$f(m, v) = \frac{1}{2}mv^2,$$

where f is measured in Joules if the mass m is in kilograms and the velocity v is in meters per second.

(a) Fix the mass of the object to be 2 kilograms. Write f as a function of the velocity alone. Then find the derivative of f with respect to the velocity (keeping the mass fixed at 2 kg) when the velocity is 3 meters per second. Include the units. Explain in detail what this tells us about kinetic energy.

(b) Now let the mass vary but keep the velocity constant at 5 meters per second. Write f as a function of the mass alone. Then find the derivative of f with respect to the mass (keeping the velocity fixed at 5 meters per second) when the mass is 2 kilograms. Include the units. Explain in detail what this tells us about kinetic energy.

$\bowtie$

Introduction

In first semester calculus we used the derivative to describe the change in a single variable function as the independent variable changed. In fact, the derivative of a function of a single variable told us how the dependent variable changed for every unit increase in the independent variable. The kinetic energy function described in Preview Activity 2.2 has more than one independent variable, and we calculated how a change in either the mass or velocity of an object (while the other variable is constant) will alter the kinetic energy. As we proceed this semester we will develop analogs of the derivative that will allow us to describe the change in kinetic energy given changes in mass and velocity. We begin with the first order partial derivatives in this section.

First Order Partial Derivatives

As Preview Activity 2.1 shows, if f is a function of two variables and we hold one constant, we can then treat f as a function of only one variable and apply our concepts and techniques from single variable calculus to f. For example, if $f = f(x, y)$ is a function of the two independent variables x and y and we hold y constant and allow x to vary, then we can differentiate the resulting function of x to obtain what we call the partial derivative of f with respect to x.

Definition 2.3. The **partial derivative of f with respect to x** is

$$\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

for those values of x for which the limit exists.

IMPORTANT NOTE! It is important to recognize that the y variable does not change in this definition – the only independent variable is x. We have two different notations for this partial derivative: $\frac{\partial f}{\partial x}(x, y)$ (where the ∂ symbol is like the d symbol for the derivative from single variable calculus, but has a slightly different shape to remind us that we are taking a partial derivative) and $f_x(x, y)$ (where the subscript indicates the independent variable that is allowed to vary).

Activity 2.3.

If $f = f(x, y)$ is a function of the two independent variables x and y and we hold x constant and allow y to vary, then we can differentiate the resulting function of y to obtain what we call the partial derivative of f with respect to y. Complete the following definition:

The partial derivative of f with respect to y is

$$\frac{\partial f}{\partial y}(x, y) = f_y(x, y) = ...$$

$\triangleleft$

These partial derivatives f_x and f_y are called *first order* partial derivatives because we only differentiate once. We will discuss the second order partial derivatives later in this section.

Here are a few problems to review techniques of differentiation and practice calculating first order partial derivatives. Notice that everything we have done with partial derivatives can be applied to functions of more than two variables.

Activity 2.4.

Calculate all partial derivatives of the indicated functions.

(a) $f(x, y) = 3x^3 - 2x^2y^5$

(b) $g(r, s) = rs\cos(r)$

(c) $f(w, x, y) = (6w + 1)\cos(3x^2 + 4xy^3 + y)$

(d) $q(x, t, z) = \frac{x 2^t z^3}{1+x^2}$

$\triangleleft$

Interpretations of First Order Partial Derivatives

Recall that the derivative of a single variable function has a geometric interpretation as the slope of the line tangent to the graph at a given point. We have a similar geometric interpretation of the first order partial derivatives, and we can also describe what they represent in context. We begin with the geometric interpretation.

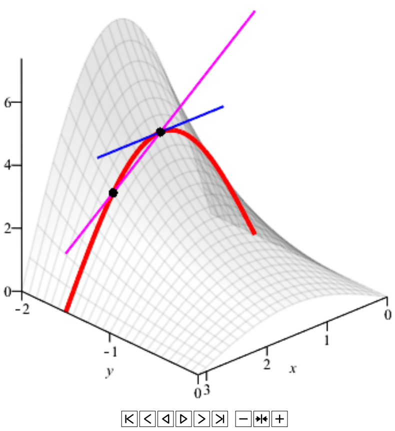

Figure 2.3: $\frac{\partial f}{\partial x}$.

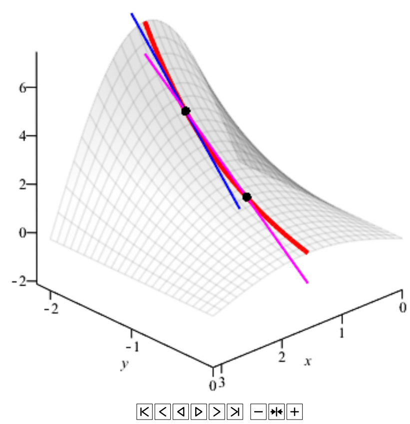

Figure 2.4: $\frac{\partial f}{\partial y}$.

Activity 2.5.

Let $f(x, y) = \sin(x)e^{-y}$. A graph of f on the domain $[0, \pi] \times [-2, 0]$ is shown in Figure 2.3.

(a) The animation in Figure 2.3 shows the difference quotients

$$\frac{f(a+h, b) - f(a, b)}{h}$$

for different values of h. Run the animation in Figure 2.3. Pick a frame and identify the point $(a, b, f(a, b))$ and the point $(a+h, b, f(a+h, b))$. Then explain as best you can, and in as much detail as you can, what the difference quotient represents, geometrically, for some appropriate value of h. Your explanation should involve the use of the words "secant" and slope.

(b) Now explain in as much detail as you can what you think the partial derivative

$$\lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

represents, geometrically, in Figure 2.3. (Hint: What do you think the blue segment represents?)

◁

Activity 2.6.

Let $f(x, y) = \sin(x)e^{-y}$. A graph of f on the domain $[0, \pi] \times [-2, 0]$ is shown in Figure 2.4.

(a) Run the animation in Figure 2.4. Pick a frame and explain as best you can, and in as much detail as you can, what the difference quotient

$$\frac{f(a, b + h) - f(a, b)}{h}$$

represents, geometrically, for some appropriate value of h.

(b) Now explain in as much detail as you can what you think the partial derivative

$$\lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

represents, geometrically, in Figure 2.4.

◁

Activities 2.5 and **??** provide geometric interpretations of the first order partial derivatives. Now we consider the first order partial derivatives in context. Recall that the difference quotient $\frac{f(a+h)-f(a)}{h}$ for a function f of a single variable x at a point where $x = a$ tells us the average rate of change of f over the interval $[a, a + h]$, while the derivative $f'(a)$ tells us the instantaneous rate of change of f at $x = a$, or, more specifically, how f changes at $x = a$ for every one unit increase in x from a. We can use these same concepts to explain the meanings of the partial derivatives in context.

Activity 2.7.

Let $f(x, y) = \frac{1}{2}xy^2$ represent the kinetic energy in Joules of an object of mass x in kilograms with velocity y in meters per second. Let (a, b) be the point $(4, 5)$ in the domain of f.

(a) Calculate $f_x(a, b)$.

(b) Explain as best you can in the context of kinetic energy what the partial derivative

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

tells us about kinetic energy.

◁

Activity 2.8.

Let $f(x, y) = \frac{1}{2}xy^2$ represent the kinetic energy in Joules of an object of mass x in kilograms with velocity y in meters per second. Let (a, b) be the point $(4, 5)$ in the domain of f.

(a) Calculate $f_y(a,b)$.

(b) Explain as best you can in the context of kinetic energy what the partial derivative

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

tells us about kinetic energy.

◁

In single variable calculus we saw how we can use the difference quotient to approximate derivatives if we have data instead of an algebraic formula. The same ideas apply to partial derivatives.

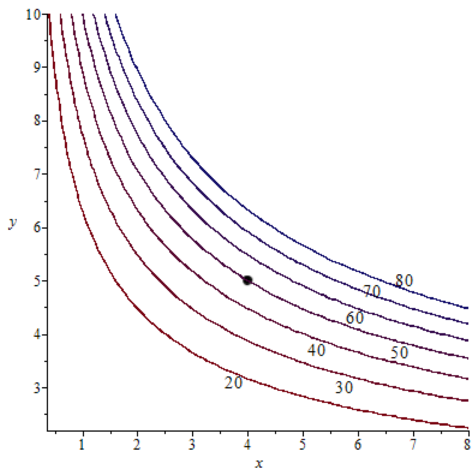

Figure 2.5: A contour plot

	4.6	4.8	5	5.2	5.4
3.8	40.204	43.776	47.5	51.376	55.404
3.9	41.262	44.928	48.75	52.728	56.862
4	42.32	46.08	50	54.08	58.32
4.1	43.378	47.232	51.25	55.432	59.778
4.2	44.436	48.384	52.5	56.784	61.236

Table 2.1: Kinetic energy

Activity 2.9.

Often we are certain graphical information about a function instead of a rule. We can use that information to approximate partial derivatives. For example, suppose that we are given a contour plot of the kinetic energy function (as in Figure 2.5) instead of a formula. Use this contour plot to approximate $f_x(4,5)$ and $f_y(4,5)$ as best you can. Compare to your calculations from Activities 2.7 and 2.8.

◁

Activity 2.10.

Often we are given numerical information about a function instead of a rule. We can use that information to approximate partial derivatives. For example, suppose that we are given a table of values of the kinetic energy function (as in Table 2.1) instead of a formula. Use this table of data to approximate $f_x(4,5)$ and $f_y(4,5)$ as best you can. Compare to your calculations from Activities 2.7 and 2.8.

◁

Summary

In this section, we encountered the following important ideas:

- There are two first order partial derivatives of f.
  - The partial derivative of f with respect to x is defined as
    $$\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$
    at those points (x, y) for which the limit exists.
  - The partial derivative of f with respect to y is defined as
    $$\frac{\partial f}{\partial y}(x, y) = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$
    at those points (x, y) for which the limit exists.
- The first order partial derivatives tell us several things.
  - The partial derivative $\frac{\partial f}{\partial x}(a, b)$ tells us how f changes for every unit increase in x from the value a, keeping y constant at b. Geometrically, the partial derivative $f_x(a, b)$ tells us the slope of the line tangent to the $y = b$ trace of the function f at the point (a, b).
  - The partial derivative $\frac{\partial f}{\partial y}(a, b)$ tells us how f changes for every unit increase in y from the value b, keeping x constant at a. Geometrically, the partial derivative $f_y(a, b)$ tells us the slope of the line tangent to the $x = a$ trace of the function f at the point (a, b).

2.3 Second Order Partial Derivatives

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How are the second order partial derivatives of a function f of the independent variables x and y defined?
- What do the second order partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y \partial x}$ of a function f of the independent variables x and y tell us about f?
- What special relationship do the second order mixed partial derivatives of $f = f(x, y)$ have with each other?

Preview Activity 2.3. Consider the kinetic energy function f from Preview Activity 2.2 defined by

$$f(m, v) = \frac{1}{2}mv^2,$$

where f is measured in Joules if the mass m is in kilograms and the velocity v is in meters per second.

(a) The first order partial derivative f_m of f is a function of both m and v. Calculate $f_{mm} = (f_m)_m$, the partial derivative of f_m with respect to m. What are the units of f_{mm}?

(b) The first order partial derivative f_m of f is a function of both m and v. Calculate $f_{mv} = (f_m)_v$, the partial derivative of f_m with respect to v. What are the units of f_{mv}?

(c) The first order partial derivative f_v of f is a function of both m and v. Calculate $f_{vm} = (f_v)_m$, the partial derivative of f_v with respect to m. What are the units of f_{vm}?

(d) The first order partial derivative f_v of f is a function of both m and v. Calculate $f_{vv} = (f_v)_v$, the partial derivative of f_v with respect to v. What are the units of f_{vv}?

(e) What, if anything, do you notice about $f_{mv}(m, v)$ and $f_{vm}(m, v)$?

$\bowtie$

Introduction

A function f of two independent variables x and y has two first order partial derivatives, f_x and f_y. As we saw in Preview Activity 2.3, each of these first order partial derivatives has two partial derivatives. This gives us a total of four seconde order partial derivatives, and it is these partial derivatives that we study in this section.

Second Order Partial Derivatives

Since there was only one derivative of a single variable function, there was only one second derivative. In the two variable case, things are more complicated. The partial derivatives f_x and f_y of a function f of the independent variables x and y are called the *first order* partial derivatives. Each of these partial derivatives f_x and f_y are functions of x and y and therefore may have partial derivatives themselves. Each of the first order partial derivatives f_x and f_y can have two partial derivatives – with respect to x and y – and so there are four *second order* partial derivatives:

- $\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = f_{xx} = (f_x)_x$,
- $\frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2} = f_{yy} = (f_y)_y$,
- $\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = (f_x)_y$,
- $\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = (f_y)_x$.

The first two are called *unmixed* second order partial derivatives while the last two are called the *mixed* second order partial derivatives.

Be careful about the notation. When using the partial derivatives symbol, e.g., $\frac{\partial^2 f}{\partial y \partial x}$, we differentiate first with respect to x first, then y. However, when using the subscript notation, e.g., f_{yx}, we differentiate with respect to y first, then x. Order matters!

Once we know how to calculate partial derivatives, then calculating second order partial derivatives is just more of the same.

Activity 2.11.

Find all second order partial derivatives of the following functions.

(a) $f(x, y) = x^2y^3$

(b) $f(x, y) = y\cos(x)$

(c) $g(s, t) = st^3 + s^4$

◁

Interpreting the Second Order Partial Derivatives

Recall that the second derivative from single variable calculus tells us the concavity of our function. We can interpret the second order partial derivatives of a two-variable function in a similar way.

The second order partial derivatives $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ are called the *unmixed* partials because we differentiate with respect to the same variable twice and don't mix the variables. In the following activity we investigate the geometric interpretation of the second order unmixed partials.

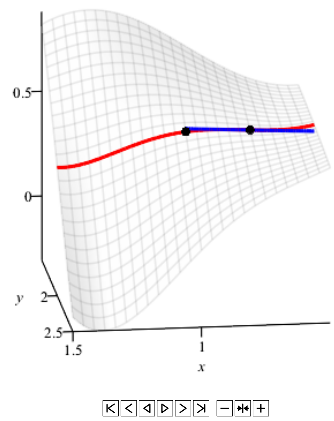

Figure 2.6: $\frac{\partial^2 f}{\partial x^2}$.

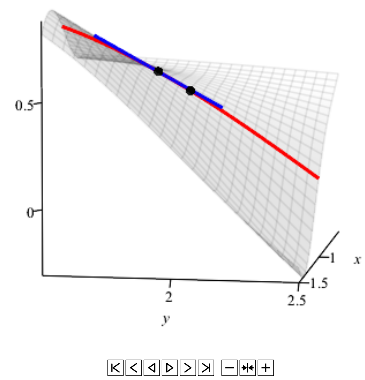

Figure 2.7: $\frac{\partial^2 f}{\partial y^2}$.

Activity 2.12.

Let $f = f(x, y)$ be a function of the independent variables x and y.

(a) Run the animation in Figure 2.6. Explain as best you can what the second order unmixed partial derivative Explain as best you can what the second order mixed partial derivative

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x}$$

tells us about the graph of f by first considering what the first order partial derivative $\frac{\partial f}{\partial x}$ represents, then describing what happens to this first order partial derivative as we increase x.

(b) Run the animation in Figure 2.7. Explain as best you can what the second order unmixed partial derivative $\dfrac{\partial^2 f}{\partial y^2}$ tells us about the graph of f.

◁

The two second order partial derivatives $\dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$ are called the second order *mixed* partials because we mix the variables, differentiating first with respect to one variable, then with respect to the other. In the next activity we investigate the geometry of the second order unmixed partials.

Activity 2.13.

Let $f = f(x, y)$ be a function of the independent variables x and y.

(a) Run the animation in Figure 2.8. Explain as best you can what the second order mixed

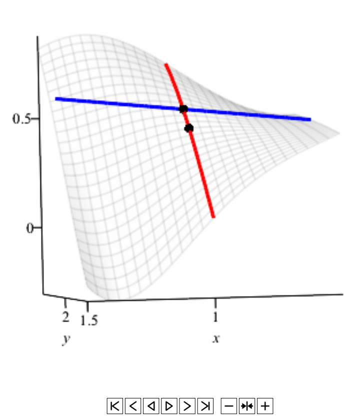

Figure 2.8: $\frac{\partial^2 f}{\partial y \partial x}$.

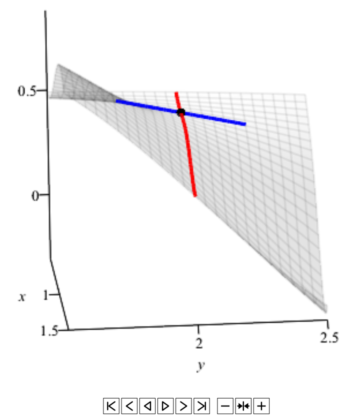

Figure 2.9: $\frac{\partial^2 f}{\partial x \partial y}$.

partial derivative

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$$

tells us about the graph of f by first considering what the first order partial derivative $\frac{\partial f}{\partial x}$ represents, then describing what happens to this first order partial derivative as we increase y.

(b) Run the animation in Figure 2.9. Explain as best you can what the second order mixed partial derivative $\dfrac{\partial^2 f}{\partial x \partial y}$ tells us about the graph of f.

$\triangleleft$

Now we consider the second order partial derivatives in context.

Activity 2.14.

Let $f(x, y) = \frac{1}{2}xy^2$ represent the kinetic energy in Joules of an object of mass x in kilograms with velocity y in meters per second. Let (a, b) be the point $(4, 5)$ in the domain of f.

(a) Calculate $\dfrac{\partial^2 f}{\partial x^2}$ at the point (a, b). Then explain as best you can what this second order partial derivative tells us about kinetic energy. (Hint: Recall that $\dfrac{\partial^2 f}{\partial x^2}$ is the first order partial derivative of f_x with respect to x. Compare to the discussion in Activity **??**.)

(b) Calculate $\dfrac{\partial^2 f}{\partial y^2}$ at the point (a, b). Then explain as best you can what this second order partial derivative tells us about kinetic energy.

(c) Calculate $\frac{\partial^2 f}{\partial y \partial x}$ at the point (a, b). Then explain as best you can what this second order partial derivative tells us about kinetic energy.

(d) Calculate $\frac{\partial^2 f}{\partial x \partial y}$ at the point (a, b). Then explain as best you can what this second order partial derivative tells us about kinetic energy.

◁

You may have noticed something interesting about the two mixed second order partial derivatives in Preview Activity 2.3 and Activity 2.14. In each of those cases we saw that the mixed second order partial derivatives were equal. This is no mistake and we state the result without proof.

Clairaut's Theorem. Let f be a function defined on a open disk that contains the point (a, b). If f_{xy} and f_{yx} are continuous on this disk, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Just as we did with the first order partial derivatives, we can approximate the second order partial derivatives even if we have only numeric information about our function.

Activity 2.15.

Consider the table of values of the kinetic energy function (Table **??**, reproduced here in Table **??** for convenience) instead of a formula. In this activity we will use this table of data to approximate $f_{xx}(4, 5)$, $f_{yy}(4, 5)$, and $f_{xy}(4, 5)$ as best we can. When done compare to your calculations from Activity **??**.

	4.6	4.8	5	5.2	5.4
3.8	40.204	43.776	47.5	51.376	55.404
3.9	41.262	44.928	48.75	52.728	56.862
4	42.32	46.08	50	54.08	58.32
4.1	43.378	47.232	51.25	55.432	59.778
4.2	44.436	48.384	52.5	56.784	61.236

Table 2.2: Kinetic energy

(a) To approximate the second order partial derivative $f_{xx}(4, 5)$ we can use the symmetric difference quotient

$$f_{xx}(4, 5) \approx \frac{f_x(4 + h, 5) - f_x(4 - h, 5)}{2h}.$$

i. For what value of h can we make the best calculations for this symmetric difference quotient?

ii. Approximate $f_x(4+h, 5)$ and $f_x(4-h, 5)$ for the value of h you found in the previous part.

iii. Use your approximations to $f_x(4+h,5)$ and $f_x(4-h,5)$ to approximate $f_{xx}(4,5)$.

(b) Repeat the process from part (a) to approximate $f_{yy}(4,5)$.

(c) Now we deal with the second order mixed partials.

i. Write the appropriate difference quotient that will allow you to approximate $f_{xy}(4,5)$.

ii. Carry out the calculations to approximate $f_{xy}(4,5)$. (It is illustrative to approximate $f_{yx}(4,5)$ to check on Clairaut's Theorem – that is left for you to do.)

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Summary

In this section, we encountered the following important ideas:

- There are four second order partial derivatives of a function f of two independent variables x and y.
  - The second order partial derivative of f with respect to x twice is defined as
  $$\frac{\partial^2 f}{\partial x^2}(x,y) = f_{xx}(x,y) = \lim_{h\to 0} \frac{f_x(x+h,y) - f_x(x,y)}{h}$$
  at those points (x,y) for which the limit exists.
  - The second order partial derivative of f with respect to y twice is defined as
  $$\frac{\partial^2 f}{\partial y^2}(x,y) = f_{yy}(x,y) = \lim_{h\to 0} \frac{f_y(x+h,y) - f_y(x,y)}{h}$$
  at those points (x,y) for which the limit exists.
  - The second order partial derivative of f with respect to x and then y is defined as
  $$\frac{\partial^2 f}{\partial y \partial x}(x,y) = f_{xy}(x,y) = \lim_{h\to 0} \frac{f_x(x,y+h) - f_x(x,y)}{h}$$
  at those points (x,y) for which the limit exists.
  - The second order partial derivative of f with respect to y and then x is defined as
  $$\frac{\partial^2 f}{\partial x \partial y}(x,y) = f_{yx}(x,y) = \lim_{h\to 0} \frac{f_y(x+h,y) - f_y(x,y)}{h}$$
  at those points (x,y) for which the limit exists.
- The second order partial derivatives tell us many things.
  - The second order partial derivative of f with respect to x twice tells us
    - how f_x changes for each unit increase in x from the point (a,b) keeping y constant at b,

  - how the slopes of the tangent lines to the x trace of f when $y = b$ are changing as we increase x,
  - the concavity of the $y = b$ trace to f at the point (a, b).
- The second order partial derivative of f with respect to y twice tells us
  - how f_y changes for each unit increase in y from the point (a, b) keeping x constant at a,
  - how the slopes of the tangent lines to the y trace of f when $x = a$ are changing as we increase y,
  - the concavity of the $x = a$ trace to f at the point (a, b).
- The second order partial derivative of f with respect to x first then y tells us
  - how f_x changes for each unit increase in y from the point (a, b) keeping x constant at a,
  - how the slopes of the tangent lines to the traces in the x direction change as y increases from the point (a, b). This tells us the "twist" in the surface at the point (a, b) in the y direction.
- The second order partial derivative of f with respect to y first then x tells us
  - how f_y changes for each unit increase in x from the point (a, b) keeping y constant at b,
  - how the slopes of the tangent lines to the traces in the y direction change as x increases from the point (a, b). This tells us the "twist" in the surface at the point (a, b) in the x direction.

- Provided that f_{xy} and f_{yx} are continuous on an open disk containing the point (a, b), then Clairaut's Theorem tells us that

$$f_{xy}(a, b) = f_{yx}(a, b).$$

2.4 Linearization:Tangent Planes and Differentials

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What does it mean for a function of two variables to be locally linear at a point?
- How do we find the equation of the plane tangent to a locally linear function at a point?
- What is the differential of a function of two variables? For what can use the differential?

Preview Activity 2.4. In this section we extend the notion of local linearity from functions of one variable to functions of two variables.

(a) Review the concept of local linearity from single variable calculus (look it up in your first semester calculus book or on-line) and explain what it means for a function of a single variable to be locally linear at a point. What is the linearization of a single variable function at a point and how do we use this linearization?

(b) Let $f(x, y) = x^2 - y^2$. Use the program CalcPlot3D[1] at http://web.monroecc.edu/manila/webfiles/calcNSF/JavaCode/CalcPlot3D.htm (or some other 3D plotter with zooming capability) to draw a graph of this surface on the domain $[-2, 2] \times [-2, 2]$ (enter your function as Function 1 and make sure the box to select this function is checked). Use the menu box at the far right to set the axes. Choose the z window coordinates so that the surface fits in the graphing box. Click on the $\oplus$ magnifying glass to zoom in on this surface around the origin. Explain what you see and how this is related to local linearity from first semester calculus.

(c) Keep the same function f from part (b). (It might be easier to visualize our work from this point on if you select "View Settings" and then "Make Surfaces Transparent". You can always toggle this option to make the surfaces opaque if you don't like it.) Now we want to explore the behavior of this function around the point $(1, 1)$. Select "Show a trace point on the surface" and move the point in the domain to $(1, 1)$. Alternatively, use the Trace Menu to enter a trace point.

   i. In the applet select "Show a fx tangent line at point" to see a picture of the trace in the x direction through the point $(1, 1)$, along with a picture of the line tangent to the surface in this direction. What is the slope of this tangent line at this point? What does this slope tell us about the relationship between x and $f(x, y)$ at this point? Explain why the vector $\langle 1, 0, f_x(1, 1)\rangle$ is a direction vector for this tangent line.

   ii. In the applet select "Show a fy tangent line at point". Find a direction vector for this tangent line to f in the y direction at the point $(1, 1)$.

[1]Many thanks to Professor Paul Seeburger at Monroe Community College for this fantastic program.

iii. The tangent lines in i. and ii. form a plane. How can we use the direction vectors for these tangent lines found in parts i. and ii. to find a normal vector for this plane? Find an equation of the plane containing these two tangent lines.

iv. The plane found in part iii. of this activity is the *tangent plane* to the surface f at the point $(1, 1)$. Select the option "Show the tangent plane at point". Then reformat the axes so that the surface is drawn over a small interval containing the point $(1, 1)$. Compare the graph of the surface to the graph of the tangent plane. What do you see?

$\bowtie$

Introduction

In single variable calculus we studied the local linearity of a function of a single variable. In this section we study local linearity of functions of two variables.

Local Linearity for Functions of Two Variables

In Preview Activity 2.4 we investigated what happened as we magnified the graph of a function f around a point. In the two variable case, the graph of a function that is locally linear at a point looks like a plane as we zoom in around that point. When this happens we say that the function is *locally linear* at that point. This plane the surface looks like as we zoom in is called the *tangent plane* to the surface at that point. The tangent plane plays the role for functions of two variables that the tangent line played for single variable functions. If we let $L(x, y)$ be the plane tangent to f at the point (a, b), then for (x, y) close to (a, b) we have

$$f(x, y) \approx L(x, y).$$

In order to use a linear approximation – the tangent plane – we need to know how to find the equation of a plane tangent to a surface at a point.

Equations of Tangent Planes

Preview Activity 2.4 contains all of the major ideas we need to find a general formula for the equation of the plane tangent to a function f at a point $(a, b, f(a, b))$. We already have a point on the tangent plane, namely the point of tangency $(a, b, f(a, b))$, so all we need to find the equation of the tangent plane is a normal vector to the plane. The partial derivatives of f provide us two vectors parallel to the tangent plane. Since $f_x(a, b)$ tells us the slope of the trace of f in the x direction at (a, b) (or approximately how the z values change for every 1 unit increase in x from a while holding b constant), the vector $\langle 1, 0, f_x(a, b)\rangle$ is a direction vector for the tangent line to f in the x direction at (a, b). So $\langle 1, 0, f_x(a, b)\rangle$ is a vector parallel to the tangent plane to f at (a, b). Similarly, the vector $\langle 0, 1, f_y(a, b)\rangle$ is a direction vector for the tangent line to f in the y direction at (a, b), and is therefore parallel to the tangent plane to f at (a, b). These two vectors are parallel to the tangent plane to f at (a, b), so their cross product will be a vector orthogonal to the tangent plane to f at (a, b). Thus, a normal vector to the tangent plane to f at (a, b) is

$$\langle 1, 0, f_x(a, b)\rangle \times \langle 0, 1, f_y(a, b)\rangle = \langle -f_x(a, b), -f_y(a, b), 1\rangle.$$

The equation of the tangent plane to f at (a, b) is then

$$-f_x(a, b)(x - a) - f_y(a, b)(y - b) + (z - f(a, b)) = 0.$$

Solving for z gives us the following.

If $f = f(x, y)$ is a function of two variables, and if the first order partial derivatives of f exist and are continuous at a point (a, b) in the domain of f, then the **equation of the plane tangent to the graph of f at the point** $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b). \tag{2.1}$$

The function L defined by

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b), \tag{2.2}$$

when it exists, is called the linearization of f at (a, b).

Activity 2.16.

Find the equation of the tangent plane to $f(x, y) = x^2y$ at the point $(1, 2)$.

◁

In single variable calculus, the existence of $f'(a)$ guaranteed that f was locally linear at $x = a$. Given the equation of the tangent plane in (2.1), it is not unreasonable to think that the existence of both $f_x(a, b)$ and $f_y(a, b)$ are enough to ensure that f is locally linear at (a, b). We explore that idea in the next activity.

Activity 2.17.

Let f be the function defined by $f(x, y) = x^{1/3}y^{1/3}$.

(a) Determine

$$\lim_{h\to 0} \frac{f(0 + h, 0) - f(0, 0)}{h}.$$

What does this limit tell us about $f_x(0, 0)$?

(b) Note that $f(x, y) = f(y, x)$, and this symmetry implies that $f_x(0, 0) = f_y(0, 0)$. So both partial derivatives of f exist at $(0, 0)$. A picture of the surface defined by f near $(0, 0)$ is shown in Figure 2.10. Based on this picture, do you think f is locally linear at $(0, 0)$? Why?

(c) Show that the curve where $x = y$ on the surface defined by f is not differentiable at 0. What does this tell us about the local linearity of f at $(0, 0)$?

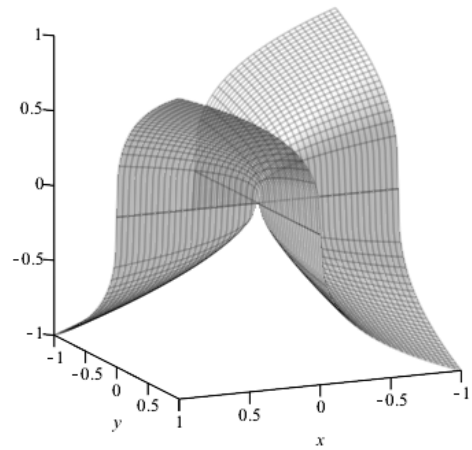

Figure 2.10: The surface for $f(x, y) = x^{1/3}y^{1/3}$.

◁

Activity 2.17 shows that it is possible for both partial derivatives $f_x(a, b)$ and $f_y(a, b)$ to exist without f being locally linear at (a, b). The problem with the example of $f(x, y) = x^{1/3}y^{1/3}$ is that for $x \neq 0$ we have $f_x(x, y) = \frac{y^{1/3}}{3x^{2/3}}$, and $\lim_{(x,y)\to(0,0)} f_x(x, y)$ doesn't exist. So the partial derivative f_x is not continuous at $(0, 0)$. This is why we have the additional condition that the partial derivatives of f be continuous at (a, b) in our description of the tangent plane. This continuity is enough to ensure that f is locally linear at (a, b). When a function is locally linear at a point, we will also say it is *differentiable* at that point.

> If f is a function of the independent variables x and y and both f_x and f_y exist and are continuous in an open disk containing the point (a, b), then f is **differentiable** at (a, b).

Activity 2.18.

Is the function f defined by $f(x, y) = \frac{x^2}{y^2+1}$ locally linear at $(0, 0)$? Explain.

◁

The Differential of a Function of Two Variables

We think of the differential dx of a single variable as representing a small change in x. Since $\frac{dy}{dx} = f'(x)$, we often write $dy = f'(x)dx$ and recognize that a small change in x causes a corresponding small change in the function values in f that is approximated by dy or $f'(x)dx$. The quantity dy is called the *differential* of y. Now we want to understand the differential for functions of two variables.

Let $f = f(x, y)$. If f has continuous partial derivatives f_x and f_y at a point (a, b), then f has a tangent plane (2.1) at (a, b). In this case, for (x, y) close to (a, b), the change in f, or the difference between $f(x, y)$ and $f(a, b)$, is approximated by $L(x, y) - f(a, b)$ using the linearization (2.2) of f. Now

$$L(x, y) - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

So if dx represents a small change in x from a (that is, $dx = x - a$) and dy represents a small change in y from b (that is $dy = y - b$), then there is a corresponding change df in f that is approximated by

$$f_x(a, b)dx + f_y(a, b)dy.$$

This quantity is called the *differential* of f and is denote df.

Definition 2.4. The **differential** df of a function $f = f(x, y)$ of the independent variables x and y is

$$df = f_x(a, b)dx + f_y(a, b)dy. \tag{2.3}$$

If dx represents a small change in x from a (that is, $dx = x - a$) and dy represents a small change in y from b (that is $dy = y - b$), then df approximated the corresponding change in f from the point (a, b) to the point $(a + dx, b + dy)$.

We use the differential to approximate the change in a quantity.

Activity 2.19.

Let f represent the vertical displacement in centimeters from the rest position of a string (like a guitar string) as a function of the distance x in centimeters from the fixed left end of the string and y the time in seconds after the string has been plucked.[2] A simple model for f could be

$$f(x, y) = \cos(x)\sin(2y).$$

Use the differential (2.3) to approximate how much more this vibrating string is vertically displaced from its position at $(a, b) = \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ if we decrease a by 0.01 cm and increase the time by 0.1 seconds. Compare to the value of f at the point $\left(\frac{\pi}{4} - 0.01, \frac{\pi}{3} + 0.1\right)$.

$\triangleleft$

Summary

In this section, we encountered the following important ideas:

- A function f of two independent variables is locally linear at a point (a, b) in its domain if the graph of f looks like a plane as we zoom in on the graph around the point (a, b).
- If a function $f = f(x, y)$ is locally linear at a point (a, b) in its domain, then the two vectors $\langle 1, 0, f_x(a, b)\rangle$ and $\langle 0, 1, f_y(a, b)\rangle$ lie in the tangent plane. The vector

$$\langle 1, 0, f_x(a, b)\rangle \times \langle 0, 1, f_y(a, b)\rangle = \langle -f_x(a, b), -f_y(a, b), 1\rangle$$

[2] An interesting video of this can be seen at https://www.youtube.com/watch?v=TKF6nFzpHBUA.

is normal to the tangent plane and so the equation of the tangent plane to f at (a, b) is

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + (z - f(a,b)) = 0.$$

- The differential df of a function $f = f(x, y)$ of the independent variables x and y is

$$df = f_x(a,b)dx + f_y(a,b)dy.$$

  We can use this differential to approximate small changes in f that result from small changes in the independent variables.

2.5 The Chain Rule

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- Under what circumstances do we need to use the Chain Rule to find a derivative?
- How can we use a tree diagram to guide us in applying the Chain Rule?

Preview Activity 2.5. There are several proposed formulas to approximate the surface area A in square meters of the human body as a function of the height h in centimeters and weight w in kilograms of the body. One model[3] uses the formula

$$A(h, w) = 0.0072h^{0.725}w^{0.425}.$$

A person's height h and weight w both change as a person ages, so h and w can both be considered as functions of time t. Let us think about what is happening to a child whose height is $a = 60$ centimeters and weight is $b = 9$ kilograms. Suppose at this point in time that h and w are changing over time so that h is increasing by 100 centimeters per year and w is increasing by 16 kg per year. Our goal is to determine the exact instantaneous change in A for every unit increase in time at the point $(a, b, A(a, b))$.

(a) What are $\frac{dh}{dt}$ and $\frac{dw}{dt}$ at (a, b)? Explain why

$$h(t) \approx L_h(t) = 60 + 100t \quad \text{and} \quad w(t) \approx L_w(t) = 9 + 16t$$

for t close to 0, and why these approximations get better as t gets closer to 0. (Hint: Think linearizations.)

(b) Explain why, for (h, w) close to (a, b) we have

$$A(h, w) \approx A(L_h(t), L_w(t)),$$

and why the approximations get better as t gets closer to 0. Find a formula for $A(L_h(t), L_w(t))$ in terms of t.

(c) Find $\frac{dA}{dt}$. Then calculate

$$\frac{\partial A}{\partial h}\frac{dh}{dt} + \frac{\partial A}{\partial w}\frac{dw}{dt}$$

and compare to what you found for $\frac{dA}{dt}$. What do you notice?

⋈

[3] DuBois D, DuBois DF. A formula to estimate the approximate surface area if height and weight be known. *Arch Int Med* 1916;17:863-71.

Introduction

In single variable calculus we encountered the Chain Rule that allowed us to differentiate composites of functions. More specifically, if y is a function of x and x is a function of t, then we can think of y as a function of t and

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

In this section, we investigate the Chain Rule for functions of two variables.

The Chain Rule

Preview Activity 2.5 contains the basic elements to understand the Chain Rule for a function of two variables. Note that we found $\frac{dA}{dt}$ by parameterizing both h and w in terms of their linearizations at the base point (a, b). We then compared the quantity $\frac{dA}{dt}$ to $\frac{\partial A}{\partial h}\frac{dh}{dt} + \frac{\partial A}{\partial w}\frac{dw}{dt}$ and saw that we obtained the same result. To see that this is true in general, let $f = f(x, y)$ be a function of the independent variables x and y, where x and y are functions of a parameter t. Let Δf, Δx, Δy, and Δt represent small changes in f, x, and y, respectively. Since f is a function of x, a small change Δx in x may force a small change in f. The partial derivative $\frac{\partial f}{\partial x}$ tells us how f changes as x changes, so the change Δx in x corresponds to a change $\frac{\partial f}{\partial x}\Delta x$ in f. Similarly, the change Δy in y corresponds to a small change $\frac{\partial f}{\partial y}\Delta y$ in f. So we approximate the change in f caused by the small changes Δx and Δy in x and y by

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

Assuming that Δt is not 0 we then have

$$\frac{\Delta f}{\Delta t} \approx \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t}.$$

So

$$\begin{aligned}
\frac{df}{dt} &= \lim_{\Delta t \to 0}\left(\frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t}\right) \\
&= \lim_{\Delta t \to 0}\left(\frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t}\right) + \lim_{\Delta t \to 0}\left(\frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t}\right) \\
&= \frac{\partial f}{\partial x}\lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right) + \frac{\partial f}{\partial y}\lim_{\Delta t \to 0}\left(\frac{\Delta y}{\Delta t}\right) \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.
\end{aligned}$$

The last equation is one form of the Chain Rule.[4]

Let $z = f(x, y)$ where f is a differentiable function of the independent variables x and y, and let x and y be differentiable functions of an independent variable t. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{2.4}$$

It is important to note the difference in the derivatives in (2.4). Since z is a function of the two variables x and y, the derivatives in the Chain Rule for z with respect to x and y are partial derivatives. However, since $x = x(t)$ and $y = y(t)$ are functions of the single variable t, their derivatives are the standard derivatives of functions of one variable, and when we compose z with $x(t)$ and $y(t)$ we then have z as a function of the single variable t making the derivative of z with respect to t a standard derivative from single variable calculus as well.

Activity 2.20.

Find $\frac{df}{dt}$ if $f(x, y) = 2x^2y$, $x = \cos(t)$, and $y = \ln(t)$.

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Visualizing the Chain Rule: Tree Diagrams

A tree diagram can help us visualize the Chain Rule by helping us keep track of the different ways in which one variable can cause a change in another. We place the dependent function at the top of the tree, its independent variables at the next level. With these independent variables now as functions, we place their independent variables at the next level. We link the dependent variables with their independent variables, and label the links with the appropriate derivatives or partial derivatives. To calculate a partial derivative with the Chain Rule, run along a path that links the dependent variable to the independent variable in question and multiply together all the derivatives or partial derivatives. Add these products along all paths and you have implemented the Chain Rule.

Figure 2.11 shows a tree diagram for z as a function of two variables x and y, with x and y functions of a single variable t. Note that the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the standard single variable derivatives from single variable calculus.

We can extend the Chain Rule to encompass the situations where x and y are multivariable functions as well. Figure 2.12 shows a tree diagram for z as a function of two variables x and y, with x and y each functions of two more variables r and s. In this situation we consider z as a function of both r and s, so the Chair Rule tells us how to calculate the partial derivatives of z with respect to either r or s. To compute $\frac{\partial z}{\partial s}$, we look for the paths that lead from z to s. There are two such paths, from z to x, then x to s; and from z to y, then y to s. To compute the change in z that results from how a change in s affects x, we multiply $\frac{\partial z}{\partial x}$ by $\frac{\partial x}{\partial s}$. To compute the change in z

[4] Although we have not presented a formal proof of the Chain Rule, we believe this argument is convincing enough.

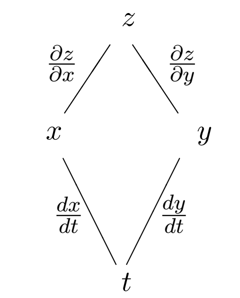

Figure 2.11: Tree diagram for $z = z(x, y)$, $x = x(t)$, and $y = y(t)$.

that results from how a change in s affects y, we multiply $\frac{\partial z}{\partial y}$ by $\frac{\partial y}{\partial s}$. The total change $\frac{\partial z}{\partial s}$ in z that comes from a change in s is then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}.$$

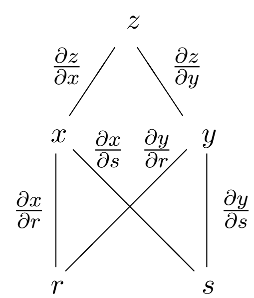

Figure 2.12: Tree diagram for $z = z(x, y)$, $x = x(r, s)$, and $y = y(r, s)$.

We conclude this section with one more example.

Activity 2.21.

The voltage V (in volts) across a circuit is given by Ohm's Law: $V = IR$, where I is the current (in amps) in the circuit and R is the resistance (in ohms). Suppose we connect two resistors with resistances R_1 and R_2 in parallel as shown in Figure 2.13. The total resistance R in the circuit is given by

$$R = \frac{R_1 + R_2}{R_1 R_2}.$$

(a) Assume that the current and the resistances R_1 and R_2 are changing over time, t. Use the Chain Rule to write a formula for $\frac{dV}{dt}$.

(b) Suppose that, at some particular point in time, we measure the current to be 3 amps and that the current is increasing at $\frac{1}{10}$ amps per second, while resistance R_1 is 2 ohms and decreasing at the rate of 0.2 ohms per second and R_2 is 1 ohm and increasing at the rate of 0.5 ohms per second. At what rate is the voltage changing at this point in time?

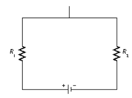

Figure 2.13: Resistors in parallel.

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Summary

In this section, we encountered the following important ideas:

- If f is a function of several variables x_1, x_2, $\ldots$, x_n, and if those variables are functions of another set of variables $y_1, y_2, \ldots, y_k$, then through composition we can consider f as a function of the variables $y_1, y_2, \ldots, y_k$, and we use the Chain Rule to find the partial derivatives of f with respect to the variables $y_1, y_2, \ldots, y_k$.
- We can form a tree that shows the dependence of f on its variables, then the dependence of those variables on other variables. To find the derivative of f with respect to a given variable t, we follow the links from f down the tree to the variable t, multiplying together all derivatives that are indicated by these links. Then we add together all of the derivatives found to calculate the derivative of f with respect to t.

2.6 Directional Derivatives and the Gradient

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What do we mean by a directional derivative?
- How do we calculate a directional derivative?
- How do we interpret a directional derivative?
- What is the gradient of a function and what does it tell us?

Preview Activity 2.6. Wind Chill f (in degrees Fahrenheit) is the term used to describe how cold it feels based on the actual temperature x (in degrees Fahrenheit) and the wind speed y (in miles per hour). For example, if it is 10 degrees and the wind is blowing at 5 miles per hour, then it feels as though the temperature is about 1.2 degrees Fahrenheit. A formula[5] for wind chill is given by

$$f(x, y) = 35.74 + 0.6215x - 35.75y^{0.16} + 0.4275xy^{0.16}.$$

Suppose we want to determine how the wind chill changes at the point (10,5) if the wind speed and temperature each increase in a 3:1 proportion. (in other words, in the direction of the vector $\mathbf{v} = \langle 3, 1\rangle$).

(a) Find a unit vector $\mathbf{u} = \langle u_1, u_2\rangle$ in the direction of $\mathbf{v}$.

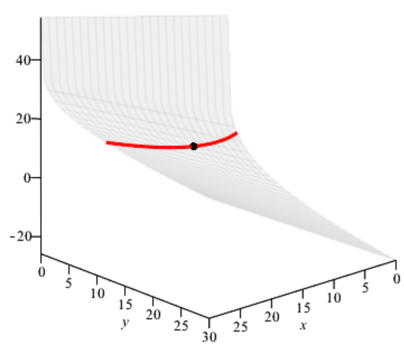

Figure 2.14: The wind chill surface.

(b) The direction of the unit vector $\mathbf{u}$ defines a curve on the surface from the point $(10, 5)$ as shown in Figure 2.14. To stay on this curve from the point $(10, 5)$, if x changes by an amount u_1, then y must change by an amount u_2. Use this idea to find a parameterization

[5] http://www.crh.noaa.gov/ddc/?n=windchill

$\langle x(t), x(t)\rangle$ of the line in the plane through the point $(10, 5)$ in the direction of the vector $\mathbf{u}$. Set up your parameterization so that $x(0) = 10$ and $y(0) = 5$.

(c) The parameterization of the line in part (a) provides a way to write f as a function of t and describe the z coordinates of the points on the curve on the surface defined by the vector $\mathbf{u}$, as shown in Figure **??**. So along this curve we can consider f as a function of t. Use the Chain rule to show that

$$\left.\frac{df}{dt}\right|_{t=0} = f_x(10, 5)u_1 + f_y(10, 5)u_2. \tag{2.5}$$

(Hint: How are $\left.\frac{dx}{dt}\right|_{(10,5)}$ and $\left.\frac{dy}{dt}\right|_{(10,5)}$ related to u_1 and u_2?)

(d) Use the result from part (c) to approximate the value of $\left.\frac{df}{dt}\right|_{t=0}$ to three decimal places. Interpret the result in the context of wind chill.

$\bowtie$

Introduction

The partial derivatives of a function tell us how the function changes as we hold all but one independent variable constant and allow the remaining independent variable to change. The directional derivatives tell us how a function changes if we change more than one independent variable at the same time. We introduce the directional derivatives and the gradient in this section.

Directional Derivatives

When we studied partial derivatives, we changed only one independent variable at a time. Now we want to consider the situation where both of the independent variables are changing at the same time, as in our Preview Activity in which both temperature and wind velocity were changing.

To formally define a derivative in the direction of motion, we want to represent the change in f for every *unit* change in the direction of motion. We can represent this unit change with a unit direction vector. Let $\mathbf{u} = \langle u_1, u_2\rangle$ be a *unit vector* in the direction of motion. If we move a distance h in the direction of $\mathbf{u}$ from a fixed point (a, b), then we will arrive at the new point $(a + u_1h, b + u_2h)$ (this is only true because $\mathbf{u}$ is a unit vector). The slope of the secant line to the curve on the surface through (a, b) in the direction of $\mathbf{u}$ through the points (a, b) and $(a + u_1h, b + u_2h)$ is

$$\frac{f(a + u_1h, b + u_2h) - f(a, b)}{h}. \tag{2.6}$$

Activity 2.22.

Figure 2.15 provides an animation to illustrate the difference quotients (2.6) for an arbitrary function f.

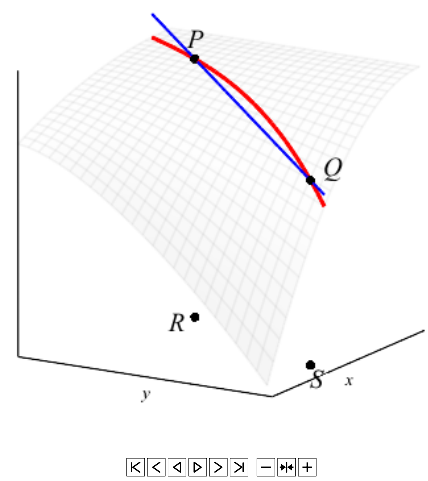

Figure 2.15: A directional derivative.

(a) Run the animation. Then identify the coordinates of the points P, Q, R, and S.

(b) What vector is $\overrightarrow{RS}$?

(c) Explain what is happening to these difference quotients as we let $h \to 0$. How can we geometrically interpret the limit

$$\lim_{h \to 0} \frac{f(a + u_1 h, b + u_2 h) - f(a, b)}{h}?$$

Explain in as much detail as you can.

$\lhd$

The limit of the difference quotients (2.6) is given a special name.

Definition 2.5. The **derivative of** $f = f(x, y)$ **in the direction of the unit vector** $\mathbf{u} = \langle u_1, u_2 \rangle$ is

$$D_{\mathbf{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + u_1 h, y + u_2 h) - f(x, y)}{h}$$

for those values of x and y for which the limit exists.

This derivative $D_{\mathbf{u}} f(x, y)$ is called a *directional derivative*. When we evaluate the directional derivative $D_{\mathbf{u}} f(x, y)$ at a point (a, b), the result $D_{\mathbf{u}} f(a, b)$ tells us how f changes for every unit increase in (x, y) from (a, b) in the direction of the vector $\mathbf{u}$. The quantity $D_{\mathbf{u}}f(a, b)$ also tells us the slope of the line tangent to the surface in the direction of $\mathbf{u}$ at the point $(a, b, f(a, b))$ (since the vector $\mathbf{u}$ is 2-dimensional, we can use the word slope here).

So our question about wind chill in Preview Activity 2.6 can be formulated as a question about a directional derivative. In this case, we want to calculate the derivative of the f in the direction of the vector $\langle 3, 1\rangle$ at the point $(10, 5)$. The directional derivative is defined using a unit vector, so if $\mathbf{u} = \langle u_2, u_2\rangle = \frac{1}{\sqrt{10}}\langle 3, 1\rangle$, then we want to calculate $D_{\mathbf{u}}f(10, 5)$. In our Preview Activity we showed that the derivative of f in the direction of the unit vector $\mathbf{u}$ could be found by $f_x(10, 5)u_1 + f_y(10, 5)u_2$ (see equation (**??**). Next we will see that this works in general.

Calculating a Directional Derivative

We can use the Chain Rule to find a formula for the directional derivative $D_{\mathbf{u}}f(x, y)$ at the point (a, b) in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2\rangle$. The variables x and y are changing in a linear manner in the direction of $\mathbf{u}$, with

$$x = a + u_1 t \quad \text{and} \quad y = b + u_2 t.$$

By the Chain Rule we must have

$$D_{\mathbf{u}}f(a, b) = f_x(a, b)\frac{dx}{dt}\bigg|_{(a,b)} + f_y(a, b)\frac{dy}{dt}\bigg|_{(a,b)} = f_x(a, b)u_1 + f_y(a, b)u_2.$$

When we evaluate the directional derivative at an arbitrary point we obtain the following formula.

We can calculate the derivative of $f = f(x, y)$ in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2\rangle$ by

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)u_1 + f_y(x, y)u_2. \tag{2.7}$$

IMPORTANT NOTE: To use equation (2.7) we MUST have a UNIT VECTOR $\mathbf{u} = \langle u_1, u_2\rangle$ in the direction of motion.

Activity 2.23.

Let $f(x, y) = 3xy - x^2y^3$.

(a) Find the derivative of f in the direction of the vector $\mathbf{v} = \langle 2, 3\rangle$ at the point $(1, -1)$.

(b) Find $D_{\mathbf{i}}f(x, y)$. What familiar function is $D_{\mathbf{i}}f(x, y)$? What familiar function is $D_{\mathbf{j}}f(x, y)$?

◁

The Gradient

In many situations (e.g., mountain climbing, skiing, heat seeking missiles) we are interested in knowing the direction in which a function is increasing or decreasing most rapidly. For example, consider how a heat-seeking missile might work.[6]

Suppose that the temperature surrounding a fighter plane can be modeled by the function f defined by

$$f(x, y) = \frac{100}{1 + (x-5)^2 + 4(y-2.5)^2},$$

where (x, y) is a point in the plane of the fighter and $f(x, y)$ is measured in degrees Celsius. Assume that a heat-seeking missile is fired from the point (2,4). The missile travels in such a way that it periodically updates its direction of motion to travel in the direction in which the temperature increases most rapidly (thus, the missile seeks heat and, hopefully, follows the plane's exhaust). This is illustrated in the animation in Figure 2.16. We want to determine the direction in which the missile travels each time it updates – the direction of the greatest increase in temperature. As we will see, the *gradient* vector contains this information.

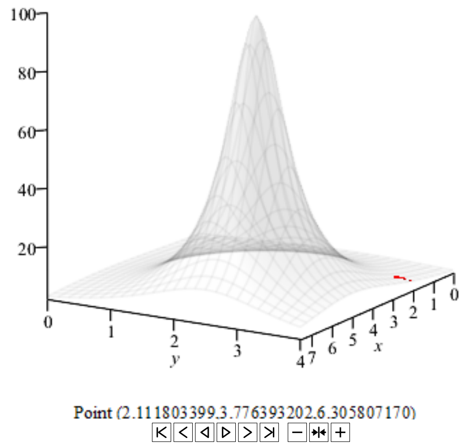

Figure 2.16: Using the gradient.

To explain what is happening in Figure 2.16, recall that the derivative of f in the direction of a unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is defined as

$$D_{\mathbf{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + u_1 h, y + u_2 h) - f(x, y)}{h}$$

and that we can calculate $D_{\mathbf{u}} f(x, y)$ by

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) u_1 + f_y(x, y) u_2.$$

Note that we can also write this directional derivative as a dot product

$$D_{\mathbf{u}} f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u}.$$

[6]This application is borrowed from United States Air Force Academy Department of Mathematical Scienceshttp://www.nku.edu/~longa/classes/mat320/mathematica/multcalc.htm.

It is the vector $\langle f_x(x,y), f_y(x,y)\rangle$ that we want to understand.

The derivative $D_{\mathbf{u}}f$ in the direction of a unit vector $\mathbf{u} = \langle u_1, u_2\rangle$ tells us how f changes if (x,y) changes in the direction of $\mathbf{u}$. Since the missile wants to find the direction of greatest increase in f, when the missile is at a point $(a,b,f(a,b))$ the missile must determine the direction $\mathbf{u}$ that will maximize the value of the directional derivative $D_{\mathbf{u}}f(a,b)$. We examine how to find this direction in the next activity.

Activity 2.24.

In this activity we determine the direction of greatest increase of a function.

(a) Explain why

$$D_{\mathbf{u}}f(a,b) = |\langle f_x(a,b), f_y(a,b)\rangle| \cos(\theta), \tag{2.8}$$

where θ is the angle between $\langle f_x(a,b), f_y(a,b)\rangle$ and $\mathbf{u}$. (Hint: Use the dot product.)

(b) At the point (a,b), the only quantity in (2.8) that can change is θ (which determines the direction $\mathbf{u}$ of travel). Explain why $\theta = 0$ makes $|\langle f_x(a,b), f_y(a,b)\rangle| \cos(\theta)$ as large as possible.

(c) When $\theta = 0$, in what direction does the vector $\mathbf{u}$ point? Why? What does this tell us about the direction of greatest increase of f at the point (a,b)? Why?

◁

As Activity 2.24 shows, the vector $\langle f_x(a,b), f_y(a,b)\rangle$ has a special property – it points in the direction of greatest increase of f. Because of this we give the vector $\langle f_x(a,b), f_y(a,b)\rangle$ a special name.

Definition 2.6. The **gradient** of the function $f = f(x,y)$ is the vector

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle.$$

We read ∇f as "grad f" or "del f".[7] The vector $\nabla f(x,y)$ tells us the direction of greatest increase of the function f at the point (x,y).

Activity 2.25.

Let $f(x,y) = \frac{100}{1+(x-5)^2+4(y-2.5)^2}$.

(a) Find $f_x(x,y)$ and $f_y(x,y)$.

(b) Find the direction of greatest increase in f at the point $(3,4)$.

◁

As we saw in Figure 2.16, our missile tracks the plane by traveling in the direction of greatest increase in f for a short while, recalculating the direction of greatest increase and traveling in that

[7] The symbol ∇ is called *nabla*, which comes from a Greek word for a certain type of harp that has a similar shape.

new direction for a short while, and repeating this process until the missile finds the plane.[8] Although we won't use this method for this purpose, we can use this process for arbitrary functions to numerically approximate local maximum (or minimum) values.

The Gradient and Contours

The gradient has a natural relationship with the level curves or contours of a function. In our missile example the level curves of f are the curves defined by

$$c = \frac{100}{1 + (x-5)^2 + 4(y-2.5)^2}$$

for constants c. Figure 2.17 shows some level curves of f along with a unit vector in the direction of $\nabla f(3,4)$ at the point $(3,4)$. Notice that it appears that $\nabla f(3,4)$ is orthogonal to the level curve

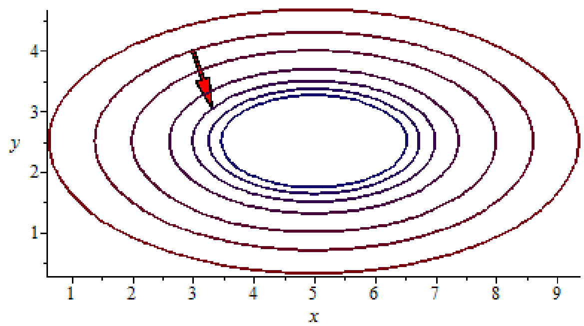

Figure 2.17: Contours and a gradient vector

of f at $(3,4)$. This is no accident, as we discover in our next activity.

Activity 2.26.

Suppose we parameterize a level curve $f(x,y) = c$ by

$$x = x(t), \quad y = y(t), \quad z = c.$$

(a) If we stay on the level curve, the z value doesn't change. So if we change t but stay on the level curve, what is $\frac{df}{dt}$?

(b) Explain each step in the following string of equalities.

$$0 = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \nabla f \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle.$$

[8]We have seen this process of periodically updating direction of travel when we investigated Newton's method for approximating roots.

(c) What does $\nabla f \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = 0$ tell us about the relationship between ∇f and $\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$? Why does this show that the gradient is always orthogonal to the level curve as indicated in Figure **??**?

◁

Activity 2.27.

Let $f(x, y) = x^2 + 3y^2$.

(a) Find ∇f.

(b) Find $\nabla f(1, 2)$.

(c) Find the direction of greatest increase in f at the point $(1, 2)$. Explain. A graph of the surface defined by f is shown in Figure 2.18. Illustrate this direction on the surface.

(d) A contour diagram of f is shown in Figure 2.19. Illustrate your calculation from (b) on this contour diagram.

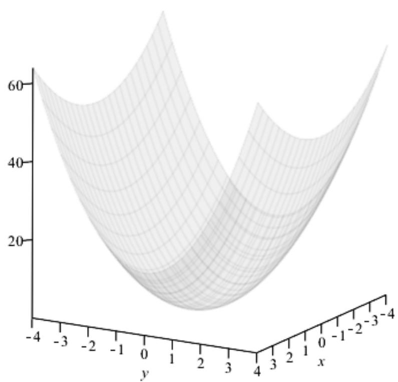

Figure 2.18: The surface for $f(x, y) = x^2 + 3y^2$.

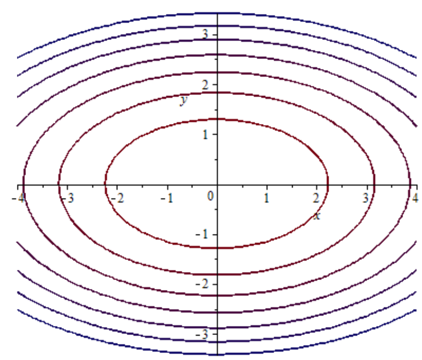

Figure 2.19: Contours for $f(x, y) = x^2 + 3y^2$.

◁

Summary

In this section, we encountered the following important ideas:

- The derivative of a function $f = f(x, y)$ in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is

$$D_{\mathbf{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + u_1 h, y + u_2 h) - f(x, y)}{h}$$

for those values of x and y for which the limit exists.

- We calculate the derivative of $f = f(x, y)$ in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ by

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)u_1 + f_y(x, y)u_2.$$

- The directional derivative $D_{\mathbf{u}} f(x, y)$ at a point (a, b) tells us how f changes for every unit increase in (x, y) from (a, b) in the direction of the vector $\mathbf{u}$. The quantity $D_{\mathbf{u}} f(a, b)$ also tells us the slope of the line tangent to the surface in the direction of $\mathbf{u}$ at the point $(a, b, f(a, b))$.

- The gradient of the function $f = f(x, y)$ is the vector $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$. The gradient tells us the direction of greatest increase of f at the point (a, b).

2.7 Optimization

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What are the differences between relative extreme values and global extreme values of a function of two variables?
- How is the process of finding the global maximum or minimum of a function over the functions entire domain different from determining the global maximum or minimum on a closed and bounded domain?

Preview Activity 2.7. The quantity of a product demanded by consumers is often a function of the price of the product. The quantity of demand for a product may also depend on the price of other products (the demand for McDonald's hamburgers may be affected by the price of Burger King hamburgers, or the demand for gas at one station can be affected by the price of gas at another). Suppose quantities demanded q_1 and q_2 of two goods are dependent on their respective prices p_1 and p_2 as follows:

$$q_1 = 150 - 2p_1 - p_2 \tag{2.9}$$
$$q_2 = 200 - p_1 - 3p_2. \tag{2.10}$$

A problem for the manufacturer and seller of both products is how to set prices in order to maximize revenue.

We assume the maximal revenue will occur when the manufacturer sells all of the products. So if we let f be the revenue obtained by selling q_1 items of good one at price p_1 per item and q_2 items of good two at price p_2 per item, then

$$f(p_1, p_2, q_1, q_2) = p_1q_1 + p_2q_2.$$

We can reduce the revenue to a function of just two variables p_1 and p_2 by using (2.9) and (2.10), giving us

$$f(p_1, p_2) = p_1(150 - 2p_1 - p_2) + p_2(200 - p_1 - 3p_2) = 150p_1 + 200p_2 - 2p_1p_2 - 2p_1^2 - 3p_2^2.$$

A graph of f as a function of p_1 and p_2 is shown in Figure **??**.

(a) Does it appear that there is a maximum value for the revenue. Why?

(b) Based on the graph of the revenue surface in Figure 2.20, describe what you think the tangent plane to the surface looks like at the point where the maximum value occurs. What should we expect the values of $\frac{\partial f}{\partial p_1}$ and $\frac{\partial f}{\partial p_2}$ to be at this maximum value?

(c) Assume your response to part (b) is correct. Use this response to find the maximum revenue.

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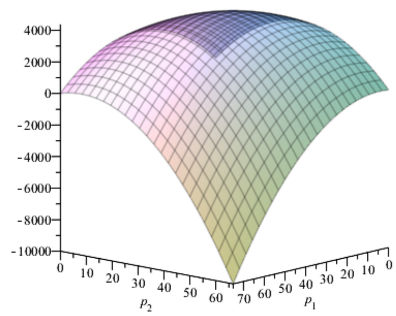

Figure 2.20: A revenue function.

Introduction

In single variable calculus we learned how to find the optimum values of a function of a single variable by finding and classifying the critical points of a function. In this section we will investigate the analogous process of finding optimal values for functions of more than one variable.

Extreme Values and Critical Points

Recall from single variable calculus that a value M of a function of one variable is a maximum value if M is greater than or equal to all other values of the function. We also had notions of local (or relative) maximum values. The same definitions work for functions of several variables.

Definition 2.7. Let $f = f(x, y)$ be a function of the independent variables x and y.

1. A value $M = f(a, b)$ is a **maximum value** (or absolute or global maximum value) of f if $M \geq f(x, y)$ for all (x, y) in the domain of f.
2. A value $M = f(a, b)$ is a **relative** (or local) maximum value of f if $M \geq f(x, y)$ for all (x, y) in the domain of f near (a, b).

Activity 2.28.

Let $f = f(x, y)$ be a function of the independent variables x and y.

(a) Write the definition of a minimum value of f.

(b) Write the definition of a relative minimum value of f.

◁

Note that maximum and minimum values of a function are z values – values of the dependent variable – and they occur at some points (a, b) in the domain of f. Also note that a maximum value is also a local maximum value. We give these values special names.

> **Definition 2.8.** The **extreme values** of a function are the local and global maximum and minimum values of the function.

Activity 2.29.

Let $f(x, y) = \sin(x) + \cos(y)$. Determine the absolute maximum and minimum values of f. At what points do these extreme values occur?

$\triangleleft$

The process of finding maximum or minimum values is called *optimization*. Next we explore how we can find maximum and minimum values of a function of two variables. Recall from single variable calculus that we found maximum and minimum values of a function of a single variable by looking at the critical points of the function – the points where the derivative is 0 or undefined. There is a similar definition of critical point for functions of two variables. In Preview Activity 2.7 we saw that at the maximum point of our revenue surface the tangent plane appears horizontal, indicating that both f_x and f_y are 0 there. That leads us to the following notion, similar to the one from single variable calculus.

> **Definition 2.9.** A **critical point** (a, b) of a function $f = f(x, y)$ is a point in the domain of f at which the gradient of f is 0 or where either f_x or f_y fail to exist.

If a function has a maximum or minimum value, it has to occur at a critical point. So just like in single variable calculus, our first step in finding optimal values of a function is to find the critical points.

Activity 2.30.

Find the critical points of the following functions. Use appropriate technology (e.g., Wolfram|Alpha or CalcPlot3D[9]) to draw the graphs of the surfaces to verify your answers.

(a) $f(x, y) = 2 + x^2 + y^2$

(b) $f(x, y) = 2x - x^2 - \frac{1}{4}y^2$

(c) $f(x, y) = |x| + |y|$

$\triangleleft$

[9] at http://web.monroecc.edu/manila/webfiles/calcNSF/JavaCode/CalcPlot3D.htm

Classifying Critical Points: The Second Derivative Test

Finding critical points is only part of the optimization process. Once we find critical points, we then need to classify them as giving maximum, minimum values or neither. In single variable calculus we had first and second derivative tests to classify critical points on open intervals and an Extreme Value Theorem for finding optimal values on closed intervals. In multivariable calculus we will have only a second derivative test to use on open domains, but will have an analog of the method we used to find optimal values on closed and bounded domains.

Recall the First Derivative Test from single variable calculus to help us classify extreme values of a function of a single variable on an open interval.

The First Derivative Test. If $(p, f(p))$ is a critical point of a continuous function f that is differentiable around $x = p$ (except possibly at p), then

- $f(p)$ is a relative maximum value of f if f' changes from positive to negative at $x = p$,
- $f(p)$ is a relative minimum value of f if f' changes from negative to positive at $x = p$.

There is no corresponding first derivative test for functions of more than one variable. To understand why, consider the function f defined by $f(x, y) = x^4 + y^4 - 4x^2y^2$. The graph of the surface defined by f is shown in Figure 2.21. A computation left to the reader shows that f has a critical point at $(0, 0)$ and that f_x changes from negative to positive in the x direction at $(0, 0)$ and f_y also changes from negative to positive in the y direction at $(0, 0)$. The changes in sign in f_x and f_y at $(0, 0)$ might lead us to believe that f has a local minimum value at $(0, 0)$, but the graph in Figure 2.21 shows that this is not the case. In fact, there are points (x, y) near $(0, 0)$ where $f(x, y) > f(0, 0)$ and points (x, y) near $(0, 0)$ where $f(x, y) < f(0, 0)$. Near $(0, 0)$, the graph of f looks like a saddle and we call such a point a *saddle point*.

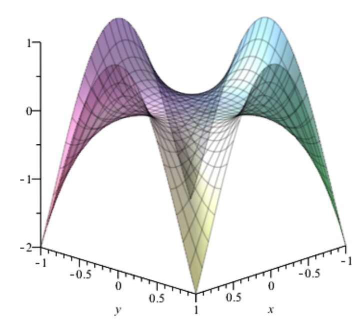

Figure 2.21: The surface defined by $f(x, y) = x^4 + y^4 - 4x^2y^2$

Definition 2.10. A point $(a, b, f(a, b))$ is a **saddle point** of $f = f(x, y)$ if (a, b) is a critical point of f and within any distance of (a, b) there are points (s, t) and (u, v) such that $f(s, t) > f(a, b)$ and $f(u, v) < f(a, b)$.

Now recall the Second Derivative Test for classifying critical points of functions of a single variable.

The Second Derivative Test. If $(p, f(p))$ is a critical point of a function f so that $f'(p) = 0$ and if $f''(p)$ exists, then

- $f(p)$ is a relative maximum value of f if $f''(p) < 0$,
- $f(p)$ is a relative minimum value of f if $f''(p) > 0$.

There is a corresponding second derivative test for functions of two variables.

The Second Derivative Test. Let $f = f(x, y)$ be a function of the variables x and y. Suppose (a, b) is a point at which f is differentiable and $f_x(a, b) = f_y(a, b) = 0$. Let

$$D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2.$$

1. (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then f has a relative minimum at (a, b).
   (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then f has a relative maximum at (a, b).
2. If $D < 0$, then f has a saddle point at (a, b).
3. If $D = 0$ we get no information about what happens at (a, b).

The value D in the Second Derivative Test is called the *discriminant* of the function f at (a, b). In the last section we provide an explanation for why the Second Derivative Test works. For now, we will assume its validity.

Activity 2.31.

Return to our revenue function $f(p_1, p_2) = p_1(150 - 2p_1 - p_2) + p_2(200 - p_1 - 3p_2)$ from our Preview Activity. Recall that

$$\frac{\partial f}{\partial p_1} = 150 - 2p_2 - 4p_1 \quad \text{and} \quad \frac{\partial f}{\partial p_2} = 200 - 2p_1 - 6p_2,$$

so the sole critical point of f is $(25, 25)$.

(a) Find the discriminant of f at its critical point.

(b) Use the Second Derivative Test to classify the critical point of f. Does the result confirm what we learned in the Preview Activity?

$\triangleleft$

Activity 2.32.

Find the critical points of the following functions and use the Second Derivative Test to classify the critical points.

(a) $f(x,y) = 3x^3 + y^2 - 9x + 4y$

(b) $f(x,y) = xy + \frac{2}{x} + \frac{4}{y}$

$\triangleleft$

Optimization on a Restricted Domain

The Second Derivative Test helps us classify critical points of a function, but it does not tell us if our function actually has an absolute maximum or minimum value. For single variable functions, the Extreme Value Theorem told us that a function that is continuous on a closed interval $[a, b]$ always has a maximum and minimum value on that interval. To find the maximum and minimum values, we tested the function at the critical points in the interval and at the endpoints. A similar thing happens for functions of two variables.

For functions of two variables, closed and bounded regions play the role that closed intervals did for functions of a single variable. A closed region is a region that contains its boundary ($x^2 + y^2 \leq 1$ as opposed to $x^2 + y^2 < 1$, for example) while a bounded region is one that does not stretch to infinity in any direction. Just as for functions of a single variable, continuous functions of several variables that are defined on closed, bounded regions must have maximum and minimum values in those regions.

The Extreme Value Theorem. Let $f = f(x,y)$ be a continuous function on a closed and bounded region R. Then f attains a maximum value at some point in R and f attains a minimum value at some point in R.

To find the optimal values of functions defined on closed regions, we check the function at the critical points in the interior of the domain and on the boundary of the region. The maximum value of f occurs at one of these points, as does the minimum value. It may be more difficult than in single variable calculus to check on the boundary in these circumstances. An example is in order.

Example 2.1. Suppose the temperature T at each point on the circular plate $x^2 + y^2 \leq 1$ is given by

$$T(x,y) = 2x^2 + y^2 - y.$$

The domain $R = \{(x,y) : x^2 + y^2 \leq 1\}$ is a closed and bounded region, so the Extreme Value Theorem assures us that T reaches a maximum and minimum value on the plate. We will find the hottest and coldest points on the plate and the corresponding hottest and coldest temperatures on the plate.

Here is the basic procedure for optimizing functions on closed and bounded domains.

Step 1: Find all critical points of the function in the interior of the domain.

Step 2: Find all the critical points of the function on the boundary of the domain. Working on the boundary of the domain reduces this part of the problem to one or more single variable optimization problems.

Step 3: The maximum value of the function is the largest value at the points obtained in Steps 1 and 2. The minimum value of the function is the smallest value at the points obtained in Steps 1 and 2.

Let's apply this procedure to our temperature function T. To help us in our analysis, a graph of the surface defined by our function T on R is shown in Figure 2.22. The figure seems to indicate that T has optimal values both in the interior of R and on the boundary of R.

Step 1: The first step in optimizing T is to find the critical points of T in the interior of R. To do this, we find the points where $\nabla T = \mathbf{0}$ or where ∇T does not exist. Since $T_x = 4x$ and $T_y = 2y - 1$, the gradient of T is defined everywhere and the only critical points of T occur where

$$\nabla T = \langle 4x, 2y - 1 \rangle = \mathbf{0}.$$

Therefore, the only critical point of T in R is $\left(0, \frac{1}{2}\right)$.

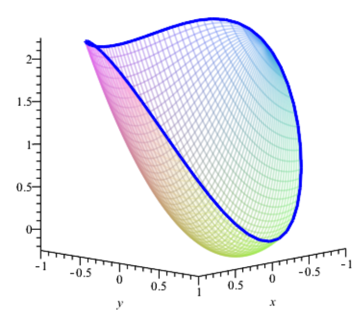

Figure 2.22: Temperature of a plate on $x^2 + y^2 \leq 1$

Step 2: Now we have to find the critical points of T on the boundary. In our problem, the boundary of R is the circle $x^2 + y^2 = 1$. To find the critical points of T on the boundary circle, we parameterize the circle. Recall that the unit circle can be parameterized by

$$x(t) = \cos(t) \qquad \text{and} \qquad y(t) = \sin(t)$$

for t in $[0, 2\pi]$. To find the critical points of T on the boundary, we substitute our parameterization for x and y in T to obtain

$$T(t) = 2x^2 + y^2 - y = 2\cos^2(t) + \sin^2(t) - \sin(t).$$

We can find $\frac{dT}{dt}$ directly now (or we could have used the Chain Rule):

$$\frac{dT}{dt} = -4\cos(t)\sin(t) + 2\sin(t)\cos(t) - \cos(t) = -2\cos(t)\sin(t) - \cos(t) = -\cos(t)(2\sin(t) + 1).$$

Now $\frac{dT}{dt}$ is defined everywhere, so we only need to find the critical points where $\frac{dT}{dt} = 0$ and also consider the endpoints $t = 0$ and $t = 2\pi$. Notice that $\frac{dT}{dt} = 0$ when $\cos(t) = 0$ or $\sin(t) = -\frac{1}{2}$. For t between 0 and 2π we have $\cos(t) = 0$ when $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$, and $\sin(t) = -\frac{1}{2}$ when $t = \frac{7\pi}{6}$ and $t = \frac{11\pi}{6}$. Now we find the corresponding (x, y) points for these values of t:

- $(x, y) = (0, 1)$ when $t = \frac{\pi}{2}$,
- $(x, y) = (0, -1)$ when $t = \frac{3\pi}{2}$,
- $(x, y) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ when $t = \frac{7\pi}{6}$,
- $(x, y) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ when $t = \frac{11\pi}{6}$,
- $(x, y) = (1, 0)$ when $t = 0$ and when $t = 2\pi$.

Step 3: Now we simply compare the values of T at all of the critical points found in Steps 1 and 2:

- $T\left(0, \frac{1}{2}\right) = -\frac{1}{4}$,
- $T(0, 1) = 0$,
- $T(0, -1) = 2$,
- $T\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{9}{4}$,
- $T\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{9}{4}$,
- $T(1, 0) = 2$.

So the maximum value of T on R is $\frac{9}{4}$ which occurs at the two points $\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the boundary, and the minimum value of T on R is $-\frac{1}{4}$ which occurs at the critical point $\left(0, \frac{1}{2}\right)$ in the interior of R.

Activity 2.33.

Let $f(x, y) = x^2 - 3y^2 - 4x + 6y$ with triangular domain R with vertices at $(0, 0)$, $(4, 0)$, and $(0, 4)$. A graph of f on the domain R appears in Figure 2.23.

(a) Find all of the critical points of f in R.

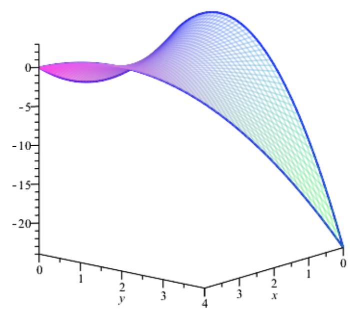

Figure 2.23: $f(x, y) = x^2 - 3y^2 - 4x + 6y$

(b) Parameterize the horizontal leg of the triangular domain. Then find the critical points of f on that leg.

(c) Parameterize the vertical leg of the triangular domain. Then find the critical points of f on that leg.

(d) Parameterize the hypotenuse of the triangular domain. Then find the critical points of f on the hypotenuse.

(e) Find the absolute maximum and absolute minimum value of f on R.

$\triangleleft$

Optimization problems often come in a descriptive form, where the reader has to use the information given to find the function to optimize. In these situations it is important to determine what quantities can change (the variables) and which do not change (constants). The information provided should be enough to find a function of two variables to which we can apply the techniques of this section. We conclude this section with an activity of this type of problem.

Justifying The Second Derivative Test

Why the Second Derivative Test is valid for functions of two variables can be seen through the use of Taylor polynomials. Recall that the second order Taylor polynomial $P_2(x)$ centered at $x = a$ for a function f of a single variable x is

$$P_2(x) = f(a) = f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

This polynomial has the properties that $P(a) = f(a)$, $P'(a) = f'(a)$, and $P''(a) = f''(a)$, so that P and f both pass through the point $(a, f(a))$, go in the same direction at $(a, f(a))$, and have the same concavity at $(a, f(a))$. This makes $P_2(x)$ the quadratic function that best fits the graph of f for x close to a.

There are also Taylor polynomials of the first and second order for a function f of two variables x and y. As in the single variable case, we look for polynomials whose derivatives agree with those of f. The complication of the additional independent variable means that our polynomials will need to contain all of the possible monomials of the indicated degrees. So if $P_1(x,y)$ is the first order Taylor polynomial for f centered at the point (a,b), then $P_1(x,y)$ has the form

$$P_1(x,y) = f(a,b) + c_1(x-a) + c_2(y-b).$$

Note that $P_1(a,b) = f(a,b)$. To make $P_1(x,y)$ the best linear approximation to f near the point (a,b) we also want to have $\left.\frac{\partial P_1}{\partial x}\right|_{(a,b)} = f_x(a,b)$, and $\left.\frac{\partial P_1}{\partial y}\right|_{(a,b)} = f_y(a,b)$. This implies that we must have

$$c_1 = \left.\frac{\partial P_1}{\partial x}\right|_{(a,b)} = f_x(a,b) \quad \text{and} \quad c_2 = \left.\frac{\partial P_1}{\partial y}\right|_{(a,b)} = f_y(a,b).$$

Thus,

$$P_1(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Note that $P_1(x,y)$ is just the linearization of f at (a,b).

Similarly, the second order Taylor polynomial $P_2(x,y)$ centered at the point (a,b) for the function f will have the form

$$P_2(x,y) = P_1(x,y) + c_3(x-a)^2 + c_4(x-a)(y-b) + c_5(y-b)^2.$$

A quick calculation shows that $P_2(a,b) = f(a,b)$, $\left.\frac{\partial P_2}{\partial x}\right|_{(a,b)} = f_x(a,b)$, and $\left.\frac{\partial P_2}{\partial y}\right|_{(a,b)} = f_y(a,b)$. To make $P_2(x,y)$ the best quadratic approximation to f near the point (a,b) we will also need to have

$$\left.\frac{\partial^2 P_2}{\partial x^2}\right|_{(a,b)} = f_{xx}(a,b), \quad \left.\frac{\partial^2 P_2}{\partial x \partial y}\right|_{(a,b)} = f_{xy}(a,b), \quad \text{and} \quad \left.\frac{\partial^2 P_2}{\partial y^2}\right|_{(a,b)} = f_{yy}(a,b).$$

Since

$$\left.\frac{\partial^2 P_2}{\partial x^2}\right|_{(a,b)} = 2c_3, \quad \left.\frac{\partial^2 P_2}{\partial x \partial y}\right|_{(a,b)} = c_4, \quad \text{and} \quad \left.\frac{\partial^2 P_2}{\partial y^2}\right|_{(a,b)} = 2c_5,$$

we must have

$$\left.\frac{\partial^2 P_2}{\partial x^2}\right|_{(a,b)} = 2c_3, \quad c_4 = f_{xy}(a,b), \quad \text{and} \quad \left.\frac{\partial^2 P_2}{\partial y^2}\right|_{(a,b)} = 2c_5.$$

Thus,

$$P_2(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2.$$

Now we can examine a critical point (a,b) of a function f of two variables for which $\nabla f(a,b) = 0$. We approximate f near the point (a,b) by its second order Taylor polynomial $P_2(x,y)$. Since $f_x(a,b) = f_y(a,b) = 0$, our $P_2(x,y)$ has the form

$$P_2(x,y) = f(a,b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2.$$

To classify this critical point of f, we just need to determine the behavior of the Taylor polynomial $P_2(x, y)$ at this critical point (a, b). To simplify the algebra we let

$$r = f_{xx}(a, b), \; s = f_{xy}(a, b), \; \text{and} \; t = f_{yy}(a, b).$$

To simplify the situation even more, we can translate the graph so that the point (a, b) is at the origin and just assume that $a = b = 0$. Since the constant term in $P_2(x, y)$ does not contribute to the shape of the graph of $P_2(x, y)$, we only need concern ourselves with the quadratic

$$Q(x, y) = rx^2 + sxy + ty^2.$$

To better understand the behavior of Q, we complete the square on the first two summands

$$\begin{aligned} rx^2 + sxy + ty^2 &= r\left(x^2 + \frac{s}{r}xy\right) + ty^2 \\ &= r\left(x^2 + \frac{s}{r}xy + \left(\frac{s}{2r}\right)^2 y^2\right) + ty^2 - r\left(\frac{s}{2r}\right)^2 y^2 \\ &= r\left(x + \frac{s}{2r}y\right)^2 + \left(t - \frac{s^2}{4r}\right) y^2 \\ &= r\left[\left(x + \frac{s}{2r}y\right)^2 + \frac{1}{4}\left(\frac{4tr - s^2}{r^2}\right) y^2\right]. \end{aligned}$$

Now $\left(x + \frac{s}{2r}y\right)^2 \geq 0$, so whether Q opens up, down, or has a saddle point is determined by the value of r and the value of $D = 4tr - s^2$. (The expression $4tr - s^2$ is called the *discriminant* – just like the discriminant in the quadratic formula). To classify the critical point, we examine cases.

- If $D > 0$, then $Q(x, y)$ has the form

  $$Q(x, y) = ru^2 + v^2,$$

  where $u = x + \frac{s}{2r}y$ and $v = \sqrt{\frac{4tr - s^2}{4r^2}}$. So we will have a paraboloid that opens up if $r > 0$ and opens down if $r < 0$.

- If $D < 0$, then $Q(x, y)$ has the form

  $$Q(x, y) = ru^2 - v^2,$$

  which will give us a saddle point.

- If $D = 0$, then $Q(x, y)$ has the form

  $$Q(x, y) = r\left(x + \frac{s}{2r}y\right)^2,$$

  which is a parabolic cylinder. This turns out to give us no information.

To summarize,

1. (a) If $D > 0$ and $r > 0$, we have a relative minimum at our critical point.
   (b) If $D > 0$ and $r < 0$, we have a relative maximum at our critical point.
2. If $D < 0$, then our critical point is a saddle point.
3. If $D = 0$, we get no information.

Recall that $r = \frac{f_{xx}(a,b)}{2}$, $s = f_{xy}(a, b)$, and $t = \frac{f_{yy}(a,b)}{2}$, so our discriminant has the form

$$D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2,$$

giving us the Second Derivative Test.

Summary

In this section, we encountered the following important ideas:

- A relative extreme value of a function is a value that a function attains at a point that is the greatest or lowest value of the function in some small open disk containing that point. A global extreme value of a function is a value that a function attains at a point that is the greatest or lowest value of the function over its entire domain.
- To find the extreme values of a function f on its entire domain, we first find the critical points, the points at which gradient of f is 0 or where either f_x or f_y fail to exist. If a critical point is in the interior of the domain, then we can possibly determine its status with the Second Derivative Test. If f is defined over a closed and bounded domain, then we find the critical points of f in the interior of the domain, and also look for critical points on the boundary of the domain. We compare the values of f at the critical points in the interior and on the boundary to determine the absolute extreme values.

2.8 Constrained Optimization:Lagrange Multipliers

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How do we use a Lagrange multiplier to solve a constrained optimization problem?
- What is the practical meaning of the Lagrange multiplier?

Preview Activity 2.8. The Cobb-Douglas production function is used in economics to model production levels based on labor and equipment. A Cobb-Douglas function has the form

$$f(x, y) = Cx^{\alpha}y^{1-\alpha}$$

where x is the dollar amount spent on labor and y the amount of capital spent on equipment. Suppose we have a specific Cobb-Douglas function of the form

$$f(x, y) = 50x^{0.4}y^{0.6}.$$

The problem we want to address is how to maximize output in this system if we have a fixed amount of money, say $1.5 million, to spend.

This is a problem where we have an external constraint on our variables, namely that $x + y$ cannot exceed $1.5 million. We assume we will use all of the capital available to maximize production.[10]

(a) Our constraint defines a function g, given by

$$g(x, y) = x + y.$$

Explain why the constraint is a level curve of g, and is therefore a two-dimensional curve.

(b) Use the applet at

http://ocw.mit.edu/ans7870/18/18.02/f07/tools/LagrangeMultipliersTwoVariables.html

to draw a contour diagram of f along with level curves of the constraint function g on the rectangle $[0, 2] \times [0, 2]$. Click off the boxes Show grad f and Show grad g. You will see the level curve $g(x, y) = b$ in yellow and level curves of f in blue. You can vary the value of b using the b slider. You should also see the level curve $f(x, y) = a$ in green, and you can control the value of a with the a slider. The intersections of the constraint curve with the

[10] We could solve this problem by solving the equation $x + y = 1.5$ for y and substituting into f to write f as a function of x alone. However, there is a general method that we want to understand in this section, and we are using a two-variable example to make the ideas easier to see.

level curves of f are the points of interest. Draw the constraint $g(x, y) = 1.5$, or at least get as close as you can to the value $b = 1.5$. Now move the a slider back and forth to see which level curves of f intersect the constraint curve $g(x, y) = 1.5$. Explain why the maximum production will occur when the graph of the constraint is tangent to a contour of f.

(c) Now check the boxes Show grad f and Show grad g. You can then move the pink point around to see how the two gradients are related at various points in the plane. To find an optimal production value, we want to understand how ∇f is related to ∇g at the point of tangency.

i. In general, how are ∇f and ∇g related to the level curves of f and g, respectively?

ii. Explain then why ∇f and ∇g will be parallel at the point where the graph of the constraint is tangent to a level curve of f.

$\bowtie$

Introduction

We previously consider how to find the optimum values of functions on unrestricted domains or on closed, bounded domains. Now we consider the problem of optimization with constraints.

Constrained Optimization and Lagrange Multipliers

In Preview Activity 2.8 we have a problem where we have an external constraint on our variables, namely that the sum of our labor and capital for equipment cannot exceed $1.5 million. We saw that we can make a function g from our constraint, where $g(x, y) = x + y$. Our constraint is just a contour $g(x, y) = c$ of g where c is a constant (in our case 1.5). Figure 2.24 illustrates that the maximum value of our Cobb-Douglas production function f is a maximum, subject to the constraint $g(x, y) = c$, when the graph of $g(x, y) = c$ is tangent to a contour of f. The value of f on this contour is the maximum value. To find this point where the graph of the constraint is tangent to a contour of f, recall that ∇f is perpendicular to the contours of f and ∇g is perpendicular to the contour of g. So we need to determine the points where ∇g and ∇f are parallel. Two vectors are parallel if one is a nonzero scalar multiple of the other, so we then look for values of a parameter λ that makes

$$\nabla f = \lambda \nabla g. \tag{2.11}$$

The constant λ is called a *Lagrange multiplier* and can be interpreted in a useful way we will see later.

To find the values of λ that satisfy (2.11) for our Cobb-Douglas function, we calculate both ∇f and ∇g. Now

$$\nabla f = \frac{20y^{0.6}}{x^{0.6}}\mathbf{i} + \frac{30x^{0.4}}{y^{0.4}}\mathbf{j} \quad \text{and} \quad \nabla g = \mathbf{i} + \mathbf{j},$$

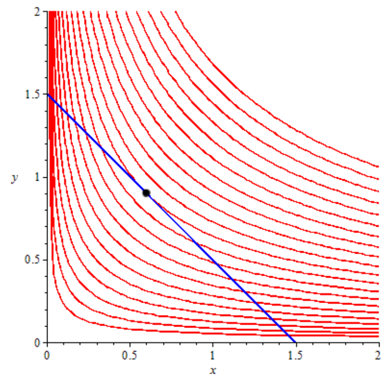

Figure 2.24: Contours of f and the constraint contour.

so we need a value of λ so that

$$\frac{20y^{0.6}}{x^{0.6}}\mathbf{i} + \frac{30x^{0.4}}{y^{0.4}}\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j}).$$

Equating components and the original constraint give us the three equations

$$20y^{0.6} = \lambda x^{0.6} \tag{2.12}$$

$$30x^{0.4} = \lambda y^{0.4} \tag{2.13}$$

$$x + y = 1.5 \tag{2.14}$$

in the three unknowns x, y, and λ. Dividing both sides of (2.12) by the corresponding sides of (2.13) yields

$$\frac{2}{3}\frac{y^{0.6}}{x^{0.4}} = \frac{x^{0.6}}{y^{0.4}}$$

$$\frac{2}{3}y = x \tag{2.15}$$

Substituting into (2.14) gives us

$$\frac{2}{3}y + y = 1.5$$

or

$$y = \frac{9}{10}.$$

Then $x = \frac{3}{5}$ and $\lambda \approx 25.51$. Therefore we maximize our output when we spend \$0.6 million on labor and \$0.9 million on equipment. We summarize the method we developed.

> The general technique for optimizing a function $f = f(x, y)$ subject to a constraint $g(x, y) = c$ is to solve the system $\nabla f = \lambda \nabla g$ for x, y, and λ. There may be more than one solution: evaluate f at each solution and the largest resulting value is the maximum value of f and the smallest resulting value is the minimum value of f subject to the constraint $g(x, y) = c$.

Activity 2.34.

A soda pop can holds about 355 cc of liquid. In this activity we want to find the dimensions of such a can that will minimize the surface area.

(a) What assumption should we make about the shape of a soda pop can? What are the variables in this problem? What restriction, if any, are there on these variables?

(b) What quantity do we want to optimize in this problem? What equation describes the constraint?

(c) Find λ and the values of your variables that satisfy equation (2.11) in the context of this problem.

(d) Determine the dimensions of the pop can that give the desired solution to this constrained optimization problem.

◁

The method of Lagrange multipliers works as well for functions of more than two variables.

Activity 2.35.

Find the dimensions of the least expensive packing crate with a volume of 240 cubic feet when the material for the top costs \$2 per square foot, the bottom is \$3 per square foot and the sides are \$1.50 per square foot.

◁

What does the Lagrange Multiplier Tell Us?

There is a useful interpretation of the Lagrange multiplier λ. Recall that our optimal solution occurs when $\nabla f = \lambda \nabla g$ and $g(x, y) = c$. If we change the constraint (the value of c), then the Chain Rule tells us that

$$\frac{df}{dc} = \frac{\partial f}{\partial x}\frac{dx}{dc} + \frac{\partial f}{\partial y}\frac{dy}{dc} = \left(\lambda\frac{\partial g}{\partial x}\right)\frac{dx}{dc} + \left(\lambda\frac{\partial g}{\partial y}\right)\frac{dy}{dc} = \lambda\left(\frac{\partial g}{\partial x}\frac{dx}{dc} + \frac{\partial g}{\partial y}\frac{dy}{dc}\right) = \lambda\frac{dg}{dc} = \lambda,$$

since $g(x, y) = c$. So λ tells us about how much f will change if we increase the value of c by 1 unit. In our Cobb-Douglas model, increasing expenditures on labor and equipment by 1 million dollars will increase our output by approximately 25.51 units.

Activity 2.36.

Assume the value of λ is 0.52 in Activity 2.34. Explain in detail the meaning of this value.

◁

Summary

In this section, we encountered the following important ideas:

- The general technique for optimizing a function $f = f(x, y)$ subject to a constraint $g(x, y) = c$ is to solve the system $\nabla f = \lambda \nabla g$ and $g(x, y) = c$ for x, y, and λ. There may be more than one solution: evaluate f at each solution and the largest resulting value is the maximum value of f and the smallest resulting value is the minimum value of f subject to the constraint $g(x, y) = c$.
- If we are optimizing a function f subject to a constraint of the form $g(x, y) = c$, then the value of the Lagrange multiplier tells us about how much f will change if we increase the value of c by 1 unit.

Chapter 3

Multiple Integrals

3.1 Double Riemann Sums and Double Integrals over Rectangles

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a double Riemann sum?
- How is the double integral of a continuous function $f = f(x, y)$ defined?
- What are two things the double integral of a function can tell us?

Preview Activity 3.1. (a) Review the concept of the Riemann sum from single variable calculus. Then explain how the definite integral $\int_a^b f(x)\,dx$ of a continuous function of a single variable x on an interval $[a, b]$ is defined. Draw a picture to illustrate your definition.

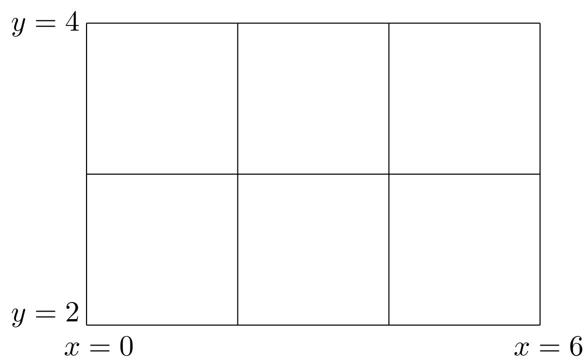

Figure 3.1: Rectangular domain R with subrectangles.

(b) In this section we will extend the idea of the definite integral to functions of two variables over rectangular domains. To do so, we will need to understand how to partition a rectangle into subrectangles. Let R be rectangular domain $R = \{(x, y) : 0 \leq x \leq 6, 2 \leq y \leq 4\}$ (we can also represent this domain as $[0, 6] \times [2, 4]$). To form a partition of R we will partition both intervals $[0, 6]$ and $[2, 4]$. Partition the interval $[0, 6]$ into three uniformly sized subintervals and the interval $[2, 4]$ into two evenly sized subintervals as shown in Figure 3.1. Here we discuss a method for identifying the endpoints of each subinterval and the resulting subrectangles.

i. Let $0 = x_0 < x_1 < x_2 < x_3 = 6$ be the endpoints of the subintervals of $[0, 6]$ after partitioning. What is the length Δx of each subinterval $[x_{i-1}, x_i]$ for i from 1 to 3?

ii. Explicitly identify x_0, x_1, x_2, and x_3. Draw a picture of Figure 3.1 and label these endpoints on your drawing.

iii. Let $2 = y_0 < y_1 < y_2 = 4$ be the endpoints of the subintervals of $[2, 4]$ after partitioning. What is the length Δy of each subinterval $[y_{j-1}, y_j]$ for j from 1 to 2? Identify y_0, y_1, and y_2 and label these endpoints on your drawing of Figure 3.1.

iv. Let R_{ij} denote the subrectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$. How does the number of subrectangles depend on the partitions of the intervals $[0, 6]$ and $[2, 4]$? Appropriately label each subrectangle in your drawing of Figure 3.1.

v. What is area ΔA of each subrectangle?

$\bowtie$

Introduction

In Preview Activity **??** we recalled from single variable calculus that we approximated the area under the graph of a positive function f on an interval $[a, b]$ by adding areas of rectangles. The process was to subdivide the interval $[a, b]$ into smaller subintervals, construct rectangles on each of these smaller intervals to approximate the region under the curve on that subinterval, then use the sum of the areas of these rectangles to approximate the area under the curve. We will extend this process in this section to its three-dimensional analogs, double Riemann sums and double integrals over rectangles.

Double Riemann Sums over Rectangles

We used closed intervals for our domains for continuous functions in single variable calculus to define definite integrals. Things will be a bit more complicated for multivariate functions, where our domains can be any closed and bounded region. We will start, though, with rectangular domains, and deal with more complicated domains in the next section.

Let $f = f(x, y)$ be a continuous function defined on a rectangular domain $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. In single variable calculus, we partitioned our domain interval into subintervals and used rectangles to approximate the values of the integral on each subinterval. Now our domain is a rectangle R and we want to partition R into subrectangles. We do this by partitioning each of the intervals $[a, b]$ and $[c, d]$ into subintervals and those subintervals create a partition of R into subrectangles. We saw a concrete example of this in our Preview Activity. To define double Riemann sums and double integrals, we will need to keep track of a lot of different objects and be comfortable with quite a bit of notation. The next activity will help in this regard.

Activity 3.1.

Let $f(x, y) = 100 - x^2 - y^2$ be defined on the rectangular domain $R = [a, b] \times [c, d]$. Partition the interval $[a, b]$ into four uniformly sized subintervals and the interval $[c, d]$ into three evenly sized subintervals as shown in Figure 3.2. As we did in Preview Activity 3.1, we will need a method for identifying the endpoints of each subinterval and the resulting subrectangles.

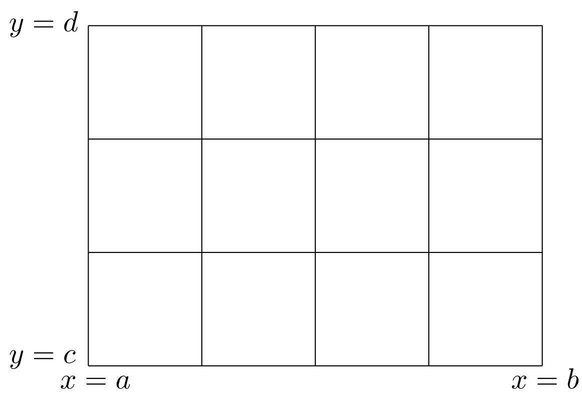

Figure 3.2: Rectangular domain with subrectangles.

(a) Let $a = x_0 < x_1 < x_2 < x_3 < x_4 = b$ be the endpoints of the subintervals of $[a, b]$ after partitioning. Label these endpoints in Figure 3.2.

(b) What is the length Δx of each subinterval $[x_{i-1}, x_i]$? Your answer should be in terms of a and b.

(c) Let $c = y_0 < y_1 < y_2 < y_3 = d$ be the endpoints of the subintervals of $[c, d]$ after partitioning. Label these endpoints in Figure 3.2.

(d) What is the length Δy of each subinterval $[y_{j-1}, y_j]$? Your answer should be in terms of c and d.

(e) The partitions of the intervals $[a, b]$ and $[c, d]$ partition the rectangle R into subrectangles. How many subrectangles are there?

(f) Let R_{ij} denote the subrectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Label each subrectangle in Figure 3.2.

(g) What is area ΔA of each subrectangle?

(h) Now let $[a, b] = [0, 8]$ and $[c, d] = [2, 6]$. Let (x_{11}^*, y_{11}^*) be the point in the upper right corner of the subrectangle R_{11}. Identify and correctly label this point in Figure 3.2.

Calculate the product
$$f\left(x_{11}^*, y_{11}^*\right) \Delta A.$$
Explain, geometrically, what this product represents.

(i) For each i and j, choose a point (x_{ij}^*, y_{ij}^*) in the subrectangle $R_{i,j}$. Identify and correctly label these points in Figure 3.2. Explain what the product
$$f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$
represents.

(j) If we were to add all the values $f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$ for each i and j, what does the resulting number approximate about the surface defined by f on the domain R? (You don't actually need to add these values.)

(k) Write a double sum using summation notation that expresses the arbitrary sum from part (j).

◁

Double Riemann Sums and Double Integrals

Now we use the process from the previous section to formally define double Riemann sums and double integrals.

The Double Riemann Sum.

Definition 3.1. Let f be a continuous function on a rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. The **double Riemann sum for f over R** is created as follows.

- Partition the interval $[a, b]$ into m subintervals of equal length $\Delta x = \frac{b-a}{m}$. Let x_0, x_1, ..., x_m be the endpoints of these subintervals, where $a = x_0 < x_1 < x_2 < \cdots < x_m = b$.
- Partition the interval $[c, d]$ into n subintervals of equal length $\Delta y = \frac{d-c}{n}$. Let y_0, y_1, ..., y_n be the endpoints of these subintervals, where $c = y_0 < y_1 < y_2 < \cdots < y_n = d$.
- These two partitions create a partition of the rectangle R into mn subrectangles R_{ij} with opposite vertices (x_{i-1}, y_{j-1}) and (x_i, y_j) for i between 1 and m and j between 1 and n. These rectangles all have equal area $\Delta A = \Delta x \cdot \Delta y$.
- Choose a point (x_{ij}^*, y_{ij}^*) in rectangle R_{ij}. Then a double Riemann sum for f over R is given by
$$\sum_{j=1}^{n} \sum_{i=1}^{m} f\left(x_{ij}^*, y_{ij}^*\right) \cdot \Delta A.$$

If $f(x, y) \geq 0$ on the rectangle R, as we let the number of subrectangles increase without bound (in other words, let both m and n in our double Riemann sums go to infinity), as illustrated in Figure 3.3, the sum of the volumes of the rectangles becomes the actual volume of the solid bounded above by f on R. This gives us the double integral.

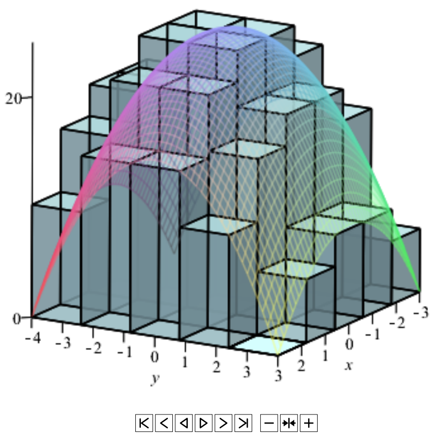

Figure 3.3: Double Riemann sums.

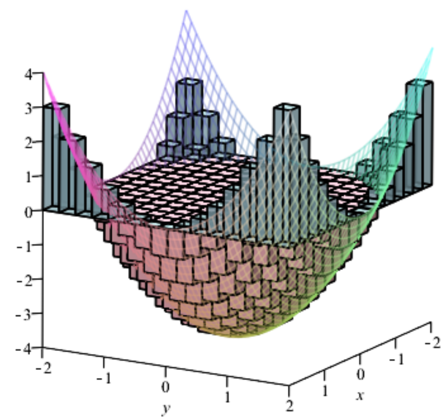

Figure 3.4: Graph of $f(x, y) = x^2 + y^2 - 4$ on a rectangle.

The Double Integral Over a Rectangle.

Definition 3.2. With terms defined as in the Double Riemann Sum, the **double integral of f over R** is

$$\int\int_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f\left(x_{ij}^*, y_{ij}^*\right) \cdot \Delta A.$$

Interpretation of Double Riemann Sums and Double integrals.

At the moment, there are two ways we can interpret the value of the double integral.

- For each i and j, the product $f\left(x_{ij}^*, y_{ij}^*\right) \cdot \Delta A$ can be interpreted as a "signed" volume of a box with base area ΔA and "signed" height $f\left(x_{ij}^*, y_{ij}^*\right)$. Since f can have negative values, this "height" could be negative. See Figure 3.4 for an illustration. The sum

$$\sum_{j=1}^{n} \sum_{i=1}^{m} f\left(x_{ij}^*, y_{ij}^*\right) \cdot \Delta A$$

can then be interpreted as a sum of "signed" volumes of cubes, with a negative sign attached to those cubes whose heights are below the xy-plane. We can then realize the double integral $\int\int_R f(x,y)\,dA$ as a difference in volumes: $\int\int_R f(x,y)\,dA$ tells us the volume of the solids the graph of f bounds above the xy-plane over the rectangle R MINUS the volume of the solids the graph of f bounds below the xy-plane under the rectangle R.

- The average of all of the values $f\left(x_{ij}^*, y_{ij}^*\right)$ is given by

$$\frac{1}{mn}\sum_{j=1}^{n}\sum_{i=1}^{m} f\left(x_{ij}^*, y_{ij}^*\right).$$

If we take the limit as m and n go to infinity, we will obtain what we call the average value of f. To view this as a double Riemann sum, note that

$$\Delta x = \frac{b-a}{m} \quad \text{and} \quad \Delta y = \frac{d-c}{n}.$$

So

$$\frac{1}{mn} = \frac{\Delta x \cdot \Delta y}{(b-a)(d-c)} = \frac{\Delta A}{A(R)},$$

where $A(R)$ denotes the area of the rectangle R. Then

$$\lim_{m,n\to\infty} \frac{1}{mn}\sum_{j=1}^{n}\sum_{i=1}^{m} f\left(x_{ij}^*, y_{ij}^*\right) = \lim_{m,n\to\infty} \frac{1}{A(R)}\sum_{j=1}^{n}\sum_{i=1}^{m} f\left(x_{ij}^*, y_{ij}^*\right)\cdot\Delta A = \frac{1}{A(R)}\int\int_R f(x,y)\,dA.$$

So the double integral of f over R divided by the area of R gives us the average value of the function f on R. If $f(x,y) \geq 0$ on R, we can interpret this average value of f on R as the height of the box with base R that has the same volume as the volume of the surface defined by f over R.

Activity 3.2.

Let $f(x,y) = x + 2y$ and let $R = [0,2] \times [1,3]$.

(a) Draw a picture of R. Partition $[0,2]$ into 2 subintervals of equal length and the interval $[1,3]$ into two subintervals of equal length. Draw these partitions on your picture of R and label the resulting subrectangles using the labeling scheme we established in the definition of a double Riemann sum.

(b) For each i and j, let $\left(x_{ij}^*, y_{ij}^*\right)$ be the midpoint of the rectangle R_{ij}. Identify the coordinates of each $\left(x_{ij}^*, y_{ij}^*\right)$. Draw these points on your picture of R.

(c) Calculate the Riemann sum

$$\sum_{j=1}^{n}\sum_{i=1}^{m} f\left(x_{ij}^*, y_{ij}^*\right)\cdot\Delta A$$

using the partitions we have described. If we let $\left(x_{ij}^*, y_{ij}^*\right)$ be the midpoint of the rectangle R_{ij} for each i and j, then the resulting Riemann sum is called a *midpoint sum*.

(d) Explain two things the sum you just calculated approximates.

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Activity 3.3.

Let $f(x,y) = \sqrt{4-y^2}$ on the rectangular domain $R = [1,7] \times [-2,2]$. Partition $[1,7]$ into 3 equal length subintervals and $[-2,2]$ into 2 equal length subintervals. A table of values of f at some points in R is given in Table 3.1, and a graph of f with the indicated partitions is shown in Figure 3.5.

	-2	-1	0	1	2
1	0	$\sqrt{3}$	2	$\sqrt{3}$	0
2	0	$\sqrt{3}$	2	$\sqrt{3}$	0
3	0	$\sqrt{3}$	2	$\sqrt{3}$	0
4	0	$\sqrt{3}$	2	$\sqrt{3}$	0
5	0	$\sqrt{3}$	2	$\sqrt{3}$	0
6	0	$\sqrt{3}$	2	$\sqrt{3}$	0
7	0	$\sqrt{3}$	2	$\sqrt{3}$	0

Table 3.1: Table of values of $f(x,y) = \sqrt{4-y^2}$.

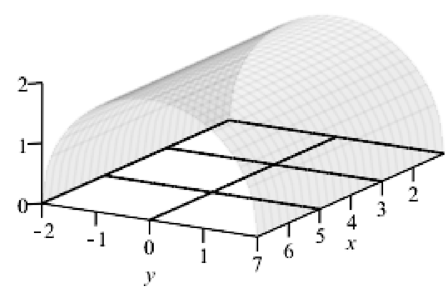

Figure 3.5: Graph of $f(x,y) = \sqrt{4-y^2}$ on R.

(a) Outline the partition of R into subrectangles on the table of values in Table 3.1.

(b) Calculate the double Riemann sum using the given partition of R and the values of f in the upper right corner of each subrectangle.

(c) Use geometry to calculate the exact value of $\int\int_R f(x,y)\,dA$ and compare it to your approximation. How could we obtain a better approximation?

◁

We conclude this section with a list of properties of double integrals. We saw that these properties were satisfied by single integrals and the arguments for double integrals are the same, so we omit them.

Properties of Double Integrals. Let f and g be continuous functions on a rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, and let k be a constant. Then

1. $\int\int_R (f(x, y) + g(x, y))\, dA = \int\int_R f(x, y)\, dA + \int\int_R g(x, y)\, dA$.
2. $\int\int_R kf(x, y)\, dA = k \int\int_R f(x, y)\, dA$.
3. If $f(x, y) \geq g(x, y)$ on R, then $\int\int_R f(x, y)\, dA \geq \int\int_R g(x, y)\, dA$.

Summary

In this section, we encountered the following important ideas:

- Let f be a continuous function on a rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. The double Riemann sum for f over R is created as follows.
  - Partition the interval $[a, b]$ into m subintervals of equal length $\Delta x = \frac{b-a}{m}$. Let x_0, x_1, $\ldots$, x_m be the endpoints of these subintervals, where $a = x_0 < x_1 < x_2 < \cdots < x_m = b$.
  - Partition the interval $[c, d]$ into n subintervals of equal length $\Delta y = \frac{d-c}{n}$. Let y_0, y_1, $\ldots$, y_n be the endpoints of these subintervals, where $c = y_0 < y_1 < y_2 < \cdots < y_n = d$.
  - These two partitions create a partition of the rectangle R into mn subrectangles R_{ij} with opposite vertices (x_{i-1}, y_{j-1}) and (x_i, y_j) for i between 1 and m and j between 1 and n. These rectangles all have equal area $\Delta A = \Delta x \cdot \Delta y$.
  - Choose a point (x_{ij}^*, y_{ij}^*) in rectangle $R_{i,j}$. Then a double Riemann sum for f over R is given by
  $$\sum_{j=1}^{n} \sum_{i=1}^{m} f\left(x_{ij}^*, y_{ij}^*\right) \cdot \Delta A.$$
- With terms defined as in the Double Riemann Sum, the double integral of f over R is
$$\int\int_R f(x, y)\, dA = \lim_{m,n \to \infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f\left(x_{ij}^*, y_{ij}^*\right) \cdot \Delta A.$$
- Two things the double integral $\displaystyle\int\int_R f(x, y)\, dA$ can tell us are:
  - The volume of the solids the graph of f bounds above the xy-plane over the rectangle R MINUS the volume of the solids the graph of f bounds below the xy-plane under the rectangle R;
  - Dividing the double integral of f over R by the area of R gives us the average value of the function f on R. If $f(x, y) \geq 0$ on R, we can interpret this average value of f on R as

the height of the box with base R that has the same volume as the volume of the surface defined by f over R.

3.2 Iterated Integrals

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How do we evaluate a double integral over a rectangle as an iterated integral, and why does this process work?

Preview Activity 3.2. Let $f(x,y) = 25 - x^2 - y^2$ on the rectangular domain $R = [-3,3] \times [-4,4]$.

(a) As we did with partial derivatives, we can treat one of the variables in f as constant and think of the resulting function as a function of a single variable. Now we investigate what happens if we integrate instead of differentiate. Choose a fixed value of x in the interior of $[-3,3]$. Let

$$A(x) = \int_{-4}^{4} f(x,y)\, dy.$$

Think about what the definite integral from single variable calculus tells us and explain what the value of $A(x)$ tells us about the surface defined by f. (Hint: Recall what a trace is and consider how $A(x)$ is related to Figure 3.6.)

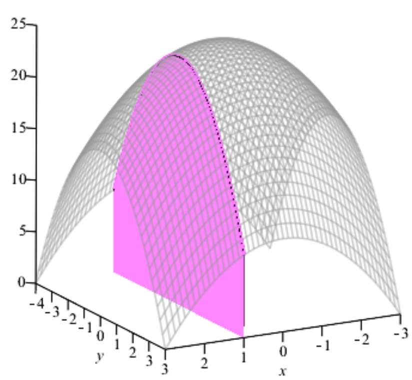

Figure 3.6: A cross section with fixed x.

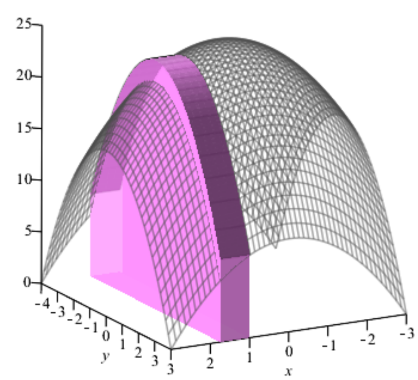

Figure 3.7: A cross section with fixed y and Δx.

(b) Since f is continuous on R, we can define the function $A = A(x)$ at every value of x in

$[-3, 3]$. Consequently, we can define the definite integral $\int_{-3}^{3} A(x)\,dx$ of $A(x)$ as

$$\int_{-3}^{3} A(x)\,dx = \lim_{m\to\infty} \sum_{i=1}^{m} A(x_i^*)\,\Delta x,$$

where $-3 = x_0 < x_1 < x_2 < \cdots < x_n = 3$ are the endpoints of m equal length subintervals of $[-3, 3]$ and x_i^* is some point in the ith subinterval. Explain as best you can what one of the products $A(x_i^*)\,\Delta x$ looks like. (Hint: Consider how $A(x_i^*)\Delta x$ is related to Figure 3.7.)

(c) Can you visualize what the sum $\sum_{i=1}^{m} A(x_i^*)\,\Delta x$ looks like? Explain as best you can. What will happen when we take the limit as m goes to infinity?

(d) Based on the above problems, why does $\int\int_R f(x, y)\,dA$ equal

$$\int_{-3}^{3} A(x)\,dx = \int_{-3}^{3} \left(\int_{-4}^{4} f(x, y)\,dy\right)\,dx$$

The latter integral is an *iterated integral*.

$\bowtie$

Introduction

Recall that we defined the double integral of a continuous function $f = f(x, y)$ over a rectangle $R = [a, b] \times [c, d]$ as

$$\int\int_R f(x, y)\,dA = \lim_{m,n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f\left(x_{ij}^*, y_{ij}^*\right) \cdot \Delta A,$$

where

- $x_0, x_1, \ldots, x_m$ are the endpoints of m equal length subintervals of $[a, b]$ of length $\Delta x = \frac{b-a}{m}$ with $a = x_0 < x_1 < x_2 < \cdots < x_m = b$;
- $y_0, y_1, \ldots, y_n$ are the endpoints of n equal length subintervals of $[c, d]$ of length $\Delta y = \frac{d-c}{n}$ with $c = y_0 < y_1 < y_2 < \cdots < y_n = d$;
- $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ is the subrectangle of R with opposite vertices (x_{i-1}, y_{j-1}) and (x_i, y_j) for i between 1 and m and j between 1 and n. The area of R_{ij} is $\Delta A = \Delta x \cdot \Delta y$;
- (x_{ij}^*, y_{ij}^*) is a point in rectangle R_{ij}.

So $\int\int_R f(x, y)\,dA$ is a limit of double Riemann sums. While this definition tells us exactly what the double integral is, it is not very helpful for determining the value of a double integral. Fortunately, there is a way to view a double integral as an *iterated integral*, which will make computations feasible in many cases.

Iterated Integrals

The general process for writing a double integral as an iterated integral over a rectangular domain works as we discussed in our example in Preview Activity 3.2. Let f be a continuous function on a rectangular domain $R = [a, b] \times [c, d]$. Let

$$A(x) = \int_c^d f(x, y)\, dy.$$

The function $A = A(x)$ tells us a cross sectional area[1] in the y direction for the fixed value of x of the solid bound between the surface defined by f and the xy-plane.

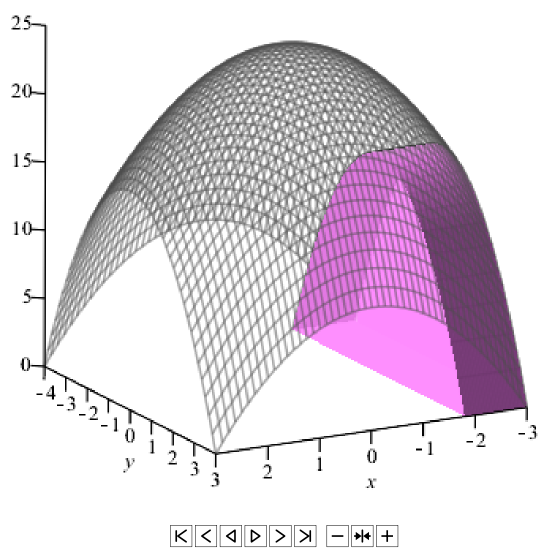

Figure 3.8: Summing cross section slices.

The cross sectional piece is determine by the input x in A. Since A is a function of x we can integrate A with respect to x. In doing so we use the partition of $[a, b]$ and make an approximation to the integral:

$$\int_a^b A(x)\, dx \approx \sum_{i=1}^{m} A(x_i^*)\Delta x,$$

where x_i^* is any number in the subinterval $[x_{i-1}, x_i]$. Each summand $A(x_i^*)\Delta x$ will represent an approximation of a fixed cross sectional slice of the surface in the y direction with a fixed width of Δx as illustrated in Figure 3.7. We add the signed volumes of these slices as shown in the animation in Figure 3.8 to obtain an approximation of the total signed volume.

[1] By area we mean "signed" area.

As we let the number of subintervals in the x direction approach infinity, we can see that the Riemann sum $\sum_{i=1}^{m} A(x_i^*)\Delta x$ approaches a limit and that limit is the sum of signed volumes bounded by the function f on R. Therefore,

$$\int\int_R f(x,y)\,dA = \lim_{m\to\infty} \sum_{i=1}^{m} A(x_i^*)\Delta x = \int_a^b A(x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy\right)\,dx.$$

So we can compute the double integral of f over R by first integrating f with respect to y on $[c,d]$, then integrating the resulting function of x with respect to x on $[a,b]$. The nested integral

$$\int_a^b \left(\int_c^d f(x,y)\,dy\right)\,dx = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

is called an *iterated integral*. Thus, each double integral can be represented by two single integrals.

There is nothing magical about integrating with respect to y first. The same argument shows that we can also find the double integral as an iterated integral integrating with respect to x first, or

$$\int\int_R f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx\right)\,dy \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

The net result of the previous arguments is called Fubini's Theorem.

Fubini's Theorem. If $f = f(x,y)$ is a continuous function on a rectangle $R = [a,b]\times[c,d]$, then

$$\int\int_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Activity 3.4.

Let $f(x,y) = 25 - x^2 - y^2$ on the rectangular domain $R = [-3,3]\times[-4,4]$.

(a) Consider x to be a constant and evaluate the integral to find $A(x)$ if

$$A(x) = \int_{-4}^{4} f(x,y)\,dy.$$

Your result should be a function of x only.

(b) Now find the value of $\int_{-3}^{3} A(x)\,dx$.

(c) What is the value of $\int\int_R f(x,y)\,dA$? Explain two things this number tells us.

$\triangleleft$

Activity 3.5.

Let $f(x, y) = x + y^2$ on the rectangle $R = [0, 2] \times [0, 3]$.

(a) Set up an iterated integral, integrating first with respect to x, then y, to evaluate $\int\int_R f(x, y)\, dA$. Evaluate this integral.

(b) Set up an iterated integral, integrating first with respect to y, then x, to evaluate $\int\int_R f(x, y)\, dA$. Evaluate this integral.

$\triangleleft$

Summary

In this section, we encountered the following important ideas:

- We evaluate a double integral $\int\int_R f(x, y)\, dA$ over a rectangle $R = [a, b] \times [c, d]$ in one of two ways:
  - $\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx$ or
  - $\int_c^d \left(\int_a^b f(x, y)\, dx \right) dy$.

  This process works because the inner integral represents a cross sectional (signed) area and then the outer integral sums up all of these cross sectional signed areas.

3.3 Double Integrals over General Regions

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How do we define a double integral over a non-rectangular region?
- What general form does an iterated integral over a non-rectangular region have?

Preview Activity 3.3. A tetrahedron is a three-dimensional figure with four faces, each of which is a triangle. A picture of the tetrahedron T with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ is shown in Figure 3.9. If we place one vertex at the origin and let vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be determined by the edges of the tetrahedron that have one end at the origin, then a formula that tells us the volume V of the tetrahedron is

$$V = \frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|. \tag{3.1}$$

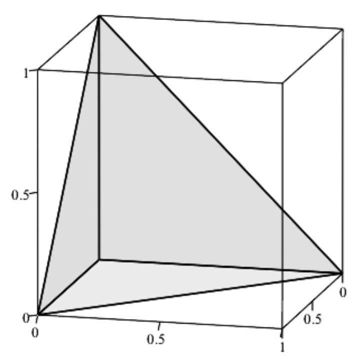

Figure 3.9: The tetrahedron T.

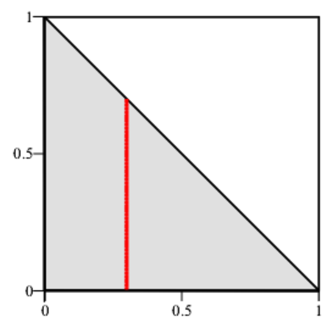

Figure 3.10: Projecting T onto the xy-plane.

(a) Use the formula (3.1) to find the volume of the tetrahedron T.

(b) Instead of memorizing or looking up the formula for the volume of a tetrahedron, we can use a double integral to calculate the volume of the tetrahedron T. To see how, notice that the top face of the tetrahedron T is the plane whose equation is

$$z = 1 - (x + y).$$

So the volume of the tetrahedron will be given by an iterated integral of the form

$$\int_?^? \int_?^? 1 - (x + y)\, dy\, dx.$$

The problem is to find the limits on the integrals. The limits on this double integral are in terms of x and y. To see what the domain is over which we need to integrate, think of standing way above the tetrahedron looking straight down on it. Alternatively, think of projecting the entire tetrahedron onto the xy-plane. The resulting domain is shown in Figure 3.10. This is not a rectangular domain but a triangular one, so we need to understand how to represent this domain in terms of x and y.

i. Explain why we can represent the triangular region with the inequalities

$$0 \leq y \leq 1 - x \quad \text{and} \quad 0 \leq x \leq 1.$$

(Hint: Consider the cross sectional slice shown in Figure 3.10.)

ii. Evaluate the iterated integral

$$\int_0^1 \int_0^{1-x} 1 - (x + y)\, dy\, dx$$

and compare to your result from part (a).

$\bowtie$

Introduction

Recall that we defined the double integral of a continuous function $f = f(x, y)$ over a rectangle $R = [a, b] \times [c, d]$ as

$$\int\int_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f\left(x_{ij}^*, y_{ij}^*\right) \cdot \Delta A,$$

where

- x_0, x_1, ..., x_m are the endpoints of m equal length subintervals of $[a, b]$ of length $\Delta x = \frac{b-a}{m}$ with $a = x_0 < x_1 < x_2 < \cdots < x_m = b$;
- y_0, y_1, ..., y_n are the endpoints of n equal length subintervals of $[c, d]$ of length $\Delta y = \frac{d-c}{n}$ with $c = y_0 < y_1 < y_2 < \cdots < y_n = d$;
- $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ is the subrectangle of R with opposite vertices (x_{i-1}, y_{j-1}) and (x_i, y_j) for i between 1 and m and j between 1 and n. The area of R_{ij} is $\Delta A = \Delta x \cdot \Delta y$;
- (x_{ij}^*, y_{ij}^*) is a point in rectangle R_{ij}.

We have also seen that we can evaluate a double integral $\displaystyle\int\int_R f(x, y)\, dA$ over R as an iterated integral of either of the forms

$$\int_a^b \int_c^d f(x, y)\, dy\, dx \qquad \text{or} \qquad \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

So far we have interpreted the double integral in one of two ways:

- $\displaystyle\int\int_R f(x,y)\,dA$ tells us the volume of the solids the graph of f bounds above the xy-plane over the rectangle R MINUS the volume of the solids the graph of f bounds below the xy-plane under the rectangle R;
- $\displaystyle\frac{1}{A(R)}\int\int_R f(x,y)\,dA$, where $A(R)$ is the area of R tells us the average value of the function f on R. If $f(x,y) \geq 0$ on R, we can interpret this average value of f on R as the height of the box with base R that has the same volume as the volume of the surface defined by f over R.

Now we see how to define and evaluate double integrals over non-rectangular regions.

Double Integrals over General Regions

A function $f = f(x,y)$ can be defined over regions other than rectangular ones. In this section we learn how to define a double integral over non-rectangular regions.

Suppose f is a continuous function on a closed, bounded domain D. For example, consider D as the hourglass domain shown in Figure 3.11. We can enclose D in a rectangular domain R

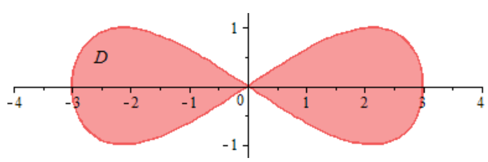

Figure 3.11: A non-rectangular domain.

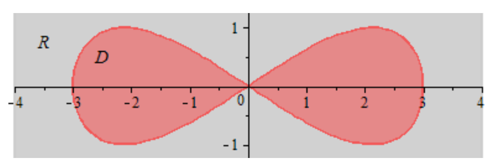

Figure 3.12: Enclosing this domain in a rectangle.

as shown in Figure 3.12. Now we extend the function f to be defined over R so that we can use our definition of the double integral over a rectangle. We extend f in such a way that its values at the points in R that are not in D contribute 0 to the value of the integral. In other words, define a function $F = F(x,y)$ on R as

$$F(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D, \\ 0, & \text{if } (x,y) \notin D \end{cases}.$$

We then say that the double integral of f over D is the same as the double integral of F over R, or

$$\int\int_D f(x,y)\,dA = \int\int_R F(x,y)\,dA.$$

In practice, we just ignore everything that is in R but not in D, since these regions contribute 0 to the value of the integral.

Just as with double integrals over rectangles, a double integral over a domain D can be evaluated as an iterated integral of the form

$$\int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

where $g_1 = g_1(x)$ and $g_2 = g_2(x)$ are functions of x only and the region D is described by the inequalities $g_1(x) \le y \le g_2(x)$ and $a \le x \le b$. Alternatively, we can evaluate a double integral over a domain D as an iterated integral of the form

$$\int_{y=c}^{d} \int_{x=h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

where $h_1 = h_1(y)$ and $h_2 = h_2(y)$ are functions of y only and the region D is described by the inequalities $h_1(y) \le x \le h_2(y)$ and $c \le y \le d$.

The forms that an iterated integral can take are worth special note.

In an iterated double integral:

- the limits on the outer integral must be constants;
- the limits on the inner integral must be constants or in terms of only the remaining variable – that is, if the inner integral is with respect to y, then its limits must be in terms of x, and vice versa.

Example 3.1. Let $f(x,y) = x^2 y$ be defined on the triangle D with vertices $(0,0)$, $(2,0)$, and $(2,3)$ as shown in Figure 3.13. To evaluate $\displaystyle\int\int_D f(x,y)\, dA$ we must be able to describe the region D in

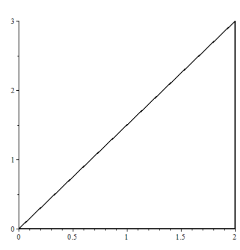

Figure 3.13: A triangular domain.

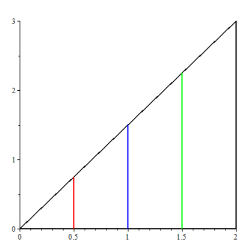

Figure 3.14: Slices in the y direction.

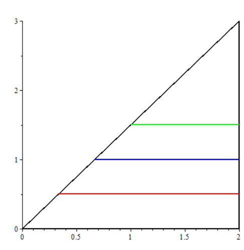

Figure 3.15: Slices in the x direction.

terms of the variables x and y. We take two approaches.

Approach 1: Integrate with respect to y first: In this case we want to evaluate our double integral as an iterated integral in the form

$$\int\int_D x^2y\,dA = \int_{x=a}^{b}\int_{y=g_1(x)}^{g_2(x)} x^2y\,dy\,dx$$

and need to describe D in terms of inequalities

$$g_1(x) \leq y \leq g_2(x) \qquad \text{and } a \leq x \leq b.$$

Since we are integrating with respect to y first, our iterated integral has the form

$$\int_a^b A(x)\,dx,$$

where $A(x)$ is a cross sectional area in the y direction. So we are slicing our domain in the y direction and we want to understand what a cross sectional area will look like. Several slices are shown in Figure 3.14. On a slice with fixed x value, the y values are bounded below by 0 and above by the y coordinate on the hypotenuse of the triangle. So $g_1(x) = 0$ and to find $g_2(x)$ we need to write this hypotenuse for y in terms of x. The hypotenuse connects the points (0,0) and (2,3) and has equation $y = \frac{3}{2}x$. This gives the upper bound on y as $g_2(x) = \frac{3}{2}x$. The leftmost vertical cross section is at $x = 0$ and the rightmost one is at $x = 2$, so we have $a = 0$ and $b = 2$. Therefore,

$$\int\int_D x^2y\,dA = \int_{x=0}^{2}\int_{y=0}^{(3/2)x} x^2y\,dy\,dx.$$

We calculate the iterated integral as follows:

$$\begin{aligned}
\int_{x=0}^{2}\int_{y=0}^{(3/2)x} x^2y\,dy\,dx &= \int_{x=0}^{2}\left[x^2\frac{y^2}{2}\right]\Bigg|_{y=0}^{(3/2)x}\,dx \\
&= \int_{x=0}^{2}\frac{9}{8}x^4\,dx \\
&= \frac{9}{8}\frac{x^5}{5}\Bigg|_{x=0}^{2} \\
&= \left(\frac{9}{8}\right)\left(\frac{32}{5}\right) \\
&= \frac{36}{5}.
\end{aligned}$$

Approach 2: Integrate with respect to x first: In this case we want to evaluate our double integral as an iterated integral in the form

$$\int\int_D x^2y\,dA = \int_{y=c}^{d}\int_{x=h_1(y)}^{h_2(y)} x^2y\,dx\,dy$$

and need to describe D in terms of inequalities

$$h_1(y) \leq x \leq h_2(y) \quad \text{ and } c \leq y \leq d.$$

Since we are integrating with respect to x first, our iterated integral has the form

$$\int_c^d A(y)\, dy,$$

where $A(y)$ is a cross sectional area in the x direction. So we are slicing our domain in the x direction and we want to understand what a cross sectional area will look like. Several slices are shown at right in Figure 3.15.

On a slice with fixed y value, the x values are bounded below by the x coordinate on the hypotenuse of the triangle and above by 2. So $h_2(y) = 2$ and to find $h_1(y)$ we need to write the hypotenuse for x in terms of y. Solving the equation we have for the hypotenuse for x gives us $x = \frac{2}{3}y$. This makes $h_1(y) = \frac{2}{3}y$. The lowest horizontal cross section is at $y = 0$ and the uppermost one is at $y = 3$, so we have $c = 0$ and $d = 3$. Therefore,

$$\int\int_D x^2 y\, dA = \int_{y=0}^{3} \int_{x=(2/3)y}^{2} x^2 y\, dx\, dy.$$

We calculate the iterated integral as follows:

$$\begin{aligned}
\int_{y=0}^{3} \int_{x=(2/3)y}^{2} x^2 y\, dx\, dy &= \int_{y=0}^{3} \left[\frac{x^3}{3}\right]\Bigg|_{x=(2/3)x}^{2} y\, dx \\
&= \int_{y=0}^{3} \left[\frac{8}{3}y - \frac{8}{81}y^4\right] dy \\
&= \frac{8}{3}\frac{y^2}{2}\Bigg|_{y=0}^{3} - \frac{8}{81}\frac{y^5}{5}\Bigg|_{y=0}^{3} \\
&= \left(\frac{8}{3}\right)\left(\frac{9}{2}\right) - \left(\frac{8}{81}\right)\left(\frac{243}{5}\right) \\
&= 12 - \frac{24}{5} \\
&= \frac{36}{5}.
\end{aligned}$$

Activity 3.6.

Consider the double integral $\int\int_D \left(\frac{3}{4}\right)(4 - x - 2y)\, dA$, where D is the triangular region with vertices (0,0), (4,0), and (0,2).

(a) Set up this integral as an iterated integral of the form $\int\int_D \left(\frac{3}{4}\right)(4-x-2y)\, dy\, dx$. Draw a picture of D with relevant cross sections.

(b) Set up this integral as an iterated integral of the form $\int \int_D \left(\frac{3}{4}\right)(4-x-2y)\,dx\,dy$. Draw a picture of D with relevant cross sections.

(c) Evaluate the two integrals. Explain the result.

◁

Activity 3.7.

Consider the iterated integral $\displaystyle\int_3^5 \int_{-x}^{x^2} (4x + 10y)\,dy\,dx$.

(a) Sketch the region of integration.

(b) Write an equivalent iterated integral (or integrals), integrating with respect to x first, then y.

(c) Evaluate the iterated integral. Explain what the value you obtained tells you.

◁

Activity 3.8.

Some integrals cannot be evaluated as one type of iterated integral, but can as the other. We do not know how to integrate e^{y^2} with respect to y, so we cannot evaluate the iterated integral $\displaystyle\int_0^4 \int_{x/2}^{2} e^{y^2}\,dy\,dx$. Set up an iterated integral to integrate with respect to x first and then y that is equal to $\displaystyle\int_0^4 \int_{x/2}^{2} e^{y^2}\,dy\,dx$. Then evaluate the latter iterated integral.

◁

Summary

In this section, we encountered the following important ideas:

- For a double integral $\int \int_D f(x,y)\,dA$ over a non-rectangular region D, we enclose D in a rectangle R and then extend integrand f to a function F so that $F(x,y) = 0$ at all points in R outside of D and $F(x,y) = f(x,y)$ for all points in D. We then define $\int \int_D f(x,y)\,dA$ to be equal to $\int \int_R F(x,y)\,dA$.
- In an iterated double integral, the limits on the outer integral must be constants while the the limits on the inner integral must be constants or in terms of only the remaining variable. In other words, an iterated double integral has one of the following forms:

$$\int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx,$$

where $g_1 = g_1(x)$ and $g_2 = g_2(x)$ are functions of x only and the region D is described by the inequalities $g_1(x) \leq y \leq g_2(x)$ and $a \leq x \leq b$ or

$$\int_{y=c}^{d} \int_{x=h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where $h_1 = h_1(y)$ and $h_2 = h_2(y)$ are functions of y only and the region D is described by the inequalities $h_1(y) \leq x \leq h_2(y)$ and $c \leq y \leq d$.

3.4 Applications of Double Integrals

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How do we use a double integral to find the area between two curves?
- How do we use the integration process to find a double integral to determine the mass of a lamina if we have a mass density function for the lamina?
- How do we find the center of mass of a lamina if we have a mass density function on the lamina?
- What is a joint probability density function? How do we determine the probability of an event if we know a probability density function?

Preview Activity 3.4. Suppose that we have a flat, thin object (called a *lamina*) whose density varies across the object. We can think of the density on a lamina as a measure of mass per unit area. As an example, consider a circular plate D of radius 1 whose density δ varies depending on the distance from its center so that the density in grams per square centimeter at point (x, y) is

$$\delta(x, y) = 10 - 2(x^2 + y^2).$$

Just as in single variable calculus when we calculated the mass of a rod, we will use the integration process to find the mass of this plate.

(a) Review how the double integral of a function $f = f(x, y)$ is defined over a rectangle and reproduce a complete definition here. Then explain two things that the double integral can tell us about f.

(b) Now we apply the integration process to our mass problem. Partition the plate into subrectangles R_{ij}, where $1 \leq i \leq m$ and $1 \leq j \leq n$, of equal area ΔA (assume the density is 0 at any point outside of the disk) and select a point $\left(x_{ij}^*, y_{ij}^*\right)$ in R_{ij} for each i and j.

i. Find a reasonable approximation of the mass of the plate on the subrectangle R_{ij}. Explain why you have a reasonable approximation. (Hint: mass is density times area if density is constant.)

ii. Using your approximation from part (a), find a double Riemann sum that provides a reasonable approximation to the mass of the plate.

iii. Explain how the approximation in part (b) shows that the double integral

$$\int\int_D \delta(x, y)\, dA$$

gives the exact mass of the plate.

iv. Find the exact mass of this plate.

$\bowtie$

Introduction

We have interpreted a double integral of a function f over a domain D in two different ways so far. First, a double integral can tell us a difference of volumes – the volume the surface defined by f bounds above the xy-plane on D minus the volume the surface bounds below the xy-plane on D, or if we divide the double integral by the area of D we find the average value of f on D. In this section we will investigate several other applications of double integrals. All of these applications use the integration process as we saw in Preview Activity **??**: we partition into small regions, approximate the quantity desired on each small region, then take the limit as the size of each region goes to 0 to obtain an integral.

Area

We have used a single integral to calculate the areas of regions in the plane. The double integral can be used to calculate areas as well.

Activity 3.9.

Suppose we want to find the area of the bounded region D between the curves

$$y = 1 - x^2 \quad \text{and} \quad y = x - 1.$$

A picture of this region is shown in Figure 3.16. Interpret the area of the region D as the volume of a solid with base D and of uniform height 1 and write a double integral that gives the area of D. Then find the area of D.

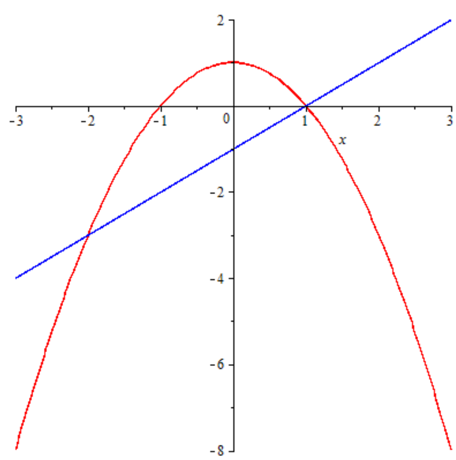

Figure 3.16: The graphs of $y = 1 - x^2$ and $y = x - 1$.

$\triangleleft$

Activity 3.10.

Find the area of the region bounded by the curves $y^2 = x$ and $4 - y^2 = x$.

◁

Mass

Density is a measure of some quantity per unit area or volume. For example, we can measure the human population density of some region as the number of humans in that region divided by the area of that region. In physics, the density of an object is the mass of the object per unit area or volume. As our preview activity showed,

> If $\delta(x, y)$ describes the density of a lamina defined by a planar region D, then the **mass** of D is given by the double integral $\int\int_D \delta(x, y)\, dA$.

Activity 3.11.

Let D be a half-disk lamina of radius 3 in quadrants IV and I, centered at the origin as shown in Figure 3.17. Assume the density at point (x, y) is equal to x. Find the mass of the lamina.

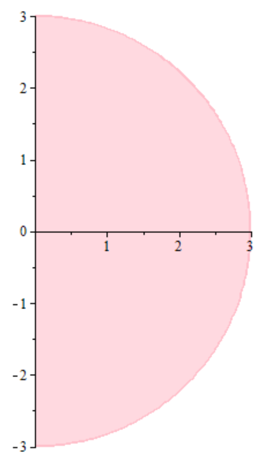

Figure 3.17: A half disk lamina.

◁

Center of Mass

The center of mass of an object is a point at which the object will balance perfectly. If we were to throw the object through the air, it would spin around its center of mass and behave as if all the mass is located at the center of mass. Let's begin by finding the center of mass $(\overline{x}, \overline{y})$ of a collection of N distinct point masses in the plane.

Let m_1, m_2, ..., m_N be N masses located in the plane. Think of these masses as connected by rigid rods of negligible weight from some central point (x, y). A picture with three masses is shown in Figure 3.18. Now imagine balancing this system by placing it on a thin pole at the point (x, y) perpendicular to the plane containing the masses. Unless the masses are perfectly balanced, the system will fall off the pole. The point $(\overline{x}, \overline{y})$ at which the system will balance perfectly is called the *center of mass* of the system. Our goal is to determine the center of mass of a system of discrete masses, then extend this to a continuous lamina.

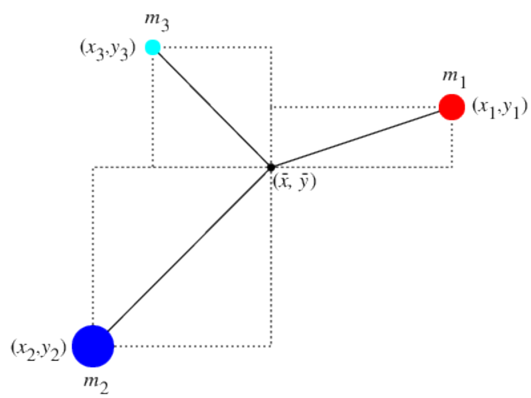

Figure 3.18: A center of mass $(\overline{x}, \overline{y})$ of three masses.

Each mass exerts a force (called a *moment*) around the lines $x = \overline{x}$ and $y = \overline{y}$ that causes the system to tilt in the direction of the mass. These moments are dependent on the mass and the distance from the given line. Let (x_1, y_1) be the location of mass m_1, (x_2, y_2) the location of mass m_2, etc. In order to balance perfectly, the moments in the x direction and in the y direction must be in equilibrium. We determine these moments and solve the resulting system to find the equilibrium point $(\overline{x}, \overline{y})$ at the center of mass.

The force that mass m_1 exerts to tilt the system from the line $x = \overline{x}$ is

$$m_1 g(\overline{y} - y_1),$$

where g is the acceleration due to gravity. Similarly, the force mass m_2 exerts to tilt the system from the line $x = \overline{x}$ is

$$m_2 g(\overline{y} - y_2).$$

In general, the force that mass m_k exerts to tilt the system from the line $x = \overline{x}$ is

$$m_k g(\overline{y} - y_2).$$

For the system to balance, we need the forces to sum to 0, or

$$\sum_{k=1}^{N} m_k g(\overline{y} - y_k) = 0.$$

Solving for $\overline{y}$ gives

$$\overline{y} = \frac{\sum_{k=1}^{N} m_k y_k}{\sum_{k=1}^{N} m_k}.$$

A similar argument shows that

$$\overline{x} = \frac{\sum_{k=1}^{N} m_k x_k}{\sum_{k=1}^{N} m_k}.$$

The value $M_x = \sum_{k=1}^{N} m_k y_k$ is called the *total moment* with respect to the x-axis and $M_y = \sum_{k=1}^{N} m_k x_k$ is the *total moment* with respect to the y-axis.

Now suppose we have a continuous lamina with a variable density $\delta(x, y)$. We partition the lamina into mn subrectangles (assuming the density is 0 at any point outside of the object) of equal area ΔA. Select a point $\left(x_{ij}^*, y_{ij}^*\right)$ on the ijth subrectangle. The quantity

$$\delta\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

is a density times area, so $\delta\left(x_{ij}^*, y_{ij}^*\right) \Delta A$ approximates the mass of the small chunk of the lamina determined by the subrectangle R_{ij}. Treat this as a point mass at the point $\left(x_{ij}^*, y_{ij}^*\right)$. The coordinates $(\overline{x}, \overline{y})$ of the center of mass of these mn point masses are given by

$$\overline{x} = \frac{\sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij}^* \delta\left(x_{ij}^*, y_{ij}^*\right) \Delta A}{\sum_{j=1}^{n} \sum_{i=1}^{m} \delta\left(x_{ij}^*, y_{ij}^*\right) \Delta A} \quad \text{and} \quad \overline{y} = \frac{\sum_{j=1}^{n} \sum_{i=1}^{m} y_{ij}^* \delta\left(x_{ij}^*, y_{ij}^*\right) \Delta A}{\sum_{j=1}^{n} \sum_{i=1}^{m} \delta\left(x_{ij}^*, y_{ij}^*\right) \Delta A}.$$

If we take the limit as m and n go to infinity, we obtain the center of mass $(\overline{x}, \overline{y})$ of the lamina.

The coordinates $(\overline{x}, \overline{y})$ of the **center of mass of a lamina** D with density $\delta = \delta(x, y)$ are given by

$$\overline{x} = \frac{\int\int_D x\delta(x, y)\, dA}{\int\int_D \delta(x, y)\, dA} \quad \text{and} \quad \overline{y} = \frac{\int\int_D y\delta(x, y)\, dA}{\int\int_D \delta(x, y)\, dA}.$$

The integrals in the numerators of the calculations of $\overline{x}$ and $\overline{y}$ are called the *moments* of the lamina about the coordinate axes. So the moment of a lamina D with density $\delta = \delta(x, y)$ about the y-axis is

$$M_y = \int\int_D x\delta(x, y)\, dA$$

and the moment of D about the x-axis is

$$M_x = \int\int_D y\delta(x, y)\, dA.$$

Activity 3.12.

In this activity we determine integrals that represent the center of mass of a lamina D described by the triangular region bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant if the density at point (x, y) is $\delta(x, y) = 6x + 6y + 6$. A picture of the lamina is shown in Figure 3.19.

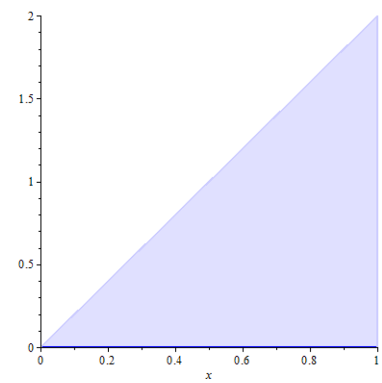

Figure 3.19: The lamina bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant.

(a) Set up an iterated integral that represents the mass of the lamina.

(b) Assume the mass of the lamina is 14. Set up two iterated integrals that represent the coordinates of the center of mass of the lamina.

$\triangleleft$

Activity 3.13.

Let D be a half-disk lamina of radius 3 in quadrants IV and I, centered at the origin as in Activity **??**. Assume the density at point (x, y) is equal to x. Find the center of mass of the lamina.

$\triangleleft$

Probability

Calculating probabilities is an incredibly important application of integration in the physical, social, and life sciences. To understand the basics, consider the game of darts in which a player throws a dart at a board and tries to hit a particular target. Let us suppose that a dart board is in the form of a disk D with radius 10 inches. If we assume that a player throws a dart at random, and is not aiming at any particular point, then it is equally probably that the dart will strike any single point on the board as any other. In other words, the probability that the dart will strike a given point is $\frac{1}{100\pi}$, or the reciprocal of the area of D (assuming that the dart thrower does not miss the board entirely). Similarly, the probability that the dart strikes a point in the disk D_3 of radius 3 inches is given by the area of D_3 divided by the area of D. In other words, the probability that the dart strikes the disk D_3 is the area of the disk D_3 divided by the area of the board, or

$$\frac{9\pi}{100\pi} = \int\int_{D_3} \frac{1}{100\pi}\, dA.$$

The integrand can be thought of as a *distribution function*, describing how the dart strikes are distributed across the board. In this case the distribution function is constant since we are assuming a uniform distribution, but that does not have to be the case. If the player is fairly good and is aiming for the bulls eye (the center of D), then the distribution function f would be skewed toward the center, say

$$f(x, y) = Ke^{-(x^2+y^2)}$$

for some constant positive K. If we assume that the player is consistent enough so that the dart always strikes the board, then the probability that the dart strikes the board somewhere is 1, and the distribution function f will have to satisfy[2]

$$\int\int_D f(x, y)\, dA = 1.$$

Then the probability that the dart strikes in the disk D_1 of radius 1 would be

$$\int\int_{D_1} f(x, y)\, dA.$$

In fact, the probability that the dart strikes in any region R is given by

$$\int\int_R f(x, y)\, dA.$$

The preceding discussion highlights the general idea behind calculating probabilities. We assume we have a *joint probability density function* f, a function of two independent variables x and y defined on a domain D that satisfies the conditions

- $f(x, y) \geq 0$ for all x and y in D,
- the probability that x is between some values a and b while y is between some values c and d is given by
$$\int_a^b \int_c^d f(x, y)\, dy\, dx,$$
- The probability that the point (x, y) is in D is 1, that is
$$\int\int_D f(x, y)\, dA = 1. \tag{3.2}$$

[2]This makes $K = \frac{1}{\pi\left(1-e^{-100}\right)}$, which you can check.

Note that it is possible that D could be an infinite region and the limits on the integral in (3.2) could be infinite. When we have a probability density function $f = f(x, y)$, the probability that the point (x, y) is in some region R contained in the domain D (the notation we use here is $P((x, y) \in R)$) is found by

$$P((x, y) \in R) = \int\int_R f(x, y)\, dA.$$

Activity 3.14.

A firm manufactures smoke detectors. Two components for the detectors come from different suppliers – one in Michigan and one in Ohio. The company studies these components for their reliability and their data suggests that if x is the life span (in years) of a randomly chosen component from the Michigan supplier and y the life span (in years) of a randomly chosen component from the Ohio supplier, then the joint probability density function f might be given by

$$f(x, y) = e^{-x}e^{-y}.$$

(a) Theoretically, the components might last forever, so the domain D of the function f is $D = \{(x, y) : x \geq 0 \text{ and } y \geq 0\}$. To show that f is a probability density function on D we need to demonstrate that

$$\int\int_D f(x, y)\, dA = 1,$$

or that

$$\int_0^\infty \int_0^\infty f(x, y)\, dy\, dx = 1.$$

Use your knowledge of improper integrals to verify that f is a probability density function.

(b) Assume that the smoke detector fails only if both of the supplied components fail. If we want to determine the probability that a randomly selected detector will fail within one year, we will need to determine the probability that the life span of each component is between 0 and 1 years. Set up an iterated integral that represents this probability.

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Activity 3.15.

Let x denote the time (in minutes) that a person spends waiting in a checkout line at a grocery store and y the time (in minutes) that it takes to check out. Set up an integral that will determine the probability that you will spend no more than 10 minutes waiting and then checking out at this grocery store if the joint probability density function for x and y is

$$f(x, y) = \frac{1}{8}e^{-x/4-y/2}.$$

◁

Summary

In this section, we encountered the following important ideas:

- We think of the area of a region D to be the volume of a solid of uniform height 1 and base D. So the area of D is given by
$$\int\int_D 1\,dA.$$
- To find the mass of a lamina D if we have a mass density function δ by first partition the lamina into subrectangles R_{ij}, where $1 \leq i \leq m$ and $1 \leq j \leq n$, of equal area ΔA (assume the density is 0 at any point outside of the disk) and select a point $\left(x_{ij}^*, y_{ij}^*\right)$ in R_{ij} for each i and j. We approximate the mass of the ijth slice by assuming constant density $\delta\left(x_{ij}^*, y_{ij}^*\right)$ on that slice. This gives us an approximate mass of the ijth slice to be $\delta\left(x_{ij}^*, y_{ij}^*\right)\Delta A$. Adding the mass approximations on each slice and taking the limit as both m and n go to infinity gives us the double integral
$$\int\int_D \delta(x,y)\,dA$$
that represents the mass of the lamina.
- To find the center of mass of a continuous lamina with a variable density $\delta(x,y)$, we partition the lamina into mn subrectangles of equal area dA, and approximate the mass on the ijth subrectangle by assuming a constant density on that subrectangle. We then use the formula for the center of mass of mn discrete masses and take the limit as m and n go to infinity. This gives us the x and y coordinates of the center of mass as
$$\overline{x} = \frac{\int\int_D x\delta(x,y)\,dA}{\int\int_D \delta(x,y)\,dA} \quad \text{and} \quad \overline{y} = \frac{\int\int_D y\delta(x,y)\,dA}{\int\int_D \delta(x,y)\,dA}.$$
- A joint probability density function f is a function of two independent variables x and y defined on a domain D that satisfies the conditions
  - $f(x,y) \geq 0$ for all x and y in D,
  - the probability that x is between some values a and b while y is between some values c and d is given by
$$\int_a^b \int_c^d f(x,y)\,dy\,dx,$$
  - $\displaystyle\int\int_D f(x,y)\,dA = 1.$

If R is some subregion of D, then the probability that (x, y) is in R is given by

$$\int\int_R f(x,y)\,dA.$$

3.5 Double Integrals in Polar Coordinates

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What are the polar coordinates of a point in two-space?
- How do we convert between polar coordinates and rectangular coordinates?
- What is the area element in polar coordinates?
- How do we convert a double integral in rectangular coordinates to a double integral in polar coordinates?

Preview Activity 3.5. The rectangular coordinate system allows us to draw pictures of graphs over a rectangular grid. The polar coordinate system is an alternate coordinate system that allows us to define and draw a different variety of functions. Polar coordinates also provide a convenient and useful way to represent complex numbers. Our application of polar coordinates will be to evaluate some double integrals that cannot be easily evaluated in rectangular coordinates.

The coordinates of a point determine the location of the point. The rectangular coordinates of a point P are given in an ordered pair (x, y), where x is the (signed) distance from the y-axis to P and y is the (signed) distance from the x-axis to P. In this section we will look at another set of coordinates for a point in the plane, the polar coordinates.

(a) Determine the rectangular coordinates of the following points:

i. The point that is a distance 1 from the origin along the positive x-axis.

ii. The point that is a distance 2 from the origin and makes an angle of $\frac{\pi}{2}$ with the positive x-axis.

iii. The point that is a distance 3 from the origin and makes an angle of $\frac{2\pi}{3}$ with the positive x-axis.

(b) Part (a) indicates that the two pieces of information, the distance r from a point to the origin along with the angle θ that the line through the origin and the point makes with the positive x-axis, completely determine the location of the point. The point (r, θ) is the polar coordinate representation for a point. Find polar coordinates for the points with the given rectangular coordinates.

i. $(0, -1)$

ii. $(-2, 0)$

iii. $(-1, 1)$

$\bowtie$

Introduction

We have defined double integrals in the rectangular coordinate system. There are many of these integrals that are difficult, if not impossible, to integrate in rectangular coordinates. It is useful, therefore, to be able to translate to other coordinate systems where the evaluation is easier. One such coordinate system is the polar coordinate system. In this section we provide a quick discussion of polar coordinates and then introduce double integrals in polar coordinates.

A Quick Overview of Polar Coordinates

The rectangular coordinate system allows us to draw pictures of graphs over a rectangular grid. The polar coordinate system is an alternative coordinate system that allows us to define and draw a different variety of functions. While a point P in rectangular coordinates is described by an ordered pair (x, y), where x is the displacement from the y-axis to P and y is the displacement from the x-axis to P, we saw in Preview Activity 3.5 that we can also describe the location of the point P with polar coordinates (r, θ) where r is the distance from the origin to P and θ is the angle the ray through the origin and P makes with the positive x-axis as shown in Figure 3.20. We can convert from rectangular to polar and polar to rectangular coordinates using a little trigonometry.

Converting from rectangular to polar. If we are given the rectangular coordinates (x, y) of a point P, then the polar coordinates (r, θ) of P satisfy

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \tan(\theta) = \frac{y}{x}, \text{ assuming } x \neq 0.$$

Converting from polar to rectangular. If we are given the polar coordinates (r, θ) of a point P, then the rectangular coordinates (x, y) of P satisfy

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta).$$

Activity 3.16.

(a) Find the rectangular coordinates of the point whose polar coordinates are $\left(2, -\frac{\pi}{3}\right)$.

(b) Find polar coordinates for the point whose rectangular coordinates are $P = (-1, 1)$. Now find a second pair of polar coordinates for the point P. How many different ways are there to represent a point in polar coordinates?

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We can draw graphs of curves in polar coordinates just as we do in rectangular coordinates. However, when plotting in polar coordinates we use a grid that is different than the rectangular coordinate grid. The angles θ and distances r partition the plane into small wedges as shown in

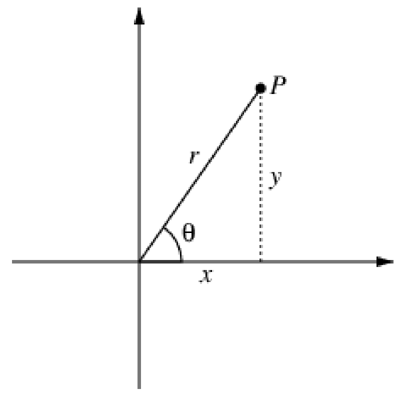

Figure 3.20: The polar coordinates of a point.

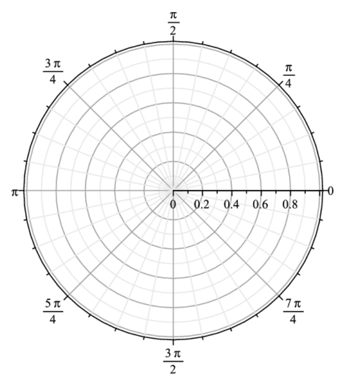

Figure 3.21: The polar coordinate grid.

Figure 3.21.

Activity 3.17.

Most polar graphing devices[3] can plot curves in polar coordinates of the form $r = f(\theta)$. Use such a device to complete this activity.

(a) Before plotting the polar graph $r = 1$, think about what shape it should have. Then use appropriate technology to draw the graph and test your intuition.

(b) The equation $\theta = 1$ does not define r as a function of θ, so we can't graph this equation on many polar plotters. What do you think the graph of the polar curve $\theta = 1$ looks like? Why?

(c) Before plotting the curve, what do you think the graph of the polar curve $r = \theta$ looks

[3] like your calculator using the POL mode, or a web applet such as http://webspace.ship.edu/msrenault/ggb/polar_grapher.html

like? Why? Draw the curve to test your intuition.

(d) Experiment with other polar curves and find one that you think is interesting.

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Integrating in Polar Coordinates

Consider the double integral

$$\int\int_D e^{x^2+y^2}\, dA,$$

where D is the unit disk. This is not an integral that we can directly evaluate in rectangular coordinates, but a change to polar coordinates will convert this integral into one that we can easily evaluate.

We have seen how to evaluate the double integral $\int\int_D f(x,y)\, dA$ as an iterated integral of the form

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

in rectangular coordinates, because we know that $dA = dy\, dx$ in rectangular coordinates. To make the change to polar coordinates, we need to represent the variables x and y in polar coordinates (which we know how to do), but we also have to understand how to write the area element dA in polar coordinates. The product $dy\, dx$ represents an element of area in rectangular coordinates (height dy times length dx), but what does an area element dA look like in polar coordinates, and how do we find the area element in terms of dr and $d\theta$? We address this question in the next activity.

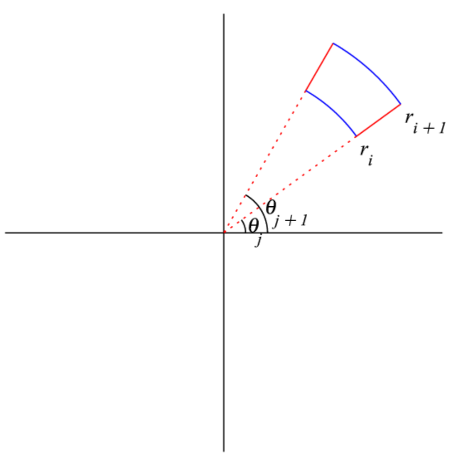

Figure 3.22: A polar rectangle.

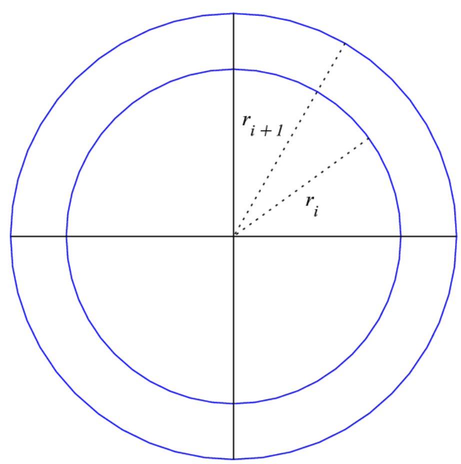

Figure 3.23: An annulus.

Activity 3.18.

Consider a polar rectangle R, with r between r_i and r_{i+1} and θ between θ_j and θ_{j+1} as shown in Figure 3.22. Let $\Delta r = r_{i+1} - r_i$ and $\Delta\theta = \theta_{j+1} - \theta_j$. Let ΔA be the area of this region.

(a) Explain why the area ΔA in polar coordinates is NOT $\Delta r\, \Delta\theta$.

(b) Now find ΔA by following these steps.

i. Find the area of the annulus (the washer-like region) between r_i and r_{i+1} as shown at right in Figure 3.23. This area will be in terms of r_i and r_{i+1}.

ii. Our region R is only a portion of the annulus, so the area ΔA of R is only a fraction of the area of the annulus. What fraction of the area of the annulus is the area ΔA? (Hint: What is the central angle of a circle? What fraction of that central angle does our region R subtend?)

iii. What is ΔA in terms of r_i, r_{i+1}, θ_j, and θ_{j+1}?

iv. Write the area ΔA in terms of r_i, r_{i+1}, Δr, and $\Delta\theta$. (Hint: Think about how to factor a difference of squares.)

(c) As we take the limit as Δr and $\Delta\theta$ go to 0, Δr becomes dr, $\Delta\theta$ becomes $d\theta$, and ΔA becomes dA, an element of area. Write dA in terms of r, dr, and $d\theta$.

$\triangleleft$

What Activity 3.18 shows is that when we want to convert an integral from rectangular coordinates to polar coordinates, we have to be sure to change the area element dA to $dA = r\,dr\,d\theta$ in polar coordinates. In other words, if we have a double integral $\int\int_D f(x,y)\,dA$ in rectangular coordinates, to convert the integral to an iterated integral in polar coordinates we replace x with $r\cos(\theta)$, y with $r\sin(\theta)$ and dA with $r\,dr\,d\theta$. Of course, we need to describe the region D in polar coordinates as well. To summarize

The double integral $\int\int_D f(x,y)\,dA$ in rectangular coordinates can be converted to an iterated double integral in polar coordinates as $\int\int_D f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta$.

Example 3.2. Let $f(x,y) = e^{x^2+y^2}$ on the disk $D = \{(x,y) : x^2+y^2 \le 1\}$. In rectangular coordinates the double integral $\displaystyle\int\int_D f(x,y)\,dA$ can be written as the iterated integral

$$\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx.$$

This is not an integral we can evaluate in rectangular coordinates. Since $r = \sqrt{x^2+y^2}$ in polar coordinates, converting to polar coordinates might simplify things. To convert to polar coordinates, we replace x with $r\cos(\theta)$, y with $r\sin(\theta)$ and $dy\,dx$ with $r\,dr\,d\theta$ to obtain

$$\int\int_D e^{r^2}\,r\,dr\,d\theta.$$

The disc D is described in polar coordinates by the constraints $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. So we have

$$\int\int_D e^{r^2}\, r\, dr\, d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} e^{r^2}\, r\, dr\, d\theta.$$

We can now evaluate this last integral in polar coordinates as follows:

$$\begin{aligned}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} e^{r^2}\, r\, dr\, d\theta &= \int_{\theta=0}^{2\pi}\left(\frac{1}{2}e^{r^2}\Big|_{r=0}^{1}\right) d\theta \\ &= \frac{1}{2}\int_{\theta=0}^{2\pi}(e-1)\ d\theta \\ &= \frac{1}{2}(e-1)\int_{\theta=0}^{2\pi} d\theta \\ &= \frac{1}{2}(e-1)\,[\theta]\Big|_{\theta=0}^{2\pi} \\ &= \pi(e-1).\end{aligned}$$

Activity 3.19.

Let $f(x, y) = x + y$ on the disk $D = \{(x, y) : x^2 + y^2 \leq 4\}$.

(a) Write the double integral of f over D as an iterated integral in rectangular coordinates.

(b) Write the double integral of f over D as an iterated integral in polar coordinates.

(c) Evaluate one of the iterated integrals. Should we have expected the final result?

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Activity 3.20.

Evaluate $\int\int_D x\, dA$ in polar coordinates, where D is the region bounded above by the line $y = x$ and below by the circle $x^2 + (y-1)^2 = 1$. The graph of the curves is shown in Figure 3.24.

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Summary

In this section, we encountered the following important ideas:

- The polar representation of a point P is the ordered pair (r, θ) where r is the distance from the origin to P and θ is the angle the ray through the origin and P makes with the positive x-axis.
- • The polar coordinates r and θ of a point (x, y) in rectangular coordinates satisfy

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan(\theta) = \frac{y}{x}.$$

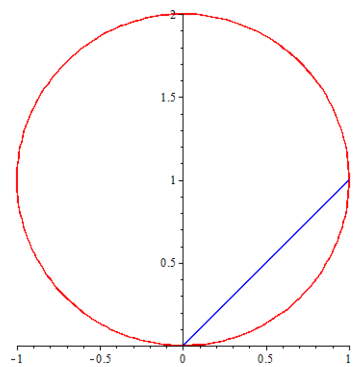

Figure 3.24: The graphs of $y = x$ and $x^2 + (y-1)^2 = 1$.

- The rectangular coordinates x and y of a point (r, θ) in polar coordinates satisfy

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta).$$

- The area element dA in polar coordinates is determined by the area of a slice of an annulus and is given by

$$dA = r\,dr\,d\theta.$$

- The convert the double integral $\int\int_D f(x,y)\,dA$ to an iterated integral in polar coordinates we substitute $r\cos(\theta)$ for x, $r\sin(\theta)$ for y, and $r\,dr\,d\theta$ for dA to obtain the iterated integral $\int\int_D f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta$ in polar coordinates.

3.6 Surfaces Defined Parametrically and Surface Area

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a parameterization of a surface?
- How do we find the surface area of a parametrically defined surface?

Preview Activity 3.6. We have seen how to define curves parametrically in the plane and in three-space using a single parameter. Now we investigate what happens if we add a second parameter.

(a) Recall that we can parameterize the unit circle with the parameterization

$$x(t) = \cos(t) \quad \text{and} \quad y(t) = \sin(t).$$

i. How would we parameterize the circle of radius 1 in 3-space that has its center at $(0, 0, 1)$ and lies in the plane $z = 1$?

ii. How would we parameterize the circle of radius 1 in 3-space that has its center at $(0, 0, -1)$ and lies in the plane $z = -1$?

iii. How would we parameterize the circle of radius 1 in 3-space that has its center at $(0, 0, 5)$ and lies in the plane $z = 5$?

iv. Taking into account your responses to the previous questions, what do you think the graph of the set of parametric equations

$$x(s, t) = \cos(t), \quad y(s, t) = \sin(t), \quad \text{and} \quad z(s, t) = s.$$

looks like? Explain.

(b) Now modify the parameterization from part (a) to construct the parameterization of a cone with vertex at the origin in the positive z-direction with base radius 4 and height 3. Use appropriate technology[4] to assist you. (Hint: The cross sections parallel to the xz plane are circles, with the radii varying linearly as z increases.)

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Introduction

We have seen how curves in space can be defined parametrically, and now we will see how we can also define surfaces parametrically. A curve in space is defined using one parameter, so a surface

[4] e.g., http://www.flashandmath.com/mathlets/multicalc/paramrec/surf_graph_rectan.html

will be defined using two parameters. If $x = x(s,t)$, $y = y(s,t)$, and $z = z(s,t)$ are functions of independent parameters s and t, then the terminal points of all vectors of the form

$$\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}$$

form a surface in space. The set $x = x(s,t)$, $y = y(s,t)$, and $z = z(s,t)$ are the *parametric equations* for the surface, or a *parametrization* of the surface. In Preview Activity 3.6 we saw how to parameterize a cylinder and a cone. Now we look consider more examples.

Activity 3.21.

Recall how the Cartesian coordinates of a point are related to the spherical coordinates of a point. Use this relationship to find a two-variable parameterization of a sphere of radius 2. Draw the surface defined by your parameterization with appropriate technology (e.g., http://web.monroecc.edu/manila/webfiles/calcNSF/JavaCode/CalcPlot3D.htm or http://www.flashandmath.com/mathlets/multicalc/paramrec/surf_graph_rectan.html).

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Activity 3.22.

Let $f = f(x,y)$ be a function of x and y. Find a two-variable parameterization of the surface defined by f. This shows that any of our familiar surfaces can be drawn as parametric surfaces.

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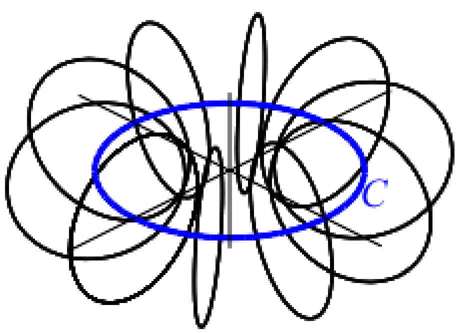

Figure 3.25: A torus by revolving circles.

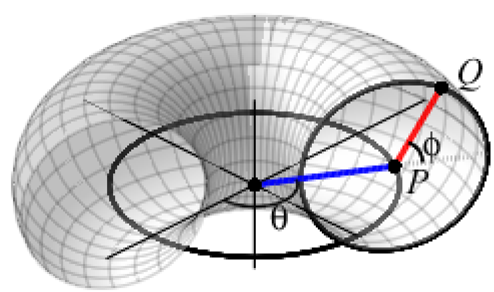

Figure 3.26: A parametrically defined torus.

Example 3.3. A more complicated example is a torus (or doughnut). We will assume our torus has its center at the origin and horizontal axis in the xy-plane. We can visualize a torus by starting with a circle C of radius b in the xy-plane centered at the origin, and then revolving a circle with radius a centered on C and perpendicular to C around the axis of C. An illustration of some of the circles obtained in this way is shown in Figure 3.25. To parameterize this torus, we first parameterize the circle C of radius b in the xy-plane. One such parameterization is

$$x(\theta) = b\cos(\theta), \quad y(\theta) = b\sin(\theta), \quad \text{and} \quad z(\theta) = 0$$

for $theta$ from 0 to 2π. Fix a point $(h, k, 0) = (b\cos(\theta), b\sin(\theta), 0)$ on the circle C and let K be the circle centered at $(h, k, 0)$ with radius a perpendicular to C. If we consider the vectors

$$\langle a\cos(\theta)\cos(\phi), a\sin(\theta)\cos(\phi), a\sin(\phi)\rangle$$

as having initial points at the center of K, then the terminal points of these vectors trace out the circle K as ϕ runs from 0 to 2π as illustrated in Figure 3.26. Adding these vectors to the vector $\langle b\cos(\theta), b\sin(\theta), 0\rangle$ will give us vectors from the origin to points on K. So if we let both θ and ϕ vary from 0 to 2π, we will obtain the parameterization

$$x(\theta, \phi) = b\cos(\theta) + a\cos(\theta)\cos(\phi), \quad y(\theta, \phi) = b\sin(\theta) + a\cos(\theta)\cos(\phi), \quad \text{and} \quad z(t) = a\sin(\phi)$$

for the torus.

The Surface Area of Parametrically Defined Surfaces

Recall that a differentiable function is locally linear – that is, if we zoom in on the surface around a point, the surface looks like its tangent plane. We exploit that idea to find surface area. We will approximate the surface with small parallelograms and the surface area by adding the areas of these approximation parallelograms. To create our approximations we use the integration process – subdividing, approximating, adding our approximations, and then taking limits.

Let

$$\mathbf{r}(s, t) = x(s, t)\mathbf{i} + y(s, t)\mathbf{j} + z(s, t)\mathbf{k}$$

be a vector-valued function that defines a surface over a rectangular domain $a \leq s \leq b$ and $c \leq t \leq d$. We define

$$\mathbf{r}_s(s, t) = x_s(s, t)\mathbf{i} + y_s(s, t)\mathbf{j} + z_s(s, t)\mathbf{k} \quad \text{and} \quad \mathbf{r}_t(s, t) = x_t(s, t)\mathbf{i} + y_t(s, t)\mathbf{j} + z_t(s, t)\mathbf{k}.$$

Partition the interval $[a, b]$ into m subintervals of length $\Delta s = \frac{b-a}{n}$ and let s_0, s_1, …, s_m be the endpoints of these subintervals, where $a = s_0 < s_1 < s_2 < \cdots < s_m = b$. Also partition the interval $[c, d]$ into n subintervals of equal length $\Delta t = \frac{d-c}{n}$ and let $t_0, t_1, \ldots, t_n$ be the endpoints of these subintervals, where $c = t_0 < t_1 < t_2 < \cdots < t_n = d$. These two partitions create a partition of the rectangle $R = [a, b] \times [c, d]$ in st-coordinates into mn sub-rectangles R_{ij} with opposite vertices (s_{i-1}, t_{j-1}) and (s_i, t_j) for i between 1 and m and j between 1 and n. These rectangles all have equal area $\Delta A = \Delta s \cdot \Delta t$.

If we increase s by a small amount Δs from the point (s_{i-1}, t_{j-1}), then $\mathbf{r}$ changes by approximately $\mathbf{r}_s(s_{i-1}, t_{j-1})\Delta s$. Similarly, if we increase t by a small amount Δt from the point (s_{i-1}, t_{j-1}), then $\mathbf{r}$ changes by approximately $\mathbf{r}_t(s_{i-1}, t_{j-1})\Delta t$. So we can approximate the surface defined by $\mathbf{r}$ on the st-rectangle $[s_{i-1}, s_i] \times [s_{i-1} + \Delta s, t_{j-1} + \Delta t]$ with the parallelogram determined by the

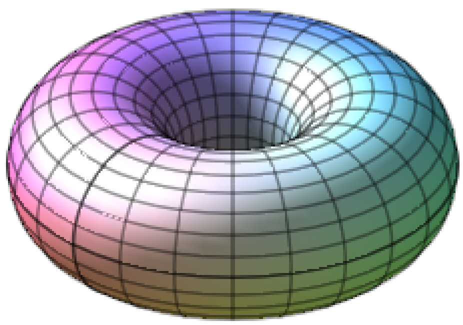

Figure 3.27: A surface.

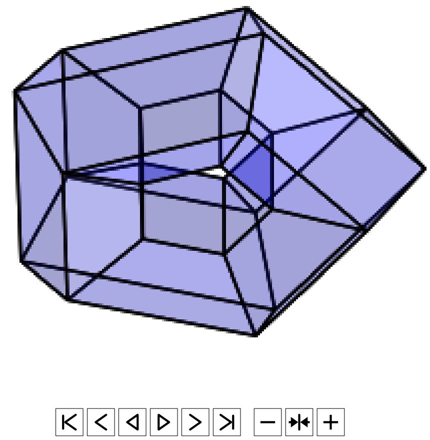

Figure 3.28: Approximating the surface with parallelograms.

vectors $\mathbf{r}_s(s_{i-1}, t_{j-1})\Delta s$ and $\mathbf{r}_t(s_{i-1}, t_{j-1})\Delta t$. An animation illustrating these surface area approximations using a torus (as in Example 3.3) is shown in Figure 3.28, which you can compare to the torus itself in Figure 3.27.

The key step is to find the area of these parallelograms.

Activity 3.23.

Explain why the area of the parallelogram determined by the vectors $\mathbf{r}_s(s_{i-1}, t_{j-1})\Delta s$ and $\mathbf{r}_t(s_{i-1}, t_{j-1})\Delta t$ is

$$|(\mathbf{r}_s(s_{i-1}, t_{j-1})\Delta s) \times (\mathbf{r}_t(s_{i-1}, t_{j-1})\Delta t)| = |\mathbf{r}_s(s_{i-1}, t_{j-1}) \times \mathbf{r}_t(s_{i-1}, t_{j-1})|\Delta s \Delta t. \tag{3.3}$$

$\triangleleft$

We sum these surface area approximations (3.3) over all sub-rectangles to obtain

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{r}_s(s_{i-1}, t_{j-1}) \times \mathbf{r}_t(s_{i-1}, t_{j-1})|\Delta s \Delta t$$

as an estimate of the surface area of the entire surface defined by $\mathbf{r}$. Taking the limit as $m, n \to \infty$ shows that the surface area of the surface defined by $\mathbf{r}$ over the domain D is

$$\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{r}_s(s_{i-1}, t_{j-1}) \times \mathbf{r}_t(s_{i-1}, t_{j-1})|\Delta s \Delta t = \int \int_D |\mathbf{r}_s \times \mathbf{r}_t| \, dA.$$

Let $\mathbf{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$ be a parameterization of a smooth surface over a domain D. The **area of the surface** defined by $\mathbf{r}$ on D is given by

$$\int\int_D |\mathbf{r}_s \times \mathbf{r}_t| \, dA.$$

Activity 3.24.

Consider the cylinder with radius r and height h defined parametrically by

$$\mathbf{r}(s,t) = r\cos(s)\mathbf{i} + r\sin(s)\mathbf{j} + t\mathbf{k}$$

for $0 \leq s \leq 2\pi$ and $0 \leq t \leq h$, as shown in Figure 3.29.

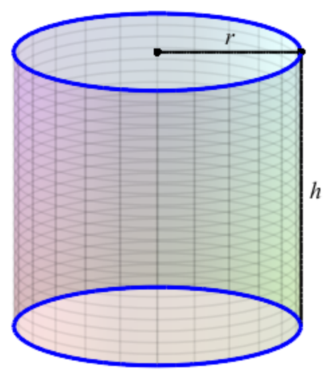

Figure 3.29: A cylinder.

(a) Set up an iterated integral to determine the surface area of this cylinder.

(b) Evaluate the iterated integral. Is the result what you expected? Explain.

◁

Activity 3.25.

Set up an iterated integral to find the surface area of a sphere of radius a (see Activity **??**). Then calculate the surface area and compare to the formula for the surface area of sphere that you can find on-line. (Hint: Exploit the symmetry.)

◁

We can apply this formula for the surface area of a parametrically defined solid to find the surface area of the solid defined by a function $f = f(x,y)$.

Activity 3.26.

Let $f = f(x, y)$ define a smooth solid.

(a) Parameterize the surface defined by f and then show that the area of the surface defined by the graph of f is

$$\int\int_D \sqrt{(f_x(x,y))^2 + (f_y(x,y))^2 + 1}\, dA.$$

(b) Use geometry to find area of the surface defined by $f(x, y) = \sqrt{4 - x^2}$ on the rectangle $D = [-2, 2] \times [0, 3]$. Then calculate this area with the formula found in part (a).

$\triangleleft$

Summary

In this section, we encountered the following important ideas:

- A parameterization of a curve describes the coordinates of a point on the curve in terms of a single parameter t, while a parameterization of a surface describes the coordinates of points on the surface in terms of two independent parameters.
- If $\mathbf{r}(s, t) = \langle x(s, t), y(s, t), z(s, t)\rangle$ describes a smooth surface in 3-space on a domain D, then the area of that surface is given by

$$\int\int_D |\mathbf{r}_s \times \mathbf{r}_t|\, dA.$$

3.7 Triple Integrals

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How is a triple Riemann sum defined?
- How is the triple integral of a continuous function $f = f(x, y, z)$ defined?
- What are two things the triple integral of a function can tell us?

Preview Activity 3.7. Consider a solid piece granite in the shape of a box $B = \{(x, y, z) : 0 \leq x \leq 4, 0 \leq y \leq 6, 0 \leq z \leq 8\}$, whose density varies from point to point. Let $\delta(x, y, z)$ represent the mass density of the piece of granite at point (x, y, z) in kilograms per cubic meter (so we are measuring x, y, and z in meters). Our goal is to find the mass of this solid. If the density was constant, then we could find the mass by multiplying by the volume, but since our density varies from point to point, we use the integration process as we have for similar problems in the past.

(a) Review the process by which we developed a double integral to find the mass of a lamina whose density is given by $\delta = \delta(x, y)$. Explain that process here.

(b) Now we apply this integration process to determine the mass of our piece of granite. Partition the interval $[0, 4]$ into 2 subintervals of equal length, the interval $[0, 6]$ into 3 subintervals of equal length, and the interval $[0, 8]$ into 2 subintervals of equal length. This partitions the box B into sub-boxes as shown in Figure 3.30.

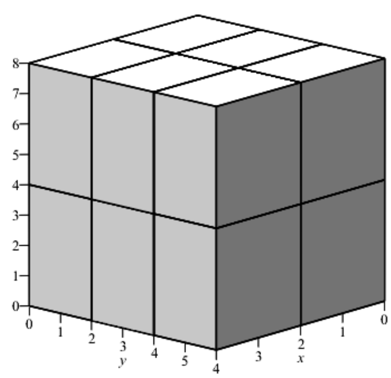

Figure 3.30: A partitioned three-dimensional domain.

i. Let $0 = x_0 < x_1 < x_2 = 4$ be the endpoints of the subintervals of $[0, 4]$ after partitioning. Identify x_0, x_1, and x_2. Draw a picture of Figure 3.30 and label these endpoints on your drawing. What is the length Δx of each subinterval $[x_{i-1}, x_i]$ for i from 1 to 2?

ii. Let $0 = y_0 < y_1 < y_2 < y_3 = 6$ be the endpoints of the subintervals of $[0, 6]$ after partitioning. Identify y_0, y_1, y_2, and y_3 and label these endpoints on your drawing of Figure 3.30. What is the length Δy of each subinterval $[y_{j-1}, y_j]$ for j from 1 to 3?

iii. Let $0 = z_0 < z_1 < z_2 = 8$ be the endpoints of the subintervals of $[0, 8]$ after partitioning. Identify z_0, z_1, and z_2 and label these endpoints on your drawing of Figure 3.30. What is the length Δz of each subinterval $[z_{i-1}, z_i]$ for i from 1 to 2?

iv. The partitions of the intervals $[0, 4]$, $[0, 6]$ and $[0, 8]$ partition the box B into sub-boxes. How many sub-boxes are there? What is volume ΔV of each sub-box?

v. Let B_{ijk} denote the sub-box $[x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$. Appropriately label each visible sub-box in your drawing of Figure 3.30 according to this labeling scheme.

vi. Choose a point $\left(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\right)$ in the i, j, kth sub-box. What physical quantity will $\delta\left(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\right)\Delta V$ approximate?

vii. What final steps should we take to determine the exact mass of our piece of granite?

$\bowtie$

Introduction

Recall that we defined the double integral of a continuous function $f = f(x, y)$ over a rectangle $R = [a, b] \times [c, d]$ as a limit of a double Riemann sum. Now we turn to the problem introduced in Preview Activity 3.7 of creating triple Riemann sums and then triple integrals for functions of three variables. This process generalizes naturally to integrals for functions of any number of variables. However, our geometric intuition tends to leave us as we proceed to higher degrees.

Triple Riemann Sums and Triple Integrals

Consider a solid box B (like a box-shaped piece of granite as in Preview Activity 3.7). Let $\delta(x, y, z)$ represent the density of solid at point (x, y, z). To find the mass of this solid we use the integration process as we did in our Preview Activity. Defining triple Riemann sums and triple integrals involves keeping track of a lot of different objects, and we further develop our abilities to deal with these objects in our next activity.

Activity 3.27.

Suppose our box B has the form $[a, b] \times [c, d] \times [r, s]$, that is $B = \{(x, y, z) : a \le x \le b, c \le y \le d, r \le z \le s\}$. We partition the box as illustrated in Figure 3.31.

(a) Let $a = x_0 < x_1 < x_2 < x_3 < x_4 = b$ be the endpoints of the subintervals of $[a, b]$ after partitioning. Label these endpoints in Figure 3.31. What is the length Δx of each subinterval $[x_{i-1}, x_i]$ for i from 1 to 4? Your answer should be in terms of a and b.

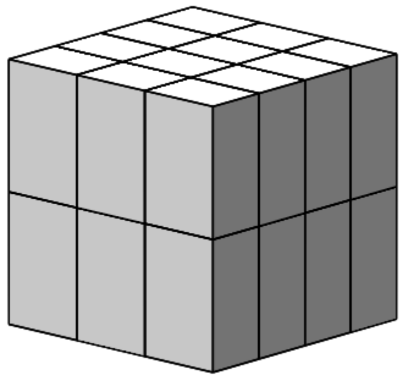

Figure 3.31: A partitioned three-dimensional domain.

(b) Let $c = y_0 < y_1 < y_2 < y_3 = d$ be the endpoints of the subintervals of $[c, d]$ after partitioning. Label these endpoints in Figure 3.31. What is the length Δy of each subinterval $[y_{j-1}, y_j]$ for j from 1 to 3? Your answer should be in terms of c and d.

(c) Let $r = z_0 < z_1 < z_2 = s$ be the endpoints of the subintervals of $[r, s]$ after partitioning. Label these endpoints in Figure 3.31. What is the length Δz of each subinterval $[z_{k-1}, z_k]$ for k from 1 to 2? Your answer should be in terms of r and s.

(d) The partitions of the intervals $[a, b]$, $[c, d]$, and $[r, s]$ partition the box B into sub-boxes. How many sub-boxes are there? What is the volume ΔV of each sub-box?

(e) Let B_{ijk} denote the sub-box $[x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$. Label each visible sub-box in Figure 3.31.

(f) Now let $\left(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\right)$ be an arbitrary point in the i, j, kth sub-box. Explain what the product

$$\delta\left(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\right) \Delta V$$

represents.

(g) If we were to add all the values $\delta\left(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\right) \Delta V$ for each i, j, and k, what does the resulting number approximate about the piece of granite?

(h) Write a triple sum using summation notation that expresses the arbitrary sum from part (k).

◁

This process of partitioning that we carried out in Activity 3.27 is the idea behind the triple Riemann sum.

The Triple Riemann Sum.

Definition 3.3. Let $f = f(x, y, z)$ be a continuous function on a box $B = [a, b] \times [c, d] \times [r, s]$. The **triple Riemann sum of** f **over** B is created as follows.

- Partition the interval $[a, b]$ into m subintervals of equal length $\Delta x = \frac{b-a}{m}$. Let x_0, x_1, ..., x_m be the endpoints of these subintervals, where $a = x_0 < x_1 < x_2 < \cdots < x_m = b$.
- Partition the interval $[c, d]$ into n subintervals of equal length $\Delta y = \frac{d-c}{n}$. Let y_0, y_1, ..., y_n be the endpoints of these subintervals, where $c = y_0 < y_1 < y_2 < \cdots < y_n = d$.
- Partition the interval $[r, s]$ into l subintervals of equal length $\Delta z = \frac{s-r}{l}$. Let z_0, z_1, ..., z_l be the endpoints of these subintervals, where $r = z_0 < z_1 < z_2 < \cdots < z_l = s$.
- Let B_{ijk} be the sub-box of B with opposite vertices $(x_{i-1}, y_{j-1}, z_{l-1})$ and (x_i, y_j, z_k) for i between 1 and m, j between 1 and n, and k between 1 and l. The volume of B_{ijk} is $\Delta V = \Delta x \cdot \Delta y \cdot \Delta z$.
- Let $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ be a point in box B_{ijk} for each i, j, and k. Then a triple Riemann sum for f on B is
$$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} f\left(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\right) \cdot \Delta V.$$

If $f(x, y, z)$ represents the mass density of the box B, then our previous activities indicate that the triple Riemann sum approximates the total mass of the box B. If we let the number of sub-boxes increase without bound (in other words, let m, n, and l in our triple Riemann sum go to infinity), the sum of the mass approximations becomes the actual mass of the solid B. This gives us the triple integral.

The Triple Integral Over a Box.

Definition 3.4. With terms defined as in the Triple Riemann Sum, the **triple integral of** f **over** B is
$$\int\int\int_B f(x, y, z)\, dV = \lim_{m,n,l\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} f\left(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\right) \cdot \Delta V.$$

As we argued above, if $f(x, y, z)$ represents the density of the solid B at each point (x, y, z), then

$$\int\int\int_B f(x, y, z)\, dV$$

represents the mass of B.

Also, as with double integrals, if we want to integrate over a region that is not a box, we enclose the region in a rectangular box and assume the density is 0 everywhere inside the box that is outside the region. Similarly, we can use triple integrals to find the volume of a solid, the average value of a function, and the center of mass of a solid with variable density:

- The triple integral
$$\int\int\int_S 1\,dV$$
represents the **volume** of the solid S,
- The **average value** of the function $f = f(x, y, x)$ over a solid domain S is given by
$$\left(\frac{1}{V(S)}\right)\int\int\int_S f(x, y, z)\,dV,$$
where $V(S)$ is the volume of the solid S, and
- The **center of mass** of the solid S with density $\delta = \delta(x, y, z)$ is $(\overline{x}, \overline{y}, \overline{z})$, where
$$\overline{x} = \frac{\int\int\int_S x\,\delta(x, y, z)\,dV}{M}, \quad \overline{y} = \frac{\int\int\int_S y\,\delta(x, y, z)\,dV}{M}, \quad \overline{z} = \frac{\int\int\int_S z\,\delta(x, y, z)\,dV}{M},$$
and $M = \displaystyle\int\int\int_S \delta(x, y, z)\,dV$ is the mass of the solid S.

In the Cartesian coordinate system, the area element dV is $dz\,dy\,dx$, and, as a consequence, a triple integral of a function f over a box $B = [a, b] \times [c, d] \times [r, s]$ in Cartesian coordinates can be evaluated as an iterated integral of the form

$$\int\int\int_B f(x, y, x)\,dV = \int_a^b \int_c^d \int_r^s f(x, y, z)\,dz\,dy\,dx.$$

Activity 3.28.

Evaluate the triple integral of $f(x, y, z) = x - y + 2z$ over the box $B = [0, 2] \times [1, 4] \times [-2, 3]$.

$\lhd$

If we want to evaluate a triple integral as an iterated integral over a solid S that is not a box, then we need to describe the solid in terms of variable limits.

Activity 3.29.

Let S be the solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$. A picture of S is shown in Figure

3.32. In this activity we set up an iterated integral of the form

$$\int_{?}^{?}\int_{?}^{?}\int_{?}^{?} \delta(x,y,z)\,dz\,dy\,dx \tag{3.4}$$

to represent the mass of S if $\delta(x,y,z)$ tells us the density of S at the point (x,y,z). Our job is to find the limits on the integrals.

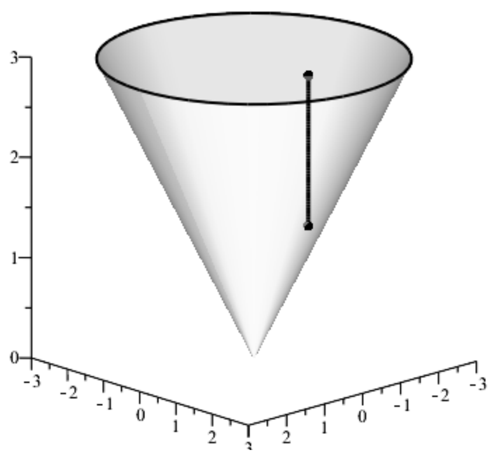

Figure 3.32: A cone.

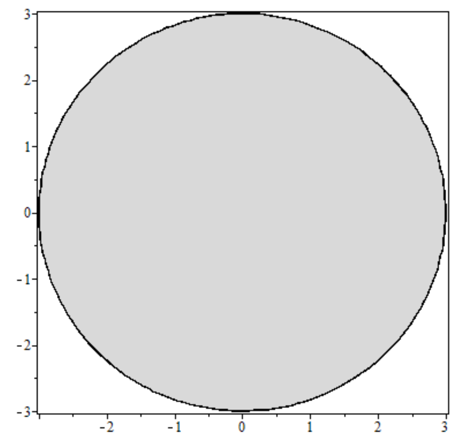

Figure 3.33: The projection of the cone.

(a) What are the limits on z in the innermost integral? A slice in the z direction with x and y fixed is pictured in Figure 3.32. Note that at least one of these limits is not constant but depends on x and y.

(b) The projection of the cone S onto the xy-plane is shown in Figure 3.33. What are the limits on the middle integral? Note that at least one of these limits is not constant and depends on x.

(c) Write an iterated integral of the form (3.4) that represents the mass of the cone S.

◁

IMPORTANT NOTE: When setting up iterated integrals, the limits on a given variable can be ONLY in terms of the remaining variables. For example, a triple integral $\int\int\int_S f(x,y,z)\,dV$ over a solid S can be represented by iterated integrals of the form

- $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$
- $\displaystyle\int_c^d \int_{g_1(y)}^{g_2(y)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dx\,dy$

- $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,z)}^{h_2(x,z)} f(x,y,z)\,dy\,dz\,dx$
- $\displaystyle\int_r^s \int_{g_1(z)}^{g_2(z)} \int_{h_1(x,z)}^{h_2(x,z)} f(x,y,z)\,dy\,dx\,dz$
- $\displaystyle\int_c^d \int_{g_1(y)}^{g_2(y)} \int_{h_1(y,z)}^{h_2(y,z)} f(x,y,z)\,dx\,dz\,dy$
- $\displaystyle\int_r^s \int_{g_1(z)}^{g_2(z)} \int_{h_1(y,z)}^{h_2(y,z)} f(x,y,z)\,dx\,dy\,dz$

where g_1, g_2, h_1, and h_2 are functions of the indicated variables.

Example 3.4. Let us find the mass of the tetrahedron in the first octant bounded by the coordinate planes and the plane $x+2y+3z=6$ if the density at point (x,y,z) is given by $\delta(x,y,z)=x+y+z$. A picture of the solid tetrahedron is shown in Figure 3.34.

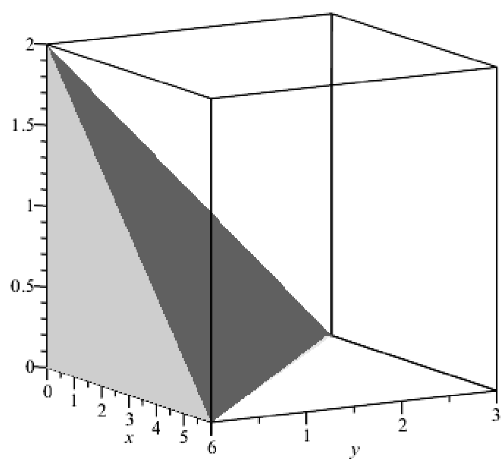

Figure 3.34: A tetrahedron.

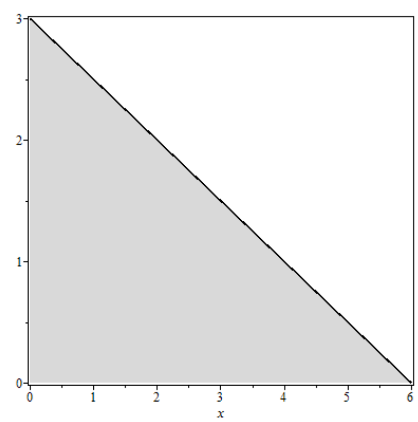

Figure 3.35: The projection of the tetrahedron.

We find the mass of the tetrahedron by the triple integral

$$\int\int\int_S \delta(x,y,z)\,dV,$$

where S is the solid tetrahedron described above. In this example, we will integrate with respect to z first. The top of the tetrahedron is given by the equation

$$x+2y+3z=6$$

and solving for z yields

$$z = \frac{1}{3}(6 - x - 2y).$$

The bottom of the tetrahedron is the xy-plane, so the limits on z in our iterated integral are $0 \leq z \leq \frac{1}{3}(6 - x - 2y)$.

To find the bounds on x and y we project the tetrahedron onto the xy-plane (or set $z = 0$). The resulting relation between x and y is

$$x + 2y = 6.$$

Figure 3.35 shows the projection of the tetrahedron onto the xy-plane.

If we choose next to integrate with respect to y, then the lower limit on y is the x-axis and the upper limit is the hypotenuse of the triangle. Note that the hypotenuse joins the points $(6, 0)$ and $(0, 3)$ and so has equation $y = 3 - \frac{1}{2}x$. Thus, the bounds on y are $0 \leq y \leq 3 - \frac{1}{2}x$. Finally, the x values run from 0 to 6, so our iterated integral that gives the mass of the tetrahedron is

$$\int_0^6 \int_0^{3-(1/2)x} \int_0^{(1/3)(6-x-2y)} x + y + x \, dz \, dy \, dx. \tag{3.5}$$

Evaluating the triple integral gives us

$$\begin{aligned}
\int_0^6 \int_0^{3-(1/2)x} \int_0^{(1/3)(6-x-2y)} x + y + x \, dz \, dy \, dx &= \int_0^6 \int_0^{3-(1/2)x} \left[xz + yz + \frac{z}{2} \right] \Bigg|_0^{(1/3)(6-x-2y)} \, dy \, dx \\
&= \int_0^6 \int_0^{3-(1/2)x} \frac{4}{3}x - \frac{5}{18}x^2 - \frac{7}{9}xy + \frac{2}{3}y - \frac{4}{9}y^2 + 2 \, dy \, dx \\
&= \int_0^6 \left[\frac{4}{3}xy - \frac{5}{18}x^2 y - \frac{7}{18}xy^2 + \frac{1}{3}y^2 - \frac{4}{27}y^3 + 2y \right] \Bigg|_0^{3-(1/2)x} \, dx \\
&= \int_0^6 5 + \frac{1}{2}x - \frac{7}{12}x^2 + \frac{13}{216}x^3 \, dx \\
&= \left[5x + \frac{1}{4}x^2 - \frac{7}{36}x^3 + \frac{13}{864}x^4 \right] \Bigg|_0^6 \\
&= \frac{33}{2}.
\end{aligned}$$

Activity 3.30.

There are several other ways we could have set up the integral to give the mass of the tetrahedron in Example 3.4.

(a) How many different iterated integrals could be set up that are equal to the integral in (3.5)?

(b) Set up an iterated integral, integrating first with respect to z, then x, then y that is equivalent to the integral in (3.5).

(c) Set up an iterated integral, integrating first with respect to y, then z, then x that is equivalent to the integral in (3.5).

(d) Set up an iterated integral, integrating first with respect to x, then y, then z that is equivalent to the integral in (3.5).

$\triangleleft$

Setting up limits on iterated integrals can be a challenge, and to become proficient at it requires some practice. Here is another activity to help develop your skills.

Activity 3.31.

Consider the solid S that is bounded by the parabolic cylinder $y = x^2$ and the planes $z = 0$ and $z = 1 - y$ as shown in Figure 3.36. Assume the density of S is given by $\delta(x, y, z) = z$

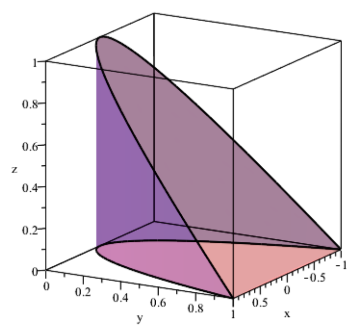

Figure 3.36: The solid S in Activity **??**.

(a) Set up an iterated integral, integrating first with respect to z, then y, then x that represents the mass of S. A picture of the projection of S onto the xy-plane is shown in Figure 3.37.

(b) Set up an iterated integral, integrating first with respect to y, then z, then x that represents the mass of S. A picture of the projection of S onto the xz-plane is shown in Figure 3.38.

(c) Set up an iterated integral, integrating first with respect to x, then y, then z that represents the mass of S. A picture of the projection of S onto the yz-plane is shown in Figure 3.39.

$\triangleleft$

We conclude with two additional activities in which we will apply triple integrals.

Activity 3.32.

A solid S is bounded below by the square $z = 0$, $-1 \le x \le 1$, $-1 \le y \le 1$ and above by the surface $z = 2 - x^2 - y^2$. A picture of the solid is shown in Figure 3.40.

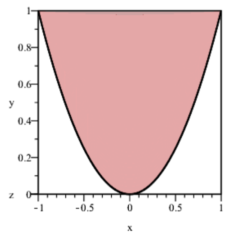

Figure 3.37: Projecting S onto the xy-plane.

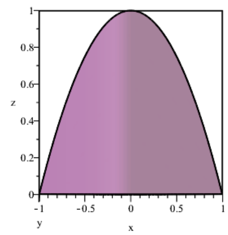

Figure 3.38: Projecting S onto the xz-plane.

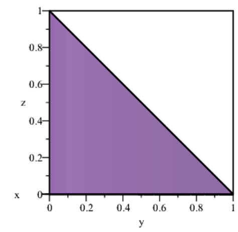

Figure 3.39: Projecting S onto the yz-plane.

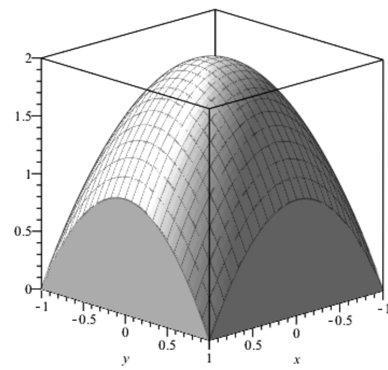

Figure 3.40: The solid bounded by the surface $z = 2 - x^2 - y^2$.

(a) Set up an iterated integral to find the volume of the solid S.

(b) Set up iterated integrals to find the center of mass of S if the density at point (x, y, z) is $\delta(x, y, z) = x^2 + 1$.

(c) Set up an iterated integral to find the average density on S using the density function from part (b).

◁

Activity 3.33.

Set up an iterated integral or integrals that will find the average sum of the numbers x, y, and z if y is between 0 and 2, x is greater than or equal to 0 but cannot exceed $2y$, and z is greater than or equal to 0 but cannot exceed $x + y$.

◁

Summary

In this section, we encountered the following important ideas:

- Let $f = f(x, y, z)$ be a continuous function on a box $B = [a, b] \times [c, d] \times [r, s]$. The triple Riemann sum of f over B is created follows.
  - Partition the interval $[a, b]$ into m subintervals of equal length $\Delta x = \frac{b-a}{m}$. Let x_0, x_1, ..., x_m be the endpoints of these subintervals, where $a = x_0 < x_1 < x_2 < \cdots < x_m = b$.
  - Partition the interval $[c, d]$ into n subintervals of equal length $\Delta y = \frac{d-c}{n}$. Let y_0, y_1, ..., y_n be the endpoints of these subintervals, where $c = y_0 < y_1 < y_2 < \cdots < y_n = d$.
  - Partition the interval $[r, s]$ into l subintervals of equal length $\Delta z = \frac{s-r}{l}$. Let z_0, z_1, ..., z_l be the endpoints of these subintervals, where $r = z_0 < z_1 < z_2 < \cdots < z_l = s$.
  - Let B_{ijk} be the sub-box of B with opposite vertices $(x_{i-1}, y_{j-1}, z_{l-1})$ and (x_i, y_j, z_k) for i between 1 and m, j between 1 and n, and k between 1 and l. The volume of B_{ijk} is $\Delta V = \Delta x \cdot \Delta y \cdot \Delta z$.
  - Let $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ be a point in box B_{ijk} for each i, j, and k. Then a triple Riemann sum for f on B is
  $$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} f\left(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\right) \cdot \Delta V.$$
- The triple integral of a continuous function $f = f(x, y, z)$ on a box $B = [a, b] \times [c, d] \times [r, s]$ is defined as
$$\int\int\int_B f(x, y, z)\, dV = \lim_{\Delta V \to 0} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} f\left(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\right) \cdot \Delta V,$$
with the triple Riemann sum as defined in the first bullet. If f is defined over some solid S, we enclose S in a box B and define f to have the value 0 outside of B. In this case,
$$\int\int\int_S f(x, y, z)\, dV = \int\int\int_B f(x, y, z)\, dV.$$
- The triple integral $\int\int\int_S f(x, y, z)\, dV$ can tell us
  - the volume of the solid S if $f(x, y, z) = 1$,
  - the mass of the solid S if f represents the density of S at the point (x, y, z).

3.8 Triple Integrals in Cylindrical and Spherical Coordinates

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What are the cylindrical coordinates of a point?
- How do we convert from Cartesian coordinates to cylindrical coordinates and from cylindrical coordinates to Cartesian coordinates?
- What is the volume element in cylindrical coordinates? How does this inform us about evaluating a triple integral as an iterated integral in cylindrical coordinates?
- What are the spherical coordinates of a point?
- How do we convert from Cartesian coordinates to spherical coordinates and from spherical coordinates to Cartesian coordinates?
- What is the volume element in spherical coordinates? How does this inform us about evaluating a triple integral as an iterated integral in spherical coordinates?

Preview Activity 3.8. We have encountered two different coordinate systems in 2 dimensions: the rectangular and polar coordinates systems. Just as in 2D, there are different coordinate systems in 3D. We are already familiar with the Cartesian coordinate system in 3D, now we study the cylindrical and spherical coordinate systems.

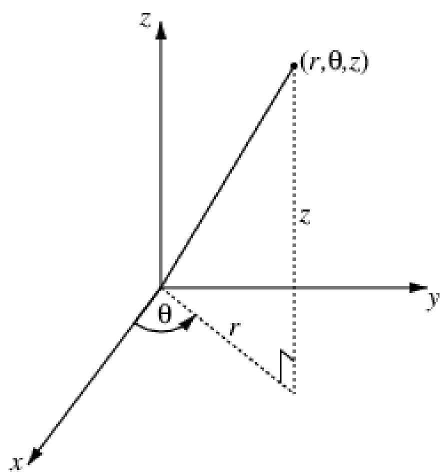

Figure 3.41: The cylindrical coordinates of a point.

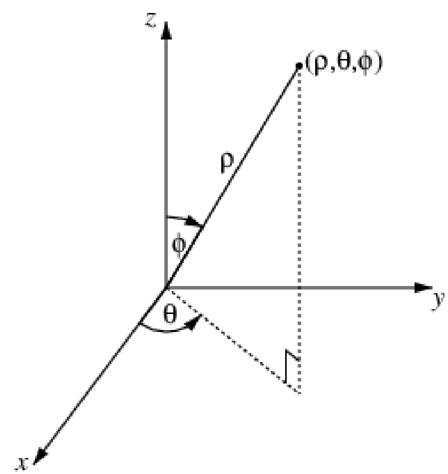

Figure 3.42: The spherical coordinates of a point.

Part A. The cylindrical coordinates of a point are (r, θ, z) where r and θ are the polar coordinates of the point (x, y) and z is the same z coordinate as in Cartesian coordinates. An illustration is given in Figure 3.41.

(a) Find a set of cylindrical coordinates of the point whose Cartesian coordinates are $(-1, \sqrt{3}, 3)$. Draw a picture to illustrate all of the coordinates.

(b) Find the Cartesian coordinates of the point whose cylindrical coordinates are $\left(2, \frac{5\pi}{4}, 1\right)$. Draw a picture to illustrate all of the coordinates.

Part B. The spherical coordinates of a point in 3-space are ρ (rho), θ, and ϕ (phi), where ρ is the distance from the point to the origin, θ is the same as in polar coordinates, and ϕ is the angle between the positive z axis and the vector from the origin to the point as illustrated in Figure 3.42. Now we consider the point P whose Cartesian coordinates are $(-2, 2, \sqrt{8})$.

(a) What is the distance from P to the origin (this is the value of ρ in the spherical coordinates of P)?

(b) What is the projection of P onto the xy-plane? Use this projection to find the value of θ in the spherical coordinates of P.

(c) Based on the picture in Figure 3.42, how is the angle ϕ related to ρ and the z coordinate of P? Use this idea to find ϕ in the spherical coordinates of P. Draw a picture to illustrate the meanings of ρ, θ, and ϕ in this example.

(d) Based on your responses to (a), (b), and (c), if we are given the Cartesian coordinates (x, y, z) of a point Q, how are the values of ρ, θ, and ϕ in the spherical coordinates of Q determined by x, y, and z?

$\bowtie$

Introduction

We have encountered two different coordinate systems in 2 dimensions: the rectangular and polar coordinates systems. Just as in 2D, there are different coordinate systems in 3D. We are already familiar with the Cartesian coordinate system in 3D, now we study the cylindrical and spherical coordinate systems, and determine how to evaluate triple integrals using cylindrical and spherical coordinates.

Cylindrical Coordinates

The cylindrical coordinates of a point are (r, θ, z) where r and θ are the polar coordinates of the point (x, y) and z is the same z coordinate as in Cartesian coordinates as shown in Figure 3.41.

Figure 3.41 indicates how to translate between Cartesian and cylindrical coordinates. Since the z coordinate is the same in both systems, the conversions of the first two coordinates are the same as the formulas for polar coordinates:

$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad z = z$$

$$r^2 = x^2 + y^2, \qquad \tan(\theta) = \frac{y}{x}, \qquad z = z.$$

The reason that the (r, θ, z) coordinates are called cylindrical coordinates can be seen in the next activity.

Activity 3.34.

In this activity we draw graphs of some surfaces using cylindrical coordinates. To increase your intuition and facility for these surfaces, you should think about what these graphs look like before you graph them using technology.[5]

(a) Plot the graph of the cylindrical equation $r = 2$ for $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 2$. What does this surface look like? Why might this be a reason we call these coordinates *cylindrical coordinates*?

(b) Plot the graph of the cylindrical equation $\theta = 2$ for $0 \leq r \leq 2$ and $0 \leq z \leq 2$. What does this surface look like?

(c) Plot the graph of the cylindrical equation $z = 2$ for $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. What does this surface look like?

(d) Plot the graph of the cylindrical equation $z = r$ for $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. What does this surface look like?

(e) Plot the graph of the cylindrical equation $z = \theta$ for $0 \leq \theta \leq 4\pi$. What does this surface look like?

$\triangleleft$

As the name and as Activity 3.34 might imply, cylindrical coordinates are useful for describing surfaces that are cylindrical in nature.

Triple Integrals in Cylindrical Coordinates

To evaluate a triple integral $\int\int\int_S f(x, y, z)\, dV$ as an iterated integral in Cartesian coordinates, we use the fact that the volume element dV is equal to $dz\, dy\, dx$ (the volume of a box). To evaluate a triple integral in cylindrical coordinates, we have to know the volume element dV in cylindrical coordinates.

Activity 3.35.

[5] e.g., `http://www.math.uri.edu/~bkaskosz/flashmo/cylin/` – to plot $r = 2$, set r to 2, θ to s, and z to t – to plot $\theta = \pi/3$, set $\theta = \pi/3$, $r = s$, and $z = t$, for example. Thanks to Barbara Kaskosz of URI and the Flash and Math team.

A picture of a cylindrical box,

$$B = \{(r, \theta, z) : r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2, z_1 \le z \le z_2\}$$

is shown in Figure 3.43. Let $\Delta r = r_2 - r_1$, $\Delta\theta = \theta_2 - \theta_1$, and $\Delta z = z_2 - z_1$. We need to determine the volume ΔV of B in terms of Δr, $\Delta\theta$, Δz, and r, θ, and z.

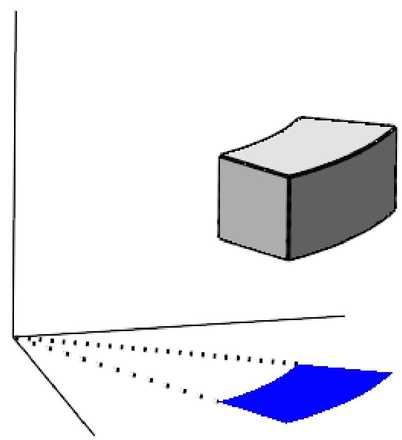

Figure 3.43: A cylindrical box.

(a) Appropriately label Δr, $\Delta\theta$, and Δz in Figure 3.43.

(b) We previously determined the area ΔA in polar coordinates (this is the area of the projection of the box onto the xy-plane, shaded blue in Figure 3.43. What is that area ΔA? Given that z is the standard z coordinate from Cartesian coordinates, how is the volume ΔV in cylindrical coordinates related to the area ΔA in polar coordinates?

$\triangleleft$

Activity 3.35 shows that the volume element dV in cylindrical coordinates is given by $dV = r\,dz\,dr\,d\theta$, which gives us the following.

The triple integral $\int\int\int_S f(x, y, z)\,dV$ becomes the *iterated integral*

$$\int\int\int_S f(r\cos(\theta), r\sin(\theta), z)\,r\,dz\,dr\,d\theta$$

in cylindrical coordinates.

Activity 3.36.

Let S be the solid bounded above by the graph of $z = x^2 + y^2$ and below by $z = 0$ on the unit circle. Set up and evaluate an iterated integral in cylindrical coordinates that gives the volume of S.

◁

Activity 3.37.

Suppose the density of the cone defined by $r = 1 - z$, with $z \geq 0$, is given by $\delta(r, \theta, z) = z$. A picture of the cone is shown in Figure 3.44, and the projection of the cone onto the xy-plane in given in Figure 3.45. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.

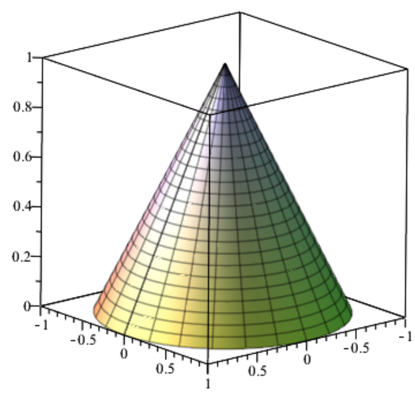

Figure 3.44: The cylindrical cone $r = 1 - z$.

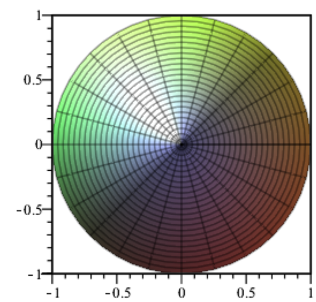

Figure 3.45: The projection into the xy-plane.

◁

Activity 3.38.

Set up an iterated integral in cylindrical coordinates to find the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the cone $z = 4 - \sqrt{x^2 + y^2}$. A picture is shown in Figure 3.46.

◁

Spherical Coordinates

There are many different coordinate systems, each of which has its own utility. We have already encountered the Cartesian and cylindrical coordinate systems in 3D, and now we introduce the spherical coordinate system. The spherical coordinates of a point in 3-space are ρ (rho), θ, and ϕ (phi), where ρ is the distance from the point to the origin, ϕ is the angle between the positive z axis

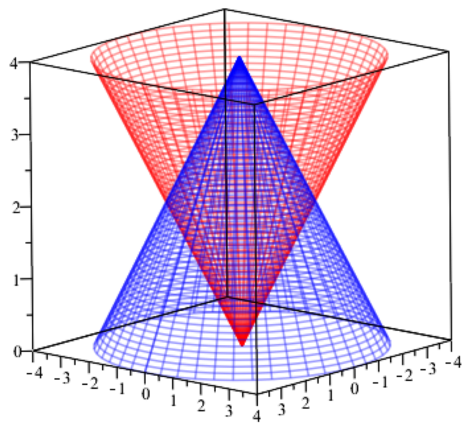

Figure 3.46: A solid bounded by the cones $z = \sqrt{x^2+y^2}$ and $z = 4 - \sqrt{x^2+y^2}$.

and the vector from the origin to the point, and θ is the same as in polar coordinates as illustrated in Figure 3.42.

In Preview Activity 3.8 we saw how to find the spherical coordinates of a point if we know the Cartesian coordinates:

$$\rho = \sqrt{x^2+y^2+z^2}, \qquad \tan(\theta) = \frac{y}{x} \text{ if } x \neq 0, \qquad \cos(\phi) = \frac{z}{\rho} \text{ if } \rho \neq 0.$$

To convert from spherical coordinates to Cartesian coordinates, Figure 3.47 shows that

$$x = \rho\sin(\phi)\cos(\theta), \qquad y = \rho\sin(\phi)\sin(\theta), \qquad z = \rho\cos(\phi).$$

Activity 3.39.

(a) Find a set of spherical coordinates of the point whose Cartesian coordinates are $(-1, 1, 1)$. Draw a picture to illustrate all of the coordinates.

(b) Find the Cartesian coordinates of the point whose spherical coordinates are $\left(2, \frac{\pi}{6}, \frac{\pi}{3}\right)$. Draw a picture to illustrate all of the coordinates.

$\triangleleft$

Activity 3.40.

In this activity we draw graphs of some surfaces using spherical coordinates. To increase your intuition and facility for these surfaces, you should think about what these graphs look like before you graph them using technology.[6]

[6] e.g., http://www.flashandmath.com/mathlets/multicalc/paramsphere/surf_graph_sphere.html – to plot $\rho = 2$, set ρ to 2, θ to s, and ϕ to t, for example. Thanks to Barbara Kaskosz of URI and the Flash and Math team.

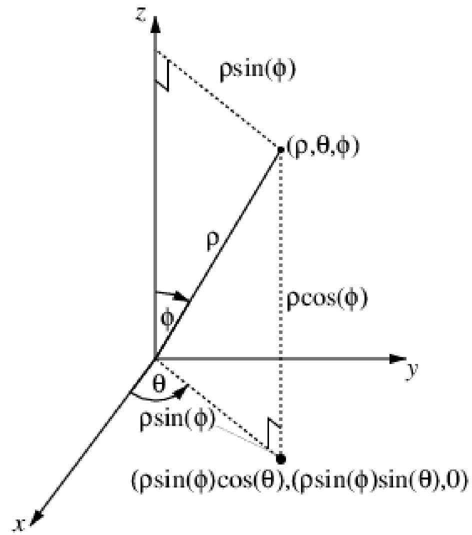

Figure 3.47: Converting from spherical to Cartesian coordinates.

(a) Plot the graph of $\rho = 1$ for $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. What does this surface look like? Why might this be a reason we call these coordinates *spherical coordinates*?

(b) Plot the graph of $\phi = \frac{\pi}{3}$ for $0 \leq \rho \leq 1$ and $0 \leq \theta \leq 2\pi$. What does this surface look like?

(c) Plot the graph of $\theta = \frac{\pi}{6}$ for $0 \leq \rho \leq 1$ and $0 \leq \phi \leq \pi$. What does this surface look like?

(d) Plot the graph of $\rho = \theta$ for $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. What does this surface look like?

◁

As the name and as Activity 3.40 might imply, spherical coordinates are useful for describing surfaces that are spherical in nature.

Triple Integrals in Spherical Coordinates

To evaluate a triple integral $\int\int\int_S f(x,y,z)\,dV$ in spherical coordinates, we have to know the volume element dV in spherical coordinates.

Activity 3.41.

To find the volume element dV in spherical coordinates, we need to understand how to determine the volume of a spherical box of the form $\rho_1 \leq \rho \leq \rho_2$ (with $\Delta\rho = \rho_2 - \rho_1$), $\phi_1 \leq \phi \leq \phi_2$ (with $\Delta\phi = \phi_2 - \phi_1$), and $\theta_1 \leq \theta \leq \theta_2$ (with $\Delta\theta = \theta_2 - \theta_1$). An illustration of such a box is given in Figure 3.48. This spherical box is a bit more complicated than the cylindrical box we encountered earlier. In this case, it is easier to approximate the volume ΔV than to compute it directly. Here we can approximate the volume ΔV of this spherical box with the volume of a

Cartesian box whose sides have the lengths of the sides of this spherical box. In other words,

$$\Delta V \approx |PS|\,|\widehat{PR}|\,|\widehat{PQ}|.$$

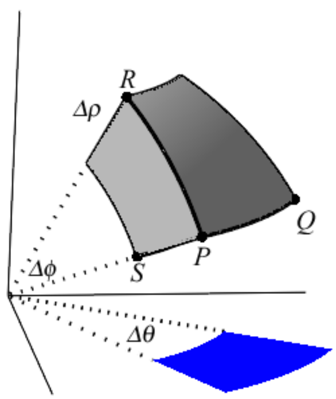

Figure 3.48: A spherical box.

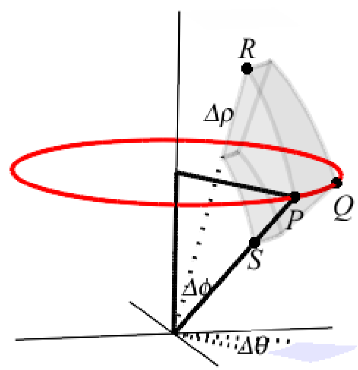

Figure 3.49: A spherical volume element.

(a) What is the length $|PS|$ in terms of $\Delta\rho$, $\Delta\theta$, $\Delta\phi$, r_1, r_2, θ_1, θ_2, ϕ_1, and ϕ_2?

(b) What is the length of the arc $\widehat{PR}$? (Hint: The arc $\widehat{PR}$ is an arc of a circle of radius ρ_2.)

(c) What is the length of the arc $\widehat{PQ}$? (Hint: The arc $\widehat{PQ}$ lies on a horizontal circle as illustrated in Figure 3.49. What is the radius of this circle?)

(d) What is our approximation of ΔV in spherical coordinates? What happens to ΔV as we let $\Delta\rho$, $\Delta\phi$ and $\Delta\theta$ go to 0?

◁

Activity 3.41 shows that $dV = \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$ in spherical coordinates, and gives us the following.

The triple integral

$$\int\int\int_S f(x,y,z)\, dV$$

can be represented as an **iterated integral in spherical coordinates** as

$$\int\int\int_S f(x,y,z)\, dV = \int\int\int_S f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

Activity 3.42.

We can use spherical coordinates to derive the formula for the volume of a sphere of radius a.

(a) How can we describe a sphere in spherical coordinates? (Hint: Refer to Activity 3.40, part (a).)

(b) Use the result of (a) to set up and evaluate an iterated integral in spherical coordinates to find the volume of a sphere of radius a.

$\triangleleft$

Activity 3.43.

Set up an iterated integral in spherical coordinates to find the mass of the solid cut from the sphere $\rho = 2$ by the cone $\phi = \frac{\pi}{4}$ if the density δ at the point (x,y,z) is $\delta(x,y,z) = \sqrt{x^2+y^2+z^2}$.[7] A picture is shown in Figure 3.50.

$\triangleleft$

Summary

In this section, we encountered the following important ideas:

- The cylindrical coordinates of a point P are (r,θ,z) where r is the distance from the origin to the projection of P onto the xy-plane, θ is the angle that the projection of P onto the xy-plane makes with the positive x-axis, and z is the vertical distance from P to the projection of P onto the xy-plane.
  - The cylindrical coordinates r, θ, and z of a point (x,y,z) in the Cartesian coordinate system are given by

    $$r^2 = x^2 + y^2, \qquad \tan(\theta) = \frac{y}{x}, \qquad z = z.$$

    The Cartesian coordinates x, y, and z of a point (r,θ,z) in the cylindrical coordinate system are given by

    $$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad z = z.$$

[7] Does this look kind of like the Death Star?

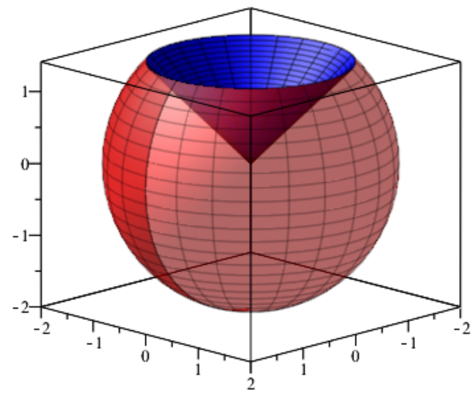

Figure 3.50: The solid cut from the sphere $\rho = 2$ by the cone $\phi = \frac{\pi}{4}$.

- The volume element dV in cylindrical coordinates is $dV = r\,dz\,dr\,d\theta$. This means that a triple integral $\int\int\int_S f(x, y, z)\,dA$ can be evaluated as the iterated integral

$$\int\int\int_S f(r\cos(\theta), r\sin(\theta), z)\,r\,dz\,dr\,d\theta.$$

- The spherical coordinates of a point P in 3-space are ρ (rho), ϕ (phi), and θ, where ρ is the distance from P to the origin, ϕ is the angle between the positive z axis and the vector from the origin to P, and θ is the angle that the projection of P onto the xy-plane makes with the positive x-axis.
  - The spherical coordinates ρ, ϕ, and θ of a point (x, y, z) in the Cartesian coordinate system are given by

$$\rho^2 = x^2 + y^2 + z^2, \qquad \tan(\theta) = \frac{y}{x}, \qquad \cos(\phi) = \frac{z}{\rho}.$$

    The Cartesian coordinates x, y, and z of a point (ρ, ϕ, θ) in the spherical coordinate system are given by

$$x = \rho\sin(\phi)\cos(\theta), \qquad y = \rho\sin(\phi)\sin(\theta), \qquad z = \rho\cos(\phi).$$

- The volume element dV in spherical coordinates is $dV = \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$. This means that a triple integral $\int\int\int_S f(x, y, z)\,dA$ can be evaluated as the iterated integral

$$\int\int\int_S f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\,\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta.$$

3.9 Change of Variables

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a change of variables?
- What is the Jacobean, and how is it related to a change of variables?

Preview Activity 3.9. In this section we will discuss change of variables in multivariable integrals. We have already seen examples of this, namely the conversion of double integrals to iterated integrals in polar coordinates. We investigate this idea in more detail in this preview activity. Consider the double integral

$$\int\int_D x^2 + y^2 \, dA, \tag{3.6}$$

where D is the upper half of the unit circle.

(a) Write the double integral (3.6) as an iterated integral in rectangular coordinates.

(b) Write the double integral (3.6) as an iterated integral in polar coordinates.

(c) When we write the double integral (3.6) as an iterated integral in polar coordinates we make a change of variables, namely

$$x = r\cos(\theta), \qquad \text{and} \qquad y = r\sin(\theta). \tag{3.7}$$

We also then have to change dA to $r\,dr\,d\theta$. What this process does is identify a polar rectangle $[r_1, r_2] \times [\theta_1, \theta_2]$ with a Cartesian rectangle, under the transformations in (3.7). The vertices of the polar rectangle are transformed into the vertices of a closed and bounded region in rectangular coordinates. To work with a more concrete example, consider the polar rectangle P with $r_1 = 1 \leq r \leq 2 = r_2$ and $\theta_1 = \frac{\pi}{6} \leq \theta \leq \frac{\pi}{4} = \theta_2$.

i. Use the transformations in (3.7) to find the rectangular vertices that correspond to the polar vertices. In other words, find the corresponding x and y coordinates for the vertices of the polar rectangle P. Label the point that corresponds to the polar vertex (r_1, θ_1) as (x_1, y_1), the point corresponding to the polar vertex (r_2, θ_1) as (x_2, y_2), the point corresponding to the polar vertex (r_1, θ_2) as (x_3, y_3), and the point corresponding to the polar vertex (r_2, θ_2) as (x_4, y_4).

ii. Draw a picture of the figure in rectangular coordinates that has the points (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) as vertices. What does your picture look like? What is the area of this region in rectangular coordinates? How does this area relate to dA?

$\bowtie$

Introduction

In single variable calculus, we encountered the idea of a change of variable in a definite integral through the method of substitution. For example, the definite integral

$$\int_0^2 2x(x^2+1)^3\,dx$$

can be transformed into the definite integral

$$\int_1^5 u^3\,du$$

with the change of variable $u = x^2 + 1$. (Note that $du = 2x\,dx$.)

This change of variable works directly because the elements of length, du and dx, are measured in the same way. In this section we will see that a change of variables in a multiple integral is a bit more complicated.

Change of Variables in Polar Coordinates

The idea behind a change of variables can be seen through Preview Activity 3.9. We saw that in a change of variables from rectangular coordinates to polar coordinates, a polar rectangle $[r_1, r_2] \times [\theta_1, \theta_2]$ gets mapped to a Cartesian rectangle under the transformation

$$x = r\cos(\theta), \quad \text{and} \quad y = r\sin(\theta).$$

The vertices (r_1, θ_1), (r_2, θ_1), (r_1, θ_2),and (r_2, θ_2) of the polar rectangle are transformed into the vertices (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4), respectively, of a closed and bounded region in rectangular coordinates. If we lived in polar coordinates, the polar rectangle P would look to us as shown in Figure 3.51. The image P' of the polar rectangle P under the transformations in (3.7) is shown in Figure 3.52. We have seen that the area of the transformed rectangle P' is given by $\frac{r_2+r_1}{2}\Delta r\Delta\theta$, and as Δr and $\Delta\theta$ go to 0 this area becomes the area element $dA = r\,dr\,d\theta$. This is the general idea of a change of variables in multiple integrals.

General Change of Coordinates

In this section we focus on double integrals, and make the connection to triple integrals in a later section. We may be able to simplify a double integral of the form

$$\int\int_D f(x,y)\,dA$$

by making a change of variable of the form

$$x = x(s,t) \quad \text{and} \quad y = y(s,t)$$

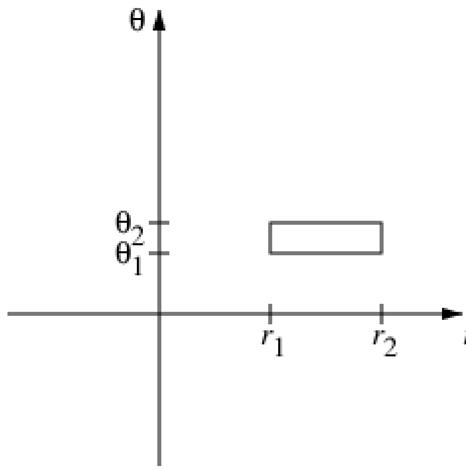

Figure 3.51: Rectangle P in the polar world.

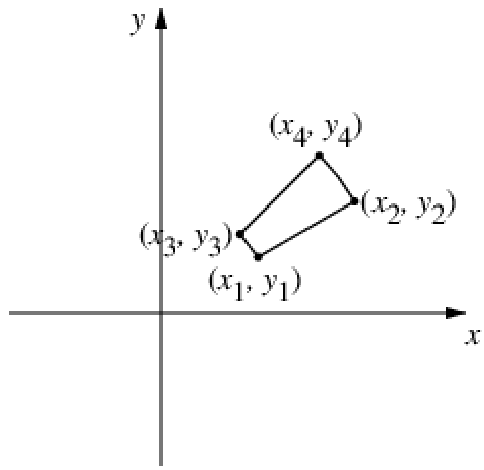

Figure 3.52: Image P' in the Cartesian world.

where x and y are functions of new variables s and t. This translates our problem from the st-plane into the xy-plane. The equations $x = x(s,t)$ and $y = y(s,t)$ convert s and t to x and y by the given formulas the *change of variable* formulas. To complete the change to the new s, t variables, we need to know our element of area dA in this new system. An example should help to illustrate the idea.

Activity 3.44.

Consider the change of variables

$$x = s + 2t \quad \text{and} \quad y = 2s + \sqrt{t}.$$

Let's see what happens to the rectangle $T = [0,1] \times [1,4]$ in the st-plane under this change of variable.

(a) Draw a picture of T in the st-plane.

(b) Find the image of the st-vertex $(0,1)$ in the xy-plane.

(c) Find the image of the st-vertex $(0,4)$ in the xy-plane.

(d) Find the image of the st-vertex $(1,1)$ in the xy-plane.

(e) Find the image of the st-vertex $(1,4)$ in the xy-plane.

(f) Draw a picture of the image T' of the st-rectangle T in the xy-plane. What does the image look like?

(g) To translate an integral with a change of variable, we need to find the area element dA for this type of translated rectangle. How would find the area of the xy-figure T'?

$\triangleleft$

Activity 3.44 presents the general idea of how a change of variables work. We partition a rectangular domain in the st system into subrectangles. Let $T = [a, b] \times [a + \Delta s, b + \Delta t]$ be one of these subrectangles. Then we transform this into a region T' in the standard xy Cartesian coordinate system. The region T' is called the *image* of T and the region T is the *pre-image* of T'. Although the sides of this xy region T' aren't necessarily straight (linear), we will approximate the element of area dA for this region with the area of the parallelogram whose sides are given by the vectors

- $\mathbf{v}$ from $(x(a, b), y(a, b))$ to $(x(a + \Delta s, b), y(a + \Delta s, b))$
- $\mathbf{w}$ from $(x(a, b), y(a, b))$ to $(x(a, b + \Delta t), y(a, b + \Delta t))$.

An example of a translated image T' of T in the xy-plane is shown in Figure 3.53, using the polar coordinate transformations.

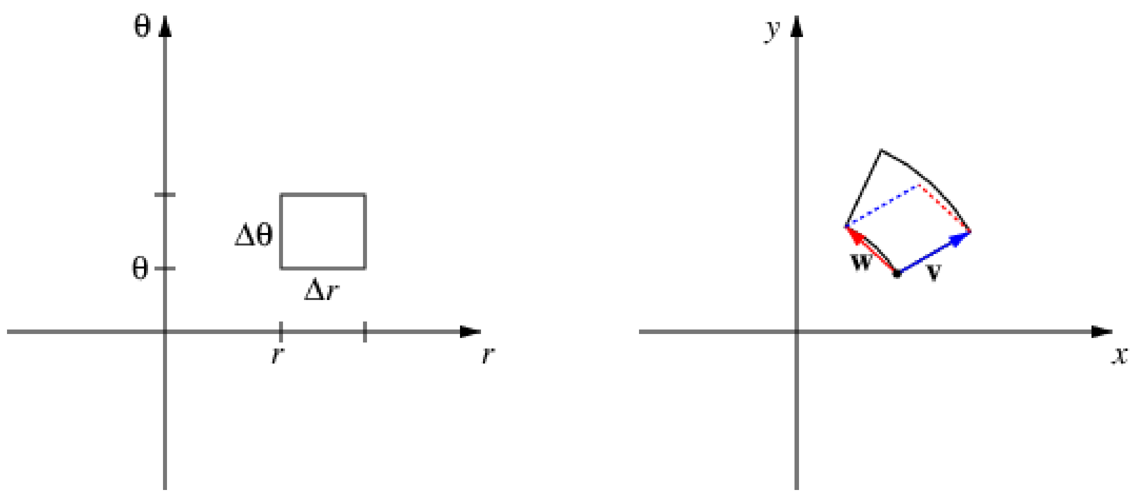

Figure 3.53: Approximating an area in polar coordinates.

The components of the vectors $\mathbf{v}$ and $\mathbf{w}$ are

$$\mathbf{v} = \langle x(a + \Delta s, b) - x(a, b), y(a + \Delta s, b) - y(a, b), 0\rangle$$
$$\mathbf{w} = \langle x(a, b + \Delta t) - x(a, b), y(a, b + \Delta s) - y(a, b), 0\rangle \,.$$

We can rewrite $\mathbf{v}$ and $\mathbf{w}$ as

$$\mathbf{v} = \left\langle \frac{x(a + \Delta s, b) - x(a, b)}{\Delta s}, \frac{y(a + \Delta s, b) - y(a, b)}{\Delta s}, 0 \right\rangle \Delta s$$
$$\mathbf{w} = \left\langle \frac{x(a, b + \Delta t) - x(a, b)}{\Delta t}, \frac{y(a, b + \Delta s) - y(a, b)}{\Delta t}, 0 \right\rangle \Delta t.$$

For small Δs and Δt we have

$$\mathbf{v} \approx \left\langle \frac{\partial x}{\partial s}(a,b), \frac{\partial y}{\partial s}(a,b), 0 \right\rangle \Delta s \quad \text{and} \quad \mathbf{w} \approx \left\langle \frac{\partial x}{\partial t}(a,b), \frac{\partial y}{\partial t}(a,b), 0 \right\rangle \Delta t.$$

Recall that the area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$ is $|\mathbf{v} \times \mathbf{w}|$. Now

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &\approx \left\langle \frac{\partial x}{\partial s}(a,b), \frac{\partial y}{\partial s}(a,b), 0 \right\rangle \Delta s \times \left\langle \frac{\partial x}{\partial t}(a,b), \frac{\partial y}{\partial t}(a,b), 0 \right\rangle \Delta t \\ &= \left\langle 0, 0, \frac{\partial x}{\partial s}(a,b)\frac{\partial y}{\partial t}(a,b) - \frac{\partial x}{\partial t}(a,b)\frac{\partial y}{\partial s}(a,b) \right\rangle \Delta s \, \Delta t. \end{aligned}$$

So

$$\begin{aligned} |\mathbf{v} \times \mathbf{w}| &\approx \left| \left\langle 0, 0, \frac{\partial x}{\partial s}(a,b)\frac{\partial y}{\partial t}(a,b) - \frac{\partial x}{\partial t}(a,b)\frac{\partial y}{\partial s}(a,b) \right\rangle \Delta s \, \Delta t \right| \\ &= \left| \frac{\partial x}{\partial s}(a,b)\frac{\partial y}{\partial t}(a,b) - \frac{\partial x}{\partial t}(a,b)\frac{\partial y}{\partial s}(a,b) \right| \Delta s \, \Delta t. \end{aligned}$$

Therefore, as the number of subdivisions goes to infinity in each direction we have

$$dA = \left| \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \right| ds \, dt.$$

This gives us the change of variable formula in a double integral

$$\int\int_T f(x,y)\, dA = \int\int_R f(x,y)\, dy\, dx = \int\int_{T'} f(x(s,t), y(s,t)) \left| \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \right| ds\, dt.$$

The quantity

$$\left| \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \right|$$

is called the *Jacobian*. There is also a useful shorthand notation for the Jacobean:

$$\frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}.$$ [8]

Activity 3.45.

[8] If you are familiar with determinants of matrices, we can can also represent the Jacobian as the determinant of a 2×2 matrix

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}.$$

Find the Jacobian when changing from rectangular to polar coordinates.

◁

The change of variable formula that we have just derived is the following.

Change of Variables in a Double Integral. Suppose a change of variables $x = x(s,t)$ and $y = y(s,t)$ transforms a closed and bounded region R in the st-plane into a closed and bounded region R' in the xy-plane. Under nice[9] conditions we have

$$\int\int_R f(x,y)\,dA = \int\int_{R'} f(x(s,t), y(s,t)) \left| \frac{\partial(x,y)}{\partial(s,t)} \right| \,ds\,dt.$$

A natural question to ask is, given a particular double integral, how can we find a useful change of variables to make? There are two general factors to consider: if the integrand is particularly difficult, we might choose a change of variables (possibly indicated by the form of the integrand) to make the integrand easier; or we might choose a change of variables that transforms the region of integration into one that has a simpler form. These ideas are illustrated in the next activities.

Activity 3.46.

Consider the problem of finding the area of the region D defined by the ellipse $x^2 + \frac{y^2}{4} = 1$. Here we will make a change of variables so that the pre-image of the domain is a circle.

(a) Let $x(s,t) = s$ and $y(s,t) = 2t$. Show that the pre-image of the the ellipse in the st-plane is the circle $s^2 + t^2 = 1$.

(b) Recall that the area of the ellipse D is equal to the double integral $\int\int_D 1\,dA$. Show that

$$\int\int_D 1\,dA = \int\int_{D'} 2\,ds\,dt$$

where D' is the disk bounded by the circle $s^2 + t^2 = 1$.

(c) Without doing any integration, explain why the area of the ellipse D is 2π.

◁

Activity 3.47.

Let D be the region in the xy-plane bounded by the lines $y = 0$, $x = 0$, and $x + y = 1$. We will evaluate the double integral

$$\int\int_D \sqrt{x+y}(x-y)^2\,dA \tag{3.8}$$

with a change of variables.

(a) Sketch the xy domain D.

(b) We would like to make the integrand a bit easier to integrate. Let $s = x+y$ and $t = x-y$. Explain why this might make integration easier.

(c) Solve the equations $s = x + y$ and $t = x - y$ for x and y. This will give us the change of variables we want.

(d) To make this change of variables, we will need to know the st-region S that corresponds to the xy-region D.

i. What st equation corresponds to the xy equation $x + y = 1$?
ii. What st equation corresponds to the xy equation $x = 0$?
iii. What st equation corresponds to the xy equation $y = 0$?
iv. Sketch the st region S that corresponds to the xy domain D.

(e) Make the change of variables indicated by $s = x+y$ and $t = x-y$ in the double integral (**??**) and set up an iterated integral in st variables whose value is the double integral. Evaluate the iterated integral.

$\triangleleft$

Change of Variables in a Triple Integral

The same types of arguments can be used to show that a change of variables $x = x(s,t,u)$, $y = y(s,t,u)$, and $z = z(s,t,u)$ in a triple integral

$$\int\int\int_S f(x,y,z)\,dV$$

leads to the integral

$$\int\int\int_S f(x,y,z)\,dV = \int\int\int_{S'} f(x(s,t,u),y(s,t,u),z(s,t,u)) \left|\frac{\partial(x,y,z)}{\partial(s,t,u)}\right| ds\,dt\,du,$$

where

$$\frac{\partial(x,y,z)}{\partial(s,t,u)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u} \end{vmatrix}.$$

In expanded form,

$$\frac{\partial(x,y,z)}{\partial(s,t,u)} = \frac{\partial x}{\partial s}\left[\frac{\partial y}{\partial t}\frac{\partial z}{\partial u} - \frac{\partial y}{\partial u}\frac{\partial z}{\partial t}\right] - \frac{\partial x}{\partial t}\left[\frac{\partial y}{\partial s}\frac{\partial z}{\partial u} - \frac{\partial y}{\partial u}\frac{\partial z}{\partial s}\right] + \frac{\partial x}{\partial u}\left[\frac{\partial y}{\partial s}\frac{\partial z}{\partial t} - \frac{\partial y}{\partial t}\frac{\partial z}{\partial s}\right].$$

The expression $\frac{\partial(x,y,z)}{\partial(s,t,u)}$ is again called the Jacobian.

Activity 3.48.

Verify that the change of variables

$$x(\rho, \theta, \phi) = \rho \sin(\phi) \cos(\theta), \quad y(\rho, \theta, \phi) = \rho \sin(\phi) \sin(\theta), \quad z(\rho, \theta, \phi) = \rho \cos(\phi)$$

from spherical coordinates to rectangular coordinates has Jacobian $\rho^2 \sin(\phi)$.

$\triangleleft$

Summary

In this section, we encountered the following important ideas:

- If an integral is described in terms of one set of variables, we may write that set of variables in terms of another set of the same number of variables. If the new variables are chosen appropriately, the transformed integral may be easier to evaluate.
- The Jacobian is a scalar function that relates the area or volume element in one coordinate system to the corresponding element in a new system determined by a change of variables.

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